Lecture 12 10/19/17

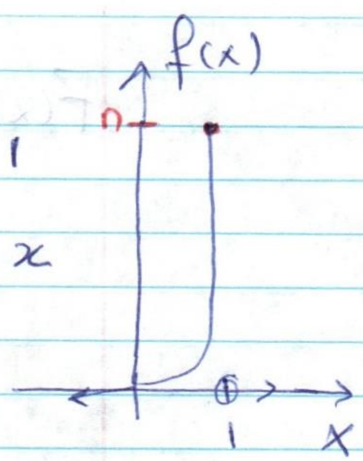
Order Statistics

$X_1, \dots X_n \overset{iid}{\sim} f(x)$ (continuous) last time

Today

$X_1 \dots X_n \overset{iid}{\sim} U(0,1) \Rightarrow f(x) = 1$

$\Rightarrow F(x) = x$



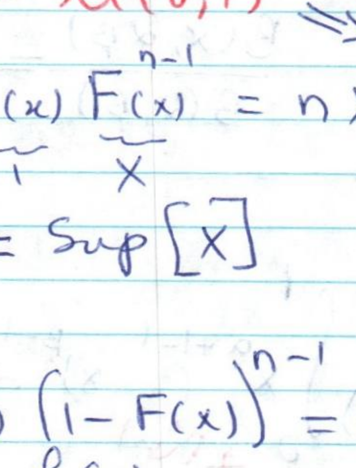
$$f_{X_n}(x) = n \underbrace{f(x)}_{1} \underbrace{F(x)}_{x}^{n-1} = n x^{n-1}$$

$$\text{Supp}[X_{(k)}] = \text{Sup}[X]$$

$$f_{X_{(1)}} = n f(x) (1 - F(x))^{n-1} = n (1-x)^{n-1}$$

$\text{Sup}[x] = (0,1)$

For $n = 10$, $k = 5$



$X \sim \text{Beta}(\alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $\alpha > 0, \beta > 0$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \underbrace{f(x)}_{1} (\underbrace{F(x)}_{x})^{k-1} (1 - \underbrace{F(x)}_{x})^{n-k} \propto \underbrace{x^{k-1}(1-k)^{n-k}}_{k(x)}$$

$$f(x) = \frac{1}{c} k(x) \quad \int_{\text{Sup}[x]} k(x)\, dx = c \int_0^1 x^{k-1} (1-x)^{n-k+1-1}\, dx = \beta(k, n-k+1)$$

$$f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} x^{k-1}(1-x)^{n-k} = \text{Beta}(k, n-k+1)$$

$$\int_{Sup[x]} f(x) = 1 \ ?? \quad \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{B(\alpha,\beta)} \frac{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}{B(\alpha,\beta)} = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\overbrace{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt}^{\text{incomplete beta Funct}}}{B(\alpha,\beta)} =$$

$$\frac{B(x,\alpha\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$

regularized incomplete beta Function

$$E(x) = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\dfrac{\overbrace{\Gamma(\alpha+1)}^{=\alpha\Gamma(\alpha)}\Gamma(\beta)}{\Gamma(\alpha+\beta+1) = (\alpha+\beta)\Gamma(\alpha+\beta)}}{\dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

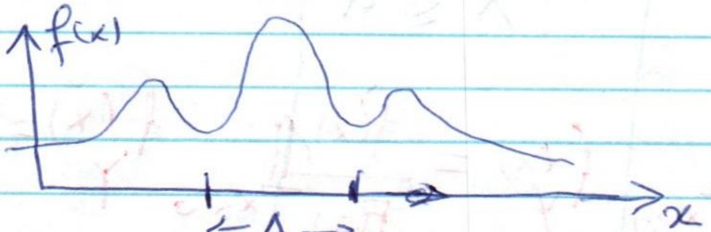
$$= \frac{\alpha}{\alpha+\beta}$$

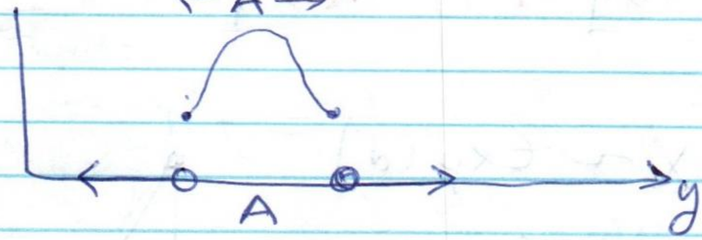
Truncations

$X \sim f(x)$

what if we know $X \in A$ where $A \subseteq \text{support}[X]$

Call this condi distribution Y. $f_Y(y) = ?$

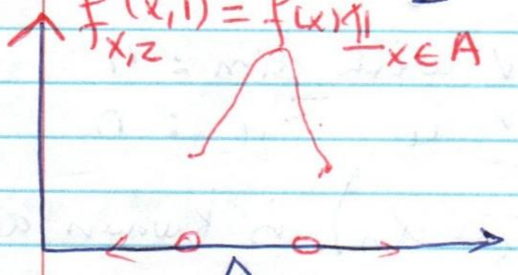




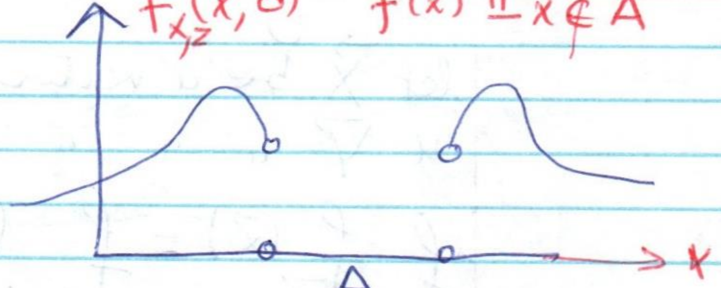
$$Z = \mathbb{1}_{X \in A} \sim \text{Bern}(p(X \in A))$$

$$f_{X|Z} = \frac{f_{X,Z}(x,z)}{p_Z(z)}$$

$f_{X,Z}(x,1) = f(x)\mathbb{1}_{x \in A}$



$f_{X,Z}(x,0) = f(x)\mathbb{1}_{x \notin A}$



$$f_{X,Z}(x,z) = f(x)\mathbb{1}_{x \in A}^{z}\,\mathbb{1}_{x \notin A}^{1-z}$$

$$f_{X/Z}(x,z) = \frac{f(x)\,\mathbb{1}_{x \in A}^{z}\,\mathbb{1}_{x \notin A}^{1-z}}{P(x \in A)^{z}\,(1 - p(x \in A))^{1-z}}$$

$Y = X/Z = 1$

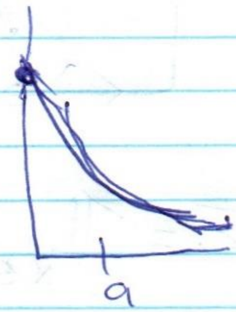
$f_Y(x) = f_{X/Z}(x,1) = \frac{f(x)}{P(X\in A)} \mathbb{1}_{x\in A}$

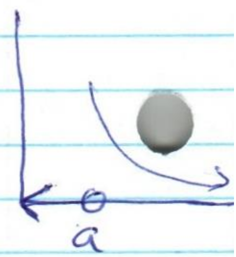
$X \geq a$	$X \leq a$	$X \in (a,b)$
$f_Y(x) = \frac{f(x)}{1-F(a)} \mathbb{1}_{x\geq a}$	$f_Y(x) = \frac{f(x)}{F(a)} \mathbb{1}_{x\leq 0}$	$f_Y(x) = \frac{f(x)}{F(b)-F(b)} \mathbb{1}_{x\in(a,b)}$

<u>Example</u> $X \sim \text{Exp}(\lambda)$

we know $X \geq a$

$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}} \mathbb{1}_{x\geq a}$





let $g: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ g is 1:1

let $\vec{X}$ be a vector r.v with dim : n

" $\vec{Y}$ " " " rv " " : n

if $f_{\vec{X}}(\vec{x}) = f_{X_1,\dots X_n}(x_1,\dots x_n)$ is known and

$\vec{Y} = g(\vec{X})$ find $f_{\vec{Y}}(\vec{y}) = f_{Y_1 \dots Y_n}(y_1, \dots y_n)$

$$Y_1 = g_1(X_1 \ldots X_n)$$
$$Y_2 = g_2(X_1, \ldots X_n)$$
$$\vdots$$
$$Y_n = g_n(X_1, \ldots, X_n)$$

Since g is 1:1, $\exists\, h_1 \ldots h_n$ where

$$X_1 = h_1(Y_1, \ldots, Y_n)$$
$$X_2 = h_2(Y_1, \ldots, Y_n)$$
$$\vdots$$
$$X_n = h_n(Y_1, \ldots Y_n)$$

$$f_{Y_1, \ldots Y_n}(y_1, \ldots y_n) = f_{X_1, \ldots X_n}(h_1(y_1 \ldots y_n), \ldots h_n(y_1 \ldots y_n))$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right| \qquad |J_h(y_1, \ldots y_n)|$$

Jacobian

$$J_h = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_n} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}\right)$$

one-dim case

$$J_h = \det\left(\left[\frac{\partial g^{-1}(y)}{\partial y}\right]\right) = \frac{\partial g^{-1}(y)}{\partial y}$$

$$f_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = f_{X_1, \ldots, X_n}(h_1(y_1 \ldots y_n) \ldots h_n(y_1, \ldots, y_n)) \, |J_h(y_1 \ldots y_n)|$$

given $X_1, X_2 \sim$ and iid

$$Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$$

See p(150-151)

$$Y_2 = X_2 = g_2(X_1, X_2)$$

$$X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$X_2 = Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = Y_2, \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = 0, \frac{\partial h_2}{\partial y_2} = 1$$

$$J_h = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}\right) = y_2 \cdot 1 - 0 y_1$$
$$= y_2$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2) |y_2| \Rightarrow$$

$$f_{Y_1}(y_1) = \int_{\substack{Sup[Y_2] \\ \parallel \\ Sup[X_2]}} f_{X_1, X_2}(y_1 y_2, y_2) |y_2| \, dy_2$$

If $X_1\, X_2$ are independent and positive

$$f_Y(y) = \int_{Sup[X_2]} x_2 f_{X_1}(y_1, x_2) f_{X_2}(x_2)\, dx_2$$

Example given X_1, X_2 and ii jdf

$$Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2) \quad X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$Y_2 = X_1 + X_2 = g_2(X_1, X_2) \quad X_2 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2, \quad \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = -y_2, \quad \frac{\partial h_2}{\partial y_2} = 1 - y_1$$

$$J_h = \det\left(\begin{bmatrix} y_2, & y_1 \\ -y_2, & 1-y_1 \end{bmatrix}\right) = y_2(1-y_1) - y_1(-y_2) = y_2$$

$$J_h = y_2$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2(1-y_1))\,|y_2| \Rightarrow$$

$$f_{Y_1}(y_1) = \int_{\substack{Sup[Y_2] \\ = Sup[X_2]}} f_{X_1, X_2}(y_1 y_2, y_2(1-y_1))\,|y_2|\, dy_2$$

if X_1, X_2 are independent and positive

$$f_Y(y_1) = \int_{Sup[Y_2]} y_2 \, f_{X_1}(y_1 y_2) \, f_{X_2}(y_2(1-y_1)) \, y_2$$

Example

Let $X_1 \sim \text{Gamma}(\alpha, \lambda)$ independent of $X_2 \sim \text{Gamma}(\beta, \lambda)$

$$Y_1 = \frac{X_1}{X_1 + X_2} \sim ??$$

$Sup[Y_1] = (0, 1)$

$Sup[Y_2] = (0, \infty)$