HW B 1(d), 1(e) Solution

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$P(\theta, \sigma^2) = P(\theta)\, P(\sigma^2)$

$P(\theta) = N(\mu_0, \tau^2), \quad P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0 \sigma_0^2}{2}\right)$

$$P(\theta, \sigma^2 | x) \propto P(x | \theta, \sigma^2)\, P(\theta)\, P(\sigma^2)$$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2} (\sigma^2)^{-\left(\frac{n_0}{2}+1\right)} e^{-\frac{n_0\sigma_0^2}{2\sigma^2}}$$

$$\propto (\sigma^2)^{-n/2} (\sigma^2)^{-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)} e^{-\frac{1}{2\tau^2}\left(\theta^2 - 2\theta\mu_0 + \mu_0^2\right)}$$

$$\propto (\sigma^2)^{-\left(\frac{n+n_0}{2}+1\right)} e^{-\frac{(n-1)s^2 + n_0\sigma_0^2}{2\sigma^2}} \underbrace{e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}} e^{-\frac{1}{2\tau^2}\theta^2} e^{\frac{\theta\mu_0}{\tau^2}}$$

$$e^{-\frac{n}{2\sigma^2}\bar{x}^2} e^{\frac{n\bar{x}}{\sigma^2}\theta} e^{-\frac{n\theta^2}{2\sigma^2}} e^{-\frac{1}{2\tau^2}\theta^2} e^{\frac{\theta\mu_0}{\tau^2}}$$

$$= (\sigma^2)^{-\left(\frac{n+n_0}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} e^{\underbrace{\left(-\frac{n}{2\sigma^2} - \frac{1}{2\tau^2}\right)}_{a}\theta^2 + \underbrace{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)}_{b}\theta}$$

$$-\frac{1}{2v}(\theta - c)^2 = -\frac{1}{2v}\left(\theta^2 - 2c\theta + c^2\right) \propto -\frac{1}{2v}\theta^2 + \frac{c\theta}{v}$$

$$\Rightarrow a = -\frac{1}{2v} \Rightarrow v = -\frac{1}{2a} = -\frac{1}{\cancel{2}\left(-\frac{n}{\cancel{2}\sigma^2} - \frac{1}{\cancel{2}\tau^2}\right)} = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$b = \frac{c}{v} \Rightarrow c = bv = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\propto (\sigma^2)^{-\left(\frac{n+n_0}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} e^{-\frac{1}{2\left(\frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)}\left(\theta - \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)^2}$$

$$\propto (\sigma^2)^{-\left(\frac{n+n_0}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} \sqrt{2\pi\frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}\; \underbrace{\frac{1}{\sqrt{2\pi\left(\frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)}}}_{N(\theta_p, \sigma_p^2)}$$

$$\propto N(\theta_p, \sigma_p^2)\, (\sigma^2)^{-\left(\frac{n+n_0}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n\theta_0\right)} \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1/2}$$

Invgamma but this is not part of it

$$\Rightarrow k(\sigma^2 | x) =$$

let $g(\sigma^2|x) := \ln\left(k(\sigma^2|x)\right)$

Since unknown distribution, can't sample from it

Approximate its distribution via $P(\sigma^2|x) \approx N\left(\hat{\sigma}^2_{MAP}, \left(\frac{1}{\sqrt{g''(\hat{\sigma}^2_{MAP}|x)}}\right)^2\right)$

$$\hat{\sigma}^2_{MAP} = \operatorname{argmax} P(\sigma^2|x) = \operatorname{argmax} k(\sigma^2|x) = \operatorname{argmax} g(\sigma^2|x)$$

$$g(\sigma^2|x) = -\left(\frac{n}{2}+1\right)\ln(\sigma^2) - \frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n\theta_0^2\right) - \frac{1}{2}\ln\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)$$

$$g'(\sigma^2|x) = \frac{-\left(\frac{n+n_0}{2}+1\right)}{\sigma^2} + \frac{(n-1)s^2 + n\bar{x}^2 + n\theta_0^2}{2(\sigma^2)^2} + \frac{n}{2(\sigma^2)^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)} \overset{\text{set}}{=} 0$$

$$2\sigma^2(n + \sigma^2/\tau^2)$$

$$\Rightarrow -\left(\frac{n+n_0}{2}+1\right) + \frac{(n-1)s^2 + n\bar{x}^2 + n\theta_0^2}{2\sigma^2} + \frac{n}{2\left(n+\frac{\sigma^2}{\tau^2}\right)} = 0$$

$$\Rightarrow -n - n_0 - 2 + \frac{(n-1)s^2 + n\bar{x}^2 + n\theta_0^2}{\sigma^2} + \frac{n}{n+\frac{\sigma^2}{\tau^2}} = 0$$

$$\Rightarrow -(n+n_0+2)\sigma^2 + (n-1)s^2 + n\bar{x}^2 + n\theta_0^2 + \frac{n\sigma^2\tau^2}{n\tau^2+\sigma^2} = 0$$

$$\Rightarrow \underbrace{(n\tau^2}_{a} + \sigma^2)\left(\underbrace{(n-1)s^2 + n\bar{x}^2 + n\theta_0^2}_{b} - \underbrace{(n+n_0+2)}_{c}\sigma^2\right) + \underbrace{n\tau^2}_{a}\sigma^2 = 0$$

$$(a+\sigma^2)(b+c\sigma^2) + a\sigma^2 = ab + (b+ac+a)\sigma^2 + c(\sigma^2)^2$$

$$\sigma^2 = \frac{-(b+ac+a) \pm \sqrt{(b+ac+a)^2 - 4cab}}{2c}$$

I imagine the (+) root is the solution...