Lecture 8 9/26/17

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ $\quad (Y = -X_2)$

$D = X_1 - X_2 = X_1 + Y$

Transformation of R.V's

let $X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in [0,1]}$

$$Y = 3 + X \sim \begin{cases} 4 & \text{wp } p \\ 3 & \text{wp } 1-p \end{cases} = P_Y(y) =$$

$$P_Y(y) = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y \in [3,4]}$$

$\text{support}[Y] = \{y : y-3 \in \text{sup}[X]\}$

$$Y = c + aX \overset{= g(x)}{\Rightarrow} X = \frac{Y-c}{a} = g^{-1}(y)$$

$$\text{Sup}[Y] = \{y : \frac{y-c}{a} \in \text{Sup}[X]\}$$

$$= \{y : \frac{y-c}{a} \in [0,1]\}$$

$$= \{c, c+a\}$$

$X \sim \text{Bern}(p)$

$$P_Y(y) = p^{\frac{y-c}{a}}(1-p)^{1-\frac{y-c}{a}} \mathbb{1}_{y \in \{c, c+a\}} =$$

$$= P_X(g^{-1}(y)) \quad \text{modeling support}$$

$X \sim Bin(n, p)$

$Y = aX + c$

$$P_Y(y) = \binom{n}{g^{-1}(y)} p^{g^{-1}(y)} (1-p)^{n - g^{-1}(y)} \underset{g^{-1}(y) \in Sup[X]}{\mathbb{1}}$$

$Y = X^3$

$$P_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n - \sqrt[3]{y}} \underset{Y \in \{0^3, 1^3, 2^3 \ldots n^3\}}{\mathbb{1}}$$

$X \sim Geom(p)$

$Y = \max\{\cancel{x}\, 3, X\} = g(x)$ is not 1:1

X	Y
0	3
1	3
2	3
3	3
4	4

$\}$ Not 1-1

$$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x) = \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$$

$\{x : x \in g^{-1}(y)\}$

$\uparrow$ inverse

Formula for discrete r.v hours function

$$P_Y(y) = (\underbrace{P_X(0)}_{p} + \underbrace{P_X(1)}_{p(1-p)} + \underbrace{P_X(2)}_{p(1-p)^2} + \underbrace{P_X(3)}_{p(1-p)^3}) \mathbb{1}_{y=3} + p(1-p)^{y} \underset{y \in \{4, 5, \ldots\}}{\mathbb{1}}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2 = X_1 + Y$, $Y = -X_2$

$$P_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} \quad -y \in \{0,-1,-2\dots\}$$

$\text{Sup}[D] = \mathbb{Z}$ nave

$$P_D(d) = \sum_{x \in \text{Sup}[X_1]} P_{X_1}(x)\, P_Y(d-x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{(-(d-x))!}$$

$d-x \in \{0,-1,-2\dots\}$

$x-d \in \{0,1,2\dots\}$

$x \in \{d, d+1, d+2 \dots\}$

$$= e^{-2\lambda} \sum_{x=\max(0,d)}^{\infty} \frac{\lambda^{2x-d}}{x!\,(x-d)!}$$

$$= e^{-2\lambda} \begin{cases} \displaystyle\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!\,(x-d)!} & \text{if } d \geq 0 \quad (\text{upper}) \\ \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!\,(x-d)!} & \text{if } d < 0 \quad (\text{lower}) \end{cases}$$

let $x' = x - d \Rightarrow x = x' + d$

$$\sum_{x'=0}^{\infty} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!\,x'!} = \sum_{i=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2i+d}}{\Gamma(i+d-1)\,\Gamma(i+1)}$$

$\underline{\underline{\text{Note}}}$ $\Gamma(n-1) = n!$

Modified Bessel function of the 1st Kind denoted $I_d(2\lambda)$

$I_d(2\lambda) = I_{|d|}(2\lambda)$

let $d' = -d$ $\displaystyle\sum_{x=0}^{\infty} \frac{d^{2x+d'}}{x!(x+d')} = I_{d'}(2d) = I_{|d|}(2d)$

if $d < 0$
$\Rightarrow d' = |d|$

$P_D(d) = e^{-2d} I_d(2d) =$ Skellam (d, d)

Point spreads, baseball, soccer .. 1946

let $X \sim \mathcal{U}(0, 1)$

$Y = aX = g(X)$ 1:1

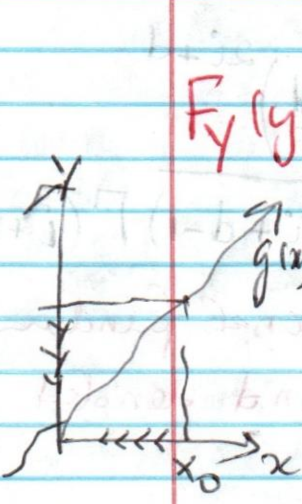
$$\underbrace{P_Y(y) = P_X(g^{-1}(y))}_{\text{Not useful}}$$

let $Y = g(x)$, 1:1 Find $f_Y(y)$ given $f_X(x)$?

if g is 1:1 $\Rightarrow$

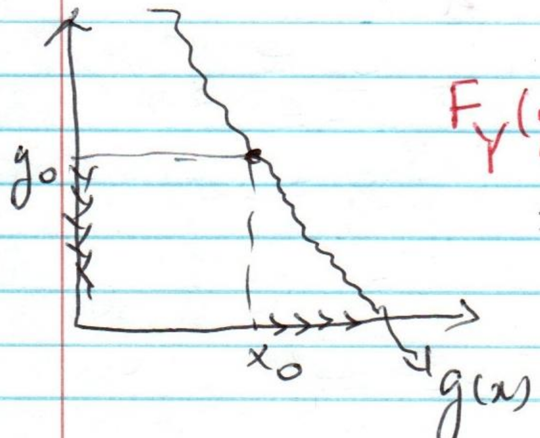
a) g is strictly increasing
b) g is strictly decreasing

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$



$$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right] = \frac{d}{dy}\left[F_X(g^{-1}(y))\right] =$$

$$= f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$



$$F_Y(y) = P(Y \leq y) = P(g(x) \leq y)$$
$$= P(X \geq g^{-1}(y)) = 1 - P(X < g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))]$$
$$= -f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$
$$= f_X(\underbrace{g^{-1}(y)}_{+}) \left(-\frac{d}{dy}\underbrace{[g^{-1}(y)]}_{-}\right)$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

$$\text{sup}(Y) = g(\text{sup}[X]) = \{y : g^{-1}(y) \in \text{sup}[X]\}$$

if $Y = aX + c = g(x)$

$\Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-c}{a}\right)$

if $Y = -X$

$\Rightarrow f_Y(y) = f_X(-y)$

if $Y = X + c$

$\Rightarrow f_Y(y) = f_X(y-c)$ shifted distribution

if $Y = aX \Rightarrow f_Y(x) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$

if: $X \sim \mathcal{U}(0,1)$

$Y = aX + c \quad f_Y(y) = \frac{1}{|a|} \overbrace{f_X\left(\frac{y-c}{a}\right)}^{1} = \frac{1}{|a|}$

$\mathrm{supp}[Y] = [c, c+a]$

if $X \sim Exp(d)$

$Y = aX + c$

$f_Y(y) = \frac{1}{|a|} f_Y\left(\frac{y-c}{a}\right) = \frac{1}{|a|} d e^{-d\left(\frac{y-c}{a}\right)} = \frac{d}{|a|}\left(e^{\frac{dc}{a}}\right) e^{-\frac{dy}{a}}$

$\text{sup}[y] = (c, \infty)$

let $c = 0$ $\quad = \frac{d}{|a|} e^{-\frac{d}{a} y}$

let $a > 0$ $\quad = \frac{d}{a} e^{-\frac{d}{a} y} = Exp\left(\frac{d}{a}\right)$

$X \sim U(0,1)$

$Y = 1 - X \sim U(0,1) = f_Y(y) = \underbrace{f_Y(y-1)}_{1} = 1$

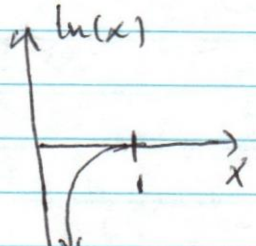
$\text{sup}[Y] = 1 - [0,1]$

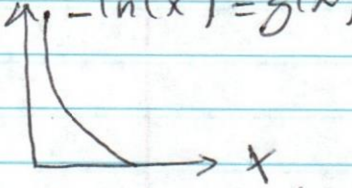
$= [0,1]$

$Y = aX \Rightarrow f_Y(x) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$

$X \sim U(0,1)$

$Y = -\ln(x)$





$y = g(x) = -\ln(x)$

$-y = \ln x$

$\Rightarrow e^{-y} = x = g^{-1}(y)$

$f_Y(y) = \underbrace{f_X(g^{-1}(y))}_{1} \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$

$= \frac{d}{dy}\left[-e^{-y}\right] = e^{-y} = Exp(1)$

$X \sim \text{Exp}(1)$

$$Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right)$$