MATH 621 Fall 2017 Homework #5

Professor Adam Kapelner

Due KY604 (under the door) 11:59PM, Thursday, November 30, 2017

(this document last updated Wednesday 29th November, 2017 at 2:39am)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, read about the normal distribution, the χ^2 distribution, Student's T, the F, Cochran's Theorem, the multivariate normal distribution and characteristic functions.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 10 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

The χ^2 r.v. within Cochran's Theorem.

(a) [easy] Given $X_1, \ldots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$, a density with finite variance, state the classic estimator (a r.v.) and the estimate (a scalar value) for μ.

(b) [easy] Prove this estimator is unbiased i.e $\mathbb{E}[\cdot] = \mu$.

(c) [easy] Given $X_1, \ldots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$, a density with finite variance, state the classic estimator S^2 (a r.v.) and the estimate (a scalar value) for σ^2.

(d) [difficult] Prove this estimator is unbiased i.e $\mathbb{E}[\cdot] = \sigma^2$. The answer is online but try to do it yourself.

(e) [easy] State Cochran's Theorem.

(f) [easy] Given $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ Show that $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$.

(g) [easy] Express $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2$ in vector notation.

(h) [easy] Express $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2$ as a quadratic form. What is the matrix that determines this quadratic form?

(i) [easy] What is the rank of the determining matrix?

(j) [easy] When computing $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2$, how many independent pieces of information go into the calculation?

(k) [easy] Define degrees of freedom.

(l) [easy] Show that $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2}$.

(m) [easy] Show that $\frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2_1$.

(n) [easy] Express $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$ in vector notation.

(o) [easy] Express $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$ as a quadratic form. What is the matrix that determines this quadratic form? Call it B_2.

(p) [easy] What is the rank of the determining matrix?

(q) [easy] When computing $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$, how many independent pieces of information go into the calculation?

(r) [easy] Express $\frac{(n-1)S^2}{\sigma^2}$ in vector notation.

(s) [harder] Express $\frac{(n-1)S^2}{\sigma^2}$ as a quadratic form. What is the matrix that determines this quadratic form? Call it B_1.

(t) [harder] What is the rank of the determining matrix?

(u) [easy] When computing $\frac{(n-1)S^2}{\sigma^2}$, how many independent pieces of information go into the calculation?

(v) [easy] What is $B_1 + B_2$? Why should this be?

(w) [easy] What is $\text{rank}(B_1) + \text{rank}(B_2)$?

(x) [easy] Show that B_1 is positive semi-definite (PSD).

(y) [easy] Show that B_2 is positive semi-definite (PSD).

(z) [harder] Using Cochran's Theorem, show that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ and that $\frac{(n-1)S^2}{\sigma^2}$ is independent of $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$.

(aa) [difficult] What is $B_1 B_2$? Why should this be?

(bb) [harder] Using the answer in (z), show that $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$.

(cc) [difficult] In (d), you proved that $\mathbb{E}\left[S^2\right] = \sigma^2$. What is $\mathbb{E}\left[S\right]$ under the assumption of $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu,\, \sigma^2\right)$? You will need to read online about the χ distribution.

(dd) [difficult] Create a new estimator S' that is unbiased for estimating σ. You can use a function of the original S.

Problem 2

Some questions about ch.f.'s and the MVN really quickly

(a) [easy] Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \overset{iid}{\sim} \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$. Write the PDF of $\boldsymbol{X}_i$.

(b) [easy] Find the mean $\boldsymbol{\mu}_{\bar{x}}$ of $\bar{\boldsymbol{X}} := \frac{\boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n}{n}$.

(c) [harder] Find the variance matrix $\Sigma_{\bar{x}}$ of $\bar{\boldsymbol{X}} := \frac{\boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n}{n}$.

(d) [difficult] Use the ch.f. of the MVN to prove that $\bar{\boldsymbol{X}} \sim \mathcal{N}_n(\boldsymbol{\mu}_{\bar{x}}, \Sigma_{\bar{x}})$ and substitute in the mean vector and variance covariance matrix answers for (b) and (c).

(e) [difficult] Show that $\left(\bar{\boldsymbol{X}} - \boldsymbol{\mu}_{\bar{x}}\right)^\top \Sigma_{\bar{x}}^{-1} \left(\bar{\boldsymbol{X}} - \boldsymbol{\mu}_{\bar{x}}\right) \sim \chi^2_n$. This amounts to repeating a proof from class.

(f) [harder] The parameter space for the multivatiate normal distribution is $\boldsymbol{\mu} \in \mathbb{R}^n$ but what is the valid space for Σ? You can get this from wikipedia. Make sure you explain the answer.