Lecture 14 Math 621 10/26/17

$X \sim \text{Gamma}(\alpha,\beta)$ $\quad Y|X=x \sim \text{Poisson}(x) \Rightarrow f_Y(y) =$

one param

$= \dfrac{\Gamma(y+k)}{\Gamma(k)\,\Gamma(y-1)}\, p^k (1-p)^y \quad = \text{ExtNegBin}(p,k)$ ← Support same

two params

↑

"more freedom"

If $k \in \mathbb{N}$

$= \binom{y+k+1}{k} p^k (1-p)^y = \text{NegBin}(p,k)$

AKA "overdispersion"

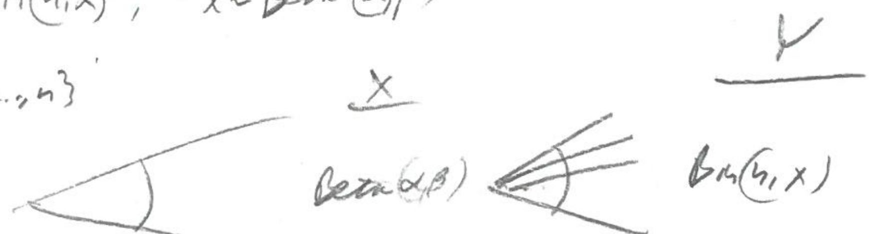
Poisson – count data

NegBin – also count data but now overdispersed

Another Example of this...

$Y|X \sim \text{Bin}(n,x)$, $\quad X \sim \text{Beta}(\alpha,\beta)$

$\text{Supp}(Y) = \{0,\ldots,n\}$ still!



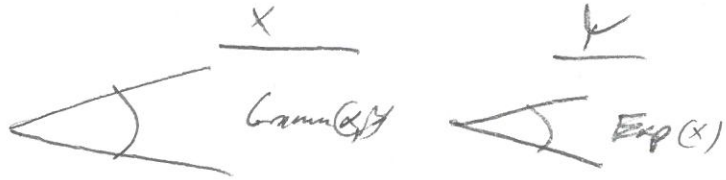
compound distr.

$$p_Y(y) = \int_{\text{Supp}(X)} p_{Y|X}(y,x)\, f_X(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\,dx = \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha,\, n-y+\beta) = \text{BetaBinomial}(n,\alpha,\beta)$$

"Overdispersed binomial"

$Y|X \sim Exp(x)$, $\quad X \sim Gamma(\alpha,\beta)$ $\qquad Supp(Y) = (0,\infty)$ still



$$f_Y(y) = \int_{Supp(X)} f_{Y|X}(y,x)\, f_X(x)\,dx = \int_0^\infty x e^{-xy} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x(\beta+y)}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u^\alpha}{(\beta+y)^\alpha} e^{-u} \frac{1}{\beta+y}\,du$$

let $u = x(\beta+y)$ $\quad \frac{du}{dx} = \beta+y \Rightarrow dx = \frac{1}{\beta+y}\,du$

$\Rightarrow x = \frac{1}{\beta+y} u$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)(\beta+y)^{\alpha+1}} \int_0^\infty u^{\alpha+1-1} e^{-u}\,du = \frac{\beta^\alpha \overbrace{\Gamma(\alpha+1)}^{\alpha!}}{\underbrace{\Gamma(\alpha)}_{(\alpha-1)!}(\beta+y)^{\alpha+1}} = \frac{\alpha}{\beta}\left(\frac{\beta}{\beta+y}\right)^{\alpha+1}$$

$$= \frac{\alpha}{\beta}\left(1+\frac{y}{\beta}\right)^{-(\alpha+1)} = Lomax(\beta,\alpha)$$

Another type of survival distr. "overdispersed exponential"

$\Rightarrow$ New r.v.'s can be created via mixes and compounds

$\boxed{\text{NEW TOPIC}}$

Recall Complex #'s

field of complex #'s

$a,b \in \mathbb{R}$ $\quad z := a+bi \in \mathbb{C}$ $\quad$ where $i := \sqrt{-1} \Rightarrow i^2=-1,\ i^3=-i,\ i^4=1$

$Re[z]=a, \quad Im[z]=b$

Recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$$\Rightarrow e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \frac{(itx)^4}{4!} + \frac{(itx)^5}{5!} + \dots$$

$$= 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \dots$$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

pattern: i is in the odd terms and not in even terms

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

2 terms ⊕, 2 terms ⊖, etc...

$i\sin(tx) = itx - i\frac{t^3x^3}{3!} + i\frac{t^5x^5}{5!}$

$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} \dots$

$\Rightarrow e^{itx} = \cos(tx) + i\sin(tx)$

$\Rightarrow e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Note: if $tx = \pi \Rightarrow e^{i\pi} = -1 + 0 \Rightarrow e^{i\pi} + 1 = 0$

Euler's identity

If you imagine

$z = |z|e^{i\theta}$ where $|z| = \sqrt{a^2+b^2}$ the "complex norm"

and θ is the complex argument

$Arg[z] := \theta = \arctan\left(\frac{b}{a}\right) \in [-\pi, \pi]$

~~Recall Complex #'s~~

~~$a, b \in \mathbb{R}$~~

~~$z = a + bi \in \mathbb{C}$ where $i := \sqrt{-1} \Rightarrow i^2 = -1$, $i^3 = -\sqrt{-1} = -i$, $i^4 = 1$~~

~~$Re(z) := a$, $Im(z) := b$~~

Define $L^1 := \left\{ f : \int_{\mathbb{R}} |f(x)|\, dx < \infty \right\}$ "L^1 integrable" or absolutely integrable

are all PDF's $\in L^1$? YES!

are all $f \in L^1$ PDF's? NO... But... they can be scaled to be

If $f \in L^1$ then $\exists \hat{f}$ known as the "Fourier transform of f" defined by

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\, dx$$

If $\hat{f} \in L^1$ (not guaranteed!), then the Fourier transform can be inverted back to f via:

$$f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\, dt$$

$f(x)$ is called the "time domain", $\hat{f}(t)$ is called the frequency domain

Why? $f(x)$ can be decomposed into a sum of sines and cosines called a "Fourier Series"!

$\mathrm{Re}\left[\hat{f}(t)\right]$ gives the amplitude of a frequency t and

$\mathrm{Arg}\left[\hat{f}(t)\right]$ '' '' phase shift of the '' ''.

let $\phi(t) = \hat{f}\left(-\frac{t}{2\pi}\right) = \int_{\mathbb{R}} e^{itx} f(x)\,dx$

Inversion ... if $\hat{\phi}(t) \in L^1$ then ...

$f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\,dt$ let $u = -2\pi t \Rightarrow \frac{du}{dt} = -2\pi \Rightarrow dt = -\frac{1}{2\pi}du$

$\Rightarrow t = -\frac{u}{2\pi}$

$$= \int_{u=\infty}^{u=-\infty} e^{2\pi i \left(-\frac{1}{2\pi}u\right)x} \underbrace{\hat{f}\left(-\frac{u}{2\pi}\right)}_{\phi(u)} du = -\int_{\mathbb{R}} e^{-iux} \phi(u)\left(\frac{1}{2\pi}\right)du = \frac{1}{2\pi}\int e^{-iux}\phi(u)\,du$$

$\phi_X(t) = E\left[e^{itX}\right] = \int_{\mathrm{supp}(X)} e^{itx} f(x)\,dx$ if f is a PDF of r.v. X

Also

$\phi_X(t) = E\left(e^{itX}\right) = \underbrace{\sum_{x \in \mathrm{supp}(X)}}_{\text{technically a Lebesgue integral}} e^{itx} p(x)$ if p is a PMF of r.v. X

$\phi_X(t)$ is called the characteristic function of r.v. X (ch. f.)

Lots of cool properties of ch.f's ① $\phi(0)=1$ since $E[e^{i(0)X}] = E(1) = 1$

lets say X_1, X_2 ind. $Y = X_1 + X_2$ ~ convolution

② $\phi_{X_1+X_2}(t) = E\left[e^{it(X_1+X_2)}\right] = E\left[e^{itX_1} e^{itX_2}\right] = E\left[e^{itX_1}\right] E\left[e^{itX_2}\right]$ by indep.

$$= \phi_{X_1}(t)\,\phi_{X_2}(t)$$

let $Y = aX + b$

③ $\phi_Y(t) = E\left[e^{itY}\right] = E\left[e^{it(aX+b)}\right] = E\left[e^{i\overset{t'}{(ta)}X} e^{itb}\right] = e^{itb}\,\phi_X(at)$

④ $\phi_X(t)$ is bounded by 1 and $\Rightarrow$ it always exists

$$|\phi_X(t)| = \left|E\left[e^{itX}\right]\right| = \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| \le \int_{\mathbb{R}} |e^{itx} f(x)|\,dx \le \int_{\mathbb{R}} |e^{itx}|\,|f(x)|\,dx$$

$$= \int_{\mathbb{R}} |\cos(tx) + i\sin(tx)|\,|f(x)|\,dx$$

$$\Downarrow$$

$$= \int_{\mathbb{R}} \underbrace{(\cos^2(tx) + \sin^2 tx)}_{1}\,|f(x)|\,dx$$

since if $f(x)$ is PDF its always $+$

$$= \int |f(x)|\,dx = \int f(x)\,dx = 1 \checkmark$$

Define $M_X(t) = \phi\left(\frac{t}{i}\right) = E\left[e^{tX}\right]$ the moment gen. function

is not guaranteed to exist! Just one example why ch.f.'s are more powerful...

~~⑤ $E(X^k) = \phi_X^{(k)}(0)$ (i.e. it can generate moments)~~