Math 621 Lec 1 8/29/17

A discrete r.v. X has prob. mass function (PMF) $p(x) := P(X=x)$ notated $X \sim p(x)$ and cumulative distr. function (CDF) $F(x) := P(X \leq x)$.
The r.v. X has support $Supp(X) := \{x : p(x) > 0, x \in \mathbb{R}\}$
Since X is discrete $|Supp(X)| \leq |\mathbb{N}|$ i.e. either finite or c+bly infinite. The support and PMF are related via $\sum_{x \in Supp[X]} p(x) = 1$

The most fundamental discrete r.v. is the Bernoulli

$$X \sim Bern(p) = p^x(1-p)^{1-x}$$

$$p(x) = \begin{cases} 1 & wp\ p \\ 0 & wp\ 1-p \end{cases} \qquad Supp(X) = \{0, 1\}$$

What is p? A parameter that is an element of the parameter space, $p \in (0,1)$.
$p \neq 0$ and $p \neq 1$ because those cases are degenerate!

$X \sim Deg(c) := c$ wp $1 \Rightarrow p(x) = 1$

Strange... you assume support is implied.

What if support was not implied? $\Rightarrow X \sim Deg(c) = \mathbb{1}_{x=c}$

Indicator function $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$.

If $X_1, X_2 \overset{ind}{\sim}$
$\Rightarrow \underbrace{p_{X_1,X_2}(x_1,x_2)}_{\text{joint mass function}} = p_{X_1}(x_1)\, p_{X_2}(x_2)$ $\forall x_1, x_2$

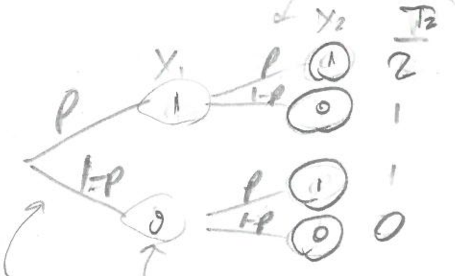
If $X_1 \overset{d}{=} X_2$
$p_{X_1}(x) = p_{X_2}(x)$
$\forall x$

If $X_1, X_2 \overset{iid}{\sim} \Rightarrow X_1, X_2 \overset{ind}{\sim}$ & $X_1 \overset{d}{=} X_2$.

Consider

$T_2 = X_1 + X_2$ s.t. $X_1, X_2 \overset{iid}{\sim} \text{Ber}(p)$

How do we find the PMF? First pass. Draw a tree and note the sums (probs.)



branches after states represent cond. probs

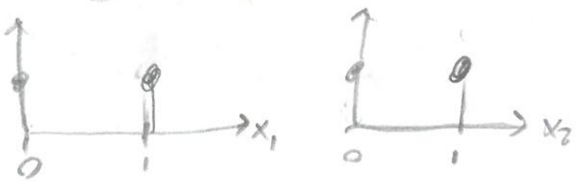
e.g. $P(X_2 = 1 \mid X_1 = 1)$

numbers on branches represent probs

circles represent possible sums

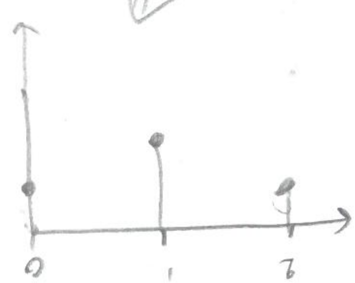
Now tally

$P(T=0) = (1-p)^2$, $P(T=1) = 2p(1-p)$, $P(T=2) = p^2$. This can be thought of as moving one dist over another



if $p = \frac{1}{2}$

$\Downarrow$



this is the tallying that goes on in the tree

$$P(T=t) \overset{?}{=} \sum_{x_1 \in \text{Supp}(X_1)} P(X_1 = x_1)\, P(X_2 = \overbrace{t - x_1}^{x_2}) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right) p^{t-x} (1-p)^{1-t+x}$$

$$= p^t \sum_{x \in \{0,1\}} (1-p)^{2-t} = 2p^t (1-p)^{2-t}$$

if $t = 0 \Rightarrow x \neq 1$!!

$$P(T=0) \overset{?}{=} 2p^0 (1-p)^{2-0} = 2(1-p)^2 \neq (1-p)^2$$

wrong... what happened ???

$\Rightarrow$ we have to selecting sum!

How do we do this? We use the "full pmf"!

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

$$P(T_2 = t) = \sum_{x \in \{0,1\}} P(X_1 = x)\, P(X_2 = t - x)$$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}\, p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$\left(\mathbb{1}_{0 \in \{0,1\}} \mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{1 \in \{0,1\}} \mathbb{1}_{t-1 \in \{0,1\}} \right)$$

$$\| $$

$$\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} = \begin{cases} 1 \text{ if } t=0 \\ 2 \text{ if } t=1 \\ 1 \text{ if } t=2 \end{cases}$$

$$\| \qquad \swarrow$$

$$\binom{2}{t} \quad \Rightarrow \quad = \binom{2}{t} p^t (1-p)^{2-t}$$

What is $\text{supp}(T_2)$? $\text{Supp}[T_2] = \text{Supp}(X_1) + \text{Supp}(X_2) = \{0, 1, 2\}$

$A + B = \{a + b : a \in A, b \in B\}$ $\quad \Rightarrow T_2 \sim \text{Bin}(2, p)$

Another alternative derivation...

$$X_1, X_2 \overset{iid}{\sim} Bern(p) := \binom{1}{x} p^x (1-p)^{1-x}$$

$$P(T=t) = \sum_{x \in \{0,1\}} P(X_1 = x)\, P(X_2 = y-x)$$

$$= \sum_{x \in \{0,1\}} \binom{1}{x} p^y (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{x}\binom{1}{y-x}$$

$$= p^y (1-p)^{2-y} \left(\binom{1}{0}\binom{1}{t} + \binom{1}{1}\binom{1}{t-1} \right)$$

$$= p^y (1-p)^{2-y} \left(\binom{1}{t} + \binom{1}{t-1} \right) = \binom{2}{t} p^t (1-p)^{2-t}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$T_2 = X_1 + X_2 \sim$

$P_{X_1}(x_1) * P_{X_2}(x_2)$

$= Binomial(2, p)$

convolution operator

$$P_{X_1}(x_1) * P_{X_2}(x_2) := \sum_{x \in supp(X_1)} P(x_1)\, P(t - x_2)$$

$T_3 = X_1 + X_2 + X_3 \sim ?$ $\quad X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$

$T_2 = X_1 + X_2 \sim Bin(2, p)$

$$p_{T_3}(t) = p_{X_3}(x_3)\, p_{T_2}(t_2) = \sum_{x \in Supp[X_3]} p_{X_3}(x)\, p_{T_2}(t - x)$$

$$= \sum_{x \in \{0,1\}} \left(\binom{1}{x} p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{y-x} (1-p)^{2-t+x}\right)$$

$$= p\ (1-p)^{3-} \sum_{x \in \{0,1\}} \binom{1}{x}\binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{1}{0}\binom{2}{t} + \binom{1}{1}\binom{2}{t-1}\right) \qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

lec 1

$$= \binom{3}{t} p\ (1-p)^{3-t}$$

lec 2 ↓

$Y = X_1 + X_2$ s.t. $X_1 \sim Bin(n, p)$, $X_2 \sim Bin(n, p)$

$$p(y) = \sum_{x \in Supp[X_1]} \left(\binom{n}{x} p^x (1-p)^{n-x}\right) \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x}$$

$$= p^y (1-p)^{2n-y} \underbrace{\sum_{x=0}^{n} \binom{n}{x}\binom{n}{y-x}}_{\binom{2n}{y} \text{ by Vandermonde's identity}} \Rightarrow Y \sim Bin(2n, p)$$