Let $T \sim \text{Exp}(\lambda) = \lambda e^{\lambda t}$ which describes the time between Poisson events. In fact, $F_T(t) = 1 - e^{-\lambda t}$.
Let $N \sim \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$ which describes the number of events occurring within a time interval. In fact, $F_N(n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$.
What is the probability that no events have occurred by $t = 1$?

$$\mathbb{P}(T > 1) = e^{-\lambda} = \mathbb{P}(N = 0) = e^{-\lambda}$$

What is the probability that at least one event occurred before $t = 1$?

$$\mathbb{P}(T < 1) = 1 - e^{-\lambda} = \mathbb{P}(N > 0) = 1 - e^{-\lambda}$$

What is the probability of no successes or one success by $t = 1$?

$$\mathbb{P}(N \leq 1) = F_N(1) = e^{-\lambda}(1 + \lambda)$$

If $T \sim \text{Erlang}(2, \lambda)$, this scenario can be computed as

$$\mathbb{P}(T > 1) = 1 - F_T(1)$$

Let $X \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$. Then $F_X(x) = \frac{\gamma(k, \lambda x)}{(k-1)!}$. This comes from

$$\underbrace{\Gamma(x)}_{\text{gamma function}} = \int_0^\infty t^{x-1} e^{-t}\, dt = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\underbrace{\gamma(x,a)}_{\text{lower incomplete gamma function}}} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\underbrace{\Gamma(x,a)}_{\text{upper incomplete gamma function}}}$$

The gamma function is known as an extension of the factorial function to all real numbers.

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{1-1} e^{-t}\, dt && = -e^{-t}\Big|_0^\infty = -(0-1) = 1 \\
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\, dt = [-t^x e^{-t}]\Big|_0^\infty - \int_0^\infty -e^{-t} x t^{x-1}\, dt = x\Gamma(x) \\
\Gamma(2) &= 1 \cdot 1 \\
\Gamma(3) &= 2\Gamma(2) = 2 \cdot 1 \\
\Gamma(4) &= 3\Gamma(3) = 3 \cdot 2 \cdot 1 \\
&\vdots \\
\Gamma(n) &= (n-1)!
\end{aligned}$$

Thus

$$F_{T_k}(x) = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

which is called the normalized gamma function.

$$1 - F_{T_k}(x) = 1 - \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k, \lambda x)}{\Gamma(k)} = Q(l, \lambda x)$$

which is called the regularized gamma function, a proportion of the entire gamma.

We know that $k \in \mathbb{N}$, then

$$\begin{aligned}
\Gamma(k, \lambda x) &= \int_{kx}^{\infty} t^{k-1} e^{-t}\, dt \\
&= -t^{k-1} e^{-t} \Big|_{\lambda x}^{\infty} - \int_{\lambda x}^{\infty} (k-1) t^{k-2} (-e^{-t})\, dt \\
&= (\lambda x)^{k-1} e^{-\lambda x} + (k-1)\Gamma(k-1, \lambda x) \\
&= (\lambda x)^{k-1} e^{-\lambda x} + (k-1)\Big((\lambda x)^{k-2} e^{-\lambda x} + (k-2)\Gamma(k-2, \lambda x)\Big) \\
&= e^{-\lambda x}\Big((\lambda x)^{k-1} + (k-1)(\lambda x)^{k-2} + (k-2)(k-1)\frac{\Gamma(k-2, \lambda x)}{e^{-\lambda x}}\Big) \\
&= e^{-\lambda x}\Big(\frac{(\lambda x)^{k-1}}{(k-1)!} + \frac{(\lambda x)^{k-2}}{(k-2)!} + \cdots + \underbrace{1}_{\Gamma(1, \lambda x) = \int_{\lambda x}^{\infty} t^{1-1} e^{-t}\, dt = e^{-\lambda x}}\Big) \\
&= e^{-\lambda x} (k-1)! \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}
\end{aligned}$$

Then

$$1 - F_{T_k}(x) = \frac{e^{-\lambda x}(k-1)! \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}{(k-1)!} = e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}$$

Let $T \sim \text{Erlang}(2, \lambda)$, then

$$\mathbb{P}(T > 1) = 1 - F_{T_2}(1) = e^{-\lambda} \sum_{i=0}^{1} \frac{(\lambda \cdot 1)^i}{i!} = e^{-\lambda}(1 + \lambda)$$

What is the probability of k successes or less by $t = 1$?

$$\mathbb{P}(N \leq k) = F_X(k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

If successes come exponentially, what is the probability of seeing k or fewer successes by 1 hr? Let $T \sim \text{Erlang}(k+1, \lambda)$. Then

$$\mathbb{P}(T > 1) = 1 - F(1) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

Poisson Process: in every unit time, there are $X \sim \text{Poisson}(\lambda)$ "hits" and each hit occurs after $T \sim \text{Exp}(\lambda)$.

$$e^{-\lambda} \sum_{i=0}^{l} \frac{\lambda^i}{i!} = \frac{\Gamma(k+1, \lambda)}{\Gamma(k)} = Q(k+1, \lambda)$$

If we let $k \to \infty$ and $Q \to 1$, then

$$\sum_{i=0}^{k} \frac{a^i}{i!} = e^a Q(k+1, a)$$

$$e^a = \sum_{i=0}^{k} \frac{a^i}{i!}$$

Running experiments	fixed time, measure number of successes	require at least 1 success	require 1 success
discretely	Binomial	Negative Binomial	Geometric
continuously	Poisson	Erlang	Exponential

What is the probability that there has been 2 successes or less by $t = 50$?

$$\begin{aligned} N &\sim \text{Binom}(50, p) \\ \mathbb{P}(N \le 2) = F_N(2) &= \binom{50}{0}(1-p)^{50} + \binom{50}{1} p(1-p)^{49} + \binom{50}{2} p^2(1-p)^{48} \\ T &\sim \text{NegBinom}(3, p) \\ \mathbb{P}(T \ge 48) &= 1 - F_T(47) \\ &= 1 - \sum_{i=0}^{47} \binom{i+2}{2} p^3 (1-p)^{46} \end{aligned}$$

Let $N \sim \text{Binom}(n, p)$ and $T \sim \text{NegBinom}(k+1, p)$, then

$$\begin{aligned} F_N(K) &= 1 - F_T(n-k-1) \\ &= 1 - \sum_{i=0}^{n-k-1} \binom{i+k}{k} p^{k+1}(1-p)^i \end{aligned}$$

Let $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$. What is $\mathbb{P}(X_1 \mid X_1 + X_2)$?
What is $\mathbb{P}(X_1)$? This is $\mathbb{P}(X_1 = x) = \mathbb{P}_X(x)$. What is $\mathbb{P}(X_1 + X_2)$? This is the same as $\mathbb{P}(X_1 + X_2 = n)$. Then

$$\begin{aligned} \mathbb{P}(X_1 = x \mid X_1 + X_2 = n) &= \frac{X_1 = x \text{ and } X_1 + X_2 = n}{\mathbb{P}(X_1 + X_2 = n)} \\ &= \frac{\mathbb{P}_{X_1, X_2}(x, n-x)}{\mathbb{P}_Y(n)} \\ &= \frac{\frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{e^{-\lambda}\lambda^{n-x}}{(n-x)!}}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} \\ &= \binom{n}{x}\left(\frac{\lambda}{2\lambda}\right)^n \\ &= \binom{n}{x}\left(\frac{1}{2}\right)^2 \\ &= \text{Binom}\left(n, \frac{1}{2}\right) \end{aligned}$$