2/17/16

5-1)

$$P(\theta \mid X_1, \ldots, X_n) = \frac{P(X_1, X_2, \ldots, X_n \mid \theta)\, P(\theta)}{P(X_1, \ldots, X_n)}$$

$$= \frac{\prod_{i=1}^{n} P(X_i \mid \theta)\, P(\theta)}{\sum P(X_1, X_2, \ldots, X_n \mid \theta')\, P(\theta')}$$

Turns out if $P(X_1, \ldots, X_n \mid \theta) = \prod_{i=1}^{n} P(x_i \mid \theta)$

$$P(X_1, \ldots, X_n) = P(X_{\pi(1)}, \ldots, X_{\pi(n)})$$

Exchangeibility "definettis Theorem"

order of $X_1, X_2, \ldots, X_n$ doesn't matter

If we observe 0,1,1 or 1,0,1 or 1,1,0

$\hat{\theta}_{map} = \operatorname{argmax}\{P(\theta \mid X)\}$

$\hat{\theta}_{MMSE} = E(\theta \mid X)$

$\hat{\theta}_{MAE} = \operatorname{med}(\theta \mid X)$

$P(\theta | X_1, X_2, X_3) = P(X_1, X_2, X_3 | \theta) P(\theta)$ $\quad X_1 = 0,\ X_2 = 1,\ X_3 = 1,$

$= 12(1-\theta)\theta^2$

so what does X^* look like?

$$P(X^* | X_1, X_2, X_3) = \int_{\theta \in \Theta_0} P(X_* | \theta)\, p(\theta | \overset{0,1,1}{X_1, X_2, X_3})\, d\theta$$

$$= \int_0^1 \theta^{X_*} (1-\theta)^{1-X_*}\ 12(1-\theta)\theta^2\, d\theta$$

$$= 12 \int_0^1 \underbrace{\theta^{X_*+2} (1-\theta)^{2-X_*}\, d\theta}_{\text{Beta } B(X_*+3,\, 3-X_*)}$$ Beta function

$$B(X,Y) = \frac{\Gamma(X)\Gamma(Y)}{\Gamma(X+Y)} \qquad \Gamma(X) = (X-1)!$$

$$\Gamma(X) = \int_0^\infty x^t e^{-x}\, dx$$

$$= 12 \frac{\Gamma(X_*+3)\Gamma(3-X_*)}{\Gamma(6) = 5! = 120} = \frac{1}{10}(X_*+2)!(2-X_*)!$$

$$= \begin{cases} 0 & \text{if } \frac{1}{10} 2!\, 2! = 0.4 \\ 1 & \text{up } \frac{1}{10} 3!\, 1! = 0.6 \end{cases} = \text{Bern}(0.6)$$

390 late

5-2)

$X_i = \sum X_i$

$$P(\theta|X_1,\dots X_n) = \frac{\overbrace{P(X_1,\dots X_n|\theta)}^{Bin(n,\theta)}\ \overbrace{P(\theta)}^{1}}{P(X_1,\dots,X_n)} = \frac{\theta^{X}(1-\theta)^{n-X}}{\int_0^1 \theta^{X}(1-\theta)^{n-X}\,d\theta}$$

$$\int P(X_1,\dots,X_n|\theta)\,\cancel{P}(\theta)\theta = \int_0^1 \binom{n}{X}\theta^{X}(1-\theta)^{n-X}\,d\theta$$

(A)

$$\frac{1}{B(X+1,n-X+1)}\,\theta^{X}(1-\theta)^{n-X} = Beta\Big(\overset{\text{\# data success}}{X+1},\ \overset{\text{\# data failure}}{(n-X)+1}\Big)$$

$$\frac{1}{B(\alpha,B)}X^{\alpha-1}(1-X)^{B-1} = \frac{\Gamma(\alpha+B)}{\Gamma(\alpha)\Gamma(B)}x^{\alpha-1}(1-x)^{B-1}$$

$\Rightarrow$ Beta distribution (α, B)

$\theta \sim U(0,1)$

390

late

Conf

$Y \sim Beta(\alpha, \beta)$

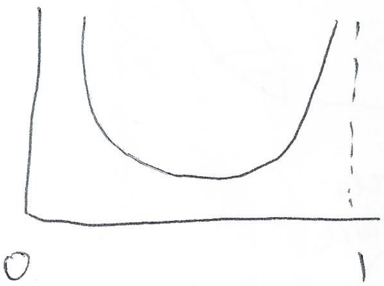
$supp(Y) = [0,1]$

$E(Y) = \frac{\alpha}{\alpha+\beta}$, $Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta-1)}$

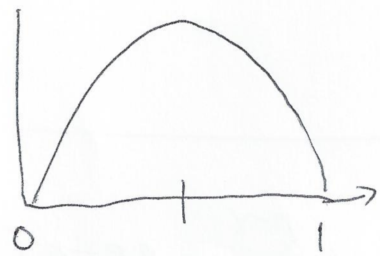
$med(Y) =$ not _____ form

$\alpha, \beta \in (0, \infty)$

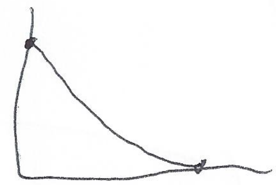
$mode(Y) = \frac{\alpha-1}{\alpha+\beta-2}$ $\forall \alpha, \beta > 1$



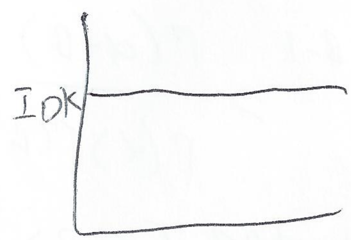
$\alpha = \beta = .5$



$\alpha = \beta = 2$



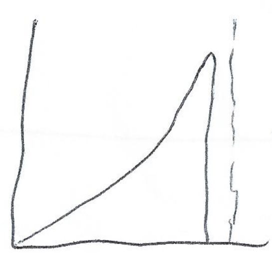
$\alpha = 1, \beta = 3$



$\alpha = \beta = 1$

$U[0,1]$

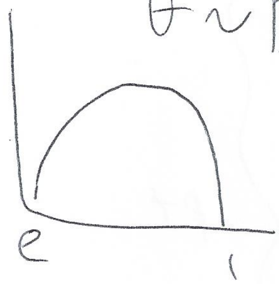
$\alpha = 3, \beta = 0.99$



$\alpha = 3$ $\beta = 1.01$

5-3)

$\theta \sim Beta(\alpha, \beta)$



after intergal $= Beta(\alpha+\beta,\ \beta+(n-x))$

$$P(\theta | X) \propto P(X|\theta)\ P(\theta)$$

Beta   Binomial   Beta

Conjugacy

$$P(x^* | X) = \int_{\Theta} P(x^* | \theta) P(\theta | X)\, d\theta$$

$$= \int_0^1 \theta^{x^*} (1-\theta)^{1-x^*} \frac{1}{B(x+\alpha, n-x+\beta)} \theta^{x+\alpha-1} (1-\theta)^{n-x+\beta-1}\, d\theta$$

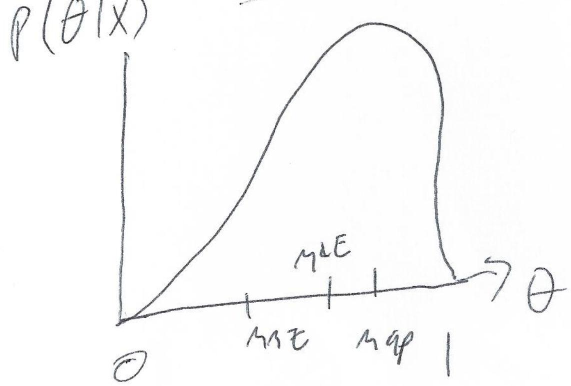
$\hat{\theta}_{MLE} = \frac{x}{n}$ $Ber\left(\underbrace{\frac{x+\alpha}{n+\alpha+\beta}}_{\hat{\theta}}\right)$

$$\hat{\theta}_{map} := \arg\max\{p(\theta|x)\} = \frac{\alpha+x-1}{\alpha+\beta+n-2}$$

$$\hat{\theta}_{\cancel{MDE}}^{MSE} := E[\theta|x] = \frac{\alpha+x}{\alpha+x+\beta+n}$$

$$\hat{\theta}_{MAE}$$

pt est



$$CI_\alpha = \left[\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}\right]$$

$$\hat{\theta}_{MLE} = \bar{X} = \hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}^2\right)$$

Want

$$P(\theta \in CI_{\theta, 1-\alpha}) = 1-\alpha$$

$$= \{0, 1\}$$

Baysian Confidace Interval (credible region)

C R

$$CR_{\theta,1-\alpha} = \left[\text{Quartile}\left[\theta|X, \tfrac{\alpha}{2}\right], \text{Quartile}\left[\theta|X, 1-\tfrac{\alpha}{2}\right] \right]$$

Beta($\alpha + x$, $\beta + n - y$)

CDF

$$F(x) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n+x)} \int_0^x t^{\alpha+x-1}(1-t)^{\beta+nx+1}\, dt \underset{\text{s-d-e}}{=} 2.55$$

set

x

Define $m(A)$ to be R "measure" set A

$m([a,b]) = b - a$

$m([a,b] \cup [c,d]) = (b-a) + (d-c)$

If $a < b < c < d$

$$CR_{\theta, 1-\alpha} := \underset{A \subset \Theta_0}{\arg\min_A} \{ M(A) : P(X \in A \mid \theta) = 1-\alpha \}$$

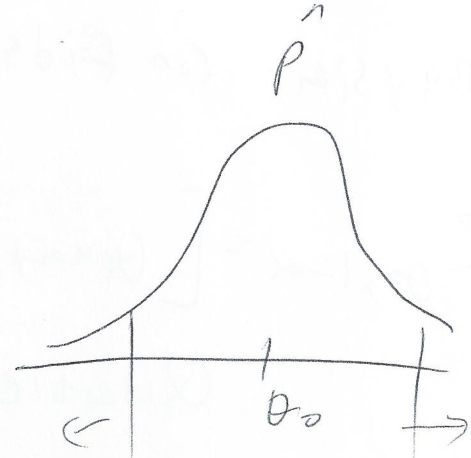
Posterior

2 side hypo test

$H_0 : \theta = \theta_0 = 0.5$

$H_a : \theta \neq \theta_0 = 0.5$

$\alpha = 5\%$



Retainment region $= \left[\theta_0 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\theta_0(1-\theta_0)}{n}} \right.$

prob $:= P\left(\begin{array}{l} \text{seeing } \hat{\theta} \text{ or} \\ \text{data } \text{nay} \\ \text{extreme} \end{array} \middle| \begin{array}{l} H_0 \text{ in the} \end{array} \right) \neq P(H_0 \mid X)$

$\theta = \theta_X$