10/17/17. Lecture #11

From last Lecture

$X \sim \text{Gamma}(k_1, \lambda)$ and of $Y \sim \text{Gamma}(k_2, \lambda)$

WTS $X + Y \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$T = X + Y \sim \dots \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1} (1-u)^{k_2-1}\, t\, du$$

$$= \frac{\lambda^{k_1+k_2} \cdot e^{-\lambda t} \cdot t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \underbrace{\int_0^1 u^{k_1-1} (1-u)^{k_2-1} \cdot du}_{\text{function of } k_1, k_2} \quad (*)$$

$V(k_1, k_2)$

$$\text{Let } X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \propto e^{-\lambda x} = k(x) \quad \text{"kernel"}$$

$\propto$: directly proportional to

- $\int_{Supp[X]} f(x)\, dx = 1$

$q \propto v \Rightarrow q = cv \quad c \in \mathbb{R}.$

So $k(x) = c f(x)$

$\Rightarrow \int_{Supp[X]} k(x)\, dx = ?$

$\int \frac{1}{c}(k(x)) = 1 \quad \therefore \int_{Supp[X]} k(x)\, dx = c$

$*$ If $X \sim \text{Bin}(n, p) := \binom{n}{p} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^n (1-p)^{-x}$

Since $n!$ & $(1-p)^n$ has no x, it is $\propto (x!(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x$

$*$ $X \sim \text{weibull}(k, \lambda) := k\lambda(x\lambda)^{k-1} \cdot e^{-(x\lambda)^k}$

$\propto x^{k-1} \cdot e^{-(x\lambda)^t}$

$*$ $X \sim \text{Gamma}(k, \lambda) = \frac{\lambda^k e^{\lambda x} \cdot x^{k-1}}{\Gamma(k)} \propto e^{\lambda x} \cdot x^{k-1}$

So Going back to $(*)$

$\propto e^{\lambda t} \cdot t^{k_1+k_2-1} \propto \text{Gamma}(k_1 + k_2, \lambda)$

PDF

If $T \sim \text{Gamma}(k_1+k_2, \lambda) = \dfrac{\lambda^{k_1+k_2} \cdot e^{-\lambda t} \cdot t^{k_1+k_2-1}}{\Gamma(k_1+k_2)}$

must be equal to $\dfrac{\lambda^{k_1+k_2}\, e^{-\lambda t} \cdot t^{k_1+k_2-1}}{\Gamma(k_1)\,\Gamma(k_2)} \displaystyle\int_0^1 u^{k_1-1}(1-u)^{k_2-1} \cdot du$

$$\Rightarrow \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \int_0^1 u^{k_1-1}(1-u)^{k_2-1} \cdot du = B(k_1, k_2)$$

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Beta function

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\int_0^\infty t^{\alpha-1} \cdot e^{-t}dt \int_0^\infty t^{\beta-1} e^{-t} \cdot dt}{\int_0^\alpha t^{\alpha+\beta-1} \cdot e^{-t}dt}$$

page 160

Order Statistics.

$X_1, X_2, \dots X_n$ are a sequence of continuous r.v's.

$X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are called "order statistics" where $X_{(1)} < X_{(2)} < \dots < X_{(n)}$. "sorted order"

$X_{min} = X_{(1)} = \min\{X_1, \dots, X_n\}$

$2 = \min\{2, 7, 9, 12\}$

$X_{max} = X_{(n)} = \max\{X_1, \dots, X_n\}$

r.v

range $R = X_{max} - X_{min}$.

In the case of $X_1, \dots X_n$ iid $f(x)$ with CDF $F(x)$,

$\boxed{\text{Max}}$ we want $F_{X_{(n)}}(x) = P(X_{(n)} \le x)$ by def.

$= P(X_1 \le x, X_2 \le x, \dots X_n \le x)$

$= P(X_1 \le x) \dots\dots P(X_n \le x)$ by indep.

$= F_{X_1}(x) \dots\dots F_{X_n}(x)$

$= F(x)^n$

$$f_{X_{(n)}}(x) = F'(x)^n = n f(x) F(x)^{n-1}$$

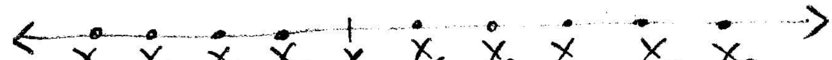
$\boxed{\text{MIN}}$

$$F_{X_{(1)}}(x) := P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$
$$= 1 - P(X_1 > x, \dots X_n > x)$$
$$= 1 - P(X_1 > x) \dots P(X_n > x) \text{ by independance}$$
$$= 1 - (1 - F_{X_1}(x)) \dots (1 - F_{X_n}(x)) \text{ def. of CDF}$$
$$= 1 - (1 - F(x))^n$$

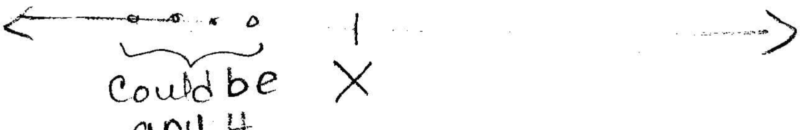
$$f_{X_{(1)}}(x) = F'_{X_{(1)}}(x) = -n(1 - F(x))^{n-1}(-f(x))$$
$$= n f(x) (1 - F(x))^{n-1}$$

<u>Goal:</u> $F_{X_{(k)}}(x) = ?$

Consider $n = 10$



$$P(X_1, X_2, X_3, X_4 \in (-\infty, x] \text{ AND } X_5, \dots, X_{10} \in (x, \infty))$$
$$= P(X_1 \le x) \dots P(X_4 \le x) P(X_5 > x) \dots P(X_{10} > x)$$
$$= F(x)^4 (1 - F(x))^6$$

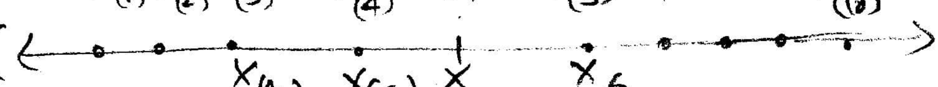


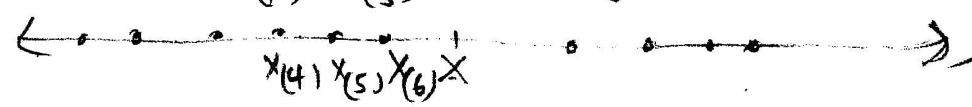
$P($any 4 are $\in -\infty, x]$, the other 6 are in $(x, \infty))$

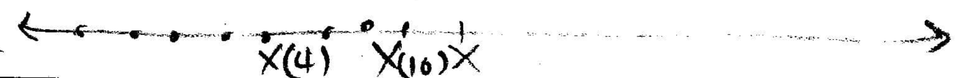
$$= \binom{10}{4} F(x)^4 (1 - F(x))^6$$

$$F_{X_{(4)}}(x) = P(X_{(4)} \le x)$$

$P($  $)$

$+ P($  $)$

$+ P($  $)$

$+ P($  $)$

$$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$$

PDF
$f(x)$
$f(x) = F'(x)$

$$\boxed{F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}}$$ CDF

$$F_{X_{(n)}}(x) = \sum_{j=\cancel{k}}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$= \cancel{\binom{n}{n}}^{1} F(x)^n \cancel{(1-F(x))^{n-n}}^{1} = F(x)^n$$

$$F_{X_{(1)}}(x) := \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$= \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} \right) - (1-F(x))^n$$

$$= \underbrace{\left(F(x) + (1-F(x))\right)^n}_{1^n = 1} - (1-F(x))^n$$

$$f_{X_{(k)}}(x) = F'_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \frac{d}{dx}\left[F(x)^j (1-F(x))^{n-j}\right]$$

$$(1-F(x))^{n-j} \cdot j F(x)^{j-1} f(x) - F(x)^j (n-j)(1-F(x))^{n-j-1} f(x)$$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \cdot (1-F(x))^{n-j} \cdot j F(x)^{j-1} \cdot f(x) - \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} F(x)^j (n-j) \cdot (1-F(x))^{n-j-1} \cdot f(x)$$

$$= \sum (j-1)! \;\ldots\ldots\; - \sum_{j=k}^{n-1} (n-j-1)!$$ reindexing

$$= f(x) \left(\sum_{j=k}^{n} \;\; - \sum_{j=k+1}^{n} \frac{n!}{(j-1)!(n-j)!} \cdot F(x)^{j-1} (1-F(x))^{n-j} \right)$$

$$(a_k + a_{k+1} + \cdots + a_n) - (a_{k+1} + \cdots + a_n) = a_k$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x) \cdot F(x)^{k-1} (1 - F(x))^{n-k}$$