Lecture 10 : 10/10/17

$$X \sim Exp(1),\ Y = -\ln(x) \sim Gumbel(0,1) = e^{-(y+e^{-y})}$$

$$= e^{-y} e^{-e^{-y}}$$

$$Y = Gumbel(0,1),\ x = e^{-Y} \sim Exp(1)$$

$$F_Y(y) = P(Y \le y) = P(-Y \geq -y) = P(e^{-Y} \geq e^{-y})$$

$$= P(X \geq e^{-y}) = 1 - F_X(e^{-y}) = e^{-e^{-y}}$$

$$X \sim Gumbel(0,1),\ Y = \mu + \beta X \sim Gumbel(\mu, \beta)$$

$$= \frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$$

$$F_Y(y) = P(Y \le y) = P\left(\frac{Y-\mu}{\beta} \le \frac{y-\mu}{\beta}\right) = P\left(X \le \frac{y-\mu}{\beta}\right)$$

$$= F_X\left(\frac{y-\mu}{\beta}\right) = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}}$$

Valid for any linear transformation

$$X \sim Gumbel(0,1),\ Y = e^{-X} \sim Exp(1)$$

Support $[X] = \mathbb{R}$

$$X \sim Gumbel(\mu, \beta),\ Y = \underbrace{e^{-X}}_{g(x)} \sim ?$$

Support $[Y] = (0, \infty)$

$$g(x) = e^{-X} \Rightarrow X = -\ln(y) = g^{-1}(y)$$

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$\left|\frac{d}{dy}\left[-\ln(y)\right]\right| = y^{-1}$$

$$= f_X\left(-\ln(y)\right)y^{-1} = \frac{1}{\beta}e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}e^{-e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}}y^{-1}$$

<u>Note:</u> $-\left(\frac{-\ln(y)-\mu}{\beta}\right) = k\left(\ln(y)+\mu\right) = k\left(\ln(y)+\ln(d)\right)$

$k = \frac{1}{\beta}$ let $\mu = \ln(d)$

$$= \ln(dy)^k$$

$$e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)} = e^{\ln(dy)^k} = (dy)^k \quad \left.\begin{array}{l}\beta\in(0,\infty)\Rightarrow k\in(0,\infty)\\ \mu\in\mathbb{R}\Rightarrow d\in(0,\infty)\end{array}\right\}$$

$$f_Y(y) = k(dy)^k e^{-(dy)^k} y^{-1}$$

$$= \underbrace{(kd)(dy)^{k-1}}_{\substack{d^k y^k \\ \underbrace{}_{d\, d^{k-1} y^k}}} e^{-(dy)^k} = \text{Weibull}(k, d)$$

CDF: $F_Y(y) = P(Y\le y) = P(\ln(Y)\le \ln(y)) =$

$$P(-\ln(Y)\ge -\ln(y)) = P(X\ge -\ln(y)) =$$

$$1 - F_X(-\ln(y)) = 1 - e^{-e^{\left(\frac{-\ln(y)-\mu}{\beta}\right)}}$$

$$= 1 - e^{-(dy)^k}$$

If $\beta = 1, \Rightarrow k = 1$ $\quad$ Weibull$(1, \lambda) = \lambda e^{-\lambda y}$

$= Exp(\lambda)$

$\Rightarrow$ Weibull$(1,1) = Exp(1)$ $\quad$ "memoryless"

<u>Case 1</u> $\quad k > 1$ eg $k = 2$

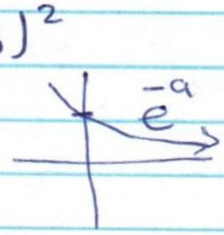
$X \sim$ Weibull$(2, \lambda) \Rightarrow F_X(x) = 1 - e^{-(\lambda x)^2}$

$\Rightarrow 1 - F_X(x) = e^{-(\lambda x)^k}$

WTS: $P(X \geq b) > P(X \geq a+b \mid X \geq a)$

$$e^{-(\lambda b)^2} = 1 - F_X(b) > \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{1 - F_X(a+b)}{1 - F_X(a)}$$

$$= \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}} = \frac{\cancel{e^{-(\lambda a)^2}}\, \overset{<1}{e^{-2\lambda^2 ab}}\, e^{-(\lambda b)^2}}{\cancel{e^{-(\lambda a)^2}}}$$



<u>Case 2</u>

$k < 1$ eg $k = \frac{1}{2}$

WTS: $P(X \geq b) < P(X \geq a+b \mid X \geq a)$

$$e^{-(\lambda b)^{1/2}} < \frac{e^{-(\lambda(a+b))^{1/2}}}{e^{(-\lambda a)^{1/2}}} = e^{-(\lambda(a+b))^{1/2} + (\lambda a)^{1/2}}$$

$$= e^{-\lambda^{1/2}((a+b)^{1/2} - a^{1/2})}$$

(take ln both side)

$$\Rightarrow \frac{-\lambda^{1/2} b^{1/2}}{-\lambda^{1/2}} < \frac{-\lambda^{1/2}}{-\lambda^{1/2}}(\cdots) \Rightarrow b^{1/2} > (a+b)^{1/2} - a^{1/2}$$

$$a^{1/2}+b^{1/2} > (a+b)^{1/2} \Rightarrow \left(a^{1/2}+b^{1/2}\right)^2 > a+b$$

$$\Rightarrow a+b+2a^{1/2}b^{1/2} > a+b$$

$$\Rightarrow 2a^{1/2}b^{1/2} > 0 \checkmark$$

let $X \sim \text{Weibull}(k,d)$, $Y = \frac{1}{X} \sim ?$ support $[Y]=(0,\infty)$

$$g^{-1}(y) = \frac{1}{y}, \quad \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \frac{1}{y^2}$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\frac{1}{y^2} = (kd)\left(\frac{d}{y}\right)^{k-1} e^{-\left(\frac{d}{y}\right)^k}\frac{1}{y^2}$$

$$f_Y(y) = (kd)\left(\frac{d}{y}\right)^{k-1} e^{-\left(\frac{d}{y}\right)^k}\frac{1}{y^2}$$

$$= kd^k \frac{1}{y^{k-1}}\frac{1}{y^2} e^{-\left(\frac{d}{y}\right)^k} = \frac{k}{d} d^{k+1}\frac{1}{y^{k+1}} e^{-\left(\frac{y}{d}\right)^{-k}}$$

$$= \frac{k}{d}\left(\frac{y}{d}\right)^{-(k+1)} e^{-\left(\frac{y}{d}\right)^{-k}} =$$

$$= \frac{k}{d}\left(\frac{y}{d}\right)^{-(k+1)} e^{-\left(\frac{y}{d}\right)^{-k}} \quad \text{Frechet}(k,d,0)$$

Centered

{Gumbell, Weibull, Frechet} belong to a special family called the Generalized extreme value distribution

$$X \sim \text{Erlang}(k,\lambda) := \overbrace{\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}}^{\text{Gamma}(k,\lambda)} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad \begin{array}{l} k \in \mathbb{N} \\ \lambda \in (0,\infty) \end{array}$$

$$X \sim \text{Negbin}(k,p) := \binom{x+k-1}{k-1} p^k (1-p)^x \quad \begin{array}{l} k \in \mathbb{N} \\ p \in (0,1) \end{array}$$

what if $k \in (0,\infty)$? $= \dfrac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)} p^k (1-p)^x$

$$X \sim \text{Negbin}(k,p) = \underbrace{\frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(x)} p^k (1-p)^x}_{\text{Extended negative binomial}}$$

Recall $X \sim \text{Erlang}(k_1,\lambda)$ independent $Y \sim \text{Erlang}(k_2,\lambda)$

$\Rightarrow X+Y \sim \text{Erlang}(k_1+k_2,\lambda)$

WTS if $X \sim \text{Gamma}(k_1,\lambda)$ independent of $Y \sim \text{Gamma}(k_2,\lambda) \Rightarrow X+Y \sim \text{Gamma}(k_1+k_2,\lambda)$

Gamma (k_1+k_2,λ)

$$X+Y \sim \int_0^t f_X(x) f_Y(t-x)\,dx = \int_0^t \quad e^{-\lambda t} \cancel{e^{\lambda x}}$$

$$= \int_0^t \frac{\lambda^{k_1} x^{k_1-1} \cancel{e^{-\lambda x}}}{\Gamma(k_1)} \frac{\lambda^{k_2} (t-x)^{k_2-1}}{\Gamma(k_2)} e^{-\lambda(t-x)}\,dx$$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1}(t-x)^{k_2-1}\,dx$$

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

$x_l = 0 \Rightarrow u_l = 0$ (x_l x_{lower}

$x_u = t \Rightarrow x_u = 1$ $\quad$ x_u x_{upper})

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 (ut)^{k_1-1}(t-ut)^{k_2-1}\,t\,du$$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k-1}\, t^{k_2-1}\, u^{k_1-1}(1-u)^{k_2-1}\,t\,du$$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)}\, t^{k_1+k_2-1} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du$$