Lec. 16 Nov. 2

standard normal

$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \iff \phi_X(t) = e^{-\frac{t^2}{2}}$

Central Limit Theorem $X_1, X_2, \cdots X_n \sim P$

$Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow N(0,1)$

$E[X] = 0 \qquad SE(X) = 1$

$Y = \mu + \sigma X \Rightarrow X = \frac{Y-\mu}{\sigma}$

$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right) = f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$

$= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{y-\mu}{\sigma}\right)^2}$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)} = N(\mu, \sigma^2)$ Normal

$X_1, \ldots, X_n \overset{iid}{\sim} p \quad \mu, \sigma^2$

$T_n = X_1 + \cdots + X_n \overset{d}{\sim} N(n\mu, n\sigma^2)$

$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} \overset{d}{\sim} N(\mu, \frac{\sigma^2}{n})$

if n is large enough

$\phi_Y(t) = e^{it\mu} \phi_X(\sigma t) = e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ ind of $X_2 \sim N(\mu_2, \sigma_2^2)$

Convo $T = X_1 + X_2 = \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx$

$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(t-x-\mu_2)}$

No indicator functs because everything is Normal,

$$\phi_Y(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$$

$$= e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

$$\Longrightarrow Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

$$\mathrm{Supp}(Y) = (0,\infty)$$

$$X \sim N(\mu, \sigma^2), \quad Y = e^X$$

$$g(x) = e^x$$

$$X = g^{-1}(y) = \ln(y).$$

$$\left|\frac{d}{dy}(g^{-1}(y))\right| = \frac{1}{Y}$$

Go back to the formula, we have:

$$Y \sim f(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}\, y} \cdot e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}$$

$\text{Log}N(\mu, \sigma^2)$

log normal

*

On begin of Y_0

$$Y_1 = Y_0(1+R_1)$$

$$Y_2 = Y_1(1+R_2) = Y_0(1+R_1)(1+R_2)$$

$$\vdots$$

$$Y_t = Y_0 \prod_{i=1}^{t}(1+R_i) = Y_0 e^{\ln\left(\prod_{i=1}^{t}(1+R_i)\right)}$$

$$= Y_0 e^{\sum_{i=1}^{t}\ln(1+R_i)} = Y_0 e^{\sum_{i=1}^{t} X_i}$$

$$\sum_{i=1}^{t} X_i \sim N(t\mu_x, t\sigma_x^2)$$

Compounded formula

$X \sim \log N(\mu, \sigma^2)$ $\quad Y = aX$

$X = \frac{Y}{a}$

$f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right) = \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{\frac{y}{a}} e^{-\frac{1}{2\sigma^2}\left(\ln\left(\frac{y}{a}\right)-\mu\right)^2}$

$= \log N(\mu + \ln(a), \sigma^2)$

$\ln(y) - (\mu + \ln(a)) = \ln(y) - \ln(a) - \mu$

Given $R = 0.03 = 3\%$

$X_i = \ln(1 + R_i)$

$X = \ln(1 + 0.03) = .02\%$

$R = -0.03 = -3\%$

$X = \ln(1 + -.03) = -0.051$

$\Rightarrow X \simeq R \Rightarrow Y_t = Y_0 e^{\sum X_i} \sim \log N(t\mu_R + \ln(Y_0), t\sigma_R^2)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \cdots \simeq x$$

* $R \overset{iid}{\sim} N(10\%, 10\%^2)$

Start \$1000

In = 5 yrs

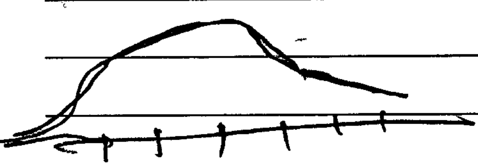
If a quantity experiences normal percentage/proportional changes, then the

$Z \sim N(0,1)$ $\quad$ $Y = Z^2$

Y is Not 1 to 1 $\quad$ $\text{supp}(Y) = (0,\infty)$

$F_Y(y) = P(Y \le y) = P(Z^2 \le y)$

$= P(Z \in [-\sqrt{y}, \sqrt{y}])$



$= 2P(Z \in [0, \sqrt{x}])$

$= 2\left(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{\frac{1}{2}}\right) = 2F_Z(\sqrt{y}) - 1$

$f_Y(y) = F_Y'(y) = \cancel{2} f_Z(\sqrt{y})\, \frac{y^{-1/2}}{\cancel{2}}$

$= \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2} = \chi^2_1$

$$\chi_1^2 = \chi^2(1)$$

chi-squared with degree of freedom = 1
parameter

$$\boxed{\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt}$$

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty \frac{1}{u} e^{-u^2} \cdot 2u\,du$$

with $u = \sqrt{t} \Rightarrow t = u^2$

$$\frac{du}{dt} = \frac{1}{2} \cdot \frac{1}{\sqrt{t}}$$

$$= 2 \underbrace{\int_0^\infty e^{-u^2} du}_{\frac{\sqrt{\pi}}{2}} = \sqrt{\pi}$$

- You squared Normal, you have Gamma

Go back $\frac{\left(\frac{1}{2}\right)^{1/2} y^{-1/2} e^{-y/2}}{\Gamma(1/2)} = \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2_1$$

$z_1, z_2 \overset{iid}{\sim} N(0,1)$ $\quad \text{Exp}(1/2)$

$z_1^2 + z_2^2 \sim \text{Gamma}\left(1, \frac{1}{2}\right) = \chi^2_2$

Convo of χ^2_1 with χ^2_1

= Convo of $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ with $\text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$z_1, \ldots, z_k \overset{iid}{\sim} N(0,1)$

$\sum_{i=1}^{k} z_i^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

ME TO WE

$$= \chi_k^2 = \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{k}{2}\right)}$$

$X \sim \chi_k^2 \qquad Y = \sqrt{X} \rightarrow X = Y^2$

$\text{supp}(Y) = (0, \infty)$

$$f_Y(y) = f_X(y^2)\,2y$$

$$= \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \left(y^2\right)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}}\, 2y$$

$$= \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}} \sim \chi_k$$

Chi-distribut-