# Math 621 (Probability)

Tue – Thu
8:00pm – 9:15pm K: 258
Prof: Kapelner

## Lecture 1

A discrete random variable (r.v) X has probability mass function (PMF)

$p(x) := P(X = x)$ and cummulative dis function (CDF)

$F(x) := F(X \leq x)$. The rv X has "support"

$Sup[X] := \{ X = p(x) \geq 0,\ x \in \mathbb{R} \}$

since X is discrete $|Sup(x)| \leq |\mathbb{N}|$

support and PMF are vectoral ie

$$\sum_{x \in support(x)} p(x) = 1$$

The most fundamental discrete random variable is the Bernoulli

$$X \sim Bern(p) := \begin{cases} 1 & \text{w.p } p \\ 0 & \text{w.p } 1-p \end{cases}$$

what is p?

P is a parameter, parameters have parameter spaces eg $p \in (0, 1)$

$p \neq 0$, and $p \neq 1$

$X \sim \text{Deg}(c) = \underbrace{\mathbb{1}}_{\text{indicator function}}{}_{x=c}$

"Degenerate"

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$$

$X \sim \text{Ber}(p) := p^x (1-p)^{1-x}$

<u>Independence</u>

$X_1 X_2$ independent r.v.s $X_1, X_2 \overset{ind}{\sim}$

"$X_1$ and $X_2$ are independent"

By definition $\rightarrow$ joint mass function

$p(x_1, x_2) = p_{X_1}(x_1)\, p_2(x_2) \quad \forall x_1, x_2$ in their support

$X_1 \overset{d}{=} X_2$ the r.v.'s $X_1, X_2$ are equal in distribution if $p_{X_1}(x) = p_{X_2}(x)$

$X_1, X_2 \overset{iid}{\sim}$ the r.v.'s $X_1, X_2$ are independent and identically distributed

Def: $X_1 X_2 \overset{iid}{\sim}$ and $X_1 \overset{d}{=} X_2$

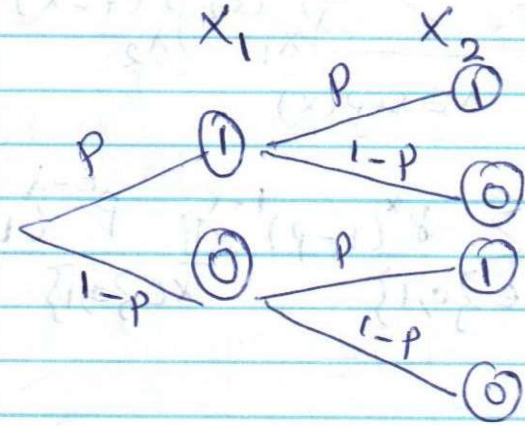
les $T_2 = X_1 + X_2$ where $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$\text{support}[T_2] = \{0, 1, 2\}$

$= \text{sup}[X_1] + \text{sup}[X_2]$

$A + B = \{a + b : a \in A, b \in B\}$

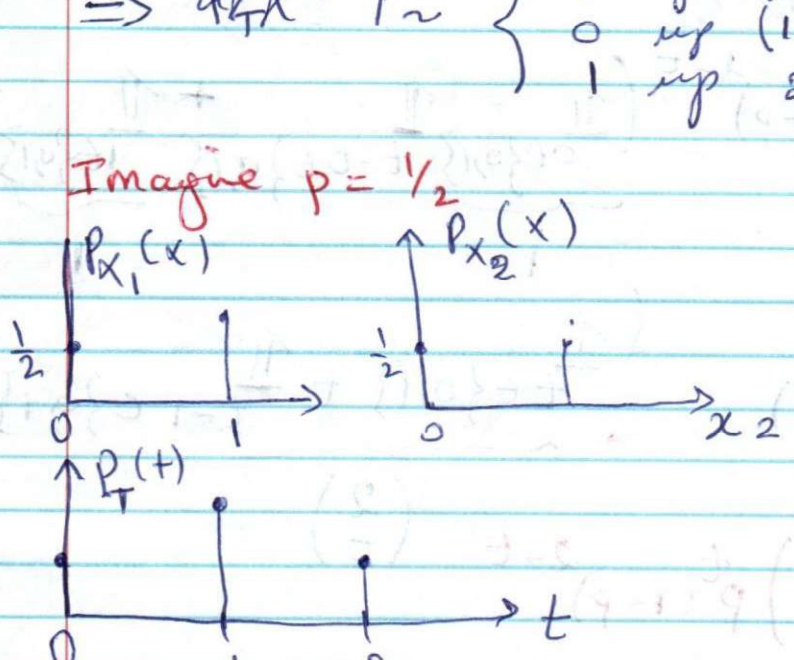
Probability tree




| $P_{X_1,X_2}(x_1,x_2)$ | $T_2$ |
|---|---|
| $p^2$ | 2 |
| $p(1-p)$ | 1 |
| $(1-p)p$ | 1 |
| $(1-p)^2$ | 0 |

$\Rightarrow$ ~~PMF~~ $\cdot\; T \sim \begin{cases} 2 \text{ wp } p^2 \\ 0 \text{ wp } (1-p)^2 \\ 1 \text{ wp } 2p(1-p) \end{cases}$

Imagine $p = 1/2$



$P_{T_2}(t) =$

$$P(T_2 = t) = \sum_{x \in \sup[X]} P_{X_1}(x) P_{X_2}(t-x)$$

$T_2 = X_1 + X_2$

<u>Example</u>: For 2 Bern

$$p(t) = P(T_2 = t) = \sum_{x \in \sup(x)} P_{X_1}(x) P_{X_2}(t-x)$$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} \, p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{(t-x) \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \Big( \underbrace{\mathbb{1}_{0 \in \{0,1\}}}_{1} \mathbb{1}_{t-0 \in \{0,1\}} + \underbrace{\mathbb{1}_{1 \in \{0,1\}}}_{1} \mathbb{1}_{t-1 \in \{0,1\}} \Big)$$

$P(0) = (1-p)^2$

$P(2) = p^2$

$P(1) = 2p(1-p)$

$$\underbrace{\Big( \mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t-1 \in \{0,1\}} \Big)}_{\binom{2}{t}}$$

$$\Rightarrow \quad P(t) = \binom{2}{t} p^t (1-p)^{2-t}$$

$X \sim Bern(p) = Bin(1,p) = \binom{1}{x} p^x (1-p)^{1-x}$

note: $\binom{n}{k}$ is only valid if $k \le n$ otherwise 0

$$\sum_{x \in sup[X_1]} p_{X_1}(x)\, p_{X_2}(x) = \sum_{x \in \{0,1\}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$p^t (1-p)^{2-t} \underbrace{\sum_{x \in [0,1]} \binom{1}{x}\binom{1}{t-x}} = \cancel{\binom{1}{0}} \cdot \binom{1}{t} + \cancel{\binom{1}{1}} \binom{1}{t-1}$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

$$p(t) = P(T_2 = t) = \underbrace{P_{X_1}(x) * P_{X_2}(x) := \sum_{x \in sup(X_1)} p_{X_1}(t)\, p_{X_2}(t-x)}$$

Convalution of two independent PMF's

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 = p_{X_3}(x) * p_{T_2}(x)$$

$$\sum_{x \in sup[X_3]} p_{X_3}(x)\, p_{T_2}(t-x) =$$

$$\sum_{x \in \{0,1\}} \binom{1}{x} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x} =$$

3b

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{1}{x} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left( \binom{2}{t} + \binom{2}{t-1} \right)$$

$$= \binom{3}{t} p^t (1-p)^{3-t}$$