Lec 19

Nov. 16, 2017

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix} = \vec{\mu}$$

$$E[X] = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

$$\Sigma = Var[\vec{X}] = E\left[(\vec{X} - \vec{\mu})(\vec{X} - \vec{\mu})^T\right]$$

$$\sigma^2 = Var[X] = E\left[(X-\mu)(X-\mu)\right]$$

$$= E\left[(X-\mu)^2\right] = Var[\quad]$$

$$\begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ \vdots \\ X_n - \mu_n \end{bmatrix} \begin{bmatrix} X_1 - \mu_1 & X_2 - \mu_2 & \cdots & X_n - \mu_n \end{bmatrix}$$

$$E\begin{bmatrix} (x_1-\mu_1)^2 & (x_1-\mu_1)(x_2-\mu_2) & \cdots \\ (x_2-\mu_2)(x_1-\mu_1) & (x_2-\mu_2)^2 & \cdots \\ \vdots & & \\ \vdots & \ddots & \\ \vdots & \cdots & (x_n-\mu_n)^2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1^2 & \sigma_{1,2}^2 & \\ \sigma_{21} & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_n^2 \end{bmatrix}$$

$$(a+b)^T = a^T + b^T$$

Symetric

$$\Sigma = E\left[(x-\mu)(x-\mu)^T\right] = E\left[(x-\mu)(x^T-\mu^T)\right]$$

$$= E\left[xx^T - \mu x^T - x\mu^T + \mu\mu^T\right]$$

$$= E\left[xx^T\right] + E\left[-\mu x^T\right] + E\left[-x\mu^T\right] + E\left[\mu\mu^T\right]$$

$$\boxed{\Sigma = E(xx^T) - \mu\mu^T} \cancel{- \mu\mu^T + \mu\mu^T}$$

$$\boxed{\sigma^2 = E(x^2) - \mu^2}$$

$X \in \mathbb{R}^{n \times m}$ $\quad A \in \mathbb{R}^{p \times n}$

$AX \in \mathbb{R}^{p \times m}$

$$E[AX] = E\left[\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{p1} & \cdots & a_{pn} \end{bmatrix}\begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}\right]$$

$$= E\begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} + \cdots + a_{1n}x_{n1} & \cdots \\ \vdots & \ddots \\ & \end{bmatrix} \begin{bmatrix} a_{1\cdot}\mu_{\cdot 1} & a_{1\cdot}\mu_{\cdot 2} \\ a_{2\cdot}\mu_{\cdot 1} & a_{2\cdot}\mu_{\cdot 2} \\ & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ \vdots \\ a_{n\cdot} \end{bmatrix} \begin{bmatrix} \mu_{\cdot 1} & \mu_{\cdot 2} & \cdots & \mu_{\cdot n} \end{bmatrix} = A\,E[X]$$

$XB, \quad E[XB] = E[X] \cdot B$

$$A = \begin{bmatrix} \leftarrow a_{1\cdot} \rightarrow \\ \leftarrow a_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow a_{p\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_{\cdot 1} & a_{\cdot 2} & \cdots & a_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$X + C = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \begin{bmatrix} c_{11} & & c_{1m} \\ \vdots & & \\ c_{n1} & \cdots & c_{nm} \end{bmatrix}$$

$$E[X+C] = E[X] + C$$

$$\mathrm{Var}[A^T X] = E[(A^T X)(A^T X)^T] - E[A^T X]E[A^T X]^T$$

$$E[A^T X X^T A] \qquad A^T E[X](A^T E[X])^T$$

$$A^T E[XX^T]A - A^T \mu\mu^T A$$

$$A^T(E[XX^T] - \mu\mu^T)A = \boxed{A' \Sigma A}$$

$$Z_1, \ldots, Z_n \overset{iid}{\sim} N(\overset{E[Z_i]}{0}, \overset{\mathrm{Var}[Z_i]}{1})$$

$$\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \sim N_n(\underset{E[Z]}{\vec{0}}, \underset{\mathrm{Var}[Z]}{I_n})$$

$\vec{\mu} \in \mathbb{R}^n$ Multivariate normal (MVN) of dim. k

$\text{supp}[\vec{z}] = \mathbb{R}^n$ $\quad N_n(\vec{0}, I_n)$

$$f_{z_1, \dots, z_n}(z_1, \dots z_n) = f(z_1) \cdot f(z_2) \cdots f_{z_n}(z_n)$$

$$= \prod_{i=1}^{n} f_z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$\vec{X} = \vec{z} + \vec{c}$, $E[X] = \vec{0} + c$,

$\text{Var}[\vec{X}] = I_n \Rightarrow \vec{X} \sim N(\vec{c}, I_n)$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-c_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum(x_i-c_i)} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\vec{x}-\vec{c})^T(\vec{x}-\vec{c})}$$

$\vec{X} = A\vec{Z}$ $\qquad E[X] = AE[\vec{Z}]$

$\vec{Z} \in \mathbb{R}^{n\times 1}$ $\qquad = A\vec{0}_n = \underline{\vec{0}_m}$

$A \in \mathbb{R}^{n\times n}$

$$\Sigma = Var[\vec{X}] = A\,Var[\vec{Z}]\,A^T$$

$$= AIA^T = \underline{AA^T}$$

ME to WE

$$\vec{X} = A\vec{z} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n \\ \cdots \\ a_{m1}z_n + \cdots + a_{mn}z_n \end{bmatrix} \begin{matrix} \sim N(0, \Sigma a_{1i}^2) \\ \sim N(0, \Sigma a_{2i}^2) \\ \vdots \\ \sim N(0, \Sigma a_{mi}^2) \end{matrix}$$

let find the PDF of X

- Assume $A \in \mathbb{R}^{n \times m}$, A is full rank

$\vec{X} = A\vec{z} = g(\vec{z})$

$\rightarrow \vec{z} = h(\vec{X}) = A^{-1}\vec{X}$