LECTURE-13 (03-28-16)

(1)

$$\theta \mid X, \sigma^2 \sim N\left(\underbrace{\frac{\frac{\bar{x}n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\theta_p}, \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\sigma_p^2} \right)$$

Likelihood Model

$x \mid \theta, \sigma^2 \sim N(\theta, \sigma^2)$

$\theta \sim N(\mu_0, \tau^2)$

over-dispersed Normal

$$X^* \mid X, \sigma^2 \sim N(\theta_p, \sigma_p^2 + \sigma^2)$$

$\rightarrow$ convolution

$$P(\sigma^2 \mid X, \theta) \propto P(X \mid \sigma^2, \theta)\, P(\sigma^2 \mid \theta)$$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2} P(\sigma^2 \mid \theta)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2} P(\sigma^2 \mid \theta)$$

Let $\hat{\sigma}^2 := \frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)^2$

MLE for σ^2 when θ is known.

$$-\frac{\sum x_i^2}{2\sigma^2} + \frac{\cancel{2}\theta\bar{x}_n}{\cancel{2}\sigma^2} - \frac{\theta^2}{2\sigma^2}$$

$$= (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}} P(\sigma^2 \mid \theta)$$

'Prior' $P(\sigma^2 \mid \theta) \propto (\sigma^2)^a e^{-b/\sigma^2}$

$$Y \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha - 1} e^{-\beta y}$$

'Change of variable' $W = \frac{1}{Y} = t(Y) \sim f_Y(t^{-1}(w)) \left| \frac{d}{dw}[t^{-1}(w)] \right|$

$t^{-1}(w) = \frac{1}{w}$

$$= f_Y\left(\frac{1}{w}\right) \left| \frac{d}{dw}\left[\frac{1}{w}\right] \right|$$

(2)

Then to Gamma $- \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha+1)} e^{-\frac{\beta}{w}}$

Then it becomes $Y^{-1} \sim \text{Inv Gamma}(\alpha, \beta)$

$$:= \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha+1)} e^{-\beta/w}$$

$$\sigma^2 \mid \theta = \sigma^2 \sim \text{Inv Gamma}(\alpha, \beta)$$

$$E[W] = \frac{\beta}{\alpha - 1} \text{ if } \alpha > 1$$

$$\text{Mode}[W] = \frac{\beta}{\alpha + 1}$$

$$SE[W] = \frac{\beta}{\alpha - 1} \frac{1}{\sqrt{\alpha - 2}} \text{ for } \alpha > 2$$

$$P(\sigma^2 \mid \theta) \propto (\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2/2}{\sigma^2}} (\sigma^2)^{-(\alpha+1)} e^{-\beta/\sigma^2}$$

$$= (\sigma^2)^{-(\alpha + n/2 + 1)} e^{\frac{(n\hat{\sigma}^2/2 + \beta)}{\sigma^2}}$$

$$\propto \text{Inv Gamma}\left(\alpha + \frac{n}{2}, \beta + \frac{n}{2}\hat{\sigma}^2\right)$$

pseudo data [Beta, Gamma]

Let $\alpha = \frac{\nu_0}{2}$ Let $\beta = \frac{\nu_0 \sigma_0^2}{2}$ $\alpha, \beta \in (0, \infty)$

$$\propto \text{Inv Gamma}\left(\frac{\nu_0 + n}{2}, \frac{\nu_0\sigma_0^2 + n\hat{\sigma}^2}{2}\right)$$

Inv Gama (online) Inverse χ^2 (Chi-square)

$\hat{\theta}_{MMSE} = E[\sigma^2 \mid X, \theta]$

$$E[\sigma^2] = \frac{\nu_0\sigma_0^2/\cancel{2}}{\nu_0 - 2/\cancel{2}}$$

$$= \frac{\nu_0}{\nu_0 - 2}\sigma_0^2$$

$$= \frac{1}{\nu_0 - 2}\sum_{i=1}^{\nu_0}(x_i - \theta)^2$$

$Y_1, \ldots, Y_{\nu_0}$

$$\sigma_0^2 := \frac{1}{\nu_0}\sum_{i=1}^{\nu_0}(x_i - \theta)^2$$

'prior n' $\rightarrow \nu_0$

$$\mathcal{L}(\sigma^2; x, \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}$$

$$\ln(\sigma^2; x, \theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{n\hat{\sigma}^2}{2\sigma^2}$$

$$\ln'(\sigma^2; x, \theta) = -\frac{n}{2\sigma^2} + \frac{n\hat{\sigma}^2}{2(\sigma^2)^2} \overset{\text{set}}{=} 0 \Rightarrow \frac{\cancel{n}}{\cancel{2}\sigma^2} = \frac{\cancel{n}\hat{\sigma}^2}{\cancel{2}(\sigma^2)^2}$$

$$\ln''(\sigma^2; x, \theta) = \frac{n}{2(\sigma^2)^2} - \frac{n\hat{\sigma}^2}{(\sigma^2)^3}$$

$$I(\theta) = E[-\ln''(\sigma^2; \cancel{x}, \theta)] = -\frac{n}{2(\sigma^2)^2} - \frac{nE[\hat{\sigma}^2]}{(\sigma^2)^3}$$

$$\frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\theta)^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(x_i-\theta)^2$$

if frequentist

$x_1, \ldots, x_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$\frac{1}{\cancel{n}} \cancel{n} E(x-\theta)^2$

$E[x^2 - 2\theta x + \theta^2]$

$E[x^2] - 2\theta E[x] + \theta^2$

$E[x^2] - \theta^2$

$= \sigma^2 \; \{var\}$

$$-\frac{n}{2(\sigma^2)^2} - \frac{n}{(\sigma^2)^2} = \frac{n}{2}\frac{1}{(\sigma^2)^2}$$

$$j(\sigma^2) \propto \sqrt{I(\theta)}$$

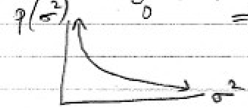
$$\sqrt{\frac{n}{2}\frac{1}{(\sigma^2)^2}}$$

$$\propto 1/\sigma^2$$

$\sigma^2 \in (0, \infty)$

$\int_0^\infty p(\sigma^2)\, d\sigma^2 = \infty$

$\Rightarrow j(\theta)$ improper



$$\propto (\sigma^2)^{-1} = \text{InvGamma}(\overset{\nu_0 = 0}{0}, \overset{\sigma_0^2}{0})$$

(Heldane)

$$\text{InvGamma}\left(\frac{\nu_0 + n}{2}, \frac{\nu_0\sigma_0^2 + n\hat{\sigma}^2}{2}\right)$$

under $j(\sigma^2)$

$$= \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

$$E[\sigma^2 | x, \theta] = \frac{n\hat{\sigma}^2/2}{\frac{n}{2} - 1} = \frac{1}{n-2}\sum(x_i-\theta)^2 \approx \frac{1}{n-1}\sum(x_i-\theta)^2$$

$$\begin{bmatrix}\theta \\ \sigma^2\end{bmatrix} \sim \underbrace{P(\theta, \sigma^2 \mid X)}_{\text{joint posterior}} \propto P(X \mid \theta, \sigma^2) \underbrace{P(\theta, \sigma^2)}_{\text{joint prior}}$$

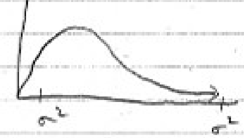
Assume

"if σ^2, then θ"

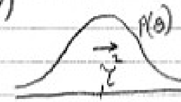
$$P(\theta, \sigma^2) = P(\theta)\, P(\sigma^2) \text{ jeffrey's}$$

$$\propto (1)\left(\frac{1}{\sigma^2}\right)$$

$$= \frac{1}{\sigma^2}$$



$$(\sigma^2)^{-(\frac{n}{2}+1)} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$



$$\sum(x_i - \bar{x} + \bar{x} - \theta)^2$$

$$= \sum(x_i - \bar{x})^2 + \sum 2(x_i - \bar{x})(\bar{x} - \theta)$$

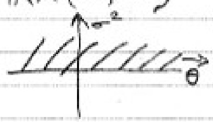
$$+ \sum(\bar{x} - \theta)^2$$

, InvGama has 'one' supp

$\mathbb{R} \times (0, \infty)$

$$= 2\sum x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta$$

$$= 2(n\bar{x}^2 - n\bar{x}^2 - n\bar{x}\theta + n\bar{x}\theta)$$



$$s^2 := \frac{1}{n-1}\sum(x_i - \bar{x})^2$$

E[ ]

$$= (n-1)s^2 + n(\bar{x} - \theta)^2$$

$$f(x,y) = f(x|y)\, f(y)$$

$$f(x,y|z) = f(x|y,z)\, f(y|z)$$

$$p(\theta, \sigma^2|X) = p(\theta|X, \sigma^2)\, p(\sigma^2|X)$$

$$p(\theta|x,\sigma^2) \propto e^{1/2\sigma^2(\bar{x}-\theta)^2}$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$ posteriro under jeffrey prior

$$= (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)}$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)\left(\sigma^{2^{-\left(\frac{n}{2}+1\right)}} e^{-\frac{(n-1)s^2}{2\sigma^2}}\right.$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$$\propto \text{InvGamma}\left(\frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$

$$f(x) = \int_{supp[Y]} f(x,y)\,dy$$

$$f(x|z) = \int_{supp[Y]} f(x,y|z)\,dy$$

$$\underline{p(\sigma^2|x)} = \int_{supp[\theta]} p(\theta, \sigma^2|x)\,d\theta$$