Let $X \sim \text{Exp}(1)$ and $Y = -\ln(X) = \ln(\frac{1}{X}) \sim \text{Gumbel}(0,1) = e^{-(y+e^{-y})} = e^{-y}e^{-e^{-y}}$ which is the standard Gumbel. Find the CDF of Gumbel. Let $Y \sim \text{Gumbel}(0,1)$.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-Y \geq -y) = \mathbb{P}(e^{-Y} \geq e^{-y}) = \mathbb{P}(X \geq e^{-y}) = 1 - F_X(e^{-y}) = e^{-e^{-y}}$$

If $X \sim \text{Gumbel}(0,1)$, then

$$Y = \mu + \beta X \sim \text{Gumbel}(\mu,\beta) = \frac{1}{\beta}e^{-(\frac{y-\mu}{\beta}+e^{-(\frac{y-\mu}{\beta})})}$$

Find the CDF of Gumbel.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(\frac{Y-\mu}{\beta} \leq \frac{y-\mu}{\beta}) \\ &= \mathbb{P}(X \leq \frac{y-\mu}{\beta}) \\ &= F_X(\frac{y-\mu}{\beta}) \\ &= e^{-e^{-(\frac{y-\mu}{\beta})}} \end{aligned}$$

Let $X \sim \text{Gumbel}(\mu,\beta)$ and $Y = e^{-X}$

$$\begin{aligned} \text{Supp}[Y] &= (0,\infty) \\ x &= -\ln(y) = g^{-1}(y) \\ |\frac{d}{dy}g^{-1}(y)| &= y^{-1} \\ f_Y(y) &= f_X(-\ln(y))y^{-1} \\ &= \frac{1}{\beta}\exp\Big(-\Big(\frac{-\ln(y)-\mu}{\beta}\Big)\Big)\exp\Big(-\exp\Big(-\Big(\frac{-\ln y-\mu}{\beta}\Big)\Big)\Big) \\ &\text{Note } -(\frac{-\ln(y)-\mu}{\beta}) = \frac{\ln(y)+\mu}{\beta} \\ &\text{Let } k = \frac{1}{\beta} \text{ and } \mu = \ln(\lambda) \text{ where } \lambda \in (0,\infty) \\ \frac{\ln(y)+\mu}{\beta} &= k(\ln(y)+\ln(\lambda)) = \ln((y\lambda)^k) \\ f_Y(y) &= k\underbrace{(y\lambda)^k}_{y^k\underbrace{\lambda^k}_{\lambda\lambda^{k-1}}} e^{-(y\lambda)^k}y^{-1} \\ &= (k\lambda)(y\lambda)^{k-1}e^{-(y\lambda)^k} \\ &= \text{Weibull}(k,\lambda) \end{aligned}$$

Note: If $k = 1$, (thus $\beta = 1$ on the Gumbel), Weibull$(1, \lambda) = \lambda e^{-\lambda y} = \text{Exp}(\lambda)$. In addition,

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(\ln(Y) \leq \ln(y)) \\
&= \mathbb{P}(-\ln(Y) \geq -\ln(y)) \\
&= \mathbb{P}(X \geq -\ln(y)) \\
&= 1 - F_X(-\ln(y)) \\
&= 1 - \exp\Big(-\exp\Big(-(\frac{-\ln(y) - \mu}{\beta}\Big)\Big)\Big) \\
&= 1 - \exp\Big(-\exp\Big(\frac{\ln(y) + \mu}{\beta}\Big)\Big) \\
&= 1 - e^{-e^{\frac{\mu}{\beta}} y^{\frac{1}{\beta}}} \\
&= 1 - e^{-e^{\ln(y)^k}} \\
&= 1 - e^{-(\lambda x)^k}
\end{aligned}$$

If $\lambda = 1$ ($n = 0$ on the Gumbel), Weibull(1,1) = Exp(1).

The Weibull distribution is used to model survival time / failure times; it's a generalization of the exponential.

- If $k \neq 1$, then it is not memoryless
- If $k > 1$, $\mathbb{P}(X \geq a + b \mid X \geq a)$ gets smaller with a (dies quicker)
- If $k < 1$, $\mathbb{P}(X \geq a + b \mid X \geq a)$ gets larger with a (dies slower)
- If $k = 1$, no change

Let's say $k > 1$ (e.g. $k = 2$):
If $X \sim$ Weibull$(2, \lambda)$, then $F_X(x) = 1 - e^{-(\lambda x)^2}$.

$$\mathbb{P}(X \geq b) > \mathbb{P}(X \geq a + b \mid X \geq a) = \frac{\mathbb{P}(X \geq a + b)}{\mathbb{P}(X \geq a)} = \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}} = \frac{e^{-(\lambda a)^2} e^{-2\lambda^2 ab} e^{-(\lambda b)^2}}{e^{-(\lambda a)^2}}$$

Then

$$e^{-\lambda^2 b^2} > e^{-2\lambda^2 ab} e^{-(\lambda b)^2}$$

This is

$$-\lambda b^2 > -\lambda(2ab + b^2) \to b^2 < 2ab + b^2$$

which is valid.

Let's say $k < 1$ (e.g. $k = \frac{1}{2}$), then $F_X(x) = 1 - e^{-(\lambda x)^{\frac{1}{2}}}$. Then

$$\mathbb{P}(X \geq b) < \mathbb{P}(X \geq a + b \mid X \geq a) = \frac{\mathbb{P}(X \geq a + b)}{\mathbb{P}(X \geq a)} = \frac{e^{-(\lambda(a+b))^{\frac{1}{2}}}}{e^{-(\lambda a)^{\frac{1}{2}}}} = e^{-(\lambda(a+b))^{\frac{1}{2}} + (\lambda a)^{\frac{1}{2}}}$$

Then

$$\begin{aligned}
e^{-(\lambda b)^{\frac{1}{2}}} = e^{-\lambda^{\frac{1}{2}} b^{\frac{1}{2}}} &< e^{-\lambda^{\frac{1}{2}}((a+b)^{\frac{1}{2}} - a^{\frac{1}{2}})} \\
-\lambda^{\frac{1}{2}} b^{\frac{1}{2}} &< -\lambda^{\frac{1}{2}}((a+b)^{\frac{1}{2}} - a^{\frac{1}{2}}) \\
b^{\frac{1}{2}} &> (a+b)^{\frac{1}{2}} - a^{\frac{1}{2}} \\
a^{\frac{1}{2}} + b^{\frac{1}{2}} &> (a+b)^{\frac{1}{2}} \\
(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 &> a+b \\
a+b+2a^{\frac{1}{2}}b^{\frac{1}{2}} &> a+b
\end{aligned}$$

which is valid.

Let $X \sim$ Weibull and $Y = \frac{1}{X}$ (inverse waiting time).

$$\begin{aligned}
x &= \frac{1}{y} = g^{-1}(y) \\
|\frac{d}{dy} g^{-1}(y)| &= \frac{1}{y^2} \\
\text{Supp}[Y] &= (0, \infty) \\
f_Y(y) &= f_X(\frac{1}{y})\frac{1}{y^2} = (k\lambda)(\frac{\lambda}{y})^{k-1} e^{-(\frac{\lambda}{y})^k} \\
&= k\lambda^k \frac{1}{y^{\underbrace{k-1+2}_{k+1}}} e^{-\frac{\lambda^k}{y^k}} \\
&= \frac{k}{\lambda}(\frac{y}{\lambda})^{-(k+1)} e^{-(\frac{y}{\lambda})^{-k}} \\
&= \text{Frechet}(k, \lambda, \underbrace{0}_{\text{centered}})
\end{aligned}$$

Parameter space: $k \in (0, \infty)$, $\lambda \in (0, \infty)$.
If $X \sim \text{Frechet}(k, \lambda, 0)$, then $Y = X + c \sim \text{Frechet}(k, \lambda, c)$.

Note: Gumbel, Weibull and Frechet belong to a special family called the Generalized Extreme Value Distribution.
Units:

- Weibull: waiting time
- Frechet: inverse waiting time
- Gumbel: log inverse waiting time

Recall that $X \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$ and $X \sim \text{NegBinom}(k, p) = \underbrace{\binom{x+k-1}{k-1}}_{\frac{(x+k-1)!}{x!(k-1)!}} p^k (1-p)^x = \frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)} p^k (1-p)^x$. For both distributions, $k \in \mathbb{N}$ since it is a number of successes. What's wrong with allowing $k \in (0, \infty)$ i. e. all positive reals? You can show that the PDF of Erlang and PMF of negative binomial would still be valid. Conceptually? Wait for a fractional number of successes? Imagine "success" is initially continuous (such as success measured in dollars). If $k \in (0, \infty)$ these distributions got different names.

$$X \sim \text{Gamma}(k, \lambda) \text{ useful due to flexible waiting time ditribution}$$
$$X \sim \text{ExtNegBinom}(k, \lambda) \text{ ignore this}$$

The supports are $(0, \infty)$.

Let $X \sim \text{Gamma}(k_1, \lambda)$ and $Y \sim \text{Gamma}(k_2, \lambda)$. Then

$$\begin{aligned}
f_{X+Y}(t) &= \int_0^\infty \frac{\lambda^{k_1} x^{k_1-1} e^{-\lambda x}}{\Gamma(k_1)} \frac{\lambda^{k_2} (t-x)^{k_2-1} \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{\lambda x}}}{\Gamma(k_2)} \mathbb{1}_{\underbrace{t-x}_{x \le t} \in (0,\infty)} \, dx \\
&= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1}(t-x)^{k_2-1} \, dx \\
&\text{Let } u = \frac{x}{t} \to \frac{du}{dx} = \frac{1}{t} \to dx = t \, du \\
&x = ut \to x_l = 0 \to u_l = 0, \; x_u = t \to u_u = 1 \\
&= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 (ut)^{k_1-1}(t-ut)^{k_2-1} \, du \\
&= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1}(1-u)^{k_2-1} t \, du
\end{aligned}$$