$BA := \dfrac{\#\text{HITS}}{\#\text{AT BAT}} = \dfrac{X}{n}$

$X_1, \dots, X_n \sim \text{Bern}(\theta)$

$\frac{1}{1} = 1, \ \frac{0}{1} = 0$ ← frequentist fails

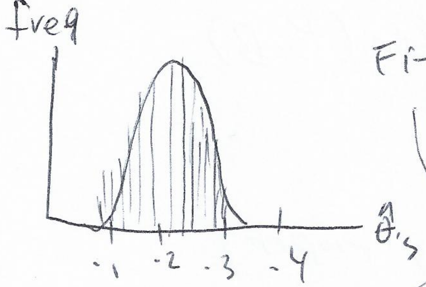
slap prior $\theta \sim \text{BetA}(\alpha, B) \longrightarrow \theta | X \sim \text{BetA}(\alpha + x, B + n - x)$

$$\theta_{MMSE} = \frac{\alpha + x}{\alpha + B + n}$$

$$= \underbrace{\rho}_{\frac{\alpha+B}{\alpha+B+n}} E(\theta) + [1 - \rho]\hat{\theta}_{MLE}$$

9,256 Baseball players

plot all < 500 at bats



Fit Beta

$\hat{\alpha}_{MLE} = 78.7$

$\hat{B}_{MLE} = 224.8$

"Empirical Baye's model"

$\theta \sim \text{BetA}(78.7, 224.8)$

$$E(\theta) = \frac{78.7}{78.7 + 224.8} = .2593$$

$X_1, \dots, X_n \sim \underset{\text{\# failure}}{Geo(\theta)} = \underbrace{(1-\theta)^x}_{\text{prob of x indep failure}} \theta$ } - Prob of final Success

0 0 0 1

$\text{supp}[X_i] = \{0, 1, 2, \dots\} = \mathbb{N}_0$

Parameter Space $\theta \in (0, 1)$

$p(x|\theta) = \prod_{i=1}^{n} p(x_i|\theta)$ if θ is given, it's a Constant.

$= \prod_{i=1}^{n} (1-\theta)^{x_i} \theta$

$= (1-\theta)^{\sum x_i} \theta^n$ is constant of intergration, factor

Kernel → $\propto (1-\theta)^{\sum x_i}$ porportional likelihood

$p(\theta|x) \propto p(x|\theta) p(\theta)$

$\propto (1-\theta)^{\sum x_i} \theta^n \underbrace{(1-\theta)^{\beta-1} \theta^{\alpha-1}}_{}$ Beta(α, β)

$= \theta^{n+\alpha-1} (1-\theta)^{\beta + \sum x_i - 1}$

$\propto \text{Beta}(n+\alpha, \beta + \sum x_i)$ → ~~not~~ (always proper)

Posterior Expectation $\hat{\theta}_{MMSE} = \dfrac{n+\alpha}{n+\alpha+\sum x_i + \beta}$

$\hat{\theta}_{map} = \dfrac{n+\alpha-1}{n+\alpha+\sum x_i - 2}$

If halden prior

$\alpha = 0, \beta = 0$

$\hat{\theta}_{MMSE} = \hat{\theta}_{MLE}$

MLE

$\ell(\theta; x) = n \ln \theta + \sum x_i \ln(1-\theta)$

$\ell'(\theta; x) = \dfrac{n}{\theta} - \dfrac{\sum x_i}{(1-\theta)}$

$\dfrac{n}{\theta} = \dfrac{\sum x_i}{(1-\theta)} = n - n\theta = \theta \sum x_i$

$n = \theta(\sum x_i + n)$

$\hat{\theta}_{MLE} = \dfrac{n}{n + \sum x_i} = \dfrac{1}{1+\bar{x}}$

If Laplace prior

$\hat{\theta}_{MAP} = \hat{\theta}_{MLE}$

If Halden prior

$\hat{\theta}_{MMSE} = \hat{\theta}_{MLE}$

$$l''(\theta;x) = -\frac{n}{\theta^2} - \frac{\sum x_i}{(1-\theta)^2}$$ ⇐ Jeffery prior

$$I(\theta) = E(-l'') = E\left[\frac{n}{\theta^2} + \frac{\sum x_i}{(1-\theta)^2}\right] = \frac{n}{\theta^2} + \frac{n\,E(x)}{(1-\theta)^2}$$

$$\Rightarrow n\left(\frac{1}{\theta^2} + \frac{\frac{\cancel{(1-\theta)}}{\theta}}{(1-\theta)^{\cancel{2}}}\right) = n\left(\frac{1-\cancel{\theta}+\cancel{\theta}}{\theta^2(1-\theta)}\right) \propto \theta^{-2}(1-\theta)^{-1}$$

Jeffery's prior

$$j(\theta) \propto \sqrt{I(\theta)} \propto \theta^{-1}(1-\theta)^{-\frac{1}{2}} \propto Beta\left(0, \tfrac{1}{2}\right)$$ (improper)

$$\hat{\theta}_{MMSE} \neq e\,E(\theta) + (1-e)\,\hat{\theta}_{MLE}$$

$= \frac{n}{n+\alpha}$ $\quad \frac{n+\sum x_i}{n}$ $\quad + \frac{\alpha}{n+\alpha}$ $\quad \frac{\alpha+\beta}{\alpha}$

$$\Rightarrow \frac{1}{\hat{\theta}_{MMSE}} = e\,\frac{1}{E(\theta)} + (1-e)\,\frac{1}{\hat{\theta}_{MLE}}$$

$$P(x) = \int_{\Theta} \underbrace{P(X|\theta)}_{\text{Geom}}\ \underbrace{P(\theta)}_{\text{Beta}}\ d\theta$$

Bayseian Conditionalism

$$P(X^*|X) = \int_{\Theta} \underbrace{P(X|\theta)}_{\text{Geom}}\ \underbrace{P(\theta|X)}_{\text{Beta}}\ d\theta$$

Conjugacy - Start with BetA end with beta

dim 1

end of geometric BetA model

$X_1, \dots, X_n \sim$ Negative Bin (r, θ) r is Known

# of failure until success

$supp(X) = \{0, 1, \dots, \}$

$\theta \in (0,1)$ $\qquad r \in \mathbb{N}$

$$P(X_i|\theta) = \prod_{i=1}^{n} \theta^r (1-\theta)^{X_i} \binom{X_i + r - 1}{X_i} \quad iid$$

$- \; - \; - \; - \; - \; - \; - \; - \; - \; \underline{1}$

X failure

$r-1$ success

$$\left(\prod_{i=1}^{n} \binom{X_i + r - 1}{X_i}\right) \theta^{rn} (1-\theta)^{\Sigma X_i}$$

$\propto$

$$P(\theta|X) \propto P(X|\theta) P(\theta)$$

$$= \left(\prod_{i=1}^{n} \binom{X_i + r - 1}{X_i}\right) \theta^{rn} (1-\theta)^{\Sigma X_i}$$

$$\propto \theta^{rn} (1-\theta)^{\Sigma X_i} \quad \theta^{\alpha - 1} (1-\theta)^{\beta - 1}$$

$$\propto \text{Beta}(rn + \alpha, \Sigma X_i + \beta)$$

Always proper

3)

$$\int_0^1 \left(\binom{x^*+r-1}{x^*} \theta^r (1-\theta)^{x^*} \right) \frac{1}{B(rn+\alpha,\ \Sigma x_i+\beta)} \theta^{rn+\alpha-1} (1-\theta)^{\Sigma x_i+\beta-1} \, d\theta$$

$\vdots$ LOTS of math

$$= \text{BetaNegBin}(r,\ rn+\alpha,\ \Sigma x_i+\beta) := p(x^* \mid x)$$

$$Y \sim \text{BetaNegBin}(r, \alpha, \beta) := \frac{\Gamma(r+x)}{x!\,\Gamma(r)} \frac{B(\alpha+r,\ \beta+x)}{B(\alpha, \beta)}$$

$\text{BetaGeo}(\alpha, \beta) :=$ H.W

(Speial case $r=1$)