# Lecture 13 10/24/17

$$Y = \frac{X_1}{X_1+X_2} \sim ?$$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$
independent of
$X_2 \sim \text{Gamma}(\beta, \lambda)$

Support$[Y] = [0,1]$
because if $X_1 \simeq 0$ $X_2 \simeq \infty \Rightarrow Y \simeq 0$
if $X_1 \simeq \infty$, $X_2 \simeq 0 \Rightarrow Y = 1$
if $X_1 \simeq X_2 \Rightarrow Y \simeq 1/2$

$$Y = \frac{X_1}{X_1+X_2} \sim \int_0^\infty f_{X_1}(y y_2) f_{X_2}(\underbrace{y_2 - y y_2}_{y_2(1-y)}) \underbrace{y_2}_{|y_2|} \, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (y y_2)^{\alpha-1} e^{-\lambda y y_2}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta (y_2(1-y))^{\beta-1} e^{-\lambda y_2(1-y)}}{\Gamma(\beta)} y_2 \, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} \underbrace{e^{-\lambda y y_2 - \lambda y_2(1-y)}}_{e^{-\lambda y_2}} \, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} e^{-\lambda y_2} \, dy_2$$

let $u = \lambda y_2 \Rightarrow \frac{du}{dy_2} = \lambda$
$\Rightarrow y_2 = \frac{1}{\lambda} u \Rightarrow dy_2 = \frac{1}{\lambda} du$

$\int_0^\infty u^{\alpha+\beta-1} e^{-u} du = \Gamma(\alpha+\beta)$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \underbrace{\frac{u^{\alpha+\beta-1}}{\lambda^{\alpha+\beta-1}} e^{-u} \frac{1}{\lambda}}_{\lambda^{\alpha+\beta}} du$$

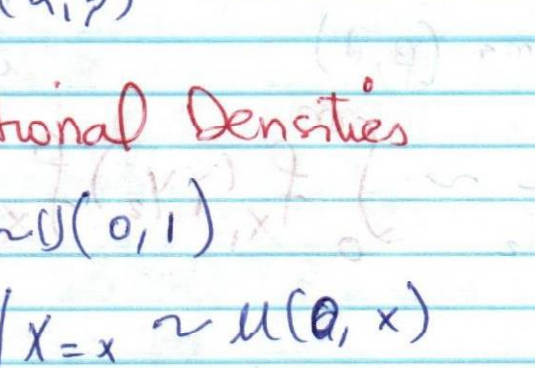
$$= \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\frac{1}{B(\alpha,\beta)}} y^{\alpha-1}(1-y)^{\beta-1} = \text{Beta}(\alpha,\beta)$$

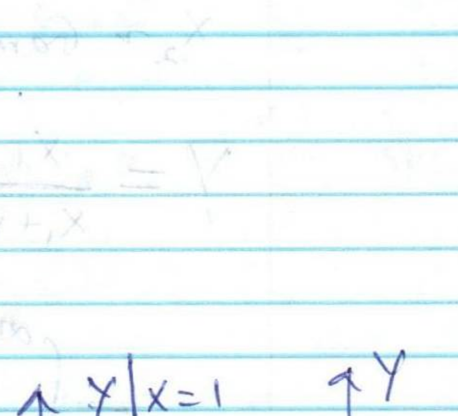
## Conditional Densities

let $X \sim U(0,1)$

let $Y|X=x \sim U(0,x)$

$\text{Supp}[Y]=[0,1]$

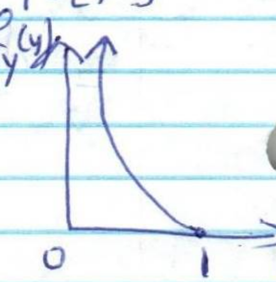



$$f_Y(y) = \int_{\text{Sup}[X]} f_{X,Y}(x,y)\,dx = \int_{\text{Sup}[X]} f_{Y|X}(y,x)\,f_X(x)\,dx$$

$$f_Y(y) = \int_0^1 f_{Y|X}(y,x)\,f_X(x)\,dx = \int_0^1 \frac{1}{X}\underbrace{\mathbb{1}_{Y\in[0,x]}}_{\substack{X\geq Y \\ 0\leq Y\leq X}}\,\mathbb{1}_{X\in[0,1]}\,dx$$

$$= \int_y^1 \frac{1}{x}\,dx = \ln(x)\Big|_y^1 = -\ln y$$



$$\int_0^1 -\ln y\, dy = -\left[y\ln y - y\right]_0^1 = y - y\ln y\Big|_0^1 = 1 \checkmark$$

Example: A download is either exponential with mean 10 min with no network traffic or exponential with mean 20 min with network traffick. How long does download take.
Network traffic is $\frac{2}{3}$ of the time

$$Y \sim \begin{cases} Exp\left(\frac{1}{10}\right) & wp\ 1/3 \\ Exp\left(\frac{1}{20}\right) & wp\ 2/3 \end{cases}$$

$$X = \underset{\text{Network Traffic}}{\mathbb{1}} \sim Ber\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{1-x}$$

$$Y \mid X=x \sim Exp\left(\left(\frac{1}{20}\right)^x\left(\frac{1}{10}\right)^{1-x}\right) = \left(\frac{1}{20}\right)^x\left(\frac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x\left(\frac{1}{10}\right)^{1-x} y}$$

$$f_Y(y) = \sum_{x \in Sup[X]} f_{Y|X}(y.x)\, p_X(x) =$$

$$= \sum_{x \in (0,1)} \left(\frac{1}{20}\right)^x\left(\frac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^x\left(\frac{1}{10}\right)^{1-x} y}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{1-x}$$

$$= \frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}y}\right) + \frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}y}\right) = \frac{1}{3} Exp\left(\frac{1}{10}\right) + \frac{2}{3} Exp\left(\frac{1}{20}\right)$$

$$X \begin{cases} 1 \to Y \sim Exp(1/20) \\ 0 \to Y \sim Exp(1/10) \end{cases}$$

mixture distribution

Example: If the download took 25 min what is the probability there was network traffic?

$$P_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y,x)\, P_X(x)}{f_Y(y)}$$

$$P_{X|Y}(1,25) = \frac{f_{Y|X}(25,1)\, P_X(1)}{f_Y(25)} =$$

$$= \frac{\frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}(25)}\right)}{\frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}(25)}\right) + \frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}(25)}\right)}$$

$$\simeq 98\% \neq 2/3$$

Example

but then rate parameta

Car accidents are poisson distn ~~for Poisson(λ)~~

~~but~~ λ is not the same for all drivers.

λ is gamma distn. $\lambda \sim \text{Gamma}(\alpha, \beta)$.

what is the distribution of accidents?

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

$$Y \,|\, X=x \sim \text{Poisson}(x) = \frac{e^{-x} x^{y}}{y!}$$

Compound Distribution ~~Compound~~

$X \qquad Y$

$\text{Gamma}(\alpha,\beta) \Leftarrow \text{poisson}(x)$

$$P_Y(y) = \int_{sup[x]} P_{Y|X}(y,x) f_X(x)\, dx$$

$supp[Y] = \{0, 1, 2 \dots\}$

$$P_Y(y) = \int_0^\infty \frac{e^{-x} x^y}{y!} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} dx$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} dx$$

let $u = (\beta+1)x \Rightarrow x = \frac{1}{\beta+1} u$

$\frac{du}{dx} = \beta + 1 \Rightarrow dx = \frac{du}{\beta+1}$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty \frac{u^{y+\alpha-1}}{(\beta+1)^{y+\alpha-1}} e^{-u} \frac{1}{\beta+1} du$$

$(\beta+1)^{y+\alpha}$

$$= \frac{\beta^\alpha \Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}} = \left(\frac{\beta}{\beta+1}\right)^\alpha \frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^y}$$

let $k = \alpha$, $p = \frac{\beta}{\beta+1} \Rightarrow 1 - p = \frac{1}{1+\beta}$

$$P_{Y|X}(y,x) = \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y-1)} p^k (1-p)^y = Ext\ NegBin(k, p)$$

if $y, k \in \mathbb{N}$

$$= \binom{y+k-1}{k} \cdot p^k (1-p)^y$$