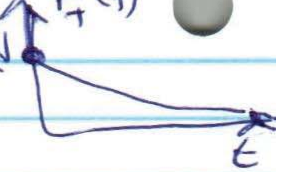
Lecture 7 : 9/19/17

$T \sim Exp(\lambda) = \lambda e^{-\lambda t}$, $F_T(t) = 1 - e^{-\lambda t}$



time until even $\lambda = np$ $n \to \infty \Rightarrow p \to 0$

$N \sim Poisson(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$, $F_N(n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{n}\frac{\lambda^i}{i!}$

# of Even

What is the probability the even did not happen by $t=1$? $\Rightarrow P(T>1) = e^{-\lambda}$

What is the probability Zero Even occure?

$P(N=0) = e^{-\lambda}$

$X \sim Erlang(k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$, $F_X(x) = \frac{\gamma(k, \lambda x)}{(k-1)!}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt = \underbrace{\int_0^a t^{x-1}e^{-t}\,dt}_{\gamma(x,a)} + \underbrace{\int_0^\infty t^{x-1}e^{-t}\,dt}_{\Gamma(x,a)}$$

$\gamma(x,a)$ lower incomplete gamma function

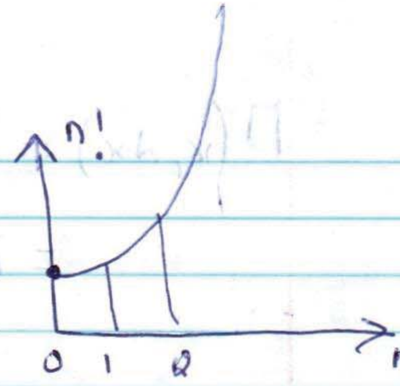
$\Gamma(x,a)$ upper incomplete gamma function

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\underbrace{e^{-t}dt}_{dv} = \underbrace{\left[-t^x e^{-t}\right]_0^\infty}_{0} + x\int_0^\infty (e^{-t})\, t^{x-1}\,dt$$

$$= x\,\Gamma(x)$$

$$\Gamma(2) = 1\,\Gamma(1) = 1$$
$$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1$$
$$\Gamma(4) = 3\,\Gamma(3) = 3\cdot 2\cdot 1$$



$$\Gamma(n) = (n-1)!\quad n \in \mathbb{N}$$

$$F_X(x) = \frac{\gamma(k,\lambda x)}{(k-1)!} = \frac{\gamma(k,\lambda x)}{\Gamma(k)}$$

$$1 - F_X(x) = 1 - \frac{\gamma(k,\lambda x)}{\Gamma(k)} = \frac{\Gamma(k) - \gamma(k,\lambda x)}{\Gamma(k)}$$

$$= \frac{\Gamma(k,\lambda x)}{\Gamma(k)}$$ regularized $\varphi(k,\lambda x)$

$$\Gamma(k,\lambda x) = \int_{\lambda x}^{\infty} \underset{u}{t^{k-1}}\, \underset{dv}{e^{-t}}\,dt$$

$$= -t^{k-1}e^{-t}\Big|_{\lambda x}^{\infty} - \int_{\lambda x}^{\infty} (k-1)t^{k-2}(-e^{-t})\,dt$$

$$= (\lambda x)^{k-1}e^{-\lambda x} + (k-1)\int_{\lambda x}^{\infty} t^{k-1}e^{-t}\,dt$$

$$= (\lambda x)^{k-1}e^{-\lambda x} + (k-1)\underbrace{\int_{\lambda x}^{\infty} t^{k-2}e^{-t}\,dt}_{\Gamma(k-1,\lambda x)}$$

$$\Gamma(k-1,\lambda x) = (\lambda x)^{k-2}e^{-\lambda x} + (k-2)\,\Gamma(k-2,\lambda x)$$

$$\Gamma(1,\lambda x) = \int_{\lambda x}^{\infty} t^{1-1}e^{-t}\,dt = e^{-\lambda x}$$

$$\Gamma(k,\lambda x) = e^{-\lambda x}\left((\lambda x)^{k-1} + (k-1)(\lambda x)^{k-2} + \cdots + (k-1)!\right)$$

$$= e^{-\lambda x}(k-1)!\left(\frac{(\lambda x)^{k-1}}{(k-1)!} + \frac{(\lambda x)^{k-2}}{(k-2)!} + \cdots + 1\right)$$

$$= e^{-\lambda x}(k-1)!\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!} = 1 - F_{T_x}(x) = \frac{\Gamma(k,\lambda x)}{\Gamma(\alpha)}$$

$$= \frac{e^{-\lambda x}\cancel{\Gamma(k)}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}{\cancel{\Gamma(k)}}$$

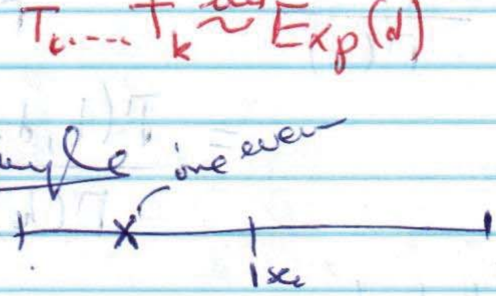
$T_k = T_1 + T_2 + \cdots + T_k$

such that

$T_1, \ldots, T_k \overset{iid}{\sim} Exp(\lambda)$

$$\Rightarrow 1 - F_{T_x}(x) = e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}$$

Example one even

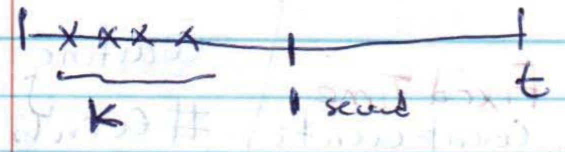


N # of evens by $t = 1$

What is the probability of no success or one success? $P(N \le 1)$

$= F_N(1) = e^{-\lambda}(1+\lambda)$

$T \sim Erlang(2,\lambda)$, $P(T > 1) = e^{-\lambda}(1+\lambda)$



P(less than or equal to k even by t = 1)

N : # of event by 1 second

$$P(N \leq k) = F_N(k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

$$T \sim \text{Erlang}(k+1, \lambda)$$

$$P(T > 1)\ e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

$$e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!} = \frac{\Gamma(k+1, \lambda)}{\Gamma(k)} = Q(k+1, \lambda)$$

$$\sum_{i=0}^{k} \frac{\lambda^i}{i!} = e^{\lambda} Q(k+1, \lambda)$$

$$\boxed{\underline{\text{Nolé}}:\ e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}$$

$$\sum_{i=0}^{k} \frac{\lambda^i}{i!} = e^{\lambda} Q(k+1, \lambda)$$?

$$\lim_{k \to \infty} Q(k, \lambda) = 1$$

Running Experiments

	Fixed Time Count events	Waiting to # events
Discretely	Binomial Bernoulli (1 event)	Negative Binomial Geometric (1 event)
Continuously	Poisson	Erlang exponential (1 event)

Poisson Process

What is the probability there has being 2 or less successes by the 50^{th} experiment?

$N \sim Bin(50, p)$, $P(N \le 2) = F_N(2)$

$$= \binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} + \binom{50}{2} p^2 (1-p)^{48}$$

$T \sim NegBin(3, p)$ $\quad P(T > 47) = P(T=48) + P(T=49) + P(T=50) + P(\ldots$ ---

$$= 1 - P(T \le 47)$$

$$= 1 - F_T(47)$$

what is the probability there has been K or less successes by experiment n?

$N \sim Bin(n, p) \quad P(N \leq k) = F_N(k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$

$T \sim NegBin(k+1, p) \quad P(T > n-(k-1)) =$

$1 - P(T \leq n-(k+1)) = 1 - F_T(n-k-1)$

$= 1 - p^{k+1} \sum_{i=0}^{n-k-1} \binom{i+k}{k} (1-p)^i$

$= 1 - p^{k+1} \sum_{i=0}^{n-k-1} \binom{i+k}{k} (1-p)^i$

let $X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$

what is $P(X_1 \mid X_1 + X_2)$? $\quad P(X_1 = x \mid X_1 + X_2 = n)$

$P_X(x) = P(X = x)$

PmF

Remember $X_1 + X_2 \sim Poisson(2\lambda)$: $= \frac{(2\lambda)^x e^{-2\lambda}}{x!}$

$P(X_1 = x \mid X_1 + X_2 = n) = \frac{P(X_1 = x \text{ and } X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$ $\quad X_2 = n - x$

$P(X_1 + X_2 = n) = \frac{e^{-2\lambda} (2\lambda)^n}{n!}$

$= \frac{P_{X_1, X_2}(x, n-x)}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} = \frac{P_{X_1}(x)\, P_{X_2}(n-x)}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}}$

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{e^{-\lambda}\lambda^{n-x}}{(n-x)!}}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} = \binom{n}{x}\frac{\lambda^n}{(2\lambda)^n}$$

$$= \binom{n}{x}\left(\frac{1}{2}\right)^n$$

$$= Bin\left(n, \frac{1}{2}\right)$$

$Y = X_1 - X_2 \sim ?$

Find $p_Y(y)$ — Transformation of r.v

$Y = X_1 + (-X_2) = X_1 + Z$ where $Z = -X_2$

$Y = g(X)$

$p_X(x)$ known $p_Y(y) = ?$

$f_X(x)$ known $f_Y(y) = ?$