Lecture-11 (03/16/16) 1

$X|\theta \sim \text{Poisson}(\theta) := \frac{e^{-\theta}\theta^{x}}{x!} \propto \frac{\theta^{x}}{x!}$

$\text{Supp}\{X\} = \{0, 1, \ldots\}$

$\theta \in (0, \infty)$

$x|\theta \sim \text{Exp}(\theta) := \theta e^{-\theta x}$

$X|\theta \sim \text{Bin}(n,\theta) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x}$

$x|\theta \sim \text{Geo}(\theta) = (1-\theta)^{x-1}\theta$

$P(\theta|X) = \text{Poisson}(\theta) = \frac{e^{-\theta}\theta^{x}}{x!}$

$\propto e^{-\theta}\theta^{x}$

$\theta \in (0, \infty)$

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$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} \propto P(X|\theta)P(\theta)$

$P(X)$ const, integ.

$\theta \in (0, \infty)$

$\propto \theta^{x} e^{-\theta}$ $\quad P(\theta) \leftarrow$

graphical matching $\quad \theta^{a} e^{-b\theta}$

$\int_0^\infty \underbrace{\theta^{a} e^{-b\theta}\, d\theta}_{=c} =$

$t = b\theta$

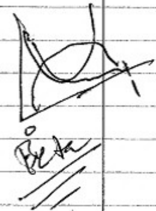
$\theta = \frac{t}{b}$

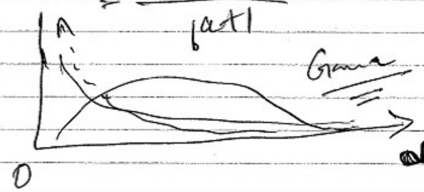
$d\theta = \frac{dt}{b}$

$= \frac{1}{b^{a+1}}\int_0^\infty t^{a} e^{-t}\, dt$

$= \frac{\Gamma(a+1)}{b^{a+1}}$

$\frac{1}{c}\int \cdots = 1$





$\theta \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$

$\text{Mode}(\theta) = \frac{\alpha-1}{\beta}$ for $\alpha \geq 1$ $\quad E(\theta) = \frac{\alpha}{\beta}$

$\text{Var}(\theta) = \frac{\alpha}{\beta^{2}}$

(3)

Param Space

$\alpha \in (0,\infty)$ $\beta \in (0,\infty)$

$\propto \theta^{\alpha-1} e^{-\beta\theta}$

$\propto (\theta^{x} e^{-\theta})(\theta^{\alpha-1} e^{-\beta\theta})$

$\propto \theta^{x+\alpha-1} e^{-(\beta+1)\theta}$

$n=1$ case!!!

"x is a single data"

$\text{Gamma}(x+\alpha, \beta+1)$

$\mathcal{L}(\theta; x_i) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} p(\theta)$

$\propto e^{-n\theta} \theta^{\sum x_i} \theta^{\alpha-1} e^{-\beta\theta}$

$= \theta^{\sum x_i + \alpha - 1} e^{-(n+\beta)\theta}$

$\propto \text{Gamma}(\sum x_i + \alpha, n+\beta)$ "x_i is a multiple data"

$\hat{\theta}_{MMSE} = E(\theta|x) = \frac{\sum x_i + \alpha}{n+\beta}$

$\hat{\theta}_{MAP} = \text{Mode}[\theta|x] = \frac{\sum x_i + \alpha - 1}{n+\beta}$

for $\sum x_i + \alpha \geq 1$

qgamma$(.5, \sum x_i + \alpha, n+\beta)$

(4) $\theta \sim \text{Gamma}(\alpha, \beta)$ $\propto \theta^{\alpha-1} e^{-\beta\theta}$

Principle of Indiff.

$p(\theta) \propto 1$

$\propto \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} (1)$

$\int_0^\infty = 1$

$p(\theta) \propto 1 = \text{Gamma}(1, 0)$ (α hyperpara, β hyperpara)

$\Rightarrow l_n(\theta; x) = -n\theta + \sum x_i \ln(\theta) - \sum \ln(x_i!)$

$l_n'(\theta; x) = -n + \frac{\sum x_i}{\theta} + 0 \overset{set}{=} 0$

$\hat{\theta}_{MLE} = \bar{X}$

Under $p(\theta) \propto 1$

$\hat{\theta}_{MMSE} = \frac{\sum x_i + 1}{n} = \frac{\sum x_i}{n} + \frac{1}{n}$

$= \hat{\theta}_{MLE} + \frac{1}{n} \to \hat{\theta}$ ($n \to \infty$)

$\hat{\theta}_{MAP} = \frac{\sum x_i + 1 - 1}{n} = \frac{\sum x_i}{n} = \hat{\theta}_{MLE}$

(5)

$\theta \sim \text{Gamma}(0,0)$ improper;

$\hat{\theta}_{MMSE} = \hat{\theta}_{MLE}$

$\hat{\theta}_{MAP} = \hat{\theta}_{MLE} - \frac{1}{n}$

$E(\theta|X) = \frac{\sum x_i + \alpha}{n+\beta}$

$P(\cancel{A})$ more important

$\frac{\sum x_i}{n+\beta}\left(\frac{n}{n}\right) + \frac{\alpha}{n+\beta}\left(\frac{\beta}{\beta}\right)$

$= \frac{n}{n+\beta} \frac{\sum x_i}{n} + \frac{\beta}{n+\beta} \frac{\alpha}{\beta}$

$= \rho E(\theta) + (1-\rho)\hat{\theta}_{MLE}$

$l''(\theta; x) = 0 - \frac{\sum x_i}{\theta^2} = \frac{-n\bar{x}}{\theta^2}$

$I(\theta) := E[-l''(\theta; X)] = \frac{n}{\theta^2} E[\bar{X}] = \frac{n}{\theta}$

$j(\theta) \propto \sqrt{I(\theta)}$

$= \sqrt{n/\theta}$

$\propto \theta^{-1/2}$

(6)

$\frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$ $\quad \boxed{\beta = 0}$

$\propto \theta^{\alpha-1} e^{-\beta\theta}$

$j(\theta) \propto \text{Gamma}(\tfrac{1}{2}, 0)$

$E(\theta)$
$SE(\theta)$
ρ

$\text{Gamma}(\alpha, \beta)$	general	$\text{Beta}(\alpha, \beta)$
$\text{Gamma}(1, 0)$ – no idea before –	Laplace	$\text{Beta}(1,1)$
$\text{Gamma}(0, 0)$	Hadame	$\text{Beta}(0,0)$
$\text{Gamma}(\tfrac{1}{2}, 0)$	Jeffrey	$\text{Beta}(\tfrac{1}{2}, \tfrac{1}{2})$

If you get a lot of data, who cares!

$p(X) = \int_\Theta p(X|\theta) p(\theta) d\theta$ Bayesian Cond.

prior pred dist.

$\text{Beta Bin}(n, \alpha, \beta)$

with updated parameters

~~Beta Bin~~

$p(X^*|X) = \int_\Theta p(X^*|\theta) p(\theta|X) d\theta$

posterior pred dist

$\text{Beta Bin}(n, \alpha+x, \beta+n-x)$

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$$p(x) = \int_0^\infty \frac{e^{-\theta}\theta^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \, d\theta$$

$\boxed{n=1}$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{x!} \int_0^\infty \theta^{x+\alpha-1} e^{-(\beta+1)\theta} \, d\theta$$

not α anymore

exactly (p.not)

α Gamma just applying

$$\boxed{\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{x!} \frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}} \int_0^\infty \frac{(\beta+1)^{x+\alpha}}{\Gamma(x+\alpha)} \theta^{x+\alpha-1} e^{-(\beta+1)\theta} \, d\theta = 1$$

$$\rightarrow \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^x \frac{1}{x!}$$

If $\alpha \in \mathbb{N}$ $\quad \frac{(x+\alpha-1)!}{(\alpha-1)!\,x!} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^x$

Let $p = \frac{\beta}{\beta+1} \in (0,1)$, $\alpha = r$

$\frac{1}{\beta+1} = 1-p$

$$p(x) = \binom{x+r-1}{x} p^r (1-p)^x = \text{NegBin}(r, p)$$

$\alpha, \frac{\beta}{\beta+1}$

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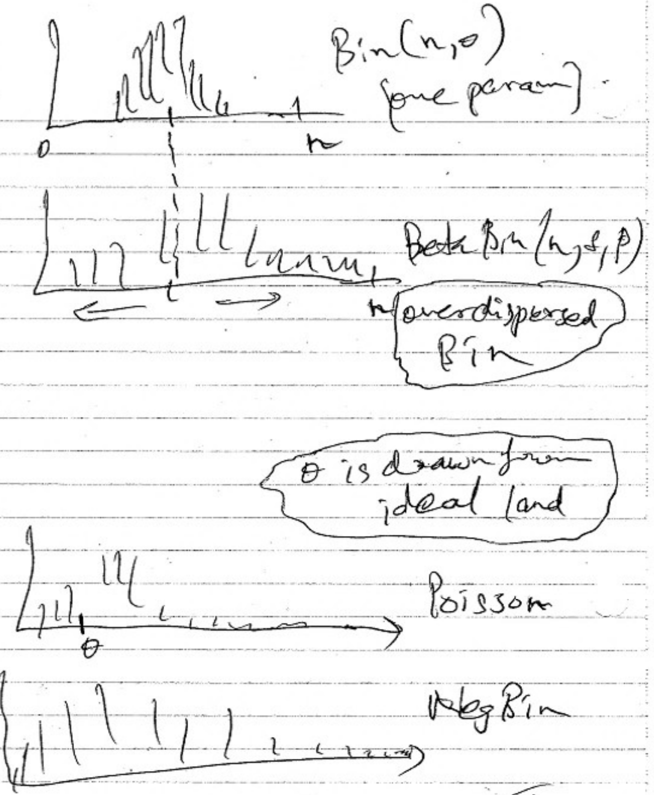

$n=1$ $\quad x^* \mid x \sim \text{NegBin}\left(\sum x_i + \alpha, \frac{n+\beta}{n+\beta+1}\right)$

$p(x^* \mid x)$

$n \to \infty \quad 1$

$X_1, \ldots, X_n \sim Exp(\theta) := \theta e^{-\theta x}$

$$\mathcal{L}(\theta; x) = \prod_{i=1}^{n} \theta e^{-\theta x} = \theta^n e^{-\theta \sum x_i}$$

$$p(\theta|X) \propto p(x|\theta)\, p(\theta)$$

$$= \theta^n e^{-\theta \sum x_i}\, p(\theta) \qquad \theta \sim \text{Gamma}(\alpha, \beta)$$

$$\propto \theta^n e^{-\theta \sum x_i}\, \theta^{\alpha-1} e^{-\beta\theta}$$

$$\propto \text{Gamma}(n+\alpha, \sum x_i + \beta)$$

All the Work ---- $\hat{\theta}_{MMSE}$

Jeffs

$$\ln(\mathcal{L}(\theta; x)) = n \ln\theta - \theta \sum x_i$$

$$\frac{d}{d\theta} \ln' = \frac{n}{\theta} - \sum x_i = 0 \to \hat{\theta} = n \frac{1}{\sum x_i} = \cancel{n}\frac{1}{\cancel{n}\bar{x}} = \frac{1}{\bar{x}}$$

$$\ln'' = -\frac{n}{\theta^2}$$

$$I(\theta) = E(-\ln''(\;)) = \frac{n}{\theta^2}$$

$$j(\theta) \propto \sqrt{I(\theta)}$$

$$= \sqrt{n/\theta^2} \propto \frac{1}{\theta} \propto \theta^{-1}$$

$$\propto \theta^{\alpha-1} e^{-\beta\theta}$$

$$j(\theta) \propto \text{Gamma}(0, 1)$$