Math 621 Lec10 10/10/17

Recall

$X \sim \text{Exp}(1)$, $Y = -\ln(X) = \ln\left(\frac{1}{X}\right) \sim \text{Gumbel}(0,1) = e^{-(y+e^{-y})} = e^{-y}e^{-e^{-y}}$

Standard Gumbel

$\Rightarrow Y \sim \text{Gumbel}$, $X = e^{-Y} \sim \text{Exp}(1)$

Find CDF of std Gumbel. Let $Y \sim \text{Gumbel}(0,1)$

$F_Y(y) = P(Y \le y) = P(-Y \ge -y) = P(e^{-Y} \ge e^{-y}) = P(X \ge e^{-y}) = 1 - F_X(e^{-y})$

$= e^{-e^{-y}}$

if $X \sim \text{Gumbel}(0,1)$, $Y = \mu + \beta X \sim \text{Gumbel}(\mu,\beta) := \frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$

Find CDF of Gumbel:

$$\underbrace{F_Y(y) = P(Y \le y) = P\left(\frac{Y-\mu}{\beta} \le \frac{y-\mu}{\beta}\right) = P\left(X \le \frac{y-\mu}{\beta}\right) = F_X\left(\frac{y-\mu}{\beta}\right)}_{\text{valid for all shifts \& scales}} = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}}$$

~~$X \sim \text{Gumbel}(0,1) \Rightarrow Y = e^{-X} \sim \text{Exp}(1)$ (inverse of our previous example)~~

$X \sim \text{Gumbel}(\mu, \beta) \Rightarrow \underline{Y = e^{-X}} \sim ?$ $\quad Supp[Y] = (0,\infty)$ same

$x = -\ln(y) = g^{-1}(y) \quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = y^{-1}$

$$f_Y(y) = f_X(-\ln(y))\, y^{-1} = \frac{1}{\beta} e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)} e^{-e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}}$$

Note $-\left(\frac{-\ln(y)-\mu}{\beta}\right) = \frac{\ln(y)+\mu}{\beta} = k(\ln(y)+\ln(\lambda)) = \ln((y\lambda)^k)$

Let $k = \frac{1}{\beta}$, $\mu = \ln(\lambda)$ $\quad \lambda \in (0,\infty)$

$\Rightarrow \lambda = e^{\mu}$

convention

$$f_Y(y) = k \underbrace{(y\lambda)^k}_{y^k \lambda^k} e^{-(y\lambda)^k} y^{-1} = (k\lambda)(y\lambda)^{k-1} e^{-(y\lambda)^k}$$

$\lambda \lambda^{k-1}$

$= \text{Weibull}(k, \lambda)$

$F_Y(y) = P(Y \le y) =$
$= P(\ln(Y) \le \ln(y))$
$= P(-\ln(Y) \ge -\ln(y))$
$= P(X \ge -\ln(y)) = 1 - F_X(-\ln(y))$
$= 1 - e^{-e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}} = 1 - e^{-e^{\frac{\ln(y)+\mu}{\beta}}}$
$= 1 - e^{-e^{\mu/\beta} y^{\frac{1}{\beta}}} = 1 - e^{-e^{\ln \lambda^k} y^k}$
$= 1 - e^{-(\lambda x)^k}$

if $k = 1$ $\quad \text{Weibull}(1, \lambda) = \lambda e^{-\lambda y} = \text{Exp}(\lambda)$

i.e. $\beta = 1$ on the Gumbel

if $\lambda = 1 \Rightarrow \text{Weibull}(1,1) = \text{Exp}(1)$

i.e. $\mu = 0$ on the Gumbel

Used to model survival times / failure times, it's a <u>generalization of the exponential</u>. If $k \ne 1 \Rightarrow$ not memoryless

if $k > 1$ $\quad P(X \ge a+b \mid X \ge a)$ gets smaller with a (die quicker) e.g.?

if $k < 1$ $\quad P(X \ge a+b \mid X \ge a)$ gets larger with a (die slower) e.g.?

if $k = 1$ $\quad$ no change!

Let's take a look at the two cases...

$K > 1$ e.g. $k = 2$

$$X \sim \text{Weibull}(2, \lambda) \Longrightarrow F_X(x) = 1 - e^{-(\lambda x)^2}$$

$\lambda^2(a^2+2ab+b^2)$

$\lambda, a, b > 0$

$\Rightarrow < 1$

WTS

$$P(X \geq b) > P(X \geq a+b \mid X \geq a) = \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}} = \frac{\cancel{e^{-(\lambda a)^2}}\, e^{-2\lambda^2 ab}\, e^{-(\lambda b)^2}}{\cancel{e^{-(\lambda a)^2}}}$$

$$= e^{-\lambda b^2} > e^{-2\lambda ab}\, {}^{\lambda b^2}$$

$$\Rightarrow -\lambda b^2 > -\lambda(2ab+b^2) \Rightarrow b^2 < 2ab+b^2 \checkmark$$

$K < 1$ e.g. $k = \frac{1}{2}$ $\quad F_X(x) = 1 - e^{-(\lambda x)^{\frac{1}{2}}}$

WTS

$$P(X \geq b) < P(X \geq a+b \mid X \geq a) = \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{e^{-(\lambda(a+b))^{\frac{1}{2}}}}{e^{-(\lambda a)^{\frac{1}{2}}}} = e^{-(\lambda(a+b))^{\frac{1}{2}} + (\lambda a)^{\frac{1}{2}}}$$

$$e^{-(\lambda b)^{\frac{1}{2}}}$$

$$= e^{-\lambda^{\frac{1}{2}} b^{\frac{1}{2}}} < e^{-\lambda^{\frac{1}{2}}\left((a+b)^{\frac{1}{2}} - a^{\frac{1}{2}}\right)}$$

$$\Rightarrow -\lambda^{\frac{1}{2}} b^{\frac{1}{2}} < -\lambda^{\frac{1}{2}}\left((a+b)^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$$

$$\Rightarrow b^{\frac{1}{2}} > (a+b)^{\frac{1}{2}} - a^{\frac{1}{2}}$$

$$\Rightarrow a^{\frac{1}{2}} + b^{\frac{1}{2}} > (a+b)^{\frac{1}{2}} \Rightarrow \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2 > a+b \Rightarrow a+b+2a^{\frac{1}{2}}b^{\frac{1}{2}} > a+b \checkmark$$

$X \sim \text{Weibull}(k, \lambda)$ $\quad Y = \frac{1}{X} \sim ?$ $\quad x = \frac{1}{y} = g^{-1}(y) \Rightarrow \left|\frac{d}{dy}(g^{-1}(y))\right| = \frac{1}{y^2}$

(inverse waiting time) $\quad \text{Supp}(Y) = (0, \infty)$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\frac{1}{y^2} = (k\lambda)\left(\frac{\lambda}{y}\right)^{k-1} e^{-\left(\frac{\lambda}{y}\right)^k}\frac{1}{y^2}$$

$$= k\,\lambda^k \frac{1}{\underbrace{y^{k-1+2}}_{k+1}} e^{-\frac{\lambda^k}{y^k}}$$

$$= \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{-(k+1)} e^{-\left(\frac{y}{\lambda}\right)^{-k}} \sim \text{Frechet}(k, \lambda, 0)$$

centered

Parameter Space

$k \in (0, \infty)$, $\lambda \in (0, \infty)$

$X \sim \text{Frechet}(k, \lambda, 0)$, $Y = X + c \Rightarrow Y \sim \text{Frechet}(k, \lambda, c)$

{Gumbel, Weibull, Frechet} belong to a special family called the Generalized Extreme Value Distr. Hopefully we can return to this and survival modeling soon.

Units:
Weibull : waiting time
Frechet : inverse waiting time
Gumbel : log inverse waiting time

Recall.

$$X \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

$$X \sim \text{NegBin}(k, p) = \underbrace{\binom{x+k-1}{k-1}}_{\frac{(x+k-1)!}{x!\,(k-1)!}} p^k (1-p)^x = \frac{\Gamma(x+k)}{\Gamma(x+1)\,\Gamma(k)} p^k (1-p)^x$$

Recall

$\Gamma(k) = (k-1)!$

$\Gamma(k+1) = k!$

for both distr's $k \in \mathbb{N}$ since it is a # of successes

What's wrong with allowing $k \in (0,\infty)$ i.e. all positive reals?

You can show that the PDF of Erlang & PMF of neg. binomial would still be valid... Conceptually?

Wait for a fractional # of successes?

Imagine "success" is really continuous. Success measured in dollars e.g.

If $k \in (0,\infty)$ these distr's get different names

$X \sim \text{Gamma}(k, \lambda)$ ⇐ very useful! More flexible waiting time distr.

$X \sim \text{ExtNegBin}(k, \lambda)$ ⇐ ignore this   Supports? Unchanged

"extended negative binomial"

Since $X+Y \sim \text{Erlang}(k_1+k_2, \lambda)$ we expect the same here... p148

$X \sim \text{Gamma}(k_1, \lambda)$, $Y \sim \text{Gamma}(k_2, \lambda)$

$$X+Y \sim \int_0^\infty \frac{\lambda^{k_1} x^{k_1-1} e^{-\lambda x}}{\Gamma(k_1)} \frac{\lambda^{k_2} (t-x)^{k_2-1} \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{+\lambda x}}}{\Gamma(k_2)} \mathbb{1}_{t-x \in (0,\infty)}\, dx$$

$t - x \in (0,\infty)$: $t \geq x$, $x \leq t$, and since $x \geq 0$...

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1}(t-x)^{k_2-1}\, dx$$

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

$\Rightarrow x = ut$

$x_\ell = 0 \Rightarrow u_\ell = 0$, $x_u = t \Rightarrow u_u = 1$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 (ut)^{k_1-1}(t-ut)^{k_2-1}\, du = \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1}(1-u)^{k_2-1}\, t\, du$$