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Lec 15 Math 621 10/31/17

$$\phi_X(t) = E\left[e^{itX}\right] = \sum_{x \in Supp(X)} e^{itx} p(x) \quad \text{for } X \text{ discrete}$$

(the ch.f. for r.v. X)

$$= \int_{Supp(X)} e^{itx} f(x)\,dx \quad \text{for } X \text{ cont.}$$

If $\phi(t) \in L^1$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)\,dt$$

Properties

1. $\phi(0) = 1$
2. $Y = X_1 + X_2 \quad \phi_Y(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$ for X_1, X_2 indep.
3. $Y = aX + b \quad \phi_Y(t) = e^{itb}\,\phi_X(at)$
4. $|\phi_X(t)| \leq 1 \quad \forall X, \forall t \Rightarrow \phi_X$ always conv.

Consider $\phi_X'(t) = \frac{d}{dt}\left[E\left[e^{itX}\right]\right] = \frac{d}{dt}\int_{\mathbb{R}} e^{itx} f(x)\,dx \stackrel{?}{=} \int_{\mathbb{R}} f(x) \frac{\partial}{\partial t}\left[e^{itx}\right] dx$

Does...

$$\frac{d}{dt}\left[\int_{\mathbb{R}} g(x,t)\,dx\right] \stackrel{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}\left[g(x,t)\right] dx$$

Conditions (a) $g(x,t)$ continuous $\forall x \in \mathbb{R}$ (b) $g(x,t)$ continuous $\forall t \in A = (a,b) \subset \mathbb{R}$

(c) $\exists t \in A$ s.t. $\int_{\mathbb{R}} g(x,t)$ converges (d) $\forall t \in A$ $\int_{\mathbb{R}} \frac{\partial}{\partial t}\left[g(x,t)\right] dx$ converges uniformly

These are satisfied

$$\Rightarrow \phi_X'(t) = \int_{\mathbb{R}} f(x)\, ix\, e^{itx}\,dx \Rightarrow \phi_X'(0) = i\int x f(x)\,dx = i\,E(X)$$

$$\phi_X''(t) = \int_{\mathbb{R}} f(x)\, i^2 x^2 e^{itx}\,dx \Rightarrow \phi_X''(0) = i^2 \int x^2 f(x)\,dx = i^2 E(X)$$

$\vdots$

5) $\Rightarrow E(X^n) = \dfrac{\phi_X^{(n)}(0)}{i^n}$ i.e. you can use the ch.f. to compute moments for X

Inversion thm. (2

(6) $P(X\in(a,b)) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita}-e^{-itb}}{it}\,\phi_X(t)\,dt$ for any ch.f.

Motivation. If $\phi_X(t)\in L^1$

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\,\phi_X(t)\,dt$$

$$P(X\in(a,b)) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\,\phi_X(t)\,dt\,dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_a^b e^{-itx}\,dx\right)\phi_X(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-itb}-e^{-ita}}{it}\,\phi_X(t)\,dt$$

However... $\phi_X(t)$ does not need to be integrable for this to work!

Since this is valid $\forall a<b$, $F(x)$ is determined uniquely by $\phi_X(t)$

(7) If $\phi_X(t)=\phi_Y(t) \Leftrightarrow F_X(x)=F_Y(x)$

(8) Consider $\phi_{X_n}(t)$ a ch.f. corresponding to $F_{X_n}(x)$ by #7

If $\forall t \lim_{n\to\infty}\phi_{X_n}(t)=\phi_X(t) \Rightarrow \forall x \lim_{n\to\infty}F_{X_n}(x)=F_X(x)$ or $\lim_{n\to\infty} X_n \overset{d}{=} X$

the limiting r.v. is the same

Def: " $X_n \xrightarrow{d} X$ "

Convergence in distribution

e.g...

$$X \sim Gamma(k, \lambda) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)}$$

$$\phi_X(t) = E(e^{itX}) = \int_0^\infty e^{itx} \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{(it-\lambda)x} dx = \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(it-\lambda)^{k-1}} e^{-u} \frac{1}{\lambda - it} du$$

let $u = (\lambda - it)x \Rightarrow x = \frac{1}{\lambda - it} u$

$\frac{du}{dx} = \lambda - it \Rightarrow dx = \frac{1}{\lambda - it} du$

other forms

$$= \frac{\lambda^k}{\Gamma(k)(\lambda - it)^k} \underbrace{\int_0^\infty u^{k-1} e^{-u} du}_{\Gamma(k)} = \left(\frac{\lambda}{\lambda - it}\right)^k = \left(\frac{\lambda - it}{\lambda}\right)^{-k} = \left(1 - \frac{it}{\lambda}\right)^{-k}$$

Prove $X \sim Gamma(k_1, \lambda)$, ind. of $Y \sim Gamma(k_2, \lambda)$

$\Rightarrow X + Y \sim Gamma(k_1 + k_2, \lambda)$

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k_1}\left(\frac{\lambda}{\lambda - it}\right)^{k_2} = \underbrace{\left(\frac{\lambda}{\lambda - it}\right)^{\overbrace{k_1 + k_2}^{k'}}}_{\text{Ch.f. of Gamma}} \Rightarrow X + Y \sim Gam(k_1 + k_2, \lambda)$$

$X \sim Poisson(\lambda)$

$$\phi_X(t) = E(e^{itX}) = \sum_{x=0}^\infty e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \sum \frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}} \qquad \lambda' = \lambda e^{it}$$

Note $e^{itx} = (e^{it})^x$

$$= \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \underbrace{\sum_{x=0}^\infty \frac{(\lambda e^{it})^x e^{-\lambda e^{it}}}{x!}}_{1} = e^{-\lambda + \lambda e^{it}} = e^{\lambda(e^{it} - 1)}$$

Since this is the pmf of a $Y \sim Poisson(\lambda e^{it})$

$X \sim Poisson(\lambda_1)$ ind. of $Y \sim Poisson(\lambda_2)$. Prove $X+Y \sim Poisson(\lambda_1+\lambda_2)$

Hard convolution before... but now

$$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t) = e^{\lambda_1(e^{it}-1)}\,e^{\lambda_2(e^{it}-1)} = e^{\lambda_1(e^{it}-1)+\lambda_2(e^{it}-1)}$$

$$= \underbrace{e^{\overset{\lambda'}{(\lambda_1+\lambda_2)}(e^{it}-1)}}_{\text{ch.f. of Poisson}}$$

p2/2 Consider:

$X_1, \ldots X_n \overset{iid}{\sim}$ distr unknown but $\mu = E(X)$, $\sigma^2 = Var[X]$

$\bar{X} = \frac{1}{n}\sum X_i$ the "avg r.v." $E[\bar{X}] = \frac{1}{n} n\mu = \mu$

let $Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ Z_n is the standardized average since $Var(\bar{X}) = \frac{1}{n^2}\sum\sigma^2 = \frac{\sigma^2}{n}$

$E(Z) = 0$, $Var(Z) = 1$

$SE(Z) =$

$$\phi_{\bar{X}_n}(t) = \phi_{\sum_{i=1}^n X_i}\left(\frac{t}{n}\right) = \left(\phi_X\left(\frac{t}{n}\right)\right)^n$$

$$\phi_{Z_n}(t) = \phi_{\bar{X}_n}\left(\frac{t}{\frac{\sigma}{\sqrt{n}}}\right) e^{it\left(\frac{-\mu}{\frac{\sigma}{\sqrt{n}}}\right)} = \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu\sqrt{n}}{\sigma}\cdot\frac{n}{n}}$$

$$= \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu n}{\sigma\sqrt{n}}}$$ ← put an n in the top

$$\phi_{Z_n}(t) = \left(\phi_X\left(\frac{\frac{t\sqrt{n}}{\sigma}}{n}\right)\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}} = \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}$$

We want to see what happens when $\lim_{n\to\infty} \phi_Z(t)$, the standard avg. of many, many iid r.v.'s

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$$\phi_{Z_n}(t) = e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}\right)}$$

$$= e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right) - \frac{it\mu n}{\sigma\sqrt{n}}}$$

$$= e^{n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu n}{\sigma\sqrt{n}}}$$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim_{n\to\infty} n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{1}{n}} \frac{\left(\frac{t^2}{\sigma^2}\right)}{\left(\frac{t^2}{\sigma^2}\right)}}$$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}}$$

let $u = \frac{t}{\sigma\sqrt{n}}$

L'Hopital's Rule

$$= e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\phi(u)) - i\mu u}{u^2}} = e^{\frac{t^2}{2\sigma^2} \lim_{u\to 0} \frac{\frac{1}{\phi(u)}\phi'(u) - i\mu}{u}}$$

$$= e^{\frac{t^2}{2\sigma^2} \lim_{u\to 0} \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right]} = e^{\frac{t^2}{2\sigma^2} \lim_{u\to 0} \frac{\phi(u)\phi''(u) - \phi'(u)^2}{\phi(u)^2}} = e^{\frac{t^2}{2\sigma^2} \frac{\phi(0)\phi''(0) - \phi'(0)^2}{\phi(0)^2}}$$

$$= e^{\frac{t^2}{2\sigma^2} \frac{(1)\left(i^2 E(X^2) - (iE(X))^2\right)}{(1)^2}} = e^{\frac{t^2}{2\sigma^2} \overset{-1}{i^2} \overset{\sigma^2}{\left(E(X^2) - E(X)^2\right)}} = e^{-\frac{t^2}{2}}$$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{-\frac{t^2}{2}}$$

Let's use the inversion thm. to get the density of the r.v. which the limiting ch.f. represents

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itx}\,\phi(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} e^{-\frac{t^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itx + \frac{t^2}{2}\right)}dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 + \frac{x^2}{2}\right)}dt = \frac{1}{2\pi}e^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2}dt$$

Note: $\frac{t^2}{2} + itx = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 - \left(\frac{\sqrt{2}\,ix}{2}\right)^2 = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 + \frac{x^2}{2}$

"Completing the square"

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,dy$

Famous Integral: Gaussian Integral $= \sqrt{\pi}$

$$= \frac{1}{2\pi}e^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-y^2}\sqrt{2}\,dy = \frac{\sqrt{2}}{2\pi}e^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-y^2}dy = \frac{\sqrt{2\pi}}{2\pi}e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

FINAL↑

MIDTERM 2↓

This density corresponds to the standard normal r.v.

$$X \sim N(0,1) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad \phi_X(t) = e^{-\frac{t^2}{2}} \quad E(X) = 0,\ SE(X) = 1$$

Why? Proved above!

let $Y = \mu + \sigma X$

$$f_Y(y) = \frac{1}{|\sigma|} f_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{|\sigma|}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} = N(\mu, \sigma^2)$$

Same as many other r.v.'s under linear transformation

$$\phi_Y(t) = e^{it\mu}\phi_X(\sigma t) = e^{it\mu}e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$