10/17/17

$X \sim \text{Gamma}(k_1, \lambda)$

indep of

$Y \sim \text{Gamma}(k_2, \lambda)$

W.T.S

$X + Y \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$T = X + Y \sim \ldots \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1} (1-u)^{k_2-1} t \, du$$

$$= \frac{\lambda^{k_1+k_2} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} e^{-\lambda t} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} du$$

$$X \sim \text{Exp}(\lambda): \lambda e^{-\lambda x} \propto e^{-\lambda x} = k(x) = c f(x)$$

"kernel" $\Rightarrow f(x) = \frac{1}{c} k(x)$

directly proportion

$$1 = \int_{\text{supp}(X)} f(x) \, dx$$

$$X \sim \text{Bin}(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \propto (x!(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x$$

$$X \sim \text{Weibull}(k, \lambda) := k\lambda (x\lambda)^{k-1} e^{-(x\lambda)^k} \propto x^{k-1} e^{-(x\lambda)^k}$$

$$X \sim \text{Gamma}(k, \lambda) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)} \propto e^{-\lambda x} x^{k-1}$$

$$T \sim \text{Gamma}(k_1+k_2,\lambda) = \frac{\lambda^{k_1+k_2} t^{k_1+k_2-1} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du$$

$$\propto$$

$$\propto e^{-\lambda t} t^{k_1+k_2-1} \propto \text{Gamma}(k_1+k_2,\lambda)$$

$$T \sim \text{Gamma}(k_1+k_2,\lambda) = \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1+k_2)}$$

$$\underset{\text{be}}{\text{must}} = \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du$$

$$\Rightarrow \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du = \beta(k_1,k_2)$$

$$\beta(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

in the book "Beta function".

$$\beta(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

extra
$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\int_0^\infty t^{\alpha-1}e^{-t}\,dt \int_0^\infty t^{\beta-1}e^{-t}\,dt}{\int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dt}.$$

p160 Order Statistics

$X_1, X_2, \dots, X_n$ are a sequence of continuous r.v.'s

$X_{(1)}, X_{(2)}, \dots X_{(n)}$ are called "order statistics"

Where $X_{(1)} < X_{(2)} < \dots < X_{(n)}$

"sorted order"

$X_{min} = X_{(1)} = \min\{X_1, \dots X_n\}$

$2 = \min\{2, 7, 9, 12\}$.

$X_{max} = X_{(n)} = \max\{X_1, \dots, X_n\}$.

Range $R = X_{max} - X_{min}$

In the case of $X_1, \dots, X_n \overset{iid}{\sim} f(x)$ with CDF $F(x)$

~~X_{max}~~

We want $\underset{CDF}{F_{X_{(n)}}(x)} = P(X_{(n)} \le x) = P(X_1 \le x, X_2 \le x, \dots, X_n \le x)$

$$= P(X_1 \le x) \cdots P(X_n \le x)$$

$$= F_{X_1}(x) \cdots F_{X_{(n)}}(x)$$

$$= F^n(x)$$

PDF $f_{X_{(n)}}(x) = F'_{X_{(n)}}(x) = n f(x) F^{n-1}(x)$

$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$

CDF

$= 1 - P(X_1 > x, \dots, X_{(n)} > x)$

$= 1 - P(X_1 > x) \cdots P(X_n > x)$

$= 1 - (1 - F_{X_1}(x)) \cdots (1 - F_{X_{(n)}}(x))$

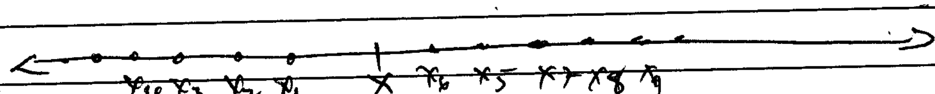
$= 1 - (1 - F(x))^n$

PDF $\quad f_{(X_1)}(x) = F'_{X_{(1)}}(x) = -n(1 - F(x))^{n-1}(-f(x))$ #

$= n f(x)(1 - F(x))^{n-1}$

Goal: $F_{X_{(n)}}(x)$

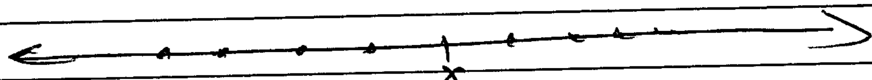
consider $n = 10$



$P(X_1, X_2, X_3, X_4 \in (-\infty, x]$ and $X_5, X_6, \dots X_{10} \in (x, \infty))$

$= P(X_1 \le x) \cdots P(X_4 \le 4)\ P(X_5 > x) \cdots P(X_{10} > x)$

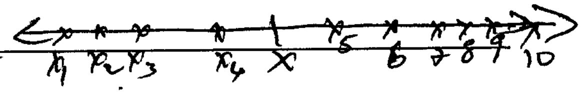
$= F(x)^4 (1 - F(x))^6$

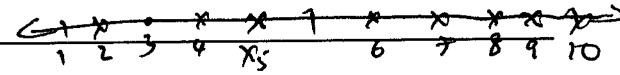


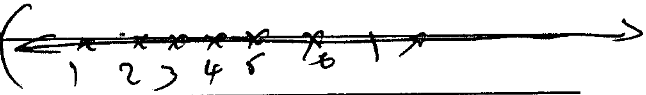
$P($any 4 are $\in (-\infty, x]$, the other 6 are $\in (x, \infty))$

$= \binom{10}{4} F(x)^4 (1 - F(x))^6$

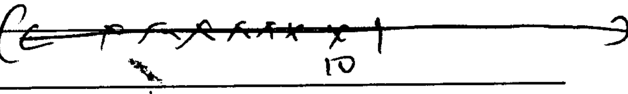
CDF

$F_{X_{(4)}}(x) = P(X_4 \le x) = P($  $)$

$+ P($  $)$

$+ P($  $)$

$\vdots$

$+ P($  $)$

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^{j} (1 - F(x))^{n-j}$$

$$F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} F(x)^{j} (1 - F(x))^{n-j}$$

$$= \cancel{\binom{n}{n}} F(x)^{n} \cancel{(1 - F(x))^{n-n}} \checkmark$$

cdf

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^{j} (1 - F(x))^{n-j}$$

$$= \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^{j} (1 - F(x))^{n-j} \right) - (1 - F(x))^{n}$$

$$= \underbrace{\left(\left(F(x)\right) + \left(1 - F(x)\right)\right)^n} - \left(1 - F(x)\right)^n$$

$$1^n - \left(1 - F(x)\right)^n$$

$$1 - \left(1 - F(x)\right)^n \checkmark$$

PDF $f_{X_{(k)}}(x) = F'_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F^j(x)\left(1 - F(x)\right)^{n-j}\right]$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \frac{d}{dx}\left[F^j(x)\left(1 - F(x)\right)^{n-j}\right]$$

$$= \left(\left(1 - F(x)\right)^{n-j} j F(x)^{j-1} f(x) - F(x)^j (n-j)\left(1 - F(x)\right)^{n-j-1} f(x)\right)$$

$$= f(x)\left(\sum_{j=k}^{n} \frac{n!}{\cancel{j!}_{(j-1)}(n-j)!}\left(1 - F(x)\right)^{n-j} \cancel{j} F(x)^{j-1} - \sum_{j=k+1}^{n-1} \frac{n!}{j!\cancel{(n-j)!}_{\cancel{(n-j-1)!}}} F^{j}(x)\cancel{(n-j)}\left(1 - F(x)\right)^{n-j-1}\right)$$

$$(j-1)!(n-j)!$$

$$\left(a_k + a_{k+1} + a_{k+2} \cdots + a_n\right) - \left(a_{k+1} + \cdots + a_n\right) = a_k$$

$$\frac{n!}{(k-1)!\,(n-k)!}\, f(x)\,(1-F(x))^{n-k}\, F(x)^{k-1}$$