Let $X_1, X_2 \overset{iid}{\sim} \text{Bern}(\frac{1}{2})$ and $T_2 = X_1 + X_2$. For Bern(p):

$$\begin{aligned}
\mathbb{P}_{T_2}(x) &= \mathbb{P}_{X_1}(x) \cdot \mathbb{P}_{X_2}(x) \\
&= \sum_{x \in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x) \\
&= \sum_{x \in \{0,1\}} p^x(1-p)^{1-x}p^{t-x}(1-p)^{1-t+x} \\
&= \sum_{x \in \{0,1\}} p^t(1-p)^{2-t} \\
&= p^t(1-p)^{2-t} \underbrace{\sum 1}_{2} \\
&= 2p^t(1-p)^{2-t}
\end{aligned}$$

This was wrong.

$$p(2) = p^0(1-p)^{1-0} \qquad \underbrace{p^{2-0}(1-p)^{t-2}}_{\text{turned off using indicator function}} \qquad +p^1(1-p)^{t-1}p^{2-1}(1-p)^{1-2+1}$$

Let $X_1, X_2 \overset{iid}{\sim} \text{Binom}(n, p)$. Let $Y = X_1 + X_2$. Then

$$\begin{aligned}
\mathbb{P}_Y(x) &= \mathbb{P}_{X_1}(x) \cdot \mathbb{P}_{X_2}(x) \\
&= \sum_{x \in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(y-x) \\
&= \sum_{x=0}^{n} \binom{n}{x} p^x(1-p)^{n-x} \underbrace{\mathbb{1}_{x \in \{0,1,\ldots,n\}}}_{\text{not needed}} \binom{n}{y-x} p^{y-x}(1-p)^{1-y+x} \underbrace{\mathbb{1}_{y-x \in \{0,1,\ldots,n\}}}_{\text{not needed}} \\
&= \sum_{x \in \{0,1,\ldots,n\}} \binom{n}{x} p^x(1-p)^{n-x} \binom{n}{y-x} p^{y-x}(1-p)^{n-y+x} \\
&= p^y(1-p)^{2n-y}\binom{2n}{y} \text{ by Vandermonde's Identity} \\
&= \text{Binom}(2n, p)
\end{aligned}$$

Consider $B_1, B_2, \cdots \overset{iid}{\sim} \text{Bern}(p)$. Let $X = \overset{\min}{t}\ \{B_t = 1\} - 1$. This is called a geometric random variable. So $X \sim \text{Geom}(p)$. $\text{Supp}[X] = \{0, 1, \ldots\} = \mathbb{N}$. Parameter Space: $0 < p < 1$. In fact

$$\begin{aligned}
\mathbb{P}(X = 0) &= p \\
\mathbb{P}(X = 1) &= (1-p)p \\
\mathbb{P}(X = 2) &= (1-p)^2 p \\
\mathbb{P}(X = x) &= (1-p)^x p
\end{aligned}$$

Now, for the convolution of Geom(p). Let $T_2 = X_1 + X_2$.

$$\begin{aligned} p(t) &= \mathbb{P}_{X_1}(x) \cdot \mathbb{P}_{X_2}(x) \\ &= \sum_{x \in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x) \\ &= \sum_{x \in \mathbb{N}_0} (1-p)^x p(1-p)^{t-x} p \mathbb{1}_{t-x \in \mathbb{N}_0} \\ &= (1-p)^t p^2 (t+1) \end{aligned}$$

Now Supp$[T_2] = \{0, 1, \dots\}$. Let $T_3 = X_1 + X_2 + X_3 = X_3 + T_2$.

$$\begin{aligned} p(t) &= \mathbb{P}_{X_3}(x) \cdot \mathbb{P}_{T_2}(x) \\ &= \sum_{x \in \text{Supp}[X_3]} \mathbb{P}_{X_3}(x)\mathbb{P}_{T_2}(t-x) \\ &= \sum_{x \in \mathbb{N}_0} (1-p)^x p(t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \text{Supp}[T_2] = \mathbb{N}_0} \\ &= p^3(1-p)^t \sum_{x \in \mathbb{N}_0} (t-x+1)\mathbb{1}_{x \le t} \\ &= (1-p)^t p^3 \Big((t+1) \sum_{x \in \mathbb{N}_0} \mathbb{1}_{x \le t} - \sum_{x \in \mathbb{N}_0} x \mathbb{1}_{x \le t} \Big) \\ &= (1-p)^t p^3 \Big((t+1) \underbrace{\sum_{x=0}^{t} 1}_{t+1} - \underbrace{\sum_{x=0}^{t} x}_{\frac{t(t+1)}{2}} \Big) \\ &= (1-p)^t p^3 \Big(\frac{t^2 + 3t + 2}{2} \Big) \end{aligned}$$

In fact, $T_3 =$ number of failures until 3 successes.

$$\mathbb{P}(T_3 = t) = \binom{t+2}{2} (1-p)^t p^3$$

Note that

$$\binom{t+2}{2} = \frac{(t+2)!}{2!t!} = \frac{(2+t)(1+t)}{2} = \frac{t^2+3t+2}{2}$$

These have a name. $T_2 \sim \text{NegBinom}(2, p)$. $T_3 \sim \text{NegBinom}(3, p)$.

Let $X \sim \text{Binom}(n, p)$ where Supp$[X] = \{0, \dots, n\}$. What if n is really big? What if p is really small? Let n and p be related such that $\lambda = np$ or $p = \frac{\lambda}{n}$. What is the pmf if

$n \to \infty$?

$$\begin{aligned}
\lim_{n\to\infty} p(x) &= \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \\
&= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \\
&= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}}_{1} \\
&= \frac{\lambda^x e^{-\lambda}}{x!} = \text{Poisson}(\lambda)
\end{aligned}$$

Let $X \sim \text{Poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$. $\text{Supp}[X] = \{0, 1, \dots\} = \mathbb{N}_0$. Parameter Space: $\lambda \in (0, \infty)$.

Convolution of Poisson: Let $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$. Let $T = X_1 + X_2$.

$$\begin{aligned}
p(t) &= \sum_{x\in \text{Supp}[X]} \mathbb{P}_{X_1}(x)\mathbb{P}_{X_2}(t-x) \\
&= \sum_{x\in\mathbb{N}_0} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{x\le t} \\
&= \lambda^t e^{-2\lambda} \sum_{x\in\mathbb{N}_0} \frac{1}{x!(t-x)!} \mathbb{1}_{x\le t} \frac{t!}{t!} \\
&= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x\in\mathbb{N}_0} \binom{t}{x} \mathbb{1}_{x\le t} \\
&= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x=0}^{t} \binom{t}{x} \\
&= \frac{\lambda^t e^{-2\lambda}}{t!} \cdot 2^t \\
&= \frac{(2\lambda)^t e^{-2\lambda}}{t!} \\
&= \text{Poisson}(2\lambda)
\end{aligned}$$