# Lecture 5 9/12/17

$\vec{X}$ is a vector of rv's such that $\dim[X] = k$

$$\vec{\mu} = E[\vec{X}] = \begin{bmatrix} E(X_1] \\ \vdots \\ \vdots \\ E[X_n] \end{bmatrix}, \quad \Sigma = Var[X] = \begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \cdots \\ Cov[X_2,X_1] & \ddots & \\ \vdots & & \ddots \\ & & Var[X_n] \end{bmatrix}$$

$$= \{Cov[X_i, X_j]\}_{\substack{i=1\ldots k \\ j=1\ldots k}}$$

$$\Sigma_0 = Corr[\vec{X}] = \begin{bmatrix} 1 & & \\ & 1 & \cdots \\ Corr[X_i,X_j] & & 1 \end{bmatrix} = Corr[X_i, X_j]_{\substack{i=1\ldots k \\ j=1\ldots k}}$$

$$T = X_1 + \cdots + X_k = \vec{1}^T \vec{X}$$

$$\vec{1}_k = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad = [1 \; \cdots \; 1] \begin{bmatrix} X_1 \\ \vdots \\ \vdots \\ X_k \end{bmatrix}$$

$$E[T] = \sum_{i=1}^{k} \mu_i = \vec{1}^T \vec{\mu}$$

$$Var[T] = Var[\vec{1}^T \vec{X}] = Var[X_1 + \cdots + X_k]$$

$$= Cov[X_1 + \cdots + X_k ; X_1 + \cdots + X_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} Cov[X_i, X_j]$$

let $Y = \vec{c}^t \cdot \vec{X}$ where $\vec{c} \in \mathbb{R}^k$

$E[Y] = \vec{c}^T \vec{\mu}$

$Var[Y] = Var[\vec{c}^T \vec{X}]$

if $A \in \mathbb{R}^{m \times m}$, $\vec{c} \in \mathbb{R}^m$

$\vec{c}^T A \vec{c} = ?$ "Quadratic Form"

$$= \vec{c}^T \begin{bmatrix} c_1 a_{11} + \cdots + c_m a_{1m} \\ c_1 a_{21} + \cdots + c_m a_{2m} \\ \vdots \\ c_1 a_{n1} + \cdots \ c_m a_{nn} \end{bmatrix} =$$

$$c_1^2 a_{11} + c_1 c_2 a_{12} + \cdots + c_1 c_m a_{1m} + c_2 c_1 a_{21} + c_2^2 a_{22} + \cdots + c_2 c_m a_{2m} + \cdots$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j a_{ij}$$

$$= Var[c_1 X_1 + \cdots + c_k X_k] = Cov[c_1 X_1 + \cdots + c_k X_k, c_1 X_1 + \cdots + c_k X_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} Cov[c_i X_i, c_j X_j]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j Cov[X_i, X_j]$$

$$Var[\vec{X}] = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots \\ Cov[X_2, X_1] & Var[X_2] & \cdots \\ \vdots & & \ddots \\ \vdots & & & \ddots \end{bmatrix}$$

$$= \vec{c}^T Var[\vec{X}] \vec{c}$$

"Variance - Covariance matrix"

Markowitz Optimal Portfolio

let $X_1, \dots X_k$ be r.v models for the returns on k assets.
let $w_1, \dots w_k$ be the weights / allocations for each. Note $\vec{1}^T\vec{w} = 1$

$$V = \vec{w}^T\vec{X}, \quad E[V] = \vec{w}^T\vec{\mu} = \mu_0, \quad Var[V] = \vec{w}^T\Sigma\vec{w}$$

Given $\mu_0$, minimize $\vec{w}^T\Sigma\vec{w}$ such that $\vec{1}^T\vec{w} = 1$
$\{\vec{w} : \vec{1}^T\vec{w} = 1\}$

If $\vec{X} \sim$ Multinomial $(n, \vec{p})$

$$E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & Cov[X_1, X_2] & & \\ & np_2(1-p_2) & & \\ & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

$$Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j$$

$$= \sum_{x_i \in Supp[X_1]} \sum_{x_j \in Supp[X_2]} x_i x_j \underbrace{p_{X_i X_j}(x_i x_j)}_{\text{we don't know this}} - \mu_i \mu_j$$

Recall $X_1 \sim Bin(n, p_1)$

$\vdots$

$X_k \sim Bin(n, p_k)$

$X_1 = \sum_{i=1}^{n} X_{i1}$ such that $X_{11}, \ldots X_{n1} \overset{iid}{\sim} Bern\, p_1$

$\vdots$

$X_k = \sum_{i=1}^{n} X_{ik}$ such that $X_{1k}, \ldots X_{nk} \overset{iid}{\sim} Bern(p_k)$

$$\begin{bmatrix} X_{11} & X_{21} & \cdots & X_{n1} \\ & & & \\ & & & \\ X_{1k} & X_{2k} & \cdots & X_{nk} \end{bmatrix}$$

If $\vec{X} \sim multin(n, \vec{p}) \Rightarrow \vec{X} = \sum_{i=1}^{n} \vec{X}_i$ such that $\vec{X}_1, \vec{X}_2 \ldots \vec{X}_n \overset{iid}{\sim} multinom(1, \vec{p})$

$$Cov[X_i, X_j] = Cov\left[\sum_{\ell=1}^{n} X_{\ell i}, \sum_{h=1}^{n} X_{hj}\right]$$

$$= \sum\sum Cov[X_{\ell i}, X_{nj}]$$

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$$= \sum_{l=1}^{n} \sum_{h=1}^{n} E\left[X_{li}, X_{hj}\right] - p_i p_j$$

$$= \sum_{l=1}^{n} E[X_{li}, l_j] - p_i p_j$$

| $l = h$ | $l \neq h$ |
|---|---|
| $\sum_{y \in [0,1]} \sum_{x \in [0,1]} xy\, p(x,y) = 0$ | $E[\cancel{X}_{li} X_{hj}] = E[X_{li}] E[X_{hj}]$ |
| | $= p_i p_j$ |
| $\sum_{l=1}^{n} - p_i p_j = -n\, p_i p_j$ | $= p_i p_j$ |

Continuous rv's X have CDF, F(x) and PDF:

$$f(x) = F'(x)$$

$$\text{Supp}[X] := \{x : f(x) > 0\}$$

$$|\text{Supp}[X]| = |\mathbb{R}|$$

**Note**

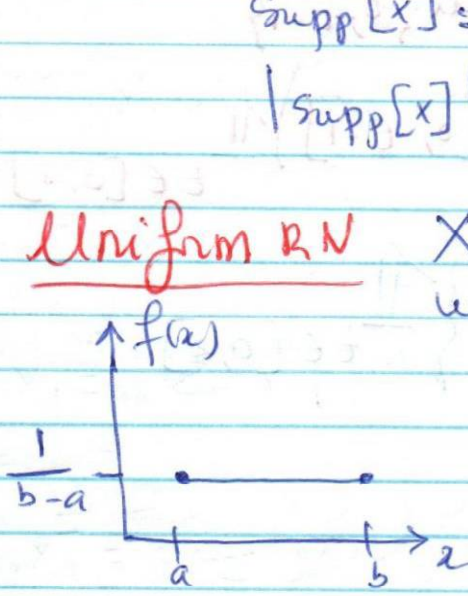
$$\underbrace{p(x)}_{\text{PMF}} = 0 \ \forall x$$

**Uniform RN** $X \sim \mathcal{U}(a,b) := \underbrace{\frac{1}{b-a}}_{f(x) \neq p(x)}$

where $a, b \in \mathbb{R}$

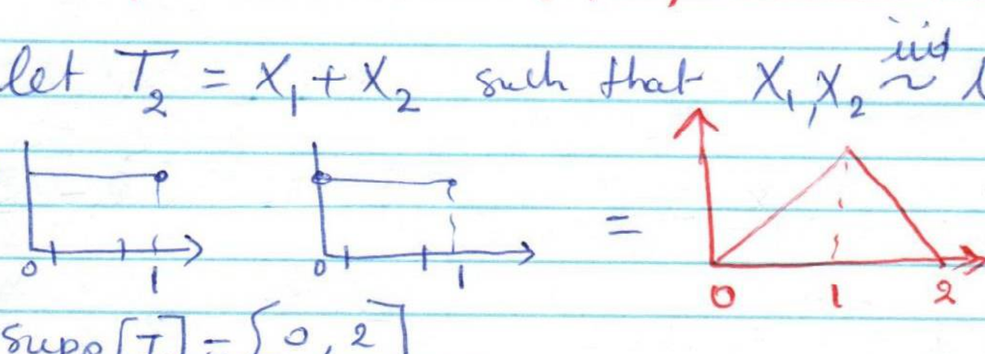
$b > a$

$\text{Supp}[X] = [a,b]$




$a=0, b=1 \Rightarrow X \sim (0,1) := 1$ called standard uniform

let $T_2 = X_1 + X_2$ such that $X_1, X_2 \overset{iid}{\sim} \mathcal{U}(0,1)$




$\text{supp}[T] = [0, 2]$

How often does $T = 0$? $X_1 = 0, X_2 = 0 \Rightarrow$ rare

" " " $T = 2$? $X_1 = 1, X_2 = 1 \Rightarrow$ rare

" " " $T = 1$? $X_1 = 0, X_2 = 1$ or $X_1 = \frac{1}{3}, X_2 = \frac{2}{3}$ common

$$f_T(t) = \int_{x \in \text{Supp}[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{t-x \in [0,1]}\, dx$$

$t - x \in [0,1]$

$x - t \in [-1, 0]$

$x \in [t-1, t]$

$$= \int_0^1 \mathbb{1}_{x \in [t-1, t]} \, dx = \int_{\max\{0, t-1\}}^{\min[1,t]} (1)\, dx =$$

$$\left(\min[1,t] - \max[0, t-1]\right) \mathbb{1}_{t \in [0,2]}$$

$$= \left\{ \begin{array}{ll} t & \text{if } t < 1 \\ \underbrace{1-(t-1)}_{2-t} & \text{if } t \geq 1 \end{array} \right\} \mathbb{1}_{t \in [0,2]}$$