LECTURE - 14 (03-30-16)

①

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\underbrace{P(\theta,\sigma^2)}_{\propto \frac{1}{\sigma^2} \propto N\,InvGama} \propto (\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)}$$

$$s^2 := \frac{1}{n-1}\overbrace{\sum (x_i-\bar{x})^2}^{SSE}$$

$$= \underbrace{e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}}_{\substack{ker(\theta|\sigma^2,x) \\ \propto N\left(\bar{x},\frac{\sigma^2}{n}\right)}} \underbrace{(\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-(n-1)s^2/\sigma^2}}_{ker(\sigma^2|X) \propto InvGama\left(\frac{n}{2},\frac{(n-1)s^2}{2}\right)}$$

$$= P(\theta|\sigma^2,x)\,P(\sigma^2|X)$$

Bayes Rule

$$P(\sigma^2|X) = \int_{supp[\theta]} P(\theta,\sigma^2|x)\,d\theta$$

$$\propto \frac{1}{d}\int_{\mathbb{R}} \underline{e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}} \underbrace{\left\{(\sigma^2)^{-\left(\frac{n}{2}+1\right)} e^{-\left(\frac{(n-1)s^2}{2}\right)/\sigma^2}\right\}}_{Const.} d\theta$$

$$\propto d\; Ker(\theta|\sigma^2,x)$$

$$\propto N\left(\bar{x},\frac{\sigma^2}{n}\right)$$

$$k(\theta,\sigma^2|X) \propto N\,InvGama$$

$$Z_1,\dots,Z_\nu \overset{iid}{\sim} N(0,1)$$

$$\sum_{i=1}^{\nu} Z_i^2 \sim \chi^2_\nu := \frac{Z^{-\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)}\theta^{\frac{\nu}{2}-1}e^{-\theta/2}$$

$$= Gamma\left(\frac{\nu}{2},\frac{1}{2}\right)$$

②

$$x_1,\dots,x_n \overset{iid}{\sim} N(\quad)$$

$$\frac{n-1}{\sigma^2}s^2 \sim \chi^2_{n-1}$$

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N\left(\mu,\left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$s \approx \sigma$

$$\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim T_{n-1} \to \frac{Z_0}{\sqrt{\frac{\chi^2_{n-1}}{n}}}$$ Standard T (Student's T)

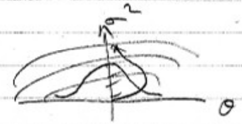
$$\xrightarrow{n\to\infty} \sim N\left(\mu,\frac{\sigma^2}{n}\right)$$

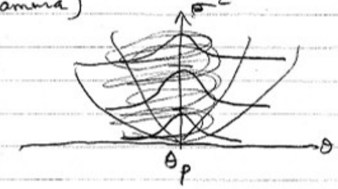
$$\bar{X} \sim T_{n-1}\left(\mu,\frac{s}{\sqrt{n}}\right)$$ general T

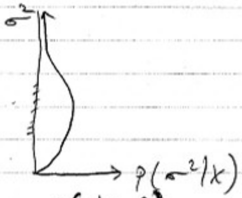
$$T_{n-1} := \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)\sqrt{\nu\pi}}\left(1+\frac{1}{\nu}\sigma^2\right)$$

Steps:

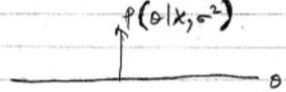
① Draw σ^2 from $\sigma^2|X$ (InvGamma)







② Draw θ from $\theta|\sigma^2,x$ (Normal using σ^2 from step 1.



③ Draw X^* from Normal using θ,σ^2 from step 1, 2.

③

In other way,

$$p(\theta,\sigma^2|X) \propto p(X|\theta,\sigma^2)\,p(\theta,\sigma^2) \propto (\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\left((n-1)s^2+n(\bar{x}-\theta)^2\right)}\, p(\theta,\sigma^2)$$

$p(\theta,\sigma^2) = p(\theta|\sigma^2)\,p(\sigma^2)$ not based on X

$\gamma^2 \approx \infty$ if 'σ' smaller, better 'θ'

$\theta|\sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{m}\right)$ if 'm' large, better idea of 'θ'

$\sigma^2 \sim \text{InvGamma}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$ ν - pseudo

'Shrinkage Estimator'

$$= N\left(\frac{n}{n+m}\bar{x} + \frac{m}{n+m}\mu_0,\ \frac{\sigma^2}{n+m}\right)$$

'n' large - $\bar{x}$
'n' small - μ_0 (prior)

$$\text{InvGamma}\left(\frac{\nu_0+n}{2},\ \frac{\overbrace{\nu_0\sigma_0^2}^{\text{prior SSE}} + \overbrace{(n-1)s^2}^{\text{SSE}} + \frac{mn}{n+m}(\bar{x}-\mu_0)^2}{2}\right)$$

Draw

$$\text{Var}\begin{bmatrix}\theta\\ \sigma^2\end{bmatrix} = \begin{bmatrix}\text{Var}(\theta) & \text{Cov}(\theta,\sigma^2)\\ & \text{Var}(\sigma^2)\end{bmatrix}$$

p.241
$X \sim N(15, 3.2^2)$
$U \sim U(0,1)$
$F^{-1}(U)$
Computer Simulation

$$p(\theta|X) = \int \overbrace{p(\theta|\sigma^2,x)\,p(\sigma^2|x)}^{p(\theta,\sigma^2|X)}\,d\sigma^2$$

Finding 'θ'

$$p(x^*|x) = \int p(x^*|\theta)\,p(\theta|x)\,d\theta$$

marginalization - elaborate average

④

$$N\left(\underbrace{\frac{n}{n+m}\bar{x} + \frac{m}{n+m}\mu_0}, \frac{\sigma^2}{n+m}\right)\ \text{InvGamma}(\ ,\ )$$

$$p(\theta|x) = \int_0^\infty \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n+m}}}\, e^{-\frac{1}{2}\frac{\sigma^2}{n+m}\left(\theta-\left(\frac{n}{n+m}\bar{x}+\frac{m}{n+m}\mu_0\right)\right)^2}\left(\ \right)\ \propto (\sigma^2)^{-\left(\frac{\nu_0+n}{2}+1\right)}\, e^{-\frac{\nu_0\sigma_0^2+(n-1)s^2+\frac{mn}{n+m}(\bar{x}-\mu_0)^2}{2}\Big/\sigma^2}\, d\sigma^2$$

$$p(\theta,\sigma^2|x) \propto p(x|\theta,\sigma^2)\,p(\theta,\sigma^2) \propto (\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\left((n-1)s^2+n(\bar{x}-\theta)^2\right)}\, p(\theta,\sigma^2)$$

$$\propto \frac{1}{\sigma^2}$$

$\mu_0 = 0$ $\quad m \to 0$

$\nu_0 = 0$ $\quad \sigma_0^2 \to 0$

$A = (n-1)s^2 + n(\bar{x}-\theta)^2$

$t = \frac{A}{2\sigma^2}$ $\qquad \sigma^2 = \frac{A}{2t}$

$\frac{d\sigma^2}{dt} = -\frac{A}{2t^2}$ $\qquad d\sigma^2 = \frac{A}{2t}\,dt$

$$= \int_\infty^0 \left(\frac{A}{2t}\right)^{-\left(\frac{n}{2}+1\right)} e^{-(t)}\left(-\frac{A}{2t^2}\,dt\right)$$

$$= -\int \frac{A^{-n/2}}{2^{-n/2-1}\, t^{-n/2-1+2}}\, e^{-t}\,dt$$

$$\propto A^{-n/2}\int_0^\infty t^{n-2/2}\, e^{-t}\,dt$$

$$\propto A^{-n/2} = \left((n-1)s^2 + n(\bar{x}-\theta)^2\right)^{-n/2}$$

$$\propto \left(1 + \frac{n(\bar{x}-\theta)}{(n-1)s^2}\right)^{-n/2}$$

(5)

$X \sim T_N(\mu, \sigma^2)$

$$:= \frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\sqrt{2\pi\sigma^2}}\left(1+\frac{1}{N}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{N+1}{2}}$$

$$\propto \left(1+\frac{1}{N}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-N+1/2}$$

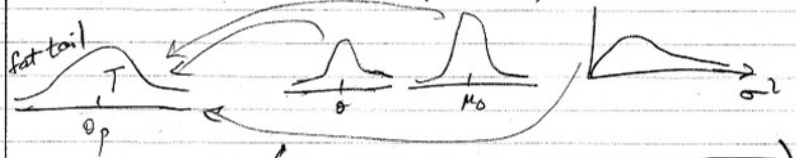
$$\frac{1}{N}\frac{n(\bar{x}-\theta)}{s^2}$$

$$\frac{1}{N}\left(\frac{\bar{x}-\theta}{s/\sqrt{n}}\right)^2 \quad \bar{x}, \frac{s}{\sqrt{n}}$$

$$T_{n-1}\left(\bar{x}, s/\sqrt{n}\right) \propto \int_0^\infty \cdots$$

$$P\left(\frac{\theta-\bar{x}}{s/\sqrt{n}} \,\Big|\, x\right) = T_{n-1}$$

$$x^* \mid x \sim N\left(\theta_p, \sigma_p^2 + \tau^2\right)$$



$$T_{N_0+n}\left(\frac{n}{n+m}\bar{x}+\frac{m}{n+m}\mu_0, \sqrt{\frac{N_0\sigma_0^2}{n+m}+\frac{mn}{(n+m)^2}(n-1)s^2}\right)$$

→ $P(\sigma^2 \mid x) = \text{InvGamma}\left(\frac{N_0+n}{2}, \frac{1}{2}(- - -)\right)$

$$\underset{\text{'m=1'}}{P(x^*} \mid \underset{\text{'n data points'}}{x)} = \int_0^\infty \int_{\mathbb{R}} \underbrace{P(x^* \mid \theta, \sigma^2)}_{(N)} \underbrace{P(\theta, \sigma^2 \mid x)}_{P(\theta \mid \sigma^2, x)\, P(\sigma^2 \mid x)} \, d\theta \, d\sigma^2$$

$N(\bar{x},)$

(6)

$$= T_{n-1}\left(\bar{x}, s\sqrt{1+\frac{1}{n}}\right)$$

$$\theta \mid X \sim T\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$