Thu 09/14/2017

X, Y are continuous rv's with
Jdf = Joint density Functi $f_{X,Y}(x,y)$
and $Z = g(X,Y)$

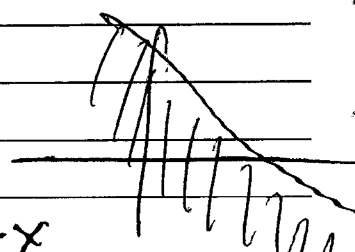
$$\underbrace{F_Z(z)}_{cdf} = P(\underbrace{Z \le z}_{cdf}) = P(\underbrace{g(X,Y) \le z}_{cdf})$$

$$\iint_{\{(x,y):\, g(x,y) \le z\}} f_{X,Y}(x,y)\,dx\,dy$$

$T = X + Y$

$$\int_{-\infty}^{z} f_Z(t)\,dt \text{ must be PDF of } Z$$

$$F_T(t) = \iint_{\{(x,y)\ x+y \le t\}} f_{X,Y}(x,y)\,dx\,dy \qquad y \le t - x$$



$$\int_{\mathbb{R}} \left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy \right) dx$$

let $t = x + y$

$\Rightarrow y = t - x$

$y_{lower} = -\infty = t_l - x \Rightarrow t_l = -\infty$

$y_{upper} = z - x = t - x \Rightarrow t_u = z$

$$\int_{\mathbb{R}} \left(\int_{-\infty}^{z} f_{X,Y}(x, t-x)\, dt \right) dx = \int_{-\infty}^{z} \left(\int_{\mathbb{R}} f_{X,Y}(x, t-x)\, dx \right) dt$$

convo of dependent

$$f_T(t) = f_X(x) \cdot f_Y(x) = (f_X \cdot f_Y)(x)$$

Convo of ind. X, Y $= f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

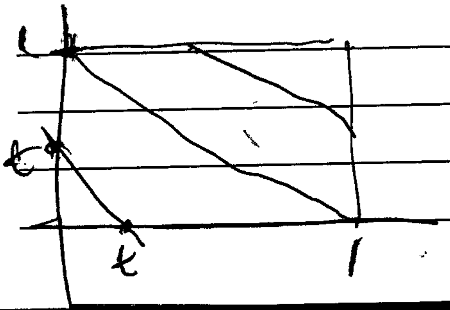
$$f_T(t) = f_X(x) f_Y(x) = \int_{\mathbb{R}} f(x) f_Y(t-x)\, dx$$

$$\int_{supp(X)} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in support}$$

$X, Y \overset{iid}{\sim} U(0,1)$

$T = X + Y \sim f(t)$

$f_{X,Y}(x,y) = 1 \cdot \mathbb{1}_{x \in [0,1]}$ α $y \in [0,1]$



$$\underset{CDF}{F_T(t)} = \begin{cases} \frac{t^2}{2} & \text{if } t \in [0,1] \\ \frac{1}{2} + \left(\frac{1}{2} - \frac{(2-t)^2}{2} \right) & \text{if } t \in [1,2] \end{cases}$$

PDF = derivative of CdF

$$f_T(t)_{PDF} = \begin{cases} t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in [1,2] \end{cases}$$

$X_1, X_2 \overset{iid}{\sim} U(a,b) = \frac{1}{b-a}$

$T = X_1 + X_2$

$\text{Supp}(X_1) = \text{Supp}(X_2) = [a,b]$

$\text{Supp}(T) = [2a, 2b]$

$$f_T(t) = \int_{\text{supp}} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in \text{Supp}(X_2)} dx$$

$$= \int_a^b \left(\frac{1}{b-a}\right)^2 \mathbb{1}_{t-x \in [a,b]} dx = \frac{1}{(b-a)^2} \int dx = \frac{1}{(b-a)^2} \left(\min\{b, t-a\} - \max\{a, t-b\} \right)$$

$$f_T(t) = \frac{\mathbb{1}_{t \in [2a, 2b]}}{(b-a)^2} \begin{cases} t-2a & \text{if } t \le a+b \\ 2b-t & \text{if } t \ge a+b \end{cases}$$

$X \sim Geom(p) = (1-p)^x p$

$F(x) = P(X \leq x) = 1 - P(X > x) = 1 - (1-p)^x \Rightarrow$

$1 - F(x) = (1-p)^x$

$p(t) = (1-p)^{tn} p$ if $n \to \infty$, $p \to 0$ but $\lambda = np$

$p(t) = \frac{\lambda}{n}^{nt}$ $\lim_{n\to\infty} p(t) = \lim_{n\to\infty} (1 - \frac{\lambda}{n})^{nt} \lim_{n\to\infty} \frac{\lambda}{n} = 0$

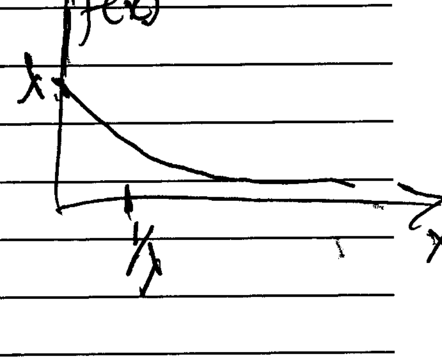
$\forall t$, $p(t) = 0 \Rightarrow$ PMF doesn't exist.

~~PDF~~ $F_n(t) = 1 - (1 - \frac{\lambda}{n})^{nt}$

$F(t) = \lim_{n\to\infty} (1 - \frac{\lambda}{n})^{nt} = 1 - e^{-\lambda t} \Rightarrow 1 - F(t) = e^{-\lambda t}$

~~PDF~~ $f(t) = F'(t) = \lambda e^{-\lambda t}$

$X \sim Exp(\lambda) = \lambda e^{-\lambda x}$
Exponential rule



$$P(X>a+b \mid X>b) = \frac{P(X>a+b \ \& \ (X>b))}{P(X>b)}$$

$$= \frac{P(X>a+b)}{P(X>b)} = \frac{1-F(a+b)}{1-F(b)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}e^{-\lambda b}}{e^{-\lambda b}}$$

$$= e^{-\lambda a}$$

$X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

$$T_2 = X_1 + X_2 \sim f(t) = \int_{supp(X_1)} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{t-x \in (0,\infty)} dx = \lambda^2 e^{-\lambda t} \int_0^t dx = \lambda^2 e^{-\lambda t} t$$

$$T_3 = X_3 + T_2 \sim \int_{supp(X_2)} f_{X_3}(x) f_{T_2}(t-x) \mathbb{1}_{t-x \in supp(T_2)} dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \lambda^2 e^{-\lambda(t-x)} (t-x) \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$\lambda^3 e^{-\lambda t} \int_0^t (t-x) dx = \lambda^3 e^{-\lambda t} \left(t \int_0^t dx - \int_0^t x dx \right) = \lambda^3 e^{-\lambda t} \frac{t^2}{2}$$

$T_4 = X_4 + T_3$

$$\sim \int_{supp(X_4)} f_{X_4}(x) f_T(t-x) \mathbb{1}_{t-x \in supp(T_3)} dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \lambda^3 e^{-\lambda(t-x)} \frac{(t-x)^2}{2} \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$\lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t (t-x)^2 dx = t^2 \int_0^t \cancel{\lambda e}\, dx - 2t \int_0^t x dx + \int_0^t x^2 dx$$

$$= \lambda^4 e^{-\lambda t} \frac{t^3}{3 * 2}$$

In general

$$\boxed{T_k = X_1 + \cdots + X_k \sim \underbrace{\frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!}}_{f_T(t)}}$$ Erlang (k

PDF

PDF $F_X(x) = \int PDF = \int_0^x \frac{\lambda^2 y^{k-1} e^{-\lambda y}}{(k-1)!} dy = \frac{1}{(k-1)!} \int_{y_0=0}^{y_1=x} \lambda (\lambda y)^{k-1} e^{-\lambda y} dy$

$= \frac{1}{(k-1)!} \int_0^{\lambda x} u^{k-1} e^{-u} du$ let $u = \lambda y \Rightarrow \frac{du}{dy} = \lambda \Rightarrow$

$dy = \frac{du}{\lambda}$