HW 7 partial solution set

$$X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$$Y = cX \sim f_Y(y) = f_X(t^{-1}(y)) \left| \frac{d}{dy} [t^{-1}(y)] \right| = f_X\left(\frac{y}{c}\right) \left| \underbrace{\frac{d}{dy}\left[\frac{1}{c} y\right]}_{\frac{1}{c}} \right| = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{y^{\alpha-1}}{c^{\alpha-1}} e^{-\frac{\beta}{c} y} \frac{1}{c}$$

$Y = t(X)$

$X = t^{-1}(Y) = \frac{1}{c} Y$

$$\Rightarrow = \frac{\left(\frac{\beta}{c}\right)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta}{c} y} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

Note: this is different from the hint on the HW due to the different parameterization

$$X^{-1} \sim \text{InvGamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\frac{\beta}{x}}$$

(derived in class, lec 13 p2)

HW 7 2(h) $\underbrace{\frac{n-1}{\sigma^2} S^2}_{2(g)} \sim \chi^2_{n-1} := \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} x^{\frac{n-1}{2}-1} e^{-\frac{1}{2} x} = \text{Gamma}\left(\frac{n-1}{2}, \frac{1}{2}\right)$

2(i)

$$S^2 = \underset{\text{scale}}{\frac{\sigma^2}{n-1}} \underset{\text{gamma}}{\left(\frac{n-1}{\sigma^2} S^2\right)} = \text{Gamma}\left(\frac{n-1}{2}, \frac{\frac{1}{2}}{\frac{\sigma^2}{n-1}}\right) = \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1/\sigma^2}{2}\right)$$

2(j) $\frac{\sigma^2}{(n-1) S^2} = \underset{\text{Gamma}}{\left(\frac{n-1}{\sigma^2} S^2\right)}^{\overset{\text{inverse}}{-1}} \sim \text{InvGamma}\left(\frac{n-1}{2}, \frac{1}{2}\right)$

4(k) See next page

4(l) $X \sim \text{InvGamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\frac{\beta}{x}}$

Note: $Y = cX \sim f_Y(y) = f_X(t^{-1}(y)) \left| \frac{d}{dy} [t^{-1}(y)] \right| = f_X\left(\frac{y}{c}\right) \left| \frac{d}{dy} \left(\frac{y}{c}\right) \right| = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{y^{-(\alpha+1)}}{c^{-(\alpha+1)}} e^{-\frac{c\beta}{y}} \frac{1}{c}$

$$= \frac{(c\beta)^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\frac{c\beta}{y}} = \text{InvGamma}(\alpha, c\beta)$$

Note: this is different from the hint on the HW due to the different parameterization

$$\sigma^2 = ((n-1) S^2) \underbrace{\left(\frac{\sigma^2}{(n-1) S^2}\right)}_{\substack{\text{InvGamma} \\ \text{from 2j}}} \sim \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1) S^2}{2}\right)$$

$\Rightarrow$ Same exact distr. as 2(j)

Under θ known, uninformative prior,

$$P(\sigma^2 | x, \theta) \propto P(x | \theta, \sigma^2) P(\sigma^2 | \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} \left(\frac{1}{\sigma}\right)$$

$$\propto (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{1}{2\sigma^2} \underbrace{\sum (x_i-\theta)^2}_{n\hat{\sigma}^2}}$$

$$= (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

Note how $\alpha \neq \frac{n-1}{2}$ here but does so later

But if θ unknown & uninformative prior,

4(k)

$$P(\sigma^2 | x) \propto \int_{\mathbb{R}} P(x | \theta, \sigma^2) P(\theta, \sigma^2) \, d\theta$$

$$= \int_{\mathbb{R}} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} (1)\left(\frac{1}{\sigma^2}\right) d\theta$$

use subst from class

$$\propto \int_{\mathbb{R}} (\sigma^2)^{-n/2-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)} d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2}{2\sigma^2}} \underbrace{\int_{\mathbb{R}} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2} d\theta}_{\sqrt{2\pi\frac{\sigma^2}{n}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2} d\theta}_{1}}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1+\frac{1}{2}} e^{-\frac{(n-1)s^2}{2\sigma^2}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2}{2\sigma^2}}$$

$$= (\sigma^2)^{-\left(\frac{n+1}{2}-1+1\right)} e^{-\frac{(n-1)s^2}{2\sigma^2}}$$

$$= (\sigma^2)^{-\left(\frac{n-1}{2}+1\right)} e^{-\frac{(n-1)s^2}{2\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$