lecture 21 11/30/17

Let X be non-negative r.v. with finite expectation μ.
Consider $a > 0$, a constant.
Consider the inequality.

$a \cdot \mathbb{1}_{X \geq a} \leq X$ Is this true?

real-value

Yes

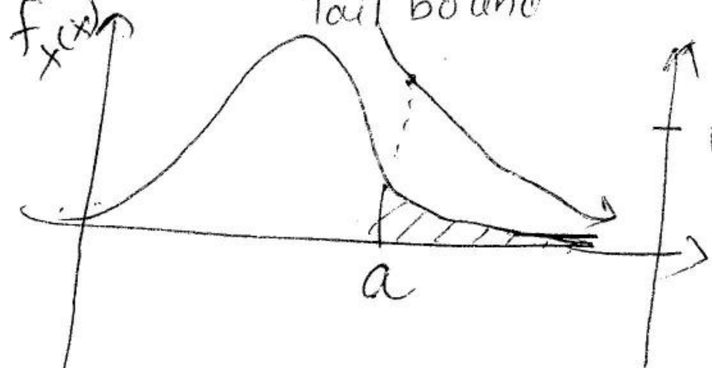
If $x \geq a$ ... $a(1) \leq x \Rightarrow x \geq a$ ✓
$x < a$ $\quad a(0) \leq x \Rightarrow x \geq 0$ true by assumption.

$E[a \mathbb{1}_{X \geq a}] \leq \mu$

how often that is 1

$\Rightarrow a\, E[\mathbb{1}_{X \geq a}] \leq \mu$.
$\Rightarrow a\, P(X \geq a) \leq \mu$
$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$ (*) Markov's Inequality

Tail bound



Tons of Corallaries.

* Let $a^* = a'\mu$. $\quad \frac{\mu}{a'\mu} = \frac{1}{a'}$

$P(X \geq a'\mu) \leq \frac{1}{a'}$

* Let h be a strickly monotonically increasing function.

$h(a)\, \mathbb{1}_{h(x) \geq h(a)} \leq h(x) \Rightarrow$

$$P(h(x) \geq h(a)) \leq \frac{E[h(x)]}{h(a)}$$

strickly monoto. increasing so invertible 1-1

$$\Rightarrow P(X \geq a) \leq \frac{E[h(x)]}{h(a)}$$

x in nonneg: def:

* $h(x) = |X|^p$ s.t. $p > 1$ — yes this is monot... increasing to $(0,\infty)$

$$P(X \geq a) \leq \frac{E[|X|^p]}{a^p}$$

* Recall Quantile $[X, P] = F_X^{-1}(P)$ if continuous.

$$P(X \geq a) \leq \frac{\mu}{a}$$

$$\Rightarrow 1 - P(X \leq a) \leq \frac{\mu}{a}$$

$\Rightarrow 1 - F(a) \leq \frac{\mu}{a}$.

Let $a = F_X^{-1}(a)$

$1 - F(F_X^{-1}(a)) \leq \frac{\mu}{F_X^{-1}(p)}$

$\Rightarrow 1 - p \leq \frac{\mu}{F_X^{-1}(p)}$

$\Rightarrow \text{Quantile}[X, p] \leq \frac{\mu}{1-p}$. relationship of quantile & mean.

$\Rightarrow \text{Med } X \leq 2\mu$ median verses mean.

* Consider any r.v. X. $|X|$ is non-negative.
$P(|X| \geq a) \leq \frac{E[|X|]}{a}$ more general
both tails

* Let X be any r.v. with finite μ, finite σ^2.
Let $Y := (X-\mu)^2$ Note: Y is non-negative.
$P(Y \geq a^2) \leq \frac{E[Y]}{a^2} = \frac{E[(X-\mu)^2]}{a^2}$
$P((X-\mu)^2 \geq a^2) \leq \frac{\sigma^2}{a^2}$

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ Chebyshev's Inequality

* let X be any r.v. , Let $Y = e^{tX}$

Note: Y is non neg:

$$P(Y \geq c) \leq \frac{E[X]}{c}$$

$$P(e^{tX} \geq c) \leq \frac{E[e^{tX}]}{c}$$

let $c = e^{ta}$

$$\Rightarrow P(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX}]}{e^{ta}} = \frac{M_X(t)}{e^{ta}}$$

$$\boxed{\text{Note: } M_X(t) : E(e^{tX})}$$

Moment generating function.

If $t>0$

$$P(X \geq a) \leq e^{-ta} M_X(t)$$

no more t inside

If $t<0$

$$P(X \leq a) \leq e^{-ta} M_X(t)$$

$$\Rightarrow P(X \geq a) \leq \min_{t>0} \{ e^{-ta} M_X(t) \}$$

$$\Rightarrow P(X \le a) \le \min_{t<0} \{ e^{-ta} M_X(t) \}$$

Chernoff's Inequality

— // —

$$X \sim Bin\left(n, \tfrac{1}{4}\right) \Rightarrow \mu = \tfrac{1}{4} n \quad , \sigma^2 = \tfrac{3}{16} n$$

$P\left(X \ge \frac{3}{4} n\right)$

np

If n is large $X \approx N\left(\frac{1}{4} n , \left(\sqrt{\frac{3}{16} n}\right)^2\right)$

$$P\left(X \ge \tfrac{3}{4} n\right) = P\left(\frac{X - \frac{1}{4} n}{\sqrt{\frac{3}{16} n}} > \frac{\frac{3}{4} n - \frac{1}{4} n}{\sqrt{\frac{3}{16} n}}\right)$$

$$= P\left(Z > \frac{2}{\sqrt{3}} \sqrt{n}\right)$$ close to zero

Now checking using the following 3 which one is closer

Markov's $P\left(X > \frac{3}{4} n\right) \le \dfrac{\frac{1}{4} n}{\frac{3}{4} n} = \frac{1}{3}$

Chebyshen's

~~$P\left(\left|X - \frac{1}{4} n\right| \ge\right.$~~

Chebyshev's

$P(X \geq \frac{3}{4}n) = P(X - \frac{1}{4}n \geq \frac{3}{4}n - \frac{1}{4}n)$

$\leq P(X - \frac{1}{4}n \geq \frac{1}{2}n) + P(\frac{1}{4}n - X \geq \frac{1}{2}n)$

$= P(X - \frac{1}{4}n \geq \frac{1}{2}S \text{ OR } \frac{1}{4}n - X \geq \frac{1}{2}n)$

$= P(|X - \frac{1}{4}n| \geq \frac{1}{2}n) \leq \dfrac{\frac{3}{16}n}{\frac{1}{4}n^2} = \dfrac{3}{4n}$

hyperb...

Chernoff's

$X \sim Bin(n, p)$

$m_X(t) := E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \overset{\text{Binom Thm}}{=} (1-p+pe^t)^n$

$X \sim Bin(n, \frac{1}{4}) \Rightarrow m_X(t) = (\frac{3}{4} + \frac{1}{4}e^t)^n$

$P(X \geq \frac{3}{4}n) \leq \min_{t>0} \{ e^{-t \cdot \frac{3}{4}n} (\frac{3}{4} + \frac{1}{4}e^t)^n \}$

$= \min_{t>0} \{ (\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t})^n \}$

Take $\frac{d}{dt}[\;] \overset{\text{set}}{=} 0$

always + so forget abt it

$$\Rightarrow n\left(\underset{\downarrow}{—} \right)^{n-1}\left(-\frac{9}{16}e^{-\frac{3}{4}t}+\frac{1}{16}e^{\frac{1}{4}t}\right)=0$$

$$\Rightarrow \quad e^{\frac{1}{4}t}=9e^{-\frac{3}{4}t}$$

$$\Rightarrow \quad \frac{1}{4}t=\ln 9-\frac{3}{4}t$$

$$t_{min}=\ln(9)$$

minimum t

So $P(X\geq \frac{3}{4}n)=\left(\frac{3}{4}e^{-\frac{3}{4}\ln(9)}+\frac{1}{4}e^{\frac{1}{4}\ln(9)}\right)^n$

$$=\left(\frac{3}{4}\cdot 9^{-3/4}+\frac{1}{4}9^{1/4}\right)^n=\frac{\sqrt[4]{9}}{4^n}\left(\frac{3}{9^3}+1\right)^n$$

$$=\sqrt[4]{9}\left(\frac{1.004}{4}\right)^n$$

$\rightarrow 0$ exponential fast

So this is the best one !