Lecture 19 11/16/17

If $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$, $E[x] = \begin{bmatrix} E[x_1] \\ \vdots \\ E[x_n] \end{bmatrix} = \vec{\mu}$

if $X = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$, $E[X] = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$

$$\Sigma: \text{Var}[\vec{X}] = E\left[(\vec{X} - \vec{\mu})(\vec{X} - \vec{\mu})^T\right]$$

$$\sigma^2 = \text{Var}[X] = E\left[(X-\mu)(X-\mu)\right]$$
$$= E\left[(X-\mu)^2\right]$$

Note: if $q \in \mathbb{R}^n$: $\vec{q}^T q \in \mathbb{R}$ inner product
$\vec{q}\vec{q}^T \in \mathbb{R}^{n \times n}$ is outer product

$$\begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_n - \mu_n \end{bmatrix} \begin{bmatrix} x_1 - \mu_1, & x_2 - \mu_2 & \cdots & x_n - \mu_n \end{bmatrix} = \begin{bmatrix} (x_1-\mu_1)^2 & (x_1-\mu_1)(x_2-\mu_2) & \cdots \\ & (x_2-\mu_2)^2 & \\ & & \ddots \\ & & & (x_n-\mu_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \\ \sigma_{21} & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_n^2 \end{bmatrix} = \Sigma$$
symmetric

$$(a+b)^T = a^T + b^T$$

$$\Sigma := E\left[(X-\mu)(X-\mu)^T\right] = E\left[(X-\mu)\left(X^T-\mu^T\right)\right]$$

$$= E\left[\underbrace{\underset{n\times 1}{X}\,\underset{1\times n}{X^T}}_{n\times n} - \mu X^T - X\mu^T + \mu\mu^T\right]$$

$$= E\left[XX^T\right] + E\left[-\mu X^T\right] + E\left[-X\mu^T\right] + E\left[\mu\mu^T\right]$$

let $X \in \mathbb{R}^{n\times m}$, $A \in \mathbb{R}^{p\times n}$

$AX \in \mathbb{R}^{p\times m}$

$$E[AX] = E\left[\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{p1} & \cdots & a_{pn} \end{bmatrix}\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}\right]$$

$$= E\begin{bmatrix} a_{11}x_{11} + a_{12}x_{12} + \cdots a_{1n}x_{n1} + \cdots & & \\ \vdots & \vdots & \vdots \\ & & \end{bmatrix}$$ = See page 59

$$\text{let } A = \begin{bmatrix} \leftarrow a_{1\cdot} \rightarrow \\ \leftarrow a_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow a_{p\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_{\cdot 1} & a_{\cdot 2} & \cdots & a_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$E\left[a_{1\cdot}X_{\cdot 1}\right] = E\left[a_{11}x_{11} + \cdots + a_{1n}x_{n1}\right] = a_{11}\mu_{11} + \cdots a_{1n}\mu_{n1}$$

$$= a_{1\cdot}\mu_{\cdot 1}$$

<u>Note</u>: $(AB)^T = B^T A^T$

$$= \begin{bmatrix} a_{1\cdot}\mu_{\cdot 1} & a_{1\cdot}\mu_{\cdot 2} & & \\ a_{2\cdot}\mu_{\cdot 1} & a_{2\cdot}\mu_{\cdot 2} & & \\ & & \ddots & \\ & & & \end{bmatrix} = \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ \vdots \\ a_{n\cdot} \end{bmatrix} \left[\mu_{\cdot 1}, \mu_{\cdot 2} \cdots \mu_{\cdot n}\right]$$

$$= A\,E[X] \qquad \left(XB,\; E[XB] = E[X]B \right)$$

For $X + C = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} + \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ c_{n1} & \cdots & c_{nm} \end{bmatrix}$

$$E[X + C] = E[X] + C$$

So $\Sigma = E[XX^T] - \mu\mu^T - \mu\mu^T + \mu\mu^T$ (see P58b)

$$\sigma^2 = E[X^2] - \mu^2$$

<u>Consider</u> $\cancel{Var[A^T X]} = E\left[(A^T X)(A^T X)^T\right]$

$n \times p$ $p \times m$

$$= E[A^T X X^T A]$$

$$= A^T E[XX^T] A$$

$$Var[A^T X] = E[(A^T X)(A^T X)^T] - \underbrace{E[A^T X]\,E[A^T X]^T}_{A^T E[X]\,(A^T E[X])^T}$$

$$= A^T E[X X^T] A - A^T \mu \mu^T A$$

$$= A^T (E[X X^T] - \mu \mu^T) A = A^T \Sigma A$$

$$Var[AX] = A \Sigma A^T$$

dleming matrix

$$Var[\vec{a}^T \vec{X}] = \underset{1\times n}{\vec{a}^T}\ \underset{n \times n}{Var(X)}\ \underset{n\times 1}{\vec{a}}$$

$E[z_i]$ $Var[z_i]$

Let $Z_1 \ldots Z_n \overset{iid}{\sim} N(0,1)$

$E[\vec{Z}] = \vec{\mu}$

$Var[\vec{Z}] = \Sigma$

$$\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \sim N_n(\vec{0}, I_n) := f_{z_1, \ldots z_n}(z_1, \ldots z_n)$$

multivariate Normal (mvN) of dimention n

$\vec{\mu} \in \mathbb{R}^n$

$Supp[\vec{Z}] = \mathbb{R}^n$

$$E[\vec{Z}] = 0$$

$$Var[\vec{Z}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} = I_n$$

$$f_{Z_1,\dots Z_n}(z_1,\dots z_n) = f_{Z_1}(z_1)\cdot\ \cdot\ \cdot f_{Z_n}(z_n) = \cancel{f_{\vec{Z}}(\vec{z})}$$

$$= \prod_{i=1}^{n} f_Z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

Let $\vec{X} = \vec{Z} + \vec{c}$

$$E[\vec{X}] = E[\vec{Z}+\vec{c}] = E[Z] + E[\vec{C}]$$

$$= \vec{0} + \vec{C}.$$

$$Var[\vec{X}] = I_n \Rightarrow \vec{X} \sim N(\vec{c}, I_n)$$

$$f_{X_1\dots X_n}(x_1\dots x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-c_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum(x_i-c_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\vec{x}-\vec{c})^T(\vec{x}-\vec{c})}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\vec{x}-\vec{c})^T(\vec{x}-\vec{c})}$$

Let $\vec{X} = A\vec{Z}$

$\vec{Z} \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{m\times n}$

$$E[\vec{X}] = AE[\vec{Z}] = A\vec{O}_n = \vec{O}_m$$

$$\Sigma = Var[\vec{X}] = A\,Var[\vec{Z}]\,A^T$$

$$= A\underline{I}A^T$$

$$= AA^T$$

$$\underset{\in \mathbb{R}^{m\times 1}}{\vec{X}} = A\vec{Z} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}z_1 + a_{12}z_2 + \cdots a_{1n}z_n \\ a_{21}z_1 + a_{22}z_2 + \cdots a_{2n}z_n \\ \vdots \\ a_{m1}z_1 + a_{n2}z_2 \cdots a_{mn}z_n \end{bmatrix} \begin{matrix} \sim N(0, \Sigma a_{1i}^2) \\ \sim N(0, \Sigma a_{2i}^2) \\ \vdots \\ \sim N(0, \Sigma a_{mi}^2) \end{matrix}$$

Assume $A \in \mathbb{R}^{n \times n}$ (square), also A full rank (or invertible)

$$\vec{x} = A\vec{z} = g(\vec{z}), \quad \vec{z} = h(\vec{x}) = A^{-1}\vec{x}$$

$$\vec{z} = h(\vec{x}) = A^{-1}\vec{x}$$