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• $X \sim \text{Gamma}(k_1, \lambda)$ ind. of $Y \sim \text{Gamma}(k_2, \lambda)$

$X+Y \sim \text{Gamma}(k_1+k_2, \lambda)$

$$T = X+Y \sim \cdots \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} a^{k_1-1} (1-a)^{k_2-1} t\, da$$

$$= \frac{\lambda^{k_1+k_2} t^{k_1+k_2-1} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} du$$

$$\propto e^{-\lambda t} t^{k_1+k_2-1} \propto \text{Gamma}(k_1+k_2, \lambda)$$

"kernel"

Proportional • $X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \propto e^{-\lambda x} = k(x) = cf(x)$

$\propto$ directly proportional to

$$1 = \int_{\text{supp}(x)} f(x)\, dx$$

$$\int_{\text{supp}(x)} k(x)\, dx = c$$

$$\int \frac{1}{c} k(x) = 1$$

• $X \sim \text{bin}(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \propto (x!(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x$$

• $X \sim \text{Weibull}(k, \lambda) = k\lambda(x\lambda)^{k-1} e^{-(x\lambda)^k} \propto x^{k-1} e^{-(x\lambda)^k}$

$$X \sim \text{Gamma}(k, \lambda) = \frac{\lambda^k e^{\lambda x} x^{k-1}}{\Gamma(k)} \propto e^{\lambda x} x^{k-1}$$

$\cdot\ X \sim \text{Gamma}(k_1+k_2, \lambda) = \dfrac{\lambda^{k_1+k_2} e^{-\lambda t}\, t^{k_1+k_2-1}}{\Gamma(k_1+k_2)}$

$= \dfrac{\lambda^{k_1+k_2} e^{-\lambda t}\, t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \displaystyle\int_0^1 u^{k_1-1} (1-u)^{k_2-1}\, du$

$\Rightarrow \dfrac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} = \displaystyle\int_0^1 u^{k_1-1} (1-u)^{k_2-1}\, du$

$\cdot\ B(\alpha, \beta)$ "Beta f-n"

$:= \displaystyle\int_0^1 t^{\alpha-1} (1-t)^{\beta-1}\, dt$

$\beta(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$\displaystyle\int_0^1 t^{\alpha-1} (1-t)^{\beta-1}\, dt = \dfrac{\int_0^\infty t^{\alpha-1} e^{-t}\, dt \int_0^\infty t^{\beta-1} e^{-t}\, dt}{\int_0^\infty t^{\alpha+\beta-1} e^{-t}\, dt}$

- Order statistics:

$X_1\ X_2 \dots X_n$ are a sequence of continuous r.v's

$X_{(1)}\ X_{(2)} \dots X_{(n)}$ are called "order statistic's

where $X_{(1)} < X_{(2)} < \dots < X_{(n)}$

- $X_{min} = X_{(1)} = \min\{X_1 \dots X_n\}$

$X_{max} = X_{(n)} = \max\{X_1 \dots X_n\}$

- $R = X_{max} - X_{min}$ (range)

In the case of $X_1 \dots X_n \sim f(x)$ will CDF $F(x)$

Ex max. survival

- X_{max}:

maximum is less or equal than it

one of them is max

We want $F_{X(n)}(x) = P(X_{(n)} \le x)$

$= P(X_1 \le x,\ X_2 \le x \dots X_n \le x)$

$= P(X_1 \le x) \dots P(X_n \le x)$

Def. of CDF

$= F_{X_1}(x) \dots F_{X_n}(x)$

$= F(x)^n$

U-substitution

- $f_{x(n)}(x) = n F'_{X_n}(x) F(x)^{n-1}$

- $F_{X(1)}(x) := P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$

$= 1 - P(X_1 > x \dots X_n > x)$

$= 1 - P(X_1 > x) \dots P(X_n > x)$

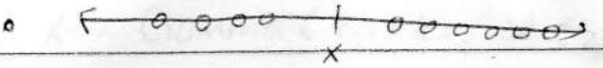
$= 1 - (1 - F_{X_1}(x)) \cdots (1 - F_{X_1}(x))$

$= 1 - (1 - F(x))^n$

$f_{X(1)}(x) = F'_{X(1)}(x) = -n(1-F(x))^{n-1}(-f(x))$

$= n f(x)(1-F(x))^{n-1}$

let
$n=10$



$P(X_1, X_2, X_3, X_4 \in (-\infty, x) \; \& \; X_5 \ldots X_{10} \in (x, \infty))$

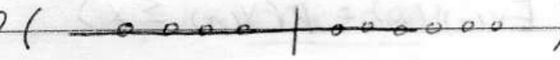
$= P(X_1 \leq x) \cdots P(X_4 \leq x) \, P(X_5 > x) \cdots P(X_{10} > x)$

$= F(x)^4 (1-F(x))^6$

$P(\text{any } 4 \in (-\infty, x), \text{ the other } 6 \in (x, \infty))$

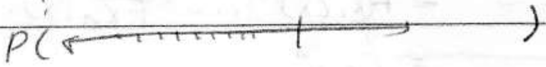
$= \binom{10}{4} F(x)^4 (1-F(x))^6$

$F_{X_{(4)}}(x) := P(X_{(4)} \leq x) =$

$P($  $)$

$+ P($  $)$

$\vdots$

$+ P($  $)$

$$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$$

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} F(x)^j (1-F(n))^{n-j}$$

$$= \cancel{\binom{n}{n}} F(x)^n \cancel{(1-F(x))^{n-n}}$$

$$= F(x)^n$$

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} \right) - (1-F(x))^n$$

$1^n \rightarrow 1$

$= ((F(x)) + (1-F(x)))^n - (1-F(x))^n$

Product Rule

$$\cdot f_{X(k)}(x) = F'_{X(k)}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n}\frac{n!}{j!(n-j)!}\,\frac{d}{dx}\underbrace{\left[F(x)^j(1-F(x))^{n-j}\right]}$$

$$(1-F(x))^{n-j}\, j\, F(x)^{j-1} f(x)$$

$$- F(x)^j (k-j)(1-F(x))^{k-j-1} f(x)$$

$$= f(x)\left(\sum_{j=k}^{n}\frac{n!}{\cancel{j}!(n-j)!}\left((1-F(x))^{k-j}\cancel{j}F(x)^{j-1}f(x)\right)\right.$$

$$\left.- \sum_{j=k+1}^{n-1}\frac{n!}{j!\underbrace{\cancel{(n-j)!}}_{(n-j-1)!}} F(x)^j (\cancel{k}-j)(1-F(x))^{k-j-1} f(x)\right)$$

$$= (a_k + a_{k+1} + \dots + a_n) - (a_{k+1} + \dots + a_n)$$

$$= a_k$$

$$= \frac{n!}{(k-1)!(n-k)!} f(x)(1-F(x))^{n-k} F(x)^{k-1}$$