Lect 23 Math 621 12/7/17

$X_n \xrightarrow{d} X$ means $\forall x \ \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ the CDF's converge pt-wise

we showed that $X_n \xrightarrow{d} X \iff \forall x \ \lim_{n\to\infty} p_{X_n}(x) = p_X(x)$

for discrete r.v.'s w/ support $\subset \mathbb{N}$.
Does the proof work for support $\subset \mathbb{Z}$?
I believe so.

let $X_n \sim \text{Binomial}\left(n, \frac{\lambda}{n}\right)$

$X_n \xrightarrow{d} X \sim \text{Poisson}(\lambda)$

---

let $X_n \sim \text{Geom}(n\lambda)$ $\quad Supp[X_n] = \{\frac{1}{n}, \frac{2}{n}, \ldots\}$

$X_n \xrightarrow{d} X \sim \text{Exp}(\lambda)$

$$X_n \sim \begin{cases} \frac{1}{n+1} & \text{wp } 1-p \\ 1-\frac{1}{n+1} & \text{wp } p \end{cases}$$

Prove $X_n \xrightarrow{d} X \sim \text{Bern}(p)$

$$\lim p_{X_n}(x) = \lim_{n\to\infty} (1-p)^{\mathbb{1}_{x=\frac{1}{n+1}}} (p)^{\mathbb{1}_{x=1-\frac{1}{n+1}}} \mathbb{1}_{x \in \{\frac{1}{n+1}, 1-\frac{1}{n+1}\}}$$

$$= p^{\mathbb{1}_{x=0}} (1-p)^{\mathbb{1}_{x=1}} \mathbb{1}_{x\in\{0,1\}} = \text{Bern}(p)$$

$X_n \sim \text{Binom}(n,p)$ let $Y_n = \frac{X_n - np}{\sqrt{np(1-p)}}$

$Y_n \xrightarrow{d} N(0,1)$ HARD PROOF

Consider...

$X_n \xrightarrow{d} c$ s.t. $c \in \mathbb{R}$, what is this? Recall $c \sim Deg(c)$

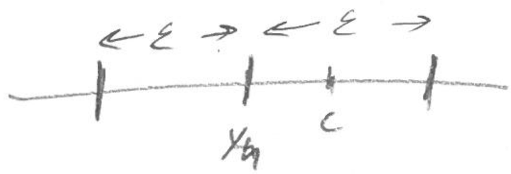
$$\Rightarrow \forall x, \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$$

(On HW)

(II) Convergence in Probability

X_n converges in prob. to a constant c denoted $X_n \xrightarrow{P} c$ iff

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$$



e.g. let $X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right)$

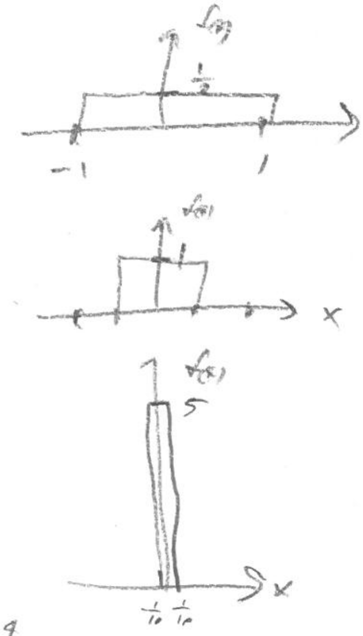
$\Rightarrow f_{X_n}(x) = \frac{n}{2}$

$\Rightarrow$

for $n=1$

for $n=2$

for $n=10$



Prove $X_n \xrightarrow{P} 0$

WTS $\lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} P(|X_n| \geq \varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} P(X_n < -\varepsilon) + P(X_n > \varepsilon) = 0$

$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n} - \varepsilon\right)\frac{n}{2} \mathbb{1}_{\varepsilon < \frac{1}{n}} + \left(\frac{1}{n} - \varepsilon\right)\frac{n}{2} \mathbb{1}_{\varepsilon < \frac{1}{n}} = 0$

$\Rightarrow \lim_{n\to\infty} (1 - \varepsilon n) \mathbb{1}_{\varepsilon < \frac{1}{n}} = 0$

the prob. cannot escape a small region!

Consider $X_1, X_2, \ldots \overset{iid}{\sim}$ w/ mean μ and variance σ^2.

Define $\overline{X}_n = \frac{X_1 + \ldots + X_n}{n}$

Consider $\overline{X}_1, \overline{X}_2, \overline{X}_3, \ldots$ mean? all μ. Variance? $\frac{\sigma^2}{n}$ They are not iid!

Prove $\overline{X}_n \xrightarrow{P} \mu$ Weak Law of Large Numbers

WTS $\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \varepsilon) = 0$ Note: $P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \frac{\left(\frac{\sigma^2}{n}\right)}{\varepsilon^2}$

$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$ ✓ Thus easy.

We assumed finite variance. Not needed! (HW).

(III) Convergence in "L^r norm". For $r \geq 1$

$X_n \xrightarrow{L^r} c$ means $\lim_{n\to\infty} E[|X_n - c|^r] = 0$

eg. $X_n \xrightarrow{L^1} c$ means $\lim_{n\to\infty} E(|X_n - c|) = 0$ "Convergence in mean"

eg $X_n \xrightarrow{L^2} c$ means $\lim_{n\to\infty} E((X_n - c)^2) = 0$ "mean square convergence"

If $X_n \sim U(0, \frac{1}{n})$ Prove that $X_n \xrightarrow{L^r} 0 \; \forall r$

WTS $\lim_{n\to\infty} E[|X - 0|^r] = 0$

$\Rightarrow \lim_{n\to\infty} E[|X|^r] = 0$

$\Rightarrow \lim_{n\to\infty} E[X^r] = 0$

$\Rightarrow \lim_{n\to\infty} \int_0^{\frac{1}{n}} |x|^r (n)\, dx = \lim_{n\to\infty} \left[\frac{|x|^{r+1}}{r+1}\right]_0^{\frac{1}{n}} n$

$= \lim_{n\to\infty} \frac{1}{n^{r+1}(r+1)} n = \lim_{n\to\infty} \frac{1}{n^r(r+1)} = 0$ ✓

Let $1 \le r < s$

Prove $X_n \xrightarrow{L^s} c \Rightarrow X_n \xrightarrow{L^r} c$

Recall we used Holder's inequality to show

$$E\left[|X|^r\right] \le \left(E\left[|X|^s\right]\right)^{\frac{r}{s}}$$

$$\Rightarrow \lim_{n\to\infty} E\left[|X_n - c|^r\right] \le \lim_{n\to\infty} \left(E\left[|X_n - c|^s\right]\right)^{\frac{r}{s}} = \left(\lim_{n\to\infty} E\left[|X_n - c|^s\right]\right)^{\frac{r}{s}} = 0^{\frac{r}{s}} = 0$$

Note

$E\left[|X|\right] \ge 0$ since $|X|$ has positive support

$$\Rightarrow \lim_{n\to\infty} E\left[|X_n - c|^r\right] \ge 0 \Rightarrow \lim_{n\to\infty}\left[|X_n - c|^r\right] = 0 \Rightarrow X_n \xrightarrow{L^r} c$$

Prove: $X_n \xrightarrow{L^r} c \Rightarrow X_n \xrightarrow{P} c$

Markov's Ineq

$$\lim_{n\to\infty} P\left(|X_n - c| \ge \varepsilon\right) = \lim_{n\to\infty} P\left(|X_n - c|^r \ge \varepsilon^r\right) \le \lim_{n\to\infty} \frac{E\left[|X_n - c|^r\right]}{\varepsilon^r} = 0 \checkmark$$

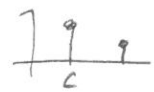
$X_n \xrightarrow{P} c \not\Rightarrow X_n \xrightarrow{L^r} c$ Contradiction?

$X_n \sim \begin{cases} n^2 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1-\frac{1}{n} \end{cases}$

Note: $X_n \xrightarrow{P} 0$ Why? $\lim_{n\to\infty} P\left(|X_n - 0| \ge \varepsilon\right) = \lim_{n\to\infty} P\left(X_n \ge \varepsilon\right) = \lim_{n\to\infty} P\left(X_n = n^2\right) = \lim_{n\to\infty} \frac{1}{n} = 0 \checkmark$

Note $X_n \overset{L^r}{\not\to} 0$ Why? $\lim_{n\to\infty} E\left[|X_n - 0|^r\right] = \lim_{n\to\infty} E\left(X_n^r\right) = \lim_{n\to\infty} \sum_{\text{Supp}(X)} x_n^r\, p_{X_n}(x_n) = \lim_{n\to\infty} (n^2)^r \frac{1}{n} = \lim n^{2r-1} = \infty \checkmark$

Conv. in mean is stronger than conv in prob! Why?



can be sent off

but expansion will remain!!

Let $X_n \sim N\left(0, \left(\frac{1}{n}\right)^2\right)$ Prove $X_n \xrightarrow{P} 0$ $\lim_{n\to\infty} P\left(|X_n - 0| \ge \varepsilon\right) \le \lim_{n\to\infty} \frac{\frac{\sigma^2}{n}}{\varepsilon^2} = 0 \checkmark$

Prove $X_n \xrightarrow{L^4} 0$ $\lim_{n\to\infty} E\left[|X_n - 0|^4\right] = \lim_{n\to\infty} E\left[X_n^4\right] = \lim_{n\to\infty} \frac{3}{n^2} = 0 \checkmark$

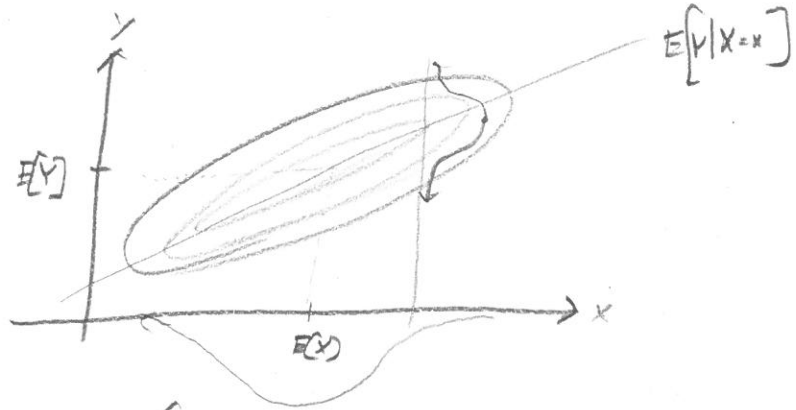
$$\phi_{X_n}(t) = e^{-\frac{1}{2}\sigma^2 t^2} = e^{-\frac{t^2}{2n}} \quad \phi_{X_n}^{(4)}(t) = e^{-t^2/2n}\left(\frac{3n^2 - 6nt^2 + t^4}{n^4}\right), \quad \phi_{X_n}^{(4)}(0) = \frac{3n^2}{n^4} = \frac{3}{n^2} = E\left(X_n^4\right)$$

↑ final

End of course

↓ not on final

Imagine two r.v.'s creating a joint density $f_{X,Y}(x,y)$



$$E[Y] = \int_{Supp(Y)} y \, f_Y(y) \, dy$$

$$= \int_{Supp(Y)} y \int_{Supp(X)} f_{X,Y}(x,y) \, dx \, dy$$

$$= \int_{Supp(Y)} y \int_{Supp(X)} f_{Y|X}(x,y) f_X(x) \, dx \, dy = \int_{Supp(Y)} \int_{Supp(X)} y \, f_{Y|X}(x,y) f_X(x) \, dx \, dy$$

$$= \int_{Supp(X)} \left(\int_{Supp(Y)} y \, f_{Y|X}(x,y) \, dy \right) f_X(x) \, dx$$

$$= \int_{Supp(X)} E[Y|X] \, f_X(x) \, dx = E(g(X))$$

$$\Rightarrow \boxed{E(Y) = E_X\left[E_Y[Y|X]\right]}$$ Law of Iterated expectation

$$Var_Y(Y) = E(Y^2) - E(Y)^2$$

$$= E_X\left[E_Y(Y^2|X)\right] - E_X\left[E_Y(Y|X)\right]^2$$

Note $Var_Y(Y|X) = E(Y^2|X) - E(Y|X)^2$

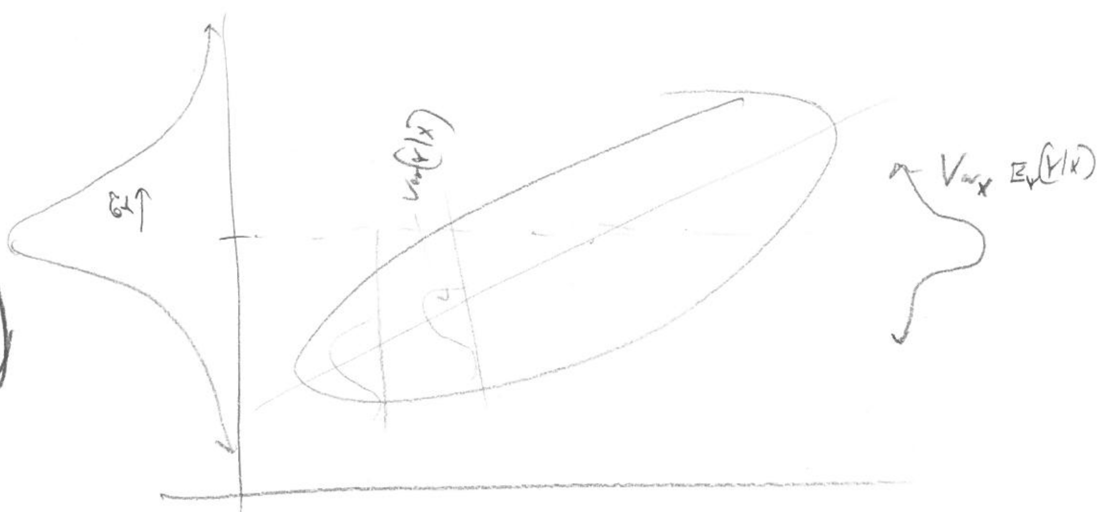
$$= E_X\left[Var(Y|X) + E(Y|X)^2\right] - E_X\left(E_Y(Y|X)\right)^2$$

$$= E_X\left(Var_Y(Y|X)\right) + \underbrace{E_X\left[E(Y|X)^2\right] - E_X\left(E_Y(Y|X)\right)^2}_{E(Q^2) - E(Q)^2 = Var(Q)}$$

$$\boxed{Var_Y(Y) = E_X\left(Var_Y(Y|X)\right) + Var_X\left[E_Y(Y|X)\right]}$$

Law of Total Variance

II



I