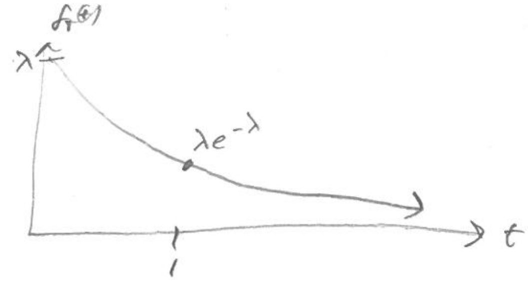
Lec 7 Math 621 9/13/17

$T \sim Exp(\lambda) = \lambda e^{-\lambda t}$

ans: time to error




$F_T(t) = 1 - e^{-\lambda t}$

$N \sim Poisson(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$

ans: number of errors

$F_N(n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$

no closed form exists

What is prob that no errors have occurred by $t=1$?

$\Rightarrow P(T > 1) = e^{-\lambda} = P(N = 0) = e^{-\lambda}$

Coincidence? We shall see...

What is the prob that at least one error occurs before $t = 1$?

$P(T < 1) = 1 - e^{-\lambda} = P(N > 0) = 1 - e^{-\lambda}$

$\uparrow$ error occured

Coincidence?

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Recall from last class...

$X \sim Erlang(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$

$F_X(x) = \ldots = \frac{\gamma(k, \lambda x)}{(k-1)!}$

What is the prob of no success or one success by $t=1$?

$P(N \leq 1) = F_N(1) = e^{-\lambda}(1 + \lambda)$

$T \sim Erlang(2, \lambda)$

$P(T > 1) = 1 - F_T(1)$

$\uparrow$ either no successes or one success by 1.

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,\,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,\,a)}$$

gamma function

lower incomplete gamma function

upper incomplete gamma function

The gamma function is known as the continuous extension of the factorial function to all real #'s.

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = -e^{-t}\Big]_0^\infty = -(0-1) = 1$$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u} \underbrace{e^{-t}\, dt}_{dv} = uv - \int v\, du = \left[-t^x e^{-t}\right]_0^\infty - \int_0^\infty -e^{-t}\, x\, t^{x-1}\, dt = x\,\Gamma(x)$$

$$\Rightarrow \Gamma(2) = 1 \cdot 1$$

$$\Gamma(3) = 2\,\Gamma(2) = 2 \cdot 1$$

$$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1$$

$$\vdots$$

$$\Gamma(n) = (n-1)!$$

$$\Rightarrow \quad F_{T_k}(x) = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

Sometimes called the "standardized" gamma function

$$1 - F_{T_k}(x) = 1 - \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k, \lambda x)}{\Gamma(k)} = Q(k, \lambda x)$$

regularized gamma function (prop. of whole gamma)

We know that $k \in \mathbb{N}$

$$\Gamma(k, \lambda x) = \int_{\lambda x}^{\infty} \underbrace{t^{k-1}}_{u} \underbrace{e^{-t}\,dt}_{dv} = uv\Big]_{\lambda x}^{\infty} - \int_{\lambda x}^{\infty} v\,du$$

$$= -t^{k-1}e^{-t}\Big]_{\lambda x}^{\infty} - \int_{\lambda x}^{\infty} (k-1)t^{k-2}(-e^{-t})\,dt$$

$$= (\lambda x)^{k-1}e^{-\lambda x} + (k-1)\,\Gamma(k-1, \lambda x)$$

$$= (\lambda x)^{k-1}e^{-\lambda x} + (k-1)\left((\lambda x)^{k-2}e^{-\lambda x} + (k-2)\,\Gamma(k-2, \lambda x)\right)$$

$$= e^{-\lambda x}\left((\lambda x)^{k-1} + (k-1)(\lambda x)^{k-2} + (k-2)(k-1)\frac{\Gamma(k-2,\lambda x)}{e^{-\lambda x}}\right) = e^{-\lambda x}(k-1)!\left(\frac{(\lambda x)^{k-1}}{(k-1)!} + \frac{(\lambda x)^{k-2}}{(k-2)!} + \dots + 1\right)$$

factor this out

Since $k \in \mathbb{N}$, then ultimately we are left with: $\Gamma(1, \lambda x) = \int_{\lambda x}^{\infty} \cancel{t^{1-1}}\, e^{-t}\,dt = e^{-\lambda x}$

$$= e^{-\lambda x}(k-1)!\sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}$$

$$\Rightarrow 1 - F_{T_k}(x) = \frac{e^{-\lambda x}\cancel{(k-1)!}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}{\cancel{(k-1)!}}$$

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$T \sim \text{Erlang}(2, \lambda)$

$$P(T > 1) = 1 - F_{T_2}(1) = e^{-\lambda}\sum_{i=0}^{1}\frac{(\lambda(1))^i}{i!} = e^{-\lambda}(1+\lambda)$$ Same!

If you still don't see the pattern...


What is the prob of $k$ successes or less ~~by~~ $t=1$?

$$P(N \le k) = F_X(k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

If successes come exponentially, what's the prob of seeing $k$ or fewer by 1 hr?

$$T \sim Erlang(k+1, \lambda)$$

$$P(T > 1) = 1 - F(1) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

Poisson process: in every unit time, there are $X \sim Poisson(\lambda)$ "hits" and each hit occurs after $T \sim Exp(\lambda)$.

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if $k \to \infty$ $Q \to 1$

Identity:

$$e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!} = \frac{\Gamma(k+1, \lambda)}{\Gamma(k)} = Q(k+1, \lambda) \Rightarrow \sum_{i=0}^{k} \frac{a^i}{i!} = e^a Q(k+1, a)$$

$$\Rightarrow e^a = \sum_{i=0}^{k} \frac{a^i}{i!}$$

| Running experiments | fixed time<br>measure # of successes | require # successes $\geq 1$ | require 1 success |
|---|---|---|---|
| discretely | Binomial | Neg Binomial | Geometric |
| continuously | Poisson | Erlang | Exponential |

The same relationship exists between the Bin. & Neg Bin.

What's the prob that there has been 2 successes or less by $t=50$?

$N \sim Bin(50, p)$ $\quad P(N \leq 2) = F_N(2) = \binom{50}{0}(1-p)^{50} + \binom{50}{1}p^1(1-p)^{49} + \binom{50}{2}p^2(1-p)^{48}$

$T \sim NegBin(3, p)$ $\quad P(T \geq 48) = 1 - F_T(\cancel{48}\ 47)$

$= 1 - \sum_{i=0}^{\cancel{48}\ 47} \binom{i+2}{2} p^3 (1-p)^i$

2 successes or less means 48 failures, 49 failures or 50 failures

$\Rightarrow$ Another combinatorial identity

$N \sim Bin(n, p)$
$T \sim NegBin(k+1, p)$

$F_N(k) = 1 - F_T(n-k-1)$

$$\sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{n-k-1} \binom{i+k}{k} p^{k+1} (1-p)^i$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

What is $P(X_1 \mid X_1 + X_2)$?

Notation is weird...

What is $P(X_1)$? This is $P(X_1 = x) = p_X(x)$

What is $P(X_1 + X_2)$? ... $P(X_1 + X_2 = n) =$ or some value...

$X_2 = n - X_1 = n - x$

$$\Rightarrow P(X_1 = x \mid X_1 + X_2 = n) = \frac{P(X_1 = x \ \& \ X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$ represents a joint event

$$= \frac{p_{X_1, X_2}(x, n-x)}{p_Y(n)}$$

let $Y = X_1 + X_2 \sim \text{Poisson}(2\lambda)$ (see class notes)

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{n-x}}{(n-x)!}}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} = \binom{n}{x}\left(\frac{\lambda}{2\lambda}\right)^n = \binom{n}{x}\left(\frac{1}{2}\right)^n = \text{Binomial}\left(n, \frac{1}{2}\right)$$

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$X_1 - X_2 \sim$ ?

Can be written as $X_1 + Y$ s.t. $Y = -X_2$ and then we can use convolution.

How do we find the PMF $p_Y(y)$?