Lecture 22 12/5/17

$$\left| E[X \cdot Y] \right| \leq \sqrt{E[X^2] E[Y^2]}$$

Equal only if $X = cY, c \in \mathbb{R}$

$$\text{Corr}(X,Y) = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$$

WTS $\text{Corr}[X,Y] \in [-1,1]$

$$Z_x = \frac{X - \mu_x}{\sigma_x}, \quad Z_Y = \frac{Y - \mu_Y}{\sigma_Y}$$

$$E[Z_x] = E[Z_Y] = 0$$

$$SE[Z_x] = SE[Z_Y] = 1$$

$$\hookrightarrow E[Z_x^2] = E[Z_Y^2] = 1$$

$$\left| E[Z_x Z_Y] \right| \leq \sqrt{E[Z_x^2] E[Z_Y^2]} = 1$$

$$\Rightarrow E[Z_x, Z_Y] \in [-1,1]$$

$$\text{Corr}[Z_x, Z_Y] = \frac{\text{cov}[Z_x, Z_Y]}{SE[Z_x] SE[Z_Y]} =$$

$$\frac{E[Z_x Z_Y] - \overset{0}{E[Z_x]} \overset{0}{E[Z_Y]}}{\underbrace{SE[Z_x]}_{1} \underbrace{SE[Z_Y]}_{1}} = E[Z_x Z_Y] \in [-1,1]$$

$$\text{Corr}[X,Y] = \frac{E[X,Y] - \mu_x \mu_y}{SE[X]\, SE[Y]}$$

$$= \frac{E\left[(\sigma_x Z_x + \mu_x)(\sigma_y Z_y + \mu_y)\right] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

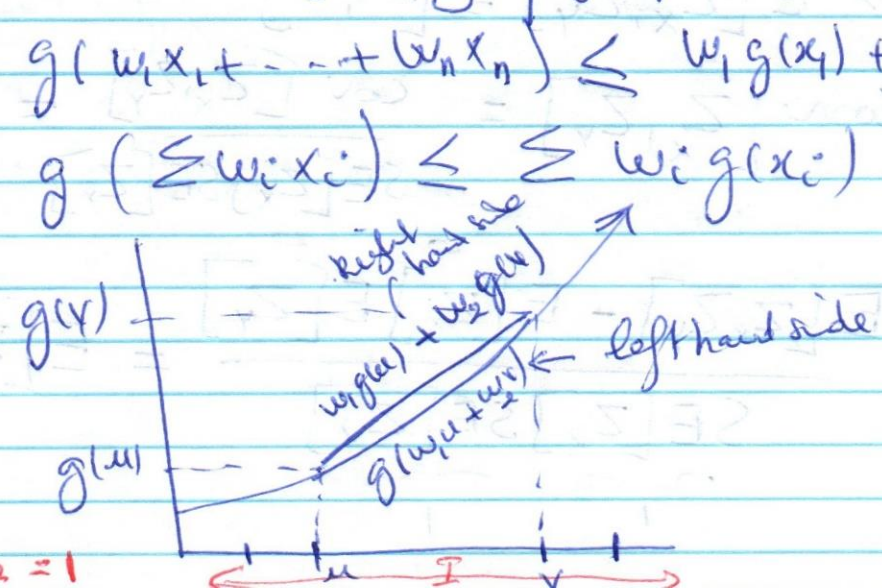
$$= \frac{E[\sigma_x \sigma_y Z_x Z_y] + \overset{0}{\cancel{E[\mu_x \sigma_y Z_y]}} + \overset{0}{\cancel{E[\mu_y \sigma_x Z_x]}} + \overset{0}{\cancel{E[\mu_x \mu_y] - \mu_x \mu_y}}}{\sigma_x \sigma_y}$$

$$= \frac{\cancel{\sigma_x \sigma_y} E[Z_x Z_y]}{\cancel{\sigma_x \sigma_y}} = E[Z_x Z_y] \in [-1, 1]$$

Defn: A function g is "convex" on the interval $I \subset \mathbb{R}$ if $\forall \{x_1 \dots x_n\} \in I$ and $\forall\ w_1 \dots w_n$ such that $w_i > 0$, $\sum w_i = 1$ (weights / proportion) then

$$g(w_1 x_1 + \dots + w_n x_n) \leq w_1 g(x_1) + \dots w_n g(x_n)$$

$$g\left(\sum w_i x_i\right) \leq \sum w_i g(x_i)$$



$w_1 + w_2 = 1$

Theorem if g is twice differentiable in I
g is convex if $g''(x) \geq \forall x \in I$

Consider discrete rv X, and support
$\mathrm{Sup}(X) = \{X_1, \ldots, X_n\} \subset I$ and
pmF $p(x_i) = w_i$ such that $\sum w_i = 1$
if g is convex in I the by definition of convexity
$$g(w_1 x_1 + \cdots w_n x_n) \leq w_1 g(x_1) + \cdots + w_n g(x_n)$$
$$\Rightarrow g(E[X]) \leq E[g(X)]$$ Jensens inequality (true for all rv)

Consider $g(x) = x^2$ for X positive
$$E[X]^2 \leq E[X^2]$$
$$\mu^2 \leq \sigma^2 + \mu^2 \Rightarrow \sigma^2 \geq 0$$

Example $g(x) = -\ln x$ for $x > 0$

g Convex? $g''(x) = \frac{1}{x^2} \geq 0 \quad \forall x \geq 0 \Rightarrow$ yes

let $X \sim \begin{cases} a^p & \text{wp } \frac{1}{p} \\ b^q & \text{wp } \frac{1}{q} \end{cases}$

Note $\frac{1}{p}+\frac{1}{q}=1$, $p,q>0$ $a,b>0 \Rightarrow$

X is positive

$$E[X] = \frac{a^p}{p}+\frac{b^q}{q}$$

$$g(X) \sim \begin{cases} -p\ln(a) & \text{wp } \frac{1}{p} \\ -q\ln(b) & \text{wp } \frac{1}{q} \end{cases}$$

$$E\left[g[X]\right] = \frac{-p\ln(a)}{p} - \frac{q\ln(b)}{q}$$

$$= -\ln(ab)$$

By Jensen's Inequality

$$-\ln\left(\frac{a^p}{p}+\frac{b^q}{q}\right) \leq -\ln(ab)$$

$$\ln\left(\frac{a^p}{p}+\frac{b^q}{q}\right) \geq \ln(ab)$$

$ab \leq \frac{a^p}{p}+\frac{b^q}{q}$ Young's Inequality

Let $a = X$, $b = Y$ $\Rightarrow$ X, Y positive

$$XY \leq \frac{X^p}{p} + \frac{Y^2}{q}$$

$$E[XY] \leq \frac{E[X^p]}{p} + \frac{E[Y^q]}{q}$$

let $a = \frac{X}{A}$, $b = \frac{Y}{B}$ A, B constants > 0

$$\Rightarrow \frac{E[XY]}{AB} \leq \frac{E[X^p]}{pA^p} + \frac{E[Y^2]}{qB^q}$$

let $A = E[X^p]^{\frac{1}{p}}$ let $B = E[Y^q]^{\frac{1}{q}}$

$$\frac{E[XY]}{E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}} \leq \frac{\cancel{E[X^p]}}{p(\cancel{E[X^p]^{\frac{1}{p}}})^p} + \frac{\cancel{E[Y^q]}}{q(\cancel{E[Y^q]^{\frac{1}{q}}})^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow E[XY] \leq E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}$$

Hölders Inequality

generalization of Cauchy-Schwartz

$$\forall X, Y \quad E[|XY|] \leq E[|X|^p]^{\frac{1}{p}} E[|Y|^q]^{\frac{1}{q}}$$

$p = \frac{s}{r}$

Example: let $0 < r < s$ $\quad q = \frac{p}{p-1} = \frac{s}{s-r}$

let $X = V^r$, $Y \sim \text{Deg}(1)$

$$E\left[|V^r|\right] \leq E\left[|V^r|^{\frac{s}{r}}\right]^{\frac{r}{s}}$$

$$= \left(E\left[|V|^s\right]\right)^{\frac{r}{s}}$$

$\forall\, r<s$

if $E\left[|V|^s\right] < \infty \Rightarrow E\left[|V|^r\right] < \infty$

$E\left[|V|^r\right] = \infty \Rightarrow E\left[|V|^s\right] = \infty \;\; \forall\, s > r$

Note: $E\left[X^k\right]$ is the k^{th} moment of X

Convergence : many type of convergence

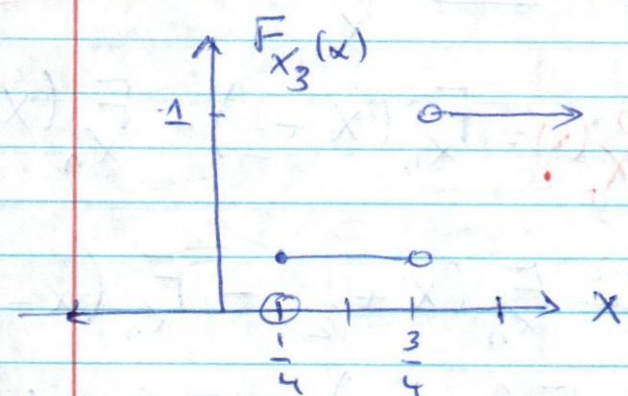
Consider sequences of r.v.'s $X_1, X_2 \cdots$

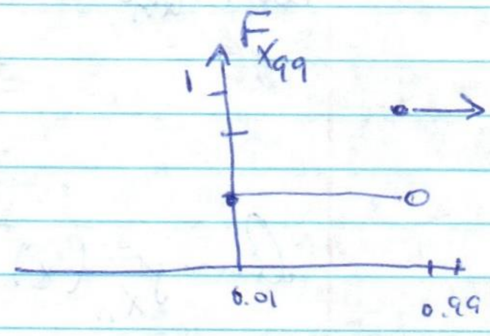
I Convergence in distribution

let $X_n \sim \begin{cases} \frac{1}{n+1} & \text{wp } \frac{1}{3} \\ 1-\frac{1}{n+1} & \text{wp } \frac{2}{3} \end{cases}$

eg $X_3 \sim \begin{cases} \frac{1}{4} & \text{wp } \frac{1}{3} \\ \frac{3}{4} & \text{wp } \frac{2}{3} \end{cases}$

$X_{99} \sim \begin{cases} 0.01 & \text{wp } \frac{1}{3} \\ 0.99 & \text{wp } 2/3 \end{cases}$

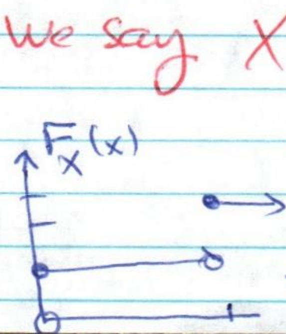




$\forall_x \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ converge pointwise

We say $X_n \xrightarrow{d} X$ ie X_n converge in distribution to X if

$\forall_x \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$



$\Rightarrow X \sim \text{Bern}\left(\frac{2}{3}\right)$

Note 'Chf'

$$\boxed{\forall t \quad \lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X}$$

Levy's Continuity theorem

If $\mathrm{Supp}[X_n] \subseteq \mathbb{N}$ then

and $\mathrm{Supp}[X] \subseteq \mathbb{N}$

Theorem

$$X_n \xrightarrow{d} X \iff \lim_{n\to\infty} p_{X_n}(x) = p_X(x)$$

Proof: $\Rightarrow$

~~$\lim F_{X_n}(x) = F_X(x)$~~

Note $p_{X_n}(x) =$

$$p_{X_n}(x) = F_{X_n}\left(x+\tfrac{1}{2}\right) - F_{X_n}\left(x-\tfrac{1}{2}\right)$$

$$\lim p_{X_n}(x) = \lim F_{X_n}\left(x+\tfrac{1}{2}\right) - F_{X_n}\left(x-\tfrac{1}{2}\right)$$

$$= \lim F_X\left(x+\tfrac{1}{2}\right) - F_X\left(x-\tfrac{1}{2}\right)$$

$$= p_X(x) \checkmark$$

$\Leftarrow \lim_{n\to\infty} F_{X_n}(x) = \lim P(X_n \le x)$

$= \lim_{n\to\infty} \sum_{i=1}^{x} p_{X_n}(i) = \sum_{i=1}^{x} \lim_{n\to\infty} p_{X_n}(i)$

$= \sum_{i=1}^{x} p_X(i) = P(X \le x) = F_X(x)$

$X_n \xrightarrow{d} X \quad X \sim Deg(c)$

$F_X(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & x < c \end{cases}$

If I write $X_n \xrightarrow{d} c$ that mean

$\lim F_{X_n}(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$

We Say: (X_n Converge in Probability)

$X_n \xrightarrow{P} c$, X_n Converge to constant c if

$\forall\, \varepsilon > 0 \ \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$