Let $X \sim U(0,1)$ and $Y = \ln(\frac{1}{x} - 1) = g(x)$.

$$\begin{aligned} x &\in [0,1] \\ \frac{1}{x} &\in (1,\infty) \\ \frac{1}{x} - 1 &\in (0,\infty) \\ \ln(\frac{1}{x} - 1) &\in \mathbb{R} \\ -\ln(\frac{1}{x} - 1) &\in \mathbb{R} \\ \text{Supp}[Y] &= \mathbb{R} \end{aligned}$$

If $y = -\ln(\frac{1}{x} - 1)$ then $g^{-1}(y) = \frac{1}{1+e^{-y}}$
Let

$$f(x) = \frac{L}{1 + e^{-k(x - x_0)}}$$

be the logistic function where L is the max, k is the steepness and x_0 is the midpoint. If we let $L = 1$, $x_0 = 0$ and $k = 1$, we get the standard logistic function

$$f(x) = \frac{1}{1 + e^{-x}} = \frac{e^x}{1 + e^x} = g(x)$$

Thus

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| = f_X\left(\frac{1}{1 + e^{-y}}\right)\frac{e^{-y}}{(1 + e^{-y})^2} = \frac{e^{-y}}{(1 + e^{-y})^2} = \text{Logistic}(0,1)$$

By integrating this to get the CDF, we get

$$F_Y(y) = \frac{1}{1 + e^{-y}}$$

This distribution looks like the normal distribution but has heavier tails.

Let $X \sim \text{Exp}(1)$ and $Y = ke^X$ such that $k \in (0,\infty)$. $\text{Supp}[X] = (0,\infty)$. If $k = 1$, $\text{Supp}[Y] = (1,\infty)$; otherwise for general k, $\text{Supp}[Y] = (k,\infty)$.

$$y = ke^x \to g^{-1}(y) = \ln\left(\frac{y}{k}\right)$$

Then

$$\begin{aligned} f_Y(y) &= f_X\left(\ln\frac{y}{k}\right)y^{-1} \\ &= \lambda e^{-\lambda \ln \frac{y}{k}} y^{-1} \\ &= \lambda e^{\ln\left(\frac{k}{y}\right)^{\lambda}} y^{-1} \\ &= \lambda\left(\frac{k}{y}\right)^{\lambda}\frac{1}{y} \\ &= \frac{\lambda k^d}{y^{d+1}} \\ &= \text{Pareto}(k,d) \end{aligned}$$

Then

$$F_Y(y) = \int_k^y \frac{\lambda k^d}{t^{d+1}}\,dt = 1 - \left(\frac{k}{y}\right)^d$$

This distribution is used to model

- population spreads - towns/cities
- survivals, hard drive failures
- surge of sand particles
- file size/ packet size in Internet traffic
- "Pareto Principle" - 1896 - 80% of the land in Italy was owned by 20% of the population

Let $X \sim \text{Pareto}(1, \overbrace{\log_4(5)}^{1.16})$.

What values of x has $p = \mathbb{P}(X \leq x)$ if continuous if $F_X^{-1}(p)$? Quantile$[x,p] = \overset{inf}{x}\{F(x) \geq p\}$.

$$\begin{aligned} p = F_Y(p) &= 1 - (\frac{k}{y})^\lambda \\ 1 - p &= \left(\frac{k}{y}\right)^\lambda \\ \Big(1-p\Big)^{\frac{1}{\lambda}} &= \frac{k}{y} \\ y &= k(1-p)^{-\frac{1}{\lambda}} = F_Y^{-1}(p) \end{aligned}$$

For $X \sim \text{Pareto}(1, \log_4 5)$,

$$\begin{aligned} F_X^{-1}(p) &= (1-p)^{-0.86} \\ F_X^{-1}(0.8) = (1-0.8)^{-0.86} &= 4 \\ 1 - F_X(4) = 1 - \left(\frac{1}{4}\right)^{1.16} &= 0.8 \end{aligned}$$

Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$ and $D = X - Y$. Let $Z = -Y$ such that $f_Z(z) = f_Y(-z) = e^z$. Then

$$\begin{aligned} D &= X + Z \\ &\sim \int_{\text{Supp}[X]} f_X(x) f_Z(d-x)\,dx \\ &= \int_0^\infty e^{-x} e^{d-x} \mathbb{1}_{d-x\in(-\infty,0)}\,dx \\ &= e^d \int_0^\infty e^{-2x}\,dx \\ &= e^d \Big[-\frac{1}{2}e^{-2x}\Big]_{\max\{0,d\}}^\infty \\ &= \frac{1}{2}\begin{cases} e^d & \text{if } d \leq 0 \\ e^{-d} & \text{if } d > 0 \end{cases} \\ &= \frac{1}{2}e^{-|d|} = \text{Laplace}(0,1) \end{aligned}$$

The Laplace distribution is a "double Exponential" distribution.
1774 - "First Law of.. " - Imagine you're measuring a value V. Your measuring instrument is not perfect so you measure $Y \neq V$ but close so $Y = V + \varepsilon$ where ε is the error. It seems reasonable that $\mathrm{E}[\varepsilon] = 0$ and so $\mathrm{E}[Y] = V$. If $\mathrm{Med}(\varepsilon) = 0$ then $\mathrm{Med}(Y) = V$.

$$f_\varepsilon(\varepsilon) = f_\varepsilon(-\varepsilon)$$

Over/under numbers of the same magnitude are equiprobable.

$$f'(\varepsilon) < 0 \text{ if } \varepsilon > 0$$

and so

$$f'(\varepsilon) = f'(-\varepsilon) \to f(\varepsilon) = ce^{-mx}$$

It was figured out that $f(\varepsilon) \propto e^{-\varepsilon^2}$ = Normal when Gauss was 2 years old. This became the Second Law of Errors.

Let $X \sim \mathrm{Exp}(1) = e^{-x}$ and $Y = -\ln X$ where $\mathrm{Supp}[Y] = \mathbb{R}$.

$$y = \ln \frac{1}{x} \to g^{-1}(y) = e^{-y}$$

Then

$$\begin{aligned}
\left|\frac{d}{dy}[g^{-1}(y)]\right| &= e^{-y} \\
f_Y(y) &= f_X(e^{-y})e^{-y} \\
&= e^{-e^{-y}}e^{-y} \\
&= \exp\Big(-(y + e^{-y})\Big) \\
&= \mathrm{Gumbel}(0,1)
\end{aligned}$$

This is the standard Gumbel distribution.

Let $X \sim \mathrm{Gumbel}(0,1)$ and

$$Y = \mu + \beta X \sim \frac{1}{|\beta|} f_X\Big(\frac{y-\mu}{\beta}\Big) = \frac{1}{|\beta|}\exp\Big(-\Big(\frac{y-\mu}{\beta} + e^{-\frac{y-\mu}{\beta}}\Big)\Big) = \mathrm{Gumbel}(\mu,\beta)$$

Parameter Space: $\beta > 0$, $\mu \in \mathbb{R}$.

$$\mathrm{Gumbel}(\mu,\beta) = \frac{1}{\beta}\exp\Big(-\Big(\frac{y-\mu}{\beta} + e^{-\frac{(y-\mu)}{\beta}}\Big)\Big)$$