Lecture 16 11/2/17

$$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \iff \phi_X(t) = e^{-\frac{t^2}{2}}$$

Standard Normal $\quad E[X] = 0, \ SE[X] = 1$

$$X_1 \ldots X_n \overset{iid}{\sim}, \ Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

(CLT)

Example:

$$Y = \mu + \sigma X, \ \sigma \in (0, \infty)$$

$$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} = N(\mu, \sigma^2)$$

"Normal"

$$E[Y] = \mu + \sigma E[X] = \mu$$

$$SE[Y] = \sigma SE[X] = \sigma$$

$$X_1, \ldots X_n \overset{iid}{\sim} P, \underset{\text{mean}}{\mu}, \underset{\text{variance}}{\sigma^2}$$

$$T_n = X_1 + \cdots + X_n \overset{d}{\simeq} N(n\mu, n\sigma^2)$$

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \overset{d}{\simeq} N\left(\mu, \frac{\sigma^2}{n}\right) \text{ if } n \text{ is "large enough"}$$

Practice: $\phi_Y(t) = e^{it\mu} \phi_X(\sigma t) = e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

using convolution

Let $X_1 \sim N(\mu_1, \sigma_1^2)$ indep of $X_2 \sim N(\mu_2, \sigma_2^2)$

$$T = X_1 + X_2 \sim \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(t-x-\mu_2)^2} dx = \ldots \text{Bad}$$

using convolution is hard

Now let use characteristic funct

$$\phi_Y(t) = \phi_{X_1}(t)\phi_{X_2}(t) = e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{it\overset{\mu'}{(\mu_1+\mu_2)} - \frac{\overset{\sigma'^2}{(\sigma_1^2+\sigma_2^2)}t^2}{2}} \Rightarrow$$

$$Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

Example

$$\cancel{X} \sim N(\mu, \sigma^2),\ Y = e^X = g(x) \nearrow x = g^{-1}(y) = \ln(y)$$

$$\mathrm{Sup}[Y] = (0, \infty)$$

$$\left|\frac{d}{dx}[g^{-1}(y)]\right| = \frac{1}{y}$$

$$Y \sim f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{y} e^{-\frac{1}{2\sigma^2}(\ln y - \mu)^2}$$

$$= \mathrm{Log}N(\mu, \sigma^2)$$

"Log normal"

Example: You begin with y_0 amount of \$
you invest it for t time points
with μ_R, σ_R^2 (iid)
R_t is the rate of return (varies)

$$Y_1 = y_0(1+R_1)$$
$$Y_2 = Y_1(1+R_2) = Y_0(1+R_1)(1+R_2)$$
$$Y_t = Y_0 \prod_{i=1}^{t}(1+R_i) = Y_0 e^{\ln\left(\prod_{i=1}^{t}(1+R_i)\right)}$$
$$= Y_0 e^{\sum_{i=1}^{t}\ln(1+R_i)} \quad \text{let } X_i = \ln(1+R_i) \text{ iid}$$
$$= Y_0 e^{\sum_{i=1}^{t} X_i}$$

$$\sum_{i=1}^{t} X_i \overset{d}{\simeq} N(t\mu_X, t\sigma_X^2)$$

Note $e^{\sum x_i} \simeq \log N(t\mu_x, t\sigma_x^2)$

Let $X \sim \log N(\mu, \sigma^2)$, $Y = aX \Rightarrow$
$a \in (0, \infty)$

$$Y \sim f_Y(y) = \frac{1}{a} f_X\left(\frac{Y}{a}\right) = \cancel{\frac{1}{a}\frac{1}{\sqrt{2\pi\sigma^2}}\frac{1}{\frac{Y}{a}} e^{-\frac{1}{2\sigma^2}\ln(\ln}}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{\frac{Y}{a}} e^{-\frac{1}{2\sigma^2}\left(\underbrace{\ln\left(\frac{Y}{a}\right)-\mu}_{\ln(y)-\overbrace{(\mu+\ln(a))}^{\mu'}}\right)^2}$$

$$= \log N(\mu + \ln(a), \sigma^2)$$

$X_i = \ln(1+R_i)$, μ_R, σ_R^2

$R = 0{,}03 = 3\%$

$\Rightarrow X = \ln(1+0{,}03) = .0296$

$R = -0.05 = -5\% \Rightarrow X = \ln(1+ -.05) = -0.051$

$\Rightarrow X \simeq R$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \simeq x$ is x small

$\Rightarrow Y_t = Y_0 e^{\sum x_i} \simeq \log N(t\mu_R + \ln(y_0), t\sigma_R^2)$

Example: $R \overset{iid}{\sim} N(10\%, 10\%^2)$

Start with \$1000, in 5 years what is the Probability you have more than \$1650?

$\log(1000) = 6.91$

$P(Y_t > 1650) = 1 - F_{Y_t}(1650) = 1 - \text{plnorm}(1650, 7.41, \sqrt{.05})$

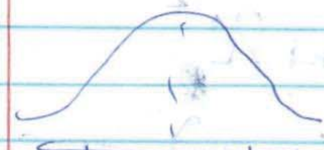
$Y_t \sim \log N(\underbrace{.5 + 6.91}_{7.41}, \underbrace{5 \cdot 10\%^2}_{.05})$

If a quantity experiences normal percentage/Proportional changes then the resultant quantity $\sim \log N$

let $Z \sim N(0,1)$, $Y = Z^2$, $\text{Sup}[Y] = (0, \infty)$

Not 1.1 so we can't use our formula

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$



$$= 2P(Z \in [0, \sqrt{y}])$$

$$= 2\left(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{=1/2}\right)$$

$$= 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = F_Y'(y) = 2 f_Z(\sqrt{y}) \frac{y^{-1/2}}{2}$$

$$= \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}}$$

$$= \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2} = \chi^2_1$$

Chi-Squared with degree of freedom = 1

$$= \chi^2(1)$$

Parameter = degree of freedom

Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt$ let $u = \sqrt{t}$, $\frac{du}{dt} = \frac{1}{2}\frac{1}{\sqrt{t}}$

$dt = 2\sqrt{t}\, du$

$$\Gamma(1/2) = \int_0^\infty \frac{1}{u} e^{-u^2} \cdot 2u\, du = 2\underbrace{\int_0^\infty e^{-u^2}\, du}_{\sqrt{\pi}/2}$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

(52b)

$$f_Y(y) = \frac{\left(\frac{1}{2}\right)^{1/2} y^{-1/2} e^{-y/2}}{\Gamma(1/2)} = \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

Let $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$Z_1^2 + Z_2^2$ Convolution of χ_1^2 and χ_1^2

$= \text{Conv of Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$Z_1^2 + Z_2^2 \sim \text{Gamma}\left(1, \tfrac{1}{2}\right) = \underset{\overset{\shortparallel}{Exp(1/2)}}{\chi_2^2}$$

$Z_1, \ldots, Z_k \overset{iid}{\sim} N(0,1)$

$$\sum_{i=1}^{k} Z_i^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi_k^2 =$$

$$\chi_k^2 = \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{k}{2}\right)}$$

$$= \frac{1}{2^{\frac{k}{2}}} \frac{\left(x^{\frac{k}{2}-1} e^{-\frac{x}{2}}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

$Supp[Y] = (0, \infty)$

Let $X \sim \chi^2_k$, $Y = \sqrt{X}$, $g^{-1}(y) = y^2$

$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = 2y$

$$f_Y(y) = f_X(y^2) \cdot 2y = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} (y^2)^{\frac{k}{2}-1} e^{-y^2/2}\, 2y$$

$\left((y^2)^{1/2}\right)^{k-2} = y^{k-2} \cdot y = y^{k-1}$

$$= \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}} \sim \chi_k$$

Chi-dist'n