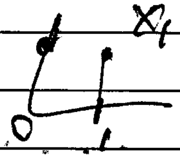
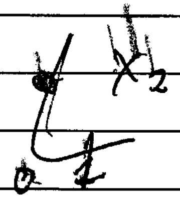
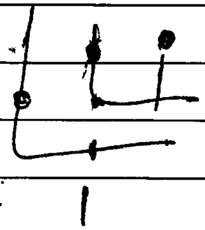
08/31/2017

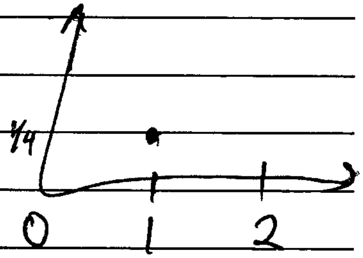
$X_1, X_2 \sim Bern\left(\frac{1}{2}\right)$

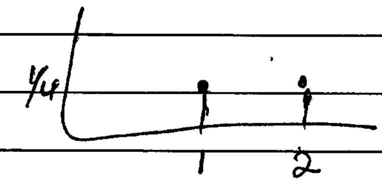
$T_2 = X_1 + X_2$

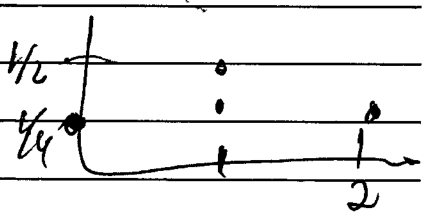



$X_1 + X_2 \sim P_{X_1}(x) * P_{X_2}(x)$

$= \sum P_{X_1}(x) P_{X_2}(t-x)$

$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x}$

$P(2) = p^0 (1-p)^{1-0} p^{2-0} (1-p)^{1+2-0} + p^1 (1-p)^{1-1} p^{2-1} (1-p)$

$X_1, X_2 \sim Bin(n, p)$

$Y = X_1 + X_2 \quad P_{X_1}(x) \cdot P_{X_2}(x) = \sum_{x \in \{x_i\}} P_{X_1}(x) P_{X_2}(Y - x)$

$= \sum_{x \in \{0,1,\dots n\}}^{n} \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots n\}} \cdot \binom{n}{Y-x} p^{Y-x} (1-p)^{n-Y+x} \mathbb{1}_{Y \in \{\}}$

not need     not need

$= \sum \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{Y-x} p^{Y-x} (1-p)^{n-Y+x}$

$= p^Y (1-p)^{2n-Y} \sum \binom{n}{x}\binom{n}{Y-x} = \binom{2n}{Y}$

PMF

Consider $B_1\ B_2 \sim Bern(p)$

let $X = \min\{B_t = 1\} - 1$

$X \sim Geom(p)$

$P(X=0) = p \qquad P(X=2) = (1-p)^2 p$

$P(X=1) = (1-p)p \qquad P(X=n) = (1-p)^n p$

$$Supp(X) = \{0,1,2,\dots\} = \mathbb{N}_0$$

$$T_2 = X_1 + X_2 \sim p(t) = P_{X_1}(x) * P_{X_2}(x)$$

$$= \sum_{x \in Supp(X_1)} P_{X_1}(x)\, P_{X_2}(t-x) = \sum_{x \in Supp\{0,1,2,\dots\}} (1-p)^x p (1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$\mathbb{1}_{t-x \in \{0,1,2,\dots\}}$$

$$\mathbb{1}_{t \in \{x,0,1,\dots\}}$$

$$\mathbb{1}_{t \geq x} = \mathbb{1}_{x \leq t}$$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \leq t}$$

$$= \sum_{x=0}^{t} 1 = t+1$$

$$= (t+1)(1-p)^t p^2$$

$$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim p(t)$$

$$= P_{X_3}(x) * P_{T_2}(x) = \sum_{Supp(X_3)} P_{X_3}(x)\, P_{T_2}(t-x)$$

$$= \sum_{x \in \{0,1,\dots\}} (1-p)^x p\, (t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}_{t-x \in Supp(T_2)}$$

$$= (1-p)^t p^3 \sum_{x\in\{0,1,\dots\}} (t-x+1)\,\mathbb{1}_{x\le t}$$

$$= (1-p)^t p^3 \left( (t+1)\sum_{x=0}^{t} \mathbb{1}_{x\le t} - \sum_{x=0}^{t} x\,\mathbb{1}_{x\le t} \right)$$

$$= (1-p)^t p^3 \left( (t+1)(t+1) - \frac{t(t+1)}{2} \right)$$

$$\sum_{x=0}^{t} 1 \qquad \sum_{x=0}^{t} x$$

$$= (1-p)^t p^3 \left( \frac{t^2+3t+2}{2} \right)$$

$$= \binom{t+2}{2}(1-p)^t p^3$$

$T_2 \sim NegBin(2, p)$
$T_3 \sim NegBin(3, p)$ | Negative Binomial

$$X = NegBin(k, p) = \binom{x+k-1}{k-1}(1-p)^x p^k$$

$X \sim Bin(n, p)$  $Supp = \{0, 1, \ldots n\}$

What if $n$ is very big?

is $p$ is very small?

$\lambda = np$

PMF  if $n \to \infty$?

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\lambda\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \underbrace{\lim \frac{n!}{(n-x)!\,n^x}}_{} \underbrace{\lim \left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\lim \left(1-\frac{\lambda}{n}\right)^{-x}}_{1}$$

$$\boxed{X \sim Poisson(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad Supp(X) = \{0, 1, \ldots\} \quad \lambda \in (0, \infty)}$$

$X_1, X_2 \sim \text{Poisson}(\lambda)$

$T = X_1 + X_2 \sim p_{X_1}(x) * p_{X_2}(x)$

$$= \sum_{x \in \text{supp}(x)} p_{X_1}(x)\, p_{X_2}(t-x) = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{x} e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t \in \{0,1,\dots\}}$$

$$= \lambda^{t} e^{-2\lambda} \sum \frac{1}{x!(t-x)!} \mathbb{1}_{x \le t} \qquad \mathbb{1}_{x \le}$$

$$= \frac{\lambda^{t} e^{-2\lambda}}{t!} \sum \binom{t}{x} = \frac{(2\lambda)^{t} e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$