Lecture 13 3/28/16 Math 390.03

we found the post. pred. distr. was a convolution

$$\theta | X, \sigma^2 \sim N\left(\underbrace{\frac{\frac{\bar{x}n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\theta_p}, \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\sigma_p^2}\right)$$

variance known beforehand

If $P(\theta) \propto 1$, $\theta_p = \bar{x}$, $\sigma_p^2 = \frac{\sigma^2}{n}$

overdispersed normal is itself normal

$$X^* | X, \sigma^2 \sim N(\theta_p, \sigma_p^2 + \sigma^2)$$

What about?

$\sigma^2 | X, \theta$ → now inference for variance / SE considering mean is known

same lik always

$$P(\sigma^2 | x, \theta) \propto P(x | \sigma^2, \theta)\, P(\sigma^2 | \theta)$$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2}\right) P(\sigma^2 | \theta)$$

$$\propto \frac{1}{\sqrt{\sigma^2}^{\,n}} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}\, P(\sigma^2 | \theta)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}\, P(\sigma^2 | \theta)$$

$$= (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}}\, P(\sigma^2 | \theta)$$

let $\hat{\sigma}^2 := \frac{1}{n}\sum(x_i - \theta)^2$

MLE for σ^2 if θ known (and consistent estimator)

What does it look like? Pattern matching

$$(\sigma^2)^a e^{-\frac{b}{\sigma^2}}$$

Recall

$$Y \sim \text{Gamma}(\alpha,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \Rightarrow \frac{1}{Y} \sim \text{InvGamma}(\alpha,\beta)$$

$$W = \frac{1}{Y} \sim ? \qquad W = t(Y) = \frac{1}{Y} \Rightarrow t^{-1}(w) = \frac{1}{w}$$

$$f_W(w) = f_Y(t^{-1}(w)) \left| \frac{d}{dw}[t^{-1}(w)] \right|$$

$$= f_Y\left(\frac{1}{w}\right) \left| \frac{d}{dw}\left[\frac{1}{w}\right] \right|$$

$$= f_Y\left(\frac{1}{w}\right) \frac{1}{w^2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha-1)} e^{-\frac{\beta}{w}} w^{-2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha+1)} e^{-\frac{\beta}{w}}$$

"inverse"

$$E[W] = \frac{\beta}{\alpha-1} \quad \text{if } \alpha > 1$$

$$\text{Mode}[W] = \frac{\beta}{\alpha+1}$$

$$SE[W] = \frac{\beta}{\alpha-1} \frac{1}{\sqrt{\alpha-2}} \quad \text{for } \alpha > 2$$

$$p(w) \propto w^{-(\alpha+1)} e^{-\frac{\beta}{w}} \qquad \text{InvGamma}(\alpha,\beta) := \qquad \alpha, \beta \in (0,\infty)$$

so $\sigma^2 \sim \text{InvGamma}(\alpha,\beta)$

$$p(\sigma^2 \mid X, \theta) \propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}} (\sigma^2)^{-(\alpha+1)} e^{-\frac{\beta}{\sigma^2}}$$

Conjugate

$$\propto (\sigma^2)^{-(\alpha+\frac{n}{2}+1)} e^{-\frac{1}{\sigma^2}\left(\frac{n\hat{\sigma}^2}{2}+\beta\right)}$$

$$\propto \text{InvGamma}\left(\alpha+\frac{n}{2}, \beta+\frac{n\hat{\sigma}^2}{2}\right)$$

Correct but nonstandard

$$\alpha = \frac{\nu_0}{2} \qquad \beta = \frac{\nu_0\sigma_0^2}{2}$$

Hyper params

ν_0: How many samples did I see beforehand? $Y_1, \ldots, Y_{\nu_0}$

$$\sigma_0^2 = \frac{1}{\nu_0}\sum_{i=1}^{\nu_0}(y_i-\theta)^2$$

guess as its prior knowledge

$$E[\sigma^2] = \frac{\frac{\nu_0\sigma_0^2}{2}}{\frac{\nu_0}{2}-1} = \frac{\nu_0\sigma_0^2}{\nu_0-2} = \frac{1}{\nu_0-2}\sum_{i=1}^{\nu_0}(y_i-\theta)^2$$

$$\Rightarrow \quad = \text{InvGamma}\left(\frac{\nu_0+n}{2}, \frac{\nu_0\sigma_0^2+n\hat{\sigma}^2}{2}\right)$$

Jeffreys Prior?

$$\ell(\sigma^2; X, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\overbrace{\Sigma(X_i-\theta)^2}^{n\hat{\sigma}^2}$$

only thing that's a function of X!

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{n\hat{\sigma}^2}{2\sigma^2}$$

$$\ell'(\sigma^2; X, \theta) = -\frac{n}{2\sigma^2} + \frac{n\hat{\sigma}^2}{2(\sigma^2)^2}$$

$$\ell''(\sigma^2; X; \theta) = \frac{n}{2(\sigma^2)^2} - \frac{n\hat{\sigma}^2}{(\sigma^2)^3}$$

using iid

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\theta)^2\right] = \frac{1}{n} n E(X-\theta)^2$$

$$= E[X^2] - 2\theta E[X] + \theta^2$$

$\sigma^2 = E(X^2) - \theta^2$

$$= \cancel{\sigma^2+\theta^2-2\theta^2+\theta^2} = \sigma^2$$

unbiased estimator if θ known!

$$I(\sigma^2) = E\left[-\ell''(\sigma^2; X; \theta)\right] = -\frac{n}{2(\sigma^2)^2} + \frac{n}{(\sigma^2)^3}\underbrace{E[\hat{\sigma}^2]}_{=\sigma^2} = \frac{1}{(\sigma^2)^2}\left(-\frac{n}{2} + n\right) = \frac{n}{2}\frac{1}{(\sigma^2)^2}$$

$$j(\sigma^2) \propto \sqrt{I(\sigma^2)} \propto \frac{1}{\sigma^2}$$

Q: proper? $\int_0^\infty \frac{1}{\sigma^2}d\sigma^2 = \infty$

$\Rightarrow$ No!

$$\frac{1}{\sigma^2} = (\sigma^2)^{-1} e^{-\frac{0}{\sigma^2}}$$

$\propto$ InvGamma$(0, 0)$ $\Rightarrow$ conjugate

$\nu_0 = 0$, $\sigma_0^2 = 0$

$$\Rightarrow \sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

$\hat{\sigma}_{MLE} = \frac{1}{n}\Sigma(X_i-\theta)^2$

$$\hat{\sigma}^2_{MMSE} = \frac{\frac{n\hat{\sigma}^2}{2}}{\frac{n}{2}-1} = \frac{n}{n-2}\hat{\sigma}^2 = \frac{1}{n-2}\Sigma(X_i-\theta)^2 \approx \frac{1}{n-1}\Sigma(X_i-\theta)^2$$

$\approx$ (to MLE)

unbiased classic est.

Both θ, σ^2 unknown

$$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2) \underbrace{P(\theta, \sigma^2)}$$

joint prior

if you consider

θ, σ^2 ind

$= P(\theta) P(\sigma^2)$

we will return to this...

For now, use Jeffreys prior

$$P(\theta, \sigma^2) \propto (1) \left(\frac{1}{\sigma^2}\right)$$

Jeffreys prior for $\theta | \sigma^2$

Jeffreys prior for $\sigma^2 | \theta$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} \left(\frac{1}{\sigma^2}\right)$$

Note:
$\Sigma (X_i - \theta)^2$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(X_i - \theta)^2} \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{X} - \theta)^2\right)}$$

Is this InvGamma?
No dim of $P(\theta, \sigma^2 | X) = 2$
InvGamma is univariate.

See next page

$$\sum (X_i - \theta)^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\sum (X_i - \bar{X} + \bar{X} - \theta)^2$$

$$\sum (X_i - \bar{X})^2 + \sum 2(X_i - \bar{X})(\bar{X} - \theta) + \sum (\bar{X} - \theta)^2$$

$$2\left(x_i \bar{x} - \bar{x}^2 - x_i \theta + \bar{x}\theta\right)$$

$$2 \quad \sum x_i \bar{x} - \bar{x}^2 - \sum x_i \theta + \sum \bar{x}\theta$$

$$2\left(\cancel{n\bar{x}^2 - n\bar{x}^2} \cancel{- n\bar{x}\theta + n\bar{x}\theta}\right)$$

$$= \sum (X_i - \bar{X})^2 + \sum (\bar{X} - \theta)^2 = (n-1)s^2 + n(\bar{X} - \theta)^2$$

let $s^2 := \frac{1}{n-1} \sum (X_i - \bar{X})^2$ = <u>purely</u> a function of X

Math 633 unbiased est for σ^2

$E(s^2) = \sigma^2$ Bessel's correction?

$$p(\theta, \sigma^2 | x) \propto \ldots$$

What do we do now?

If we knew this distr, we could take 2-dim expectation, matrix of variance, etc...

Recall $f(x,y) = f(x|y)\, f(y)$ (Bayes Rule)

and $f(x,y|z) = f(x|y,z)\, f(y|z)$ (Ditto)

Here

$$P(\theta, \sigma^2 | x) \propto P_1(\theta | \sigma^2, x)\, P_2(\sigma^2 | x)$$

$$P(\theta | \sigma^2, x) \propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}((n-1)s^2)} e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2} n(\bar{x}-\theta)^2}$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$P(\sigma^2 | x)$?

Recall..

$$f(x) = \int_{Supp(Y)} f(x,y)\, dy \quad , \quad f(x|z) = \int_{Supp(Y)} f(x,y|z)\, dy$$

$$P(\sigma^2|x) = \int_{Supp(\theta)} P(\theta, \sigma^2 | x)\, d\theta \underset{\text{Why}}{\propto} \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}((n-1)s^2)} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2} d\theta$$

$$= (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{1}{2\sigma^2}(n-1)s^2} \int_{\mathbb{R}} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2} d\theta$$

Kernel of Gaussian $\Rightarrow$ It's integral is a constant

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}(n-1)s^2}$$

$$\propto \text{InvGamma}\left(\frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$