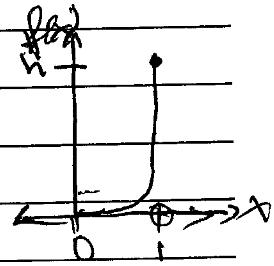
10/19/17

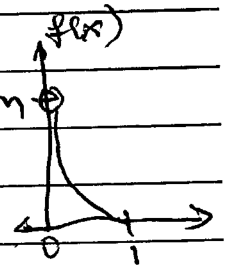
Order Statistics

$X_1, X_2, \dots X_n \overset{iid}{\sim} U(0,1) \Rightarrow f(x) = 1$, $\Rightarrow F(x) = x$



$$f_{X_{(n)}}(x) = n \underbrace{f(x)}_{1} \underbrace{F^{n-1}(x)}_{x} = n x^{n-1}$$

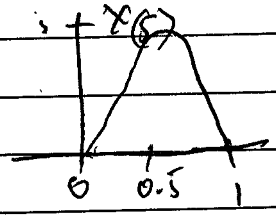
$$\text{Supp}[X_{(k)}] = \text{Supp}[X]$$



$$f_{X_{(1)}}(x) = n \underbrace{f(x)}_{1} (1 - F(x))^{n-1} = n(1-x)^{n-1}$$

$n = 10$

$k = 5$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \underbrace{f(x)}_{1} (\underbrace{F(x)}_{x})^{k-1} (1 - \underbrace{F(x)}_{x})^{n-k}.$$

$$\propto \underbrace{x^{k-1}(1-x)^{n-k}}_{k(x)}$$

$$f(x) = \frac{1}{c} k(x). \qquad \int_{\text{Supp}[x]} k(x)\,dx = c$$

$$\int_0^1 x^{k-1}(1-x)^{n-k+1-1}\,dx = \beta(k, n-k+1)$$

$$X \sim \mathrm{Beta}(\alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$f(x) = \frac{1}{B(k, n+k+1)} x^{k-1}(1-x)^{n-k} = \mathrm{Beta}(k, n-k+1)$$

$$\mathrm{Supp}[X] = (0,1) \qquad \alpha > 0,\ \beta > 0$$

$$\int_{\mathrm{Supp}[X]} f(x) = 1 \qquad \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{\cancel{B(\alpha,\beta)}} \frac{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}{\cancel{B(\alpha,\beta)}} = 1$$

$$F(x) = \int_0^1 \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\overbrace{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt}^{\text{incomplete beta function}}}{B(\alpha,\beta)}$$

$$= \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta) \rightarrow \text{regul incomplete beta function}$$

$$E(X) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

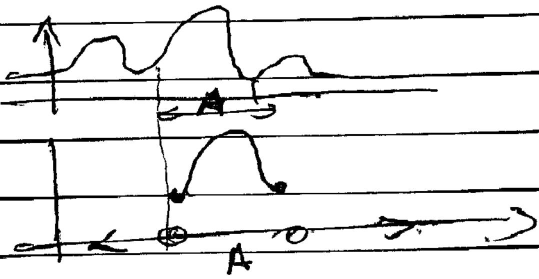
$$= \frac{\dfrac{\overset{\alpha\cancel{\Gamma(\alpha)}}{\Gamma(\alpha+1)}\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1) = (\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}}{\dfrac{\cancel{\Gamma(\alpha)\Gamma(\beta)}}{\cancel{\Gamma(\alpha+\beta)}}} = \frac{\alpha}{\alpha+\beta}$$

Truncations

$X \sim f(x)$

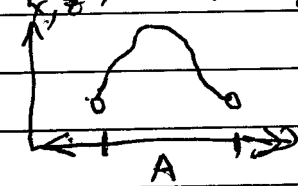
What is if we know $X \in A$ where $A \subseteq supp[X]$

Call this cond. distr. Y $f_Y(y) = ?$



$Z = \mathbb{1}_{X \in A} \sim Ber(P(X \in A))$

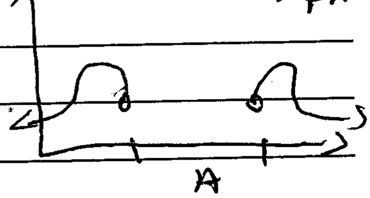
$f_{X,Z}(x,1) = f(x)\,\mathbb{1}_{x \in A}$



$$f_{X|Z}(x,z) = \frac{f_{X,Z}(x,z)}{P_Z(z)}$$

$$= \frac{f(x)\,\mathbb{1}_{x \in A}^{z}\,\mathbb{1}_{x \notin A}^{1-z}}{P(X \in A)^{z}\,(1-P(X \in A))^{1-z}}$$

$f_{X,Z}(x,0) = f(x)\,\mathbb{1}_{x \notin A}$



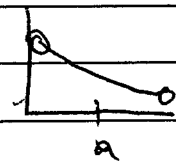
$Y = X|Z=1$

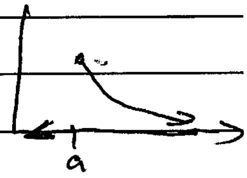
$$f_Y(x) = f_{X|Z}(x,1) = \frac{f(x)}{P(X \in A)}\,\mathbb{1}_{x \in A}$$

$X \geq a$	$X \leq a$	$X \in (a,b)$
$f_Y(x) = \frac{f(x)}{1-F(a)}\,\mathbb{1}_{x \geq a}$	$f_Y(x) = \frac{f(x)}{F(a)}\,\mathbb{1}_{x \leq a}$	$f_Y(x) = \frac{f(x)}{F(b)-F(a)}\,\mathbb{1}_{x \in (a,b)}$

$X \sim Exp(\lambda)$

~~We know $X \geq a$~~





$$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}} \mathbb{1}_{x \geq a}$$

Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\quad$ g is 1-1

let $\vec{X}$ be a Vector r.v with dim = n

let $\vec{Y}$ be a Vector r.v with dim n.

If $f_{\vec{X}}(\vec{x}) = f_{X_1, \dots, X_n}(x_1, \dots, x_n)$ is know and

$\vec{Y} = g(\vec{X})$ find $f_{\vec{Y}}(\vec{y}) = f_{Y_1, \dots, Y_n}(y_1, \dots, y_n)$

$Y_1 = g_1(X_1, \dots, X_n)$ $\quad$ since g is 1-1

$Y_2 = g_2(X_1, \dots, X_n)$ $\quad$ $\exists\, h_1, \dots, h_n$ $\quad$ $X_1 = h_1(Y_1, \dots, Y_n)$

$\vdots$ $\quad$ $X_2 = h_2(Y_1, \dots, Y_n)$

$Y_n = g_n(X_1, \dots, X_n)$ $\quad$ $\vdots$

$X_n = h_n(Y_1, \dots, Y_n)$.

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = f_{X_1, \dots, X_n}\big(h_1(y_1, \dots, y_n), \dots, h_n(y_1, \dots, y_n)\big). \, |J_n(y_1, \dots, y_n)|$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

$$J_n = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}\right)$$

one-dim case

$$J_n = \det\left(\left[\frac{\partial g^{-1}(y)}{\partial y}\right]\right) = \frac{\partial g^{-1}(y)}{\partial y}$$

given X_1, X_2 and its jdf

$$Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2) \qquad X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

p150-151 $$Y_2 = X_2 = g_2(X_1, X_2) \qquad X_2 = Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2 \qquad \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = 0, \quad \frac{\partial h_2}{\partial y_2} = 1$$

$$J_n = \det\left(\begin{bmatrix} y_2, & y_1 \\ 0 & 1 \end{bmatrix}\right) = y_2 \cdot 1 - 0 \cdot y_1 = y_2$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2)\,|y_2|$$

$$\Rightarrow f_{Y_1}(y_1) = \int_{\substack{\text{Supp}\{Y_2\} \\ \shortparallel \\ \text{Supp}\{X_2\}}} f_{X_1, X_2}(y_1 y_2, y_2)\,|y_2|\, dy_2$$

If X_1, X_2 are independent and positive,

$$f_Y(y) = \int_{Supp[X_2]} x_2 f_{X_1}(y x_2) f_{X_2}(x_2)\, dx_2 .$$

Given X_1, X_2 and their pdf

$$Y_1 = \frac{X_1}{X_1+X_2} = g_1(x_1, x_2) \quad X_1 = Y_1 Y_2 = h_1(y_1, y_2)$$

$$Y_2 = X_1 + X_2 = g_2(x_1, x_2) \quad X_2 = Y_2 - Y_1 Y_2 = h_2(y_1, y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2 , \quad \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = -y_2 , \quad \frac{\partial h_2}{\partial y_2} = 1 - y_1$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1 X_2}(y_1 y_2, y_2(1-y_1)) \, |y_2|$$

$$\Rightarrow f_{Y_1}(y_1) = \int_{\substack{Supp[X_1] \\ \cup \\ Supp[X_2]}} f_{X_1, X_2}(y_1 y_2, y_2(1-y_1)) \, |y_2| \, dy_2$$

If X_1, X_2 are independent and positive

$$f_Y(y_1) = \int_{supp[Y_2]} y_2 f_{X_1}(y_1 y_2) f_{X_2}(y_2(1-y_1))\, dx_2.$$

$X_1 \sim Gamma(\alpha, \lambda)$ ind of $X_2 \sim Gamma(\beta, \lambda)$

$$Y_1 = \frac{X_1}{X_1 + X_2} \sim$$

$supp[Y_1] = (0,1)$

$supp[Y_2] = (0, \infty)$