10/10/17

$X \sim Exp(1)$; $Y = -\ln(X)$ PDF

$\sim Gumbel(0,1) = e^{-(y+e^{-y})} = e^{-y}e^{-e^{-y}}$

$Y \sim Gumbel(0,1)$, $X = e^{-Y} \sim Exp(1)$

$F_Y(y) = P(Y \leq y) = P(-Y \geq -y) = P(e^{-Y} \geq e^{-y})$

$= P(X \geq e^{-y}) = 1 - F_X(e^{-y}) = e^{-e^{-y}}$

CDF

$X \sim Gumbel(0,1)$, $Y = \mu + \beta X \sim Gumbel(\mu, \beta)$

$= \frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$

$$\boxed{F_Y(y) = P(Y \leq y) = P\left(\frac{Y-\mu}{\beta} \leq \frac{y-\mu}{\beta}\right) = P\left(X \leq \frac{y-\mu}{\beta}\right) = F_X\left(\frac{y-\mu}{\beta}\right) = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}} \quad \text{CDF}}$$

Valid for any linear transformation.

$X \sim Gumbel(0,1)$, $Y = e^{-X} \sim Exp(1)$ $supp[X] = \mathbb{R}$

$X \sim Gumbel(\mu, \beta)$, $Y = e^{-X} \sim ?$ $supp[Y] = (0, \infty)$

$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$ $\left|\frac{d}{dy}(-\ln y)\right| = y^{-1}$

$= f_X(-\ln y)\, y^{-1} = \frac{1}{\beta} e^{-\left(\frac{-\ln y - \mu}{\beta}\right)} e^{-e^{-\left(\frac{-\ln y - \mu}{\beta}\right)}} \cdot y^{-1}$

let $k = \frac{1}{\beta}$

Note: $-\left(\frac{-\ln(y)-\mu}{\beta}\right) = \frac{\ln(y)+\mu}{\beta} = k\left(\ln(y) + \cancel{\mu}\right)$

$= k\left(\ln y + \ln(\lambda)\right) = k\ln(\lambda y) = \ln(\lambda y)^k$.

let $\mu = \ln(\lambda)$

$$e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)} = e^{\ln(\lambda y)^k} = (\lambda y)^k$$

$\beta \in (0,\infty) \Rightarrow k \in (0,\infty)$, $\mu \in \mathbb{R} \Rightarrow \lambda \in (0,\infty)$

$$f_Y(y) = k(\lambda y)^k e^{-(\lambda y)^k} y^{-1} = (k\lambda)(\lambda y)^{k-1} e^{-(\lambda y)^k}$$

$$= \text{Weibull}(k,\lambda)$$

$$F_Y(y) = P(Y \le y) = P(\ln(Y) \le \ln(y))$$

CDF

$$= P(-\ln(Y) \ge -\ln(y)) = P(X \ge -\ln(y))$$

$$= 1 - F_X(-\ln(y))$$

$$= 1 - e^{-e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}} = 1 - e^{-(\lambda y)^k}$$

with $e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)} = (\lambda y)^k$

"memoryless"

If $\beta = 1 \Rightarrow k = 1$ $\quad$ $\text{Weibull}(1,\lambda) = \lambda e^{-\lambda y} = \text{Exp}(\lambda)$

$\text{Weibull}(1,1) = \text{Exp}(1)$

Wife of $P(X \geq 7 \text{ yr}) > P(X \geq 7+3 \mid X \geq 3)$ to last 7 more years

Case #1 $k > 1$ eg $k = 2$

$$X \sim \text{Weibull}(2, \lambda) \Rightarrow F_X(x) = 1 - e^{-(\lambda x)^2}$$

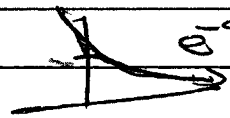
$$\Rightarrow 1 - F_X(x) = e^{-(\lambda x)^k}$$

WTS: $P(X \geq b) > P(X \geq a+b \mid X \geq a)$

$$e^{-(\lambda b)^2} = 1 - F_X(b) > \frac{P(X \geq a+b)}{P(X \geq a)}$$

$$= \frac{1 - F_X(a+b)}{1 - F_X(a)} = \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}}$$

$$= \frac{\cancel{e^{-(\lambda a)^2}} e^{-2\lambda^2 ab} e^{-(\lambda b)^2}}{\cancel{e^{-(\lambda a)^2}}} = e^{-2\lambda^2 ab} e^{-(\lambda b)^2}$$



manufacturer..

Case #2 $k < 1$ eg. $k = \frac{1}{2}$

WTS $P(X \geq b) < P(X \geq a+b \mid X \geq a)$

$$e^{-(\lambda b)^{\frac{1}{2}}} < \frac{e^{-(\lambda(a+b))^{1/2}}}{e^{-(\lambda a)^{1/2}}} = e^{-(\lambda(a+b))^{1/2} + (\lambda a)^{1/2}}$$

$$= e^{-\lambda^{1/2}\left((a+b)^{1/2} - a^{1/2}\right)}$$

log..

$\Rightarrow$

$$\Rightarrow \frac{-\lambda^{1/2} b^{1/2}}{-\lambda^{1/2}} < \frac{-\lambda^{\frac{1}{2}}}{-\lambda^{1/2}}(\cdots) \Rightarrow b^{\frac{1}{2}} > (a+b)^{\frac{1}{2}} - a^{1/2}$$

$$\Rightarrow a^{1/2} + b^{1/2} > (a+b)^{\frac{1}{2}} \Rightarrow (a^{1/2} + b^{1/2})^2 > a+b$$

$$\Rightarrow \cancel{a}+\cancel{b}+2a^{1/2}b^{1/2} > \cancel{a}+\cancel{b}$$

$$2a^{1/2}b^{1/2} > 0.$$

$X \sim \text{Weibull}(k,\lambda)$, $Y = \frac{1}{X} \sim ?$ $\quad \text{Supp}(Y) = (0,\infty)$

for eoa

$$g^{-1}(y) = \frac{1}{y} \qquad \left|\frac{d}{dy}\left(g^{-1}(y)\right)\right| = \frac{1}{y^2}$$

$$\boxed{f_Y(y) = f_X\left(\frac{1}{y}\right)\cdot\frac{1}{y^2} = (k\lambda)\left(\frac{\lambda}{y}\right)^{k-1} e^{-\left(\frac{\lambda}{y}\right)^k}\frac{1}{y^2}}$$

$$= k\lambda^k \underbrace{\frac{1}{y^{k-1}}\cdot\frac{1}{y^2}}_{\frac{y^{k-1+2}}{y^{k+1}}} e^{-\left(\frac{\lambda}{y}\right)^k} = \frac{k}{\lambda}\lambda^{k+1}\frac{1}{y^{k+1}} e^{-\left(\frac{y}{\lambda}\right)^{k}}$$

$$= \frac{k}{\lambda}\left(\frac{Y}{\lambda}\right)^{-(k+1)} e^{-\left(\frac{y}{\lambda}\right)^{-k}}$$

$$= \text{Frechet}(k, \lambda, 0)$$

centered.

log, inverse time $\quad$ time $\quad$ inverse time

{ Gumbell, Weibull, Frechet }

belong to a special family call the generalized extrem value distribution.

waiting for thing to happen → $\exp(\lambda)$. $\text{Gamma}(k,\lambda)$

$$X \sim \text{Erlang}(k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

$k \in \mathbb{N},\ \lambda \in (0,\infty)$

waiting discrete time

$$X \sim \text{Negbino}(k,p) = \binom{x+k-1}{k-1} p^k (1-p)^x$$

$k \in \mathbb{N},\ p \in (0,1)$

$$= \underbrace{\frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)}}_{\substack{\text{Extended} \\ \text{negative} \\ \text{binomial.}}} \cdot p^k (1-p)^x$$

what if $k \in (0,\infty)$

not mean

Recall

$X \sim \text{Erlang}(k_1,\lambda)$ ind of
$Y \sim \text{Erlang}(k_2,\lambda)$
$\Rightarrow X+Y \sim \text{Erlang}(k_1+k_2,\lambda)$

WTS if $X \sim \text{Gamma}(k_1,\lambda)$ ind of
$Y \sim \text{Gamma}(k_2,\lambda)$
$\Rightarrow X+Y \sim \text{Gamma}(k_1+k_2,\lambda)$ 8/48

$$X+Y \sim \int_0^t f_X(x) f_Y(t-x)\,dx$$

$$= \int_0^t \frac{\lambda^{k_1} x^{k_1-1} \cancel{e^{-\lambda x}}}{\Gamma(k_1)} \frac{\lambda^{k_2}(t-x)^{k_2-1} \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}}}{\Gamma(k_2)}\,dx$$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1} (t-x)^{k_2-1} \, dx.$$

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

$x_l = 0 \Rightarrow u_l = 0$

$x_u = t \Rightarrow u_u = 1$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \underbrace{\int_0^1 (ut)^{k_1-1} \underbrace{(t-ut)}_{t(1-u)}^{k_2-1} t\,du}_{\int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1}(1-u)^{k_2-1} t\,du}$$