11/9/17 $X_1 \sim N(0,1)$ and $X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} = \int_{\text{supp}[X_2]} |x| f_{X_1}(xr) f_{X_2}(x)\,dx$$

$$= \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 r^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} |x|\, e^{-\frac{1}{2}x^2(r^2+1)}\,dx = \frac{1}{2\pi}\left(\int_{-\infty}^{0}(-x)e^{-\frac{1}{2}x^2(r^2+1)} + \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\,dx\right)$$

$$= \frac{1}{\pi}\int_0^{\infty} x\, e^{-\frac{1}{2}x^2(r^2+1)}\,dx = \frac{1}{\pi}\int_0^{-\infty} x e^{u} \frac{-1}{x(r^2+1)}\,du$$

$u = -\frac{1}{2}x^2(r^2+1)$

$$= \frac{1}{\pi(r^2+1)}\left[-e^{u}\right]_0^{-\infty}$$

$\frac{du}{dx} = -x(r^2+1)$

$$= \frac{1}{\pi(r^2+1)} = \text{Cauchy}(0,1)$$

$dx = -\frac{1}{x(r^2+1)}\,du$

$x = 0 \Rightarrow u = 0$

$x = \infty \Rightarrow u = -\infty$

midterm 2 // Final

$X_1, \dots, X_n \overset{iid}{\sim} f(\mu, \delta^2)$ unknown distr

unknown and we wish to estimate them (inference)

$\bar{X}$ is the Average r.v "estimation" $\bar{x} = \frac{x_1 + \dots + x_n}{n}$ is a realization. "estimate"

$E(\bar{X}) = \mu$ unbiased.

~~S^2 is the squar variance r.v.~~

$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$ is a realization $E(S^2) = \delta^2$

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \delta^2)$

$\bar{X} \sim N(\mu, \frac{\delta^2}{n})$ $S^2 = \frac{1}{n-1}\left((X_1 - \bar{X})^2 + \dots + (X_n - \bar{X})^2\right)$

$S^2 \sim ?$

let $\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

$$\sum_{i=1}^{n} z_i^2 = \vec{z}^T \vec{z} \sim \chi^2_n$$

$$\sum z_i^2 = \sum \left(\frac{x_i - \mu}{\delta}\right)^2 = \frac{\sum (x_i - \mu)^2}{\delta^2} \sim \chi^2_n$$

$$\sum (x_i - \mu)^2 = \sum \left((x_i - \bar{x}) + (\bar{x} - \mu)\right)^2$$
$$= \sum (x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - \mu) + (\bar{x} - \mu)^2$$

$$= \sum (x_i - \bar{x})^2 + 2\left(\sum x_i\bar{x} - \mu \sum x_i - \sum \bar{x}^2 + \sum \bar{x}\mu\right) + n(\bar{x}-\mu)^2$$

$$\cancel{n\bar{x}^2 - n\mu\bar{x} - n\bar{x}^2 + n\mu\bar{x}}$$

$$\Rightarrow \frac{\sum (x_i - \bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\frac{(n-1)s^2}{\sigma^2} \qquad \left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_1$$

Conjecture $\sim \chi^2_{(n-1)}$

Cochran's Theor (1934).

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

let $Q_1, \dots, Q_k$ be scalar r.v.s ...

via the quadratic forms

$Q_j = \vec{Z}^T B_j \vec{Z}$ where $B_1, \dots, B_k$ are positive semi-definite matrices

s.t $\vec{Z}^T\vec{Z} = Q_1 + \dots + Q_k$

$= \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z} + \dots + \vec{Z}^T B_k \vec{Z}$

$= \vec{Z}^T (B_1 + \dots B_k)\vec{Z}$

$\Rightarrow I_n = B_1 + \dots + B_k.$

a) $n = \sum \text{rank}(B_j)$

$\Updownarrow$

b) Q_j's are independent

$\Updownarrow$

c) $Q_j \sim \chi^2_{\text{rank}(B_j)}$

$$\sum z_i^2 = \sum\left((z_i-\bar z)+(\bar z)\right)^2$$

$$= \sum(z_i-\bar z)^2 + 2\underbrace{\sum(z_i-\bar z)}\bar z + \sum \bar z^2$$

$$2\left(\sum z_i \bar z - \sum \bar z^2\right)$$

$$2\left(n\bar z^2 - n\bar z^2\right)$$

$$\sum z_i^2 = \underbrace{\sum(z_i-\bar z)^2}_{Q_1} + \underbrace{n\bar z^2}_{Q_2}$$

$$Q_2 = n\bar z^2 = \vec z^T B_2 \vec z = \vec z^T\left(\frac{1}{n}J_n\right)\vec z$$

$$J_n = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{n\times n} \qquad = z^T \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} z = z^T \begin{bmatrix} \bar z \\ \vdots \\ \bar z \end{bmatrix}$$

$$= z_1\bar z + \cdots + z_n \bar z = \bar z\left(\sum z_i\right)$$

$$= \bar z n \bar z = n\bar z^2$$

Theorem; matrix A is symmetric, indep
$\Rightarrow tr(A) = \cancel{m}$ rank(A)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \qquad \left(\frac{1}{n}J_n\right)\left(\frac{1}{n}J_n\right) = \frac{1}{n^2}J_nJ_n = \frac{1}{n^2}nJ_n = \frac{1}{n}J_n$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2J_n \qquad \Rightarrow tr\left(\frac{1}{n}J_n\right) = 1 = \text{rank}\left(\frac{1}{n}J_n\right)$$

$$\sum (z_i - \bar{z})^2 = \vec{z}^T B_1 \vec{z}$$

$$= \sum z_i^2 - 2 z_i \bar{z} + \bar{z}^2$$

$$= \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2$$

$$= \sum z_i^2 - n\bar{z}^2$$

$$= \vec{z}^T I_n \vec{z} - \vec{z}^T \tfrac{1}{n} J_n \vec{z}$$

$$= \vec{z}^T \underbrace{\left(I_n - \tfrac{1}{n} J_n\right)}_{B_2} \vec{z} = \cancel{B_2}$$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ & 1-\frac{1}{n} & & \\ -\frac{1}{n} & & \ddots & 1-\frac{1}{n} \end{bmatrix}$$

symmetric

indepen

$$\left(I_n - \tfrac{1}{n}J_n\right)\left(I_n - \tfrac{1}{n}J_n\right) = I_n - 2\tfrac{1}{n}J_n + \tfrac{1}{n}J_n = I_n - \tfrac{1}{n}J_n$$

$$tr\left(I_n - \tfrac{1}{n}J_n\right) = n-1 = \text{rank}(B_2)$$

Proven

$n\bar{z}^2 \sim \chi^2_1$

indep of

$\sum (z_i - \bar{z})^2 \sim \chi^2_{(n-1)}$

Definition

Matrix A is positive semi-def if

$\forall \vec{v},\ \vec{v}^T A \vec{v} \geq 0.$

$$\frac{n(\bar{x}-\mu)^2}{\sigma^2} = \underbrace{\frac{1}{\sigma}(\vec{x}-\vec{\mu})^T}_{\vec{z}^T} \left(\tfrac{1}{n}J_n\right) \frac{1}{\sigma}(\vec{x}-\vec{\mu}) \sim \chi^2_1$$

ind of

$$\frac{\sum (x_i - \bar{x})^2}{\sigma^2} = \frac{1}{\sigma}(\vec{x}-\vec{\mu})^T \left(I_n - \tfrac{1}{n}J_n\right) \frac{1}{\sigma}(\vec{x}-\vec{\mu}) \sim \chi^2_{n-1}$$

$\Rightarrow \dfrac{n(\bar{X}-\mu)^2}{\sigma^2}$ ind of $\dfrac{(n-1)S^2}{\sigma^2}$

$\Rightarrow \bar{X}, S^2$ are independent.
Fisher 1925

Geary, 1936 proved this is unique to the normal.

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$$

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{"z test"}$$

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2}}$$

Student 1908

$$= \frac{\overbrace{\dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{N(0,1)}}{\sqrt{\dfrac{\underbrace{\frac{n-1}{\sigma^2}S^2}_{\chi^2_{n-1}}}{n-1}}} = \frac{Z}{\sqrt{\dfrac{\chi^2_{n-1}}{n-1}}} = T_{n-1}$$

both independ