9/26/17

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$① = X_1 - X_2 = X_1 + Y$

$Y = -X_2$

Transformation of RV'S

$X \sim \text{Bern}(p) = p^x(1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

$Y = 3 + X \sim \begin{cases} 4 & wp\ p \\ 3 & wp\ 1-p \end{cases} = P_Y(y) = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y \in \{}$

$\text{Supp}[Y] = \{y : y-3 \in \text{Supp}[X]\}$

$Y = c + aX = g(X) \Rightarrow X = \frac{y-c}{a} = g^{-1}(y)$

$\text{Supp}(Y) = \{y : \frac{y-c}{a} \in \text{Supp}[X]\}$

$= \{y : \frac{y-c}{a} \in \{0,1\}\}$

$= \{c, c+a\}$

$X \sim \text{Bern}(p)$

Working theory

$P_Y(y) = p^{\frac{y-c}{a}}(1-p)^{1-\frac{y-c}{a}} \mathbb{1}_{y \in \{c, c+a\}} = P_X(g^{-1}(y))$

modify support

$Y = aX + c$

$P_Y(y) = \binom{n}{g^{-1}(y)} p^{g^{-1}(y)}(1-p)^{n-g^{-1}(y)} \mathbb{1}_{g^{-1}(y) \in \text{Supp}[X]}$

$Y = X^3$

$$P_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{y \in \{0, 1^3, 2^3, 3^3, \dots, n^3\}}.$$

~~This is 1-1~~

$X \sim \text{Geom}(P)$

$Y = \max\{3, X\} = g(X)$ is not 1-1

X	Y
0	3
1	3
2	3
3	3
4	4
5	5
⋮	⋮

~~Formula~~ for discrete r.v. transformation

$$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x) = \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$$

$$= \{x : x \in g^{-1}(y)\}$$

inverse relation

$$P_Y(y) = (\underbrace{P_X(0)}_{p} + \underbrace{P_X(1)}_{p(1-p)} + \underbrace{P_X(2)}_{p(1-p)^2} + \underbrace{P_X(3)}_{p(1-p)^3}) \mathbb{1}_{y=3} + p(1-p)^y \mathbb{1}_{y \in \{4,5,\dots\}}.$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2$

$= X_1 + Y$

$Y = -X_2$

$$P_Y(y) = \frac{e^{-\lambda} \lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{0, -1, -2, \dots\}}.$$

$\text{Supp}[D] = \mathbb{Z}$ ~~reval~~

$$P_D(d) = \sum_{x \in \mathrm{Supp}\{X_1\}} P_{X_1}(x) P_Y(d-x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{e^{-\lambda}\lambda^{\overline{x-d}}}{(-(d-x))!}$$

$\Uparrow$ $d-x \in \{0,-1,-2,\dots\}$

$\Downarrow$

$x-d \in \{0,1,2,\dots\}$

$\Downarrow$

$x \in \{d, d+1, d+2, \dots\}$.

$$= e^{-2\lambda} \sum_{x \geq \max\{0,d\}}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!}$$

$$= e^{-2\lambda} \begin{cases} \displaystyle\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \geq 0 \text{ (upper)} \\ \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \text{ (lower)}. \end{cases}$$

let $x' = x-d \implies x = x'+d$

$$\sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\, x'!} = \underbrace{\sum_{i=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2i+d}}{\Gamma(i+d-1)\Gamma(i+1)}}$$

$\Gamma(n-1) = n!$

modified Bessel Function of the 1st kind denoted $I_d(2\lambda)$

$$= I_d(2\lambda) = I_{|d|}(2\lambda)$$

let $d' = -d$

$$\sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!} = I_{d'}(2\lambda) = I_{|d|}(2\lambda)$$

if $d<0 \Rightarrow d' = |d|$

$P_D^{(d)} = e^{-2\lambda} I_d(2\lambda) = \text{Skellam}(\lambda, \lambda)$ 1946

let $X \sim U(0,1)$

$Y = aX = g(x)$, 1-1

$P_Y^{(y)} \stackrel{?}{=} P_X(g^{-1}(y))$.

Not useful

$Y = g(x)$, 1-1. Find $f_Y(y)$ given $f_X(x)$.

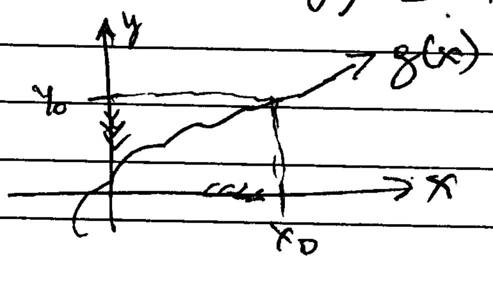
If g is 1-1

$\Rightarrow$ a) g is strictly increasing

b) g " " decreasing.

$F_Y(y) = P(Y \leq y) = P(g(x) \leq y) = P(X \leq g^{-1}(y))$

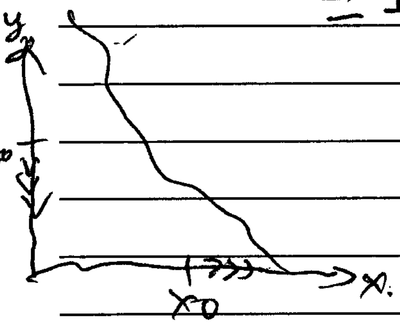
$= F_X(g^{-1}(y))$



$$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right] = \frac{d}{dy}\left[F_X(g^{-1}(y))\right] = f_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y))$$

$$= 1 - P(X < g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$



$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -f_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$= \underset{+}{f_X(g^{-1}(y))}\left(\underset{-}{-\frac{d}{dy}\left[g^{-1}(y)\right]}\right)$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$$

$$\mathrm{Supp}[Y] = g(\mathrm{Supp}[X]) = \{y : g^{-1}(y) \in \mathrm{Supp}[X]\}$$

let $Y = aX + c = g(X)$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-c}{a}\right)$$

Special cases

$Y = -X$

$f_Y(y) = f_X(-y)$

$Y = X + c$

$f_Y(y) = f_X(y-c)$ shifted distribution

$Y = aX$

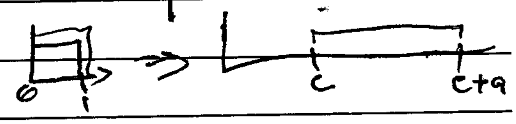
$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$

<u>Uniform</u>

$X \sim U(0,1)$

$Y = aX + c \qquad f_Y(y) = \frac{1}{|a|} \underbrace{f_X\left(\frac{y-c}{a}\right)}_{1} = \frac{1}{|a|}$

Support$(Y) = [c, c+a]$



$X \sim Exp(\lambda)$

$Y = aX + c \qquad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-c}{a}\right) = \frac{1}{|a|} \lambda e^{-\lambda\left(\frac{y-c}{a}\right)}$

$= \frac{1}{|a|} e^{\lambda c} \lambda e^{-\frac{\lambda y}{a}}$

$= \frac{\lambda}{|a|} \left(e^{\frac{\lambda c}{a}}\right) e^{-\frac{\lambda y}{a}} \qquad \text{supp}\{Y\} = (c, \infty)$

let $c = 0 \Rightarrow \frac{\lambda}{|a|} e^{-\frac{\lambda y}{a}}$ with $\lambda \in (0, \infty)$

let $a > 0 \Rightarrow \frac{\lambda}{a} e^{-\frac{\lambda y}{a}} \qquad \lambda' = \frac{\lambda}{a}$

$Exp\left(\frac{\lambda}{a}\right)$

$X \sim U(0,1)$

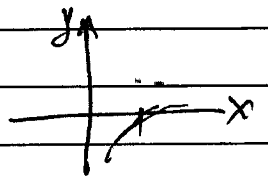
$Y = 1 - X \sim U(0,1) = f_Y(y) = \underbrace{f_X(y-1)}_{1} = 1$

$\text{Supp}(Y) = 1 - [0,1]$

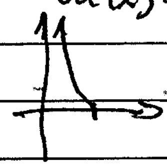
$= [0,1]$

$X \sim U(0,1)$

$Y = -\ln(x)$



$-\ln(x) = g(x)$



$y = g(x) = -\ln(x)$

$-y = \ln(x)$

$\Rightarrow e^{-y} = x = g^{-1}(y)$

$f_Y(y) = \underbrace{f_X(g^{-1}(y))}_{1} \left|\frac{d}{dy}[g^{-1}(y)]\right|$

$= \frac{d}{dy}[-e^{-y}] = e^{-y} = \text{Exp}(1)$ $\quad$ $\text{Supp}[Y] = (0, \infty)$

$X \sim \text{Exp}(1)$

$Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right)$