Lecture 18 11/9/17

$X_1 \sim N(0,1)$
ind of
$X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} = \int_{\sup[X]} |x| f_{X_1}(xr) f_{X_2}(x)\, dx$$

$$= \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 r^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} |x| e^{-\frac{1}{2}x^2(r^2+1)}\, dx$$

$$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} (-x) e^{-\frac{1}{2}x^2(r^2+1)}\, dx + \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\, dx \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\, dx = \frac{1}{\pi} \int_0^{-\infty} x e^{u} \frac{-1}{x(r^2+1)}\, du$$

$u = -\frac{1}{2}x^2(r^2+1)$ $\quad x = 0 \Rightarrow u = 0$

$\frac{du}{dx} = -x(r^2+1)$ $\quad x = \infty \Rightarrow u = -\infty$

$\Rightarrow dx = \frac{-1}{x(r^2+1)}\, du$

$$= \frac{1}{\pi(r^2+1)} \left[-e^{u} \right]_0^{-\infty}$$

$$= \frac{1}{\pi(r^2+1)} = \text{Cauchy}(0,1)$$

Midterm 2 End Here

$X_1, \dots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$ unknown dist

$\overline{X}$ is the average r.v $\bar{x} = \frac{x_1 + \dots x_n}{n}$ realisation

"Estimator" "estimate"

$E[\overline{X}] = \mu$ unbiased

S^2 is a sample variance random variable $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$ is realisation $E[S^2] = \sigma^2$

"Estimator" "estimate"

Let assume $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$

What is $S^2 \sim$?

$$S^2 = \frac{1}{n-1}\left((X_1 - \overline{X})^2 + \dots + (X_n - \overline{X})^2\right)$$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}, \quad \sum_{i=1}^{n} Z_i^2 = \vec{Z}^T \vec{Z} \sim \chi_n^2$$

$$\sum Z_i^2 = \sum \left(\frac{X - \mu}{\sigma}\right)^2 = \frac{\sum (X_i - \mu)^2}{\sigma^2} \sim \chi_n^2$$

$$\sum (x_i-\mu)^2 = \sum\left((x_i-\bar{x})+(\bar{x}-\mu)\right)^2$$

$$= \sum (x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2$$

$$= \sum (x_i-\bar{x})^2 + 2\left(\sum x_i\bar{x} - \mu\sum x_i - \sum \bar{x}^2 + \sum \bar{x}\mu\right)$$

$$\cancel{n\bar{x}^2} - \cancel{n\mu\bar{x}} - \cancel{n\bar{x}^2} + \cancel{n\bar{x}\mu} \quad + n(\bar{x}-\mu)^2$$

$$= \frac{\sum (x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \qquad \left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_1$$

Cochran's Theorem (1934)

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

let $Q_1, \dots, Q_k$ be scalar r.v.'s with the quadratic form

$Q_j = \vec{Z} B_j \vec{Z}$ where $B_1, \dots, B_k$ are positive semi define matrices such that

$\vec{Z}^T\vec{Z} = Q_1 + \dots + Q_k$

Defn: Positive semi-definite Matrix A is positive semi-definite if $\forall \vec{v}$ $\vec{v}^T A \vec{v} \geq 0$

(a) $n = \sum \text{rank}(B_j)$ χ^2_n

$\Updownarrow$

(b) Q_j's are independent

$\Updownarrow$

(c) $Q_j \sim \chi^2_{\text{rank}(B_j)}$

$= \vec{Z}^T B_1 \vec{Z} + \dots + \vec{Z}^T B_k \vec{Z}$

$= \vec{Z}^T (B_1 + \dots + B_k)\vec{Z}$

$\Rightarrow I_n = B_1 + \dots + B_k$

$$\sum z_i^2 = \sum\left((z_i - \bar{z}) + (\bar{z})\right)^2$$

$$= \sum (z_i - \bar{z})^2 + \underbrace{2\sum (z_i - \bar{z})\bar{z}}_{\substack{2\left(\sum z_i\bar{z} - \sum \bar{z}^2\right) \\ 2\left(\cancel{n\bar{z}^2} - \cancel{n\bar{z}^2}\right)}} + \underbrace{\sum \bar{z}^2}_{n\bar{z}^2}$$

$$\Rightarrow \sum z_i^2 = \underbrace{\sum (z_i - \bar{z})^2}_{Q_1} + \underbrace{n\bar{z}^2}_{Q_2}$$

$$Q_2 = n\bar{z}^2 = \vec{z}^T B_2 \vec{z}$$

let $J_n = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ $n \times n$

Example $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2J_2$

$$Q_2 = \vec{z}^T\left(\tfrac{1}{n}J_n\right)\vec{z} = z^T \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ & & \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$$

$$= z^T \begin{bmatrix} \bar{z} \\ \vdots \\ \bar{z} \end{bmatrix} = z_1\bar{z} + \cdots + z_n\bar{z} = \bar{z}\left(\sum z_i\right)$$

$$= \bar{z}n\bar{z}$$

$$= n\bar{z}^2$$

Theorem: Matrix A is symmetric, idempotent

$$\Rightarrow tr(A) = rank(A)$$

$$\left(\tfrac{1}{n}J_n\right)\left(\tfrac{1}{n}J_n\right) = \tfrac{1}{n^2}J_nJ_n = \tfrac{1}{n^2}nJ_n = \tfrac{1}{n}J_n \Rightarrow$$

$$tr\left(\tfrac{1}{n}J_n\right) = 1 = rank\left(\tfrac{1}{n}J_n\right)$$

57b

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow$$

$$S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$$

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$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{"z test"}$$

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2}}$$

"student 1908"

$$= \frac{\dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\dfrac{\frac{n-1}{\sigma^2}S^2}{n-1}}}$$

$N(0,1)$

χ^2_{n-1}

$$= \frac{Z}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}} = T_{n-1}$$

Both independent

$$\sum (z_i - \bar{z})^2 = \vec{Z}^T B_1 \vec{Z}$$

$$= \sum z_i^2 - 2 z_i \bar{z} + \bar{z}^2$$

$$= \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2$$

$$= \sum z_i^2 - n\bar{z}^2$$

$$= \vec{z}^T I_n \vec{z} - \vec{z}^T \tfrac{1}{n} J_n \vec{z} = \vec{z}^T \underbrace{(I_n - \tfrac{1}{n} J_n)}_{= B_2} \vec{z}$$

$$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & & & \vdots \\ -\frac{1}{n} & & & 1-\frac{1}{n} \end{bmatrix}$$

Proven: $n\bar{z}^2 \sim \chi^2_1$ iid of $\sum (z_i - \bar{z})^2 \sim \chi^2_{n-1}$

$$\frac{n(\bar{x}-\mu)^2}{\sigma^2} = \underbrace{\frac{1}{\sigma}(\vec{x}-\vec{\mu})}_{\vec{z}^T}\left(\frac{1}{n}J_n\right)\frac{1}{\sigma}(\vec{x}-\vec{\mu}) \sim \chi^2_1$$

$$\frac{\sum (x_i - \bar{x})^2}{\sigma^2} = \frac{1}{\sigma}(\vec{x}-\vec{\mu})^T\left(I_n - \frac{1}{n}J_n\right)\frac{1}{\sigma}(\vec{x}-\vec{\mu}) \sim \chi^2_{n-1}$$

$\Rightarrow \dfrac{n(\bar{x}-\mu)^2}{\sigma^2}$ ind of $\dfrac{(n-1)S^2}{\sigma^2} \Rightarrow \bar{X}, S^2$ are indpdt

"Fisher 1935"

Gerny, 1936 Proved this is unique to the normal