Lec. 18

Nov. 09

$X_1 \sim N(0,1)$ ind of $X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} = \int_{\text{supp}\,x} |x|\, f_{X_1}(xr)\, f_{X_2}(x)\, dx$$

$$= \int_{\mathbb{R}} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2r^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} |x| e^{-\frac{1}{2}x^2(r^2+1)}\, dx = \frac{1}{2\pi}\left(\int_{-\infty}^{0} (-x) e^{-\frac{1}{2}x^2(r^2+1)}\, dx + \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\, dx \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} x e^{-\frac{1}{2}x^2(r^2+1)}\, dx = \frac{1}{\pi} \int_0^{-\infty} x e^{u} \frac{-1}{x(r^2+1)}\, du$$

with $u = -\frac{1}{2}x^2(r^2+1)$

$\frac{du}{dx} = x(r^2+1)$

$dx = \frac{1}{x(r^2+1)}\, du$

$$= \frac{1}{\pi(r^2+1)} \left[-e^{u} \right]_0^{-\infty}$$

$$= \frac{1}{\pi(r^2+1)} = \text{Cauchy}(0,1)$$

Mid term 4

Final

$X_1, X_2, \dots, X_n \sim f(\mu, \beta)$

α, β are unknown and we want to estimate them (inference

$\bar{X}$ is the average r.v. (Estimator)

$\bar{x}$ is a realizati (Estimate)

$E[\bar{X}] = \mu$

S^2 is the single variance $\quad s^2 = \frac{1}{n-1}\sum(x_i-\bar{x})$ is a realizati

$E[S^2] = \sigma^2$

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \quad s^2 = \frac{1}{n-1}\left((x_1-\bar{x})^2 + \dots + (x_n-\bar{x})^2\right)$

$S^2 \sim ?$

$$\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \quad \sum_{i=1}^{n} z_i^2 = \vec{z}^T \vec{z} \sim \chi^2_n$$

$$\sum z_i^2 = \sum \left(\frac{x_i-\mu}{\sigma}\right)^2 = \frac{\sum(x_i-\mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\cancel{\sum(x_i-\mu)^2 = \sum((x_i-\mu)+(x_i)}$$

$$\sum(x_i-\mu)^2 = \sum((x_i-\bar{x})+(\bar{x}-\mu))^2$$

$$= \sum (x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2$$

$$= \sum(x_i-\bar{x})^2 + 2\left(\sum x_i\bar{x} - \mu\sum x_i - \sum \bar{x}^2 + \sum\bar{x}\mu\right) + n(\bar{x}-\mu)^2$$

$$\Rightarrow \frac{\sum(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\left(\frac{x-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim N^2(\ ) \sim \chi^2_1$$

Conj. $\dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

Cochran's Theorem

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

let $Q_1, \ldots, Q_k$ be scalar r.v. in the quadratic forms.

$Q_j = \vec{Z}^T B_j \vec{Z}$ when $B_1, \ldots B_k$ are positive semi-defn.

a) $n = \sum \text{Rank}(B_j)$

b) Q_j's are independent

c) $Q_j \sim \chi^2_{\text{Rank}(B_j)}$

$$\vec{Z}^T\vec{Z} = Q_1 + \cdots + Q_k$$

$$= \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z} + \cdots + \vec{Z}^T B_k \vec{Z}$$

$$\sim \chi^2_n \quad \vec{Z}^T I \vec{Z} = \vec{Z}^T (B_1 + \cdots + B_k) \vec{Z}$$

$$\Longrightarrow I_n = B_1 + \cdots + B_k$$

$$\sum z_i^2 = \sum\left((z_i - \bar{z}) + (\bar{z})\right)^2$$

$$= \sum (z_i - \bar{z})^2 + \underbrace{2\sum (z_i - \bar{z})\bar{z}}_{2\left(\sum z_i \bar{z} - 2\bar{z}^2\right)} + \sum \bar{z}^2$$

$$\sum z_i^2 = \underbrace{\sum (z_i - \bar{z})^2}_{Q_1} + \underbrace{n\bar{z}^2}_{Q_2}$$

$$Q_2 = n\bar{z}^2 = \vec{z}^T B_2 \vec{z} = \vec{z}^T \left(\frac{1}{n} J_n\right) \vec{z}$$

$$= \vec{z}^T \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} z = z$$

$$J_n = \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix}}_{n \times n}$$

<u>theorem</u>: Matrix A is symmetric independent $\longrightarrow$ $tr(A) = rank(A)$

Matrix A is positive semi def. if $\forall \vec{v}, \ldots, \vec{v}^T A \vec{v} = 0$

$n\bar{z}^2 \sim \chi^2_1$

ind. of

$\sum(z_i-\bar{z})^2 \sim \chi^2_{n-1}$

$$\frac{n(\bar{X}-\mu)^2}{\sigma^2} = \frac{1}{\sigma}(\vec{X}-\vec{\mu})^T\left(\frac{1}{n}J_n\right)\frac{1}{\sigma}(\vec{X}-\vec{\mu}) \sim \chi^2_1$$

$$\frac{\sum(X_i-\bar{X})^2}{\sigma^2} = \frac{1}{\sigma}(\vec{X}-\vec{\mu})^T\left(I_n-\frac{1}{n}J_n\right)\frac{1}{\sigma}(\vec{X}-\vec{\mu}) \sim \chi^2_{n-1}$$

$$= \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

with $S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$

$S^2, \bar{X}$ are independent

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \text{"z test"}$$

T-test

$$\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{s^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}s^2}}$$

$$= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{n-1}{\sigma^2}s^2}{n-1}}} = \frac{z}{\sqrt{\frac{\chi^2}{n-1}}} = T\text{-test}$$