Lecture 20 Math 621 11/28/17

$\vec{Z} \sim N_n(\vec{0}_n, I_n)$

Is Σ symmetric?

$\Sigma \stackrel{?}{=} \Sigma^T = (AA^T)^T = A^{T^T} A^T = AA^T$ ✓

$E[\vec{Z}] = \vec{0}_n$, $Var[\vec{Z}] = I_n$

$\vec{\mu} \in \mathbb{R}^n$

if $\vec{X} = \vec{Z} + \vec{\mu} \sim N(\vec{\mu}, I_n)$

if $\vec{X} = A\vec{Z}$, $\mu = E[\vec{X}] = A\,E[\vec{Z}] = A\,\vec{0}_n = \vec{0}_m$, $\Sigma = Var[\vec{X}] = AA^T \in \mathbb{R}^{m \times m}$

$A \in \mathbb{R}^{m \times n}$

$f_{\vec{X}}(\vec{x})$? Note $\vec{X} = g(\vec{Z}) = A\vec{Z}$. Assume $A \in \mathbb{R}^{n \times n}$ for now.

$\vec{Z} = h(\vec{X}) = A^{-1}\vec{X}$, g is only 1:1 if A is full rank.

Let's do multivariable change of variables like before...

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}\big(\underbrace{h(\vec{x})}_{A^{-1}x}\big)\,|J_h(\vec{x})|$$

$h(\vec{x}) = h_1(\vec{x}), \dots, h_n(\vec{x})$

$$J_h = \begin{bmatrix} \frac{\partial}{\partial x_1} h_1(\vec{x}) & \cdots & \frac{\partial}{\partial x_n} h_1(\vec{x}) \\ \vdots & & \\ \frac{\partial}{\partial x_1} h_n(\vec{x}) & \cdots & \frac{\partial}{\partial x_n} h_n(\vec{x}) \end{bmatrix}$$

$$h(\vec{X}) = \begin{bmatrix} h_1(\vec{X}) \\ h_2(\vec{X}) \\ \vdots \\ h_n(\vec{X}) \end{bmatrix} = B\vec{X}$$

let $B = A^{-1}$ just for notational simplicity

$$B = \begin{bmatrix} \vec{b}_{1\cdot} \\ \vdots \\ \vec{b}_{n\cdot} \end{bmatrix} \text{ or } \begin{bmatrix} \vec{b}_{\cdot 1} & \cdots & \vec{b}_{\cdot n} \end{bmatrix} \text{ or } \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$$h_1(\vec{X}) = \vec{b}_{1\cdot}\vec{X} = b_{11}X_1 + \dots + b_{1n}X_n$$

$$\left.\begin{aligned} \frac{\partial}{\partial x_1}[h_1(\vec{x})] &= b_{11} \\ \frac{\partial}{\partial x_2}[h_1(\vec{x})] &= b_{12} \\ &\vdots \\ \frac{\partial}{\partial x_n}[h_1(\vec{x})] &= b_{1n} \end{aligned}\right\} \text{top row of } J_h$$

$$h_2(\vec{X}) = \vec{b}_{2\cdot}\vec{X} = b_{21}X_1 + b_{22}X_2 + \dots + b_{2n}X_n$$

$$\frac{\partial}{\partial x_1}[h_2(\vec{x})] = b_{21}$$

$$J_h = \det\left(\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix}\right) = \det(B) = \det(A^{-1})$$

Sidebar ... vector derivatives ...

$$\frac{\partial}{\partial \vec{x}}[C\vec{x}] = C$$

Many different definitions ... no time for this ...

$$\Rightarrow f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(A^{-1}X)\,|\det(A^{-1})|$$

Recall $\det(A^{-1}) = \dfrac{1}{\det(A)}$ since $AA^{-1} = I$, $\det(AA^{-1}) = \det(I)$, $\det(A)\det(A^{-1}) = 1$

$$= \frac{1}{(2\pi)^{n/2}|\det(A)|} e^{-\frac{1}{2}\underbrace{(A^{-1}\vec{x})^T(A^{-1}\vec{x})}_{\vec{x}^T(A^{-1})^T A^{-1}\vec{x}}}$$

why? $(AB)^{-1} = B^{-1}A^{-1}$

since $(AB)(AB)^{-1} = I$

$A\underbrace{BB^{-1}}_{I}A^{-1} = AA^{-1} = I$ ✓

Recall: $\Sigma = AA^T \Rightarrow \Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$

Does $(A^T)^{-1} = (A^{-1})^T$?

$$\begin{cases} AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I \Rightarrow (A^{-1})^T A^T = I \\ (A^T)^{-1} A^T = I \end{cases}$$

by def of matrix inverse $\Rightarrow (A^T)^{-1} = (A^{-1})^T$

$\Rightarrow \Sigma^{-1} = (A^{-1})^T A^{-1}$ symmetric? $\Sigma^{-1\,T} \stackrel{?}{=} \Sigma^{-1} = ((A^{-1})^T A^{-1})^T = (A^{-1})^T (A^{-1})^{TT} = (A^{-1})^T A^{-1}$ ✓

$$\Rightarrow f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2} |\det(A)|} e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}}$$

Just to get into canonical form...

Note $|\det(\Sigma)| = |\det(AA^T)| = |\det(A)\det(A^T)| = |\det(A)^2|$

$\det(A) = \det(A^T)$

$$\Rightarrow \sqrt{|\det(\Sigma)|} = |\det(A)|$$

$$\Rightarrow f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |\det(\Sigma)|}} e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}} = N_n(\vec{0}, \Sigma)$$

This is the generalization of $\left(\frac{x-\mu}{\sigma}\right)^2$ in the univariate normal i.e. "standardized distance" sqd

This "standardized distance" sqd is called "Mahalanobis distance" sqd (1936) by P.C. Mahalanobis (very famous Indian statistician). This is super useful!! I used this in my thesis to measure distance between two people for a clinical trial

$$\text{If } \vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n |\det(\Sigma)|}} e^{-\frac{1}{2} (\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu})}$$

If $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$, $B \in \mathbb{R}^{m \times n}$, $B\vec{X} \sim$? We can't use multi. change of var's too easily...

How to prove?

Recall $\phi_X(t) = E[e^{itX}]$. What about for vector $\vec{X}$? $\phi_{\vec{X}}(\vec{t}) = E[e^{i\vec{t}^T\vec{X}}]$

for ch.f.'s

Rules.. again.. Same..

if indep.

$$\phi_{\vec X_1+\vec X_2}(\vec t) = E\left[e^{i\vec t^T(\vec X_1+\vec X_2)}\right] = E\left[e^{i\vec t^T\vec X_1 + i\vec t^T\vec X_2}\right] = E\left[e^{i\vec t^T\vec X_1}e^{i\vec t^T\vec X_2}\right] = \phi_{\vec X_1}(\vec t)\,\phi_{\vec X_2}(\vec t)$$

$$\phi_{A\vec X+\vec c}(\vec t) = E\left[e^{it^T(A\vec X+\vec c)}\right] = E\left[e^{it^TA\vec X + it^T\vec c}\right] = e^{i\vec t^T\vec c}\,E\left[e^{it^TA\vec X}\right]$$

for appropriately sized $\vec c$, A

$$= e^{i\vec t^T\vec c}\,\phi_X(A^Tt)$$

let $t'^T = t^TA$

$\Rightarrow t' = t'^{TT} = (t^TA)^T = A^Tt$

What is ch.f. for mvN: $\vec Z \sim N_n(\vec 0, I_n)$

$$E\left[e^{it\vec Z}\right] = \underbrace{\int_{\mathbb R}\cdots\int_{\mathbb R}}_{\mathbb R^n} e^{i\vec t^T\vec z} f_{\vec Z}(\vec z)\,d\vec z = \int\cdots\int e^{i(t_1z_1+t_2z_2+\dots+t_nz_n)}\frac{1}{(\sqrt{2\pi})^n}e^{-\frac12(z_1^2+\dots+z_n^2)}\,dz_1\cdots dz_n$$

$$= \prod_{i=1}^n \int_{\mathbb R}\frac{1}{\sqrt{2\pi}}e^{it_iz_i - \frac12 z_i^2}\,dz_i$$

we did a very similar exercise to invert the limiting ch.f. during the CLT proof

$$-\frac12 z^2 + itz = -\frac12(z^2 - 2itz) = -\frac12\left((z-it)^2 - i^2t^2\right) = -\frac12\left((z-it)^2 + t^2\right) = -\frac12(z-it)^2 - \frac{t^2}{2}$$

$$\prod_{i=1}^n e^{-t_i^2/2}\underbrace{\int_{\mathbb R}\frac{1}{\sqrt{2\pi}}e^{-\frac12(z_i-it_i)^2}\,dz_i}_{\substack{\text{PDF for}\\ N(it_i,1)\\ =1}} = e^{-\frac12\sum t_i^2} = \boxed{e^{-\frac12\vec t^T\vec t} = \phi_{\vec Z}(t)}$$

$A\in\mathbb R^{n\times n}$

if $X = A\vec Z + \vec\mu$

$\Rightarrow X \sim N_n(\vec\mu, \Sigma)$

s.t. $\Sigma = AA^T$

$$\phi_{\vec X}(t) = e^{i\vec t^T\vec\mu}\,\phi_{\vec Z}(A^Tt) = e^{i\vec t^T\vec\mu}\,e^{-\frac12 t^TAA^T\vec t}$$

$$= e^{i\vec t^T\vec\mu - \frac12 t^T\Sigma\vec t}$$

$\vec Y = B\vec X$. How is $Y\sim$? $\phi_{\vec Y}(t) = \phi_{\vec X}(B^Tt) = e^{it^TB\mu - \frac12 t^TB\Sigma B^Tt} \Rightarrow Y \sim N_m(B\mu, B\Sigma B^T)$

$B\in\mathbb R^{m\times n}$ $m\neq n$ necessarily

Let's do another one... $\vec{Z} \sim N_n(\vec{0}, I_n)$

Now $X = B\vec{Z} + \vec{c}$ $\quad \phi_X(\vec{t}) = e^{i\vec{t}^T\vec{c}}\phi_Z(B^T\vec{t}) = e^{i\vec{t}^T\vec{c} - \frac{1}{2}\vec{t}^T BB^T\vec{t}}$

where $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^m$ $\quad \Rightarrow X \sim N_m(\vec{c}, \Sigma)$

dimension $\neq n$ necessarily

but Σ must be full rank!

$\Sigma = BB^T$

Given $\vec{X}$ how do we standardize back to $\vec{Z}$?

$X = AZ + \mu \Rightarrow X - \mu = AZ \Rightarrow A^{-1}(X - \mu) = Z$ $\quad$ You can only get back if A is invertible!

$Z = A^{-1}X - A^{-1}\mu$

Why? Imagine $A = \vec{1}^T$, $\vec{c} = \vec{0}$

$X = \Sigma z_i \sim N(0, n\sigma^2)$

$X \in \mathbb{R}^1 \Rightarrow$ no way to know what all n z_i's would be! Makes sense...

$$\vec{Z}^T\vec{Z} = \sum_{i=1}^n z_i^2 \sim \chi^2_n$$

$$\Rightarrow \left(A^{-1}(X - \vec{\mu})\right)^T\left(A^{-1}(X - \mu)\right) = \vec{Z}^T\vec{Z} \sim \chi^2_n$$

$$(\vec{X}^T - \vec{\mu}^T)\underbrace{(A^{-1})^T A^{-1}}(\vec{X} - \vec{\mu})$$

$$\underbrace{(A^T)^{-1}A^{-1}}_{\Sigma^{-1}} \Rightarrow (\vec{X} - \vec{\mu})^T\Sigma^{-1}(\vec{X} - \vec{\mu}) \sim \chi^2_n$$

Distr. of Mahalanobis distance squared

Standardizing and squaring! "Sphering"

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \vec{X} \sim N_n(\mu\vec{1}, \sigma^2 I_n)$$

$\Sigma = \sigma^2 I = AA^T \Rightarrow A = \sigma I$

$\vec{X} = \sigma I\vec{Z} + \vec{\mu} = \sigma\vec{Z} + \vec{\mu}$ $\quad$ Same as univariate case!

Same as before...

$$(\vec{X} - \vec{\mu})^T\frac{1}{\sigma^2}(\vec{X} - \mu) \sim \chi^2_n$$

$$= \frac{1}{\sigma^2}(\vec{X} - \vec{\mu})^T(\vec{X} - \vec{\mu})$$

$$\frac{1}{\sigma^2}\Sigma(X_i - \mu)^2 \sim \chi^2_n$$