Lecture 20 11/28/17

$\vec{Z} \sim N_n(\vec{0}_n, I_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$

$Z_1, \dots Z_n \overset{iid}{\sim} N(0,1)$

$E(\vec{Z}) = 0, \quad Var(Z) = I_n$

let $\vec{X} = \vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, I_n)$

$\vec{X} = A\vec{Z}, \quad A \in \mathbb{R}^{m \times n}$

$E[\vec{X}] = AE[\vec{Z}] = A \cdot \vec{0}_n = \vec{0}_m$

Cholesky decomposition ($B = AA^T$)

$\Sigma = Var[X] = A \underbrace{Var(\vec{Z})}_{I_n} A^T = AA^T = \Sigma$

what is $f_X(\vec{x})$?

we know $X = g(\vec{Z}) = A\vec{Z}$

$\vec{Z} = h(\vec{X}) = A^{-1}\vec{X}$ where h is the inverse function

we need $m = n$ ie A is a square and A is full rank

$\vec{Z} = h(\vec{X}) = \begin{bmatrix} h_1(\vec{x}) \\ h_2(\vec{x}) \\ \vdots \\ h_n(\vec{x}) \end{bmatrix}, \quad f_X(\vec{x}) = f_{\vec{Z}}(h(\vec{x})) \left| J_h(\vec{x}) \right|$

let $B = A^{-1} = \begin{bmatrix} \leftarrow \vec{b}_{1\cdot} \rightarrow \\ \leftarrow \vec{b}_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow \vec{b}_{n\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b_{\cdot 1} & b_{\cdot 2} & \cdots & b_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} =$

$f_X(\vec{x}) = f_Z(B\vec{x}) \left| J_h(\vec{x}) \right| \quad \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

$= f_Z(A^{-1}X) \det(A^{-1})$

$$\vec{Z} = B\vec{X} = \begin{bmatrix} h_1(\vec{x}) = \vec{b}_{1.}\vec{x} = b_{11}x_1 + \cdots + b_{1n}x_n \\ h_2(\vec{x}) = \vec{b}_{2.}\vec{x} = b_{21}x_1 + \cdots + b_{2n}x_n \\ \vdots \\ h_n(\vec{x}) = \vec{b}_{n.}\vec{X} = b_{n1}x_1 + \cdots \cancel{b}_{nn}x_n \end{bmatrix}$$

$$\overset{\shortparallel}{h(\vec{x})}$$

$$J_h(\vec{x}) = \det\left(\begin{bmatrix} \frac{\partial}{\partial x_1}[h_1(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_1(\vec{x})] \\ \vdots & & \\ \frac{\partial}{\partial x_1}[h_n(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_n(\vec{x})] \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right)$$

$$= \det(B) = \det(A^{-1})$$

<u>Note</u> $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$AA^{-1} = I$

$\det(AA^{-1}) = \det(I) = 1$

$\det(A) \cdot \det(A^{-1}) = 1$

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

also $\det(A \cdot B) = \det(A) \cdot \det(B)$

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}\vec{x})^T(A^{-1}\vec{x})} \cdot \frac{1}{|\det(A)|}$$

Note

$$A A^{-1} = I$$

$$(A A^{-1})^T = I^T = I \qquad \text{also} \quad A^T (A^T)^{-1} = I = (A^T)^{-1} A^T = I$$

$$(A^{-1})^T A^T = I \qquad \Rightarrow \quad (A^T)^{-1} = (A^{-1})^T$$

$$\Rightarrow f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \vec{x}^T (A^{-1})^T A^{-1} \vec{x}} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \vec{x}^T (A^T)^{-1} A^{-1} \vec{x}} \cdot \frac{1}{|\det(A)|}$$

Note $\Sigma = A A^T$, $\quad (AB)^{-1} = B^{-1} A^{-1}$

$\Sigma^{-1} = (A A^T)^{-1}$, $\quad (AB)^{-1}(AB) = I$

$= (A^T)^{-1} A^{-1}$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}} \cdot \frac{1}{|\det(A)|}$$

Note $\det(\Sigma) = \det(A) \det(A^T) = \det(A)^2 \Rightarrow$

$\det(A) = \sqrt{\det(\Sigma)}$ (also $\det(A) = \det(A^T)$)

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}}$$

$$= N_n(\vec{0}_n, \overset{\overset{AA^T}{\shortparallel}}{\Sigma})$$

general MVN

general multi variance Normal

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma) =$$

$$Y = \vec{x} - \vec{\mu} = A\vec{Z}$$

$$\frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

Recall $\phi_X(t) = E\left[e^{itX}\right]$ generalize to

$$\phi_{\vec{X}}(t) = E\left[e^{i\vec{t}^T\vec{X}}\right]$$

$$\phi_{\vec{X}_1+\vec{X}_2}(t) = E\left[e^{it^T(\vec{X}_1+\vec{X}_2)}\right]$$

$\boxed{\vec{X}_1 \text{ ind of } \vec{X}_2}$

$$= E\left[e^{it^T\vec{X}_1} e^{i\vec{t}^T\vec{X}_2}\right]$$

$$= E\left[e^{it^T X_1}\right] E\left[e^{it^T X_2}\right]$$

$$= \phi_{\vec{X}_1}(\vec{t})\, \phi_{X_2}(\vec{t})$$

$$\vec{Y} = A\vec{X} + \vec{C}$$

$A \in \mathbb{R}^{m \times n}$, $\vec{C} \in \mathbb{R}^{m}$, $\vec{X} \in \mathbb{R}^{n}$, $\vec{Y} \in \mathbb{R}^{m}$

$$\phi_{\vec{Y}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{Y}}\right] = E\left[e^{i\vec{t}^T(A\vec{X}+\vec{C})}\right]$$

$$= E\left[e^{i\vec{t}^TA\vec{X}}\, e^{i\vec{t}^T\vec{C}}\right] = e^{i\vec{t}^T\vec{C}}\, E\left[e^{i\vec{t}^TA\vec{X}}\right]$$

let $t'^T = t^TA \to t' = (t^TA)^T = A^T\vec{t}$

$$= e^{i\vec{t}^T\vec{C}}\, E\left[e^{i\vec{t}'^T\vec{X}}\right]$$

$$= e^{i\vec{t}^T\vec{C}}\, \phi_{\vec{X}}(\vec{t}')$$

$$= e^{i\vec{t}^T\vec{C}}\, \phi_{\vec{X}}(A^T\vec{t})$$

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{Z}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{Z}}\right]$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i\vec{t}^T\vec{z}}\, f_{\vec{Z}}(\vec{z})\, d\vec{z}$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i(t_1z_1+\cdots+t_nz_n)}\, \frac{1}{(\sqrt{2\pi})^n}\, e^{-\frac{1}{2}(z_1^2+\cdots+z_n^2)}\, dz_1 \cdots dz_n$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{n} e^{it_j z_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_j^2} \, dz_i$$

$$= \prod_{j=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{it_j z_j - \frac{1}{2} z_j^2} \, dz_j$$

Note $-\frac{1}{2} z^2 + itz = -\frac{1}{2}\left(z^2 - 2itz\right) = -\frac{1}{2}\left((z-it)^2 - i^2 t^2\right)$

$= -\frac{1}{2}\left((z-it)^2 + t^2\right)$

$$= \prod_{j=1}^{n} e^{-\frac{1}{2} t_j^2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \underline{e^{-\frac{1}{2}(z_j - it_j)^2} \, dz_j}}_{N(it_j,\, 1^2)}$$

$= 1$

$$= e^{-\frac{1}{2} \sum_{j=1}^{n} t_j^2}$$

$$= e^{-\frac{1}{2} \vec{t}^T \vec{t}}$$

$$\boxed{\phi_{\vec{Z}}(\vec{t}) = e^{-\frac{1}{2} \vec{t}^T \vec{t}}}$$

$A \in \mathbb{R}^{n\times n}$

$$\vec{X} = A\vec{Z} + \vec{\mu} \Rightarrow \phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu}}\phi_{\vec{Z}}(A^T\vec{t})$$

$$= e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\vec{t}^T A A^T t}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T \Sigma \vec{t}}$$

$$\vec{Y} = B\vec{X} \Rightarrow \phi_{\vec{Y}}(\vec{t}) = \phi_{\vec{X}}(B^T\vec{t})$$

$B \in \mathbb{R}^{m\times n}$

$$= e^{i\vec{t}^T \underbrace{B\vec{\mu}}_{\mu'} - \frac{1}{2}\vec{t}^T \underbrace{B\Sigma B^T}_{\Sigma'}\vec{t}} \Rightarrow$$

$$\vec{Y} \sim N_m(B\mu, B\Sigma B^T)$$

If $\vec{X} = A\vec{Z} + \vec{\mu} \Rightarrow \vec{Z} = A^{-1}(\vec{X} - \vec{\mu})$

(Note $\vec{Z}^T Z \sim \chi^2_n$)

$$(A^{-1}(\vec{X}-\vec{\mu}))^T(A^{-1}(\vec{X}-\vec{\mu})) \Rightarrow (\vec{X}-\vec{\mu})^T\Sigma^{-1}(\vec{X}-\mu)$$

$$(\vec{X}-\vec{\mu})^T \underbrace{(A^{-1})^T A^{-1}}_{(A^T)^{-1}A^{-1}} (\vec{X}-\mu) \qquad \sim \chi^2_n$$

$$\underbrace{(A^T)^{-1}A^{-1}}_{(AA^T)^{-1}}$$

$$(AA^T)^{-1} = \Sigma^{-1}$$

"Mahalanobis Distance"

"sphering"

65b

$$X \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{X}-\vec{\mu})^T \Sigma^{-1} (\vec{X}-\vec{\mu})}$$

$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

<u>Example</u>

$$X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\Rightarrow \vec{X} \sim N_n(\underbrace{\mu 1_n}_{\vec{\mu}}, \sigma^2 I_n)$$

$$\Sigma = \sigma^2 I \Rightarrow \underbrace{\sigma I}_{A} \cdot \underbrace{(\sigma I)^T}_{A^T}$$

$$\vec{X} = \sigma I \vec{Z} + \vec{\mu} = \sigma \vec{Z} + \vec{\mu}$$

$$(\vec{X}-\vec{\mu})^T \frac{1}{\sigma^2} I (\vec{X}-\vec{\mu}) \sim \chi^2$$

$$\sum \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$