9/12/17

$\vec{X}$ is a vector of r.v's s.t $\dim[\vec{X}] = k$

$$\vec{\mu} = E(\vec{X}) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix},$$

$$\Sigma := Var(X) = \begin{bmatrix} Var(X_1) & cov(X_1, X_n) & \\ cov(X_2, X_1) & Var(X_2) & \\ & & \ddots & \\ & & & Var(X_k) \end{bmatrix} = \{cov(X_i, X_j)\}_{\substack{i=1\dots k \\ j=1\dots k.}}$$

$$\Sigma_0 = Corr(\vec{X}) = \begin{bmatrix} 1 & & \\ & \ddots & \\ cor(X_i, X_j) & & 1 \end{bmatrix} = \{corr(X_i, X_j)\}_{\substack{i=1\dots k \\ j=1\dots k.}}$$

$$T = X_1 + \dots + X_k = 1^T \vec{X}$$

$$1_k = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad = [1 \cdots 1] \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix},$$

$$E(T) = \sum_{i=1}^{k} \mu_i = 1^T \vec{\mu}$$

$$Var(T) = Var[1^T \vec{X}] = Var[X_1 + \dots + X_k]$$

$$= cor[X_i + \dots X_k, X_1 + \dots + X_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} cor[X_i, X_j]$$

let $Y = \vec{c}^T \vec{X}$, where $\vec{c} \in \mathbb{R}^k$

$E(Y) = \vec{c}^T \vec{\mu}$ $\qquad Var(Y) = Var[\vec{c}^T \vec{X}]$

If $A \in \mathbb{R}^{n\times m}$, $\vec{c} \in \mathbb{R}^{n}$

$\vec{c}^T A \vec{c}$ "Quadratic Form"

$$= \vec{c}^T \begin{bmatrix} c_1 a_{11} + \cdots + c_n a_{1m} \\ c_1 a_{21} + \cdots + c_m a_{2m} \\ \vdots \qquad \vdots \\ c_1 a_{m1} + \cdots + c_m a_{nm} \end{bmatrix} = \begin{matrix} c_1^2 a_{11} + c_1 c_2 a_{12} + \cdots c_1 c_m a_{1m} + \\ c_2 c_1 a_{21} + c_2^2 a_{22} + \cdots c_2 c_m a_{2m} + \end{matrix}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} c_i c_j a_{ij}$$

$$= \text{Var}\left[c_1 X_1 + \cdots + c_k X_k\right]$$

$$= \text{Cor}\left[c_1 X_1 + \cdots + c_k X_k,\ c_1 X_1 + \cdots c_k X_k\right]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \text{Cor}\left[c_i X_i,\ c_j X_j\right]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \cancel{\text{Cor}}\, c_i c_j\, \text{Cor}\left[X_i, X_j\right]$$

$$= \vec{c}^T \text{Var}\left[\vec{X}\right] \vec{c} \qquad \text{Var}(\vec{X}) = \begin{bmatrix} \text{Var}(X_1) & \text{Cor}(X_1, X_2) & \cdots \\ \text{Cor}(X_2, X_1) & \text{Var}(X_2) & \cdots \\ & & \ddots \end{bmatrix}$$

"Variance – Covariance matrix"

Markowitz optimal Portfolio

let $X_1, \dots, X_k$ be r.v models for the returns on k assets.
let $w_1, \dots, w_k$ be the weights / allocations for each. Note $\vec{1}^T \vec{w} = 1$

$$V = \vec{w}^T \vec{X} \qquad E[V] = \vec{w}^T \vec{\mu} = \mu_0$$

$$Var(V) = \vec{w}^T \Sigma \vec{w}$$

Given a μ_0, min $\vec{w}^T \Sigma \vec{w}$ st $\vec{1}^T \vec{\mu} = 1$

or $\{\vec{\mu} ; \vec{1}^T \vec{w} = 1\}$.

$\vec{X} \sim$ multinomial $(n, \vec{p})$

$$E(\vec{X}) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_k) \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}.$$

$$Var(\vec{X}) = \begin{bmatrix} np_1(1-p_1) & Cov(X_1, X_2) & & \\ & np_2(1-p_2) & & \\ & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

$$\mathrm{cov}(X_i, X_j) = E[X_i, X_j] - \mu_i \mu_j$$

$$= \sum_{x_i \in \mathrm{supp}(X_i)} \sum_{x_j \in \mathrm{supp}(X_j)} x_i x_j P_{X_i X_j}(x_i, x_j) - \mu_i \mu_j$$

We don't know this.

Recall

$X_1 \sim \mathrm{Bin}(n, P_1)$ $\quad X_1 = \sum_{i=1}^{n} X_{i1}$ s.t $X_{11}, X_{12}, \dots, X_{n1} \overset{iid}{\sim} \mathrm{Ber}(P_1)$

$\vdots$

$X_k \sim \mathrm{Bin}(n, P_k)$ $\quad X_k = \sum_{i=1}^{n} X_{ik}$ s.t $X_{1k}, \dots, X_{nk} \overset{iid}{\sim} \mathrm{Bern}(P_k)$

$$\begin{bmatrix} X_{11} & X_{21} & \cdots & X_{n1} \\ \\ X_{1k} & X_{2k} & \cdots & X_{nk} \end{bmatrix}.$$

$\vec{X} \sim \mathrm{multinomial}(n, \vec{P})$

$\vec{X} = \sum_{i=1}^{n} X_{i\cdot}$ s.t $\vec{X}_{1\cdot}, \dots, \vec{X}_{n\cdot} \overset{iid}{\sim} \mathrm{multinom}(1, \vec{P})$

$\mathrm{cov}[X_i, X_j]$

$= \mathrm{cov}\left[\sum_{l=1}^{n} X_{li}, \sum_{h=1}^{n} X_{hj}\right]$

$= \sum \sum \mathrm{cov}[X_{li}, X_{hj}]$

$= \sum_{l=1}^{n} \sum_{h=1}^{n} E[X_{li}, X_{hj}] - P_i P_j$

$$= \sum_{y\in\{0,1\}} \sum_{x\in\{0,1\}} xy\, p(x,y) = 0 \quad \text{if } \ell = h.$$

$$= \sum E(X_{\ell i} X_{\ell j}) - p_i p_j'$$

$$= \sum_{\ell=1}^{n} -p_i p_j' = -n p_i p_j'$$

$p(x,y)$

$y \backslash x$	0	1
0	+	+
1	+	0

xy

$y \backslash x$	0	1
0	0	0
1	0	1

$\ell \neq h$.

$$E[X_{\ell i} X_{h j}] = E(X_{\ell i}) E(X_{h j}').$$

$$= p_i p_j'$$

$$Var(\vec{X}) = n \begin{bmatrix} p_1(1-p_1) & -p_1 p_2 & \cdots \\ -p_2 p_1 & p_2(1-p_2) & \\ & & \ddots \\ & & & p_k(1-p_k) \end{bmatrix}$$

For $\vec{X} \sim \text{multinom}(n, \vec{p})$.

Continuous r.v's X have CDF

$F(x)$ and PDF:

$f(x) = F'(x)$

Note $\underbrace{p(x)}_{\text{PMF}} = 0 \ \forall x$

$Supp(X) = \{x : f(x) > 0\}$.

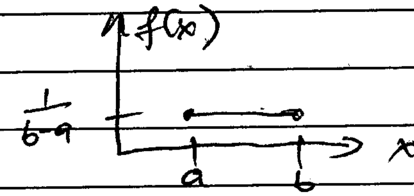
$|supp(x)| = |\mathbb{R}|$

$\nexists\, p$ s.t. $\sum_{x \in supp[x]} p(x) < \infty$

$f(x) \neq p(x)$

$X \sim U(a,b) = \frac{1}{b-a}$

where $a, b \in \mathbb{R}$, $b > a$

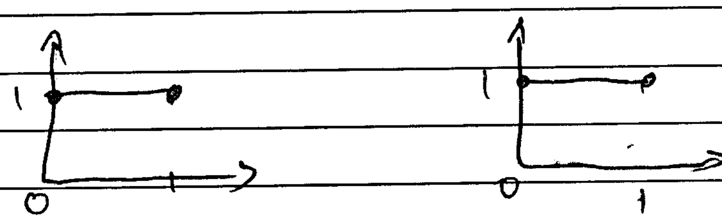


$supp[X) = [a,b]$

let

$a = 0,\ b = 1 \Rightarrow X \sim U(0,1)$ (standard uniform)

let $T_2 = X_1 + X_2$ s.t. $X_1, X_2 \overset{iid}{\sim} U(0,1)$.



$Supp(T) = [0,2]$.

How far does $T = 0$? $X_1 = 0, X_2 = 0 \Rightarrow$ rare

$T = 2$? $X_1 = 1\ X_2 = 1 \Rightarrow$ rare

$T = 1$? $X_1 = 0$ or $X_1 = \frac{1}{3}, X_2 = \frac{2}{3} \Rightarrow$ com

$X_2 = 1$

$$f_T(t) = \int_{x \in \text{Supp}[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{\underbrace{t-x \in [0,1]}_{\underbrace{x-t \in [-1,0]}_{x \in [t-1,\, t]}}}\, dx$$

$$= \int_0^1 \mathbb{1}_{x \in [t-1,\, t]}\, dx = \int_{\max\{0,\, t-1\}}^{\min\{1,\, t\}} 1\, dx = \left(\min(1,t) - \max\{0, t-1\}\right) \mathbb{1}_{t \in [0,2]}$$

$$= \begin{cases} t & \text{if } t < 1 \\ \underbrace{1-(t-1)}_{2-t} & \text{if } t \geq 1 \end{cases} \mathbb{1}_{t \in [0,2]}$$