Oct. 31

Lec. 15

$\phi = E[e^{itx}] = \sum_{x \in supp(X)} e^{itx} p(x)$ if X discrete

$= \int_{supp(X)} e^{itx} f(x)\,dx$ if X contin.

Consider $\phi_X'(t) = \frac{d}{dt}\left(E[e^{itx}]\right)$

$= \frac{d}{dt}\left[\int_{\mathbb{R}} e^{itx} f(x)\,dx\right] \stackrel{?}{=} \int_{\mathbb{R}} f(x) \frac{d}{dt}\left[e^{itx}\right] dx$

Does $\frac{d}{dt}\left[\int g(x,t)\,dx\right] \stackrel{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}\left[g(x,t)\right] dx$

Continuous

(a) $\exists\, t \in A$ such that $\int_{\mathbb{R}} g(t,x)\,dx$ converge

$A = [a,b) \in \mathbb{R}$

(b) $g(x,t)$ ~~[illegible]~~ cont, $\forall\, t \in A$.

(c) $g(x,t)$ Cont. $\forall x \in \mathbb{R}$

d) $\forall\ t \in A$ $\int_{\mathbb{R}} \frac{\partial}{\partial t} g(x,t)\, dt$ Conv. uniformly

$\phi'(t) = \int_{\mathbb{R}} f(x)\, ix e^{itx}\, dx$

Consider $\phi_X'(0) = \int_{\mathbb{R}} f(x)\, ix\, dx = i \int_{\mathbb{R}} x f(x)\, dx = iE(X)$

$\phi''(t) = \int_{\mathbb{R}} f(x)\, i^2 x^2 e^{itx}\, dx$

$\phi''(0) = i^2 \int_{\mathbb{R}} x^2 f(x)\, dx = i^2 E(x^2)$

$\phi'''(0) = i^3 E(x^3)$

$\Rightarrow$ $E[X^n] = \dfrac{\phi^{(n)}(0)}{i^n}$ #5

$\forall\ a < b$ For any characteristic fct we can find its proba

(6) $$P(X \in (a,b)) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}\,\phi_X(t)\,dt$$

Inversion theorem

Motivation if $\phi_X \in L^1$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\,\phi_X(t)\,dx$$

$$P(X \in (a,b)) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\,\phi_X(t)\,dt\,dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_a^b e^{-itx}\,dx\right)\phi_X(t)\,dt$$

(7) $$\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$$

(8) $\phi_{X_n}(t)$ = characteristic fct of X_n

If $\forall\, t$ $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$

$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ also

$\lim_{n\to\infty} X_n = X$ or $X_n \xrightarrow{d} X$

Let do some Examples

$X \sim \text{Gamma}(k, \lambda)$

$$\phi_X(t) = \int_0^\infty e^{itx} \frac{\lambda e^{-\lambda x} x^{k-1}}{\Gamma(k)} dx$$

$$= \frac{\lambda}{\Gamma(k)} \int_0^\infty x^{k-1} e^{(it-\lambda)x} dx$$

Let $u = (\lambda - it)x \Rightarrow x = \frac{1}{\lambda - it} u$

$$dx = \frac{du}{\lambda - it}$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(\lambda - it)^{k-1}} e^{-u} \cdot \frac{1}{\lambda - it} du$$

$$= \frac{\lambda^k}{\Gamma(k)(\lambda - it)^k} \underbrace{\int_0^\infty u^{k-1} e^{-u} du}_{\Gamma(k)} = \left(\frac{\lambda}{\lambda - it}\right)^k$$

$$= \left(1 - \frac{it}{\lambda}\right)^{-k}$$

$X_1 \sim \text{Gamma}(k_1, \lambda)$ ind. of $X_2 \sim \text{Gamma}(k_2, \lambda)$

$X_1 + X_2 \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k_1+k_2}$$

<u>other example</u>

$X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \sum \frac{(\lambda e^{it})^x e^{-\lambda e^{it}}}{x!}$$

$$= e^{-\lambda + \lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$$\frac{(e^{it})^x \lambda^x e^{-\lambda}}{x!} = \frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda it}}$$

Another example

$X_1 \sim \text{Poisson}(\lambda_1)$ ind - of

$X_2 \sim \text{Poisson}(\lambda_2)$

$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t) = e^{\lambda_1(e^{it}-1)} e^{\lambda_2(e^{it}-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

Another example

$X_1, \dots, X_n \overset{iid}{\sim}$ some distr. with finite μ & finite variable σ^2

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E[\bar{X}] = \mu$$

$$Var[\bar{X}] = \frac{\sigma^2}{n}$$

Define $Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ $\qquad E(Z_n) = 0$

$\qquad Var(Z_n) = 1$

$$\phi_{\bar{X}_n}(t) = \phi_{\Sigma X_i}\left(\frac{t}{n}\right) = \left(\phi_X\left(\frac{t}{n}\right)\right)^n$$

Using rule # 2, and #3

$$\phi_{Z_n}(t) = \phi_{\bar{X}_n}\left(\frac{t}{\frac{\sigma}{\sqrt{n}}}\right) e^{it\left(\frac{-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}$$

$$= \phi_{\bar{X}}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu\sqrt{n}}{\sigma}\cdot\frac{n}{n}}$$

$$= \phi_X\left(\frac{t\sqrt{n}}{\sigma}\right) e^{\frac{it\mu n}{\sigma\sqrt{n}}}$$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \lim_{n\to\infty} \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}$$

$$= \lim_{n\to\infty} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}\right)}$$

Recall $y = e^{\ln(y)}$

$$= \lim e^{n\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu n}{\sigma\sqrt{n}}}$$

$$= \lim e^{n\left(\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim n\left(\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

let $u = \frac{t}{\sigma\sqrt{n}}$ $\quad n \to \infty, u \to 0$

$$= e^{\frac{t^2}{\sigma \cdot n} \lim_{u \to 0} \frac{\ln(\phi_x(u)) - i\mu \cdot u}{u^2}}$$

using l'Hopital rule

$$= e^{\frac{t^2}{2\sigma \cdot n} \lim \frac{\frac{\phi'(u)}{\phi(u)} - i\mu}{u}}$$

$$= e^{\frac{t^2}{2\sigma^2} \lim \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right]}$$

$$\text{at} \lim_{u \to 0} \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right] = \frac{\phi''(0)\cdot\phi(0) - (\phi'(0))^2}{\phi^2(u)}$$

$$= i^2\left(E(X^2) - \mu^2\right) = -\sigma^2$$

with $\phi(0) = 1$

$\phi'(0) = (i\mu)^2$

Go back to the e, we have

$$= e^{\frac{t^2}{2\sigma^2}(-\sigma^2)} = e^{-\frac{t^2}{2}} = \phi_Z(t)$$

Now we will find the pdf

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} e^{-\frac{t^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itx + \frac{t^2}{2}\right)}\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}}{2}ix\right)^2 + \frac{x^2}{2}\right)}\,dt$$

$$\frac{t^2}{2} itx = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, ix}{2}\right)^2 + \frac{x^2}{2}$$

$$\frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, ix}{2}\right)^2} dt$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy \quad \text{with}$$

$$y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, ix}{2}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\, dy$$