Lec. 14 Oct. 26, 2017

$X \sim$ Gamma(β), $Y \sim$ Poisson(X)

$f_Y(y) = $ NegBin

$Q \sim$ Poisson (λ)

$E(Q) = \lambda$

$SE(Q) = \lambda$

$Q \sim$ Neg. Bin (k, p)

$E(Q) = kp$

$SE(Q) \neq E(Q)$

"overdispersed" poisson

Example

$X \sim$ Beta(α, β) $\quad Y \sim$ Bin(n, x)

$X \sim$ Beta(α, β), $Y \sim$ Bin(n, x)

$$P_Y(y) = \int P_{Y|X}(y, x) f_X(x)\, dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

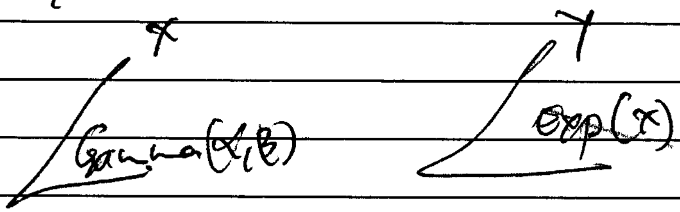
$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1} dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \text{Beta Binomial}(\alpha, \beta, n)$$

= overdispersed binomial

other example

$X \sim \text{Gamma}(\alpha, \beta)$ $\quad$ $Y|X \sim \text{Exp}(x)$



$$f_Y(y) = \int \underbrace{f_{Y|X}(y,x)}_{\text{Exp}(X)} f_X(x)\,dx$$

$$= \int_0^\infty x e^{-xy} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1-1} e^{-(\beta+y)x}\,dx$$

let $u = (\beta+y)x \Rightarrow x = \frac{1}{\beta+y} u$

$$\Rightarrow \frac{du}{dx} = \beta+y \Rightarrow du = (\beta+y)\,dx$$

$$dx = \frac{du}{(\beta+y)}$$ We replace everything and get

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha+1-1}}{(\beta+y)^\alpha} e^{-u} \cdot \frac{1}{\beta+y} du$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{\beta^{\alpha+1}}{\beta} \cdot \frac{1}{(\beta+y)^{\alpha+1}} = \frac{\alpha}{\beta}\left(1+\frac{y}{\beta}\right)^{(\alpha+1)}$$

$$= \underline{\underline{\text{Lomax}(\beta, \alpha)}}$$

Idea of complex numbers

$a, b \in \mathbb{R}$ $\quad z = a + bi \in \mathbb{C}$

$i = \sqrt{-1} \Rightarrow i^2 = -1 \quad i^4 = 1$

$Re(z) = a \quad Im(z) = b$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{k!} + \cdots +$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} +$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

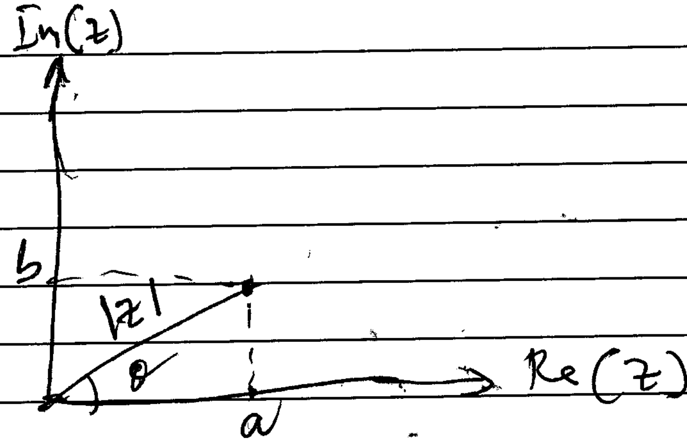
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^3x^5}{5!} - + \cdots$$

$$i\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + \cdots$$

$$e^{itx} = \cos(tx) + i\sin(tx) \quad \text{if } \pi = tx$$

$$e^{i\pi} = -1 \rightarrow e^{i\pi} + 1 = 0$$



$$|z| = \sqrt{a^2+b^2} \in (0, \infty)$$

$$\mathrm{Arg}(z) = \theta = \arctan\left(\frac{b}{a}\right) \in [-\pi, \pi]$$

$z = |z| e^{i\theta}$

$= \sqrt{a^2+b^2}\left(\cos\left(\arctan\left(\frac{b}{a}\right)\right) + i\sin\left(\arctan\left(\frac{b}{a}\right)\right)\right)$

Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)|\, dx < \infty \}$

All pdfs (PDFs) $\in L^1$

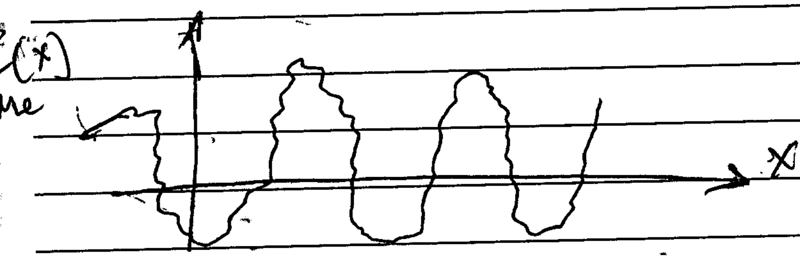
If $f \in L^1$, then $\exists\, \hat{f}$ defines as

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\, dx$$ known as the Fourier transformat. of f.

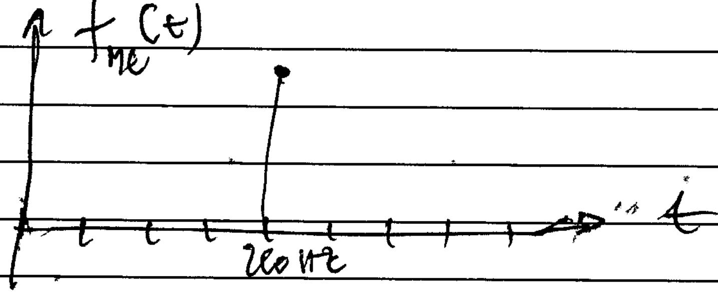
Note $\hat{f}$ doesn't necessary $\in L^1$

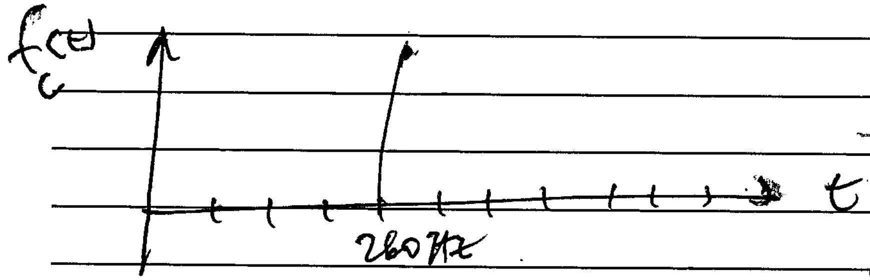
$f(x)$ is called the time domain

$\hat{f}(t)$ is called frequency domain



time domain is x





frequency domain is t

$Re[\hat{f}(t)] \rightarrow$ Amplitude of seq.

$Arg.[\hat{f}(t)] \rightarrow$ phase

Let $\phi(t) = \hat{f}\left(\frac{t}{2\pi}\right) = \int_{\mathbb{R}} e^{itx} f(x)\,dx = E(e^{itx})$

If $\phi(t) \in L'$ if $f(x)$ is a PDF

Recall: If $\hat{f} \in L'$ then $f(x) = \int_{\mathbb{R}} e^{i\pi 2tx}\hat{f}(t)\,dt$

Now let $u = -2\pi t$ and rewrite $f(x)$

$\Rightarrow t = -\frac{u}{2\pi} \Rightarrow dt = \frac{-1}{2\pi}du$

If $t = \infty \rightarrow u = -\infty$ but $t = -\infty$, $u = \infty$

$$f(x) = \int_{\infty}^{+\infty} e^{2\pi i\left(\frac{-u}{2\pi}\right)x} \hat{f}\left(\frac{-u}{2\pi}\right) \frac{1}{2\pi}\, du$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux} \hat{f}\left(\frac{-u}{2\pi}\right) du = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux} \phi(u)\, du$$

$\phi_X(t)$ = characteristic funct- of r. v. X

$$E(e^{itx}) = \begin{cases} \sum_{x \in supp(x)} e^{itx} p(x) & \text{if } X \text{ discrete} \\ \int_{supp(x)} e^{itx} f(x)\, dx & \text{if } X \text{ continuous} \end{cases}$$

(1) $\phi(0) = 1$ if X_1, X_2 ind. $Y = X_1 + X_2$

(2) $\phi_Y(t) = \phi_{X_1+X_2}(t) = E\left[e^{it(x_1+x_2)}\right] = E\left[e^{itx_1} \cdot e^{itx_2}\right]$

$$= E\left[e^{itx_1}\right] E\left[e^{itx_2}\right] = \phi_{X_1}(t)\, \phi_{X_2}(t)$$

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③ If $Y = aX + b$, $a, b \in \mathbb{R}$

$$\phi_Y(t) = E[e^{itY}] = E[e^{it(aX+b)}]$$

$$= E[e^{itX} e^{itb}] = e^{itb} E(e^{itb} \phi_X(at)$$

④ $\phi_X(t)$