START OF MIDTERM 2 MATERIAL

Transformation of Discrete Random Variables
Let $X \sim \text{Bern}(p) = p^x(1-p)^{1-x}\mathbb{1}_{x\in[0,1]} = \mathbb{P}_X(x)$. Let

$$Y = 3 + x \sim \begin{cases} 4 & \text{up } p \\ 3 & \text{up } 3-p \end{cases} = p^{y-x}(1-p)^{1-(y-3)}\mathbb{1}_{y\in[3,4]} = \mathbb{P}_Y(y)$$

$\text{Supp}[Y] = \{y : y-3 \in \text{Supp}[x]\}$ The pmf of Y looks like the pmf of X is replaced with $y-3$.
Let $Y = c + aX = g(x)$. Then $x = \frac{y-c}{a} = g^{-1}(y)$.

$$\begin{aligned} \text{Supp}[Y] &= \{y : \frac{y-c}{a} \in \text{Supp}[X]\} \\ &= \{y : \frac{y-c}{a} \in [0,1]\} \\ &= \{c, a+c\} \end{aligned}$$

Let $\mathbb{P}_Y(y) = p^{\frac{y-c}{a}}(1-p)^{1-\frac{y-c}{a}}\mathbb{1}_{y\in\{c,a+c\}} = \mathbb{P}_X(g^{-1}(y))$. This is the modeling support.

Let $X \sim \text{Binom}(n,p)$. Let $Y = a + cX$. Then

$$\begin{aligned} \mathbb{P}_Y(y) &= \binom{n}{g^{-1}(y)} p^{g^{-1}(y)}(1-p)^{n-g^{-1}(y)}\mathbb{1}_{y\in g(\text{Supp}[X])} \\ &= \binom{n}{\frac{y-c}{a}} p^{\frac{y-c}{a}}(1-p)^{n-\frac{y-c}{a}}\mathbb{1}_{y\in\{c,a+c,2a+c,\ldots,na+c\}} \end{aligned}$$

Let $X \sim \text{Binom}(n,p)$ and $Y = X^3$. Then

$$\mathbb{P}_Y(y) = \binom{4}{\sqrt[3]{y}} p^{\sqrt[3]{y}}(1-p)^{1-\sqrt[3]{y}}\mathbb{1}_{y\in\{0,1,2^3,3^3,\ldots,n^3\}}$$

Let $X \sim \text{Geom}(p)$ and $Y = \max\{3, x\}$. This looks like

X	Y
0	3
1	3
2	3
3	3
4	4
5	5
$\vdots$	$\vdots$

There is no $g^{-1}(y)$ function because g is not 1-1. Note that $\mathbb{P}_Y(4) = \mathbb{P}_X(4)$, $\mathbb{P}_Y(5) = \mathbb{P}_X(5)$, but $\mathbb{P}_Y(3) \neq \mathbb{P}_X(3)$. In fact $\mathbb{P}_Y(3) = \mathbb{P}_X(0) + \mathbb{P}_X(1) + \mathbb{P}_X(2) + \mathbb{P}_X(3)$. Thus $\mathbb{P}_Y(y) \neq$

$\mathbb{P}_X(g^{-1}(y))$. From this, conclude that this only works for g functions which are 1-1. In general, the formula for discrete random variable function

$$\mathbb{P}(Y)y = \sum_{\{x:g(x)=y} \mathbb{P}_X(x) = \sum_{\{x:x\in g^{-1}(y)} \mathbb{P}_X(x) = \mathbb{P}_X(g^{-1}(y))$$

In this example,

$$\begin{aligned}\mathbb{P}_Y(y) &= \Big(\mathbb{P}_X(0)+\mathbb{P}_X(1)+\mathbb{P}_X(2)+\mathbb{P}_X(3)\Big)\mathbb{1}_{y=3} + p(1-p)^y\mathbb{1}_{y\in\{4,5,\dots\}} \\ &= \Big(p+(1-p)p+(1-p)^2p+(1-p)^3p)\mathbb{1}_{y=3} + \underbrace{p(1-p)^y}_{\text{Geom}(p)}\mathbb{1}_{y\in\{4,5,\dots\}}\end{aligned}$$

Note that $F_Y(y) = \sum_{x:g(x)\le y}\mathbb{P}_X(x)$.

Let $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ and $Y = -X_2$. Then $\mathbb{P}_Y(y) = \mathbb{P}_X(-y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}\mathbb{1}_{y\in\{0,-1,-2,\dots\}}$. Let $D = X_1 - X_2 = X_1 + Y$. Supp$[D] = \mathbb{Z}$. Then

$$\mathbb{P}_D(d) = \sum_{x\in \text{Supp}[X_1]} \mathbb{P}_{X_1}(x)\mathbb{P}_Y(d-x) = \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}\frac{e^{-\lambda}\overbrace{\lambda^{-(d-x)}}^{\lambda^x\lambda^{-d}}}{\underbrace{(-(d-x))!}_{(x-d)!}}\mathbb{1}_{\underbrace{\underbrace{d-x\in\{0,-1,-2,\dots\}}_{x-d\in\{0,1,2,\dots\}}}_{x\in\{d,d+1,d+2,\dots\}}}$$

If $d>0$, the sum begins at d; if $d\le 0$, the sum begins at 0. Thus $\max\{0,d\}$.

$$\mathbb{P}_D(d) = e^{-2\lambda}\begin{cases}\sum_{x=d}^{\infty}\frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d\le 0 \text{ (upper)}\\ \sum_{x=0}^{\infty}\frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d<0 \text{ (lower)}\end{cases}$$

Let $x' = x-d \to x = x'+d$

$$= \sum_{x'=0}^{\infty}\frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'-d}}}{(x'+d)!x'!} = \sum_{i=0}^{\infty}\frac{\left(\frac{2\lambda}{2}\right)^{2i-d}}{\Gamma(i+d-1)\Gamma(i-1)}$$

This is the modified Bessel function of the 1st kind denoted $I_D(2\lambda)$

Let $d' = -d$

$$= \sum_{x=0}^{\infty}\frac{\lambda^{2x+d'}}{x!(x+d')!} = \underbrace{\sum_{i=0}^{\infty}\frac{\left(\frac{2\lambda}{2}\right)^{2i-d'}}{\Gamma(i+d'-1)\Gamma(i-1)}}_{I_{d'}(2\lambda)}$$

If $d<0 \to d' = |d|$

If $d>0 \to d = |d|$

Thus

$$\mathbb{P}_D(d) = e^{-2\lambda}I_{|d|}(2\lambda) = \text{Skellam}(\lambda,\lambda)$$

This distribution is used to model point spreads in baseball, soccer, hockey, differences in photon noise, etc.

Let $X \sim U(0,1)$ and $Y = aX + c = g(X)$ such that g is 1-1. Can we use the formula $\mathbb{P}_Y(y) = \mathbb{P}_X(g^{-1}(y))$? No because there is no $\mathbb{P}_X(x)$ (pmf). It will not generalize for continuous random variables..
Consider $Y = g(X)$ where g is 1-1. Find $f_Y(y)$ given $f_X(x)$. If it's 1-1, it's either strictly increasing or strictly decreasing.

If g is increasing,

$$F_y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

To find the cdf of Y, just differentiate!

$$f_Y(y) = F'_y(y) = \frac{d}{dy}[F_X(g^{-1}(y))] = F'_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)] = f_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

On the other hand, if g is decreasing,

$$F_Y(y) = \mathbb{P}(g^{-1}(y) \leq y) = \mathbb{P}(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

Then

$$f_Y(y) = F'_Y(y) = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = -\underbrace{f_X(g^{-1}(y))}_{\geq 0}\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\leq 0}$$

In general,

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dt}[g^{-1}(y)]\right|$$

Note: $\text{Supp}[Y] = g(\text{Supp}[X]) = \{g(x) : x \in \text{Supp}[Y]\} = \{y : g^{-1}(y) \in \text{Supp}[X]\}$.

If $Y = aX + c = g(X)$ where $a, c \in \mathbb{R}$ and $a \neq 0$ (the linear tranformation), then

$$y = ax + c \to x = \frac{y-c}{a} = g^{-1}(y) \to \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|a|}$$

Thus

$$f_Y(y) = \frac{1}{|a|}f_X(\frac{y-c}{a})$$

Common Linear Transformations:

$$\text{If } Y = -X \to f_Y(y) = f_X(-y)$$

$$\text{If } Y = X + c \to f_Y(y) = f_X(y - c)$$

Let $X \sim U(0,1)$ and $Y = aX + c$. Then

$$f_Y(y) = \frac{1}{|a|}f_X(\frac{y-c}{a}) = \frac{1}{|a|}(1) = \frac{1}{|a|} \text{ where Supp}[Y] = [c, a+c] \text{ and so } Y \sim U(c, a+c)$$

Let $X \sim \text{Exp}(\lambda)$ and $Y = aX + c$. In fact, $\text{Supp}[Y] = (c, \infty)$.

$$f_Y(y) = \frac{1}{|a|} f_Y(\frac{y-c}{a}) = \frac{1}{|a|}\lambda e^{-\lambda(\frac{y-c}{a})}$$

Letting $c = 0$ and $a > 0$, this becomes

$$f_Y(y) = \frac{\lambda}{a} e^{-\frac{\lambda}{a}y} = \text{Exp}(\frac{\lambda}{a})$$

Let $X \sim U(0,1)$ and $Y = 1 - X$. Then $Y \sim U(0,1) = f_Y(y) = f_Y(y-1) = 1$ where $\text{Supp}[Y] = 1 - [0,1] = [0,1]$.

Let $Y = aX$, then $f_Y(y) = \frac{1}{|a|} f_X(\frac{y}{a})$.

Let $X \sim U(0,1)$ and $Y = -\ln(x)$. Then

$$f_Y(y) = \underbrace{f_X(g^{-1}(y))}_{1} \left| \frac{d}{dy}[g^{-1}(y)] \right| = \frac{d}{dy}[-e^{-y}] = e^{-y} = \text{Exp}(1)$$

Let $X \sim \text{Exp}(1)$ and $Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^x - 1)$. Since $x \in (0, \infty)$, then $e^x \in (1, \infty)$ and so $e^x - 1 \in (0, \infty)$ and therefore $\ln(e^x - 1) \in (-\infty, \infty)$. Thus $\text{Supp}[Y] = \mathbb{R}$. To find $g^{-1}(y)$

$$\begin{aligned}
y &= \ln(e^x - 1) \\
e^y &= e^x - 1 \\
e^x &= e^y + 1 \\
x &= \underbrace{\ln(e^y + 1)}_{g^{-1}(y)}
\end{aligned}$$

Thus

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right| \\
&= f_X(\ln(e^y + 1)) \left| \frac{e^y}{e^y + 1} \right| \\
&= e^{-\ln(e^y+1)} \frac{e^y}{e^y + 1} \\
&= e^{\ln(\frac{1}{e^y+1})} \frac{e^y}{e^y + 1} \\
&= \frac{e^y}{(e^y + 1)^2} \\
&= \text{Logistic}(0, 1)
\end{aligned}$$