10/10/17

Lecture #10

$X \sim \text{Exp}(1)$

$Y = -\ln X \sim \text{Gumbel}(0,1) = e^{-(y+e^{-y})} = e^{-y} \cdot e^{-e^{-y}}$ PDF

$Y = \text{Gumbel}(0,1)$; $X = e^{-Y} \sim \text{Exp}(1)$

Try to get a CDF

$F_Y(y) = P(Y \leq y) = P(-Y \geq -y) = P(e^{-Y} \geq e^{-y})$

$= P(X \geq e^{-y}) = 1 - F_X(e^{-y}) = e^{-e^{-y}}$

$\boxed{\begin{array}{l} \text{Exp}(\lambda) = \lambda e^{-\lambda x} \\ \text{Exp}(1) = e^{-x} \end{array}}$

$X \sim \text{Gumbel}(0,1)$, $Y = \mu + \beta X \sim \text{Gumbel}(\mu, \beta)$

$= \frac{1}{\beta} e^{-\left(\left(\frac{y-\mu}{\beta}\right) + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$

CDF of general Gumbel.

$$\underbrace{\begin{aligned} F_Y(y) &= (Y \leq y) = P\left(\frac{Y-\mu}{\beta} \leq \frac{y-\mu}{\beta}\right) = P\left(X \leq \frac{y-\mu}{\beta}\right) \\ &= F_X\left(\frac{y-\mu}{\beta}\right) = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}} \end{aligned}}_{\text{Valid for any linear transformation.}}$$

$X \sim \text{Gumbel}(0,1)$ $\quad Y = e^{-X} \sim \text{Exp}(1)$

$X \sim \text{Gumbel}(\mu, \beta)$ $\quad Y = \underbrace{e^{-X}}_{g(x)} \sim ?$

$\text{Supp}[X] = \mathbb{R}$

$\text{Supp}[Y] = (0, \infty)$

$X = -\ln(Y) = g^{-1}(y)$ $\quad \left|\frac{d}{dy}(-\ln(y))\right| = \left|-\frac{1}{y}\right| = y^{-1}$

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}(g^{-1}(y))\right| = f_X(-\ln y) \cdot y^{-1}$

$= \frac{1}{\beta} e^{-\left(\left(\frac{-\ln y - \mu}{\beta}\right) + e^{-\left(\frac{-\ln y - \mu}{\beta}\right)}\right)} \cdot y^{-1}$

$= \frac{1}{\beta} e^{-\left(\frac{-\ln y - \mu}{\beta}\right)} \cdot e^{-e^{-\left(\frac{-\ln y - \mu}{\beta}\right)}} \cdot y^{-1}$

Note that $-\left(\frac{-\ln(y) - \mu}{\beta}\right) = \frac{\ln y + \mu}{\beta}$

Let $k = \frac{1}{\beta} \Rightarrow$ $\quad = k(\ln y + \mu)$

Let $\mu = \ln(\lambda) \Rightarrow = k(\ln y + \ln \lambda)$

$$= k \ln(\lambda y)$$
$$= \ln((\lambda y)^k)$$

So $e^{\left(-\frac{\ln y - \mu}{\beta}\right)} = e^{\ln(\lambda y)^k} = (\lambda y)^k$ ⊛⊛

$\beta \in (0,\infty) \Rightarrow k \in (0,\infty)$

$\mu \in \mathbb{R} \Rightarrow \lambda \in (0,\infty)$

$$f_Y(y) = k\underbrace{\underbrace{(\lambda y)^k}_{\lambda^k y^k}}_{\lambda \cdot \lambda^{k-1} \cdot y^k} \cdot e^{(-\lambda y)^k} \cdot y^{-1}$$

$$= (k\lambda)(\lambda y)^{k-1} \cdot e^{(-\lambda y)^k} = \text{Weibull}(k, \lambda)$$

Here support of the new r.v. will be $(0,\infty)$

CDF $\Rightarrow$ $F_Y(y) = P(Y \le y) = P(\ln Y \le \ln y) = P(-\ln Y \geq -\ln y)$

$$= P(X \geq -\ln y) = 1 - F_X(-\ln y)$$

$$= 1 - e^{-e^{-\left(\frac{\ln y - \mu}{\beta}\right)}} = 1 - e^{-(\lambda y)^k}$$ From ⊛⊛

If $\beta = 1 \Rightarrow k = 1$ Weibull $(1, \lambda) = \lambda e^{-\lambda y} = \text{Exp}(\lambda)$

"memoryless"

Weibull $(1,1) = \text{Exp}(1)$

$k=1, \lambda=1$

Case # 1 $k > 1$ eg. $k = 2$ $\quad P(X \geq 7yr) > P(X \geq 7+3 \mid X \geq 3)$

$X \sim$ Weibull $(2, \lambda)$

$F_X(x) = 1 - e^{-(\lambda x)^2} \Rightarrow 1 - F_X(x) = e^{-(\lambda x)^k}$

CDF

Let's do memoryless calculation.

WTS : $P(X \geq b) > P(X \geq a+b \mid X \geq a)$

$$1 - F_X(b) > \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{1 - F_X(a+b)}{1 - F_X(a)}$$

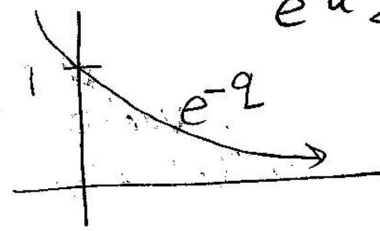
$$e^{-(\lambda b)^2} > \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}} = \frac{\cancel{e^{-(\lambda a)^2}} \cdot e^{-2\lambda^2 ab} \cdot e^{-(\lambda b)^2}}{\cancel{e^{-(\lambda a)^2}}}$$

$$e^{-(\lambda b)^2} > e^{-2\lambda^2 ab} \cdot e^{-(\lambda b)^2}$$

$$> e^{-2\lambda^2 ab}$$

$\lambda > 0 \quad a > 0 \quad b > 0 \quad q > 0$

$e^{-q} < 1$



Case # 2 $k < 1$ e.g. $k = \frac{1}{2}$

W.T.S. $P(X \geq b) < P(X \geq a+b \mid X \geq a)$

$$e^{-(\lambda b)^{1/2}} < \frac{e^{-(\lambda(a+b))^{1/2}}}{e^{-(\lambda a)^{1/2}}} = e^{-(\lambda(a+b))^{1/2} + (\lambda a)^{1/2}}$$

$$< e^{-\lambda^{1/2}\left((a+b)^{1/2} - a^{1/2}\right)}$$

log: $\Rightarrow \dfrac{-\lambda^{1/2} b^{1/2}}{-\lambda^{1/2}} < \dfrac{-\lambda^{1/2}}{-\lambda^{1/2}}(\ldots\ldots)$

$$b^{1/2} > (a+b)^{1/2} - a^{1/2} \Rightarrow a^{1/2} + b^{1/2} > (a+b)^{1/2}$$

$$(a^{1/2} + b^{1/2})^2 > a+b$$

$$\Rightarrow a + b + 2a^{1/2}b^{1/2} > a+b$$

$$2a^{1/2}b^{1/2} > 0$$

$k=1$ is the exponential

Let $X \sim \text{Weibull}(k, \lambda)$ $\quad Y = \frac{1}{X} \sim ?$

Survival time:

$g^{-1}(y) = \frac{1}{y} \qquad \text{Supp}[X] = (0, \infty] \qquad \text{Supp}[Y] = (0, \infty)$

$$\left| \frac{d}{dy}(g^{-1}(y)) \right| = |-y^{-2}| = y^{-2} = \frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{1}{y^2} = f_X\left(\frac{1}{y}\right) \cdot y^{-2}$$

$$= (k\lambda)\left(\lambda \cdot \frac{1}{y}\right)^{k-1} \cdot e^{-\left(\lambda \frac{1}{y}\right)^k} \cdot \frac{1}{y^2}$$ ⇐ Exam stop here

$$= k \cdot \lambda^k \cdot \underbrace{\frac{1}{y^{k-1}} \cdot \frac{1}{y^2}}_{y^{k+1}} \cdot e^{-\lambda k / y^k} = k\lambda^k \cdot \frac{1}{y^{k+1}} \cdot e^{-\frac{\lambda k}{y^k}}$$

$$= \frac{k}{\lambda}\,\lambda^{k+1} \cdot \frac{1}{y^{k+1}} \cdot e^{-\left(\frac{y}{\lambda}\right)^{-k}} = \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{-(k+1)} e^{-\left(\frac{y}{\lambda}\right)^{-k}}$$

$= \text{Frechet } (k, \lambda, 0)$

centered

{Gumbel (log inverse time), Weibull (time), Frechet (inverse time)} belong to a special family called the generalized extreme value dist.

Revisiting Erlang and Neg. Binomial.

Waiting for k successes

$$X \sim \text{Erlang } (k, \lambda) := \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

Waiting for ....

$$X \sim \text{NegBin } (k, p) := \binom{x+k-1}{k-1} p^k (1-p)^x = \frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)} p^k (1-p)^x$$

Erlang parameter space $k \in \mathbb{N}$, $\lambda \in (0, \infty)$

NegBin parameter space $k \in \mathbb{N}$, $p \in (0, 1)$

What if $k \in (0, \infty)$ and no longer $k \in \mathbb{N}$. Means using #'s like 2.79, 3.63

$\frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)}$ = extended negative binomial
can model # of cardiac events.

$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$ = Gamma (k, λ)
waiting time or survival time.

* Erlangs are subsets of Gamma.

<u>Recall:</u>

$X \sim \text{Erlang } (k_1, \lambda)$ independent of $Y \sim \text{Erlang } (k_2, \lambda) \Rightarrow$
$X + Y \sim \text{Erlang } (k_1 + k_2, \lambda)$.

T.S.
If $X \sim \text{Gamma } (k_1, \lambda)$ independent of
$Y \sim \text{Gamma } (k_2, \lambda) \Rightarrow X + Y \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$X+Y \sim \int_0^t f_X(x) f_Y(t-x)\, dx$$

$$= \int_0^t \frac{\lambda^{k_1} x^{k_1-1} e^{-\lambda x}}{\Gamma(k_1)} \cdot \frac{\lambda^{k_2} (t-x)^{k_2-1} \cdot \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t} \cdot e^{\lambda x}}}{\Gamma(k_2)} \cdot dx$$

$$= \frac{\lambda^{k_1+k_2} \cdot e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1} (t-x)^{k_2-1} \cdot dx$$

Let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

lower bound $x_\ell = 0 \Rightarrow u_\ell = 0$

$x_u = t \Rightarrow u_u = 1$

$$X+Y = \frac{\lambda^{k_1+k_2} \cdot e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \cdot \int_0^1 ut^{(k_1-1)} \cdot (t-ut)^{k_2-1}\, t \cdot du$$

$$= \int_0^1 t^{k_1-1} \cdot t^{k_2-1} \cdot u^{k_1-1} \cdot (1-u)^{k_2-1}\, t\, du.$$

$$= \frac{\lambda^{k_1+k_2} \cdot e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \cdot t^{k_1+k_2-1} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} \cdot du$$