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std. MVN.

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) \Leftrightarrow Z_1, Z_2, \dots, Z_n \stackrel{iid}{\sim} N(0,1)$$

$$E(\vec{Z}) = \vec{0}_n \quad , \quad Var(\vec{Z}) = I_n$$

$$\vec{X} = \vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, I_n)$$

$$\vec{X} = A\vec{Z} \; ; \; A \in \mathbb{R}^{m \times n} \qquad E(\vec{X}) = A E(\vec{Z}) = A\vec{0}_n = \vec{0}_m.$$

$$\Sigma = Var(X) = A \overbrace{Var(\vec{Z})}^{I_n} A^T = \underbrace{AA^T}_{\text{Cholesky Decomposition}} = \Sigma$$

We Want $f_{\vec{X}}(\vec{X})$.

let $\vec{X} = g(\vec{Z}) = A\vec{Z}$. We need $m = n$ ie A is square and A is full rank.

$$\vec{Z} = h(\vec{X}) = A^{-1}\vec{X}$$

h is inverse function $= \begin{bmatrix} h_1(\vec{X}) \\ \vdots \\ h_n(\vec{X}) \end{bmatrix}$

$$f_{\vec{X}}(\vec{X}) = f_{\vec{Z}}(h(\vec{X})) \, |J_h(\vec{X})|$$

$$\text{let } B = A^{-1} = \begin{bmatrix} \leftarrow \vec{b}_1 \rightarrow \\ \leftarrow \vec{b}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{b}_n \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$$\underset{\substack{\shortparallel \\ h(\vec{X})}}{\vec{Z} = B\vec{X}} = \begin{bmatrix} h_1(\vec{X}) = \vec{b}_{1\cdot}\vec{X} = b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n \\ h_2(\vec{X}) = \vec{b}_{2\cdot}\vec{X} = b_{21}x_1 + \dots + b_{2n}x_n \\ \vdots \\ h_n(\vec{X}) = b_{n\cdot}X = b_{n1}x_1 + \dots + b_{nn}x_n \end{bmatrix}$$

$$J_h(\vec{x}) = \det\left(\begin{bmatrix} \frac{\partial}{\partial x_1}[h_1(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_1(\vec{x})] \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}[h_n(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_n(\vec{x})] \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right) = \det(B) = \det(A^{-1})$$

$\boxed{\begin{array}{l} \text{A Side} \\ \frac{\partial}{\partial \vec{x}}[C\vec{x}] = C \end{array}}$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_h(\vec{x})| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}\vec{x})^T(A^{-1}\vec{x})} \cdot \frac{1}{|\det(A)|}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$AA^{-1} = I$$

$$(AA^{-1})^T = I^T = I$$

$$(A^{-1})^T A^T = I$$

$$A^T(A^T)^{-1} = I = (A^T)^{-1}A^T = I$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T(A^{-1})^T A^{-1}\vec{x}} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T(A^T)^{-1}A^{-1}\vec{x}} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x}} \cdot \frac{1}{|\det(A)|}$$

$\det(\Sigma) = \det(A)\det(A^T) = \det(A)^2 \Rightarrow \det(A) = \sqrt{\det(\Sigma)}$

$\Sigma = AA^T$

$\Sigma^{-1} = (AA^T)^{-1}$

$= (A^T)^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1}(AB) = I$

$B^{-1}A^{-1}AB = I$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}\vec{x}^T\Sigma^{-1}\vec{x}}$$

$$= N_n(\vec{0}_n, \underset{AA^T}{\underset{\parallel}{\Sigma}})$$

let $\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma)$ ← general MVN.

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

Recall $\phi_X(t) = E(e^{itX})$ generalizate to

$\phi_{\vec{X}}(\vec{t}) = E(e^{i\vec{t}^T\vec{X}})$

let X_1, X_2 ind.

$$\phi_{\vec{X}_1+\vec{X}_2}(\vec{t}) = E\left[e^{i\vec{t}^T(\vec{X}_1+\vec{X}_2)}\right] = E\left(e^{i\vec{t}^T\vec{X}_1}e^{i\vec{t}^T\vec{X}_2}\right)$$

$$= E\left(e^{i\vec{t}^T\vec{X}_1}\right)E\left(e^{i\vec{t}^T\vec{X}_2}\right) = \phi_{\vec{X}_1}(\vec{t})\,\phi_{\vec{X}_2}(\vec{t})$$

let $\vec{Y} = A\vec{X} + \vec{c}$

$A \in \mathbb{R}^{m\times n}$, $\vec{c} \in \mathbb{R}^m$

$\vec{X} \in \mathbb{R}^n$, $\vec{Y} \in \mathbb{R}^m$

$$\phi_{\vec{Y}}(\vec{t}) = E\left(e^{i\vec{t}^T Y}\right)$$

$$= E\left(e^{i\vec{t}^T(A\vec{X}+\vec{c})}\right)$$

let $\vec{t}'^T = \vec{t}^T A$

$$= E\left[e^{i\vec{t}^T A\vec{X}} e^{i\vec{t}^T\vec{c}}\right] = e^{i\vec{t}^T\vec{c}} E\left(e^{i\vec{t}^T A\vec{X}}\right)$$

$$= e^{i\vec{t}^T\vec{c}} E\left(e^{i\vec{t}'^T\vec{X}}\right) = e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(\vec{t}')$$

$$= e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(A^T\vec{t}).$$

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{Z}}(\vec{t}) = E\left(e^{i\vec{t}^T\vec{Z}}\right)$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i\vec{t}^T\vec{z}} f_{\vec{Z}}(\vec{z})\, d\vec{z}$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i(t_1 z_1 + \cdots + t_n z_n)} \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(z_1^2 + \cdots + z_n^2)}\, dz_1 \cdots dz_n$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{n} e^{i t_j z_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_j^2}\, dz_i$$

$$= \prod_{j=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{i t_j z_j - \frac{1}{2}z_j^2}\, dz_j = \prod_{j=1}^{n} e^{-\frac{1}{2}t_j^2} \overbrace{\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_j - i t_j)^2}\, dz_j}_{N(i t_j, 1)}}^{1}$$

$$-\frac{1}{2}z^2 + itz = -\frac{1}{2}(z^2 - 2itz)$$

$$= \frac{1}{2}((z - it)^2 - i^2 t^2)$$

$$= -\frac{1}{2}((z - it)^2 + t^2)$$

$$= e^{-\frac{1}{2}\sum_{j=1}^{n} t_j^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Std Charac funct MVN.

$A \in \mathbb{R}^{n \times n}$

let $\vec{X} = A\vec{Z} + \vec{\mu} \Rightarrow \phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{Z}}(A^T\vec{t})$

$= e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\vec{t}^T A A^T \vec{t}} = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T \Sigma \vec{t}}$

let $Y = B\vec{X} \Rightarrow \phi_{\vec{Y}}(\vec{t}) = \phi_{\vec{X}}(B^T\vec{t})$

$B \in \mathbb{R}^{n \times n}$

$= e^{i\vec{t}^T \underbrace{B\vec{\mu}}_{\vec{\mu}'} - \frac{1}{2}\vec{t}^T \underbrace{B \Sigma B^T}_{\Sigma'} \vec{t}}$

$\Rightarrow \vec{Y} \sim N_n(B\mu, B\Sigma B^T)$.

$\vec{X} = A\vec{Z} + \vec{\mu}$

$\Rightarrow \vec{Z} = A^{-1}(\vec{X} - \vec{\mu})$

$\vec{Z}^T\vec{Z} \sim \chi^2_n$

$(A^{-1}(\vec{X} - \vec{\mu}))^T (A^{-1}(\vec{X} - \vec{\mu}))$

$(\vec{X} - \vec{\mu})^T (A^{-1})^T A^{-1} (\vec{X} - \vec{\mu})$

$(\vec{X} - \vec{\mu})^T (A^T)^{-1} A^{-1} (\vec{X} - \vec{\mu})$

$(\vec{X} - \vec{\mu})^T (AA^T)^{-1} (\vec{X} - \vec{\mu})$

$\quad\quad\quad\quad \|$

$(\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu}) \sim \chi^2_n$

"Mahalanobis" distance

P. C. Mahalanobis (1936)

"sphering"

$$X \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

$$X \sim N(\mu, \delta^2) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\delta}\right)^2}$$

let $X_1, \dots X_n \overset{iid}{\sim} N(\mu, \delta^2)$

$$\Rightarrow \vec{X} \sim N_n(\underbrace{\mu 1_n}_{\vec{\mu}}, \delta^2 I_n)$$

$$\Sigma = \delta^2 I = \underbrace{\delta I}_{A} \underbrace{(\delta I)^T}_{A^T}$$

$$\vec{X} = \delta I \vec{z} + \vec{\mu} = \delta \vec{z} + \vec{\mu}$$

$$(\vec{X}-\vec{\mu})^T \frac{1}{\delta^2} I (\vec{X}-\vec{\mu}) \sim \chi^2$$

$$\Sigma \left(\frac{x_i - \mu}{\delta}\right)^2 \sim \chi^2_n$$