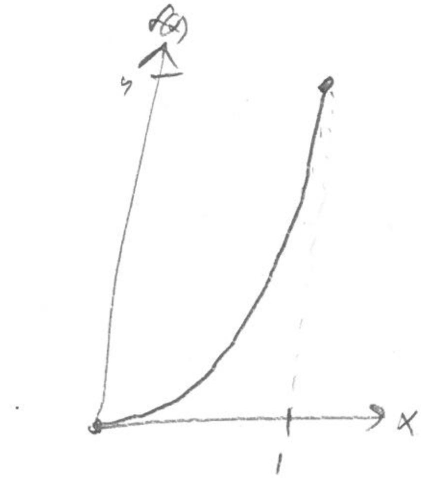
O.S. Example! $X_1, \dots, X_n \overset{iid}{\sim} U(0,1)$ What does the max look like?

$f(x) = 1,\ F(x) = x$

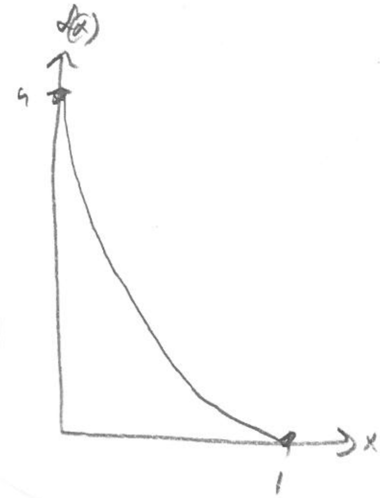
$$f_{X_{(n)}}(x) = n \underbrace{f(x)}_{1} \underbrace{F(x)}_{x}^{\,n-1} = n x^{n-1} = \text{Beta}(n,1)$$

$\text{Supp}(X_{(n)}) = (0,1)$



What does the min look like?

$$f_{X_{(1)}}(x) = n \underbrace{f(x)}_{1} \Big(1 - \underbrace{F(x)}_{x}\Big)^{n-1} = n(1-x)^{n-1} = \text{Beta}(1,n)$$



What does the k^{th} order statistic look like?

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} \underbrace{f(x)}_{1} \Big(\underbrace{F(x)}_{x}\Big)^{k-1} \Big(1 - \underbrace{F(x)}_{x}\Big)^{n-k} \propto \underbrace{x^{k-1}(1-x)^{n-k}}_{= k(x)}$$

Recall $\int_{\text{Supp}(X)} k(x)\,dx = c \Rightarrow f(x) = \frac{1}{c} k(x)$

What is $\int_0^1 x^{k-1}(1-x)^{n-k}\,dx = \int_0^1 x^{k-1}(1-x)^{(n-k+1)-1}\,dx = B(k, n-k+1)$

$$\Rightarrow f_{X_{(k)}}(x) = \frac{1}{B(k, n-k+1)} x^{k-1}(1-x)^{n-k+1-1} = \text{Beta}(k,\ n-k+1)$$

In general $X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

$\text{Supp}(X) = (0,1)$, $\alpha > 0$, $\beta > 0$

$$\int_{\text{Supp}(X)} f(x) = 1 \Rightarrow \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\cancel{B(\alpha,\beta)}}{\cancel{B(\alpha,\beta)}} = 1$$

$$F(x) = P(X \le x) = \underbrace{\frac{1}{B(\alpha,\beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt}_{B(x,\alpha,\beta)} = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$

incomplete Beta func

regularized incomplete beta function

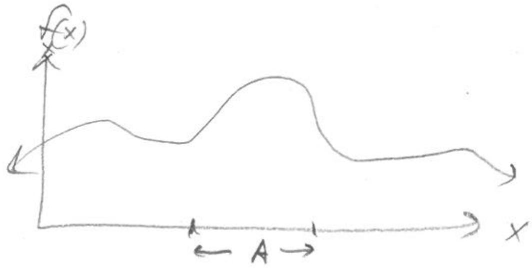
$$E(X) = \int_0^1 x \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

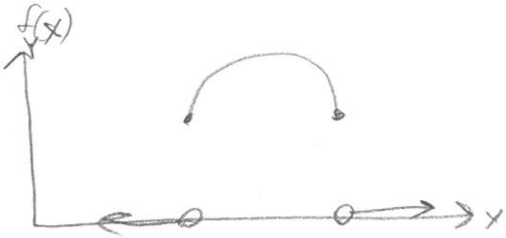
$$= \frac{\dfrac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}}{\dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\dfrac{\alpha\,\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}{(\alpha+\beta)\,\cancel{\Gamma(\alpha+\beta)}}}{\dfrac{\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}{\cancel{\Gamma(\alpha+\beta)}}}$$

Truncations

$X \sim f(x)$. What if we know that $X \in A$. What is distr of this new r.v. Y

where $A \subseteq Supp(X)$

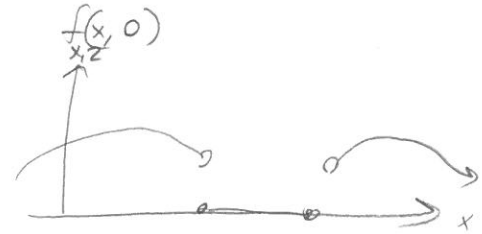


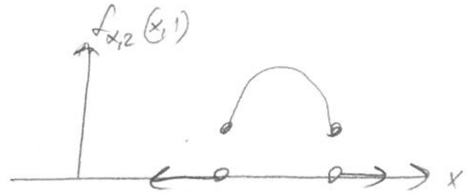


How do we get $f_Y(x)$?

let $Z = \mathbb{1}_{X \in A} \sim Bern(P(X \in A))$

$$f_{X|Z}(x,z) = \frac{f_{X,Z}(x,z)}{p_Z(z)} = \frac{f(x)\, \mathbb{1}_{x\in A}^{z}\, \mathbb{1}_{x \notin A}^{1-z}}{P(X\in A)^z (1-P(X\in A))^{1-z}}$$





$$f_{X,Z}(x,z) = f(x)\, \mathbb{1}_{x\in A}^{z}\, \mathbb{1}_{x\notin A}^{1-z}$$

$$f_Y(x) = f_{X|Z}(x,1) = \frac{f(x)}{P(X\in A)} \mathbb{1}_{x\in A}$$

Is this a PDF? $\int_{Supp(X)} \frac{f(x)}{P(X\in A)} \mathbb{1}_{x\in A}\, dx = \int_A \frac{f(x)}{P(X\in A)}\, dx = \frac{P(X\in A)}{P(X\in A)} = 1$ ✓

Typical truncations $X \geq a$, $X \leq a$ $\qquad x \in (a,b)$

$$f_Y(x) = \frac{f(x)}{1-F(a)} \mathbb{1}_{x \geq a} \qquad f_Y(x) = \frac{f(x)}{F(a)} \mathbb{1}_{x\leq a} \qquad f_Y(y) = \frac{f(x)}{F(b)-F(a)} \mathbb{1}_{x\in(a,b)}$$

$X \sim Exp(\lambda)$ $\qquad X \geq a$

$$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}} \mathbb{1} = \lambda e^{-\lambda(x-a)} \mathbb{1}_{x\geq a}$$

let $g: \mathbb{R}^n \to \mathbb{R}^n$, a one-to-one function

let $\vec{X}$ be a r.v. vector of dim n, $\vec{Y}$ be a r.v. vector of dim n

$\vec{Y} = g(\vec{X})$. We know $f_{X_1,\dots,X_n}(x_1,\dots x_n)$ and we want to find $f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)$.

$$Y_1 = g_1(X_1,\dots,X_n) \qquad X_1 = h_1(Y_1,\dots,Y_n)$$
$$Y_2 = g_2(X_1,\dots,X_n)$$
$$\vdots \qquad \qquad \vdots$$
$$Y_n = g_n(X_1,\dots,X_n) \qquad X_n = h_n(Y_1,\dots,Y_n)$$

Since g is 1:1 then $\exists h$ $\vec{X} = h(\vec{Y})$

multivariate change of variable formula is

$$f_{Y_1,\dots,Y_n}(y_1,\dots y_n) = f_{X_1,\dots,X_n}(h_1(y_1,\dots,y_n),\dots,h_n(y_1,\dots y_n)) \left| J_h(y_1,\dots,y_n) \right|$$

where J_h is the "Jacobian determinant" for function h def. by

$$\det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_n} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}\right)$$

Let's make sure the univariate case works... $Y = g(X) \Rightarrow X = \underbrace{g^{-1}}_{\text{"}h\text{"}}(Y)$

$$f_Y(y) = f_X(h(y)) \left|\det\left(\left[\frac{\partial h}{\partial y}\right]\right)\right| = f_X(g^{-1}(y)) \left|\frac{\partial g^{-1}(y)}{\partial y}\right| \checkmark$$

A typical example

$$Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$$

$$Y_2 = X_2 = g_2(X_1, X_2)$$

we really want $f_{Y_1}(y_1)$ the density of the quotient

$$X_1 = Y_1 Y_2 = \frac{X_1}{\cancel{X_2}} \cancel{X_2} = h_1(Y_1, Y_2) \qquad \frac{\partial h_1}{\partial y_1} = y_2 \qquad \frac{\partial h_1}{\partial y_2} = y_1$$

$$X_2 = Y_2 = h_2(Y_1, Y_2) \qquad \frac{\partial h_2}{\partial y_1} = 0, \qquad \frac{\partial h_2}{\partial y_2} = 1$$

$$J_h = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}\right) = (y_2)(1) - (y_1)(0) = y_2$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2)\,|y_2|$$

If X_1, X_2 independent and positive

$$f_{Y_1}(y_1) = \int_{\text{Supp}(Y_2)} f_{Y_1,Y_2}(y_1, y_2)\,dy_2 = \int_{\text{Supp}(Y_2)} f_{X_1,X_2}(y_1 y_2, y_2)\,|y_2|\,dy_2 = \int_{\text{Supp}[X_2]} x_2 f_{X_1}(x_2 y_1) f_{X_2}(x_2)\,dx_2$$

the ratio of two r.v.'s is a simple function. Is there graphic way to demonstrate this? p150-151

$$Y_1 = \frac{X_1}{X_2} \qquad F_{Y_1}(y_1) = \iint_{\{(x_1,x_2):\ \frac{x_1}{x_2} \le y_1\}} f_{X_1,X_2}(x_1, x_2)\,dx_1\,dx_2 = \dots$$

Difficult to explain in a lecture...

let $v = \frac{x_1}{x_2}$

Another example:

$$Y_1 = \frac{X_1}{X_1+X_2} \qquad X_1 = Y_1 Y_2 = h_1(y_1,y_2)$$

$$Y_2 = X_1 + X_2 \qquad X_2 = Y_2 - Y_1 Y_2 = h_2(Y_1,Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2 \qquad \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = -y_2 \qquad \frac{\partial h_2}{\partial y_2} = 1-y_1$$

$$J_h = \det\left(\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix}\right)$$

$$= y_2(1-y_1) + y_1 y_2$$

$$= y_2 - y_2 y_1 + y_2 y_1 = y_2$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1 y_2,\ y_2 - y_1 y_2)\,|y_2| \qquad f_{Y_1}(y_1) = \int_{\text{Supp}[Y_2]} \cdots\, dy_2$$

If $X_1 \sim \text{Gamma}(\alpha,\lambda)$ ind. $X_2 \sim \text{Gamma}(\beta,\lambda)$

What is distr of $Y_1 = \frac{X_1}{X_1+X_2}$?

independence

y_2 always > 0

$$f_{Y_1}(y_1) = \int_0^\infty f_{X_1}(y_1 y_2)\, f_{X_2}(y_2 - y_1 y_2)\, y_2\, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (y_1 y_2)^{\alpha-1} e^{-\lambda y_1 y_2}}{\Gamma(\alpha)} \frac{\lambda^\beta (y_2(1-y_1))^{\beta-1} e^{-\lambda y_2(1-y_1)}}{\Gamma(\beta)}\, y_2\, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y_1^{\alpha-1}(1-y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} \overbrace{e^{-\lambda y_1 y_2 - \lambda y_2(1-y_1)}}^{-\lambda(\cancel{y_1 y_2} + y_2 - \cancel{y_1 y_2})}\, dy_2$$

$$e^{-\lambda y_1 y_2 - \lambda y_2(1-y_1)} = e^{-\lambda y_2}$$

let $u = \lambda y_2 \Rightarrow \frac{du}{dy_2} = \lambda \Rightarrow dy_2 = \frac{1}{\lambda} du$

$\Rightarrow y_2 = \frac{u}{\lambda}$