Lecture 9 9/29/17

$X \sim (0,1)$, $Y = \ln\left(\frac{1}{x} - 1\right) = g(x)$

$x \in [0,1]$ $\frac{1}{x} \in (1, \infty)$, $\frac{1}{x} - 1 \in (0, \infty)$.

$\ln\left(\frac{1}{x} - 1\right) \in \mathbb{R}$, $-\ln\left(\frac{1}{x} - 1\right) \in \mathbb{R} \Rightarrow$

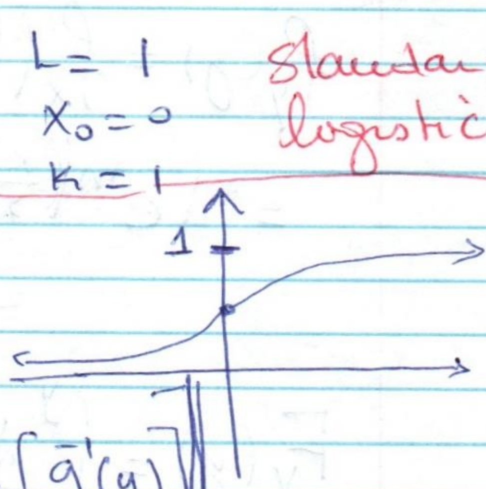
support$[Y] = \mathbb{R}$

$y = -\ln\left(\frac{1}{x} - 1\right) \Rightarrow g^{-1}(y) = \frac{1}{1+e^{-y}}$

$f(x) = \frac{L}{1 + e^{-k(x-x_0)}}$ logistic function

L : max
k : steepness
x_0 : midpoint

$$\boxed{\begin{array}{ll} f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x} & \\ L = 1 & \text{standard} \\ x_0 = 0 & \text{logistic} \\ k = 1 & \end{array}}$$



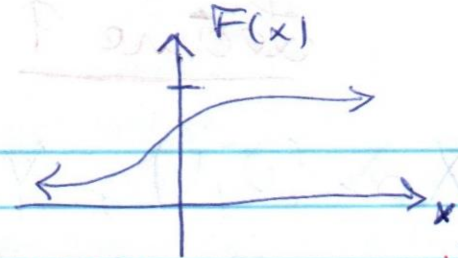
$f_Y(x) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$

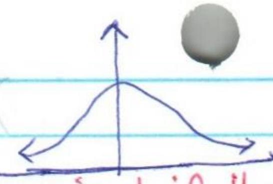
$= \underbrace{f_X\left(\frac{1}{1+e^{-y}}\right)}_{\substack{1 \\ \text{Because} \\ U(0,1)}} \frac{e^{-y}}{(1+e^{-y})^2} = \frac{e^{-y}}{(1+e^{-y})^2}$ = logistic (0,1) distribution (standard logistic)

$$\boxed{cdf = \int f_y(y)}$$

$$F_Y(y) = \frac{1}{1+e^{-y}}$$

standard logistic function





Heavier tails than normal

Hw: show $X \sim Exp(1) \Rightarrow Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right)$

let $X \sim Exp(\alpha) \Rightarrow Y = k e^{X}$ such that $k \in (0,\infty)$

support$[x] = (0,\infty)$ if $k=1$ supp$[y] = (1, \infty)$

general k sup$[y] = (k, \infty)$

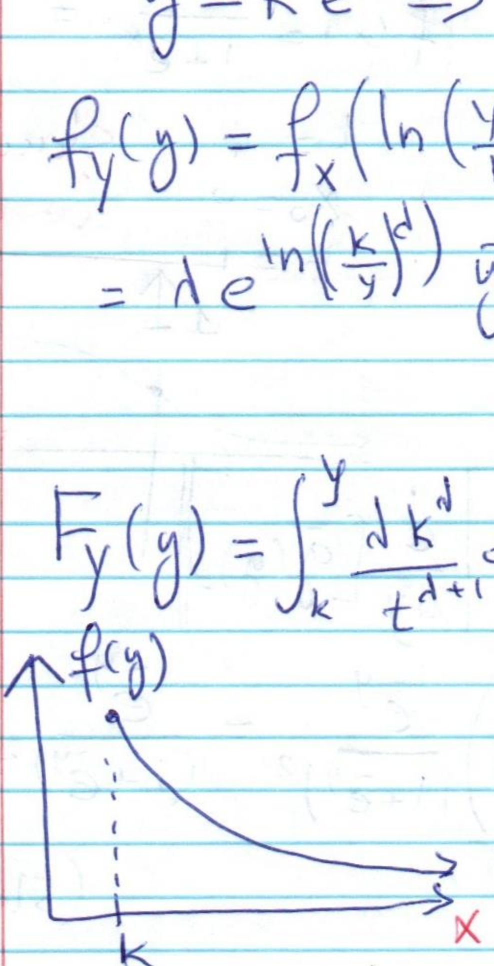
$$y = k e^{x} \Rightarrow g^{-1}(y) = \ln\left(\frac{y}{k}\right)$$

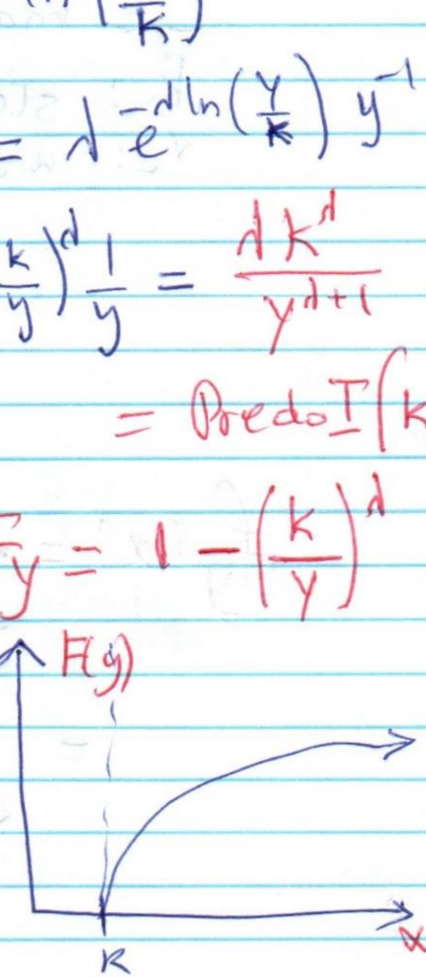
$$f_Y(y) = f_X\left(\ln\left(\frac{y}{k}\right)\right) y^{-1} = \alpha e^{-\alpha \ln\left(\frac{y}{k}\right)} y^{-1}$$

$$= \alpha e^{\ln\left(\left(\frac{k}{y}\right)^{\alpha}\right)} y^{-1} = \alpha \left(\frac{k}{y}\right)^{\alpha} \frac{1}{y} = \frac{\alpha k^{\alpha}}{y^{\alpha+1}}$$

$$= \text{Pareto I}(k, \alpha)$$

$$F_Y(y) = \int_k^y \frac{\alpha k^{\alpha}}{t^{\alpha+1}} dt \Rightarrow F_Y = 1 - \left(\frac{k}{Y}\right)^{\alpha}$$





Used to model

- Population spreads - towns / cities
- survivals, Hard drive failures
- Size of sand particles
- files size / packets size in Internet traffic
- And ... "Pareto Principle"

1896 80% of the land in Italy was own by 20% of the population.

$$X \sim \text{Pareto}\left(1, \overbrace{\log_4(5)}^{1.16}\right)$$

$\text{Quantile}[X, p] = \inf_x \{ F(x) \geq p \}$

what value of x has $p = P(X \leq x)$

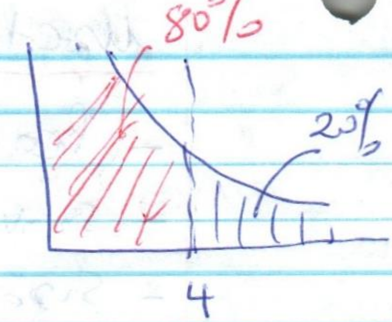
what is 99%ile of the SAT?

$\rightarrow$ if continuous is $F_X^{-1}(p)$

$$p = F_Y(p) = 1 - \left(\frac{k}{y}\right)^{\alpha} \Rightarrow 1 - p = \left(\frac{k}{y}\right)^{\alpha} \Rightarrow (1-p)^{\frac{1}{\alpha}} = \frac{k}{y}$$

$$\Rightarrow y = k(1-p)^{-\frac{1}{\alpha}} = F_Y^{-1}(p)$$

for $X \sim$ pareto $(1, \overbrace{\log_4 5}^{1.16})$



$F_X^{-1}(p) = (1-p)^{-0.86}$

$F_X^{-1}(0.8) = (1-0.8)^{-.861} = 4$

$1 - F_X(4) = 1 - \left(\frac{1}{4}\right)^{1.16} = 0.8$

let $X, Y \overset{iid}{\sim} Exp(1)$

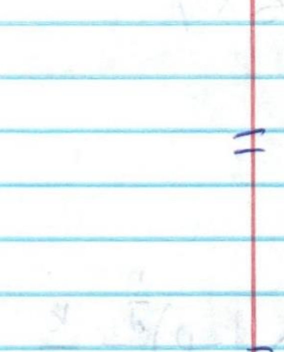
$D = X - Y$

let $Z = -Y$ $\quad f_Z(z) = f_Y(-z) = e^{z}$

$$D = X + Z \sim \int_{\sup[X]} f_X(x) f_Z(d-x)\,dx =$$

$$= \int_0^\infty e^{-x} e^{d-x} \mathbb{1}_{\substack{d-x \in (-\infty, 0) \\ x \geq d}} \, dx$$

$$= e^{d} \int_{\max[0,d]}^{\infty} e^{-2x}\,dx = e^{d}\left[-\frac{1}{2} e^{-2x}\right]_{\max[0,d]}^{\infty}$$



$$= \frac{1}{2}\begin{cases} e^{d} & \text{if } d \leq 0 \\ e^{-d} & \text{if } d > 0 \end{cases} = \frac{1}{2} e^{-|d|}$$

LaPlace(0,1).
↳ double Exponential

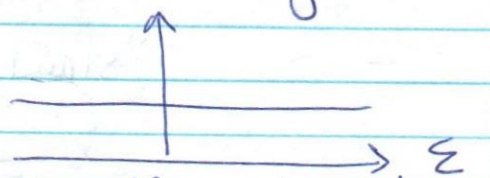
1774 "First law of :"

Imagine you measuring a value V, your measuring system is not perfect. So you measure $Y \neq V$ but close so $Y = V + \underbrace{\varepsilon}$

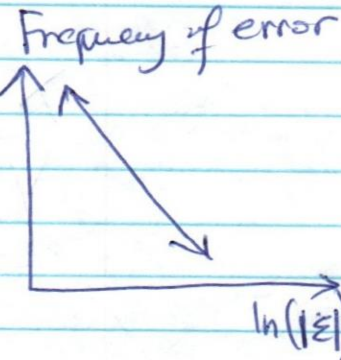
"error or noise"

It seems reasonable that $E(\varepsilon) = 0 \Rightarrow$
$E(Y) = V$. $\text{med}(\varepsilon) = 0 \Rightarrow \text{med}(Y) = V$

$f_\varepsilon(\varepsilon) = f_\varepsilon(-\varepsilon)$ over/under numbers of the same magnitude are equiprobable



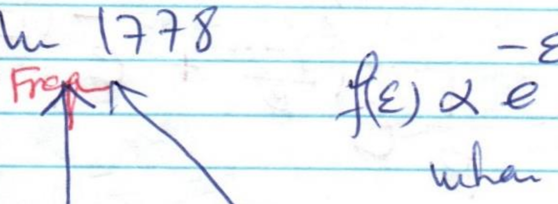
is this a good idea?



$f'(\varepsilon) < 0$ if $\varepsilon > 0$

$f''(\varepsilon) = f'(\varepsilon) \Rightarrow f(\varepsilon) = c\,e^{-mx}$

In 1778

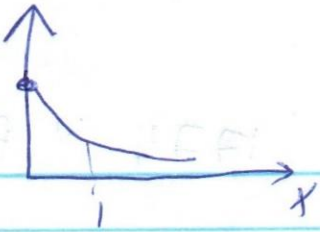


$f(\varepsilon) \propto e^{-\varepsilon^2} = \text{Normal}$

when gauss was 2 years old

"Second law of Errors"

let $X \sim \text{Exp}(1) = e^{-x}$



$y = -\ln(x)$

support$[Y] = \mathbb{R}$

$y = \ln\left(\frac{1}{x}\right) \Rightarrow g^{-1}(y) = e^{-y}$

$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = e^{-y}$

$f_y(y) = f_x(e^{-y})e^{-y} = e^{-e^{-y}}e^{-y} = e^{-(y+e^{-y})}$

$= \text{Gumbel}(0,1)$

"standard Gumbel"

$X \sim \text{Gumbel}(0,1)$

$Y = \mu + \beta X \sim \frac{1}{|\beta|} f_x\left(\frac{y-\mu}{\beta}\right) = \frac{1}{|\beta|} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$

$= \text{Gumbel}(\mu, \beta)$

param space $\beta > 0, \mu \in \mathbb{R}$

$\frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$