LECTURE - 10 (03/07/16) ①

$\hat{\theta}_{MLE} = BA := \frac{\# \text{HITS}}{\# \text{H+Bats}} = \frac{x}{n}$

$X_1, \ldots, X_n \overset{exch}{\sim} \text{Beta}(\theta)$

$\theta \sim \text{Beta}(\alpha, \beta)$

$\frac{1}{1} = 1$

$\theta | X \sim \text{Beta}(\alpha + x, \beta + n - x)$

$\frac{0}{1} = 0$

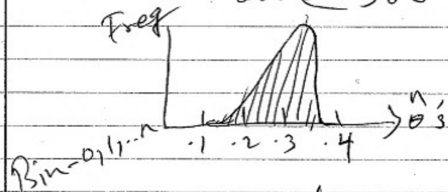
$\hat{\theta}_{MMSE} = \frac{\alpha + x}{\alpha + \beta + n}$

$= \rho E(\theta) + [1 - \rho]\hat{\theta}_{MLE}$

$\frac{\alpha + \beta}{\alpha + \beta + n}$

9,256 baseball players

all ≥ 500 + bats



Fit Beta

$\hat{\alpha}_{MLE} = 78.7$

$\hat{\beta}_{MLE} = 224.8$

$\theta \sim \text{Beta}(78.7, 224.8)$

$E(\theta) = \frac{78.7}{78.7 + 224.8} = .259$

$\frac{\alpha + \beta}{\alpha + \beta + n}$

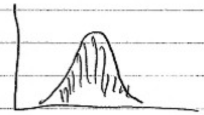
Too many pseudo counts.

②

$= \frac{303.5}{303.5 + n}(.259) + \frac{n}{303.5 + n}\bar{X} = .262$

$\rightarrow 23.7\%$ 76.7%

$\frac{303.5}{303.5 + n}(.259) + \frac{n}{303.5 + n}\bar{X} = .188$



Are you allowed to do it?

$\hat{\alpha}_{MLE} = 78.7 \rightarrow \theta \sim \text{Beta}(78.7, 224.8)$

$\hat{\beta}_{MLE} = 224.8 \rightarrow$ "Empirical Bayes Model"

$X_1, \ldots, X_n \overset{exch}{\sim} \text{Geom}(\theta) = (1-\theta)^x \theta$

0 0 0 ... 1 # failures

prob of first success

prob of x indep. failure

$\text{Supp}[X_i] = \{0, 1, 2, \ldots\} = \mathbb{N}_0$

Param space $\theta \in (0, 1)$

$p(X | \theta) = \prod_{i=1}^{n} p(x_i | \theta)$ given

e.g. $\hat{\theta} = 0.52$

$= \prod_{i=1}^{n} (1-\theta)^{x_i} \theta$ a number.

$(1-\theta)^{\sum x_i} \theta^n$ $.52^{100}$

(likelihood)

(proport) $\propto (1-\theta)^{\sum x_i}$

③

$p(\theta|x) \propto p(x|\theta)\, p(\theta)$ ← given x, not θ

$\propto (1-\theta)^{\sum x_i}\, \theta^{n}\, \overline{p(\theta)}$ → $(1-\theta)^a \theta^b$

can't take out

~~"if"~~

⇓ B conjugate

$\propto (1-\theta)^{\sum x_i}\, \theta^{n}\, \underbrace{(1-\theta)^{\beta-1}\, \theta^{\alpha-1}}_{Beta(\alpha,\beta)}$

$\propto Beta(n+\alpha,\ \beta+\sum x_i)$

Forget -1

$\propto Beta(\overset{\text{at least}}{\phi\,1+\alpha},\ \beta+1)$ (not) always proper

"Exponents are hyperparameters"

$\hat{\theta}_{MMSE} = \dfrac{n+\alpha}{n+\alpha+\sum x_i+\beta}$

$\hat{\theta}_{MAP} = \dfrac{n+\alpha-1}{n+\alpha+\sum x_i+\beta-2}$

MLE: $l(\theta;x) = n\ln\theta + \sum x_i \ln(1-\theta)$

$l'(\theta;x) = \dfrac{n}{\theta} - \dfrac{\sum x_i}{1-\theta} \overset{set}{=} 0$

$\dfrac{n}{\theta} = \dfrac{\sum x_i}{1-\theta}$

$n - n\theta = \theta \sum x_i$

$n = \theta(\sum x_i + n) \longrightarrow \hat{\theta}_{MLE} = \dfrac{n}{n+\sum x_i} = \dfrac{1}{1+\bar{x}}$

④

$\hat{\theta}_{MMSE}$

If Haldane prior $\alpha=0,\ \beta=0$

$\hat{\theta}_{MMSE} = \hat{\theta}_{MLE}$

If Laplace prior

$\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

$l''(\theta;x) = -\dfrac{n}{\theta^2} - \dfrac{\sum x_i}{(1-\theta)^2}$

$I(\theta) = E(-l''(\theta;x)) = E\left[\dfrac{n}{\theta^2} + \dfrac{\sum x_i}{(1-\theta)^2}\right]$

$= \dfrac{n}{\theta^2} + \dfrac{n\,E[x]}{(1-\theta)^2} = n\left(\dfrac{1}{\theta^2} + \dfrac{1-\theta/\theta}{(1-\theta)^2}\right)$

$= n\left(\dfrac{1-\theta+\theta}{\theta^2(1-\theta)}\right) \propto \theta^{-2}(1-\theta)^{-1}$

$E[x] = \dfrac{1-\theta}{\theta}$

$j(\theta) \propto \sqrt{I(\theta)} \propto \theta^{-1}(1-\theta)^{-1/2}$ Jeffreys Prior

$\propto Beta(0, 1/2)$

(improper)

$\hat{\theta}_{MMSE} \neq \rho E(\theta) + (1-\rho)\, \hat{\theta}_{MLE}$

$\hat{\theta}_{MMSE}^{-1} = \dfrac{n+\alpha+\sum x_i+\beta}{n+\alpha} = \dfrac{n+\sum x_i}{n+\alpha}\left(\dfrac{n}{n}\right) + \dfrac{\alpha+\beta}{n+\alpha}\left(\dfrac{\alpha}{\alpha}\right)$

$\dfrac{n}{n+\alpha}\ \dfrac{n+\sum x_i}{n} + \dfrac{\alpha}{n+\alpha}\ \dfrac{\alpha+\beta}{\alpha}$

$\dfrac{1}{\hat{\theta}_{MMSE}} = \rho\, \dfrac{1}{E(\theta)} + (1-\rho)\, \dfrac{1}{\hat{\theta}_{MLE}}$

Inverted $E[x]$

prior $\alpha/\alpha+\beta$

ρ-Rho

Harmonic mean

(5)

$$P(x) = \int_{\Theta} p(x|\theta)\, p(\theta)\, d\theta$$

Geom, Beta

Bayesian Conditional

$$P(x^*|x) = \int_{\Theta} p(x^*|\theta)\, P(\theta|x)\, d\theta$$

dim 1 — still Geom — still Beta

New ⟹ $x_1, x_n \xrightarrow{\text{exch}} \text{Neg Bin}(r, \theta)$ "r is known"

failures — r success

$\text{Supp}[X] = \{0, 1, \ldots\}$

$\theta \in (0,1)$

$r \in \mathbb{N}$

i.i.d

$$P(x|\theta) = \prod_{i=1}^{n} \binom{x_i + r - 1}{x_i} \theta^r (1-\theta)^{x_i}$$

$$\left(\prod_{i=1}^{n} \binom{x_i + r - 1}{x_i}\right) \theta^{rn} (1-\theta)^{\Sigma x_i} \propto \theta$$

x – failure, r – success (multiple) rn

$$P(\theta|x) \propto P(x|\theta)\, P(\theta)$$

proportionality constant

$$= \left(\prod_{i=1}^{n} \binom{x_i + r - 1}{x_i}\right) \theta^{rn} (1-\theta)^{\Sigma x_i}$$

$$\propto \theta^{rn} (1-\theta)^{\Sigma x_i}\, p(\theta)$$

(6)

$\theta \sim \text{Beta}(\alpha, \beta)$

$$\propto \theta^{rn} (1-\theta)^{\Sigma x_i}\, \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$\propto \text{Beta}(rn + \alpha, \Sigma x_i + \beta)$$

$$P(x) = \int P(x|\theta)\, P(\theta)\, d\theta$$ NegBin, Beta

$$p(x^*|x) = \int_0^1 p(x^*|x)\, P(\theta|x)\, d\theta$$ NegBin, Beta

$$\int_0^1 \left(\binom{x^* + r - 1}{x^*} \theta^r (1-\theta)^{x^*}\right) \frac{1}{B(rn+\alpha, \Sigma x_i + \beta)} \left(\theta^{rn+\alpha-1} \ldots^{\Sigma x_i + \beta}\right)$$

Harmonic

Beta Neg Bin (r, α, β) — $r = 3$

p.m.f. $: = \dfrac{\Gamma(r + x)}{x!\,\Gamma(r)} \dfrac{B(\alpha + r, \beta + x)}{B(\alpha, \beta)}$

Beta Geom $(1, n\alpha, \beta + \Sigma x_i)$ is

Beta NegBin (r, α, β) with $r = 1$

→ BetaNegBin $(r, \underbrace{rn + \alpha}_{\text{new } \alpha'}, \underbrace{\Sigma x_i + \beta}_{\text{new } \beta'})$