Lec. 13 Oct. 24

$$Y = \frac{X_1}{X_1 + X_2}$$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$ $\quad \text{supp}(Y) = [0, 1]$

ind. of $X_2 \sim \text{Gamma}(\beta, \lambda)$

if $X_1 \sim 0$ $\quad X_2 \sim \infty \to Y = 0$

if $X_1 \sim \infty$ $\quad X_2 \sim 0$ $\quad y = 1$

$X_1 \sim X_2$ $\quad Y \sim \frac{1}{2}$

$$Y = \frac{X_1}{X_1 + X_2} \sim \int_0^\infty \overbrace{f_{X_1}(YY_2)\, f_{X_2}(Y_2 - YY_2)}^{\text{joint pdf}}\, Y_2 \, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (YY_2)^{\alpha-1} e^{-\lambda Y Y_2}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta (Y_2(1-Y))^{\beta-1} e^{-\lambda Y_2 (1-Y)}}{\Gamma(\beta)} Y_2 \, d$$

$$\frac{\lambda^{\alpha+\beta} Y^{\alpha-1} (1-Y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty Y_2^{\alpha+\beta-1} \underbrace{e^{-\lambda Y Y_2 - \lambda Y_2(1-Y)}}_{e^{-\lambda Y_2}} \, dY_2$$

Let $u = \lambda y_2 \longrightarrow \frac{du}{dy_2} = \lambda$

$\Rightarrow Y_2 = \left(\frac{\lambda}{u}\right)^{-1} = \frac{u}{\lambda} \longrightarrow dy_2 = \frac{1}{\lambda} du$

$\int_0^\infty u^{\alpha+\beta-1} e^{-u} du = \Gamma(\alpha, \beta)$

$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{u^{\alpha+\beta-1}}{\lambda^{\alpha+\beta-1}} e^{-u} \frac{1}{\lambda} du$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} Y^{\alpha-1} (1-Y)^{\beta-1} = \text{Beta}(\alpha, \beta)$

$B(\alpha, \beta)$

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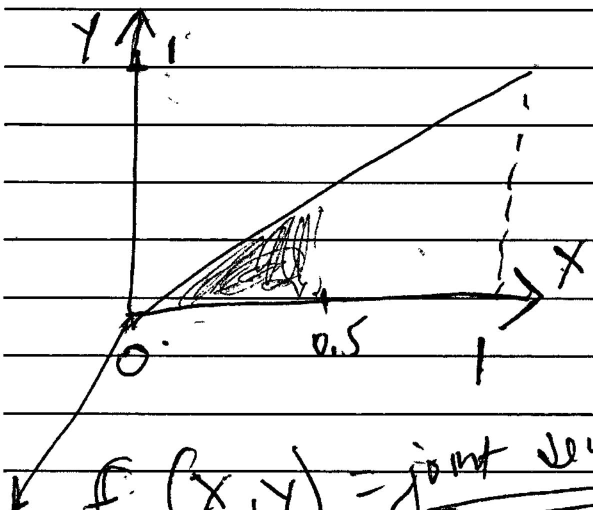
Conditional Decision

Let $X \sim U(0,1)$

Let $Y | X=x \sim U(0,x)$



$f_{X,Y}(x,y)$ = joint density

$Supp[Y] = [0,1]$

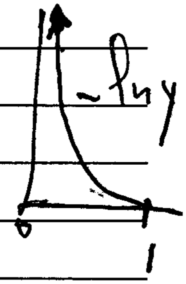
formula

$$f_Y(y) = \int_{Supp[X]} f_{X,Y}(x,y)\,dx = \int f_{Y|X}(y_0\,x) f_X(x)\,dx$$

$$f_Y(y) = \int_0^1 f_{Y|X}(y_0\,x) f_X(x)\,dx = \int_0^1 \frac{1}{x} \mathbb{1}_{\underbrace{y \in [0,x]}_{0 \le y \le x}} \mathbb{1}_{x \in [0,1]}$$

$x \geq y$

$$= \int_y^1 \frac{1}{x}\, dx = \ln(y) \Big]_y^1 = -\ln(y)$$



$$\lim_{x \to 0} x \ln x = 0$$

$$\int_0^1 -\ln y \, dy = -\Big[y\ln(y) - y\Big]_0^1 = (1-0) - (0\,0)$$

$$= 1$$

A download is either exponential with mean 10 min with no network trafic or exponential mean 20 min with network trafic. How long does the download take?

$$Y \sim \begin{cases} Exp\left(\frac{1}{10}\right) & wp\ \frac{1}{3} \\ Exp\left(\frac{1}{20}\right) & wp\ \frac{2}{3} \end{cases}$$

$$X = \mathbb{1}_{\text{network trafic}} \sim Bern\left(\tfrac{2}{3}\right) = \left(\frac{2}{3}\right)^X \left(\frac{1}{3}\right)^{1-X}$$

$$Y | X = x \sim Exp\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} = \left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} e^{-y\left(\frac{1}{20}\right)\left(\frac{1}{10}\right)^{1-x}}$$

$$f_Y(y) = \sum_{x \in \mathrm{supp}[X]} f_{Y|X}(y, x)\, p_X(x)$$

$$= \sum_{x \in [0,1]} \left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} y} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}$$

$$= \frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}y}\right) + \frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}y}\right)$$

$$= \boxed{\frac{1}{3}\,\mathrm{Exp}\left(\frac{1}{10}\right) + \frac{2}{3}\,\mathrm{Exp}\left(\frac{1}{20}\right)}$$

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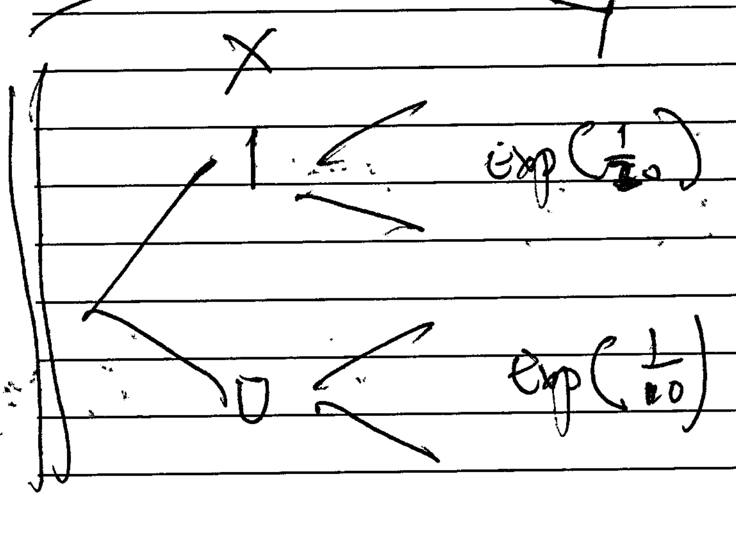

If the download took 25 min, what is the proba. there was network traffic?

$$P_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y,x)\, P_X(x)}{f_Y(y)}$$

$$P_{X|Y}(1,25) = \frac{f_{Y|X}(25,1)\, P_X(1)}{f_Y(25)}$$ ← Copy from previous exerc

$$= \frac{\overbrace{\frac{2}{3}\left(\frac{1}{30}e^{-\frac{1}{20}(25)}\right)}^{29.68}}{\underbrace{\frac{1}{2}\left(\frac{1}{10}e^{-\frac{1}{20}(25)}\right)}_{1.22} + \underbrace{\frac{2}{3}\left(\frac{1}{20}e^{-\frac{1}{20}(25)}\right)}_{29.68}} \simeq 98\% \neq \frac{2}{3}$$

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Car accidents are Poisson distr. but their rate person λ is not the same for all drivers.
λ is gamma distr. $\lambda \sim \text{Gamma}(\alpha, \beta)$
What is the distribute of accident.

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

$$Y \mid X = x \sim \text{Poisson}(x) = \frac{e^{-x} x^{y}}{y!}$$

X / Gamma(α, β) $\quad$ Y / Poisson $\quad$ Compound distr. (mix, continuous)

$P_Y(y)$? $\quad$ Support$[Y] = \{0, 1, 2, \ldots\}$

$$P_Y(y) = \int_{\text{supp}(X)} P_{Y|X}(y, x)\, f_X(x)\, dx$$

supp$[Y] = \{0, 1, \ldots\}$

$$= \int_0^{\infty} \frac{e^{-x} x^{y}}{y!} \cdot \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}\, dx$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

let use u substitute

let $u = (\beta+1)x$

$$\frac{du}{dx} = \beta+1 \Rightarrow dx = \frac{dy}{\beta+1}$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty \frac{u^{y+\alpha-1}}{(\beta+1)^{y+\alpha-1}} \cdot e^{-u} \frac{1}{\beta+1} du$$

$$\searrow\nearrow (\beta+1)^{y+\alpha}$$

$$= \frac{\beta^\alpha \Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}} = \left(\frac{\beta}{\beta+1}\right)^\alpha \frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^y}$$

let $k = \alpha$ $\quad p = \frac{\beta}{1+\beta} \Rightarrow 1-p = \frac{1}{1+\beta}$

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$$P_{Y|X}(y,x) = \frac{\Gamma(y+k)}{\Gamma(k)\,\Gamma(y-1)} p^k (1-p)^y$$

$$= \text{Exp.} + \text{NegBin}(K, p)$$

if $y, k \in \mathbb{N}$

$$= \binom{y+k-1}{k} p^k (1-p)^y = \text{NegBin}(k,p)$$