Lecture 13 3/28/16   Math 390 03

we learned that post. pred.
distr. was
a
convolution

$$\theta \mid X, \sigma^2 \sim N\left(\frac{\frac{\overline{X}n}{\sigma^2} + \frac{u_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} , \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \right)$$

↑
Variance
known
beforehand

$\underbrace{\qquad}_{\theta_p}$ $\underbrace{\qquad}_{\sigma_p^2}$

If $P(\theta) \propto 1$, $\theta_p = \overline{X}$, $\sigma_p^2 = \frac{\sigma^2}{n}$

overdispersed
normal
is
itself
normal ✓

$$X^a \mid X, \sigma^2 \sim N(\theta_p, \sigma_p^2 + \sigma^2)$$

What about?

$\sigma^2 \mid X, \theta \quad \rightarrow$ want inference for variance/SE considering
mean is known

Same like always
↓

$$P(\sigma^2 \mid X, \theta) \propto P(X \mid \sigma^2, \theta) P(\sigma^2 \mid \theta)$$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} \right) P(\sigma^2 \mid \theta)$$

$$\propto \frac{1}{\sqrt{(\sigma^2)}^n} e^{-\frac{1}{2\sigma^2} \Sigma(X_i - \theta)^2} P(\sigma^2 \mid \theta)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \Sigma(X_i - \theta)^2} P(\sigma^2 \mid \theta)$$

$$= (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}} P(\sigma^2 \mid \theta)$$

let $\hat{\sigma}^2 := \frac{1}{n} \Sigma(X_i - \theta)^2$
↑
MLE for σ^2 if θ known
(and consistent estimator)

What does it look like? Pattern matching
$$(\sigma^2)^q e^{-\frac{b}{\sigma^2}}$$

Recall

$$Y \sim \text{Gamm}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} Y^{\alpha-1} e^{-\beta Y} \qquad \Rightarrow \quad \frac{1}{Y} \sim \text{InvGamm}(\alpha, \beta)$$

$$W = \frac{1}{Y} \sim ? \qquad W = t(Y) = \frac{1}{Y} \quad \Rightarrow \quad t^{-1}(w) = \frac{1}{w}$$

$$f_W(w) = f_Y(t^{-1}(w)) \left| \frac{d}{dw}[t^{-1}(w)] \right|$$

$$E[W] = \frac{\beta}{\alpha-1} \quad \text{if } \alpha > 1$$

$$\text{Mode}[w] = \frac{\beta}{\alpha+1}$$

$$SE(W) = \frac{\beta}{\alpha-1} \frac{1}{\sqrt{\alpha-2}} \quad \text{for } \alpha > 2$$

$$= f_Y\left(\frac{1}{w}\right) \left| \frac{d}{dw}(\frac{1}{w}) \right|$$

$$= f_Y\left(\frac{1}{w}\right) \frac{1}{w^2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha-1)} e^{-\frac{\beta}{w}} w^{-2} = \frac{\beta^\alpha}{\Gamma(\alpha)} w^{-(\alpha+1)} e^{-\frac{\beta}{w}}$$

$$\underbrace{}_{\text{"Klesse"}}$$

$$P(w) \propto w^{-(\alpha+1)} e^{-\frac{\beta}{w}} \qquad \text{InvGamm}(\alpha,\beta) := \qquad \alpha, \beta \in (0, \infty)$$

$$\text{so} \quad \sigma^2 \sim \text{InvGamm}(\alpha, \beta)$$

$$P(\sigma^2 | X, \theta) \propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}} (\sigma^2)^{-(\alpha+1)} e^{-\frac{\beta}{\sigma^2}} \quad \boxed{\text{Conjugate}}$$

$$\propto (\sigma^2)^{-(\alpha+\frac{n}{2}+1)} e^{-\frac{1}{\sigma^2}\left(\frac{n\hat{\sigma}^2}{2} + \beta\right)}$$

$$\propto \text{InvGamm}\left(\alpha + \frac{n}{2}, \; \beta + \frac{n\hat{\sigma}^2}{2}\right) \qquad \text{Hyper params}$$

ν_0: How many samples did I see beforehand? $Y_1, \ldots, Y_{\nu_0}$

$$\sigma_0^2 = \frac{1}{\nu_0} \sum_{i=1}^{\nu_0} (Y_i - \theta)^2$$

guess as its prior variance

Convert to nonstandard

$$\alpha = \frac{\nu_0}{2}, \qquad \beta = \frac{\nu_0 \sigma_0^2}{2}$$

$$E[\sigma^2]: \frac{\frac{\nu_0 \sigma_0^2}{2}}{\frac{\nu_0}{2}-1} = \frac{\nu_0 \sigma_0^2}{\nu_0 - 2} = \frac{1}{\nu_0 - 2} \sum_{i=1}^{\nu_0} (Y_i - \theta)^2$$

$$\Rightarrow = \text{InvGamm}\left(\frac{\nu_0 + n}{2}, \; \frac{\nu_0 \sigma_0^2 + n \hat\sigma s^2}{2}\right)$$

The s^2 should be sigmahat^2 (sorry)

Jeffreys Prior?

$$\ell(\sigma^2; X, \theta) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \Sigma(X_i - \theta)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{n\hat{\sigma}^2}{2\sigma^2}$$

$$\ell'(\sigma^2; X, \theta) = -\frac{n}{2\sigma^2} + \frac{n\hat{\sigma}^2}{2(\sigma^2)^2}$$

$$\ell''(\sigma^2; X; \theta) = \frac{n}{2(\sigma^2)^2} - \frac{n\hat{\sigma}^2}{(\sigma^2)^3}$$

$$I(\sigma^2) = E\left[-\ell''(\sigma^2; X; \theta)\right] = -\frac{n}{2(\sigma^2)^2} + \frac{n}{(\sigma^2)^3} \underset{2}{\boxed{E[\hat{\sigma}^2]}} \underset{= \sigma^2}{} = \frac{n}{(\sigma^2)^2}\left(-\frac{n}{2} + n\right) = \frac{n}{2}\frac{1}{(\sigma^2)^2}$$

only this this's a func of X!

$\swarrow$ saying iid

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\theta)^2\right] = \frac{1}{n} n\, E(X-\theta)^2$$

$$= E[X^2] - 2\theta E[X] + \theta^2$$

$$\sigma^2 = E[X^2] - \theta^2 \qquad = \sigma^2 + \theta^2 - 2\theta^2 + \theta^2 = \sigma^2$$

unbiased estimator if θ known!

$$j(\sigma^2) \propto \sqrt{I(\sigma^2)} \propto \frac{1}{\sigma^2}$$

Q: proper? $\int_0^\infty \frac{1}{\sigma^2}d\sigma^2 = \infty$

$\Rightarrow$ NO!

$$\frac{1}{\sigma^2} = (\sigma^2)^{-1} e^{-\frac{0}{\sigma^2}} \quad '' \quad '' \quad \propto \text{Inv gamma}(0,0) \Rightarrow \text{conjugate}$$

$$\nu_0 = 0, \sigma_0^2 = 0 \qquad \hat{\sigma}^2_{MLE}$$
$$= \frac{1}{n}\Sigma(X_i-\theta)^2$$

$$\Rightarrow \sigma^2 | X, \theta \sim \text{Inv gamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

$$\hat{\sigma}^2_{MMSE} = \frac{\frac{n\hat{\sigma}^2}{2}}{\frac{n}{2} - 1} = \frac{n}{n-2}\hat{\sigma}^2 = \frac{1}{n-2}\Sigma(X_i-\theta)^2 \approx \frac{1}{n-1}\Sigma(X_i-\theta)^2$$

unbiased classic est.

Both θ, σ^2 unknown

$$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2) \, \underline{P(\theta, \sigma^2)}$$

joint prior

if you consider

θ, σ^2 ind

$$= P(\theta) P(\sigma^2)$$

he will return to this...

For *now*, use Jeffrey's prior

$$P(\theta, \sigma^2) \propto (1) \left(\frac{1}{\sigma^2} \right)$$

↑ Jeffrey's prior for $\theta | \sigma^2$

↑ Jeffrey's prior for $\sigma^2 | \theta$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} \left(\frac{1}{\sigma^2} \right)$$

Note:
$\sum (X_i - \theta)^2$

$$= \left(\frac{1}{\sqrt{2\pi}} \right)^n \left(\frac{1}{\sqrt{\sigma^2}} \right)^n e^{-\frac{1}{2\sigma^2} \sum (X_i - \theta)^2} \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{n}{2} - 1} e^{-\frac{1}{2\sigma^2} \left((n-1) S^2 + n (\bar{X} - \theta)^2 \right)}$$

Is this InvGamma?
No dim of $P(\theta, \sigma^2 | X) = 2$
InvGamma is univariate.

see next page

$$\sum (X_i - \theta)^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\sum \left(X_i - \bar{X} + \bar{X} - \theta \right)^2$$

$$\sum (X_i - \bar{X})^2 + \sum 2(X_i - \bar{X})(\bar{X} - \theta) + \sum (\bar{X} - \theta)^2$$

$$2 \left(X_i \bar{X} - \bar{X}^2 - X_i \theta + \bar{X}\theta \right)$$

$$2 \quad \sum X_i \bar{X} - \bar{X}^2 - \sum X_i \theta + \sum \bar{X}\theta$$

$$2 \left(n\bar{X}^2 - n\bar{X}^2 - n\bar{X}\theta + n\bar{X}\theta \right)$$

$$= \sum (X_i - \bar{X})^2 + \sum (\bar{X} - \theta)^2 = (n-1)s^2 + n(\bar{X} - \theta)^2$$

$$\text{let } s^2 := \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \underline{\text{purely a function of } X}$$

Proth 633   unbiased est for σ^2

$$E(s^2) = \sigma^2 \quad \text{Bessel's Correction?}$$

$$P(\theta, \sigma^2 | x) \propto \ldots$$

What do we do now?

If we knew this distr, we could take 2-dim expectation, matrix of variance, etc...

Recall $\quad f(x,y) = f(x|y)\, f(y) \quad$ (Byes Rule)

and $\quad f(x,y|z) = f(x|y,z)\, f(y|z) \quad$ (Ditto)

Here

$$P'(\theta, \sigma^2 | x) \propto P_1(\theta | \sigma^2, x)\, P_2(\sigma^2 | x)$$

$$P(\theta | \sigma^2, x) \propto (\sigma^2)^{-\frac{n}{2}-1}\, e^{-\frac{1}{2\sigma^2}\left((n-1)s^2\right)}\, e^{-\frac{1}{2\sigma^2} n (\bar{x}-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2} n (\bar{x}-\theta)^2}$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$P(\sigma^2 | x)$?

Recall..

$$f(x) = \int_{supp(y)} f(x,y)\, dy \quad , \quad f(x|z) = \int_{supp(y)} f(x,y|z)\, dy$$

$$P(\sigma^2 | x) = \int_{supp(\theta)} P(\theta, \sigma^2 | x)\, d\theta \propto \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1}\, e^{-\frac{1}{2\sigma^2}(n-1)s^2}\, e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}\, d\theta$$

why $\nwarrow$ $\mathbb{R}$

Kernel of Gaussian $\Rightarrow$ It's integral is a constant

$$= (\sigma^2)^{-\left(\frac{n}{2}+1\right)}\, e^{-\frac{1}{2\sigma^2}(n-1)s^2} \int_{\mathbb{R}} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}\, d\theta$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1}\, e^{-\frac{1}{2\sigma^2}(n-1)s^2}$$

$$\propto Inv\text{-}Gamma\left(\frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$