lec9
March 3

$n = 100$, $x = 44$, $P(\theta) = Beta(1,1)$ $= P(\theta|x) = Beta(44, 58)$

$\alpha_0 = 5\%$, $\delta = 1\%$

$H_0 : \theta \in [.49, .51]$

$P_{val} = P(H_0|X) = \int_{.49}^{.51} d\theta$ = pbeta(.51, 44, 58) − pbeta(.49, 44, 58)

= 0.06 = Retain H_0.

$F_{\theta|X}(\theta_0 + \delta) - F_{\theta|X}(\theta_0 - \delta)$

Last topic before the midterm

F: $Bin(n, \theta)$ with n fixed, $P(\theta) = Beta(\alpha, \beta) \Rightarrow P(\theta|X) = Beta(\alpha+n, \beta+n-x)$

Laplace $P(\theta) = Beta(1,1) \Rightarrow n_0 = 2$ pseudo trial $X_0 = 1$

pseudo sucess.

means fake

Laplace uniform prior is "flat" in an effort to be "objective i.e let the data speak itself and not to be "subjective i.e allow your personal biases to be part of your inferential conclusion.

Can we be more objective? Can we create a prior that has no part in the inference conclusion? This would mean $n_0 = 0$. How about alpha = β = 0.

$P(\theta) = Beta(0,0)$

↑ there is a problem w/ this. The parameter space for the beta is $\alpha > 0$ and $\beta > 0$. If $\alpha = \beta = 0$, this is not a PDF since it's integral over the support diverges. This makes it an "improper prior" since it is not true random variable. Do we care! Churning through the math we get the posterior. $P(\theta|x) = Beta(x, n-x)$

This posterior is proper, as long as $x < n$ and $x > 0$ which means you need to have at least one success and at least one failure in your data. If it's proper, you have bayesian influence (point estimates, CR's, p-value...). However, you always have $\hat{\theta}_{MMSE}$.

$\hat{\theta}_{MMSE} = \frac{x}{n} = \hat{\theta}_{MLE}$. Also $\underset{\uparrow \text{ prior}}{p} = 0$ (no shrinking).

I belive this prior was first introduced by Haldane in 1932 so we'll call it the "Haldane prior".

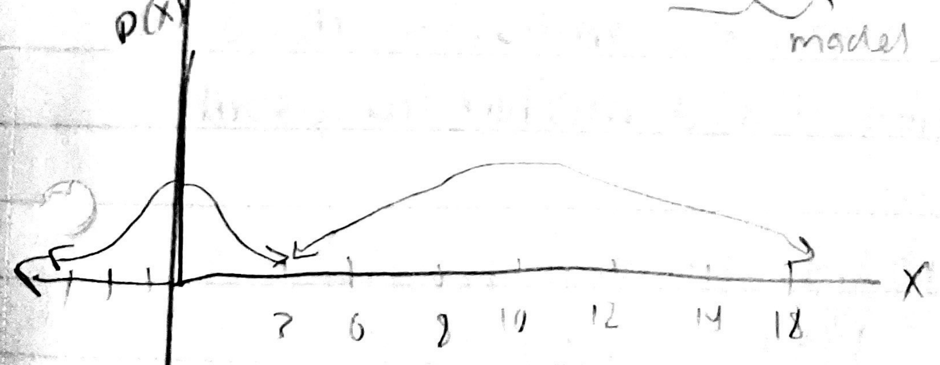
Midterm 1 ↑ problem 2 midterm 1

Midterm 2 ↓

Back to probability... We will introduce mixture/compound distribution ex.

$$X \sim \begin{cases} N(0,1^2) & \text{wp } \frac{1}{2} \\ N(10,2^2) & \text{wp } \frac{1}{2} \end{cases}$$

model — mix w/ some mixture



$$P(X) = \int_{\Theta} P(x, \vec{\theta})\, d\vec{\theta} \quad \text{if } \theta \text{ is continous}$$

or "

$$\sum_{\theta \in \Theta} P(x, \vec{\theta}) \quad \text{if } \vec{\theta} \text{ is discrete}$$

$$= \sum_{\vec{\theta} \in \Theta} P(x \mid \vec{\theta}) P(\vec{\theta})$$

$$\boxed{= \frac{1}{\sqrt{2\pi \cdot 1^2}} e^{-\frac{1}{2 \cdot 1^2}(x-0)^2} \cdot (0.5) + \frac{1}{\sqrt{2\pi \cdot 2^2}} e^{-\frac{1}{2 \cdot 2^2}(x-10)^2} (.05)}$$

$$X \sim \begin{cases} \text{Bin}(10, 0.1) & \text{mp } \frac{1}{4} \\ \text{Bin}(10, 0.8) & \text{mp } \frac{3}{4} \end{cases}$$

or, US $P(X) = \binom{10}{x} (0.1^x) (.9)^{10-x} \cdot \frac{1}{4} + \binom{10}{x} (0.8)^x (0.2)^{10-x} \cdot \frac{3}{4}$

Have we seen P(X) before that's the result of margining making P(X) a mixture/compound distribution? yes...

Beta

$$P(\theta \mid x) = \frac{P(x \mid \theta) P(\theta)}{P(x)} = \int_{\Theta} P(x \mid \theta) P(\theta)\, d\theta$$

F : Bin(n, θ) n fixed , $P(\theta) = \text{Beta}(\alpha, \beta)$

$$P(x) = \int_0^1 \binom{n}{x} (\theta)^x (1-\theta)^{n-x} \frac{1}{\beta(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \, d\theta$$

Cheat Sheet

$$= \frac{\binom{n}{x}}{\beta(\alpha,\beta)} \underbrace{\int_0^1 \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x} \, d\theta}_{\text{Beta function}}$$

$$= \frac{\binom{n}{x}}{\beta(\alpha,\beta)} \beta(\alpha+x, \beta+n-x) = \text{BetaBinomial}(n, \alpha, \beta)$$

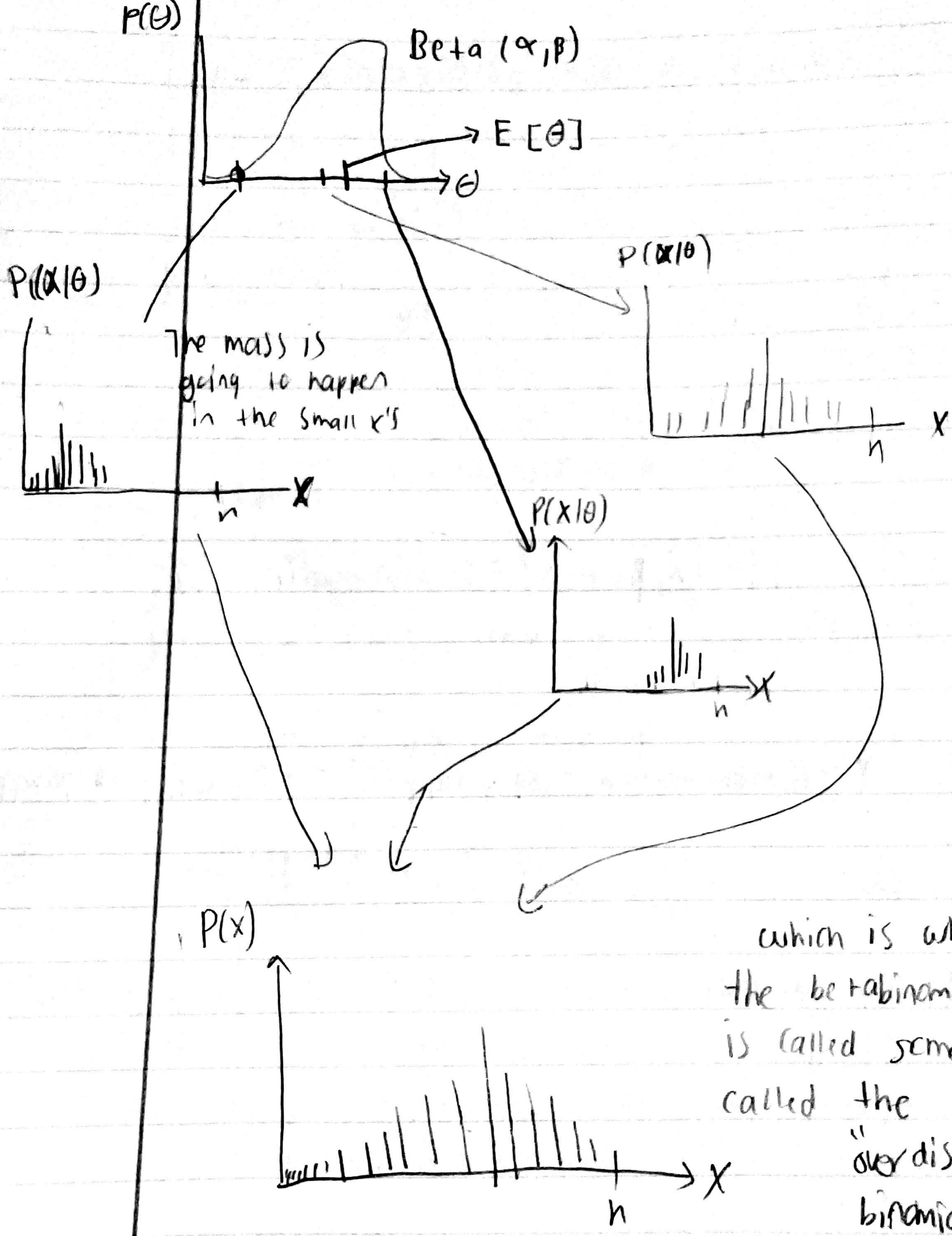
Test question

$Y \sim \text{BetaBinomial}(n, \alpha, \beta)$, $\text{supp}[Y] = \{0, 1, .., n\}, n \in \mathbb{N}, \alpha > 0, \beta > 0$

$E[Y] = \ldots = n\frac{\alpha}{\alpha+\beta}$, $\text{Var}[Y] = \ldots n \frac{\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Since the beta function is not aviable in closed form, the PMF/CFF are not available in closed form. To compute, you need a computer. Here's the notation we'll use in this class (the R notation)

$P(Y=y) = \text{dbetabinom}(y, n, \alpha, \beta)$
$P(Y \le y) = \text{pbetabinom}(y, n, \alpha, \beta)$ $\longrightarrow$



The mass is going to happen in the small x's

which is why the betabinomial rv is called sometimes called the "overdispersed binomial"

$$E[\theta] = n \frac{\alpha}{\alpha+\beta}$$

let $\theta = \frac{\alpha}{\alpha+\beta} \Rightarrow \theta\alpha + \theta\beta = \alpha = (\theta-1)\alpha = -\theta\beta = \beta = \alpha\frac{1-\theta}{\theta}$

$\downarrow$

$E[X] = n\theta$ an intuitive formula for the betabinomial expectation since it is the same as binomial expectation

let $\alpha \to \infty$ but keep $\theta = \frac{\alpha}{\alpha+\beta}$ constant

$$\lim_{\alpha\to\infty} Var[X] = \lim_{\alpha\to\infty} n \frac{\cancel{\alpha}(\alpha\frac{1-\theta}{\theta})(\alpha+\alpha(\frac{1-\theta}{\theta})+n}{(\cancel{\alpha}+\cancel{\alpha}\frac{1-\theta}{\theta})^2(\alpha+\alpha\frac{1-\theta}{\theta}+1)}$$

$$= n \lim \frac{\frac{1-\theta}{\theta}}{(1+\frac{1-\theta}{\theta})^2} \cdot \lim \cancel{\frac{\alpha+\alpha(\frac{1-\theta}{\theta})+n}{\alpha+\alpha(\frac{1-\theta}{\theta})+1}} = n \lim_{\alpha\to\infty} \frac{\theta(1-\theta)}{\theta+(1-\theta)}^2$$

$= n\theta(1-\theta)$ which is the same variance as the rv(Bin)

Review Midterm 1

H.W #2 (3E)

$$E(\theta) = 0.8 = \frac{\alpha}{\alpha+\beta}$$

$$SE[\theta] = 0.02 = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}} = \frac{1}{\alpha+\beta}\sqrt{\frac{\alpha\beta}{\alpha+\beta+1}}$$

2019 midterm

7b

$P(\theta \in HDR_{\theta, 1-\alpha} | x) = 1 - \alpha$

$P(\theta \in CR_{\theta, 1-\alpha} | x) = 1 - \alpha$

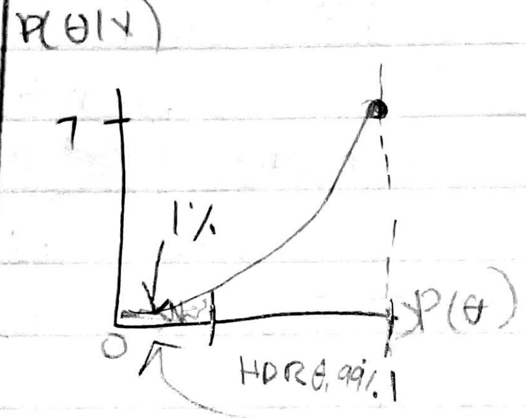
$\int_{\alpha}^{0}$

H.W 3

2j

$x = 6, \ n = 6 \qquad P(\theta) = Beta(1,1)$

$P(\theta | x) = Beta(7,1)$



$HDR_{\theta, 99\%} = [qbeta(.01, 7, 1)]$

$\gamma = pbeta(qbeta(\gamma, 0.893, 1.87))$

$= 0.893, 1.876$

$$Var[X] = n \frac{\alpha\beta}{(\alpha+\beta)^2} \frac{\alpha+\beta+n}{\alpha+\beta+1} = \underbrace{n\,\theta(1-\theta)}_{\text{var Binomial}} \underbrace{\frac{\alpha+\beta+n}{\alpha+\beta+1}}_{\substack{\text{over dispersion} \\ \in(1,n)}}$$

Thus the betabinomial is a maie more flexible model