Lec. 11 Oct. 17

$X \sim \text{Gamma}(k_1, \lambda)$

ind of $Y \sim \text{Gamma}(k_2, \lambda)$

$X+Y \sim \text{Gamma}(k_1+k_2$

convo

$$T = X+Y \searrow = \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1} (1-u)^{k_2-1} t \, du$$

$$= \frac{\lambda^{k_1+k_2} t^{k_1+k_2-1} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} du$$

$$\propto e^{-\lambda t} t^{k_1+k_2-1} \propto \text{Gamma}(k_1+k_2, \lambda)$$

pdf $f(x)$

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \propto e^{-\lambda x} = k(x) = f(x)$

"Kernel"

$$1 = \int_{\text{supp}[X]} f(x) dx \qquad \int_{\text{supp}(X)} k(x) dx = C \qquad \int \frac{1}{C} k(x) = 1$$

Some examples

$X \sim \mathrm{Bin}(n, p)$

$= \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!\,(n-x)!} p^x (1-p)^n (1-p)^{-x}$

$\propto (x!\,(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x$

$X \sim \mathrm{Weibull}(k, \lambda) = k\lambda(x\lambda)^{k-1} e^{-(x\lambda)^k} \propto x^{k-1} e^{-(x\lambda)^k}$

$X \sim \mathrm{Gamma}(k, \lambda) = \frac{\lambda^k e^{-\lambda k} x^{k-1}}{\Gamma(k)} \propto e^{\lambda x} x^{k-1}$

PDF of Gamma

$X \sim \mathrm{Gamma}(k_1 + k_2, \lambda) = \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1+k_2)}$

$= \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} \, du$

$$\Rightarrow \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du = \overset{\text{Beta}}{\overset{\downarrow}{B}}(k_1,k_2)$$

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$ Beta funct!

$\cancel{B(\alpha,\beta) = \text{Beta}}$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\int_0^\infty t^{\alpha-1}e^{-t}\,dt \cdot \int_0^\infty t^{\beta-1}e^{-t}\,dt}{\int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dt}$$

Page 160 (book) **Order Statistics**

$X_1, X_2, \ldots, X_n$ are sequence of cont. r.v.

✓ $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are order statistics
where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ sorted order

✓ $X_{min} = X_{(1)} = \min\{X_1, \ldots, X_n\}$

$2 = \min\{2, 7, 9, 12\}$

✓ $X_{max} = X_{(n)} = \max\{X_1, \ldots, X_n\}$

✓ $R = X_{max} - X_{min} = \text{Range}$

In the case of $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$
with CDF $F(x)$

✓ X_{max} = r.v. with the maxi survival

We want

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdot \ldots \cdot P(X_n \le x)$$

$$= F_X(x) \cdot \ldots \cdot F_{X_n}(x)$$

$$= F(x)^n \Longrightarrow$$

$$PDF = f_{X_{(n)}}(x) = F'_{X_n}(x) = n f(x) F(x)^{n-1}$$

Now Min

We want $F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$

$$= 1 - P(X_1 > x, \text{---}, X_n > x)$$

$$= 1 - P(X_1 > x) \cdot P(X_2 > x) \cdot \cdots \cdot P(X_n > x)$$

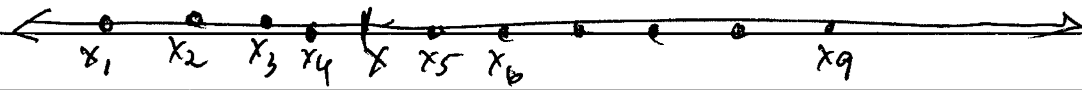
$$= 1 - (1 - F_X(x)) \cdots (1 - F_{X_1}(x))$$

$$= 1 - (1 - F(x))^n$$

$$pdf = f_{X_{(1)}}(x) = F'_{X_{(1)}}(x) = -n(1 - F(x))^{n-1}(-f(x))$$

$$= n f(x)(1 - F(x))^{n-1}$$

Goal $F_{X_n}(x)$ consider $n=10$



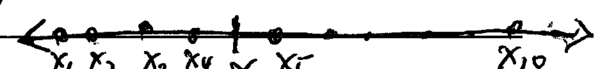
$P(X_1, X_2, X_3, X_4 \in (-\infty, x]$ AND $X_5, -- X_{10} \in (x,\infty))$

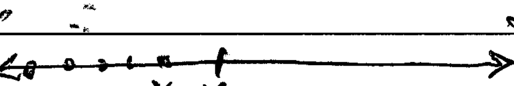
$= P(X_1 \le x) \cdot --- \cdot P(X_4 \le x)\, P(X_5 > x) \cdot --- \cdot P(X_{10} > x)$

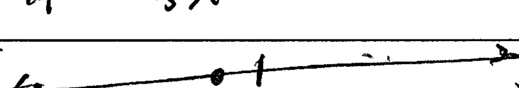
$= F(x)^4 (1 - F(x))^6$

P(any 4 are $\in (-\infty, x)$, the order 6 are $\in (x, \infty)$)

$= \binom{10}{x} F(x)^4 (1 - F(x))^6$

$F_{X_{(4)}}(x) = P(X_{(4)} \le x) = P($  $)$

$+ P($  $)$

$+ P($  $)$

$+$ --- --- --- ---

$+ P($  $)$

$$= \sum_{i=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$$

in general

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$F_{X_{(n)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$= \binom{n}{n} F(x)^n (1-F(x))^{n-n}$$

$$F_{X_1}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$= \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} \right) - (1-F(x))^n$$

$$= \left((F(x) * (1 - F(x))^n - (1 - F(x))^n \right)$$

$$1$$

PDF

$$f_{X_{(k)}}(x) = F'_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \frac{n!}{j!\,(n-j)!} \frac{d}{dx}\left[F(x)^j (1 - F(x))^{n-j}\right]$$

$$\left((1 - F(x))^{n-j} j F(x)^{j-1} f(x) - F(x)^j (n-j)(1 - F(x))^{n-j-1} f(x) \right)$$

$$= f(x) \left(\sum_{j=k}^{n} \frac{n!}{j!\,(n-1)!} (1 - F(x))^{n-1} j F(x)^{j-1} - \sum_{j=k+1}^{n} \frac{n!\, F(x)^j}{j!\,(n-j)!} (n-j)(1 \right.$$

ie $(a_k + \cancel{a_{k+1}} + \cdots + \cancel{a_n}) - (\cancel{a_{k+1}} + \cdots + \cancel{a_n}) = a_k$

$$= \frac{n!}{(k-1)!\,(n-k)!} f(x)\, F(x)^{k-1} (1 - F(x))^{n-k} = f_{(k)}(x)$$

$)^{n-j-1})$