11/9/17 Lecture #18

$X \sim N(0,1)$ ind of $X_2 \sim N(0,1)$

$R = \frac{X_1}{X_2}$ $\int_{supp[X]} |x| f_{X_1}(xr) f_{X_2}(x)\,dx$

$= \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 r^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$= \frac{1}{2\pi} \int |x| e^{-\frac{1}{2}x^2(r^2+1)}\,dx$

$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} (-x) e^{-\frac{1}{2}x^2(r^2+1)}\,dx + \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\,dx \right)$

$= \frac{1}{\pi} \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\,dx$ $\qquad u = -\frac{1}{2}x^2(r^2+1)$, $du = -x(r^2+1)dx$

$= -\frac{1}{\pi} \int_0^{-\infty} e^{u}\,du$ $\qquad x=0 \;\; u=0$, $x=\infty \;\; u=-\infty$.

$= \frac{1}{\pi(r^2+1)} \left[-e^{u}\right]_0^{\infty}$ $\qquad x\,dx = \frac{-du}{(r^2+1)}$

$= \frac{1}{\pi(r^2+1)} = \text{Cauchy}(0,1)$

Midterm

Final

$X_1, \dots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$ unknow dist. (μ unknown, σ^2 unknown)

- Goal Find mean & the variance.

$\bar{X}$ is the avg. (r.v) "estimator" $E[\bar{X}] = \mu$ unbiased.

$\bar{x}$ is a realization "estimate" $\bar{x} = \frac{x_1 + \dots + x_n}{n}$ is a #

$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$ S^2 is the sample variance r.v.

is a realization. #

$\Rightarrow \sum (X_i - \bar{X})^2 = (n-1)S^2$

$E[\tilde{S}^2] = \sigma^2$

$\uparrow$ unbiased.

$\ast \quad X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$S^2 \sim ?$

$S^2 = \frac{1}{n-1}\left((X_1 - \bar{X})^2 + \dots + (X_n - \bar{X})^2\right)$

Let $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{bmatrix} \qquad \sum_{i=1}^{n} Z_i^2 = \vec{Z}^T \vec{Z} \sim \chi^2_n$

$$\sum Z_i^2 = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum (X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\sum (X_i - \mu)^2 = \sum \left((X_i - \bar{X}) + (\bar{X} - \mu)\right)^2$$

$$= \sum (X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2$$

$$= \sum (X_i - \bar{X})^2 + 2\left(\sum X_i \bar{X} - \mu \sum X_i - \sum \bar{X}^2 + \sum \bar{X}\mu\right) + n(\bar{X} - \mu)^2$$

$$= \sum (X_i - \bar{X})^2 + 2\left(n\bar{X}^2 - n\mu\bar{X} - n\bar{X}^2 + n\bar{X}\mu\right) + n(\bar{X} - \mu)^2$$

$$\Rightarrow \frac{\sum (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 = \text{normal}^2 \sim \chi^2_1$$

conf.

$X_1 \sim \chi^2_{k_1}$ ind of $X_2 \sim \chi^2_{k_2} \Rightarrow X_1 + X_2 \sim \chi^2_{k_1+k_2}$

Cochran's Thm (1934)

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

Let $Q_1, \ldots, Q_k$ be scalar r-v's

via the quadratic forms $Q_j = \vec{z}^T B_j \vec{z}$

where $B_1, \ldots, B_k$ are positive semi-definite matrices.

(a) $n = \sum \text{rank}(B_j)$

$\Updownarrow$

(b) Q_j's are independent

$\Updownarrow$

c) $Q_j \sim \chi^2_{\text{rank}(B_j)}$

how big the dimension is

$$\sum z_i^2 = \sum\left((z_i - \bar{z}) + \bar{z}\right)^2 = \sum (z_i - \bar{z})^2 + 2\sum (z_i - \bar{z})\bar{z} + \sum \bar{z}^2$$

$$= \sum (z - \bar{z})^2 + 2\left(\sum z_i \bar{z} - \sum \bar{z}^2\right) + \sum \bar{z}^2$$

$$= 2(n\bar{z}^2 - n\bar{z}^2)$$

$$\underbrace{\sum z_i^2}_{\text{quad form } n \text{ dims}} = \underbrace{\sum (z_i - \bar{z})^2}_{Q_1 = \vec{z}^T B_1 \vec{z}} + \underbrace{n \bar{z}^2}_{Q_2 = \vec{z}^T B_2 \vec{z}}$$

Now we have to create quadratic forms for Q_1 & Q_2

Q_2 is easier.

$$Q_2 = n\bar{z}^2 = \vec{z}^T B_2 \vec{z} = \vec{z}^T \left(\tfrac{1}{n} J_n\right) \vec{z}$$

$\Rightarrow$

$$J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n}$$

$\vec{Z}^T \vec{Z} = \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z} + \cdots + \vec{Z}^T B_k \vec{Z}$

$\Rightarrow I_n = B_1 + B_2 + \cdots + B_k$

$\vec{Z}^T (B_1 + B_2 + \cdots + B_k) \vec{Z}$

$Q_1 + \cdots + Q_k = \vec{Z}^T \vec{Z}$

$$\text{rank } 1$$

$$Q_2 = z^T \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} z = z^T \begin{bmatrix} \bar{z} \\ \bar{z} \\ \bar{z} \\ \vdots \\ \bar{z} \end{bmatrix} =$$

$$= z_1\bar{z} + \cdots + z_n\bar{z} = \bar{z}\left(\sum z_i\right) = \bar{z}\, n\bar{z}$$
$$= n\bar{z}^2$$

Theorem: Matrix A is symmetric, idempotent $\Rightarrow$ $tr(A) = rank(A)$.

$$\left(\frac{1}{n}J_n\right)\left(\frac{1}{n}J_n\right) = \frac{1}{n^2} J_n . J_n \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2J_2$$

$$= \frac{1}{n^2} n . J_n = \frac{1}{n}J_n \Rightarrow tr\left(\frac{1}{n}J_n\right) = 1 = rank\left(\frac{1}{n}J_n\right)$$

$$Q_2 = \sum(z_i - \bar{z})^2 = \vec{z}^T B_1 \vec{z}$$
$$\sum z_i^2 - 2z_i\bar{z} + \bar{z}^2$$
$$= \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2$$
$$= \sum z_i^2 - n\bar{z}^2$$
$$= \vec{z}^T I_n \vec{z} - \vec{z}^T \frac{1}{n} J_n \vec{z} = \vec{z}^T \underbrace{\left(I_n - \frac{1}{n}J_n\right)}_{B_2} \vec{z}$$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ & 1-\frac{1}{n} & & \\ & & \ddots & \\ -\frac{1}{n} & & & 1-\frac{1}{n} \end{bmatrix}$$

symmetric

$$\left(I_n - \frac{1}{n}J_n\right)\left(I_n - \frac{1}{n}J_n\right) \quad \text{idempotent}$$
$$= I_n - 2\frac{1}{n}J_n + \frac{1}{n}J_n = I_n - \frac{1}{n}J_n$$

$$tr\left(I_n - \frac{1}{n}J_n\right) = n-1 = rank(B_2)$$

Def: Matrix A is pos. semi.def. if $\forall \vec{v}$,
$\vec{v}^T A \vec{v} \geq 0$

Proven: $n\bar{z}^2 \sim \chi_1^2$
ind of
$\sum (z_i - \bar{z})^2 \sim \chi^2_{n-1}$

$$\frac{n(\bar{x}-\mu)^2}{\sigma^2} = \underbrace{\frac{1}{\sigma}(\vec{x}-\vec{\mu})^T}_{\vec{z}^T} \left(\tfrac{1}{n}J_n\right) \cdot \frac{1}{\sigma}(\vec{x}-\vec{\mu})$$

χ_1^2

$$\sim \chi_1^2$$

iid of

$$\frac{\sum(x_i-\bar{x})^2}{\sigma^2} = \frac{1}{\sigma}(\vec{x}-\vec{\mu})^T \left(I_n - \tfrac{1}{n}J_n\right) \frac{1}{\sigma}(\vec{x}-\vec{\mu})$$

$$\sim \chi^2_{n-1}$$

$\Rightarrow \frac{n(\bar{x}-\mu)^2}{\sigma^2}$ ind of $\frac{(n-1)s^2}{\sigma^2}$

$\Rightarrow \bar{x}, s^2$ are independent.
Fisher proved thin 1935.

Geary, 1936 proved this is unique to the normal.

$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ $\qquad s^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1} = \text{Gam}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$

Big picture

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$ "Z test"

"T-test"

Student 1908

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}S^2}}$$

$$= \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{N(0,1)}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{\chi^2_{n-1}}}{n-1}}} = \frac{Z}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}} = T_{n-1}$$

both independent