10/31

- $\phi_X(t) = E(e^{itX}) = \sum_{x \in supp(X)} e^{itx} p(x)$ if X discrete

$= \int_{supp(X)} e^{itx} f(x)dx$ if X cont.

- ① $\phi(0) = 1$

② $Y = X_1 + X_2$ & X_1, X_2 independent $\Rightarrow \phi_Y(t) = \phi_{X_1}(t)\phi_{X_2}(t)$

③ $Y = aX + b \Rightarrow \phi_Y(t) = e^{itb}\phi_X(at)$

④ $|\phi_X(t)| \le 1, \forall X \forall t \Rightarrow \phi_X$ always exists.

- Consider $\phi_X'(t) = \frac{d}{dx}(E[e^{itX}]) = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{itx} f(x)\,dx\right]$

$\overset{?}{=} \int_{\mathbb{R}} f(x) \frac{d}{dt}[e^{itx}]dx$

- Does $\frac{d}{dt}\left[\int g(x,t)dx\right] \overset{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}[g(x,t)]dx$

Combines (a) $\exists t \in A$ s.t. $\int_{\mathbb{R}} g(x,t)dx$ converge $A = [a,b] \subset \mathbb{R}$

(b) $g(x,t)$ cont. $\forall t \in A$

(c) $g(x,t)$ cont. $\forall x \in \mathbb{R}$

(d) $\forall t \in A \int_{\mathbb{R}} \frac{\partial}{\partial t} g(x,t)dt$ conv. uniformly.

- $\phi_X'(t) = \int_{\mathbb{R}} f(x)\, ix e^{itx} dx$ | consider $\phi_X'(0) = \int_{\mathbb{R}} f(x) ix\, dx$

$= i\int_{\mathbb{R}} x f(x)dx = iE(X)$

- $\phi_X''(t) = \int_{\mathbb{R}} f(x)\, i^2 x^2 e^{itx} dx$ $\Rightarrow$ ⑤ $E(X^n) = \frac{\phi_X^{(n)}(0)}{i^n}$

- $\phi_X''(0) = i^2 \int_{\mathbb{R}} x^2 f(x)dx = i^2 E(X^2)$
- $\phi_X'''(0) = i^3 E(X^3)$

10|31

⑥ $P(X \in (a,b)) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-itx} - e^{-itb}}{it} \phi_X(x)\, dt$ Inv Thm.

Motivation if $\phi_X = L^1$

$\Rightarrow f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(x) dx$

$$P(X \in (a,b)) = \int_a^b f(x)dx = \int_a^b \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t) dt\, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_a^b e^{-itx} dx \right) \phi_X(t)\, dt$$

⑦ $\Rightarrow \phi_X(t) = \phi_Y(t) \Leftrightarrow X \overset{d}{=} Y$

⑧ $\phi_{X_n}(t)$ to the ch. f for X_n.

If $\forall t$, $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ also...

$\lim_{n\to\infty} X_n \overset{d}{=} X$ or $X_n \xrightarrow{d} X$ converges.

★ Ex: $X \sim \text{Gamma}(k, \lambda)$

$$\phi_X(x) = \int_0^\infty e^{-itx} \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{(it-\lambda)x} dx$$

U-substitution

$\rightarrow$ let $u = (\lambda - it)x$

$\Rightarrow x = \frac{u}{\lambda - it}$

$\Rightarrow dx = \frac{1}{\lambda - it} du$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(\lambda - it)^{k-1}} e^{-u} \frac{1}{\lambda - it} du$$

$$= \frac{\lambda^k}{\cancel{\Gamma(k)} (\lambda - it)^k} \underbrace{\int_0^\infty u^{k-1} e^{-u} du}_{\cancel{\Gamma(k)}} = \left(\frac{\lambda}{\lambda - it}\right)^k = \left(1 - \frac{it}{\lambda}\right)^{-k}$$

- $X_1 \sim \text{Gamma}(k_1, \lambda)$ ind. of $X_2 \sim \text{Gamma}(k_2, \lambda)$

$$X_1 + X_2 \sim \text{Gamma}(k_1 + k_2, \lambda)$$

$$\phi_{X_1 + X_2}(t) = \phi_{X_1}(t)\, \phi_{X_2}(t)$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^{k_1} \left(\frac{\lambda}{\lambda - it}\right)^{k_2} = \left(\frac{\lambda}{\lambda - it}\right)^{k_1 + k_2}$$

- $X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x e^{-\lambda e^{it}}}{x!} = e^{-\lambda + \lambda e^{it}}$$

$$= e^{\lambda(e^{it} - 1)}$$

$$\frac{(e^{it})^x \lambda^x e^{-\lambda}}{x!} = \frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}}$$

- $X_1 \sim \text{Poisson}(\lambda_1)$ ind. of $X_2 \sim \text{Poisson}(\lambda_2)$

$$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

$$\phi_{X_1 + X_2}(t) = \phi_{X_1}(t)\, \phi_{X_2}(t) = e^{\lambda_1(e^{it} - 1)} e^{\lambda_2(e^{it} - 1)}$$

$$= e^{(\lambda_1 + \lambda_2)(e^{it} - 1)}$$

- $X_1, \ldots, X_n \overset{iid}{\sim}$ some ~~discrete~~ distribution with finite μ & finite variance σ

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad E(X) = \mu, \quad V_n(X) = \frac{\sigma^2}{n}$$

Note:
$i^2(E(X^2)-u^2) = -6^2$

(Def) : $Z_n = \dfrac{\bar{X}_n - u}{\frac{6}{\sqrt{n}}}$ $E(Z_n)=0$ $Var(Z_n)=1$

$\hookrightarrow$? if $n \to \infty$

$\Phi_{\bar{X}_n}(t) = \underset{\#3}{\underset{\uparrow}{=}} \Phi_{\Sigma X_i}(\frac{t}{n}) \underset{\#2}{\underset{\uparrow}{=}} (\Phi_X(\frac{t}{n}))^n$

$\Phi_{Z_n}(t) = \Phi_{\bar{X}_n}\left(\frac{t}{\frac{6}{\sqrt{n}}}\right) e^{it\left(\frac{-u}{\frac{6}{n}}\right)} = \Phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{6}\right) e^{-\frac{itn\sqrt{n}}{6} \cdot \frac{n}{n}}$

$= \Phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{6}\right) e^{\frac{-itun}{6\sqrt{n}}}$

$\Phi_{Z_n}(t) = \left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right)^n$

$\lim_{n\to\infty} \Phi_{Z_n}(t) = \ln_{n\to\infty}\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right)^n e^{\frac{-itun}{6\sqrt{n}}}$

$= \lim_{n\to\infty} e^{\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)^n e^{\frac{-itu^n}{6\sqrt{n}}}\right)}$ $y = e^{\ln(y)}$

$= \lim e^{n\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right) - \frac{itun}{6\sqrt{n}}}$

$= \lim e^{n\left(\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right) - \frac{itu}{6\sqrt{n}}\right)}$

$= e^{\lim n\left(\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right) - \frac{itu}{6\sqrt{n}}\right)}$

$= e^{\lim \frac{\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{6^2}}{\frac{t^2}{6^2}}} = e^{\frac{t^2}{6^2}\lim_{n\to\infty}\frac{\ln\left(\Phi_X\left(\frac{t}{6\sqrt{n}}\right)\right) - \frac{i}{6\sqrt{n}}}{\left(\frac{t}{6\sqrt{n}}\right)^2}}$

Let $u = \frac{t}{6\sqrt{n}}$ $= e^{\frac{t^2}{6^2}\lim_{u\to 0}\frac{\ln(\Phi_X(u)) - iuu}{u^2}}$

$n\to\infty$
$\Rightarrow u \to 0$ $= e^{\frac{t^2}{26^2}\lim_{u\to 0}\frac{\frac{\Phi'(u)}{\Phi(u)} - iu}{u}} = e^{\frac{t^2}{26^2}\lim_{u\to\infty}\frac{d}{du}\left[\frac{\Phi'(u)}{\Phi(u)}\right]}$

$= e^{\frac{t^2}{26^2}(-6^2)} = e^{\frac{t^2}{2}} = \Phi_Z(t)$

$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} \phi(x)\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx + \frac{dt}{t}} dt$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{u}} + \frac{\sqrt{b}+x}{2}\right)^2 + \frac{b^2}{2}\right)} dt$

$= \frac{1}{\pi\sqrt{2}} e^{-\frac{x^2}{2}} \underbrace{\int_{\mathbb{R}} e^{-y^2}\, dy}_{\text{Gamma / triangle}} = \frac{1}{\sqrt{2}\pi} e^{-\frac{x^2}{6}}$ " $N(0,1)$ standard normal