09/12/2017

$\vec{X}$ is a vector of r.v. such that dim$(\vec{X})$ $= k$

$$\vec{\mu} = E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix}$$

$$Var(\vec{X}) = \begin{bmatrix} Var(X) & & Cov(X_i X_j) \\ & \ddots & \\ Cov(X_i X_j) & & Var() \end{bmatrix}$$

$$T = X_1 + \cdots + X_n = \vec{1}^T \vec{X}$$

$$T_k \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

$$E(T) = \sum_{i=1}^{k} \mu_i = \vec{1}^T \vec{\mu}$$

$$Var[T] = Var(\vec{1}^T \vec{X}) =$$

$$= Cov[X_1 + \cdots + X_k, X_1 + \cdots + X_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} Cov(X_i, X_j)$$

let $Y = c\vec{X}$ where $c \in \mathbb{R}^k$

$E(Y) = \vec{c}\vec{\mu}$

Var

If $A \in \mathbb{R}^{n \times m}$ and $\vec{c} \in \mathbb{R}^m$

$\vec{c}^T A \vec{c}$ Quadratic Form

$$= \vec{c}^T \begin{bmatrix} c_1 a_{11} + \cdots c_m a_{1m} \\ c_1 a_{21} \\ \cdots \\ c_1 a_{n1} + \cdots + c_m a_{nm} \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i c_j a_{ij}$$

$$\text{Var}[c_1 X_1 + \cdots + c_k X_k]$$

$$= \text{Cov}[c_1 X_1 + \cdots + c_k X_k, c_1 X_1 + \cdots c_k X_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \text{Cov}[c_i X_i, c_j X_j]$$

$$= \vec{c}^T \text{Var}(Y) \vec{c}$$

$Cov(x_1, x_2) = \oplus$ if both ↑ or both ↓ but $\ominus$ otherwise

Markowitz Optimal Portfolio

Let $X_1, \ldots X_k$ be r.v. made for the returns on K assets. Let $W_1, \ldots W_k$ be the weights (allocation) for each. Note $\vec{1}^T \vec{W} = 1$

$E(V) = \vec{w}\vec{\mu} = \mu_0 \qquad Var(V) = \vec{w}^T \Sigma \vec{w}$

Given a μ_0, min $\vec{w}^T \Sigma \vec{w}$

If $X \sim$ Multinomial (n, p)

$$E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$$Var(\vec{X}) = \begin{bmatrix} np_1(1-p_1) & Cov(X_1, X_2) & & \\ & np_2(1-p_2) & & \\ & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - \mu_i \mu_j$

$= \sum_{x_i \in \text{support of } X_i} \sum x_i x_j P_{X_i X_j}(x_i, x_j)$

Recall

$X_1 \sim Bin(n, p_1)$ $\quad X_1 = \sum_{i=1}^{n} X_{i1}$ — such that $X_{11}, \ldots, X_{n1} \overset{iid}{\sim} Bern(p_1)$

$X_k \sim Bin(n, p_k)$ $\quad X_k = \sum_{i=1}^{n} X_{ik}$ — such that $X_{1k}, \ldots, X_{nk} \overset{iid}{\sim} Be$

$\vec{X} = \sum_{i=1}^{n} \vec{X}_i$ such that $\vec{X}_1, \ldots, \vec{X}_n \overset{iid}{\sim} \mathrm{Multinom}(1,$

$\mathrm{Cov}[X_i, X_j]$

$= \mathrm{Cov}\left[\sum X_{li}, \sum X_{hj}\right]$

$= \sum \sum \mathrm{Cov}[X_{li}, X_{hj}]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_{li}, X_{hj}] - P_i P_j$

If $l = h \Rightarrow \sum_{x \in \{0,1\}} \sum_{x \in \{0,1\}} x y P(x,y) = 0$

X \	0	1
0	+	+
1	+	0

XY

Y		
	0	0
	0	1

If $l \neq h$ $\quad E(X_{li}, X_{hj}) = E[X_{li}]\, E[X_{hj}]$

$= P_i P_j$

Continuous r.v. X has C.D.F $F(x)$ and PDF: $f(x) = F(x)$

$\mathrm{Supp}(X) = \{x_i, f(x) > 0\}$

$|\mathrm{supp.}| = |\mathbb{R}|$

Note $P(X) = 0 \ \forall x$

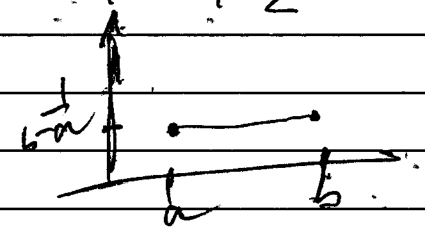
PMF

$f(x) \neq p(x)$

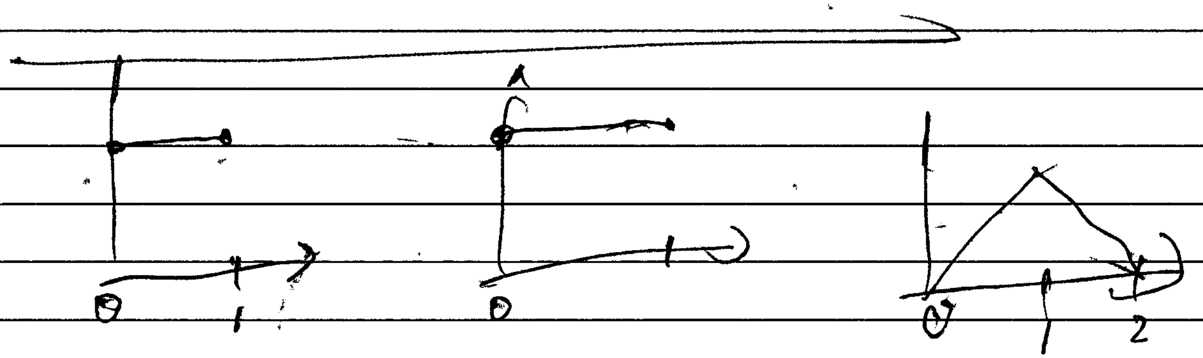
$X \sim U(a,b) = \frac{1}{b-a}$ where $a, b \in \mathbb{R}$
$b > a$

$Supp(x) = [a, b]$

$T_2 = X_1 + X_2$ such that $X_1, X_2 \overset{iid}{\sim} U(0,1)$



$a = 0, b = 1 \Rightarrow$
$X \sim U(0,1) = 1$



$Supp(T) = [0, 2]$

$T = 0$? $X_1 = 0, X_2 = 0$

$T = 2$ $X_1 = 1, X_2 = 1$

$T = 1,$ $X_1 = 0, X_2 = 1$ or $X_1 = \frac{1}{3}$ $X_2 = \frac{2}{3}$

$\Rightarrow$ continuous

$$f_T(t) = \int_0^\infty f(x) f(t-x)\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{t-x \in [0,1]}\, dx$$

$$= \int_0^1 \mathbb{1}_{x \in [t-1, t]}\, dx = \int_{\max\{0,\, t-1\}}^{\min(1,\, t)} (1)\, dx$$

$$= \int \min($$