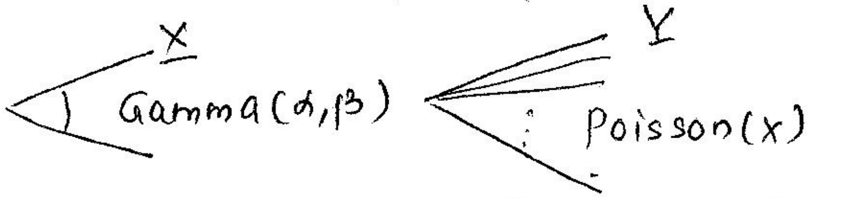
$X \sim \text{Gamma}(\alpha, \beta)$ $\quad Y|X = x \sim \text{Poisson}(x) \Rightarrow f_Y(y) = ?$

$$= \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y-1)} p^k (1-p)^Y = \text{Ext.NegBin}(p, k)$$

If $k \in \mathbb{N}$, $\quad = \binom{y+k+1}{k} p^k (1-p)^y = \text{NegBin}(p, k)$



$f_Y(y) = \text{Neg.Bin}$

$Q \sim \text{Poisson}(\lambda)$

$E(Q) = \lambda$

$SE(Q) = \lambda$

same support

$Q \sim \text{NegBin}(k, p) \leftarrow$ more flexible r.v.

$E(Q) = kp$

$SE(Q) \neq E(Q)$

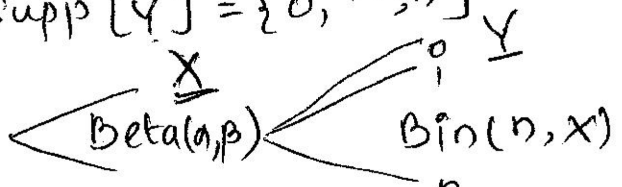
"over dispersed poisson"

Example

n fixed $\quad X \sim \text{Bern}(\alpha, \beta) \quad Y|X \sim \text{Bin}(n, x)$

x - cont.

$\text{Supp}[Y] = \{0, \dots, n\}$



$$P_Y(y) = \int_{\text{Supp}[X]} P_{Y|X} f_X(x)\, dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \cdot \frac{1}{\beta(\alpha,\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} \cdot dx$$

$$= \frac{\binom{n}{y}}{\beta(\alpha,\beta)} \cdot \int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1}\, dx = \frac{\binom{n}{y}}{\beta(\alpha,\beta)} \cdot \int_0^1 B(y+\alpha, n-y+\beta)$$

$=$ Beta Binomial (α, β, n)

"over dispersed binomial"

Example:

n-Fixed. $X \sim \text{Gamma}(\alpha,\beta)$, $Y/X \sim \text{Exp}(x)$ $\quad$ $\text{Supp}[Y]=(0,\infty)$

$\underline{X}$ $\quad\quad$ $\underline{Y}$

Gamma(α,β) $\quad$ Exp(x)

not P, b/c cont:

$$f_Y(y) = \int_{\text{Supp}[X]} f_{Y|X}(y,x)\cdot f_X(x)\,dx$$

$$= \int_0^1 x e^{-xy} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^1 x^\alpha e^{-(\beta+y)x}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^1 \frac{u^\alpha}{(\beta+y)^\alpha}\cdot e^{-u}\cdot\frac{1}{\beta+y}\cdot du$$

Let $u=(\beta+y)x$
$\Rightarrow x = \frac{1}{\beta+y}\cdot u$
$\frac{dy}{dx} = \beta+y$
$dx = \frac{1}{\beta+y}\cdot dy$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)(\beta+y)^{\alpha+1}} \underbrace{\int_0^1 u^\alpha \cdot e^{-u}\cdot du}_{=\Gamma(\alpha+1)} \quad \boxed{\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt}$$

$$= \frac{\beta^\alpha\cdot\Gamma(\alpha+1)}{\Gamma(\alpha)(\beta+y)^{\alpha+1}} \qquad \Gamma(\alpha+1)=\Gamma(\alpha)\cdot\alpha$$

$$= \frac{\beta^{\alpha+1}}{\beta}\cdot\frac{\alpha(\Gamma\cancel{(\alpha)})}{\Gamma\cancel{(\alpha)}\cdot(\beta+y)^{\alpha+1}}$$

$$= \frac{\alpha}{\beta}\left(\frac{\beta}{\beta+1}\right)^{\alpha+1} = \frac{\alpha}{\beta}\left(1+\frac{y}{\beta}\right)^{-(\alpha+1)}$$

$= \text{Lemax}(\beta,\alpha)$
Another type of survival distr.

Exponenetral
givesus one parameter
Here we get 2

Characteristic Functions.

Reviewing Complex Numbers.

$a, b \in \mathbb{R}$ $\quad z = a + bi \in \mathbb{C}$ (complex #s) $\quad i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$Re[z] := a$

$Im[z] := b$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

Taylor series.

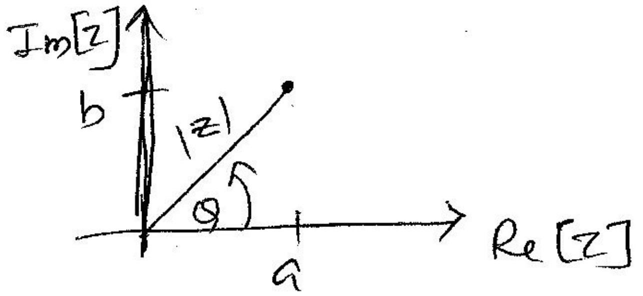
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots \qquad i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots \qquad \cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + \cdots$$

comparing terms here with the terms of e^{itx}.

$$e^{itx} = \cos(tx) + i\sin(tx)$$

if $\pi = tx$, $\quad e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$ Euler's Identity



$|z| = \sqrt{a^2+b^2} \in (0, \infty)$

Complex form.

$Arg(z) = \theta = \arctan(b/a) \in (-\pi, \pi)$

$$z = |z|e^{i\theta} = \sqrt{a^2b^2}\,(\cos\theta + i\sin\theta) = \sqrt{a^2+b^2}\,\Big(\underbrace{\cos(\arctan b/a)}_{\frac{a}{|z|}} + \underbrace{i\sin(\arctan b/a)}_{i\frac{b}{|z|}}\Big)$$

$$= |z|\left(\frac{a}{|z|} + i\frac{b}{|z|}\right)$$

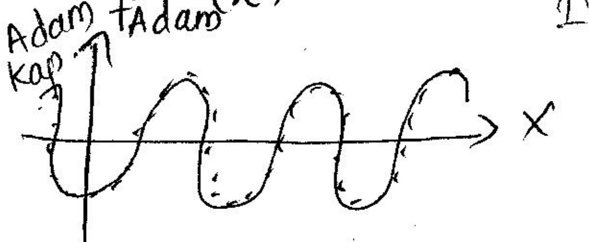
$$= a + bi$$

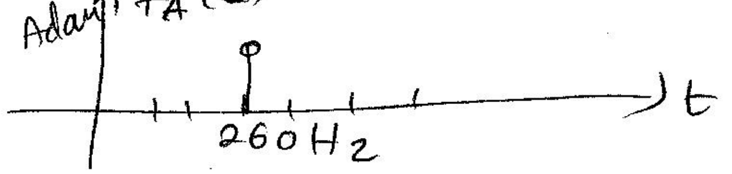
Define $L^1 := \{f : \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$ L^1 "integreble" or absolutely integreble

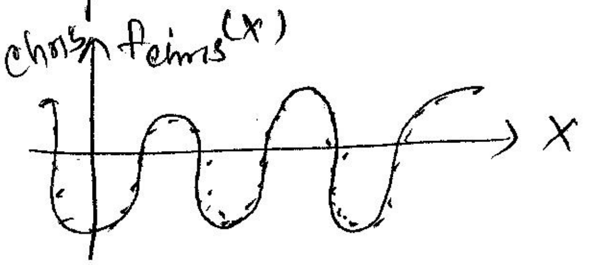
- Are all pdf's are in L^1? yes, b/c all the pdfs $\int$ to 1 and $1<\infty$.
- All functions in L^1 are not pdf's b/c they can be $\int$ to more than 1.

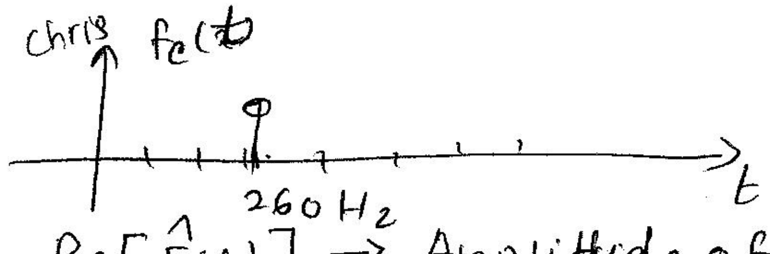
If $f \in L^1$, then $\exists$ $\hat{f}$ "f hat" defined as $\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\,dx$.
known as the Fourier Transform of f. Note $\hat{f}$ doesn't necessarily $\in L^1$. $f(x)$ is called the "time domain". $\hat{f}(t)$ is called the "frequency domain"

$f(x)$ can be noted as a sum of sins & cosines.
If $\hat{f} \in L^1$, then $f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\,dt$
Inverse fourier transform









Time domain

$Re[\hat{f}(t)] \rightarrow$ Amplitude of frequency
$Arg[\hat{f}(\theta)] \rightarrow$ phase shift of wave

Let $\phi(t) = \hat{f}\left(\frac{-t}{2\pi}\right) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$

* If $f(x)$ is a pdf, $\phi(t) = E[e^{itx}]$
* If $\phi(t) \in L^1$, then let $u = -2\pi t$

$\Rightarrow t = -u/2\pi$
$du/dt = -2\pi$
$dt = -1/2\pi \cdot du$

if $t = \infty \Rightarrow u = -\infty$
$t = -\infty \Rightarrow u = \infty$

$$f(x) = \int_{\infty}^{-\infty} e^{2\pi i \left(\frac{-u}{2\pi}\right)x} \hat{f}\left(\frac{-u}{2\pi}\right)\left(-\frac{1}{2\pi}\right) \cdot du.$$ using Inverse Fourier transform.

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \cdot \hat{f}\left(\frac{-u}{2\pi}\right) du = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-tx} \phi(u)\, du.$$

$\phi_X(t)$ is the characteristic function of r.v. X.
"Ch. f."

$$\phi_X(t) := E[e^{itX}] = \begin{cases} \sum_{x \in Supp[X]} e^{itx} p(x) & \text{if } X \text{ discrete} \\ \int_{Supp[X]} e^{itx} f(x)\, dx & \text{if } X\text{-continuous} \end{cases}$$

Properties of $\phi_X(t)$.

① $\phi(0) = 1$

② If X_1, X_2 independent, $Y = X_1 + X_2$

$$\underline{\phi_Y(t)} = \phi_{X_1+X_2}(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} \cdot e^{itX_2}]$$
$$= E[e^{itX_1}] \cdot E[e^{itX_2}] = \underline{\phi_{X_1}(t)\, \phi_{X_2}(t)}$$

③ If $Y = aX + b$, $a, b \in \mathbb{R}$.

$$\phi(t) = E[e^{itY}] = E[e^{it(aX+b)}]$$
$$= E[e^{itaX} \cdot e^{itb}] = e^{itb} E[e^{itaX}]$$
$$= e^{itb} \phi_X(at)$$

④ $\phi_X(x)$ always exists since $|\phi_X(t)| \leq 1 \ \forall t$

$$|\phi_X(t)| = |E[e^{itX}]| = \left|\int_{\mathbb{R}} e^{itx} f(x)\, dx\right| \leq \int_{\mathbb{R}} |e^{itx} f(x)|\, dx$$
$$= \int_{\mathbb{R}} \underbrace{|e^{itx}|}_{= 1} |f(x)|\, dx = \int_{\mathbb{R}} 1 \cdot |f(x)|\, dx = 1$$