Feb 22.

lec 6

①

$$\hat{\theta}_{MMAE} = \underset{\hat{\theta} \in \Theta}{\text{argmax}} \{ F$$

$$= \frac{12}{20} = 0.6$$

minimum mean absolute error

$$\hat{\theta}_{MMAE} = \text{med}[\theta | X] = a \quad \text{such that} \quad \int_{-\infty}^{a} P(\theta | X) \, d\theta = \frac{1}{2}$$

Using our model: iid bern($\theta$) and data $X = \langle 0,1,1 \rangle$, we can compute the MMAE Bayesian point estimate:

$$\int_0^a 12\theta^2(1-\theta)\,d\theta = 12\left[\frac{\theta^2}{2} - \frac{\theta^3}{3}\right] = 12\left[\frac{\theta^3}{3} - \frac{\theta^4}{4}\right]$$

$$= 12\left[\frac{a^3}{3} - \frac{a^4}{4}\right] \overset{\text{set}}{=} \frac{1}{2} \rightarrow$$

This integral in the denominator is a special intergal and is known as the "beta function":

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

The beta function has no closed form solution but can be calculated to arbitary precision using a scientific calculator.

$$= \frac{1}{\beta(\sum x_i+1, n-\sum x_i+1)} \cdot \theta^{\sum x_i+1-1}(1-\theta)^{n-\sum x_i+1-1} = $$

$$\downarrow$$

$$\text{Beta}(\sum x_i+1, n-\sum x_i+1)$$

We just derived that the posterior for the iid bernoulli likelihood is a beta distribution. Let's go back to probability class and examine the beta distribution...

$$Y \sim \text{Beta}(\alpha,\beta) \overset{pdf}{:=} \frac{1}{\beta(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} = p(y)$$

$$\text{Supp}[Y] = (0,1) \quad \int_0^1 \frac{1}{\beta(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy$$

$$= \frac{1}{\beta(\alpha,\beta)} \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy = 1$$

what does $\alpha \in$ live in and $\beta \in$ live in?

$\alpha > 0, \beta > 0$

$\alpha = 0, \beta = 1 \quad = \int_0^1 \frac{1}{y} dy = \infty$

$$E[Y] = \int_0^1 y\, p(y)\, dy = \int_0^1 y \frac{1}{\beta(\alpha,\beta)} y^{\alpha-1} (1-y)^{\beta-1} dy$$

$$= \frac{1}{\beta(\alpha,\beta)} \int_0^1 y^{\alpha+1-1} (1-y)^{\beta-1} dy$$

$$= \frac{\beta(\alpha+1,\beta)}{\beta(\alpha,\beta)}$$

To simply this, we need the gamma function:

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad \alpha > 0$$

Fact #1 $:= \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Fact #2 $:= \beta(\alpha,\beta) = \boxed{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$

$$= \frac{\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\beta(\alpha,\beta)} = \frac{\frac{\alpha\cancel{\Gamma(\alpha)\Gamma(\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}}{\cancel{\beta(\alpha,\beta)}}$$

$$= \frac{\alpha}{(\alpha+\beta)}$$

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Var [Y] = on hw.

$$\text{Mode}[Y] = \underset{y \in (0,1)}{\text{argmax}} \left\{ \frac{1}{B(\alpha,\beta)} y^{\alpha-1} (1-y)^{\beta-1} \right\}$$

this is support

$$= \text{argmax} \{ (\alpha-1)\ln(y) + (\beta-1)\ln(1-y) \}$$

derivate $\Rightarrow$ $\frac{\alpha-1}{y} - \frac{\beta-1}{1-y} \overset{set}{=} 0 = y_* = \frac{\alpha-1}{\alpha+\beta-}$

If we take the second derivate to check if it's negative, we find it's only negative if both alpha and beta are greater than 1.

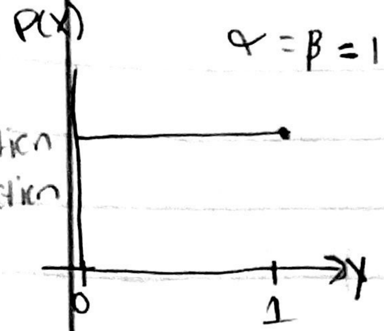
Med [Y] has no closed form expression and thus must be done with a computer. We will denote the answer to this using notation from the R programming

Let's take a look at the shapes of the beta distribution

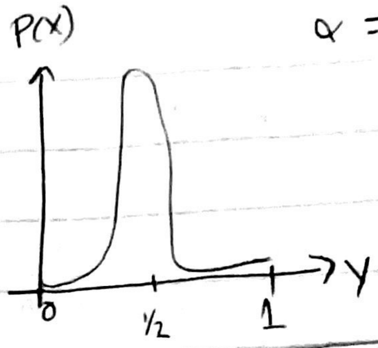
$\longrightarrow$

① $\alpha = \beta = 1$

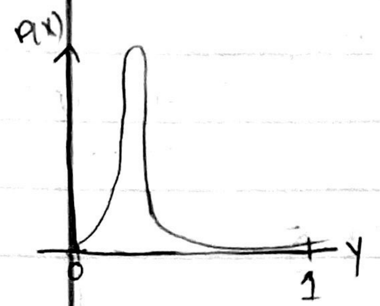
Uniform distribution function



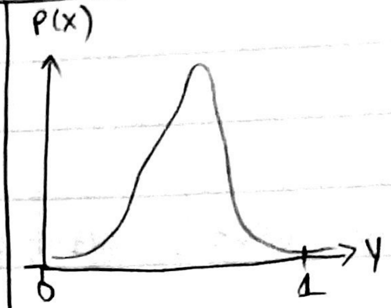

② $\alpha = \beta = 100$



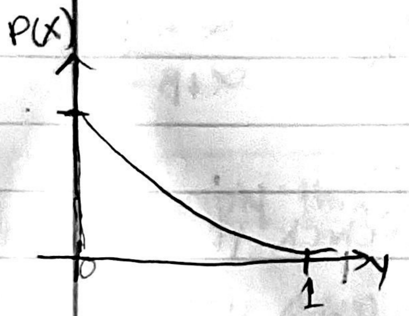

③ $\alpha = 10$, $\beta = 90$



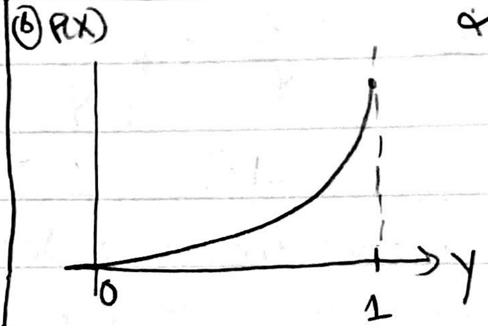

④ $\alpha = 10$, $\beta = 10$



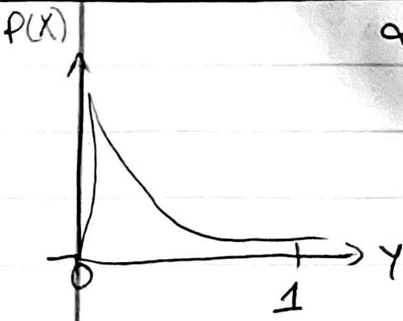

⑤ $\alpha = 1, \beta = 4$



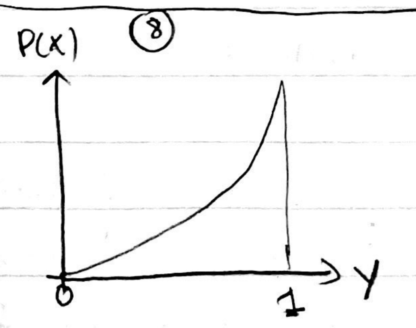

⑥ $\alpha = 4, \beta = 1$



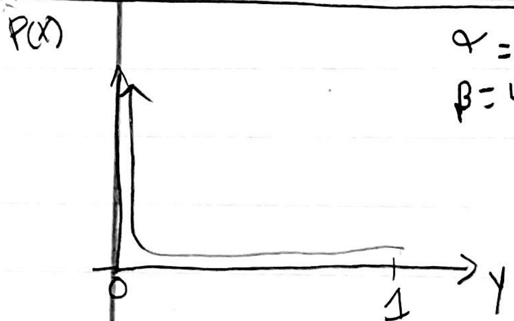

⑦ $\alpha = 1.001$, $\beta = 4$



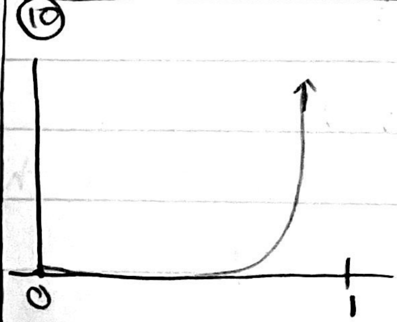

⑧ $\alpha = 4, \beta = 1.001$

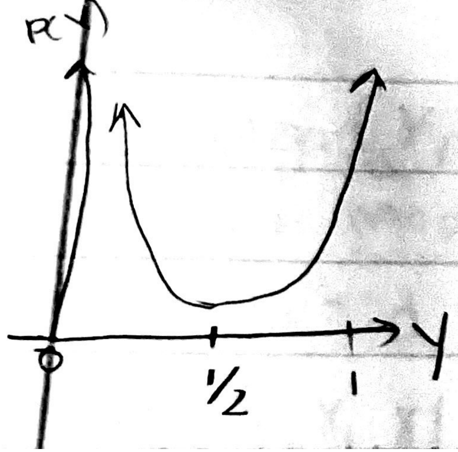



⑨ $\alpha = .999$, $\beta = 4$




⑩ $\alpha = 4, \beta = .999$






$\alpha = \beta = \frac{1}{2}$

← This is called Arcsin