Nov 28 Lec 20

$\vec{Z} \sim N_n(\vec{0}_n, I_n)$ $E(\vec{Z}) = \vec{0}$

$Z_1, \dots Z_n \sim N(0,1) \Rightarrow Var(\vec{Z}) = I_n$

$\vec{X} = \vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, I_n)$

$\vec{X} = A\vec{Z}$, $A \in \mathbb{R}^{m \times n}$ $\Rightarrow$ $\boxed{E(\vec{X}) = AE(\vec{Z}) = A\vec{0}_n = \vec{0}_n}$

$\Sigma = Var(\vec{X}) = A\,Var(\vec{Z})A^T$
$= \boxed{AA^T = \Sigma}$ because $Var(\vec{Z}) = I_n$

$\vec{X} = g(\vec{Z}) = A\vec{Z}$

$Z = h(\vec{X}) = A^{-1}\vec{X}$ h = inv. funct

We need $m = n$ i.e. A is square and A is full matrix

$\vec{Z} = A^{-1}\vec{X} = \begin{bmatrix} h_1(\vec{X}) \\ \vdots \\ h_n(\vec{X}) \end{bmatrix}$

$$\boxed{f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\left|J_h(\vec{x})\right|}$$

Let $B = A^{-1} = \begin{bmatrix} \leftarrow \vec{b}_{1\cdot} \rightarrow \\ \leftarrow \vec{b}_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow \vec{b}_{n\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{b}_{\cdot 1} & \vec{b}_{\cdot 2} & \cdots & \vec{b}_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

$$= \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

$\vec{z} = B\vec{x}$

$\Downarrow$

$h(\vec{x})$ $\begin{bmatrix} h_1(\vec{x}) = \vec{b}_{1\cdot}\vec{x} = b_{11}x_1 + \cdots + b_{1n}x_n \\ h_2(\vec{x}) = \vec{b}_{2\cdot}\vec{x} = b_{21}x_1 + \cdots + b_{2n}x_n \\ \vdots \\ h_n(\vec{x}) = \vec{b}_{n\cdot}\vec{x} = b_{n1}x_1 + \cdots + b_{nn}x_n \end{bmatrix}$

$$\vec{J}_n = \det\left(\begin{bmatrix} \frac{\partial}{\partial x_1}[h_1(x)] & \cdots & \frac{\partial}{\partial x_n}[h_1(\vec{x})] \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}[h_n(x)] & \cdots & \frac{\partial}{\partial x_n}[h_n(\vec{x})] \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right) = \det B = \det A^{-1}$$

Aside

$$\frac{\partial}{\partial \vec{x}}[C\vec{x}] = C$$

$$\det(A^{-1}) = \frac{1}{\det A}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I) = 1$$

$$\det(A) \cdot \det(A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

$\mathcal{N}(0, 1) \rightarrow \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$

$\rightarrow \frac{1}{(2\pi)^{n/2}} e^{-(A^{-1}\vec{x})^T(A^{-1}\vec{x})} \cdot \frac{1}{|\det(A)|}$

$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\vec{x}^T(A^{-1})^T A^{-1}\vec{x})}$ know that $(A^{-1})^T = (A^T)^{-1}$

$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T(A^T)^{-1}A^{-1}\vec{x}} \cdot \frac{1}{|\det A|}$ with $A^T(A^T)^{-1} = I(A^T)^{-1}A^T = I$

$\Sigma = AA^T$

$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1}(AB) = I$

$B^{-1}A^{-1}AB = I$

I

I

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T (A^T)^{-1} A^{-1} \vec{x}} \cdot \frac{1}{|\det A|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T \Sigma^{-1} \vec{x}} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}\vec{x}^T \Sigma^{-1} \vec{x}} = N_n(\vec{0}_n, \underset{AA^T}{\underset{\parallel}{\Sigma}})$$

With $\det(\Sigma) = \det(A)\det(A^T)$

$$= \det(A)^2 \Rightarrow \det(A) = \sqrt{\det(\Sigma)}$$

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

General MVN

Recall $\phi_X(t) = E[e^{itX}]$

generalize to $\phi_{\vec{X}}(\vec{t}) = E[e^{i\vec{t}^T\vec{X}}]$

$\phi_{X_1+X_2}(t) = E[e^{i\vec{t}^T(\vec{X}_1+\vec{X}_2)}]$ X_1 and X_2 are ind.

$= E[e^{i\vec{t}^T X_1} e^{i\vec{t}^T X_2}] = E[e^{i\vec{t}^T X_1}] E[e^{i\vec{t}^T X_2}]$

$= \phi_{X_1}(\vec{t})\, \phi_{X_2}(\vec{t})$

$\vec{Y} = A\vec{X} + \vec{c}$ $A \in \mathbb{R}^{m\times n}$ $\vec{c} \in \mathbb{R}^m$

$\vec{X} \in \mathbb{R}^n$, $\vec{Y} \in \mathbb{R}^m$

$\phi_Y(t) = E[e^{i\vec{t}^T\vec{Y}}] = E[e^{i\vec{t}^T(A\vec{X}+\vec{c})}]$

$$= E\left[e^{i\vec{t}^T A\vec{X}} e^{i\vec{t}^T\vec{c}}\right] = e^{i\vec{t}^T\vec{c}} E\left[e^{i\vec{t}^T A\vec{X}}\right]$$

let $t'^T = t^T A$

$$t' = (t^T A)^T = A^T\vec{t}$$

$$= e^{i\vec{t}^T\vec{c}} E\left[e^{i t'^T\vec{X}}\right] = e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(\vec{t}')$$

$$= e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(A^T\vec{t})$$

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{Z}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{Z}}\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\vec{t}^T\vec{z}} f_{\vec{Z}}(\vec{z})\, d\vec{z}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t_1 z_1 + \cdots + t_n z_n)} \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(z_1^2 + \cdots + z_n^2)}\, dz_1 \cdots dz_n$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{j=1}^{n} e^{i t_j z_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_j^2} \, dz_i$$

$$= \prod_{j=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{i t_j z_j - \frac{1}{2} z_j^2} \, dz_j$$

$$= \prod_{j=1}^{n} e^{-\frac{1}{2} t_j^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_j - i t_j)^2} \, dz_j$$

$$-\frac{1}{2} z^2 + i t z = -\frac{1}{2}\left(z^2 - 2 i t z\right)$$

$$= -\frac{1}{2}\left((z - it)^2 + t^2\right)$$

$$\boxed{\begin{aligned}
&A \in \mathbb{R}^{n \times n} \qquad \vec{X} = A\vec{z} + \vec{\mu} \\
&\vec{Y} = A'\vec{X} = B\vec{X} \qquad B \in \mathbb{R}^{m \times n} \\
\Rightarrow\ & \phi_{\vec{Y}}(\vec{t}) = \phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T B\vec{\mu} - \frac{1}{2}\vec{t}^T B \Sigma B^T \vec{t}} \\
\Rightarrow\ & \vec{Y} \sim N_m(B\mu, B\Sigma B^T)
\end{aligned}}$$

$$\vec{X} = A\vec{z} + \vec{\mu} \Rightarrow z = A^{-1}(\vec{X} - \vec{\mu})$$

$$z^T z \sim \chi^2_n$$

$$(A^{-1}(\vec{X} - \vec{\mu}))^T (A^{-1}(\vec{X} - \vec{\mu})) \Rightarrow (\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \mu) \sim \chi^2_n$$

$$(\vec{X} - \vec{\mu})(A^{-1})^T A^{-1}(\vec{X} - \vec{\mu})$$

$$(A^T)^{-1} A^{-1}$$

$$(AA^T)^{-1}$$

$$\Sigma^{-1}$$

$$X \sim N_n(\vec{\mu}, \Sigma) \Rightarrow$$

$$\frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}((\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu}))}$$

$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\Rightarrow \vec{X} \sim N_n(\mu \mathbb{1}_n, \sigma^2 I_n)$$

$$\Sigma = \sigma^2 I = \underbrace{\sigma I}_{A} \underbrace{(\sigma I)^T}_{A^T}$$

$$\vec{X} = \sigma I \vec{z} + \vec{\mu} = \sigma \vec{z} + \vec{\mu}$$

$$(\vec{X} - \vec{\mu})^T \frac{1}{\sigma^2} I (\vec{X} - \vec{\mu}) \sim \chi^2$$

$$\Sigma \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$