390 -8

2/29/16 1)

$$\hat{\theta}_{MMSE} = \underbrace{e}_{\frac{\alpha+\beta}{\alpha+\beta+n}} \underbrace{E(\theta)}_{\frac{\alpha}{\alpha+\beta}} + (1-e) \underbrace{\hat{\theta}_{MLE}}_{\frac{X}{n}}$$

$X_1, \dots, X_n \overset{exch}{\sim} Bern(\theta)$, $\theta \sim Beta(\alpha, \beta) \Rightarrow \frac{\theta}{X} \sim Beta(\alpha + x, \beta + n - x)$

$\alpha = \beta = 0$

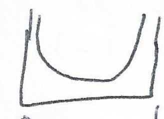
$$Beta(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{\Gamma(0)\Gamma(0)}{\Gamma(0)} \frac{1}{\theta(1-\theta)} \qquad \int_0^1 \frac{1}{\theta(1-\theta)} d\theta \overset{?}{=} 1$$

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx$$

$$\ln(\theta) - \ln(1-\theta) \Big]_0^1 = \infty$$

$$\Gamma(0) = \int_0^\infty \frac{1}{x e^{x}} dx$$

$\theta \sim Beta(0,0) \leftarrow$ "improper" "Haldare Prior" 

<u>Improper Prior</u>: intergrates to ∞.

$P(\theta | X) \propto P(X|\theta) P(\theta) \leftarrow$ not a proper distribution

Bay

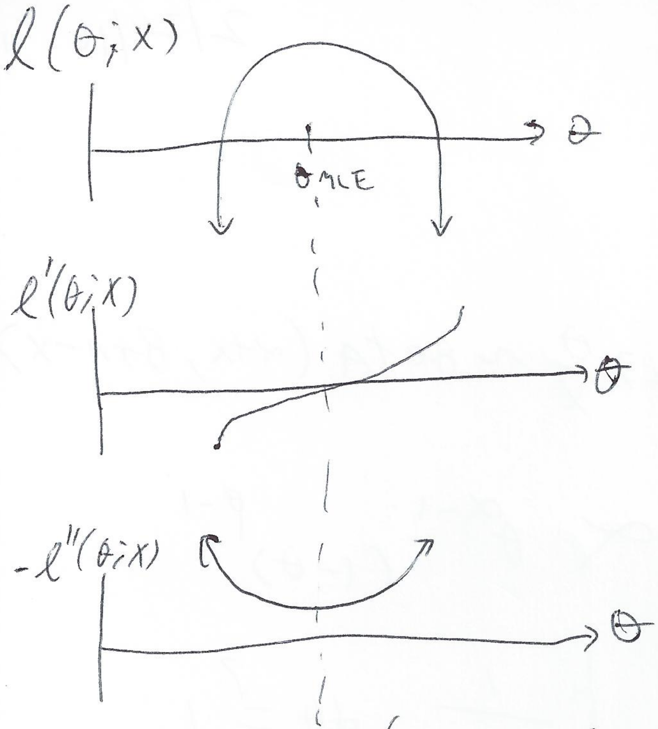
$$P(X;\theta) = \mathcal{L}(\theta; X) = \prod_{i=1}^{n} P(x_i;\theta) \qquad \text{log-likelyhood}$$

$$\hat{\theta}_{MLE} := \underset{\theta \in \Theta}{\operatorname{argmax}} \{ \mathcal{L} \} = \underset{\theta \in \Theta}{\operatorname{argmax}} \left\{ \ln\left(\prod_{i=1}^{n} P(x_i;\theta)\right) \right\}$$

$$\ln\left(\prod_{i=1}^{n} P(x_i;\theta)\right) = \ell(\theta; X)$$

$$\frac{d}{d\theta}[\ell(\theta; x)] = 0$$

$\ell'(\theta; x) = 0$ score function



$$I(\theta); = E_x\left[-\ell''(\theta;x)\right]$$

fisher information

$$= Var_x\left[\ell'(\theta;x)\right]$$

$$= E_x\left[(\ell'(\theta;x))^2\right] - \underbrace{\left[E(\ell'(\theta;x))\right]^2}_{=0}$$

$$\hat{\theta}_{MLE} \overset{Asympto}{\sim} N\left(\theta, SE(\hat{\theta}_{MLE})\right) = \left(\sqrt{\frac{1}{I(\theta)}}\right)^2$$

more info = smaller variance.

$x_1, \dots, x_n \overset{iid}{\sim} Bern(\theta)$

$$\mathcal{L}(\theta;x) = \prod_{i=1}^{n} p(x;\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$$

$$= \Sigma x_i \ln(\theta) + (n-\Sigma x_i)\ln(1-\theta)$$

$$\ell'(\theta;x) = \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{1-\theta}$$

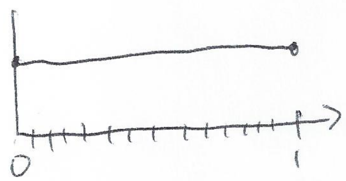
$$= \sum_{i=1}^{n} \frac{x_i}{\theta} - \frac{1-x_i}{1-\theta}$$

$$\ell''(\theta;x) = \sum_{i=1}^{n} -\frac{x_i}{\theta^2} - \frac{1-x_i}{(1-\theta)^2} = -\frac{n\bar{x}}{\theta^2} - \frac{n-n\bar{x}}{(1-\theta)^2}$$

$$-\ell''(\theta;x) = \frac{n\bar{x}}{\theta^2} + \frac{n-n\bar{x}}{(1-\theta)^2} = n\left(\frac{\bar{x}}{\theta^2} + \frac{1-\bar{x}}{(1-\theta)^2}\right)$$ 2)

$X_1,\dots,X_n \overset{ex}{\sim} Bern(\theta)$ pdf $f(\theta)$

$\theta \sim Beta(\alpha,\beta) = U(0,1) = 1$

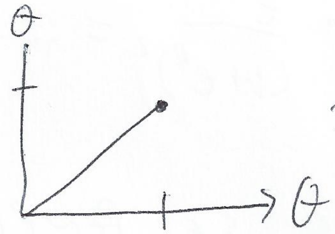


$g(\theta) :=$

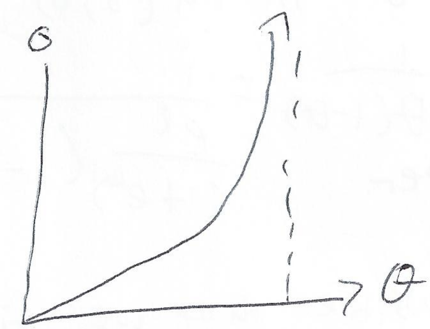
$odds(\theta) := \frac{\theta}{1-\theta}$

$(\theta \in (0,1)$

$\theta \in [0,\infty)$



$\Rightarrow$

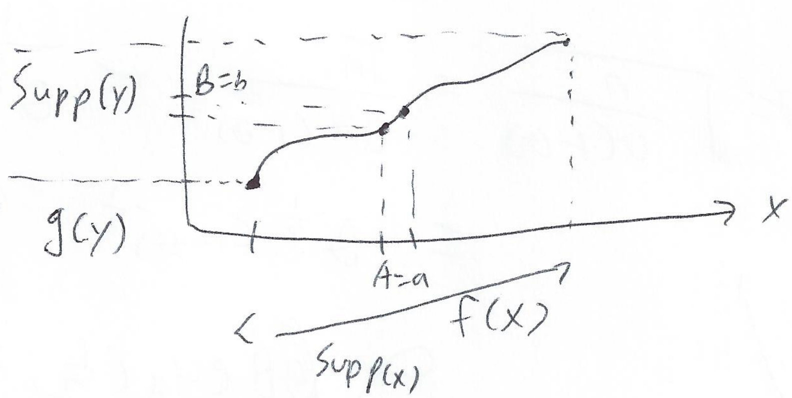


$Y = t(X)$

X is a r.v

Y is another r.v $g(y)$

t is 1:1



$P(X\in A) \approx f(a)A$

$P(Y\in B) \approx g(b)B$

$f(x)|dx| = g(y)|dy|$

$g(y) = f(x)\left|\frac{dx}{dy}\right|$

$Y = t(x)$

$\Longleftrightarrow$

$x = t^{-1}(y)$

$g(y) = f(t^{-1}(y))\frac{d}{dy}[t^{-1}(y)]$

$l := \text{logit}(\theta) := \ln(\theta) = \ln\left(\frac{\theta}{1-\theta}\right) \in \mathbb{R}$

$\iff \theta = \frac{e^l}{1+e^l} := \text{expit}(l)$

$g(l) = f\left(\frac{e^l}{1+e^l}\right)\left|\frac{d}{dl}\left[\frac{e^l}{1+e^l}\right]\right| = \frac{e^l}{(1+e^l)^2} \neq 1$

Consider $\theta \sim \text{Beta}(0,0)$

$p(\theta) \propto \frac{1}{\theta(1-\theta)} = \frac{1}{\frac{e^l}{(1+e^l)}\left(1-\frac{e^l}{1+e^l}\right)} \cdot \frac{e^l}{(1+e^l)^2} = \dots = 1$

improper

Is it possible that we find p s.t $\theta \sim p(\theta)$
which is the same across all 1:1 trasformation of θ

Binomial Case

$p(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta(1-\theta)}} = \frac{\sqrt{n}}{\theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}} \propto \frac{1}{\theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}}$

$= \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}$

$\propto \text{Beta}(\frac{1}{2}, \frac{1}{2})$

"arcsin dist"

Beta(0,0)
$\cap$
Beta($\frac{1}{2}, \frac{1}{2}$)
$\cap$
Beta(1,1)

$g(o) = f\left(\frac{o}{o+1}\right)\left|\frac{d}{do}\left[\frac{o}{o+1}\right]\right|$

$= f\left(\frac{o}{o+1}\right)\frac{1}{(o+1)^2}$

$= \frac{1}{B(\frac{1}{2},\frac{1}{2})}\left(\frac{o}{o+1}\right)^{-\frac{1}{2}}\left(1-\frac{o}{o+1}\right)^{\frac{1}{2}}\frac{1}{(o+1)^2}$

$\overset{\text{por}}{\propto} \left(\frac{o}{o+1}\,\frac{1}{o+1}\right)^{-\frac{1}{2}}\frac{1}{(o+1)^2}$

3)

$$g(\theta) = f\left(\frac{\theta}{\theta+1}\right)\left|\frac{d}{d\theta}\left[\frac{\theta}{\theta+1}\right]\right|$$

$$= f\left(\frac{\theta}{\theta+1}\right)\frac{1}{(\theta+1)^2}$$

$$= \frac{1}{B(\frac{1}{2},\frac{1}{2})}\left(\frac{\theta}{\theta+1}\right)^{-\frac{1}{2}}\left(1-\frac{\theta}{\theta+1}\right)^{-\frac{1}{2}}\frac{1}{(\theta+1)^2} \propto \left(\frac{\theta}{\theta+1}\frac{1}{\theta+1}\right)^{-\frac{1}{2}}\frac{1}{(\theta+1)^2}$$

$$\left(\frac{\theta}{(\theta+1)^2}\right)^{-\frac{1}{2}}\frac{1}{(\theta+1)^2}$$

$$\sqrt{\frac{(\theta+1)^2}{\theta}}\frac{1}{(\theta+1)^2} = \frac{\theta+1}{\sqrt{\theta}}\frac{1}{(\theta+1)^2} = \dots = \frac{1}{\theta(\theta+1)}$$

$$\theta \sim p(\theta),\ p(\theta) \propto \sqrt{I(\theta)}$$

WTS $p(\phi) \propto \sqrt{I(\theta)}$ $\forall t$ 1:1 s.t. $\phi = t(\theta)$

$$p_\phi(\phi) = p_\theta(t^{-1}(\phi))\left|\frac{d\theta}{d\phi}\right|$$

$$= p_\theta(\theta)\left|\frac{d\theta}{d\phi}\right| \propto \sqrt{I(\theta)}\left|\frac{d\theta}{d\phi}\right|$$

$$= \sqrt{I(\theta)\left(\frac{d\theta}{d\phi}\right)^2}$$

Jefferey prior.

$$\Rightarrow \sqrt{E\left[l'(\theta;X)^2\right]\left(\frac{d\theta}{d\phi}\right)^2}$$

$$= \sqrt{E\left[\left(\frac{d}{d\theta}[\ln(\mathcal{L})]\right)^2\right]\left(\frac{d\theta}{d\phi}\right)^2}$$

$$= \sqrt{\left(\frac{d\ln(\mathcal{L})}{d\theta}\frac{d\theta}{d\phi}\right)^2} = \sqrt{E\left(\frac{d}{d\phi}\ln(\mathcal{L})\right)^2} = \sqrt{I(\phi)}$$