Lecture 23: 12/07/17

Exple 1

let $X_n \sim Bin\left(n, \frac{\lambda}{n}\right)$

$\Rightarrow X_n \xrightarrow{d} X \sim pois(\lambda)$

Example 2

let $X_n \sim Geom(n\lambda)$, $supp[X_n] = \{\frac{1}{n}, \frac{2}{n} \cdots\}$

$\Rightarrow X_n \longrightarrow X \sim Exp(\lambda)$

Example 3 $X_n \sim Bin(n, p)$, $Y_n := \frac{X_n - np}{\sqrt{np(1-p)}}$

$Y_n \xrightarrow{d} Z \sim N(0,1)$? yes.

Example

$$X_n \sim \begin{cases} \frac{1}{n+1} & wp\ 1-p \\ 1 - \frac{1}{n+1} & wp\ p \end{cases}$$

$X_n \xrightarrow{d} X \sim Bern(p)$

$\lim\limits_{n \to \infty} p_{X_n}(x) = \lim\limits_{n \to \infty} p^{\mathbb{1}_{x = 1-\frac{1}{n-1}}} (1-p)^{\mathbb{1}_{x = \frac{1}{n+1}}}$.

$\mathbb{1}_{x \in \{1-\frac{1}{n-1}, \frac{1}{n+1}\}}$

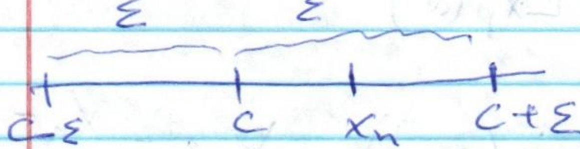
$= p^{\mathbb{1}_{x=1}} (1-p)^{\mathbb{1}_{x=0}} \mathbb{1}_{x \in \{0,1\}}$

$= Bern(p)$

Convergence in Probability

$X_n \xrightarrow{P} c$ "X_n converges in probability to constant c" will mean by definition:

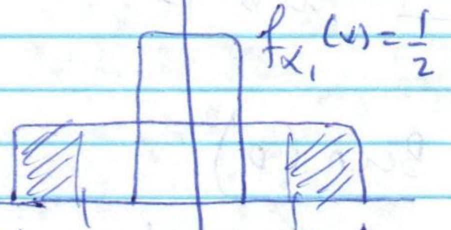
$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$$



Example: $X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right)$

$X_1 \sim U(-1, 1) \Rightarrow f_{X_1}(x) = \frac{1}{2}$

$X_3 \sim U(-3, 3) \Rightarrow f_{X_3}(x) = \frac{3}{2}$



WTS $X_n \xrightarrow{P} 0$?

$f_{X_n}(x) = \frac{n}{2}$, $\quad \forall \varepsilon > 0 \lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = 0$

$= \lim_{n\to\infty} P(|X_n| \geq \varepsilon) = 0$

$= \lim_{n\to\infty} P(X_n \geq \varepsilon) + P(X_n \leq -\varepsilon) = 0$

$F_{X_n}(x) = \frac{x-a}{b-a}$

$= \frac{nx+1}{2}$

WTS (want to show)

$$= \lim_{n\to\infty} \left(\frac{1}{n}-\varepsilon\right)\frac{n}{2}\mathbb{1}_{\varepsilon<\frac{1}{n}} + \left(\frac{1}{n}-\varepsilon\right)\frac{n}{2}\mathbb{1}_{\varepsilon<\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}-\varepsilon\right)n\,\mathbb{1}_{\varepsilon<\frac{1}{n}} = \lim_{n\to\infty} (1-\varepsilon n)\,\mathbb{1}_{\varepsilon<\frac{1}{n}} = 0$$

Consider $X_1, \ldots X_n \overset{iid}{\sim}$ with mean μ and variance σ^2

~~Define~~ Define $\overline{X}_n = \dfrac{X_1 + \cdots X_n}{n}$

WTS $\overline{X}_n \xrightarrow{P} \mu$

~~Proof~~ Prove that $\forall \varepsilon > 0$ $\lim\limits_{n\to\infty} P(|X_n - \mu| \geq \varepsilon) = 0$

$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \varepsilon \leq \lim_{n\to\infty} \frac{Var[\overline{X}_n]}{\varepsilon^2} =$$

(Chebyshev used inequality)

$$= \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$ Weak law of large numbers

(Because it requires σ^2 (finite variance))

(or in Law)

III Convergence in L^r norm for $r \geq 1$

$X_n \xrightarrow{L^r} c$ means $\lim\limits_{n\to\infty} E\left[|X_n - c|^r\right] = 0$

$X_n \xrightarrow{L^1} c$ means $\lim\limits_{n\to\infty} E\left[|X_n - c|\right] = 0$

$X_n \xrightarrow{L^2} c$ means $\lim\limits_{n\to\infty} E\left[|X_n - c|^2\right] = 0$

"mean-square convergence"

<u>Example</u>: $X_n \sim \mathcal{U}\left(0, \frac{1}{n}\right)$ (X_n positive)

Prove $X_n \xrightarrow{L^r} 0 \quad \forall r \geq 1$

$\cancel{\lim\limits_{n\to\infty} X_n \xrightarrow{L^r} 0 \; \forall}$

$$\lim_{n\to\infty} E\left[|X_n - 0|^r\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} E\left[|X_n|^r\right] = \lim_{n\to\infty} E\left[X_n^r\right] =$$

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} x^r f_{X_n}(x)\,dx = \lim_{n\to\infty} \int_0^{\frac{1}{n}} x^r n\,dx$$

$$= \lim_{n\to\infty} \left[\frac{x^{r+1}}{r+1}\right]_0^{\frac{1}{n}} n = \lim_{n\to\infty} \frac{1}{n^{r+\cancel{1}}(r+1)} \cancel{n} = 0$$

Example Let $1 \le r < s$

Prove $X_n \xrightarrow{L^s} c \Rightarrow X_n \xrightarrow{L^r} c$

Theorem from Holder

$$E\left[|V|^r\right] \le E\left[|V|^s\right]^{\frac{r}{s}}$$

$$\lim_{n\to\infty} E\left[|X_n - c|^r\right] \le \lim E\left[|X_n - c|\right]^{\frac{r}{s}}$$

$$= \left(\lim_{n\to\infty} \underbrace{E\left(|X_n - c|^s\right)}_{0}\right)^{\frac{r}{s}} = 0$$

Example $X_n \xrightarrow{L^r} c \Rightarrow X_n \xrightarrow{P} c$?

$$\lim_{n\to\infty} P(|X_n - c| \ge \varepsilon) = \lim_{n\to\infty} P\left(|X_n - c|^r \ge \varepsilon^r\right)$$

(use markov inequality) $\le \lim_{n\to\infty} \dfrac{E\left[|X_n - c|^r\right]}{\varepsilon^r} = 0$

$X_n \xrightarrow{P} c$ but not $X_n \xrightarrow{L^r} c$

Proof by counter example

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let $X_n \sim \begin{cases} n^2 & wp \ \frac{1}{n} \\ 0 & wp \ 1-\frac{1}{n} \end{cases}$

$X_n \xrightarrow{P} 0$

$\forall \varepsilon > 0 \ \lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = 0$

$\lim_{n\to\infty} P(|X_n| \geq \varepsilon)$

$= \lim_{n\to\infty} P(X_n \geq \varepsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

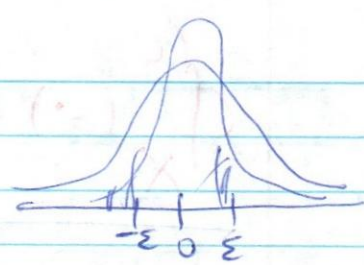
$X_n \stackrel{L^1}{\not\to} 0$

$\lim_{n\to\infty} E[|X_n - c|] = \lim_{n\to\infty} E[|X_n|]$

$= \lim_{n\to\infty} n^2 \cdot \frac{1}{n} = \lim_{n\to\infty} n = \infty$

Example 1

$X_n \sim N\left(0, \left(\frac{1}{n}\right)^2\right)$



Prove $X_n \xrightarrow{P} (0)$

$\forall \varepsilon>0 \ \lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) =$

$\lim_{n\to\infty} P(|X_n| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{Var[X_n]}{\varepsilon^2}$

$= \lim_{n\to\infty} \frac{1}{n^2\varepsilon^2} = 0$

Exam 2 Prove: $X_n \xrightarrow{L^4} 0$

$\lim_{n\to\infty} E\left[|X-0|^4\right] = \lim_{n\to\infty} E\left[X^4\right]$

$\phi_X^{(4)}(0) = E\left[X^4\right] = \frac{3}{n^2}$

$W \sim \ \phi_W(t) = e^{-\frac{t^2\sigma^2}{2}}$

$\phi^4 \quad W \sim N(\mu, \sigma^2)$

$\phi_X(t) = e^{-\frac{t^2}{2n^2}}$

$\phi_X^4(t) = e^{-\frac{t^2}{2n}} \left(\frac{3n^2 - 6nt^2 + t^4}{n^4}\right)$

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$$\phi_X^{(4)}(0) = \frac{3}{n^2}$$

Law of Iterated Expectation

$$E_Y[Y] = E_X[E_Y[Y/X]]$$

law of Total Variance

$$Var_Y[Y] = E_X[Var_Y[Y/X] + Var_X$$