Lecture 15 10/31/17

Consider $\phi_X'(t) = \frac{d}{dt}\left[E\left[e^{itX}\right]\right) = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{itx} f(x)\,dx\right] \stackrel{?}{=}$

$$= \int_{\mathbb{R}} f(x) \frac{d}{dt}\left[e^{itx}\right] dx$$

Does $\frac{d}{dt}\left[\int g(x,t)\,dx\right] \stackrel{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}\left[g(x,t)\,dx\right.$

Conditions

a) $\exists\, t \in A$ such that $\int_{\mathbb{R}} g(x,t)\,dx$ converges $A = [a,b] \subset \mathbb{R}$

b) $g(x,t)$ continuous $\forall t \in A$

c) $g'(x,t)$ continuous $\forall x \in \mathbb{R}$

d) $\forall t \in A$ $\int_{\mathbb{R}} \frac{\partial}{\partial t} g(x,t)\,dt$ converge uniformly

$$\boxed{\phi_X'(t) = \int_{\mathbb{R}} f(x)\, ix\, e^{itx}\,dx}$$

Consider $\phi_X'(0) = \int_{\mathbb{R}} f(x)\, ix\,dx = i\int_{\mathbb{R}} x f(x)\,dx$

$$= i\,E[X]$$

$$\phi''_X(t) = \int_{\mathbb{R}} f(x)\, i^2 x^2 e^{itx}\, dx$$

$$\phi''_X(0) = i^3 \int_{\mathbb{R}} x^2 f(x)\, dx = i^2 E[X^2]$$

$$\phi'''_X(0) = i^3 E[X^3]$$

(5) $E[X^n] = \dfrac{\phi_X^{(n)}(0)}{i^n}$

(6) $P(X \in (a,b)) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} \dfrac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\, dt$

$\forall a<b$

Inversion Theorem

Motivation if $\phi_X \in L^1$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ity} \phi_X(t)\, dt$$

$$P(X \in (a,b)) = \int_a^b f(x)\, dx = \int_a^b \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t)\, dt\, dx$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_a^b e^{-itx}\, dx \right) \phi_X(t)\, dt =$$

(7) $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$

(8) $\phi_{X_n}(t)$ is the characteristic Function for X_n

If $\forall t \ \lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ also.... $\lim_{n\to\infty} X_n \overset{d}{=} X$ or

$X_n \xrightarrow{d} X$ convergence in distribution

Example:

let $X \sim \text{Gamma}(k, d)$

$$\phi_X(t) = \int_0^\infty e^{itx} \frac{d^k e^{-dx} x^{k-1}}{\Gamma(k)}\, dx = \frac{d^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{(it-d)x}\, dx$$

~~$u = (it-d)x$, $dx = \frac{1}{it-d}\,dt$~~

let $u = (d-it)x \Rightarrow x = \dfrac{u}{d-it}$, $dx = \dfrac{1}{d-it}\,du$

$$\phi_X(t) = \frac{d^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(d-it)^{k-1}} e^{-u} \frac{1}{d-it}\, du$$

$$= \frac{d^k}{\Gamma(k)(d-it)^k} \underbrace{\int_0^\infty u^{k-1} e^{-u}\, du}_{\Gamma(k)} = \left(\frac{d}{d-it}\right)^k = \left(1 - \frac{it}{d}\right)^{-k}$$

Example $X_1 \sim \text{Gamma}(k_1, \lambda)$ ind of
$X_2 \sim \text{Gamma}(k_2, \lambda)$

$X_1 + X_2 \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_2(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k_1}\left(\frac{\lambda}{\lambda - it}\right)^{k}$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^{k_1+k_2} = k'$$

Example $X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx}\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty}\frac{(e^{it})^x \lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}}$$

$$= \frac{e^{-\lambda}}{e^{-\lambda e^{it}}}\underbrace{\sum \frac{(\lambda e^{it})^x e^{-\lambda e^{it}}}{x!}}_{\text{PMF of Poisson}(\lambda e^{it})} = e^{-\lambda + \lambda e^{it}}$$

$$= e^{\lambda(e^{it}-1)}$$

$X_1 \sim \text{Poisson}(\lambda_1)$ ind of $X_2 \sim \text{Poisson}(\lambda_2)$

$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2) \overset{\lambda'}{\sim}$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$$= e^{\lambda_1(e^{it}-1)}\, e^{\lambda_2(e^{it}-1)}$$

Example $X_1 \ldots X_n \overset{iid}{\sim}$ some distribution with finite mean μ finite variance σ^2

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad E[\bar{X}] = \mu$$

$$\text{Var}[\bar{X}] = \frac{\sigma^2}{n}$$

Define $Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$, $E[Z_n] = 0$

"Standardization" $\quad \text{Var}[Z_n] = 1$

"

$SE(Z_n)$

What happens $n \to \infty$?

$$\phi_{\bar{X}}(t) \overset{\text{Rule 3}}{=} \phi_{\sum X_i}\left(\frac{t}{n}\right) \overset{\text{Rule 2}}{=} \left(\phi_X\left(\frac{t}{n}\right)\right)^n \quad \text{Rule 3, Rule 2}$$

$$\phi_{Z_n}(t) = \phi_{\bar{X}_n}\left(\frac{t}{\frac{\sigma}{\sqrt{n}}}\right) e^{it\left(\frac{-\mu}{\frac{\sigma}{\sqrt{n}}}\right)} = \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu\sqrt{n}}{\sigma}}$$

Rule 3

$$\phi_{Z_n}(t) = \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{\frac{-it\mu\sqrt{n}}{\sigma}\cdot\frac{n}{n}}$$

$$= \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{\frac{-it\mu n}{\sigma\sqrt{n}}}$$

$$\phi_{Z_n}(t) = \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n e^{\frac{-it\mu n}{\sigma\sqrt{n}}}$$

Note $y = e^{\ln(y)}$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \lim_{n\to\infty} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{\frac{-it\mu n}{\sigma\sqrt{n}}}\right)}$$

$$= \lim_{n\to\infty} e^{n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu n}{\sigma\sqrt{n}}}$$

$$= \lim_{n\to\infty} e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= \cancel{e} \; e^{\lim_{n\to\infty} n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}}$$

Let $u = \frac{t}{\sigma\sqrt{n}}$, $n \to \infty \Rightarrow u = 0$

$$= e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

using hopital Rule

$$= e^{\frac{t^2}{2\sigma^2}} \lim_{u \to 0} \frac{\frac{\phi'(u)}{\phi(u)} - i\mu}{u}$$

$$= e^{\frac{t^2}{2\sigma^2}} \lim_{u \to 0} \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right]$$

$$\lim_{u \to 0} \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right] = \lim_{u \to 0} \frac{\phi''(u)\phi(u) - (\phi'(u))^2}{\phi(u)^2}$$

$$= \frac{\underbrace{\phi''(0)}_{=i^2 E[x^2]}\underbrace{\phi(0)}_{=1} - \underbrace{(\phi'(0))^2}_{=(i\mu)^2}}{\underbrace{\phi(0)^2}_{=1^2}}$$

$$= i^2(E[x^2] - \mu^2) = -\sigma^2$$

$$= e^{\frac{t^2}{2\sigma^2}(-\sigma^2)} = e^{-\frac{t^2}{2}} = \phi_Z(t)$$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} e^{-\frac{t^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itx + \frac{t^2}{2}\right)}\,dt$$

Note $\frac{t^2}{2} + itx = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 + \frac{x^2}{2}$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}ix}{2}\right)^2 + \frac{x^2}{2}\right)} dt$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}ix}{2}\right)^2} dt$$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}ix}{2}$

$\frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,dy$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\,dy$$

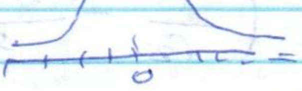
$$= \frac{1}{\pi\sqrt{2}} e^{-\frac{x^2}{2}} \underbrace{\int_{\mathbb{R}} e^{-y^2} dy}_{\text{Gaussian Integral} = \sqrt{\pi}}$$

$$= \frac{1}{\pi\sqrt{2}} e^{-\frac{x^2}{2}} \cdot \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = N(0,1)$$

Standard Normal

$N(0,1)$ Central limit Theorem



$\mathrm{Sup}(Z) = \mathbb{R}$