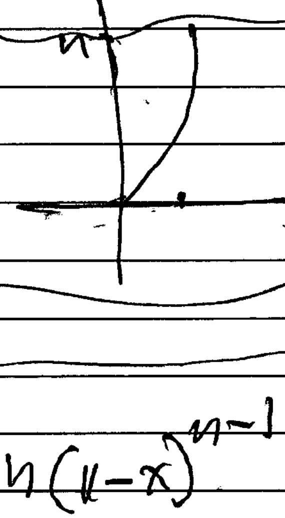
Lec-12 Oct. 19

Order statistics

$X_1, X_2, \dots, X_n \overset{iid}{\sim} f(x)$ Cont.

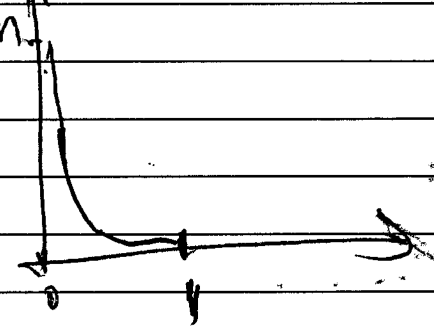
$X_1, X_2, \dots, X_n$ $U(0,1)$ $\Rightarrow$ $f(x)=1$, $F(x)=x$

pdf max $f_{X_{(n)}}(x) = n f(x) F(x)^{n-1} = n x^{n-1}$

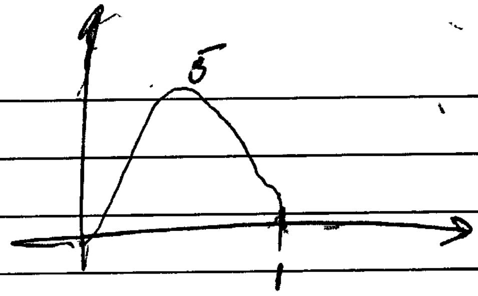


$\mathrm{Supp}(X_{(k)}) = \mathrm{Supp}(X)$

Min $f_{X_{(1)}} = n f(x) (1-F(x))^{n-1} = n(1-x)^{n-1}$



If $n=10$, $k=5$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x)\left(F(x)\right)^{k-1}\left(1-F(x)\right)^{n-k}$$

$$\propto \underbrace{X^{k-1}(1-x)^{n-k}}_{k(x)}$$

$$f(x) = \frac{1}{c} k(x)$$

$$\int_{supp(x)} k(x)\,dx = c = \int_0^1 x^{k-1}(1-x)^{n-k+1-1}\,dx = B(k, n-k+1)$$

$$\Rightarrow \frac{1}{B(k, n-k+1)} \cdot X^{k-1}(1-x)^{n-k} = Beta(k, n-k+1)$$

Recall $\boxed{X \sim Beta(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} X^{\alpha-1}(1-x)^{\beta-1} \quad Supp(x) = (0,1) \quad \alpha > 0, \ \beta > 0}$

$$\int_{\mathrm{supp}(x)} f(x) = 1 \qquad \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt}{B(\alpha,\beta)}$$

$$= \frac{B(x, \alpha\beta)}{B(\alpha,\beta)}$$

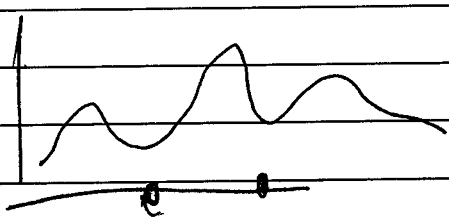
$$= I_x(\alpha,\beta)$$

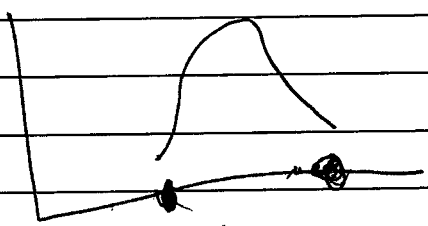
$$\boxed{\begin{aligned} F(x) &= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \\ &= \frac{\Gamma(\alpha+1,\beta)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \div \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha,\beta)} = \frac{\alpha}{\alpha+\beta} \end{aligned}}$$

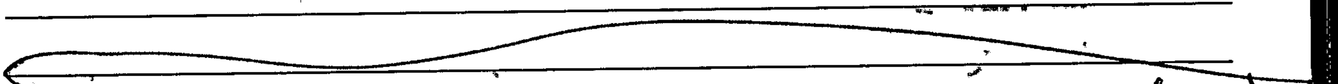
Truncations

$X \sim f(x)$

What if we know $X \in A$ where $A \subseteq supp(X)$
Call this cond. distr. Y. $f_Y(y) = ??$







$X \sim (0,1)$

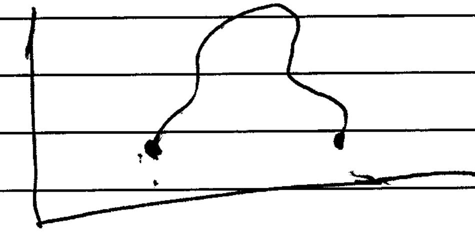
$$Z = \mathbb{1}_{X \in A} \sim Bern(p(X \in A))$$

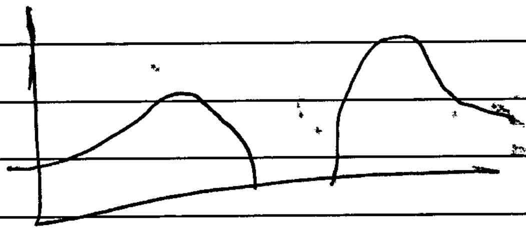
$$f_{X|Z}(x,z) = \frac{f_{X,Z}(x,z)}{p_Z(z)} = \frac{f(x)\, \mathbb{1}_{x \in A}^{z}\, \mathbb{1}_{x \notin A}^{1-z}}{P(X \in A)^{z} (1 - P(X \in A))^{1-z}}$$

$Y = X | Z = 1$

$$f_Y(x) = f_{X|Z}(x,1) = \frac{f(x)}{P(X \in A)} \mathbb{1}_{x \in A}$$

Some graphs



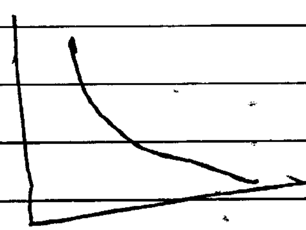


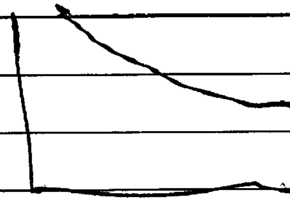
$X \geq a$	$X \leq a$	$X \in (a,b)$
$f_Y(x) = \dfrac{f(x)}{1-F(a)} \mathbb{1}_{x \geq a}$	$f_Y(x) = \dfrac{f(x)}{F(a)} \mathbb{1}_{x \leq a}$	$f_Y(x) = \dfrac{f(x)}{F(b)-F(a)} \mathbb{1}_{x \in (a,b)}$

$X \sim \mathcal{E}xp(\lambda)$

$X \geq a \Rightarrow$

$$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}} \mathbb{1}_{x \geq a}$$





Let $g : \mathbb{R}^n \to \mathbb{R}^n$ g is 1 to 1

let $\vec{X}$ be a vector r.v. with dim n

let $\vec{Y}$ " " " " "

If $f_{\vec{X}}(\vec{x}) = f_{X_1, \dots, X_n}(x_1, \dots, x_n)$ is

$\vec{Y} = g(\vec{X})$ find $f_{\vec{Y}}(\vec{y}) = f_{Y_1, \dots, Y_n}(y_1, \dots, y_n)$

$Y_1 = g_1(X_1, \dots, X_n)$ $\quad$ $X_1 = h_1(Y_1, \dots, Y_n)$

$Y_2 = g_2(X_1, \dots, X_n)$ $\quad$ since g is 1 to 1 $\quad$ $X_2 = h_2(Y_1, \dots, Y_n)$

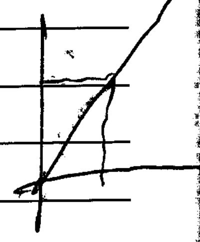
$\vdots$

$Y_n = g_n(X_1, \dots, X_n)$ $\quad$ $X_n = h_n(Y_1, \dots, Y_n)$

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = f_{X_1, \dots, X_n}\big(h_1(y_1, \dots, y_n), \dots, h_n(y_1, \dots, y_n)\big)$$

Jacobian $= |J_n(y_1, \dots, y_n)|$

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$



$$J_n = \det\left(\begin{bmatrix} \frac{dh_1}{dy_1} & \cdots & \frac{dh_1}{dy_n} \\ \vdots & & \vdots \\ \frac{dh_n}{dy_1} & \cdots & \frac{dh_n}{dy_n} \end{bmatrix}\right)$$

$$J_n = \det\left(\left[\frac{dg^{-1}(y)}{dy}\right]\right) = \frac{dg^{-1}(y)}{dy}$$

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given X_1, X_2 and

$$Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$$

$$Y_2 = X_2 = g_2(X_1, X_2)$$

$$X_1 = Y_1 Y_2 = h_1(Y_1, Y_2) \qquad \frac{dh_1}{dy_1} = Y_2 \quad \frac{dh_1}{dY_2} = Y_1$$

$$X_2 = Y_2 = h_2(Y_1, Y_2) \qquad \frac{dh_2}{dY_1} = 0, \quad \frac{dh_2}{dY_2} = 1$$

$$J_h = \det\left(\begin{bmatrix} Y_2 & Y_1 \\ 0 & 1 \end{bmatrix}\right) = Y_2$$

$$f_{Y_1, \dots, Y_n}(y_1, y_2, \dots, y_n) = f_{X_1 \dots X_n}(h_1(Y_1, \dots, Y_n), \dots, h_n(Y_1, \dots, Y_n)) \left| J_h(Y_1, \dots, Y_n) \right|$$

$$f_{Y_1, Y_2}(Y_1, Y_2) = f_{X_1, X_2}(Y_1 Y_2, Y_2) \left| Y_2 \right| \Rightarrow$$

$$f_{Y_1}(Y_1) = \int_{\text{supp}(Y_2)} f_{X_1 X_2}(Y_1 Y_2, Y_2) \left| Y_2 \right| dY_2$$

supp $[X_2]$

If X_1 X_2 ind. and (+)

$$f_{Y_1}(y) = \int_{\text{supp}(X_2)} x_2 \, f_{X_1}(y_1 y_2) \, f(x_2) \, dx_2$$

Ex

$$Y_1 = \frac{X_1}{X_1+X_2} = g(x_1, x_2) \quad \Big| \quad \frac{dh_1}{dy_1} = y_2$$

$$X_1 = Y_1 Y_2 = h(Y_1 Y_2) \quad \Big| \quad \frac{dh_1}{dY_2} = Y_1$$

$$Y_2 = X_1 + X_2 = g(X_1 X_2) \quad \Big| \quad \frac{dh_2}{dY_1} = -Y_2$$

$$X_2 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2) \quad \Big| \quad \frac{dh_2}{dY_2} = 1 - Y_1$$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$ ind. of

$X_2 \sim \text{Gamma}(\beta, \lambda)$

$Y_1 = \dfrac{X_1}{X_1 + X_2}$? Let use gt. formula for that

$\text{Supp}(Y_1) = (0, 1)$

$\text{Supp}(Y_2) = (0, \infty) \rightarrow Y_2 = X_1 + X_2$

$$Y_1 = \frac{X_1}{X_1 + X_2} \sim \int_0^\infty Y_2$$