Lec 13 Math 621 10/24/17

$$Y = \frac{X_1}{X_1 + X_2} \quad \text{where} \quad X_1 \sim \text{Gamma}(\alpha, \lambda), \overset{\text{ind. of}}{,} X_2 \sim \text{Gamma}(\beta, \lambda)$$

Supp$[Y]$ = if $X_1 \approx 0$, $X_2 \approx \infty \Rightarrow Y \approx 0$ $= [0, 1]$

if $X_1 \approx \infty$, $X_2 \approx 0 \Rightarrow Y \approx 1$

if the both about the same $\Rightarrow Y \approx \frac{1}{2}$

It can't be > 1 since $X_1 > X_1 + X_2$ is not possible since $X_2 > 0$.

Using the multivariate change of var's formula, we found ...

$$f_Y(y) = \int_0^\infty f_{X_1}(y y_2) f_{X_2}(\underbrace{y_2 - y y_2}_{= y_2(1-y)}) y_2 \, dy_2$$

$|y_2| = y_2$ since always ~~positive~~

$$= \int_0^\infty \frac{\lambda^\alpha (y y_2)^{\alpha - 1} e^{-\lambda y y_2}}{\Gamma(\alpha)} \frac{\lambda^\beta (y_2(1-y))^{\beta - 1} e^{-\lambda y_2 (1-y)}}{\Gamma(\beta)} y_2^1 \, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} e^{-\lambda y y_2 - \lambda y_2 (1-y)} dy_2$$

$$-\lambda y y_2 - \lambda y_2(1-y) = -\lambda (y y_2 + y_2 (1-y)) = -\lambda y_2 (y + 1 - y) = -\lambda y_2$$

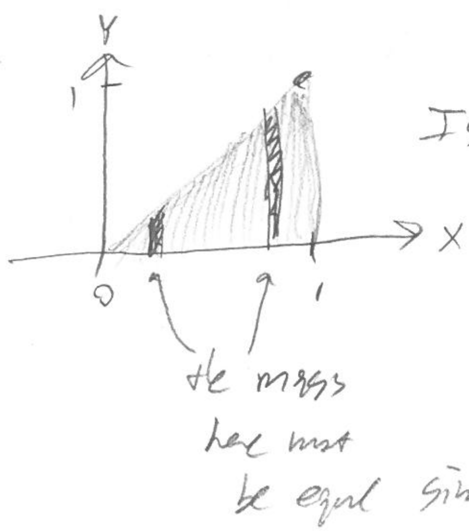
let $u = \lambda y_2 \Rightarrow \frac{du}{dy_2} = \lambda \Rightarrow dy_2 = \frac{1}{\lambda} du \Rightarrow y_2 = \frac{u}{\lambda}$

$$= \frac{\cancel{\lambda^{\alpha+\beta}}\, y_1^{\alpha-1}(1-y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^\infty \underbrace{\underbrace{\left(\frac{u}{\lambda}\right)^{\alpha+\beta-1} e^{-u}\frac{1}{\lambda}}_{\frac{u^{\alpha+\beta-1}}{\lambda^{\alpha+\beta-1}} e^{-u} \frac{1}{\lambda}}}_{\cancel{\frac{1}{\lambda^{\alpha+\beta}}} \int_0^\infty u^{\alpha+\beta-1} e^{-u}\,du}\,du}_{\Gamma(\alpha+\beta)}$$

$$= \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\frac{1}{B(\alpha,\beta)}} y_1^{\alpha_1-1}(1-y_1)^{\beta-1} = \text{Beta}(\alpha,\beta)$$

p155 cond. density... More practice with conditional densities...

let $X \sim U(0,1)$, $Y|X=x \sim U(0,x)$



Is this uniform completely throughout the Δ?

No

What does $Y \sim f_Y(y)$ look like?

$Supp(Y) = (0,1]$

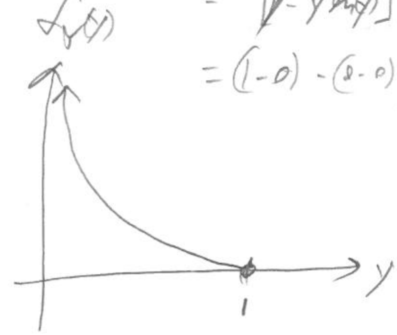
$$f_Y(y) = \int_{Supp(X)} f_{X,Y}(x,y)\,dx = \int_{Supp(X)} f_{Y|X}(y,x)\, f_X(x)\,dx$$

← hard... use indicator functions

$$= \int_{\mathbb{R}} \frac{1}{x}\, \mathbb{1}_{\underbrace{y\in[0,x]}_{\substack{0\le y\le x \\ y\le x \\ x\ge y}}}\, (1)\, \mathbb{1}_{x\in[0,1]}\, dx$$

$$= \int_y^1 \frac{1}{x}\,dx = \ln(x)\Big]_y^1 = -\ln(y)$$

$$\int_0^1 f_Y(y)\,dy \overset{?}{=} 1 = -\int_0^1 \ln(y)\,dy = -\left[y\ln(y) - y\right]_0^1 = \left[y - y\ln(y)\right]_0^1 = (1-0)-(0-0) = 1 \checkmark$$



What is $f_Y(y)$? The marginal density.

How about this. A download either takes on average 10 min without network traffic or on avg. 20 min with network traffic. Network traffic occurs w.p. $\frac{2}{3}$.

$$Y \sim \begin{cases} \text{Exp}\left(\frac{1}{10}\right) & \text{wp } \frac{1}{3} \\ \text{Exp}\left(\frac{1}{20}\right) & \text{wp } \frac{2}{3} \end{cases}$$

A fancier way we describe this is the following:

$$\text{let } X = \mathbb{1}_{\text{network traffic}} = \text{Bern}\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}$$

$$Y|X \sim \text{Exp}\left(\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x}\right) = \left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} y}$$

How long does a download take? Traffic is is measured... thus... we want the unconditional prob, $f_Y(y)$. Note: $Supp(Y) = (0,\infty)$

In general

$$f_Y(y) = \int_{Supp(X)} f_{X,Y}(x,y)\,dx$$

Here, X is discrete ... so ...

$$= \sum_{x \in Supp(X)} f_{X,Y}(x,y)\,dx$$

$$= \sum_{x \in Supp(X)} f_{Y|X}(y,x)\, p_X(x)$$

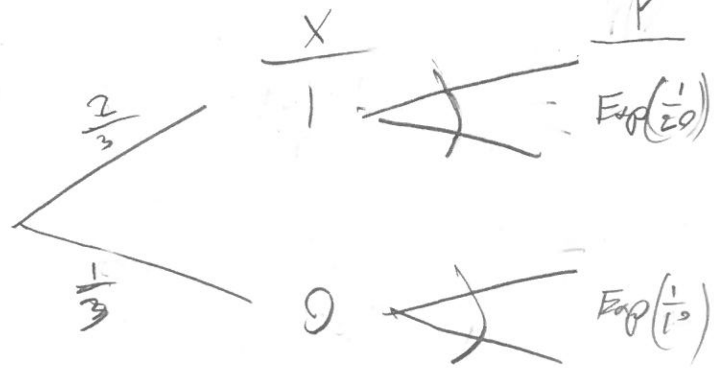
$$= \sum_{x \in \{0,1\}} \left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} y} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}$$

$$= \frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}y}\right) + \frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}y}\right)$$

... OR ...

$$= \underbrace{\frac{2}{3}\, Exp\left(\frac{1}{20}\right) + \frac{1}{3}\, Exp\left(\frac{1}{10}\right)}_{\substack{\text{mixture distribution} \\ \text{or} \\ \text{mixture model}}}$$

tree shows all



if Y is discrete: mixture distr.

A6

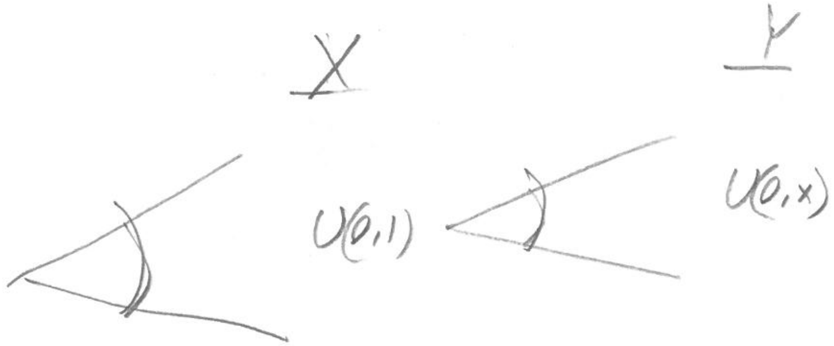
If the download took 25 min, what is the prob there was network traffic?

$P(X=1 \mid Y=25 \text{ min}) = ?$ "Flip the tree"

$$p_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{\underbrace{f_Y(y)}_{29.6826}} = \frac{f_{Y|X}(y,x)\, p_X(x)}{f_Y(y)}$$

$$p_{X|Y}(1,25) = \frac{\left(\frac{1}{20}\right) e^{-\frac{1}{20}(25)} \left(\frac{2}{3}\right)}{\frac{1}{3}\underbrace{\left(\frac{1}{10} e^{-\frac{1}{10}(25)}\right)}_{1.21825} + \frac{2}{3}\underbrace{\left(\frac{1}{20} e^{-\frac{1}{20}(25)}\right)}_{29.6826}} \approx 90\%$$

Tree for first situation



Here, since Y is a mixture of unctbly many values from X it's called a "compound distr." "Mixture distr." is for at most ctbly many elements. If X cont ⇒ compound distr.

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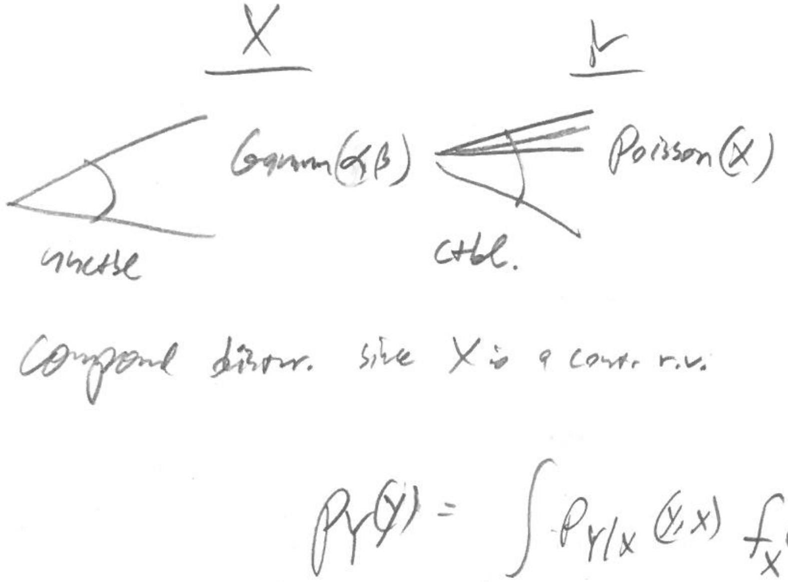
ex 9. Accidents Y ~ Poisson(λ) but λ is not the same for all drivers. It is drawn from a Gamma(α,β)



Supp(Y) = $\mathbb{N}$ still...
Since all values of λ are valid

Compound distr. since X is a cont. r.v.

$$p_Y(y) = \int_{Supp(X)} p_{Y|X}(y,x)\, f_X(x)\,dx = \int_0^\infty \frac{e^{-x}x^y}{y!}\,\frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}\,dx$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)}\int_0^\infty x^{y+\alpha-1}e^{-(\beta+1)x}\,dx$$

let $u = (\beta+1)x \Rightarrow x = \frac{u}{\beta+1}$

$\frac{du}{dx} = \beta+1 \Rightarrow dx = \frac{1}{\beta+1}\,du$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)}\int_0^\infty \left(\frac{u}{\beta+1}\right)^{y+\alpha-1} e^{-u}\,\frac{1}{\beta+1}\,du = \frac{\beta^\alpha}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}}\int_0^\infty u^{y+\alpha-1}e^{-u}\,du = \frac{\beta^\alpha\,\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}}$$

let $k=\alpha$, $p=\frac{\beta}{1+\beta} \Rightarrow 1-p=\frac{1}{1+\beta}$

$$= \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y-1)} p^k (1-p)^y = \text{ExtNegBin}(p,k)$$

If $k \in \mathbb{N}$

$$= \binom{y+k+1}{k} p^k (1-p)^y = \text{NegBin}(p,k)$$

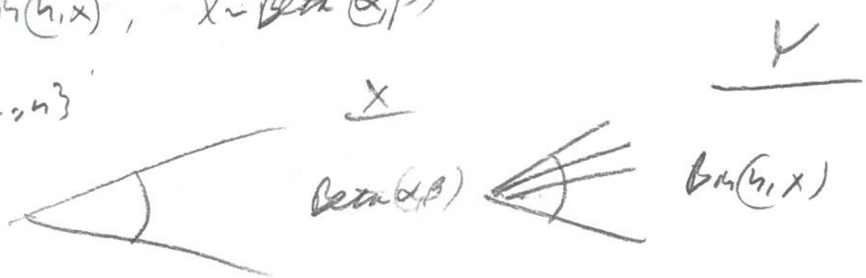
Poisson - count distn

NegBin - also count data but now overdispersed

Another Example of this..

$Y|X \sim \text{Bin}(n,x)$, $X \sim \text{Beta}(\alpha,\beta)$

$\text{Supp}(Y) = \{0,\dots,n\}$ still!



compound distr.

$$P_Y(y) = \int_{\text{Supp}(X)} P_{Y|X}(y,x) f_X(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\,dx = \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \text{BetaBinomial}(n,\alpha,\beta)$$

"Overdispersed binomial"