Lecture 21 11/30/17

Let X be a nonnegative rV with finite expectation μ. Considere $a > 0$ a constant.

Considere the inequality

$a \mathbb{1}_{X \geq a} \leq$ Intuitive? yes because

if $X \geq a$ $a(1) \leq X \Rightarrow X \geq a$ ✓

if $X < a$ $a(0) \leq X \Rightarrow X \geq 0$ true by assumption

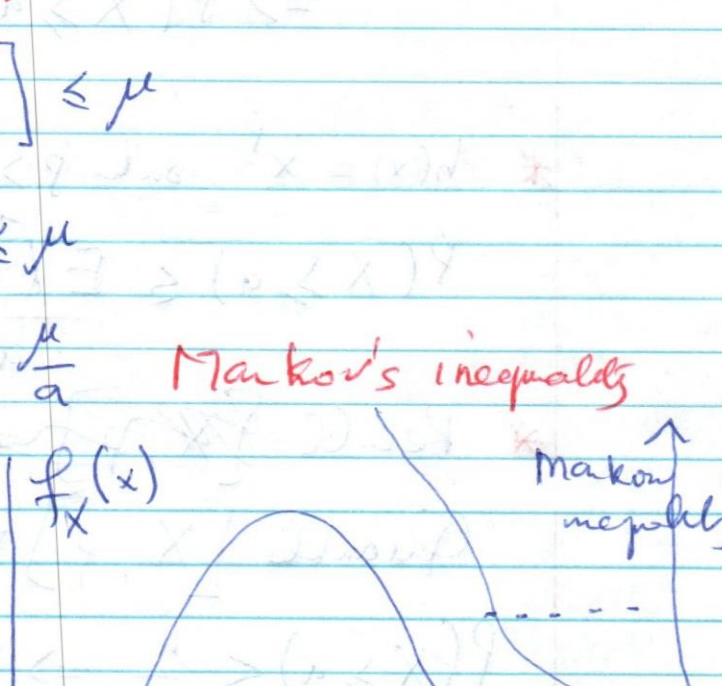
$E[a \mathbb{1}_{X \geq a}] \leq \mu$

$\Rightarrow a E[\mathbb{1}_{X \geq a}] \leq \mu$

$\Rightarrow a P(X \geq a) \leq \mu$

$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$ Markov's inequality

tail bound



~~Collo~~

Corollaries

* let $a = a'\mu$

$$P(X \geq a'\mu) \leq \frac{1}{a'}$$

* let h be monotonically increasing function

$h(a) \underline{\Downarrow}_{h(x)}$ ~~$\geq h(a)$~~ $\leq h(x) \Rightarrow$

$$P(h(x) \geq h(a)) \leq \frac{E[h(x)]}{h(a)}$$

$$\Rightarrow P(X \geq a) \leq \frac{E[h(x)]}{h(a)}$$

* $h(x) = x^p$ such $p > 1$

$$P(X \geq a) \leq \frac{E[X^p]}{a^p}$$

* Recall ~~$[X, p]$~~ =

$$\text{Quantile}[X, p] = F_X^{-1}(p) \quad (\text{if } F \text{ continuous})$$

$$P(X \geq a) \leq \frac{\mu}{a} \Rightarrow 1 - P(X \leq a) \leq \frac{\mu}{a}$$

$$\Rightarrow 1 - F(a) \leq \frac{\mu}{a} \quad \text{let } a = F_X^{-1}(p)$$

$$\Rightarrow 1 - F(F^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$$

$$\Rightarrow 1-p \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow$$

$$\text{Quantile}[X, p] \leq \frac{\mu}{1-p}$$

$$\Rightarrow \text{med}[X] \leq 2\mu$$

* Consider any rv X, $|X|$ is nonnegative

$$P(|X| \geq a) \leq \frac{E[|X|]}{a}$$

bottom tails

* Let X be any rv with finite μ, finite σ^2

let $Y = (X-\mu)^2$ Note Y is nonnegative

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2} = \frac{E[(X-\mu)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

$$\Rightarrow P((X-\mu)^2 \geq a^2) \leq \frac{\sigma^2}{a^2}$$

$$\Rightarrow P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ Chebyshew's inequality

* let X be any rv, let $Y = e^{tX}$ (Y is nonnegative)

$$P(Y \geq c) \leq \frac{E(Y)}{c}$$

$$P(e^{tX} \geq c) \leq \frac{E[e^{tX}]}{c}$$

let $c = e^{ta}$

$$\Rightarrow P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = \frac{M_X(t)}{e^{ta}}$$

Note $M_X(t) = E[e^{tX}]$ is moment generating funct.

① if $t > 0$

$$P(X \geq a) \leq e^{-ta} M_X(t) \cancel{\Rightarrow P(X}$$

② if $t < 0$

$$P(X \leq a) \leq e^{-ta} M_X(t)$$

① $\Rightarrow P(X \geq a) \leq \min_{t>0} \{ e^{-ta} M_X(t) \}$

② $\Rightarrow P(X \leq a) \leq \min_{t<0} \{ e^{-ta} M_X(t) \}$

Chernoff's Inequality

Example let $X \sim Bin\left(n, \frac{1}{4}\right) \Rightarrow$

$\mu = \frac{1}{4}n, \quad \sigma^2 = \frac{3}{16}n$

$P\left(X \geqslant \frac{3}{4}n\right)$?

if n is large $\quad X \simeq N\left(\frac{1}{4}n, \left(\sqrt{\frac{3}{16}n}\right)^2\right)$

$$P\left(X \geqslant \frac{3}{4}n\right) = P\left(\frac{X - \frac{1}{4}n}{\sqrt{\frac{3}{16}n}} > \frac{\frac{3}{4}n - \frac{1}{4}n}{\sqrt{\frac{3}{16}n}}\right)$$

$$= P\left(Z > \frac{2}{\sqrt{3}}\sqrt{n}\right) \text{ if } n \text{ large}$$

$$= 0$$

Using Markov's

$$P\left(X \geq \frac{3}{4}n\right) \leq \frac{\frac{1}{4}n}{\frac{3}{4}n} = \frac{1}{3}$$

Using Chebyshev's $\quad P\left(X \geqslant \frac{3}{4}n\right)$

$$\cancel{P\left(\left|X - \frac{1}{4}n\right| \geqslant\right.} = P\left(X - \frac{1}{4}n \geqslant \frac{3}{4}n - \frac{1}{4}n\right)$$

$$\leq P\left(X - \frac{1}{4}n \geqslant \frac{1}{2}n\right) + P\left(\frac{1}{4}n - X \geqslant \frac{1}{2}n\right)$$

$$= P\left(X - \frac{1}{4}n \geqslant \frac{1}{2}n \text{ or } \frac{1}{4}n - X \geqslant \frac{1}{2}n\right)$$

$$= P\left(\left|X - \frac{1}{4}n\right| \geq \frac{1}{2}n\right) \leq \frac{\frac{3}{16}n}{\frac{1}{4}n^2} = \frac{3}{4n}$$

Example $X \sim Bin(n, p)$

$$M_X(t) := E\left[e^{tX}\right] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} = (1-p+pe^t)^n$$

Binomial Theorem

$$X \sim Bin\left(n, \frac{1}{4}\right) \Rightarrow M_X(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n$$

using chernoff's

$$P\left(X \geq \frac{3}{4}n\right) \leq \min_{t>0} \left\{ e^{-t\left(\frac{3}{4}n\right)} \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n \right\}$$

$$= \min_{t>0} \left\{ \left(\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t}\right)^n \right\}$$

to minimize take derivative of / set $= 0$

$$\Rightarrow e^{\frac{1}{4}t} = 9e^{-\frac{3}{4}t} \Rightarrow \frac{1}{4}t = \ln(9) - \frac{3}{4}t$$

$$\Rightarrow t_{min} = \ln(9)$$

Plug t_{min} to $P\left(X \geq \frac{3}{4}n\right)$

$$\Rightarrow P\left(X \geq \frac{3}{4}n\right) = \left(\frac{3}{4} e^{-\frac{3}{4}\ln(9)} + \frac{1}{4} e^{\frac{1}{4}\ln 9}\right)^n$$

$$= \left(\frac{3}{4} \cdot 9^{-3/4} + \frac{1}{4} 9^{\frac{1}{4}}\right)^n$$

$$= \frac{\sqrt[4]{9}}{4^n}\left(\frac{3}{9^3} + 1\right)^n = \sqrt[4]{9}\left(\frac{1.004}{4}\right)^n$$

$\longrightarrow 0$ Exponentially fast.

Consider any 2 RV X and Y

with finites μ's and σ^2's.

let $W = (X - cY)^2$ such that $c \in \mathbb{R}$

(Note that W is nonnegative)

$$\Rightarrow E[W] \geq 0$$

$$\Rightarrow E[(X - cY)^2] \geq 0 \Rightarrow E[X^2 - 2cXY + c^2Y^2] \geq 0$$

$$\Rightarrow E[X^2] - 2cE[XY] + c^2 E[Y^2] \geq 0$$

let $c = \dfrac{E[XY]}{E[Y^2]}$

$$\Rightarrow E[X^2] - 2\frac{E[XY]}{E[Y^2]}E[XY] + \frac{E[XY]^2}{E[\cancel{Y^2}]}E[\cancel{Y^2}] \geq$$

$$E[X^2]E[Y^2] - 2E[XY]^2 + E[XY]^2 \geq 0$$

$$\Rightarrow E[XY]^2 \leq E[X^2]\,E[Y^2]$$

$$\Rightarrow |E[XY]| \leq \sqrt{E[X^2]\,E[Y^2]}$$

Chauchy-Schwartz Inequality

Equality if $X = cY$

what is correlation? SE (standard Error)

$$\mathrm{Corr}[X,Y] = \mathrm{Corr}[cY,Y] := \frac{\mathrm{Cov}[cY,Y]}{SE[cY]\,SE[Y]}$$

$$= \frac{c\,\mathrm{Cov}[Y,Y]}{|c|\,SE[Y]^2} = \frac{c\,\cancel{\mathrm{Var}[Y]}}{|c|\,\cancel{\mathrm{Var}[Y]}}$$

$$= \frac{c}{|c|} = \begin{cases} 1 \text{ if } c > 0 \\ -1 \text{ if } c < 0 \end{cases}$$