Final ↑

Midterm 2 ↓↓

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Lec 16 Math 241 11/2/17

$X \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $\phi_X(t) = e^{-\frac{t^2}{2}}$

$E(X) = 0$, $SE(X) = 1$ why?

Recall setup of Central Limit Thm.

$\lim_{n \to \infty} Z_n \stackrel{d}{=} X$ where $Z_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$

$E(Z_n) = 0$, $SE(Z_n) = 1$ $\forall n$

$X_1, \ldots, X_n \stackrel{iid}{\sim} f(\mu, \sigma^2)$

CLT implies

① $\sum_{i=1}^{n} X_i \stackrel{d}{\approx} N(n\mu, n\sigma^2)$ if n large enough

② $\bar{X} \stackrel{d}{\approx} N(\mu, \frac{\sigma^2}{n})$

let $Y = \mu + \sigma X$ assume $\sigma \in (0, \infty)$

$f_Y(y) = \frac{1}{|\sigma|} f_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$

$|\sigma| \to \sigma$

$= N(\mu, \sigma^2)$

general Normal r.v.

$E(Y) = \mu + \sigma \underbrace{E(X)}_{=0} = \mu$

$SE(Y) = \underbrace{|\sigma|}_{=\sigma} \underbrace{SE(X)}_{=1} = \sigma$

$\phi_Y(t) = e^{it\mu} \phi_X(\sigma t) = e^{it\mu} e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ ind. of $X_2 \sim N(\mu_2, \sigma_2^2)$, $Y = X_1 + X_2 \sim$?

$$\phi_Y(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{it\mu_1 + it\mu_2 - \left(\frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 t^2}{2}\right)}$$

$$= e^{it(\mu_1+\mu_2) - \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)} \Rightarrow Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

$$Y = X_1 + X_2 \sim f_{X_1}(x) * f_{X_2}(x) = \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(t-x-\mu_2)^2}\,dx$$

No integrable function!

$\vdots$ Lots of work

$N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X = g(X)$, $g^{-1}(y) = \ln(y) \Rightarrow \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{y}$ $\quad Supp(Y) = (0, \infty)$

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{y} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} = LogNormal(\mu, \sigma^2)$$

$X \sim LogN(\mu, \sigma^2)$, $Y = \ln(X) \sim N(\mu, \sigma^2)$

~~Change in Y reflects % change in X, so if % change $\sim N(\cdot,\cdot) \Rightarrow$ value $\sim LogN(\cdot,\cdot)$~~

The LogN distr is really cool. Consider the following situation. You have an amount of money/material Y_0.
Every time period, Y changes based on a prop. change R_t. $\mu_R = \cdot, \sigma_R = \cdot$

e.g. $Y_1 = Y_0(1+R_1)$ if $R_1 = \overset{30\%}{\overset{\shortparallel}{.30}}, Y_0 = 10 \Rightarrow Y_1 = 13$ i.e. an increase of $\underline{\underline{30\%}}$.

$$Y_2 = Y_1(1+R_2) = Y_0(1+R_1)(1+R_2)$$

$$\vdots$$

$$Y_t = Y_0 \prod_{i=1}^{t}(1+R_i) = Y_0 e^{\ln\left(\prod_{i=1}^{t} 1+R_i\right)} = Y_0 e^{\sum_{i=1}^{t} \ln(1+R_i)}$$

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let $X_i = \ln(1+R_i) \Rightarrow Y_t = Y_0 e^{\sum_{i=1}^{t} X_i}$

if t is large... $X := \sum_{i=1}^{t} X_i \overset{d}{\approx} N(t\mu_X, t\sigma_X^2)$ by the C.L.T.

$$e^X \approx \text{LogN}(t\mu_X, t\sigma_X^2) \approx \text{LogN}(t\mu_R, t\sigma_R)$$

What is μ_X? $R = 3$ $\ln(1+.03) = .0296 \approx .03$!

$R = -5$ $\ln(1+-.05) = -.051 \approx -.05$!

$\ln(1+x) \approx x$ Why? Taylor Series...

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

if x is small... these are small!

$\Rightarrow \mu_X \approx \mu_R, \; \sigma_X \approx \sigma_R$

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e.g. Start off with \$1000. Assume Stock market is iid $N(10\%, 10\%^2)$

What is prob after 5 yr you have more than \$1,650?

$Y_t = Y_0 e^X$. We need to scale the LogN!

Assm $a \in (0, \infty)$

$$X \sim \text{LogN}, \quad Y = aX \sim \frac{1}{a} f_X\left(\frac{y}{a}\right) = \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\left(\frac{y}{a}\right)} e^{-\frac{1}{2\sigma^2}\left(\ln\left(\frac{y}{a}\right) - \mu\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(\ln(y) - (\mu + \ln(a))\right)^2}$$

$$= \text{LogN}\left(\mu + \ln(a), \sigma^2\right)$$

$Y_t = \$1000 e^X \qquad X \approx N(50\%, 5(10\%)^2)$, $\ln(\$1000) = 6.91$

$$\Rightarrow Y_5 \sim \text{LogN}\left(\underbrace{50\% + 6.91}_{=7.41}, \underbrace{5(10\%)^2}_{=.5^2}\right)$$

$$P(Y_5 > \$1650) = 1 - F_{Y_5}(1650) = 1 - \text{plnorm}(1650, 7.41, 0.5) \approx 51.2\%$$

$Z \sim N(0,1)$, $Y = Z^2 \sim ?$ $\quad Supp(Y) = [0, \infty]$

Note: $g(Z)$ is not a 1:1 function

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2P(Z \in [0, \sqrt{y}])$$

$$= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2\left(F_Z(\sqrt{y}) - \frac{1}{2}\right) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = F_Y'(y) = \frac{d}{dx}\left[2F_Z(\sqrt{y}) - 1\right] = 2\frac{d}{dy}\left[F_Z(\sqrt{y})\right] = \cancel{2}\frac{1}{\cancel{2}}y^{-1/2} F_Z'(\sqrt{y}) = \frac{1}{\sqrt{y}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sqrt{y})^2}{2}}$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}\sqrt{y}} e^{-\frac{y}{2}} \sim \chi^2_1$$ "Chi-squared dist. w/ one d.o.f."

$2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

Recall $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = \int_0^\infty \frac{1}{u}e^{-u^2}2u\,du = 2\int_0^\infty e^{-u^2}du$

let $u = \sqrt{t} \Rightarrow \frac{du}{dt} = \frac{1}{2}\frac{1}{\sqrt{t}} \Rightarrow dt = 2\sqrt{t}\,du = 2u\,du$

$$f_Y(y) = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}}{\Gamma\left(\frac{1}{2}\right)} = Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\frac{1}{\sqrt{2}} = \frac{1}{2^{\frac{1}{2}}} = 2^{-\frac{1}{2}} = (2^{-1})^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$$

Recall $Gamma(\alpha, \beta) = \dfrac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$

$X_1 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \overset{ind.}{,} X_2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ when $X_1 = ?$

$X_1 + X_2 \sim Gamma\left(1, \frac{1}{2}\right)$

$X_1, \ldots, X_K \overset{iid}{\sim} Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad \sum_{i=1}^K X_i \sim Gamma\left(\frac{K}{2}, \frac{1}{2}\right) = \dfrac{\left(\frac{1}{2}\right)^{\frac{K}{2}} x^{\frac{K}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{K}{2}\right)}$

$= \dfrac{1}{2^{K/2}\Gamma\left(\frac{K}{2}\right)} x^{\frac{K}{2}-1}e^{-\frac{x}{2}} = \chi^2_K$ "Chi-Squared w/ K degrees of freedom"

Note $\chi^2_2 = Exp\left(\frac{1}{2}\right)$? Yes... How?

Put another way...

$X_1, \dots, X_k \overset{iid}{\sim} N(0,1)$

$$\sum_{i=1}^{k} X_i^2 \sim \chi^2_k$$

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let $X \sim \chi^2_k$, $Y = \sqrt{X} \sim ?$ $\quad Supp(Y) = (0, \infty)$

$g^{-1}(y) = y^2 \Rightarrow \frac{d}{dy}[g^{-1}(y)] = 2y$

$$f_Y(y) = f_X(y^2)\, 2y = \frac{1}{2^{k/2}\Gamma(\frac{k}{2})} (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}} 2y$$

its parameter

$$= \frac{1}{2^{\frac{k}{2}-1}\Gamma(\frac{k}{2})} y^{k-1} e^{-\frac{y^2}{2}} \sim \chi_k \quad \text{Chi with } k \text{ d.o.f.}$$

terminology will become clear later

$X \sim N(0,1)$

$|X| \sim ?$ well $X^2 \sim \chi^2_1$ $\quad \underset{|X|}{\sqrt{X^2}} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} = 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)$

makes sense?

$X \sim \chi^2_k$ let $Y = \frac{X}{k} \sim ?$

Let's do scales of Gammas. $c \in (0, \infty)$

$$X \sim Gamma(\alpha, \beta),\ Y = cX \sim \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\frac{\beta y}{c}}}{c\,\Gamma(\alpha)}$$

$$= \frac{\beta^\alpha y^{\alpha-1} e^{-\frac{\beta}{c}y}}{c^{\alpha-1} \cdot c\,\Gamma(\alpha)} = \frac{\left(\frac{\beta}{c}\right)^\alpha y^{\alpha-1} e^{-\left(\frac{\beta}{c}\right)y}}{\Gamma(\alpha)} = Gamma\left(\alpha, \frac{\beta}{c}\right)$$