Let $\vec{X} \sim \text{Multinorm}(n, \vec{p}) := \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$ where k is the number of categories to choose from.

$$\dim[X] = k$$

There is no indicator function since multichoose is 0 unless

$$\sum_{i=1}^{k} x_i = n \text{ and } \forall x_i \in \mathbb{N}_0$$

$$\text{Supp}[\vec{X}] = \{\vec{x} : \vec{1} \cdot \vec{x} = n \text{ and } \vec{x} \in \mathbb{N}_0^k\}$$

$$\text{Parameter Space } : \vec{p} \in \{\vec{p} : \vec{p} \in (0,1)^k \text{ and } \vec{p} \cdot \vec{1} = 1\}$$

What's the probability of getting 3 apples, 2 bananas and 5 cantaloupes if $p_A = \frac{1}{4}$, $p_B = \frac{1}{8}$ and $p_C = \frac{5}{8}$?

$$\mathbb{P}(\vec{X} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}) = \binom{10}{3, 2, 5} \left(\frac{1}{4}\right)^3 \left(\frac{1}{8}\right)^2 \left(\frac{5}{8}\right)^5$$

Let $k = 2$, then $\vec{p} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$. Thus

$$p(\vec{x}) = \mathbb{P}(x_1, x_2) = \text{Multinorm}(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

This is not binomial ($\text{Bin}(n, p)$).
Is $X_1, X_2 \overset{iid}{\sim}$? If so,

$$\mathbb{P}(x_1, x_2) = \mathbb{P}(x_1)\mathbb{P}(x_2) \to \mathbb{P}(x_1 \mid x_2) = \mathbb{P}(x_1) \text{ or } \mathbb{P}(x_2 \mid x_1) = \mathbb{P}(x_2)$$

This is true since $\forall x_1 \in \text{Supp}[X_1]$ and $\forall x_2 \in \text{Supp}[X_2]$,

$$\begin{aligned} \mathbb{P}(X_1 \mid X_2) &= \frac{\mathbb{P}(X_1, X_2)}{\mathbb{P}(X_2)} \overset{\text{if iid}}{=} \frac{\mathbb{P}(X_1)\mathbb{P}(X_2)}{\mathbb{P}(X_2)} &&= \mathbb{P}(X_1) \\ \mathbb{P}(X_2 \mid X_1) &= &&= \mathbb{P}(X_2) \end{aligned}$$

Thus are $X_1, X_2 \overset{iid}{\sim}$? No. If you know x_2, then $x_1 = n - x_2$.

They are dependent on one another.

$$\begin{aligned}
\mathbb{P}(X_1 \mid X_2) &= \frac{\mathbb{P}(X_1, X_2)}{\mathbb{P}(X_2)} \\
\mathbb{P}(X_2) &= \sum_{x_1 \in \mathrm{Supp}[X_1]} \mathbb{P}(X_1, X_2) \\
&= \sum_{x_1=0}^{n} \frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1+x_2=n} \\
&= \frac{n!}{x_2!}(1-p)^{x_2} \qquad \underbrace{\sum_{x_1=0}^{n} \frac{p^{x_1}}{x_1!} \mathbb{1}_{x_1=n-x_2}}_{\text{this is all zero except when } x_1=n-x_2} \\
&= \frac{n!}{x_2!}(1-p)^{x_2} \frac{p^{n-x_2}}{(n-x_2)!} \\
&= \binom{n}{x_2}(1-p)^{x_2} p^{n-x_2} \\
X_2 &\sim \mathrm{Binom}(n, 1-p) \\
X_1 &\sim \mathrm{Binom}(n, p)
\end{aligned}$$

This shows that the marginal distribution is a binomial distribution as well.

$$\mathbb{P}(X_1 \mid X_2) = \frac{\frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}}{\frac{n!}{x_2!(n-x_2)!}(1-p)^{x_2} p^{n-x_2}} = \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

The indicator function is 0 unless $x_1 = n - x_2$. Thus

$$\mathbb{P}(x_1 = n - x_2 \mid x_2) = \frac{(n-x_2)!}{(n-x_2)!} p^0 = 1$$

This is not the same as $\mathrm{Binom}(n, p)$.

Let $X \sim \text{Multinorm}(n, \vec{p})$. Then

$$
\begin{aligned}
\mathbb{P}(X_{-j} \mid X_j) &= \frac{\mathbb{P}(X_1, \dots, X_k)}{\mathbb{P}(X_j)} = \frac{\text{Multinorm}(n, \vec{p})}{\text{Binom}(n, p_j)} \\
&= \frac{\frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k}}{\frac{n!}{x_j!(n-x_j)!} p_j^{x_j} (1-p_j)^{n-x_j}} \\
&= \frac{(n-x_j)!}{x_1! \dots x_{j-1}! x_{j+1}! \dots x_k!} \frac{p_1^{x_1} \dots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \dots p_k^{x_k}}{(1-p_j)^{n-x_j}} \\
&\text{Let } n' = n - x_j \\
&\text{Then } \sum_{j=1}^{k} x_j = n \to x_1 + \dots + x_{j-1} + x_j + x_{j+1} + \dots + x_k = n \\
&\to x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k = n - x_j = n' \\
&= \binom{n'}{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k} \frac{p_1^{x_1} \dots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \dots p_k^{x_k}}{(1-p_j)^{n'}} \\
&\text{Let } p'_1 = \frac{p_1}{1-p_j}, p'_2 = \frac{p_2}{1-p_j}, \dots, p'_k = \frac{p_k}{1-p_j} \to p' = \begin{bmatrix} p'_1 \\ \dots \\ p'_k \end{bmatrix} \\
&= \binom{n'}{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k} \\
&\cdot \frac{(p'_1(1-p_1))^{x_1} \dots (p'_{j-1}(1-p_{j-1}))^{x_{j-1}} (p'_{j+1}(1-p_{j+1}))^{x_{j+1}} \dots (p'_k(1-p_k))^{x_k}}{(1-p_j)^{n'}} \\
&= \binom{n'}{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k} \\
&\cdot \frac{(p'_1)^{x_1} \dots (p'_{j-1})^{x_{j-1}} (p'_{j+1})^{x_{j+1}} \dots (p_k)^{x_k} (1-p_j)^{\overbrace{x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k}^{n'}}}{(1-p_j)^{n'}} \\
&= \text{Multinorm}(n', p')
\end{aligned}
$$

Recall that $\mathrm{E}[g(X_1,\dots,X_n)] = \sum_{x_1\in \mathrm{Supp}[X_1]}\cdots\sum_{x_n\in \mathrm{Supp}[X_n]} g(x_1,\dots,x_n)\mathbb{P}(x_1,\dots,x_n)$.

$$\begin{aligned}
\mathrm{E}[aX] &= a\mathrm{E}[X] \\
\mathrm{E}[X+c] &= \mathrm{E}[X]+c \\
\mathrm{E}[\sum X_i] &= \sum_{i=1}^{n}\mathrm{E}[X_i] = n\mu \\
\mathrm{E}[\prod_{i=1}^{n} X_i] &= \prod_{i=1}^{n}\mathrm{E}[X_i] = \mu^n \text{ if } X_1,\dots,X_n \overset{iid}{\sim} \\
\sigma^2 = \mathrm{Var}[X] &= \mathrm{E}[\underbrace{(X-\mu)^2}_{g(x)}] = \sum_{x\in x\in \mathrm{Supp}[X]} g(x)\mathbb{P}(x) \\
&= \sum (x-\mu)^2 p(x) \\
&= \sum x^2 p(x) + \sum(-2X\mu)p(x) + \sum \mu^2 p(x) \\
&= \mathrm{E}[X^2] - 2\mu^2 + \mu^2 = \mathrm{E}[X^2]-\mu^2 \\
\mathrm{Var}[X+c] &= \mathrm{Var}[X] \\
\mathrm{Var}[cX] &= c^2\mathrm{Var}[X] \\
\mathrm{Var}[X_1+X_2] &= \mathrm{E}[((X_1+X_2)-(\mu_1+\mu_2))^2] \\
&= \mathrm{E}[X_1^2+X_2^2+\mu_1^2+\mu_2^2-2\mu_1X_1-2\mu_1X_2-2\mu_2X_1-2\mu_2X_2+2X_1X_2+2\mu_1\mu_2] \\
&= \mathrm{E}[X_1^2]+\mathrm{E}[X_2^2]+\mu_1^2+\mu_2^2-2\mu_1^2-2\mu_1\mu_2-2\mu_1\mu_2-2\mu_2^2+2\mathrm{E}[X_1X_2]+2\mu_1\mu_2 \\
&= \mathrm{Var}[X_1]+\mathrm{Var}[X_2]+2(\mathrm{E}[X_1X_2]-\mu_1\mu_2)
\end{aligned}$$

Define covariance as follows:

$$\mathrm{Cov}[X_1,X_2] = \mathrm{E}[X_1X_2]-\mu_1\mu_2$$

In fact,

$$\mathrm{Corr}[X_1,X_2] = \frac{\mathrm{Cov}[X_1,X_2]}{\mathrm{SE}[X_1]\mathrm{SE}[X_2]} \in [-1,1]$$

$$\begin{aligned}
\mathrm{Cov}[X,X] &= \mathrm{Var}[X] \\
\mathrm{Cov}[aX_1,bX_2] &= ab\mathrm{Cov}[X_1,X_2] \\
\mathrm{Cov}[X_1+c,X_2+d] &= \mathrm{Cov}[X_1,X_2] \\
\mathrm{Cov}[X_2,X_1] &= \mathrm{Cov}[X_1,X_2] \\
\mathrm{Cov}[X+Y,Z] &= \mathrm{E}[(X+Y-\mu_X-\mu_Y)(Z-\mu_Z)] \\
&= \mathrm{E}[((X-\mu_X)+(Y-\mu_Y))(Z-\mu_Z)] \\
&= \mathrm{E}[(X-\mu_X)(Z-\mu_Z)+(Y-\mu_Y)(Z-\mu_Z)] \\
&= \mathrm{Cov}[X,Z]+\mathrm{Cov}[Y,Z]
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{Var}[X_1 + X_2] &= \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + 2\mathrm{Cov}[X_1, X_2] \\
&= \mathrm{Cov}[X_1, X_1] + \mathrm{Cov}[X_2, X_2] + \mathrm{Cov}[X_1, X_2] + \mathrm{Cov}[X_2, X_1] \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} \mathrm{Cov}[X_i, X_j] \\
\mathrm{Var}[X_1 + X_2 + \cdots + X_k] &= \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Cov}[X_i, X_k]
\end{aligned}$$

If $\vec{X}$ is a vector of random variables of dim k,

$$\mathrm{E}[\vec{X}] = \mathrm{E}[\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_k \end{bmatrix}] = \begin{bmatrix} \mathrm{E}[X_1] \\ \mathrm{E}[X_2] \\ \dots \\ \mathrm{E}[X_k] \end{bmatrix}$$

Furthermore,

$$\mathrm{Var}[\vec{X}] = \begin{bmatrix} \overbrace{\mathrm{Var}[X_1]}^{\mathrm{Cov}[X_1, X_1]} & \mathrm{Cov}[X_1, X_2] & \dots & \dots \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \mathrm{Var}[X_k] \end{bmatrix}$$

This is a symmetric $k \times k$ matrix defined by

$$\mathrm{Cov}[X_i, X_j] \;\; \forall i = 1, \dots, k \text{ and } j = 1, \dots, k$$