10/19/17 Lecture #12

Order Statistics.

$X_1, X_2, \dots, X_n \overset{iid}{\sim} f(x)$ (cont.)

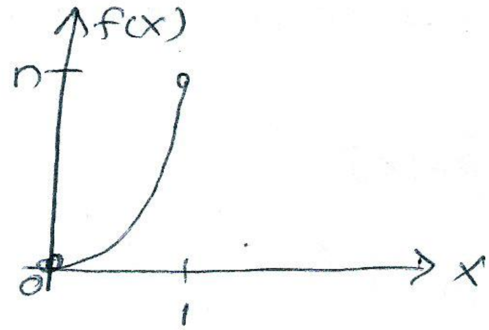
Let $X_1, \dots, X_n \overset{iid}{\sim} U(0,1) \Rightarrow f(x) = 1$

$\Rightarrow F(x) = x$

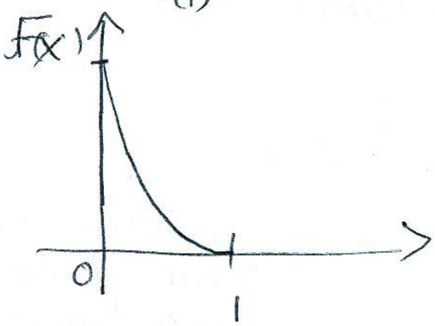
$$f_{X_{(n)}}(x) = n \underbrace{f(x)}_{=1} \cdot \underbrace{F(x)}_{x}^{n-1}$$

$$= nx$$

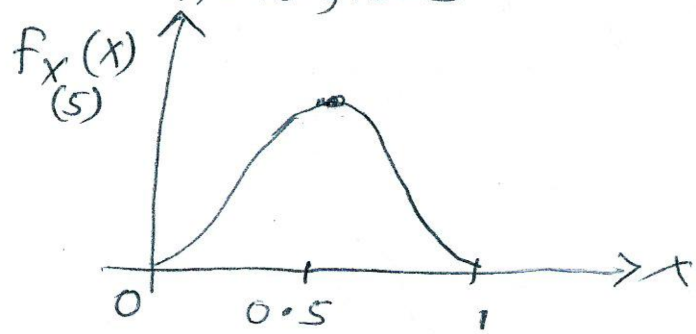
$\text{Supp}[X_{(k)}] = \text{Supp}[X]$



min $f_{X_{(1)}}(x) = n f(x)(1-F(x))^{n-1} = n(1-x)^{n-1}$



$n = 10, k = 5$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \cdot \underbrace{f(x)}_{=1} \underbrace{F(x)}_{x}^{k-1} \cdot (1-\underbrace{F(x)}_{x})^{n-k}$$

$$\propto \underbrace{x^{k-1}(1-x)^{n-k}}_{k(x)}$$

So $f(x) = \frac{1}{c} k(x)$

$$\int_{\text{supp}[x]} k(x)\,dx = c \int_0^1 x^{k-1}(1-x)^{n-k}\,dx \Rightarrow$$

$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$

$$\int_0^1 x^{k-1}(1-x)^{n-k}\,dx = B(k, n-k+1)$$

Now $f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} \cdot x^{k-1}(1-x)^{n-k}$

$= \text{Beta}(k, n-k+1)$

By $\left(X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \right)$

$\text{Supp}[X] = (0,1)$

n – positive k – positive $\alpha > 0$, $\beta > 0$

$$\int_{\text{Supp}[X]} f(x) = 1$$

$$\int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}_{B(\alpha,\beta)} = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{\overbrace{\int_0^x t^{\alpha-1}\cdot(1-t)^{\beta-1}\,dt}^{\text{incomplete beta function.}}}{B(\alpha,\beta)} = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = \underbrace{I_x}_{\text{regularized incomplete beta function}}(\alpha,\beta)$$

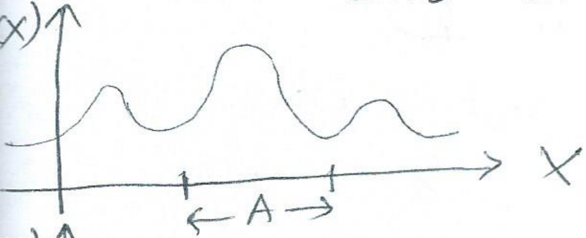
$$E(X) = \frac{1}{B(\alpha,\beta)} \cdot \int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\dfrac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha,\beta)}} = \frac{\alpha\dfrac{\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}}{\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\alpha}{\alpha+\beta}$$

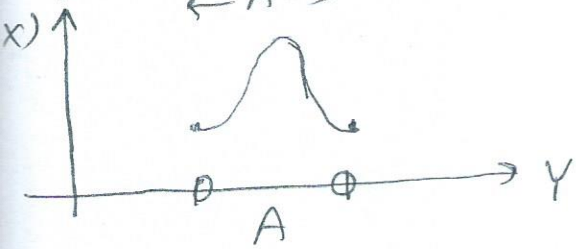
Truncations

$X \sim f(x)$. What if we know $X \in A$ where $A \subseteq \mathrm{supp}[X]$

call this cont. dist. Y. $f_Y(y) = ?$

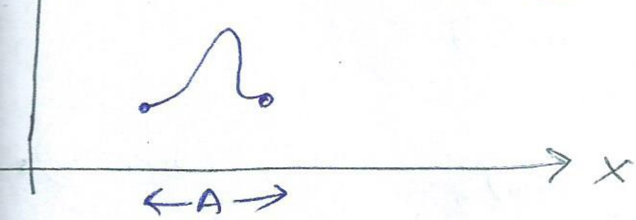


$Z = \mathbb{1}_{X \in A} \sim \mathrm{Bern}(P(X \in A))$

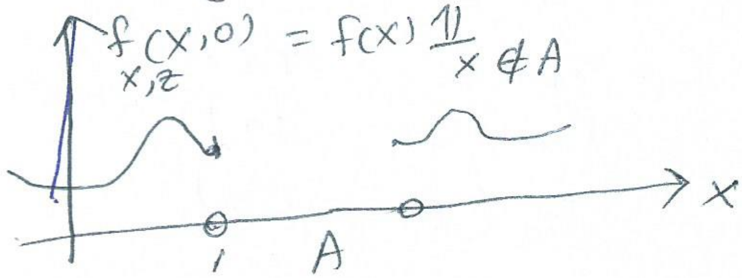


$$f_{X|Z}(x,z) = \frac{f_{X,Z}(x,z)}{P_Z(z)} = \frac{f(x) - \mathbb{1}_{X\in A}^{z}\, \mathbb{1}_{X \in A}^{1-z}}{P(X\in A)^z (1-P(X\in A))^{1-z}}$$

$f_{X,Z}(x,z) = f(x)\mathbb{1}_{X\in A}^{z}\, \mathbb{1}_{X\in A}^{1-z}$



$f_{X,Z}(x,0) = f(x)\, \mathbb{1}_{x \notin A}$



$Y = X | Z = 1 \qquad f_Y(x) = f_{X|Z}(x,1) = \dfrac{f(x)}{P(X\in A)} \cdot \mathbb{1}_{X\in A}.$

$X \geq a$

$$f_Y(x) = \frac{f(x)}{1-F(a)}\, \mathbb{1}_{X \geq a}$$

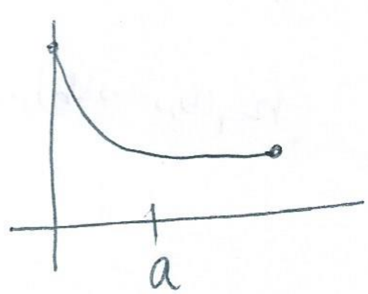
$X \leq a$

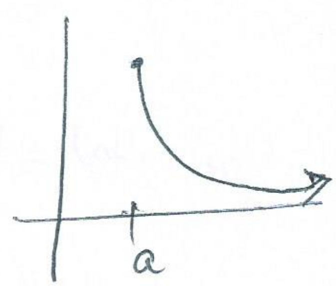
$$f_Y(x) = \frac{f(x)}{F(a)} \cdot \mathbb{1}_{X \leq a}$$

$X \in (a,b) \qquad f_Y(x) = \dfrac{f(x)}{F(b)-F(a)}\, \mathbb{1}_{X(a,b)}$

Let $X \sim \mathrm{Exp}(\lambda)$. We know $X \geq a$.

$$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}}\, \mathbb{1}_{X \leq a}$$





Let $g: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ g is 1-1

Let $\vec{X}$ be a vector r.v. with dim. n.

Let Y ,, ,, ,, ,, n

If $f_X(x) = f_{X_1, \ldots, X_n}(x_1, \ldots x_n)$ is known and $\vec{Y} = \vec{g}(\vec{x})$ find $f_{\vec{Y}}(\vec{y}) = f_{Y_1 \ldots Y_n}(y_1, \ldots, y_n)$

$$Y_1 = g_1(X_1, \ldots X_n)$$
$$Y_2 = g_2(X_1, \ldots, X_n)$$
$$\vdots$$
$$Y_n = g_n(X_1, \ldots, X_n)$$

Since p is 1-1, there exists an inverse function. ∃ $h_1, \ldots, h_n$

$$X_1 = h_1(Y_1, \ldots, Y_n)$$
$$X_2 = h_2(Y_1 \ldots Y_n)$$
$$\vdots$$
$$X_n = h_n(Y \ldots, Y_n)$$

multi dimentional mapping.

$$f_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = f_{X_1, \ldots, X_n}(h_1(y_1, \ldots, y_n), \ldots, h_n(y_1, \ldots y_n)) \cdot |J_h(y_1 \ldots y_n)|$$

Jacobian

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$$J_h := \mathrm{Det}\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_x} \end{bmatrix} \right)$$

One dimensional case:

$$J_h = \det\left(\left[\frac{\partial g^{-1}(y)}{\partial y} \right] \right) = \frac{\partial g^{-1}(y)}{\partial y}$$

$$f_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = f_{X_1, \ldots, X_n}(h_1(y_1, \ldots, y_n), \ldots, h_n(y_1, \ldots, y_n)) \cdot |J_h(y_1, \ldots, y_n)|$$

Given X_1, X_2 and it is jdf. $Y_1 = \frac{X_1}{X_2} = g(X_1, X_2)$ (5)

Goal find Y_1?

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$$Y_2 = X_2 = g_2(X_1, X_2)$$

$$X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$X_2 = Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2, \quad \frac{\partial h_1}{\partial y_2} = y_1 \qquad \frac{\partial h_2}{\partial y_1} = 0, \quad \frac{\partial h_2}{\partial y_2} = 1$$

$$J_h = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}\right) = y_2 \cdot 1 - 0 \cdot y_1 = y_2$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2) |y_2| \Rightarrow$$

$$f_{Y_1}(y) = \int_{\substack{Supp[Y_2] \\ \| \\ Supp[X_2]}} f_{X_1, X_2}(y_1 y_2, y_2) |y_2| \, dy_2$$

If X_1, X_2 are independent and positive,

$$f_Y(y) = \int_{Supp[X_2]} x_2 f_{X_1}(y_1, x_2) . f_{X_2}(x_2) dx_2$$

Example: Given X_1, X_2 and it is jdf.

$$Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2) \qquad X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$Y_2 = X_1 + X_2 = g_2(X_1, X_2) \qquad X_2 = Y_2 - Y Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = Y_2, \quad \frac{\partial h_1}{\partial y_2} = y_1 \qquad \frac{\partial h_2}{\partial y_1} = -y_2 \qquad \frac{\partial h_2}{\partial y_2} = 1 - y_1$$

$$J_h = \det\left(\begin{bmatrix} Y_2 & y_1 \\ -y_2 & 1 - y_1 \end{bmatrix}\right) = y_2(1 - y_1) - y_1(-y_2) = y_2$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2(1 - y_1)) |Y_2| \Rightarrow$$

$$f_{Y_1}(y_1) = \int_{\substack{Supp[Y_2] \\ = Supp[X_2]}} f_{X_1, X_2}(y_1 y_2, y_2(1 - y_1)) |y_2| \, dy_2$$

If X_1, X_2 are independent & positive,

$$f_Y(y_1) = \int_{Supp[Y_2]} y_2 \, f_{X_1}(y_1 y_2) \, f_{X_2}(y_2(1-y_1)) \, dy_2$$

<u>Example:</u>

Let $X_1 \sim Gamma(\alpha, \lambda)$ independent of

$X_2 \sim Gamma(\beta, \lambda)$

$Y_1 = \dfrac{X_1}{X_1 + X_2} \sim ?$

$Supp[Y_1] = (0, 1)$

$Supp[Y_2] = (0, \infty)$