390 -12

3/21/16 1)

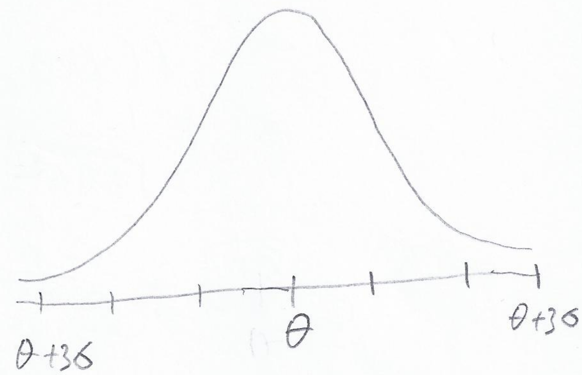
$$X \sim N(\theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\theta^2}(x-\theta)^2}$$

dim of parameter = [2]

$\theta \in \mathbb{R}$ $\quad \sigma \in (0,\infty)$, supp$(X) = \mathbb{R}$

$E(\theta) = \theta$, $E(x) = \sigma$

$$P(\theta \mid X, \sigma^2) \propto e^{\frac{x}{\sigma^2}\theta} e^{-\frac{1}{2\sigma^2}\theta^2}$$



"thin tails"

$P(X > +3\sigma) \approx 0.15\%$

Possible Inferential Targets

- $P(\theta \mid X, \sigma^2)$
- $P(\sigma^2 \mid X, \theta)$
- $P(\theta, \sigma^2 \mid X)$
- $P(\theta \mid X)$
- $P(\sigma^2 \mid X)$

$$\mathcal{L}(\theta; X, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\theta)^2}$$

$\to f(g(\theta))$

$$\ell(\theta; X, \sigma^2) = -\frac{n}{8}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum (X_i-\theta)^2$$

$$\ell'(\theta; X, \sigma^2) = -\frac{1}{2\sigma^2}(-2)\sum_{i=1}^{n} X_i - \theta = \frac{1}{\sigma^2}\left(\sum X_i - \theta\right)$$

$$\frac{1}{\sigma^2}\left(\sum X_i - n\theta\right) = \frac{n}{\sigma^2}(\bar{X} - \theta) = \text{set } 0$$

$$\Rightarrow \bar{X} = \hat{\theta}_{MLE}$$

$$\hat{\theta} = e\theta_0 + (1-e)\hat{\theta}_{MLE}$$

$$\hat{\theta}_S = \hat{\theta}_{MLE} - e(\hat{\theta}_{MLE} - \theta_0) \qquad e \in (0,1)$$

$$\hat{\theta}_{MLE} < \theta_0$$

$$P(\theta \mid X, \sigma^2) = \frac{P(X \mid \theta, \sigma^2)\, P(\theta \mid \sigma^2)}{P(X \mid \sigma^2)}$$

all n observations

$x_1, x_2, \ldots, x_n \overset{\text{exch}}{\sim} N(\theta, \sigma^2)$

$$\propto P(X \mid \theta, \sigma^2)\, P(\theta \mid \sigma^2)$$

$$\propto \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2} \Sigma (x_i - \theta)^2} \quad P(\theta \mid \sigma^2)$$

$$= \Sigma (x_i^2 - 2\theta x_i + \theta^2)$$
$$= \Sigma x_i^2 - 2\theta \Sigma x_i + n\theta^2)$$
$$= \Sigma x_i^2 - 2\theta n\bar{x} + n\theta^2$$

$$\propto e^{\boxed{\frac{\bar{x} n}{\sigma^2}}\theta}\, e^{\frac{-n}{2\sigma^2}\theta^2} \quad P(\theta \mid \sigma^2)$$

Const 1 Cons 2

$$= e^{(c_1 + a)\theta}\, e^{(c_2 + b)\theta^2}$$

$$\Rightarrow P(\theta \mid \sigma^2) \propto e^{a\theta}\, e^{b\theta^2} \propto N(M_0, \tau^2)$$

prior mean prior variance

$$\propto \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(\theta - M_0)^2}$$

$$\propto e^{-\frac{1}{2\tau^2}(\theta^2 - 2\theta M_0 + M_0^2)}$$

$$\propto e^{-\frac{1}{2\tau^2}\theta^2}\, e^{\frac{M_0}{\tau^2}\theta} \qquad \theta^2 - 2\theta M_0 + M_0^2$$

$$= e^{\underbrace{\left(\frac{\bar{x} n}{\sigma^2}\right)}_{a}\theta}\, e^{\underbrace{\left(-\frac{1}{2\tau^2} - \frac{n}{2\sigma^2}\right)}_{b}\theta^2}$$

$$e^{a\theta}\, e^{b\theta^2} = e^{a\theta + b\theta^2} = e^{b(\theta^2 + \frac{a}{b}\theta)}$$

$$\propto N\left(\frac{\frac{\bar{x} n}{\sigma^2} + \frac{M_0}{\tau^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}},\ \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

always proper

$$\hat{\theta}^n_{MMSE} = \hat{\theta}^n_{MAE} = \frac{\frac{\bar{x}n}{\sigma^2}}{\frac{1}{\tau}+\frac{1}{\sigma^2}} + \frac{\frac{\mu_0}{\tau^2}}{\frac{1}{\tau^2}+\frac{1}{\sigma^2}}$$

$$= \hat{\theta}^n_{MAP} = \mu_1$$

$$= \hat{\theta}^n_{MLE} \frac{1}{\sigma^2\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)} + E(\theta)\frac{1}{\tau^2\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right)}$$

$$= \overbrace{\left(\frac{\sigma^2}{\tau^2 n+\sigma^2}\right)}^{e} E(\theta) + \overbrace{\left(\frac{\tau^2 n}{\tau^2 n+\sigma^2}\right)}^{1-e} \hat{\theta}^n_{MLE}$$

$P(\theta) = N(\mu_0, \tau^2)$

$P(\theta) \propto 1$ by principle of indifference

improper because $\int_{\mathbb{R}} 1 \, d\theta = \infty$

$-\frac{1}{2v} = b \Rightarrow v = -\frac{1}{2b} = \frac{\sigma^2}{n}$

$-2d = \frac{a}{b} \Rightarrow d = \frac{a}{2b} = \frac{\frac{\bar{x}n}{\sigma^2}}{2\left(-\frac{n}{2\sigma^2}\right)} = \bar{x}$

$\ell'(\theta ; x, \sigma^2) = \frac{n}{\sigma^2}(\bar{x} - \theta)$

$\ell''(\theta ; x, \sigma^2) = -\frac{n}{\sigma^2}$ means fishe info is always same

$I(\theta) = E\left[-\ell''(\theta ; x, \sigma^2)\right] = \frac{n}{\sigma^2}$

$j(\theta) \propto \sqrt{I(\theta)} = \frac{\sqrt{n}}{\sigma} \propto 1$ b/c no θ

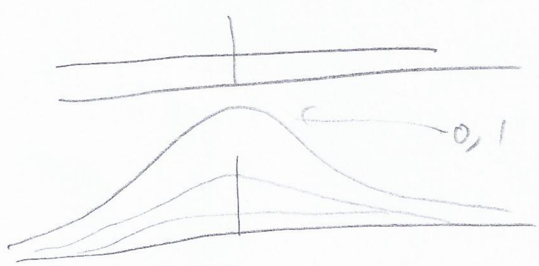
so $p(\theta) \propto 1$ improper ← principle of indiff

↳ is the Jeffery's prior

$\theta \sim Beta(0, 0)$

$\theta \mid X \sim Beta(\alpha + x, \beta + (n - x))$

$= Beta(x, n - x)$



$p(\theta) \propto$

3)

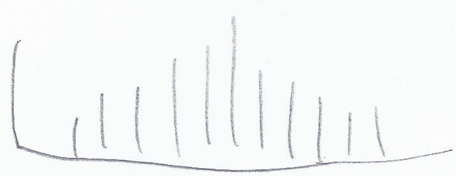
$$p(x^* \mid x, \sigma^2) = \int_{H} p(x^* \mid \theta, \sigma^2)\, p(\theta \mid x, \sigma^2)\, d\theta$$

$$= \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^*-\theta)^2}}_{N(0,\sigma^2)} \underbrace{\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta-\theta_p)^2}}_{N(\theta_p,\, \sigma_p^2)} d\theta$$

$X_1 \sim U(1, \ldots, 6)$

$X_2 \sim U(1, 2, \ldots, 6)$

$S = X_1 + X_2 \sim$? pmf



$P(S=1) = 0$

$P(S=2) = P(X_1=1) \cdot P(X_2=1)$

$P(S=3) \ldots\ldots$

$$= \sum_{x \in \operatorname{supp} X_1} P(X_1 = x)\, P(X_2 = 3 - x)$$

$$P(S=s) = \sum_{(\ )} P(X_1 = x)\, P(X_2 = s - x)$$

Convolution

$X_1 \sim f(x)$ $\quad$ $X_2 \sim g(x)$

$\text{supp}(X_1) = \text{supp}(X_2) = \mathbb{R}$

convolution

$X_1 + X_2 \sim f(x) * g(x)$

$= \int_{\mathbb{R}} f(x) g(s-x)\, dx$

$$\overbrace{N(0, \sigma^2)}^{X_1} \quad \overbrace{N(\theta, \sigma^2 p)}^{X_2} = N(\theta p, \sigma^2 + \sigma^2 p)$$

$\theta | X \sim N(\theta p, \sigma^2 p)$

variance of estimator θ

noise in likelihood model

overdispersed

Normal $\quad \sim$ Beta

Next time $\sigma^2 | X_1 \theta$