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Math 621 Lec17 11/6/17

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$$\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n$$

$$\sqrt{\sum Z_i^2} \sim \chi_n$$

$Z \sim N(0,1)$

let $Y = |Z| \sim ?$ $= \sqrt{Z^2}$

makes sense

$$Y^2 \sim \chi^2_1 \Rightarrow \sqrt{Y^2} \sim \chi_1 \quad \sqrt{Z^2} = |Z| \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} = 2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$X \sim \chi^2_k$ let $Y = \dfrac{X}{k}$

Since $\chi^2_k = \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$, we need to figure out the scaled Gamma

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$$X \sim \text{Gamma}(\alpha, \beta), \; Y = cX \sim \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\frac{\beta y}{c}}}{c\,\Gamma(\alpha)}$$

$$= \frac{\beta^\alpha y^{\alpha-1} e^{-\frac{\beta}{c}y}}{c^\alpha \Gamma(\alpha)} = \frac{\left(\frac{\beta}{c}\right)^\alpha y^{\alpha-1} e^{-\frac{\beta}{c}y}}{\Gamma(\alpha)} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$$\Rightarrow X \sim \chi^2_k, \; Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$$

$$\Rightarrow \quad X \sim \underset{\substack{\parallel \\ \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)}}{\chi^2_k}, \quad Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right) = \frac{\left(\frac{k}{2}\right)^{\frac{k}{2}} y^{\frac{k}{2}-1} e^{-\frac{k}{2}y}}{\Gamma\left(\frac{k}{2}\right)}$$

$X_1 \sim \chi^2_{k_1}$ ind of $X_2 \sim \chi^2_{k_2}$

$$R = \frac{X_1/k_1}{X_2/k_2} \sim ? \qquad \text{Supp}(R) = (0, \infty)$$

ratio of $\text{Gamma}\left(\frac{k_1}{2}, \frac{k_1}{2}\right)$ to $\text{Gamma}\left(\frac{k_2}{2}, \frac{k_2}{2}\right)$ both independent and positive...

Recall if $R = \dfrac{V_1}{V_2} \sim \displaystyle\int_{\text{Supp}[V_2]} t\, f_{V_1}(rt)\, f_{V_2}(t)\, dt$ which was proved using the Jacobian multivariate C.O.V formula

let $a = \frac{k_1}{2} \Rightarrow V_1 \sim \dfrac{a^a x^{a-1} e^{-ax}}{\Gamma(a)}$ let $b = \frac{k_2}{2} \Rightarrow V_2 \sim \dfrac{b^b x^{b-1} e^{-bx}}{\Gamma(b)}$

$$R \sim \int_0^\infty t\, \frac{a^a (rt)^{a-1} e^{-art}}{\Gamma(a)} \frac{b^b t^{b-1} e^{-bt}}{\Gamma(b)}\, dt$$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\, dt \qquad \text{let } u = (ar+b)t \Rightarrow t = \frac{1}{ar+b} u \Rightarrow dt = \frac{1}{ar+b} du$$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty \frac{u^{a+b-1}}{(ar+b)^{a+b-1}} e^{-u} \frac{1}{ar+b}\, du = \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b}} \overbrace{\int_0^\infty u^{a+b-1} e^{-u}\, du}^{\Gamma(a+b)}$$

$$= \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\substack{\parallel \\ b^{-(a+b)}\left(1+\frac{a}{b}r\right)^{-(a+b)} \\ \parallel \\ b^{-a}\cancel{b^{-b}}}} = \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)} = \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\left(\frac{k_1+k_2}{2}\right)}$$

$$= F(k_1, k_2)$$

the two parameters are called "d.o.f." - terminology from lin. mod. theory

The "F distribution", "Fisher-Snedecor" distribution "F" for "Fisher", comes up all over statistics especially when testing effects in linear models (next semester's class). k_1, k_2 here are $\in \mathbb{N}$ but the distr. is defined for $k_1, k_2 \in (0, \infty)$ due to the gamma function

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Consider $Z \sim N(0,1)$, indep. of $V \sim \chi^2_k$, let $W = \dfrac{Z}{\sqrt{\frac{V}{k}}} \sim ?$ Strange fraction... but very important (after midterm we will see why)

$Supp(W) = \mathbb{R}$

Consider $W^2 = \dfrac{Z^2}{\frac{V}{k}}$, $Z^2 \sim \chi^2_1 \Rightarrow \dfrac{Z^2}{1} \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

$\dfrac{V}{k} \sim Gamma\left(\frac{k}{2}, \frac{k}{2}\right)$

$$\Rightarrow W^2 \sim F(1,k) = \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} w^{-\frac{1}{2}} \left(1 + \frac{1}{k} w\right)^{-\frac{1+k}{2}} = \frac{1}{\sqrt{k}\, B\left(\frac{1}{2}, \frac{k}{2}\right)} w^{-\frac{1}{2}} \left(1 + \frac{w}{k}\right)^{-\frac{k+1}{2}}$$

So to get distr of W, we need to find sqrt of $F(1,k)$ and cond to $\mathbb{R}$

Why?

$X \sim F(1,k)$, $Y = \pm\sqrt{X} \Rightarrow X = g^{-1}(y) = y^2 \Rightarrow \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = 2y$

$$f_Y(y) = f_X(y^2)\, 2y = \frac{1}{\sqrt{k}\, B\left(\frac{1}{2}, \frac{k}{2}\right)} \underbrace{(y^2)^{-\frac{1}{2}}}_{\frac{1}{y}} \left(1 + \frac{y^2}{k}\right)^{-\frac{k+1}{2}} 2y$$

not simple 1:1 function!

$X \sim F(1,k)$, $Y = \pm\sqrt{X} \Rightarrow Y$ is symmetric around 0

$$F_Y(y) - F_Y(-y) = P(Y \in [-y, y]) = P(Y^2 \le y^2) = P(X \le y^2) = F_X(y^2)$$

take $\frac{d}{dy}$ both sides

$$\frac{d}{dy}\left[F_Y(y) - F_Y(-y)\right] = \frac{d}{dy}\left[F_X(y^2)\right] \Rightarrow f_Y(y) - -f_Y(-y) = f_X(y^2)\, 2y$$

$$\Rightarrow 2 f_Y(y) = f_X(y^2)\, 2y \Rightarrow f_Y(y) = f_X(y^2)\, y$$

$$f_Y(y) = f_V(y^2)\,y = \frac{1}{\sqrt{K}\,B\left(\frac{1}{2},\frac{K}{2}\right)} \cancel{(y^2)^{-\frac{1}{2}}} \left(1+\frac{y^2}{K}\right)^{-\frac{K+1}{2}} \cancel{y}$$

$$\cancel{\frac{1}{y}}$$

$$= \frac{1}{\sqrt{K}\,B\left(\frac{1}{2},\frac{K}{2}\right)} \left(1+\frac{y^2}{K}\right)^{-\frac{K+1}{2}} = T_K \Rightarrow$$

$Z \sim N(0,1)$ ind. of $V \sim \chi^2_K$

$$\frac{Z}{\sqrt{\frac{V}{K}}} \sim T_K$$

Student's T distribution w/ K d.o.f. (the parameter)

Story about Student's T...

$$\frac{V}{K} = \frac{\sum_{i=1}^{K} Z_i^2}{K} \sim N\left(1, \frac{2}{K}\right) \xrightarrow{d} \{1 \text{ w.p. } 1$$

$E[\chi^2_K] = K$

$Var(\chi^2_K) = 2K$

If $V \sim T_K$ what is $\lim_{K\to\infty} V$?

$$\lim_{K\to\infty} \frac{1}{\sqrt{K}\,B\left(\frac{1}{2},\frac{K}{2}\right)} \left(1+\frac{y^2}{K}\right)^{-\frac{K+1}{2}} = \underbrace{\lim_{K\to\infty} \frac{1}{\sqrt{K}\,B\left(\frac{1}{2},\frac{K}{2}\right)}} \underbrace{\left(\lim_{K\to\infty}\left(1+\frac{y^2}{K}\right)^{K}\right)^{-\frac{1}{2}}} \cancel{\lim_{K\to\infty}\left(1+\frac{y^2}{K}\right)^{-\frac{1}{2}}}^{\,1}$$

$$(e^{y^2})^{-\frac{1}{2}} = e^{-\frac{y^2}{2}}$$

$$\lim \frac{1}{\sqrt{K}\,B\left(\frac{1}{2},\frac{K}{2}\right)} = \lim \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{K}{2}\right)} = \frac{1}{\sqrt{\pi}} \lim \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K}\,\Gamma\left(\frac{K}{2}\right)}$$

$$\Rightarrow = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = N(0,1)$$

$$\boxed{\lim_{K\to\infty} T_K = N(0,1)}$$

Stirling Approx

If n goes large $\Gamma(n) \approx \sqrt{2\pi(n-1)} \left(\frac{n-1}{e}\right)^{n-1}$

Note $\frac{K+1}{2} - 1 = \frac{K-1}{2}$, $\frac{K}{2} - 1 = \frac{K-2}{2}$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \lim \frac{\sqrt{2\pi\left(\frac{K-1}{2}\right)} \left(\frac{K-1}{2e}\right)^{\frac{K-1}{2}}}{\sqrt{K}\sqrt{2\pi\left(\frac{K-2}{2}\right)} \left(\frac{K-2}{2e}\right)^{\frac{K-2}{2}}} = \frac{1}{\sqrt{\pi}} \lim \frac{\sqrt{K-1}}{\sqrt{K}\sqrt{K-2}} \frac{(K-1)^{\frac{K}{2}} (K-1)^{-\frac{1}{2}} (2e)^{\frac{K}{2}-1-\frac{K}{2}+\frac{1}{2}}}{(K-2)^{\frac{K}{2}} (K-2)^{-1}}$$

$$= \frac{1}{\sqrt{\pi}} \lim_{K\to\infty} \sqrt{\frac{K-2}{K}} \left(\frac{K-1}{K-2}\right)^{\frac{K}{2}} (2e)^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{e}} \underbrace{\lim_{K\to\infty} \sqrt{1-\frac{2}{K}}} \left(\lim_{K\to\infty} \left(\frac{K-1}{K-2}\right)^{K}\right)^{\frac{1}{2}}$$

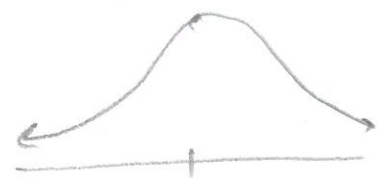
let $l = K-2 \Rightarrow K = l+2$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{e}} \left(\lim_{l\to\infty} \left(\frac{l+1}{l}\right)^{l+2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{e}} \left(\underbrace{\lim_{l\to\infty}\left(1+\frac{1}{l}\right)^{l}}_{e} \underbrace{\lim_{l\to\infty}\left(1+\frac{1}{l}\right)^{2}}_{1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{e}} \sqrt{e} = \frac{1}{\sqrt{2\pi}}$$

$X_1 \sim N(0,1)$ ind. of $X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} \sim ?$$



well... $Z \sim N(0,1)$ ind. of $V \sim \chi^2_K$

$$\Rightarrow \frac{Z}{\sqrt{\frac{V}{K}}} \sim T_K \qquad \text{Note } X_2^2 \sim \chi^2_1$$

$$\frac{X_1}{X_2} = \frac{X_1}{\sqrt{\frac{X_2^2}{1}}} \sim T_1 = \frac{\overbrace{\Gamma\left(\frac{1+1}{2}\right)}^{=\Gamma(1)=1}}{\sqrt{(1)\pi}\,\underbrace{\Gamma\left(\frac{1}{2}\right)}_{=\sqrt{\pi}}}\left(1+\frac{x^2}{1}\right)^{-\frac{1+1}{2}} = \frac{1}{\pi}\frac{1}{1+x^2} = \text{Cauchy}(0,1)$$

special case of the T dist.

$$X \sim \text{Cauchy}(0,1),\quad Y = c + \sigma X \sim \frac{1}{\sigma} f_X\left(\frac{y-c}{\sigma}\right) = \frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{y-c}{\sigma}\right)^2} = \text{Cauchy}(c,\sigma)$$

$$E(X) = \int_{\mathbb{R}} x \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{1+x^2}\,dx = \frac{1}{\pi}\left[\frac{1}{2}\ln(x^2+1)\right]_{-\infty}^{\infty} = \infty \quad \text{i.e. doesn't exist}$$

$$Var(X) = E\big[(\underbrace{X-\mu}_{\infty})^2\big] = \infty \quad \text{... no moments exist!}$$

$$M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = \infty \;\; \forall t \quad \text{since } e^x \text{ grows faster than } x^2$$

no m.g.f.

ch.f. difficult to prove here $\phi_X(t) = e^{-|t|}$ $\quad \phi'_X(t) = \frac{-t}{|t|}e^{-|t|}$ $\quad \phi'_X(0)$ dne.

(physical) 6

It is also known as the Lorentz distr. Why? Imagine you have a source of light at $y=1$ above the origin and it shines light equally in all directions. What does the light density look like on the x-axis?

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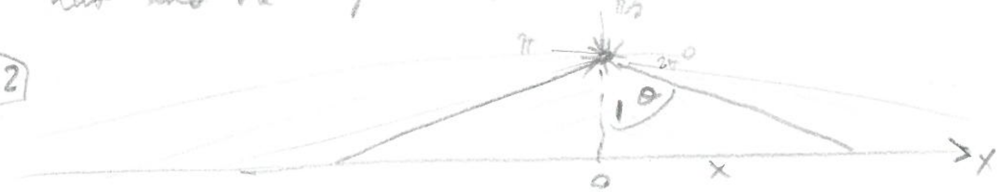

Obviously light extends to all $x \in \mathbb{R}$. If it shines uniformly ... [illegible]

Ex light shines $\theta \sim U(\pi, 2\pi) = \frac{1}{\pi}$

$\tan(\theta) = \frac{x}{1}$ $\quad x = \tan(\theta) = g(\theta) \quad \theta = \arctan(x) = g^{-1}(x) \quad \frac{d}{dx}\left[g^{-1}(x)\right] = \frac{1}{1+x^2}$

$$f_X(x) = f_\theta(g^{-1}(x)) \frac{d}{dx}\left[g^{-1}(x)\right] = \frac{1}{\pi}\frac{1}{1+x^2}$$

Proof of Cauchy using ratio

$$R = \frac{X_1}{X_2} \sim \int_{Supp[X_2]} |x_2| f_{X_1}(x_2 r) f_{X_2}(x_2)\, dx_2$$

$$= \int_{\mathbb{R}} |x_2| \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2 r^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}\, dx_2$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} |x_2| e^{-\frac{1}{2}x_2^2(r^2+1)}\, dx_2 = \frac{1}{2\pi}\left(\int_{-\infty}^{0} -x_2\, e^{-\frac{1}{2}x_2^2(r^2+1)}\, dx + \int_0^\infty x_2\, e^{-\frac{1}{2}x_2^2(r^2+1)}\, dx\right)$$

$$= \frac{1}{2\pi} \int_0^\infty \text{—}\, dx - \int_{-\infty}^0 \text{—}\, dx$$

let $u = -\frac{1}{2}x_2^2(r^2+1)$

$\frac{du}{dx_2} = -x_2(r^2+1) \Rightarrow dx = -\frac{1}{x_2(r^2+1)}$

$$\int \cancel{x_2} e^u \cdot -\frac{1}{\cancel{x_2}(r^2+1)}\, du = -\frac{1}{r^2+1}\int e^u = -\frac{1}{r^2+1} e^u$$

$x_0 = 0 \Rightarrow u = 0$
$x_0 = \infty \Rightarrow u = -\infty$
$x_0 = -\infty \Rightarrow u = -\infty$

$$= \frac{1}{2\pi}\left(-\frac{1}{r^2+1}\right)\left(\left[e^u\right]_0^{-\infty} - \left[e^u\right]_{-\infty}^0\right) = -\frac{1}{2\pi}\frac{1}{r^2+1}(-2) = \frac{1}{\pi}\frac{1}{r^2+1}$$

$(0-1) - (1-0)$