$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$\sum_{i=1}^{n} Z_i^2 \sim \chi^2_k, \quad \sqrt{\sum_{i=1}^{n} Z_i^2} \sim \chi_k$$

chi-square "Chi"

~~with~~ k degree of freedom.

let $Y = |Z|$

$$\Rightarrow Y^2 = Z^2 \sim \chi^2_1$$

$$|Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\tfrac{2}{\pi}}\, e^{-\frac{z^2}{2}} = 2\underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)}_{N(0,1)}$$

$$X \sim \chi^2_k, \quad Y = \frac{X}{k} \sim$$

$$\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$$

~~$X \sim \text{Gamma}(\alpha, \beta), \; Y = cX \; ?, \; c \in (0,\infty)$~~

$\frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)}}{\Gamma(\alpha)}$$

$$= \frac{\beta^\alpha y^{\alpha-1} e^{-\left(\frac{\beta}{c}\right) y}}{\underbrace{c\, c^{\alpha-1}}_{c^\alpha} \Gamma(\alpha)} = \frac{\left(\frac{\beta}{c}\right)^\alpha y^{\alpha-1} e^{-\frac{\beta}{c} y}}{\Gamma(\alpha)} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$\left(\frac{\beta}{c}\right)^\alpha$

let $X_1 \sim \chi^2_{k_1}$ and $X_2 \sim \chi^2_{k_2}$

$$R = \frac{\frac{X_1}{k_1}}{\frac{X_2}{k_2}} = \frac{V_1}{V_2} \sim \int_{\text{Supp}\{V_2\}} t\, f_{V_1}(rt) f_{V_2}(t)\,dt$$

$\text{Supp}\{R\} = (0, \infty)$

$$= \int_0^\infty t \frac{a^a (rt)^{a-1} e^{-art}}{\Gamma(a)} \cdot \frac{b^b t^{b-1} e^{-bt}}{\Gamma(b)}\,dt$$

let $a = \frac{k_1}{2}$ $\quad V_1 \sim \frac{a^a x^{a-1} e^{-ax}}{\Gamma(a)}$

let $b = \frac{k_2}{2}$ $\quad V_2 \sim \frac{b^b x^{b-1} e^{-bx}}{\Gamma(b)}$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\,dt \quad \cancel{= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b}}}$$

let $u = (ar+b)t$

$\Rightarrow t = \frac{1}{ar+b} u$

$dt = \frac{1}{ar+b}\,du$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b}} \underbrace{\int_0^\infty u^{a+b-1} e^{-u}\,du}_{\Gamma(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} \cdot \frac{r^{a-1}}{\underbrace{(ar+b)^{a+b}}_{b^{a+b}\left(1+\frac{ar}{b}\right)^{a+b}}}$$

$$= \frac{a^a \cancel{b^b}\, r^{a-1} \left(\frac{a}{b}r+1\right)^{-(a+b)}}{B(a,b)\, \underbrace{b^{a+b}}_{b^a \cancel{b^b}}} = \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)}$$

k_1

$=$

$$\frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{\beta\left(\frac{k_1}{2},\frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1}\left(1+\frac{k_1}{k_2}r\right)^{-\frac{k_1+k_2}{2}} = F_{k_1,k_2}$$

"F distribution"

"Fisher-Snedecor"

"F" is named after Fisher.

$Z \sim N(0,1)$ ind of $V \sim \chi^2_k$ $\quad V = Z^2 \sim \chi^2_1$

let $W = \dfrac{Z}{\sqrt{\frac{V}{k}}} \sim ?$ $\quad Y = \dfrac{Z^2}{\frac{V}{k}} = \dfrac{\frac{U}{1}}{\frac{V}{k}}$

$\to \text{supp}\{Z^2\} = (0,\infty)$

~~Supp(W) = $\mathbb{R}$~~

$Y = W^2$

$$Y \sim F_{1,k} = \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{\beta\left(\frac{1}{2},\frac{k}{2}\right)} Y^{-\frac{1}{2}}\left(1+\frac{1}{k}Y\right)^{-\frac{k+1}{2}}$$

let $X = Y^2 \Rightarrow Y = X^{1/2}$

$$f_Y(y) = \frac{1}{2}f_X(y^2)(2y) = \frac{1}{\sqrt{k}\,\beta\left(\frac{1}{2},\frac{k}{2}\right)}\left(y^2\right)^{-\frac{1}{2}}\left(1+\frac{y^2}{k}\right)^{-\frac{k+1}{2}}y$$

$f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$ $\qquad \frac{1}{y}$

$$= \frac{1}{\sqrt{k}\,\beta\left(\frac{1}{2},\frac{k}{2}\right)}\left(1+\frac{y^2}{k}\right)^{-\frac{k+1}{2}} = T_k \xrightarrow{\text{limit}} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

Student's T distr with k degree of freedom.

$k \to \infty \qquad T_k \to Z$

$$Y = \frac{Z}{\sqrt{\frac{V}{k}}} \longrightarrow Z$$

$E(\chi^2_k) = k$, $Var(\chi^2_k) = 2k$

$$\frac{V}{k} = \frac{\sum_{i=1}^{k} Z_i^2}{k} \qquad E\left(\frac{V}{k}\right) = 1, \quad V\left(\frac{V}{k}\right) = \frac{2}{k}$$

$$\approx N\left(1, \frac{2}{k}\right) \xrightarrow[k\to\infty]{} Deg(1)$$

let $X_1 \sim N(0,1)$ ind of $X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} = \frac{X_1}{\sqrt{\frac{X_2^2}{1}}} \sim T_1 = \frac{1}{B(\frac{1}{2}, \frac{1}{2})}(1+x^2)^{-1}$$

$$\frac{1}{B(\frac{1}{2}, \frac{1}{2})} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})^2} = \frac{\Gamma(1)}{(\sqrt{\pi})^2} = \frac{1}{\pi}$$

$$\Rightarrow \frac{1}{B(\frac{1}{2}, \frac{1}{2})}(1+x^2)^{-1}$$

$$= \frac{1}{\pi(1+x^2)} = \text{Cauchy}(0,1)$$

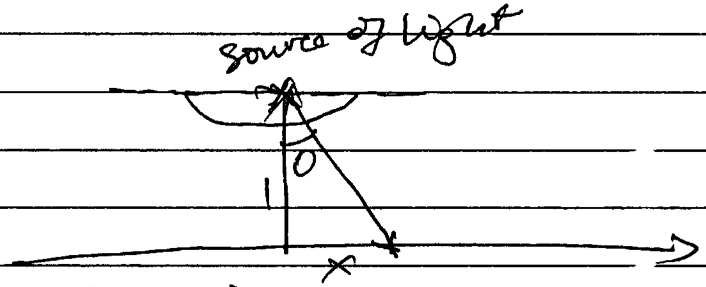
Standard.

AKA

The Lorentz distri.

$X \sim \text{Cauchy}(0,1)$, $Y = c + \delta X$, $f_Y(y) = \frac{1}{\delta} f_X\left(\frac{y-c}{\delta}\right)$

$$= \frac{1}{\pi\delta} \frac{1}{1+\left(\frac{y-c}{\delta}\right)^2} = \text{Cauchy}(c, \delta)$$



$\Theta \sim U(\pi, 2\pi) = \frac{1}{\pi}$

$X = \tan(\Theta)$, $\Theta = \arctan(x) = g^{-1}(x)$

$$\left|\frac{d}{dx}\left(g^{-1}(x)\right)\right| = \frac{1}{1+x^2}$$

$$f_X(x) = f_\Theta\left(g^{-1}(x)\right)\left|\frac{d}{dx}\left(g^{-1}(x)\right)\right| = \frac{1}{\pi}\cdot\frac{1}{1+x^2} = \text{Cauchy}.$$

$$E(X) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{1+x^2}\,dx = \frac{1}{\pi}\left[\frac{1}{2}\ln(x^2+1)\right]_{-\infty}^{\infty}$$

No Expectation $= \infty$ DNE.

$Var(X) = \infty$ DNE.

$$R = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} |x_2| \overbrace{f_{X_1}(x_2 r) f_{X_2}(x_2)}^{\text{ind.}}\,dx_2$$