10/24/17

$$Y = \frac{X_1}{X_1+X_2} \, ?$$

$\text{supp}[Y] = [0, 1]$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$

$X_1 + X_2 \geq X_1$

wide

$X_2 \sim \text{Gamma}(\beta, \lambda)$

If $X_1 \approx 0$, $X_2 \approx \infty \Rightarrow Y \approx 0$

If $X_1 \approx \infty$, $X_2 \approx 0 \Rightarrow Y \approx 1$

$X_1 \approx X_2 \Leftrightarrow Y = 1/2$

$$Y = \frac{X_1}{X_1+X_2} \sim \int_0^\infty f_{X_1}(y y_2) f_{X_2}(\underbrace{y_2 - y y_2}_{= y_2(1-y)}) \underbrace{y_2}_{= |y_2|} \, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (y y_2)^{\alpha-1} e^{-\lambda y y_2}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta (y_2(1-y))^{\beta-1} e^{-\lambda y_2(1-y)}}{\Gamma(\beta)} \, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} \underbrace{e^{-\lambda y y_2 - \lambda y_2(1-y)}} \, dy_2$$

$$= e^{-\lambda(y y_2 + y_2(1-y))}$$

$$= e^{-\lambda y_2 (y+1-y)}$$

$$= e^{-\lambda y_2}$$

let $u = \lambda y_2 \Rightarrow \frac{du}{dy_2} = \lambda$

$\Rightarrow y_2 = \frac{1}{\lambda} u \Rightarrow dy_2 = \frac{1}{\lambda} du$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{u^{\alpha+\beta-1}}{\underbrace{\lambda^{\alpha+\beta-1}}} e^{-u} \frac{1}{\lambda} du$$

$$\int_0^\infty u^{\alpha+\beta-1} e^{-u} du = \Gamma(\alpha+\beta)$$

$$= \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\frac{1}{B(\alpha,\beta)}} y^{\alpha-1}(1-y)^{\beta-1} = \text{Beta}(\alpha,\beta)$$

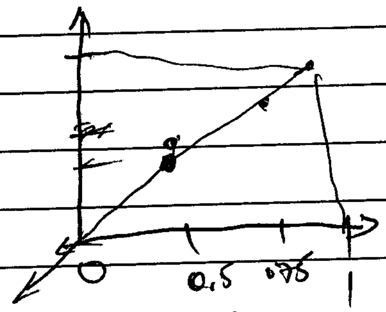
P155 Conditional Densities:

let $X \sim U(0,1)$

let $Y|X=x \sim U(0,x)$

$Y \sim ?$ $\quad$ supp$[X] = [0,x)$

Supp$[Y] = [0,1]$

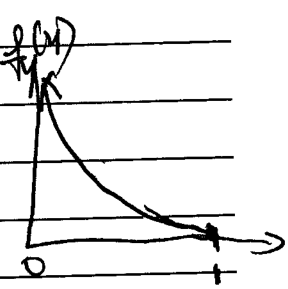


$f_{X,Y}(x,y)$ joint prob.

$$f_Y(y) = \int_{supp(X)} f_{X,Y}(x,y)\,dx = \int_{supp[X]} f_{Y|X}(y,x) f_X(x)\,dx$$

$$f_Y(y) = \int_0^1 f_{Y|X}(y,x) f_X(x)\,dx = \cancel{\int_0^1 \frac{1}{x}\mathbb{1}_{y\in[0,x)}}$$

$$= \int_0^1 \frac{1}{x}\underbrace{\underbrace{\mathbb{1}_{y\in[0,x)}}_{0\le y\le x}}_{x\ge y} 1\cdot \mathbb{1}_{x\in[0,1]}\,dx$$



$$= \int_y^1 \frac{1}{x}\,dx = \ln x\Big|_y^1 = -\ln(y)$$

$$\int_0^1 -\ln y\,dy = -[y\ln y - y]_0^1 = y - y\ln y\Big|_0^1 = (1-0)-(0-0) = 1$$

A download is either exponential mean 10 min w/ no network traffic.
Or exponential mean 20 min w/ network traffic
How long does the download take?
Network traffic is 1/3 of the time.

$$Y \sim \begin{cases} \text{Exp}\left(\frac{1}{10}\right) & \text{w.p. } \frac{1}{3} \\ \text{Exp}\left(\frac{1}{20}\right) & \text{w.p. } \frac{2}{3} \end{cases}$$

$$X = \mathbb{1}_{\text{network traffic}} \sim \text{Bern}\left(\tfrac{2}{3}\right) = \left(\tfrac{2}{3}\right)^x \left(\tfrac{1}{3}\right)^{1-x}$$

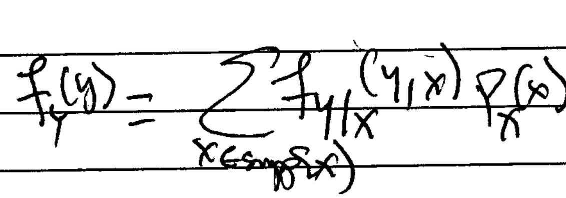
$$Y|X \sim \text{Exp}\left(\left(\tfrac{1}{20}\right)^x \left(\tfrac{1}{10}\right)^{1-x}\right) = \left(\tfrac{1}{20}\right)^x \left(\tfrac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} y}$$

$$f_Y(y) = \sum_{x \in \text{supp}(X)} f_{Y|X}(y|x)\, p_X(x)$$

$$= \sum_{x \in \{0,1\}} \left(\tfrac{1}{20}\right)^x \left(\tfrac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x \left(\frac{1}{10}\right)^{1-x} \cdot y} \cdot \left(\tfrac{2}{3}\right)^x \left(\tfrac{1}{3}\right)^{1-x}$$

$$= \frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}y}\right) + \frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}y}\right) = \frac{1}{3}\text{Exp}\left(\tfrac{1}{10}\right) + \frac{2}{3}\text{Exp}\left(\tfrac{1}{20}\right)$$

mixture distr.



If the download took 25 min. what is the prob. there was network traffic?

P.155 - 158

$$P_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y,x)\,P_X(x)}{f_Y(y)}$$

$$P_{X|Y}(1,25) = \frac{f_{Y|X}(25,1)\,P_X(1)}{f_Y(25)}$$

$$= \frac{\overbrace{\tfrac{1}{20}e^{-\frac{1}{20}(25)}}^{29.68}\cdot\left(\tfrac{2}{3}\right)}{\tfrac{1}{3}\underbrace{\left(\tfrac{1}{10}e^{-\frac{1}{10}(25)}\right)}_{1.22}+\tfrac{2}{3}\underbrace{\left(\tfrac{1}{20}e^{-\frac{1}{20}(25)}\right)}_{29.68}} \approx 98\% \neq \frac{2}{3}.$$

Car Accidents are Poisson distr.

~~$Y\sim$ Poisson(λ)~~ but λ is not the same for ~~to~~ all drivers. λ is gamma distr $\lambda \sim$ Gamma(α,β)

What is the distr of accidents?

$$X \sim \text{Gamma}(\alpha,\beta) = \frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$$

$$Y|_{X=x} \sim \text{Poisson}(x) = \frac{e^{-x}x^{y}}{y!}$$

X Gamma(α,β)

Y Poisson.

$$P_Y(y) = \int_{\mathrm{Supp}\{X\}} P_{Y|X}(y,x)\,f_X(x)\,dx = \int_0^{\infty}\frac{e^{-x}x^{y}}{y!}\cdot\frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}\,dx$$

$\mathrm{Sup}\{Y\} = \{0,1,2,\dots\}$

$$= \frac{\beta^{\alpha}}{y!\,\Gamma(\alpha)}\int_0^{\infty}x^{y+\alpha-1}e^{-(\beta+1)x}\,dx$$

621 - 02

let $u = (\beta+1)x \Rightarrow x = \frac{1}{\beta+1}u$

$\frac{du}{dx} = \beta+1 \Rightarrow dx = \frac{1}{\beta+1}du$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty \frac{u^{y+\alpha-1}}{(\beta+1)^{y+\alpha-1}} \cdot e^{u} \cdot \frac{1}{\beta+1}\, du$$

$(\beta+1)^{y+\alpha}$

$$= \frac{\beta^\alpha \Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}} = \left(\frac{\beta}{\beta+1}\right)^\alpha \frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^y}$$

let $k = \alpha$, $p = \frac{\beta}{\beta+1} \Rightarrow 1-p = \frac{1}{1+\beta}$

$$P_{Y|X}(y,x) = \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y-1)} p^k (1-p)^y = \text{Ext NegBin}(k,p)$$

If $y, k \in \mathbb{N}$

$$= \binom{y+k-1}{k} p^k (1-p)^y = \text{NegBin}(k,p)$$