Lec 22 Math 621 12/5/17

Recall

$$|\mathbb{E}[XY]| \le \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$$

Cauchy Schwarz inequality

equal if $X = cY$. If $X = cY$

$$\Rightarrow \mathrm{Corr}(X,Y) = \begin{cases} 1 & \text{if } c>0 \\ -1 & c<0 \end{cases}$$

Can we prove $\mathrm{Corr}(X,Y) \in [-1,1]$ $\forall$ r.v.'s X, Y?

with finite

Prove for any r.v.'s X, Y that $Corr(X,Y) \in [-1,1]$

let $Z_X := \frac{X-\mu_X}{\sigma_X}$, $Z_Y := \frac{Y-\mu_Y}{\sigma_Y}$ $\Rightarrow E[Z_X] = E[Z_Y] = 0$

$SE[Z_X] = SE[Z_Y] = 1 \Rightarrow E[Z_X^2] = E[Z_Y^2] = 1$

Note $\left|E[Z_X Z_Y]\right| \leq \sqrt{E[Z_X^2]\,E[Z_Y^2]} = 1$

$\Rightarrow E[Z_X Z_Y] \in [-1,1]$

$$Corr[Z_X, Z_Y] = \frac{Cov[Z_X, Z_Y]}{SE[Z_X]\,SE[Z_Y]} = \frac{E[Z_X Z_Y] - E[Z_X]\,E[Z_Y]}{SE[Z_X]\,SE[Z_Y]} = E[Z_X Z_Y] \Rightarrow Corr[Z_X, Z_Y] \in [-1,1]$$

$$Corr[X,Y] = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y} = \frac{E[(\sigma_X Z_X + \mu_X)(\sigma_Y Z_Y + \mu_Y)] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$$

$\rightarrow E[\sigma_X\sigma_Y Z_X Z_Y] + E[\mu_X \sigma_Y \overset{0}{Z_Y}] + E[\mu_Y \sigma_X \overset{0}{Z_X}] + \mu_X\mu_Y$

$= \frac{\cancel{\sigma_X\sigma_Y}\, E[Z_X Z_Y]}{\cancel{\sigma_X\sigma_Y}} \Rightarrow Corr(X,Y) \in [-1,1]$

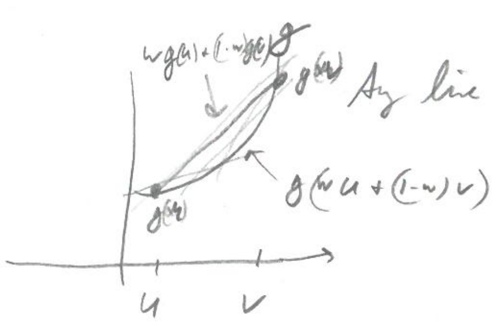
Def: g function is convex on an interval $I \subset \mathbb{R}$. If $\forall \{x_1, x_2, \dots x_n\} \subset I$ and $\forall\, w_1, w_2, \dots, w_n$ s.t. $\forall w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$ i.e. n weights,

$g(w_1 x_1 + \dots + w_n x_n) \leq w_1 g(x_1) + \dots + w_n g(x_n)$ i.e. $g(\sum w_i x_i) \leq \sum w_i g(x_i)$.

Thm: if g is twice diff., then g is convex if $g''(x) \geq 0 \ \forall x \in I$

Note $\sum w_i x_i \in I$

let u, v be the points, $w + (1-w) = 1$



Any line drawn is above function

Imagine a discrete r.v with $Supp(X) = \{x_1, \dots, x_n\}$

and PMF $p(x_i) = w_i$

$$\Rightarrow \sum w_i x_i = \sum_{x \in Supp(X)} x\, p(x) = E(X)$$

$$\sum w_i g(x_i) = \sum_{x \in Supp(X)} g(x)\, p(x) = E[g(X)]$$

$$\Rightarrow g(E(X)) \leq E[g(X)] \longrightarrow \text{Jensen's Inequality}$$

Proof for cont. r.v.'s and discrete w/ infinite support more involved but it holds for all r.v.'s

Also $g(E(X)) \geq E[g(X)]$ if g is concave (the same def above except $\geq$ instead of $\leq$)

If $g(x)$ is linear, then it is both convex and concave $\Rightarrow$

$$g(E(X)) = E[g(X)]$$

$$\Rightarrow aE(X) + b = E[aX + b]$$

$g(x) = x^2$ is convex

$$\Rightarrow E(X)^2 \leq E(X^2) \Rightarrow \mu^2 \leq \sigma^2 + \mu^2 \Rightarrow \sigma^2 \geq 0$$

~~$g(x) = e^x$ is convex~~

~~$e^{E(X)} \leq E[e^X]$~~

~~or $g(x) = e^{tX}$ $\forall t$~~

~~$e^{tE(X)} \leq E(e^{tX}) = M_X(t) \Rightarrow e^{t\mu} \leq M_X(t) \Rightarrow$ if $t>0$ $\mu \leq \frac{\ln(M_X(t))}{t} \Rightarrow \mu \leq \min_{t>0} \left\{ \frac{\ln(M_X(t))}{t} \right\}$~~

Let $g(x) = -\ln(x)$, $x>0$ Convex? $g'(x) = -\frac{1}{x}$, $g''(x) = \frac{1}{x^2} \geq 0 \ \forall x>0$

$\Rightarrow$ Yes

---

let $X \sim \begin{cases} a^p & wp \ \frac{1}{p} \\ b^q & wp \ \frac{1}{q} \end{cases}$ Note $\frac{1}{p} + \frac{1}{q} = 1$, $p,q>0$, $a,b>0$

$\Rightarrow X > 0$

$E(X) = \frac{a^p}{p} + \frac{b^q}{q}$

$g(X) \sim \begin{cases} -p\ln(a) & wp \ \frac{1}{p} \\ -q\ln(b) & wp \ \frac{1}{q} \end{cases}$

$g(E(X)) = -\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right)$

$\Rightarrow E[g(X)] = \frac{-p\ln(a)}{p} + \frac{-q\ln(b)}{q} = -\ln(ab)$

$g(E(X)) \leq E[g(X)] \Rightarrow -\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \leq -\ln(ab) \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ Young's Inequality

let $a = X$, $b = Y$

$\Rightarrow XY \leq \frac{X^p}{p} + \frac{Y^q}{q} \Rightarrow E(XY) \leq \frac{E(X^p)}{p} + \frac{E(Y^q)}{q}$

An interesting inequality...

let $a = \frac{X}{A}$, $b = \frac{Y}{B}$

$\frac{XY}{AB} \leq \frac{X^p}{pA^p} + \frac{Y^q}{qB^q} \Rightarrow \frac{E(XY)}{AC} \leq \frac{E[X^p]}{pA} + \frac{E(X^q)}{qB}$

let $A = E[X^p]^{\frac{1}{p}}$, $B = E(X^q)^{\frac{1}{q}}$

$\Rightarrow \frac{E[XY]}{E(X^p)^{\frac{1}{p}} E(X^q)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow E(XY) \leq E[X^p]^{\frac{1}{p}} E(Y^q)^{\frac{1}{q}}$

Holders Inequality

let $0<r<s$, $p=\frac{s}{r}$, $q=\frac{p}{p-1}=\frac{\frac{s}{r}}{\frac{s}{r}-1}=\frac{s}{s-r}$. let $X=V^r$, $Y=1$

$$E[V^r] \le E[(V^r)^{\frac{s}{r}}]^{\frac{1}{\frac{s}{r}}}$$

$$\Rightarrow E[V^r] \le E[V^s]^{\frac{r}{s}}$$

If $E[V^s]$ is finite $\Rightarrow E(V^r)$ is finite

For any r.v. X, if $E[|X|^s]$ is finite, any moment less than s is finite ~~too~~.

And $E(X^s) \le E(|X|^s)$ why? true

$$\int_R x^s f(x)dx \le \int_R |x^s| f(x)dx = \int_R |x^s| f(x)dx$$

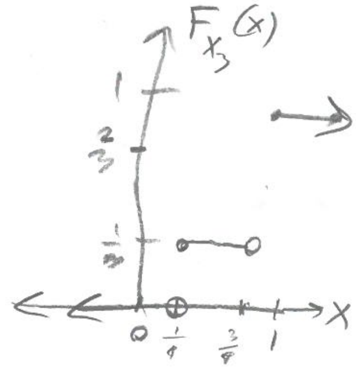
always +

Convergence of r.v.'s

Consider the seq of r.v.'s $X_1, X_2, \ldots$

Many types of convergence.

(I) Convergence in distribution



let $X_n \sim \begin{cases} \frac{1}{n+1} & \text{w.p. } \frac{1}{3} \\ 1-\frac{1}{n+1} & \text{w.p. } \frac{2}{3} \end{cases}$ for ex $X_3 = \begin{cases} \frac{1}{4} & \text{w.p. } \frac{1}{3} \\ \frac{3}{4} & \text{w.p. } \frac{2}{3} \end{cases}$

$X_{99} \sim \begin{cases} \frac{1}{100} & \text{w.p. } \frac{1}{3} \\ \frac{99}{100} & \text{w.p. } \frac{2}{3} \end{cases}$ $\quad X_n \longrightarrow \begin{cases} 0 & \text{w.p. } \frac{1}{3} \\ 1 & \text{w.p. } \frac{2}{3} \end{cases}$

$X_n \xrightarrow{d} X$ if $\forall x \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

Thm: If $Supp[X_n] \subset \mathbb{N}$ and $Supp(X) \subset \mathbb{N}$

then $X_n \xrightarrow{d} X \iff \forall_{x \in \mathbb{N}} \lim p_{X_n}(x) = p_X(x)$

Prove $\Rightarrow$

Note $p_{X_n}(x) = F_{X_n}(x + \frac{1}{2}) - F_{X_n}(x - \frac{1}{2})$

since $X_n \xrightarrow{d} X$

$$\lim_{n \to \infty} p_{X_n}(x) = \lim_{n \to \infty} F_{X_n}(x + \tfrac{1}{2}) - \lim_{n \to \infty} F_{X_n}(x - \tfrac{1}{2}) = F_X(x + \tfrac{1}{2}) - F_X(x - \tfrac{1}{2}) = p_X(x)$$

Prove $\Leftarrow$

since PMF's converge

$$\forall x \in \mathbb{N} \quad \lim_{n \to \infty} F_{X_n}(x) = \lim_{n \to \infty} P(X_n \le x) = \lim_{n \to \infty} \sum_{i=1}^{x} p_{X_n}(i) = \sum_{i=1}^{x} \lim_{n \to \infty} p_{X_n}(i) = \sum_{i=1}^{n} p_X(i) = F_X(x)$$

If $X_n \sim \begin{cases} \frac{1}{n+1} & \text{wp } \frac{1}{3} \\ 1 - \frac{1}{n+1} & \text{wp } \frac{2}{3} \end{cases}$ Prove $X_n \xrightarrow{d} \text{Bernoulli}(\frac{2}{3})$

$$p_{X_n}(x) = \left(\frac{1}{3}\right)^{\mathbb{1}_{x = \frac{1}{n+1}}} \left(\frac{2}{3}\right)^{\mathbb{1}_{x = 1 - \frac{1}{n+1}}} \mathbb{1}_{x \in \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}}$$

$$\lim p_{X_n}(x) = \left(\frac{1}{3}\right)^{\lim_{n \to \infty} \mathbb{1}_{x = \frac{1}{n+1}}} \left(\frac{2}{3}\right)^{\lim_{n \to \infty} \mathbb{1}_{x = 1 - \frac{1}{n+1}}} \lim_{n \to \infty} \mathbb{1}_{x \in \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}}$$

$$= \left(\frac{1}{3}\right)^{\mathbb{1}_{x=0}} \left(\frac{2}{3}\right)^{\mathbb{1}_{x=1}} \mathbb{1}_{x \in \{0,1\}} = \text{Bern}\left(\frac{2}{3}\right)$$

$X_n \sim \text{Binom}(n, \frac{\lambda}{n}) \xrightarrow{d} X \sim \text{Poisson}(\lambda)$ (done before)

$X_n \sim \text{Binom}(n, p)$

let $Y_n = \dfrac{X_n - np}{\sqrt{np(1-p)}}$

$Y_n \xrightarrow{d} N(0, 1)$ Hard proof!