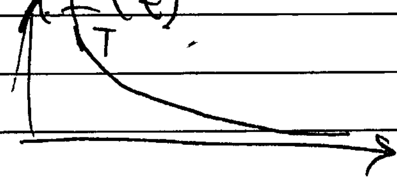
09/19/2017

$T \sim \exp(\lambda) = \lambda e^{-\lambda t}$



$F_T(t) = 1 - e^{-\lambda t}$

Time until event

$N \sim \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$ $\quad \lambda = np$

$n \to \infty \Rightarrow p \to 0$

Relation between exponential & Poisson i.e. T and N.

$$F_N(n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{n} \frac{\lambda^i}{i!}$$

What is the proba the event did not happen by $t=1$

$P(T>1) = e^{-\lambda}$

What is the proba zero event occure?

$P(N=0) = e^{-\lambda}$

$$X \sim \text{Erlang}(k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$$

$$F_X(x) = \frac{\gamma(k, \lambda x)}{(k-1)!}$$

$$= \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$ From next page

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt = \underbrace{\int_0^a t^{x-1} e^{-t}\,dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\,dt}_{\Gamma(x,a)}$$

lower incomplete gamma fctn

upper incomplete gamma fctn

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \left(-t^x e^{-t}\right)\Big]_0^\infty + x\int_0^\infty e^{-t} t^{x-1}\,dt$$

$$= x\,\Gamma(x)$$

$\Gamma(1) = 1$

$\Gamma(2) = 1\,\Gamma(1) = 1$

$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1$

$\Gamma(4) = 3\,\Gamma(3) = 3\cdot 2\cdot 1$

So $\Gamma(n) = (n-1)!$

So $1 - F_X(x) = 1 - \dfrac{\Gamma(k) - \gamma(k, \lambda x)}{\Gamma(k)}$

$$1 - \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k, \lambda x)}{\Gamma(k)}$$

$$\Gamma(k,\lambda x) = \int_{\lambda x}^{\infty} \underbrace{t^{k-1}}_{u}\, \underbrace{e^{-t}\,dt}_{dv}$$

$$= -t^{k-1}e^{-t}\Big]_{\lambda x}^{\infty} - \int_{\lambda x}^{\infty} (k-1)t^{k-2}\cdot(-e^{-t})\,dt$$

$$(\lambda x)^{k-1}e^{-\lambda x} + (k-1)\underbrace{\int_{\lambda x}^{\infty} t^{k-2}e^{-t}\,dt}_{\Gamma(k-1,\lambda x)}$$

$$\Gamma(k-1,\lambda x) = (\lambda x)^{k-2}e^{-\lambda x} + (k-2)\Gamma(k-2,\lambda x)$$

$$\Gamma(1,\lambda x) = \int_{\lambda x} t^{1-1}e^{-t}\,dt = e^{-\lambda x}$$

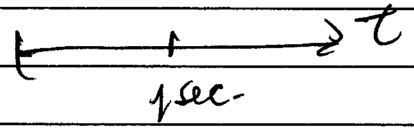
$$\Gamma(k,\lambda x) = e^{-\lambda x}\Big((\lambda x)^{k-1} + (k-1)(\lambda x)^{k-2} + (k-2)(k-1)\cdot(\lambda x)^{k-3} + \dots + (k-1)!\Big)$$

$$= e^{-\lambda x}(k-1)\left(\frac{(\lambda x)^{k-1}}{(k-1)!} + \frac{(\lambda x)^{k-2}}{(k-2)!} + \frac{(\lambda x)^{k-3}}{(k-3)!} + \dots + 1\right)$$

$$= e^{-\lambda x}\underbrace{(k-1)!}_{\Gamma(k)} \sum_{i=0}^{k-1} \frac{(\lambda x)^{i}}{i!}$$

$$1 - F(x) = \frac{\Gamma(k, \lambda x)}{\Gamma(k)} = \frac{e^{-\lambda x} \cancel{\Gamma(k)} \sum_{i=0}^{k-1} (\lambda x)^i}{\cancel{\Gamma(k)}}$$

$$= e^{-\lambda x} \sum^{k-1} (\lambda x)^i$$



N # of event by $t = 1$

Proba of no success or 1 success?

$$P(N \leq 1) = F_N(1) = e^{-\lambda}(1 + \lambda)$$

$$T \sim Erlang(2, \lambda)$$

$$P(T > 1) = e^{-\lambda}(1 + \lambda)$$

Proba (less than or equal to k event by $t = 1$)

N # even by 1 sec.

$$P(N = k) = F_N(k) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

$$T \sim Erlang(k+1, \lambda)$$

$$P(T > 1) = e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$$

$$e^{-\lambda}\sum_{i=0}^{k}\frac{\lambda^{i}}{i!} = \frac{\Gamma(k+1,\lambda)}{\Gamma(k)} = Q(k+1,\lambda)$$

$$\sum_{i=0}^{k}\frac{\lambda^{i}}{i!} = e^{\lambda}\, Q(k+1,\lambda)$$

So $\lim_{k\to\infty} Q(k,\lambda) = 1$

Relatn between Bin and Neg. Bin

Running experiments	fixed time count. events	waiting to # events
Discretely	Binomial Bern	Neg. Binomial Geo
Continuously	Poisson	Erlang

Poisson process

$$F_N(k) = 1 - F_{T_{k+1}}(1)$$

What the Prob there have been 2 or less success by the 50th experiment?

$N \sim Bin(50, p) \quad P(N \leq 2)$

$F_N(2) = \binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} + \binom{50}{2} p^2 (1-p)^{48}$

$T \sim NegBin(3, p) = P(T > 47)$

$= P(T=48) + P(T=49) + P(T=50) + P(T=51) + \cdots$

$= 1 - P(T \leq 47)$

$= 1 - F(47)$

Proba, there have been K or less success by experiment n?

$N \sim Bin(n, p) \Rightarrow P(N \leq k) = F_N(k)$

$= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$

$T \sim NegBin(k+1, p) \Rightarrow P(T > n-(k+1))$

$= 1 - P(T \leq n-(k+1)) = 1 - F_T(n-k-1)$

$$= 1 - \sum_{i=0}^{n-k-1} \binom{i+k}{k} p^{k+1} (1-p)^i$$

$$X_1 \; X_2 \sim \text{Poisson}(\lambda)$$

$$X_1 + X_2 \sim \text{Poisson}(2\lambda) \quad \frac{(2\lambda)^x e^{-2\lambda}}{x!}$$

When $P(X_1 \mid X_1 + X_2)$?

$x_2 = n - x_1$

$$P(X = x \mid X_1 + X_2 = n) = \frac{P(X_1 = x \text{ AND } X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$

$$\frac{e^{-2\lambda}(2\lambda)^n}{n!}$$

$$= \frac{P_{X_1, X_2}(x, n-x)}{\frac{e^{-2\lambda}(2\lambda)^n}{n!}} = \binom{n}{x}\left(\frac{1}{2}\right)^n = \text{Bin}\left(n, \frac{1}{2}\right)$$

2 Random variable happen once $\equiv$ JMF

$Y = X_1 - X_2$ Find $P_Y(y)$

$Y = X_1 + (-X_2) = X_1 + Z$