

Lecture 13 3/28/16 Math 390 03

11

$$\theta | X, \sigma^2 \sim N \left(\underbrace{\frac{\frac{\bar{X}n}{\sigma^2} + \frac{n_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\theta_p}, \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\sigma_p^2} \right)$$

↑
Variance
Known
 beforehand

we found that post. pred.
 distr. was
 a
 combination

If $P(\theta) \propto 1$, $\theta_p = \bar{X}$, $\sigma_p^2 = \frac{\sigma^2}{n}$

overdispersed
 normal
 is
 itself
 normal

$$X^a | X, \sigma^2 \sim N(\theta_p, \sigma_p^2 + \sigma^2)$$

What about?

$\sigma^2 | X, \theta \rightarrow$ what inference for variance/SE considering

same like always mean is known

$$\begin{aligned} P(\sigma^2 | X, \theta) &\propto P(X | \sigma^2, \theta) P(\sigma^2 | \theta) \\ &= \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} \right) P(\sigma^2 | \theta) \\ &\propto \frac{1}{\sqrt{\sigma^2}^n} e^{-\frac{1}{2\sigma^2} \sum (X_i - \theta)^2} P(\sigma^2 | \theta) \\ &\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (X_i - \theta)^2} P(\sigma^2 | \theta) \\ &= (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}} P(\sigma^2 | \theta) \end{aligned}$$

let $\hat{\sigma}^2 := \frac{1}{n} \sum (X_i - \theta)^2$
↑
MLE for σ^2 if θ known
(and column vector)

What does it look like? Pattern matching
 $(\sigma^2)^q e^{-\frac{b}{\sigma^2}}$

Recall

$$Y \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \Rightarrow \frac{1}{Y} \sim \text{InvGamma}(\alpha, \beta)$$

$$W = \frac{1}{Y} \sim ? \quad W = t(Y) = \frac{1}{Y} \Rightarrow t^{-1}(w) = \frac{1}{w}$$

$$f_W(w) = f_Y(t^{-1}(w)) \left| \frac{d}{dw} [t^{-1}(w)] \right|$$

$$= f_Y\left(\frac{1}{w}\right) \left| \frac{d}{dw} \left[\frac{1}{w}\right] \right|$$

$$= f_Y\left(\frac{1}{w}\right) \frac{1}{w^2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha-1)} e^{-\frac{\beta}{w}} w^{-2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha+1)} e^{-\frac{\beta}{w}}$$

"hiese"

$$p(w) \propto w^{-(\alpha+1)} e^{-\frac{\beta}{w}} \quad \text{InvGamma}(\alpha, \beta) := \quad \alpha, \beta \in (0, \infty)$$

$$\text{so } \sigma^2 \sim \text{InvGamma}(\alpha, \beta)$$

$$p(\sigma^2 | X, \theta) \propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}} (\sigma^2)^{-(\alpha+1)} e^{-\frac{\beta}{\sigma^2}} \quad \text{Corrigere}$$

$$\propto (\sigma^2)^{-(\alpha + \frac{n}{2} + 1)} e^{-\frac{1}{\sigma^2} \left(\frac{n\hat{\sigma}^2}{2} + \beta \right)}$$

$$\propto \text{InvGamma}\left(\alpha + \frac{n}{2}, \beta + \frac{n\hat{\sigma}^2}{2}\right)$$

Hyper params

Conver. but nonstandard

$$\alpha = \frac{\nu_0}{2}, \quad \beta = \frac{\nu_0 \sigma_0^2}{2}$$

$$E[\sigma^2] = \frac{\frac{\nu_0 \sigma_0^2}{2}}{\frac{\nu_0}{2} - 1} = \frac{\nu_0 \sigma_0^2}{\nu_0 - 2} = \frac{1}{\nu_0 - 2} \sum_{i=1}^{\nu_0} (x_i - \theta)^2$$

$$\Rightarrow = \text{InvGamma}\left(\frac{\nu_0 + n}{2}, \frac{\nu_0 \sigma_0^2 + n \hat{\sigma}^2}{2}\right)$$

ν_0 : How many samples did I see beforehand? Y_1, \dots, Y_{ν_0}

$$\sigma_0^2 = \frac{1}{\nu_0} \sum_{i=1}^{\nu_0} (Y_i - \theta)^2$$

guess as it's prior variance

Jeffrey's Prior?

$$\ell(\sigma^2; X, \theta) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \theta)^2$$

$$= -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{\sum (x_i - \theta)^2}{2\sigma^2}$$

$$\ell'(\sigma^2; X, \theta) = -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \theta)^2}{2(\sigma^2)^2}$$

$$\ell''(\sigma^2; X, \theta) = \frac{n}{2(\sigma^2)^2} - \frac{\sum (x_i - \theta)^2}{(\sigma^2)^3}$$

$$I(\theta) = E[-\ell''(\sigma^2; X, \theta)] = -\frac{n}{2(\sigma^2)^2} + \frac{n}{(\sigma^2)^2} E[\sigma^2] = \frac{n}{(\sigma^2)^2} \left(-\frac{1}{2} + 1 \right) = \frac{n}{2(\sigma^2)^2}$$

only this this is a sum of X^2

$$E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \theta)^2\right] = E[X - \theta]^2$$

$$= E[X^2] - 2\theta E[X] + \theta^2$$

$$= \sigma^2 + \theta^2 - 2\theta^2 + \theta^2 = \sigma^2$$

unbiased estimator if θ known!

$j(\sigma^2) \propto \sqrt{I(\sigma^2)} \propto \frac{1}{\sigma^2}$ \propto proper? $\int_0^\infty \frac{1}{\sigma^2} d\sigma^2 = \infty \Rightarrow \text{No!}$

$\frac{1}{\sigma^2} = (\sigma^2)^{-1} e^{-\frac{0}{\sigma^2}} \propto \text{Inverse Gamma}(0, 0) \Rightarrow \text{conjugate}$
 $\nu_0 = 0, \sigma_0^2 = 0$

$\Rightarrow \sigma^2 | X, \theta \sim \text{Inverse Gamma}\left(\frac{n}{2}, \frac{\sum (x_i - \theta)^2}{2}\right)$

$\hat{\sigma}_{\text{mpse}}^2 = \frac{\frac{\sum (x_i - \theta)^2}{2}}{\frac{n}{2} - 1} = \frac{1}{n-2} \sum (x_i - \theta)^2 \approx \frac{1}{n} \sum (x_i - \theta)^2$

unbiased classic est.

both θ, σ^2 unknown

$$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2) \underbrace{P(\theta, \sigma^2)}$$

joint prior

if you consider

θ, σ^2 ind

$$= P(\theta)P(\sigma^2)$$

We will return to this...

For now, we Jeffreys prior

$$P(\theta, \sigma^2) \propto (1) \left(\frac{1}{\sigma^2}\right)$$

↑
Jeffreys prior for θ/σ^2 ↑
Jeffreys prior for σ^2/θ

$$\propto \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} (x_i - \theta)^2} \left(\frac{1}{\sigma^2}\right)$$

Note:

$$\sum (x_i - \theta)^2$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2} \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2} ((n-1)s^2 + n(\bar{x} - \theta)^2)}$$

Is this Inv Gamma?

No dim of $P(\theta, \sigma^2 | X) = 2$

Inv Gamma is unimodal.

see next page

$$\sum (x_i - \theta)^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\sum (x_i - \bar{x} + \bar{x} - \theta)^2$$

$$\sum (x_i - \bar{x})^2 + 2 \sum (x_i - \bar{x})(\bar{x} - \theta) + \sum (\bar{x} - \theta)^2$$

$$2 \sum (x_i \bar{x} - \bar{x}^2 - x_i \theta + \bar{x} \theta)$$

$$2 \sum x_i \bar{x} - \bar{x}^2 - \sum x_i \theta + \sum \bar{x} \theta$$

$$2 \left(\cancel{n \bar{x}^2} - \cancel{n \bar{x}^2} - \cancel{n \bar{x} \theta} + \cancel{n \bar{x} \theta} \right)$$

$$= \sum (x_i - \bar{x})^2 + \sum (\bar{x} - \theta)^2 = (n-1) s^2 + n (\bar{x} - \theta)^2$$

$$\text{let } s^2 := \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \text{purely a function of } X$$

↑
Prob 633 unbiased est for σ^2

$E[s^2] = \sigma^2$ Bessel's correction?

$$p(\theta, \sigma^2 | x) \propto \dots$$

↑ What do we do now?

If we knew this dist, we could take 2-dim expectation, moments of variance, etc...

6

Recall $f(x, y) = f(x|y) f(y)$ (Bayes Rule)

and $f(x, y|z) = f(x|y, z) f(y|z)$ (Dirichlet)

Here

$$P(\theta, \sigma^2 | x) \propto P_1(\theta | \sigma^2, x) P_2(\sigma^2 | x)$$

$$\begin{aligned} P(\theta | \sigma^2, x) &\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}(n-1)s^2} e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2} \\ &\propto e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2} \\ &\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right) \end{aligned}$$

$P(\sigma^2 | x)$?

Recall...

$$f(x) = \int_{\text{supp}(Y)} f(x, y) dy, \quad f(x|z) = \int_{\text{supp}(Y)} f(x, y|z) dy$$

$$P(\sigma^2 | x) = \int_{\text{supp}(\theta)} P(\theta, \sigma^2 | x) d\theta \propto \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}(n-1)s^2} e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2} d\theta$$

why \swarrow Kernel of Gaussian \Rightarrow It's integral is a constant

$$\begin{aligned} &= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}(n-1)s^2} \int_{\mathbb{R}} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2} d\theta \\ &\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2}(n-1)s^2} \\ &\propto \text{Gamma}\left(\frac{n}{2}, \frac{(n-1)s^2}{2}\right) \end{aligned}$$