A discrete random variable (rv) X has a probability mass function (PMF)

$$p(x) := \mathbb{P}(X = x)$$

and cumulative distribution function (CDF)

$$F(x) = \mathbb{P}(X \leq x)$$

. The random variable X has "support"

$$\mathrm{Supp}[X] := \{x : p(x) > 0, x \in \mathbb{R}\}$$

Since X is discrete, $|\mathrm{Supp}(X)| \leq |\mathbb{N}|$.
Support and pmf are related as follows:

$$\sum_{x \in \mathrm{Supp}(X)} p(x) = 1$$

The most fundamental discrete random variable is the Bernoulli:

$$X \sim \mathrm{Bern}(p) := \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

What is p? p is a parameter. Parameters have parameter spaces. For example, $p \in (0, 1)$, thus $p \neq 0$ and $p \neq 1$.

$X \sim \mathrm{Deg}(c) = \{c$ with probability 1
This means that $\mathrm{Deg}(c) = \mathbb{1}_{x=c}$, where $\mathbb{1}_{x=c}$ is an indicator function.

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$$

The random variables X_1, X_2 are independent if joint mass function $\mathbb{P}(X_1, X_2) = \mathbb{P}_{X_1}(X_1)\mathbb{P}_{X_2}(X_2)$ for all x_1, x_2 in their supports.

Let $X_1 \overset{d}{=} X_2$. The random variables X_1 and X_2 are equal in distribution if $\mathbb{P}_{X_1}(X) = \mathbb{P}_{X_2}(X)$.

Let $X_1, X_2 \overset{iid}{\sim}$. The random variables X_1, X_2 are independent and identically distributed if $X_1, X_2 \overset{iid}{\sim}$ and $X_1 \overset{d}{=} X_2$.

Let $T_2 = X_1 + X_2$ where $X_1, X_2 \overset{iid}{\sim} \mathrm{Bern}(p)$. Then

$$\mathrm{Supp}[T_2] = \{0, 1, 2\} = \mathrm{Supp}[X_1] + \mathrm{Supp}[X_2]$$

In fact,

$$\mathbb{P}_{T_2}(2) = p^2$$
$$\mathbb{P}_{T_2}(0) = (1 - p)^2$$
$$\mathbb{P}_{T_2}(1) = 2p(1 - p)$$

$$
\begin{aligned}
\mathbb{P}_{T_2}(t) &= \sum_{x\in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(x) \\
&= \sum_{x\in\{0,1\}} \Big[(p^x(1-p)^{1-x})(p^{t-x}(1-p)^{1-t+x})\Big] \\
&= p^t \sum_{x\in\{0,1\}} (1-p)^{2-t} \\
&= p^t(1-p)^{2-t} \sum_{x\in\{0,1\}} 1 \\
&= 2p^t(1-p)^{2-t}
\end{aligned}
$$

But this is wrong because $\mathbb{P}_{T_2}(2) = 2p^2 \neq p^2$.

Let

$$
\begin{aligned}
p(t) = \mathbb{P}(T_2 = t) &= \sum_{x\in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x) \\
&= \sum_{x\in\{0,1\}} p^x(1-p)^{1-x}\mathbb{1}_{x\in\{0,1\}}p^{t-x}(1-p)^{1-t+x}\mathbb{1}_{t-x\in\{0,1\}} \\
&= p^t(1-p)^{2-t} \sum_{x\in\{0,1} \mathbb{1}_{x\in\{0,1\}}\mathbb{1}_{t-x\in\{0,1\}} \\
&= p^t(1-p)^{2-t}\Big(\underbrace{\mathbb{1}_{0\in\{0,1}}_{1}\mathbb{1}_{t-\in\{0,1\}} + \underbrace{\mathbb{1}_{1\in\{0,1\}}}_{1}\mathbb{1}_{t-1\in\{0,1\}}\Big) \\
&= p^t(1-p)^{2-t}\Big(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t-1\in\{0,1}\Big) \\
&= \binom{2}{t}p^t(1-p)^{2-t}
\end{aligned}
$$

This equation does satisfy $p(0), p(1), p(2)$.

Let $X \sim \text{Bern}(p) = \text{Binom}(1,p) = \binom{1}{x}p^x(1-p)^{1-x}$. Now $\binom{n}{k}$ is only valid with $k \leq n$; otherwise, it's 0. Now back to $\mathbb{P}_{T_2}(t)$.

$$
\begin{aligned}
\mathbb{P}(T_2 = t) &= \sum_{x\in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x) \\
&= \sum_{x\in\{0,1\}} \binom{1}{x}p^x(1-p)^{1-x}\binom{1}{t-x}p^{t-x}(1-p)^{1-t+x} \\
&= p^t(1-p)^{2-t}\sum_{x\in\{0,1\}}\binom{1}{x}\binom{1}{t-x} \\
&= \binom{2}{t}p^t(1-p)^{2-t} \text{ by } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\end{aligned}
$$

Convolution of Two Independent PMFs:

$$
p(t) = \mathbb{P}(T_2 = t) = \mathbb{P}_{X_1}(x)\cdot\mathbb{P}_{X_2}(x) := \sum_{x\in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x)
$$

Let $X_1, X_2, X_2 \overset{iid}{\sim} \text{Bern}(p)$. Let

$$
\begin{aligned}
T_3 &= X_1 + X_2 + X_3 = X_3 + T_2 \\
&= \mathbb{P}_{X_3}(x) \cdot \mathbb{P}_{T_2}(x) \\
&= \sum_{x \in \text{Supp}[X]} \mathbb{P}_{X_3}(x)\mathbb{P}_{T_2}(t-x) \\
&= \sum_{x \in \{0,1\}} \binom{1}{x} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x}(1-p)^{2-t+x} \\
&= p^t(1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{1}{x}\binom{2}{t-x} \\
&= \binom{3}{t} p^t (1-p)^{3-t}
\end{aligned}
$$