2/8/21

Math 341/650

Lec 03

$\hat{\theta}_{MLE} = S(x_1, \dots x_n) \overset{F: \text{iid Bern}(\theta)}{=} \bar{x} \in \mathbb{R}$ e.g 0.1984

some function of the data $x_1 \dots x_n$

By Property ②

$\hat{\theta}_{MLE} = S(X_1, X_2, \dots X_n) = \bar{X} \overset{\cdot}{\sim} N(\theta, SE[\hat{\theta}_{MLE}]^2)$

the same function except of the rv's

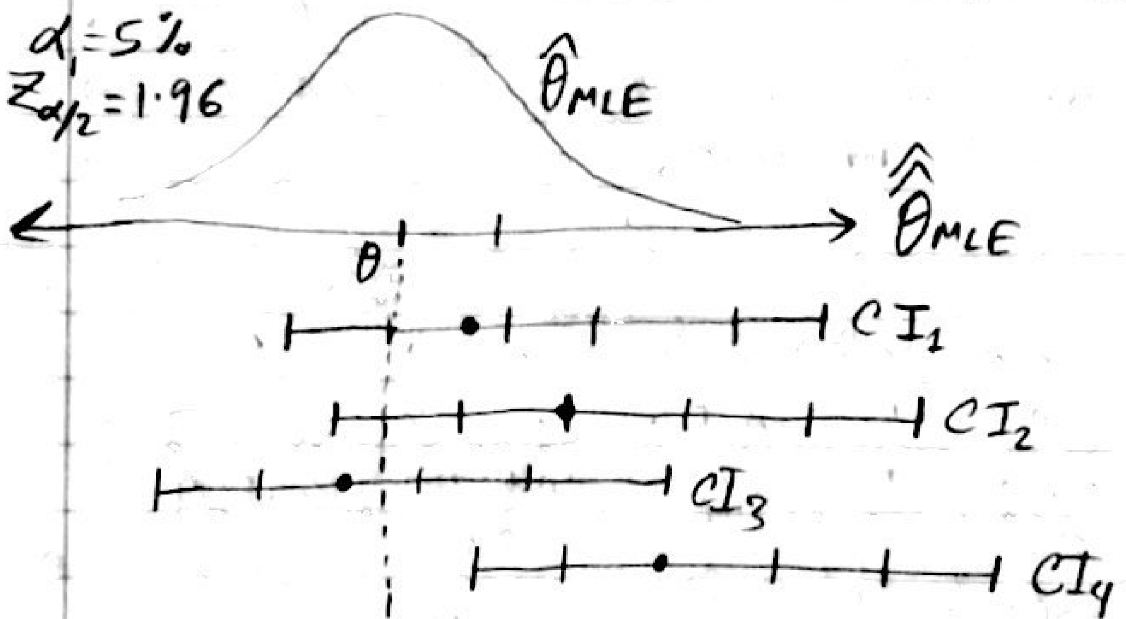
a rv, not a value

rv's have distributions

we use this normally to create the confidence interval (CI). why do CI's work?

$\alpha = 5\%$

$z_{\alpha/2} = 1.96$



CI's have 95% probability (before they're realized) of capturing (i.e. including) the real value of θ.

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Inference gole #3: "hypothesis testing" also called "theory testing"

Consider a situation where "I" am trying to convience "you" of something.

Scenario I: "I" declare aliens & UFO's exist. If they don't exist "you" need to provide "me" "sufficient" evidence that Aliens & UFO's don't exist.

Scenario II: "I" will assume for the moment that Aliens & UFO's don't exist, H_0 and I will provide "you" suffieint evidence to the point that "you're" convinced that Aliens & UFO's exists. H_a

H_0

H_a

Scenario II is more convincing, and how science generally works. The theory "I" am trying to demonstrate is called the "alternative hypothesis" (Ha) since it's alternative to maybe buisness-as-usual. In Senario II, u assume the opositive of the theory which is called the "null hypothesis" (H_0). This is the "hypothesis Testing" procedure.

In our context, theories are phrases as mathematical statments about θ, the unknown parameter. we will Study three types of Ha's:

Constant value

$H_a: \theta \neq \theta_0$ vs. $H_0: \theta = \theta_0$ two sided test/two tailed test

$H_a: \theta > \theta_0$ vs. $H_0: \theta \leq \theta_0$ right sided test/right-tailed test

$H_a: \theta < \theta^0$ vs. $H_0: \theta \geq \theta_0$ left sided test/left-tailed test.

There are two outcomes of a hypothesis test.
(A) You were not shown sufficient evidence of H_a. Thus you "fail to reject H_0" or "retain H_0".
or (B) You were indeed shown sufficient evidence of H_a. Thus you "reject H_0", or "accept H_a".

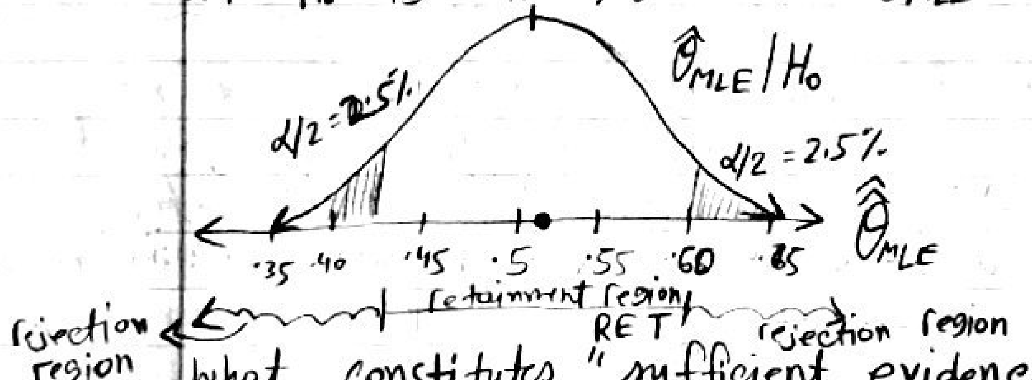
Imagine you're flipping a coin n times, & counting # of hats; F: iid Bern(θ). You want to prove the coin in unfair.

$$H_a : \theta \neq 0.5$$

$$H_0 : \theta = 0.5$$

If H_0 is true, $\theta = 0.5 \Rightarrow \hat{\theta}_{MLE} \sim N(\theta, SE[\hat{\theta}_{MLE}]^2)$

$\hat{\theta}_{MLE} = \bar{X}$, $\theta = 0.5$, $SE[\hat{\theta}_{MLE}] = \sqrt{\frac{0.5(1-0.5)}{n}} = 0.05$



What constitutes "sufficient evidence". It's a Probability of rejecting when H_0 is true. (α) Everyone is diffrent

If $\alpha = 5\%$ in a 2-tailed test, we put ½ the Probability in each tail. 5% is the most Common Scientific Standard. Thus, we retain H_0 for the most non-weird 95% of the $\hat{\theta}$ and reject H_0 for the 5% most weird $\hat{\theta}$.

$\xrightarrow{}$ Retainment region

$$RET = \left[\theta_0 + z_{\alpha/2}\, SE[\hat{\theta}_{MLE}]\right] = \left[0.5 \pm 1.96\sqrt{\frac{0.5(1-0.5)}{100}}\right]$$

0.05 above α

i.e. if $\bar{X} = \frac{61}{100} = 0.61$, $\alpha = 5\%$ $\quad = [0.402, 0.590]$

$\hat{\theta}_{MLE} = 0.61 \notin RET \Rightarrow$ Reject H_0 & conclude the coin is unfair.

e.g. if $\bar{X} = \frac{59}{100} = 0.59$

$\hat{\theta}_{MLE} = 0.59 \notin RET \Rightarrow$ fail to reject H_0 and conclude there's not enough evidence of coin being unfair.

We've covered the "frequentist" approach to statistical inference. But there are problems with it ......

① $\mathcal{F} = $ iid $Bern(\theta)$, $x = \langle 0, 0, 0\rangle$

$\hat{\theta} = \bar{X} = 0$, is that a good point estimate?

NO! You shouldn't be able to say something is absolutly impossible after $n = 3$ trials.

$$CI_{\theta,1-\alpha} = \left[\overset{0}{\theta} \pm 1.96\sqrt{\frac{0(1-0)}{3}}\right] = \{0\}$$

Is this a good confidence set? NO. This isn't a good set of "reasonable values".

② What if you have prior knowledge that Θ was ristricted to e.g. $[0.1, 0.2]$ and not the full $(0,1)$. You can't "enter that into" your inference.

③ Consider the frequentist interpretation of a CI:

① Before you do the experiment, you have a 95% probability of capturing θ. But that doesn't tell you anything about after your experiment. After your experiment you have an interval e.g. [0.37, 0.45] and you <u>can't</u> say:

$$\underbrace{P(\theta \in [0.37, 0.45]}_{\text{No Randomess!!}} = 0.95$$

② 95% of CI's will cover θ. But again, I only make one!!! So this interpretion doesn't help me!

In conclusion, any specific CI means <u>Nothing!</u>

④ Hypothesis tests result in a binary outcome: either you reject H_0 or you fail to reject H_0. What if you want to know?

$$P(H_0|X) \quad \text{or} \quad P(H_a|X) \; ?$$

You cannot!! one thing you can do is:

$$\text{P-val} := P(\text{seeing } \hat{\hat{\theta}} \text{ or more extreme} \,|\, H_0) \neq P(H_0|X)$$

⑤ $\mathcal{F}$: iid Bern (θ), $X = \langle 0,1,0 \rangle \Rightarrow \hat{\hat{\theta}}_{MLE} = 0.33$

$$CI_{\theta,95\%} = \left[0.33 \pm 1.96\sqrt{\frac{0.33(0.67)}{3}}\right] = [-.20, 0.87]$$

Is this a reasonable confidence set? No, it's outside of the legal parameter space which is $(0,1)$.

$\alpha = 0.05 = 5\%$,

$$H_0 : \theta = 0.5 \Rightarrow RET = \left[0.5 \pm 1.96\sqrt{\frac{0.5(0.5)}{3}}\right]$$
$$= [-0.066, 1.066]$$

Is this a good hypothesis test? NO b/c you NEVER Reject!!

The problem in ⑤ is b/c the asymptotic normality of the MLE doesn't "kick in" until n is large (MLE property 2 is not true yet).

we will solve all these problems with Bayesian Inference. Unfortunately, we will get other problems instead. It's a trade off and a personal decision.