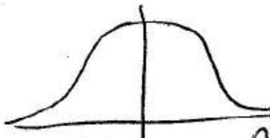
11/07/17 Lecture #17

$Z_1, \cdots Z_n \overset{iid}{\sim} N(0,1)$

$\sum_{i=1}^{n} z_i^2 \sim \chi_k^2$, chi-squared w/k degrees of freedom.

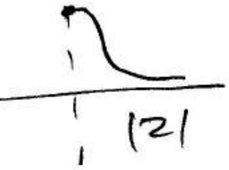
$\sqrt{\sum_{i=1}^{k} z_i^2} \sim \chi_k$ "chi"



Normal

(*) $Y = |Z| \Rightarrow Y^2 = Z^2 \sim \chi_1^2$

$|Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}} = 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)$ 2. Normal?



$|z|$

Let's do transformations of χ.

* $X \sim \chi_k^2$, $Y = \frac{X}{K} \sim ?$

$\parallel$

$\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

chi is special case for Gamma.

comparing $c = \frac{1}{K}$

Let's say $X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX \sim ?$ $c \in (0, \infty)$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)}}{\Gamma(\alpha)}$$

$$= \frac{\beta^\alpha y^{\alpha-1} e^{-\left(\frac{\beta}{c}\right)y}}{\underbrace{c \cdot c^{\alpha-1}}_{c^\alpha} \Gamma(\alpha)} = \frac{\left(\frac{\beta}{c}\right)^\alpha y^{\alpha-1} e^{-\frac{\beta}{c} y}}{\Gamma(\alpha)}$$

$$= \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$\therefore Y \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}/\frac{1}{k}\right)$

$= \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

* Support of Gamma $(0, \infty)$. Support of χ is $(0, \infty)$

(•) $X_1 \sim \chi_{k_1}^2$ ind of $X_2 \sim X_{k_2}^2$

$\frac{+}{+} = +$

$\text{Supp}[R] = (0, \infty)$

$R = \frac{X_1/k_1}{X_2/k_2} \sim ?$ ← Looks like Gamma (a, a) ← Looks like Gamma (a, b)

$$R = \frac{X_1/k_1}{X_2/k_2} = \frac{V_1}{V_2} \sim \int_{\text{supp}[V_2]} t \, f_{V_1}(rt) f_{V_2}(t) \, dt$$

$$= \int_0^\infty \frac{t \cdot a^a (rt)^{a-1} e^{-art}}{\Gamma(a)} \cdot \frac{b^b t^{b-1} e^{-bt}}{\Gamma(b)} \, dt$$

$$= \frac{a^a b^b (r)^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} \, dt$$

Let $a = \frac{k_1}{2}$

So $V_1 \sim \frac{a^a x^{a-1} e^{-ax}}{\Gamma(a)}$

Let $b = \frac{k_2}{2}$

So $V_2 \sim \frac{b^b x^{b-1} e^{-bx}}{\Gamma(b)}$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty \frac{u^{a+b-1}}{(ar+b)^{a+b-1}} e^{-u} \cdot \frac{du}{ar+b}$$

Let $u = (ar+b)t$

$du = (ar+b)dt$

$dt = \frac{du}{ar+b}$.

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b-1}} \underbrace{\int_0^\infty u^{a+b-1} e^{-u} du}_{\Gamma(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} \cdot \frac{r^{a-1}}{(ar+b)^{a+b}} = \frac{a^a b^b r^{a-1}}{B(a,b)} \cdot \underbrace{(ar+b)^{-(a+b)}}_{= b^{a+b}\left(1+\frac{a}{b}r\right)^{a+b}}$$

$$= \frac{a^a b^b}{B(a,b)} \frac{r^{a-1}}{\underbrace{b^{a+b}}_{b^a \cdot b^b}} \cdot \left(\frac{a}{b}r+1\right)^{-(a+b)}$$

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)} = \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1+\frac{k_1 r}{k_2}\right)^{-\frac{k_1+k_2}{2}}$$

$$= F_{k_1, k_2}$$

"F" distribution

"Fisher-Snecodor"

"F" is normal Fisher.

⊙ $Z \sim N(0,1)$ ind of $V \sim \chi_k^2$. Let $Y = \frac{Z}{\sqrt{\frac{V}{k}}} \sim$? pdf?

Supp: $\mathbb{R}$ supp = $\mathbb{R}$ supp $(0,\infty)$

We can find pdf by the previous method or

$$X = \frac{Z^2}{V/k} = \frac{\frac{U}{1}}{\frac{V}{k}} \qquad u = z^2 \sim \chi_1^2$$

look like the previous one.

$$Y \sim F_{1,k} = \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} x^{-\frac{1}{2}} \left(1+\frac{1}{k}x\right)^{\frac{-k+1}{2}}$$

$W = \pm\sqrt{Y}$

W is symmmetry

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$$= \frac{1}{2} f_X(y^2) \cdot 2y$$

$Y = X^2$

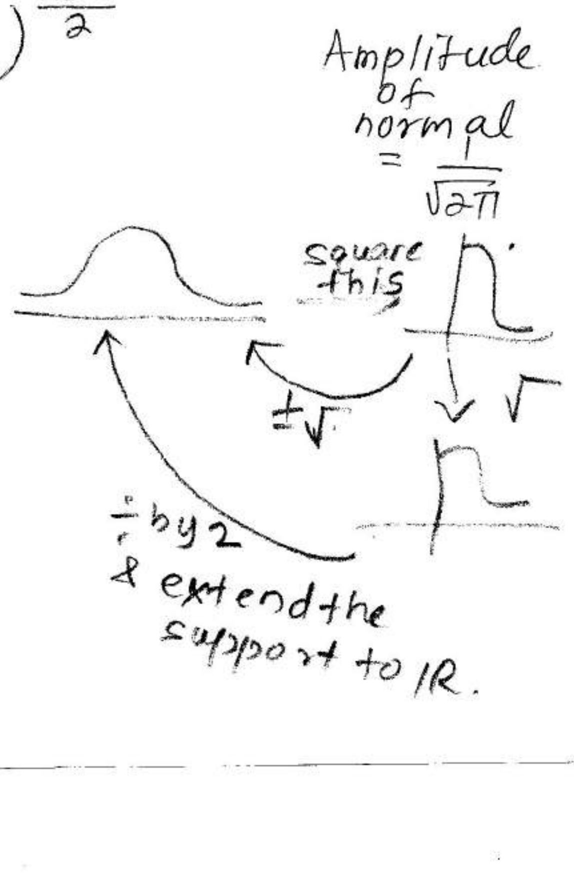
$\Uparrow$

$Y = \sqrt{X} = X^{\frac{1}{2}}$

$\cancel{\frac{d}{dy}(g^{-1}(y)) = \frac{1}{2}\frac{1}{\sqrt{x}}}$

$X =$

$\Rightarrow$



$$= \frac{1}{\sqrt{k}\,B(\frac{1}{2},\frac{k}{2})} \underbrace{(y^2)^{-\frac{1}{2}}}_{\frac{1}{y}} \left(1+\frac{y^2}{k}\right)^{-\frac{k+1}{2}} \cdot \cancel{y}$$

$$= \frac{1}{\sqrt{k}\,B(\frac{1}{2},\frac{k}{2})} \cdot \left(1+\frac{y^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

} Student's T distn. w/k degrees of freedom.

$k \to \infty \quad Y = \frac{Z}{\sqrt{\frac{V}{k}}} \Rightarrow \frac{V}{k} = \frac{\sum_{i=1}^{k} Z_i^2}{k}$

$E[\chi^2_k] = k$ } $Var[\chi^2_k] = 2k$

$E[\frac{V}{k}] = 1$ } $Var[\frac{V}{k}] = \frac{2}{k}$

from $\frac{2k}{k^2}$

$\frac{V}{k} = N(1, \frac{2}{k})$

$\underset{k\to\infty}{\Rightarrow} N(1, 0) = Deg(1)$

So $Y = \frac{Z}{\sqrt{\frac{V}{k}}} \to Z$

$T_k \to Z$

⊙ $X_1 \sim N(0,1)$ ind of $X_2 \sim N(0,1)$

$R = \frac{X_1}{X_2} = \frac{X_1}{\sqrt{\frac{X_2^2}{1}}} \sim T_1 = \frac{1}{B(\frac{1}{2},\frac{1}{2})}(1+x^2)^{-1}$

$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})} \cdot (1+x^2)^{-1} = \frac{\Gamma(1)}{(\sqrt{\pi})^2} = \frac{1}{\pi} = \frac{1}{\pi(1+x^2)}$

= Cauchy (0,1) (standard)

⊙ $X \sim$ Cauchy $(0,1)$

$Y = c + \sigma X \quad ; \; c \in (0,\infty)$

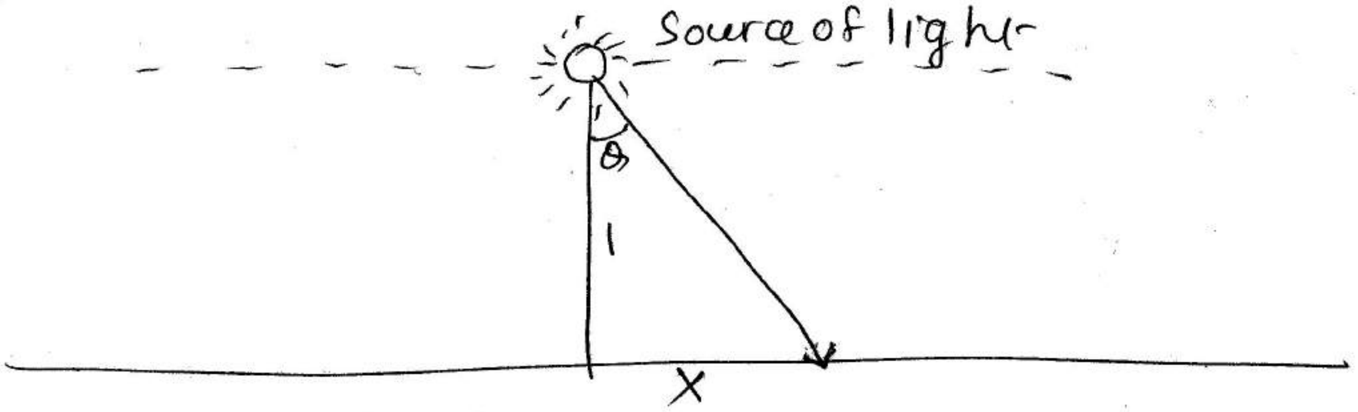
$x = \frac{Y-c}{\sigma} = g^{-1}(y)$

$(g^{-1}(y))' = \frac{1}{\sigma}$

$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-c}{\sigma}\right) = \frac{1}{\sigma} \cdot \frac{1}{\pi\left(1+\left(\frac{y-c}{\sigma}\right)^2\right)}$

$= \frac{1}{\pi\sigma} \cdot \frac{1}{1+\left(\frac{y-c}{\sigma}\right)^2}$ = Cauchy $(1, \sigma)$

AKA the Lorentz dist.



$\Theta \sim U(\pi, 2\pi) = \frac{1}{\pi}$

$X = \tan \Theta$

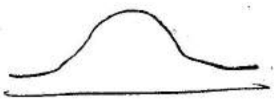
$\Theta = \arctan(x) = g^{-1}(x)$

$\frac{d}{dx}(g^{-1}(x)) = \frac{1}{1+x^2}$

$f_X(x) = f_\Theta(g^{-1}(x)) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$

$=$ Cauchy.

Cauchy looks also like normal



$E(X) = \int_{\mathbb{R}} x \cdot \frac{1}{\pi} \cdot \frac{1}{1+x^2} \, dx$

$= \frac{1}{\pi} \int_{\mathbb{R}} \frac{x}{1+x^2} \, dx = \frac{1}{\pi} \left[\frac{1}{2} \ln(x^2+1)\right]_{-\infty}^{\infty} = \infty$ does not exist

$Var[X] = \infty$

No expectation

No variance.

$R = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} |x_2| \, f_{X_1}(x_2 r) f_{X_2}(x_2) \, dx_2$