11/7

- $Z_1, \dots, Z_n \sim N(0,1)$

  $\sum_{i=1}^{b} Z_i^2 \sim \chi^2_k$ $\quad\quad \sqrt{\sum_{i=1}^{k} Z_i^2} \sim \chi_k$ "chi"

  chi with k degree of freedom

- $Y = |Z|$

  $\Rightarrow Y^2 = Z^2 \sim \chi^2_1$

  $|Z| = \sqrt{Z^2} \sim \chi_1 =$

  $$\sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} = 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)$$

- $X \sim \chi^2_k, \quad Y = \frac{X}{x} \sim ? \quad \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

  $X \sim \text{Gamma}(\alpha, \beta) \quad Y = cX \sim ? \quad c \in (0, \infty)$

  $$f_Y(x) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{\frac{1}{c}\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)}}{\Gamma(\alpha)} = \frac{\beta^\alpha y^{\alpha-1} e^{-\left(\frac{\beta}{c}\right)y}}{\underbrace{c\, c^{\alpha-1}}_{\underbrace{c^\alpha}_{\left(\frac{\beta}{c}\right)^\alpha}} \Gamma(\alpha)}$$

  $$= \frac{\left(\frac{\beta}{c}\right)^\alpha y^{\alpha-1} e^{-\frac{\beta}{c}y}}{\Gamma(\alpha)}$$

- $X_1 \sim \chi^2_{k_1}$ ind of $X_2 \sim \chi^2_{k_2}$

  $$R = \frac{\frac{X_1}{k_1}}{\frac{X_2}{k_2}} = \frac{V_1}{V_2} \sim \int_{supp(V_2)} t\, f_{V_1}(rt) f_{V_2}(t)\, dt = \int_0^\infty t \frac{a^a (rt)^{a-1} e^{-art}}{\Gamma(a)} \cdot \frac{b^b t^{b-1} e^{-bt}}{\Gamma(b)}\, dt$$

  $Supp(R) = (0, \infty)$

  Let $a = \frac{k_1}{2}$ $\quad V_1 \sim \frac{a^a x^{a-1}}{\Gamma(a)} e^{-ax}$

  $b = \frac{k_2}{2}$ $\quad V_2 \sim \frac{b^b x^{b-1}}{\Gamma(b)} e^{-bx}$

  $$\Rightarrow \frac{a^a b^b x^{a-1}}{\Gamma(a)\Gamma(\beta)} \int_0^\infty t^{a+b-1} e^{-(ar+t)t}\, dt = \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b}} \underbrace{\int_0^\infty u^{a+b-1} e^{-u}\, du}_{\Gamma(a+b)}$$

  Let $u = (ar+b)t$

  $\Rightarrow t = \frac{1}{ar+b} u$

  $dt = \frac{1}{ar+b} du$

  $$= \frac{a^a b^b}{B(a,b)} \frac{r^{a-1}}{(ar+b)^{a+b}}$$

$$= \frac{a^a b^b}{\beta(a,b)\, b^{a+b}} r^{a-1} \left(\frac{a}{b} r + 1\right)^{-(a+b)}$$

$$= \frac{\left(\frac{a}{b}\right)^a}{\beta(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)} = \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{\beta\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\frac{k_1+k_2}{2}}$$

$$= F_{k_1+k_2}$$

"F-distribution"

~~Characteristic~~
Convolution

• $Z + N(0,1)$ ind of $V \sim Z^2_k$.

Let $\cancel{y} = \frac{Z}{\sqrt{\frac{V}{k}}} \sim$? PDF?

Notation
X

$\mathrm{supp}[\cancel{y}] = \mathbb{R}$ $\quad \cancel{y}^2 = \frac{Z^2}{\frac{V}{k}} = \frac{\frac{V}{1}}{\frac{V}{k}}$

$$\cancel{y}^2 \sim \Gamma(1,k) = \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{k}, \frac{k}{2}\right)} y^{-\frac{1}{k}} \left(1 + \frac{1}{k} y\right)^{\frac{k+1}{2}}$$

$$f(\cancel{x}) = \frac{1}{\cancel{2}} f(y^2) \cancel{2} y$$

$$T_k = \frac{1}{\sqrt{k-1}\,\left(\frac{1}{k}, \frac{k}{2}\right)} \left(k + \frac{x^2}{n}\right)^{\frac{k+1}{2}}$$

Student's T-distribution with k degree of ___.

- $k \to \infty$

$Y = \frac{Z}{\sqrt{\frac{V}{b}}}$

$\frac{V}{k} = \frac{\sum_{i=1}^{k} Z_i^2}{k}$ $\quad E(X^2 k) = k \quad Var(X_k^2) = 2k$

$E(\frac{V}{k}) = 1 \quad Var(\frac{V}{k}) = \frac{-2}{k}$

$= N(1, \frac{2}{\cancel{k}}) \xrightarrow[k \to \infty]{} N(1, 0)$: Deg(1)

$\underline{T_k \to Z}$

- $X_1 \sim N(0,1)$ ind. of $X_2 \sim N(0,1)$

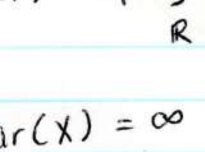
$\mathbb{R} = \frac{X_1}{X_2} = \frac{X_1}{\sqrt{\frac{X_2^2}{1}}} \sim T_1 = \frac{1}{\beta(\frac{1}{2}, \frac{1}{2})} (1 + y^2)^{-1} = \frac{1}{\pi(1+y^2)} = \text{Cauchy}(0,1)$

↑ standard

$\frac{1}{\beta(\frac{1}{2}, \frac{1}{2})} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})^2} = \frac{\Gamma(1)}{(\sqrt{\pi})^2} = \frac{1}{\pi}$

$X \sim \text{Cauchy}(0,1)$, $X = \cancel{Z} + \phi X$ $\quad f_Y(x) = \frac{1}{\phi} f_X(\frac{y}{\phi})$

$c \in (0, \infty)$

- 

$U(a_1, a_2) = \frac{1}{\pi}$

$X = \tan(\theta)$ so $\theta = \tan^{-1}(x)$

$g^{-1}(\theta) \left| \frac{d}{dy} (g^{-1}(\theta)) \right| = \frac{1}{1+x^2}$

- $E(X) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{x}{1+x^2} dx = \frac{1}{\pi} \left[\frac{1}{2} \ln(x^2+1) \right]_{-\infty}^{\infty} = \infty$ ✓

- $Var(X) = \infty$

- $\mathbb{R} = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} |x_2| f_{X_1}(x_2 + r) f_{X_1}(x_2) \, dx_2$