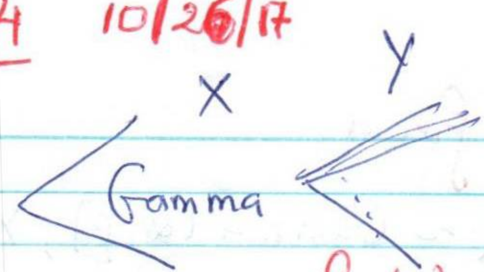
Lecture 14 10/26/17

$Q \sim \text{Poisson}(\lambda)$

$E(Q) = \lambda$

$SE[Q] = \lambda$



$f_Y(y) = \text{NegBin}$

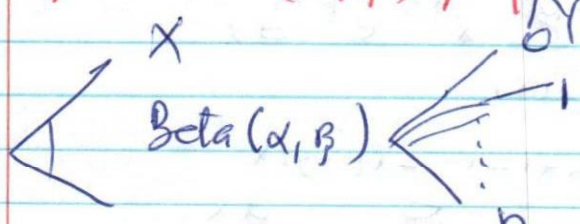
$Q \sim \text{NegBin}(K, p)$ (more flexible μ, V) call "Overdispersed Poisson"

$E(Q) = Kp$

$SE(Q) \neq E(Q)$

n fixed

$X \sim \text{Beta}(\alpha, \beta)$, $Y|X \sim \text{Bin}(n, x)$



$$P_Y(y) = \int_{Sup[X]} P_{Y|X}(y,x)\, f_X(x)\, dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1}\, dx =$$

$$\frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \text{BetaBinomial}(\alpha, \beta, n)$$

"Overdispersed binomial"

(42b)

n fixed

$X \sim \text{Gamma}(\alpha, \beta)$, $Y/X \sim Exp(x)$

$$X \quad \text{Gamma}(\alpha,\beta) \qquad Y \quad Exp(x)$$

$$f_Y(y) = \int_{\text{Sup}[x]} f_{Y/X}(y,x) f_X(x)\, dx$$

$$= \int_0^\infty x e^{-xy} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1-1} e^{-(\beta+y)x}\, dx$$

let $u = (\beta+y)x \Rightarrow x = \frac{u}{\beta+y}$ $\quad \frac{du}{dx} = \beta + y$

$$dx = \frac{1}{\beta+y}\, du$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha+1-1}}{(\beta+y)^\alpha} \frac{e^{-u} 1\, du}{\beta+y}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)(\beta+y)^{\alpha+1}} \underbrace{\int_0^\infty u^{\alpha+1-1} e^{-u}\, du}_{\Gamma(\alpha+1)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{\beta^{\alpha+1}}{\beta} \cdot \frac{1}{(\beta+y)^{\alpha+1}}$$

$$\boxed{\text{Note } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$$

$$= \cancel{\frac{\beta^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)} (\beta+y)}^{-\alpha}$$

survival (reliabi)

$$= \frac{\alpha}{\beta}\left(1 + \frac{y}{\beta}\right)^{-(\alpha+1)} = \text{Lomax}(\beta, \alpha)$$

let $a, b \in \mathbb{R}$

$z = a + bi \in \mathbb{C}$ (complex #'s)

$i = \sqrt{-1}, \Rightarrow i^2 = -1, \quad i^3 = -i \quad i^4 = 1$

$\mathrm{Re}[z] := a$

$\mathrm{Im}[z] := b$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + - \cdots$$

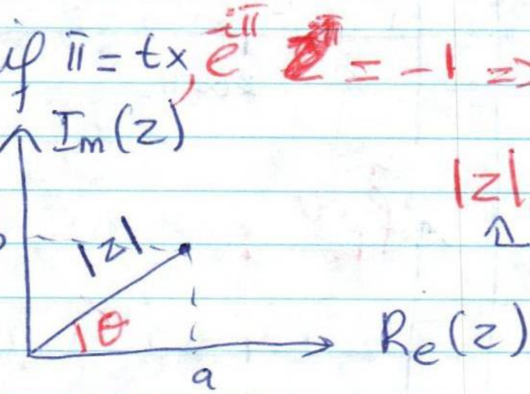
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + -$$

$$i\sin(x) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - + \cdots$$ observe that

$$\cos(x) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + \cdots$$

$$e^{itx} = \cos(tx) + i\sin(tx)$$

if $\pi = tx$, $e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$ Euler's identity



$|z| = \sqrt{a^2+b^2} \in [0, \infty)$

↑ complex norm

$\theta = \arctan\left(\frac{b}{a}\right) \in [-\pi, \pi]$

↑ Arg(z)

$$Z = |z|e^{i\theta}$$

Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \}$

all PDF's $\in L^1$ (Because they integrat to 1)

If $f \in L^1$ then $\exists\ \hat{f}$ defined as

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\,dx$$

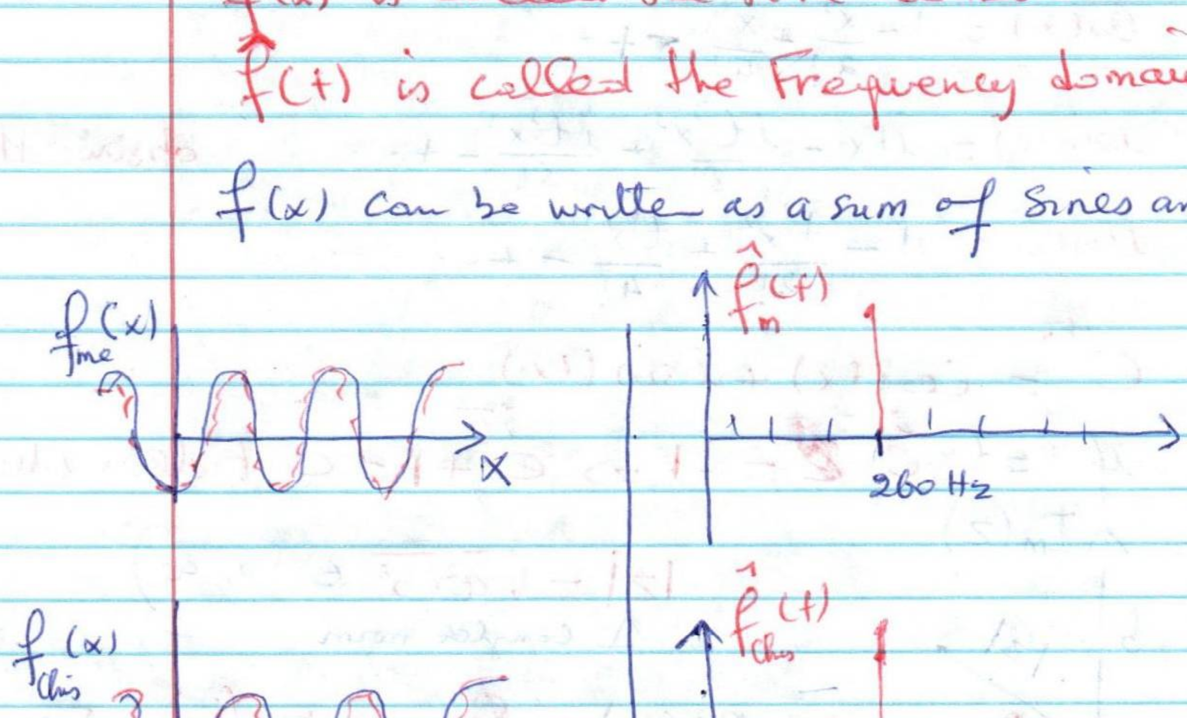
Known as the Fourier transform of f

<u>Note</u> $\hat{f}$ doesn't necessarily $\in L^1$

$f(x)$ is called the time domain

$\hat{f}(t)$ is called the Frequency domain

$f(x)$ can be writte as a sum of Sines and Cosines



Time Domain

$Re[\hat{f}(t)] \rightarrow$ Amplitude of frequency

$Arg[\hat{f}(0)] \rightarrow$ Phase shift of wave let $\phi(t) = \hat{f}\left(-\frac{t}{2\pi}\right) := \int_{\mathbb{R}} e^{itx} f(x)\, dx = \underset{\substack{\uparrow \\ \text{Expectation}}}{E}\left[e^{itx}\right]$ (if $f(x)$ is a PDF of a rv X)

$$\boxed{\begin{array}{l}\underline{\text{Note}}\ \text{If } \hat{f} \in L^1 \text{ then} \\ f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\, dt \\ \text{inverse Fourier Transform}\end{array}}$$

If $\phi(t) \in L^1$

let $u = -2\pi t \Rightarrow t = \frac{-u}{2\pi} \Rightarrow \frac{du}{dt} = -2\pi \Rightarrow dt = \frac{-1}{2\pi} du$

$$f(x) = \int_{\mathbb{R}} e^{2\pi i \left(\frac{-u}{2\pi}\right)x} \hat{f}\left(-\frac{u}{2\pi}\right) \frac{-1}{2\pi}\, du$$

$t = \infty \Rightarrow u = -\infty$
$t = -\infty \Rightarrow u = \infty$

$$f(x) = \int_{\infty}^{-\infty} e^{2\pi i \left(\frac{-u}{2\pi}\right)x} \hat{f}\left(-\frac{u}{2\pi}\right) \frac{-1}{2\pi}\, du$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \hat{f}\left(-\frac{u}{2\pi}\right) du = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi(u)\, du$$

$\phi_X(t)$ this is the characteristic function of a rv X. "Ch.f"

$$\phi_X(t) := \underset{\substack{\uparrow \\ \text{Expectation}}}{E}\left[e^{itX}\right] = \begin{cases} \sum_{x \in \text{sup}[X]} e^{itx} p(x) & \text{if } X \text{ discrete} \\ \int_{\text{sup}[X]} e^{itx} f(x)\, dx & \text{if } X \text{ continuous} \end{cases}$$

Properties

1. $\phi(0) = \underline{1}$

2. if X_1, X_2 independent, $Y = X_1 + X_2$

$$\phi_Y(t) = \phi_{X_1+X_2}(t) = E\left[e^{it(X_1+X_2)}\right] = E$$
$$= E\left[e^{itX_1} e^{itX_2}\right]$$
$$= E\left[e^{itX_1}\right] \cdot E\left[e^{itX_2}\right]$$
$$= \phi_{X_1}(t)\, \phi_{X_2}(t)$$

3. if $Y = aX + b$, $a, b \in \mathbb{R}$

$$\phi_Y(t) = E\left[e^{itY}\right] = E\left[e^{it(aX+a)}\right]$$
$$= E\left[e^{itaX} e^{itb}\right] = e^{itb} E\left[e^{itaX}\right]$$
$$= e^{itb} \phi_X(at)$$

4. $\phi_X(t)$ always exists since

$$|\phi_X(t)| \leq 1 \ \forall t$$

$$|\phi_X(t)| = |E(e^{itX})| = \left| \int_{\mathbb{R}} e^{itx} f(x)\, dx \right| \leq \int_{\mathbb{R}} \left| e^{itx} f(x) \right| dx$$

$$= \int_{\mathbb{R}} \underbrace{|e^{itx}|}_{=1} |f(x)|\, dx = \int_{\mathbb{R}} |f(x)|\, dx = \underline{1}$$

$|e^{itx}| = 1$ because $= |\cos(tx) + i\sin(tx)| = \sqrt{\cos^2 + \sin^2 tx} = 1$

45

$M_X(t) := \phi(\frac{t}{i}) = E[e^{tX}]$ does not have to always Exist

moment generating function

(5) $E[X^k] =$