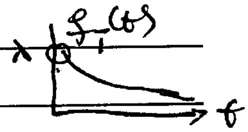
9/19/17

$T \sim Exp(\lambda) = \lambda e^{-\lambda t}$, $F_T(t) = 1 - e^{-\lambda t}$

time until event



$N \sim Poisson(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$ $\lambda = np$

# of events. $n \to \infty \Rightarrow p \to 0$

CDF $F_N(n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0} \frac{\lambda^i}{i!}$

What is the probability the event did not happen by $t = 1$?

$P(T > 1) = e^{-\lambda}$

What is the prob. zero events occurred?

$P(N = 0) = e^{-\lambda}$

$X \sim Erlang(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$, $F_X(x) = \frac{\gamma(k, \lambda x)}{(k-1)!}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \underbrace{\int_0^a t^{x-1} e^{-t} dt}_{\gamma(x,a)} + \int_a^\infty t^{x-1} e^{-t} dt$$

$\gamma(x,a)$ lower incomplete gamma function

upper incomplete gamma function

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt = -e^{-t}\Big]_0^\infty = -(0-1) = 1$$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u} \underbrace{e^{-t} dt}_{dv} = \left[-t^x e^{-t}\right]_0^\infty + x\int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x$$

$\Gamma(2) = 1\Gamma(1) = 1$

$\Gamma(3) = 2\Gamma(2) = 2\cdot 1$

$\Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1 \;\dots\; \Gamma(n) = (n-1)!\,,\ n \in \mathbb{N}.$

$$F_X(x) = \frac{\gamma(k, \lambda x)}{(k-1)!} = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

$$1 - F_X(x) = 1 - \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k) - \gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k, \lambda x)}{\Gamma(k)}.$$

$$\Gamma(k, \lambda x) = \int_{\lambda x}^{\infty} t^{k-1} e^{-t}\, dt = \underbrace{-t^{k-1} e^{-t}\Big]_{\lambda x}^{\infty}}_{(\lambda x)^{k-1} e^{-\lambda x}} - \int_{\lambda x}^{\infty} (k-1) t^{k-2} (-e^{-t})\, dt$$

$$(\lambda x)^{k-1} e^{-\lambda x} + (k-1) \underbrace{\int_{\lambda x}^{\infty} t^{k-2} e^{-t}\, dt}_{\Gamma(k-1, \lambda x)}$$

$$\Gamma(k-1, \lambda x) = (\lambda x)^{k-2} e^{-\lambda x} + (k-2)\Gamma(k-2, \lambda x)$$

$\vdots$

$$\Gamma(1, \lambda x) = \int_{\lambda x}^{\infty} t^{1-1} e^{-\lambda x}\, dt = e^{-\lambda x}$$

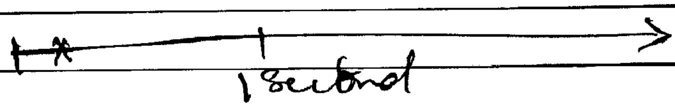
$$\Gamma(k, \lambda x) = e^{-\lambda x}\left((\lambda x)^{k-1} + (k-1)(\lambda x)^{k-2} + (k-2)(k-1)(\lambda x)^{k-3} + \dots + (k-1)!\right)$$

$$= e^{-\lambda x}(k-1)!\left(\frac{(\lambda x)^{k-1}}{(k-1)!} + \frac{(\lambda x)^{k-2}}{(k-2)!} + \frac{(\lambda x)^{k-3}}{(k-3)!} + \dots + 1\right)$$

$$= \underbrace{e^{-\lambda x}(k-1)!}_{\Gamma(k)} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} \Longrightarrow$$

$$1 - F_{T_k}(x) = \frac{\Gamma(k, \lambda x)}{\Gamma(k)} = \frac{e^{-\lambda x}\Gamma(k) \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}{\Gamma(k)}$$

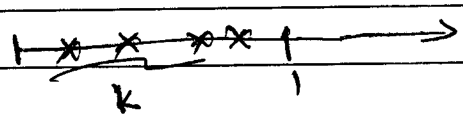
$T = T_1 + \cdots + T_k$ s.t. $T_1, \cdots T_k \overset{iid}{\sim} Exp(\lambda)$



N: # of events by $t = 1$
What is prob. of no success or one success?
$P(N \leq 1)$
$= F_N(1) = e^{-\lambda}(1+\lambda)$

$T \sim Erlang(2, \lambda) : P(T > 1) = e^{-\lambda}(1+\lambda)$

by using $1 - F_{T_k}(x) = e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}$



P(less than or equal to k event by t = 1)
N: # event by 1 sec.

$P(N \leq k) = F_N(k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$

$$T \sim \text{Erlang}(k+1, \lambda)$$

$$P(T>1) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

$$e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!} = \frac{\Gamma(k+1,\lambda)}{\Gamma(k)} = Q(k+1,\lambda)$$

$$\sum_{i=0}^{k} \frac{\lambda^i}{i!} = e^{\lambda} \underbrace{Q(k+1,\lambda)}_{\text{gamma function}}$$

$$\lim_{k\to\infty} Q(k,\lambda) = 1$$

Binomial & Neg binomial

Running experiments

	fixed time count events	Waiting to # event
Discretely	Binomial / Bernu	Neg. Binomial Geometric
continuously	Poisson	Erlang exponent

"Poisson process" in one dimension

$$\boxed{F_N(k) = 1 - F_{T_{k+1}}(1)}$$

What is the prob. there have bea 2 or less success by the 50^{th} experiment?

$N \sim Bin(50, p) \quad P(N \leq 2) = F_N(2)$

$$= \binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} + \binom{50}{2} p^2 (1-p)^{48}$$

$T \sim NegBin(3, p) \quad P(T > 47) = P(T=48) + P(T=49) + P(T=50) + \ldots$

$$= 1 - P(T \leq 47)$$

$$= 1 - F_T(47)$$

What is the prob. there have been k or less success by exponent n?

$$N \sim Bin(n,p) - P(N \leq k) = F_N(k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

$T \sim NegBin(k+1, p) \quad P(T > n-(k+1))$

$$= 1 - P(T \leq n-(k+1)) = 1 - F_T(n-k-1)$$

$$= 1 - \sum_{i=0}^{n-k-1} \binom{i+k}{k} p^{k+1} (1-p)^i$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

What is $P(X_1 | X_1 + X_2)$? ~~$P(X_1 = x)$~~

$P_X(x) = P(X = x)$
~~PMF~~

$P(X_1 = x | X_1 + X_2 = n)$

$X_1 + X_2 \sim \text{Poisson}(2\lambda) = \frac{(2\lambda)^x e^{-2\lambda}}{x!}$

$P(X_1 = x | X_1 + X_2 = n) = \frac{P(X_1 = x \,\&\, X_1 + X_2 = n)}{\underbrace{P(X_1 + X_2 = n)}_{\frac{e^{-2\lambda}(2\lambda)^n}{n!}}}$ $X_2 = n - x$

$= \frac{P_{X_1,X_2}(x, n-x)}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} = \frac{P_{X_1}(x)\, P_{X_2}(n-x)}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}}$

$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!} \cdot \frac{\cancel{e^{-\lambda}}\lambda^{(n-x)}}{(n-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^n}{n!}} = \binom{n}{x}\frac{\lambda^n}{(2\lambda)^n} = \binom{n}{x}\left(\frac{\lambda}{2\lambda}\right)^n$

$= \binom{n}{x}\left(\frac{1}{2}\right)^n = \text{Bin}\left(n, \frac{1}{2}\right)$

$Y = X_1 - X_2$? Find $P_Y(y)$

$\xrightarrow{\text{transformation}}$

$Y = X_1 + (-X_2) = X_1 + Z$ with $-X_2 = Z$