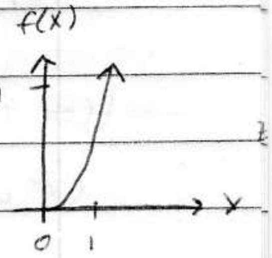
10/19

- Order Statistics:

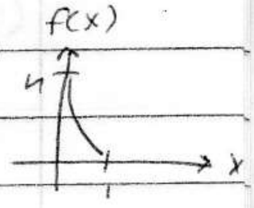
$X_1 \dots X_n \overset{iid}{\sim} f(x)$

$X_1 \dots X_n \overset{iid}{\sim} U(0,1) \rightarrow f(x) = 1,\ F(x) = x$



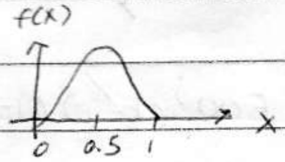
- $f_{X_{(n)}}(x) = n \underbrace{f(x)}_{1} \underbrace{F(x)}_{x}^{n-1} = n x^{n-1}$

- $Supp(X_{(n)}) = Supp(X)$



- $f_{X_{(1)}} = n f(x) (1 - F(x))^{n-1} = n(1-x)^{n-1}$

- $n = 10$
  $k = 5$



- $f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \underbrace{f(x)}_{1} (\underbrace{F(x)}_{x})^{k-1} (1 - \underbrace{F(x)}_{x})^{n-k}$

kernel trick $\propto x^{k-1}(1-x)^{n-k}$

$f(x) = \frac{1}{c} k(x)$

$\int_{supp[X]} k(x)\,dx = c$

Beta f-n

$\int_0^1 x^{n-k}(1-x)^{n-k}\,dx = \beta(k, n-k+1)$

Note: $\beta(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$

β

$\frac{1}{\beta(k, n-k+1)} x^{k-1}(1-x)^{n-k} = Beta(k, n-k+1)$

r.v X: $Beta(\alpha, \beta) = \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

$Supp[X] = (0,1) \quad \alpha > 0, \beta > 0$

$\cdot \int f(x) = 1 \qquad \int_0^1 \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx.$

$$= \frac{1}{\cancel{\beta(\alpha,\beta)}} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}_{\cancel{\beta(\alpha,\beta)}} = 1$$

$$\cdot\ F(x) = \int_0^x \frac{1}{\beta(\alpha,\beta)} \underbrace{t^{\alpha-1}(1-t)^{\beta-1}}_{\text{independence f-n}}\,dt \qquad \text{beta f-n}$$

$$= \int_0^x \frac{t^{\alpha-1}(1-t)^{\beta-1}\,dt}{\beta(\alpha,\beta)} = \frac{\beta(x,\alpha\beta)}{\beta(\alpha,\beta)} = I_x(\alpha,\beta)$$

$$\cdot\ F(x) = \frac{1}{\beta(\alpha,\beta)}\int_0^1 x^{\alpha+1-1}(1-x)^{\beta-1}\,dx = \frac{\beta(\alpha+1,\beta)}{\beta(\alpha,\beta)}$$

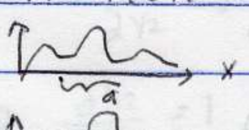
$$\alpha\cancel{\Gamma(\alpha)}$$

$$\frac{= \dfrac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)} = (\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}{\dfrac{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}{\cancel{\Gamma(\alpha+\beta)}}} \Rightarrow \frac{\alpha}{\alpha+\beta}.$$

• Transformations:

$X \sim f(x)$

What if we know $k \in A$ where $A \subseteq \text{supp}[X]$ call this cond dist. Y, $f_Y(Y) = ?$ $\qquad$ f(x) over some constant #



• $Z = \mathbb{1}_{X\in A} \sim \text{Bern}(p(X\in A))$

$$f_{X|Z}(x,z) = \frac{f_{X,Z}(x,z)}{p_Z(z)} = \frac{f(x)\,\mathbb{1}^{z}_{x\in A}\,\mathbb{1}^{1-z}_{x\notin A}}{p(X\in A)^{z}\,(1-p(X\in A))^{1-z}}$$

$$f_Y(x) = f_{X|Z}(x,1) = \frac{f(x)}{p(X\in A)}\,\mathbb{1}_{x\in A}$$

☆ •

$X \geq a$	$X \leq a$	$X \in (a,b)$
$f_Y(x) = \frac{f(x)}{1-F(a)}\,\mathbb{1}_{Y\geq a}$	$f_Y(x) = \frac{f(x)}{F(a)}\ x<a$	$f_Y(x) = \frac{f(x)}{F(b)-F(a)}\,\mathbb{1}_{x\in(a,b)}$

• $X \sim \text{Exp}(\lambda)$

We know $x \geq a$

$$f_Y(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a}}\,\mathbb{1}_{x\geq a}$$

• let $g = \mathbb{R}^a \to \mathbb{R}^a$ where g is 1-1.

let $\vec{x}$ be a vertical r.v. with dis n

$\vec{y}$ be a "

if $f_{\vec{X}}(\vec{x}) = f_{X_1 \ldots X_n}(x_1 \ldots x_n)$, &

$\vec{Y} = g(\vec{X})$, find $f_{\vec{Y}}(\vec{y}) = f_{Y_1, \ldots, Y_n}(y_1, \ldots y_n)$

$Y_1 = g_1(X_1 \ldots X_n)$

$Y_2 = g_2(X_1 \ldots X_n)$

$\vdots$

Since g is 1-1, $\exists\, h(1) \cdots h(n)$

$$X_1 = h_1(Y_1 \ldots Y_n)$$
$$X_2 = h_2(Y_1 \ldots Y_n)$$
$$\vdots$$
$$X_n = h_n(Y_1 \ldots Y_n)$$

Formula:
$$f_{Y_1 \ldots Y_K}(Y_1 \ldots Y_K) = f_{X_1 \ldots X_K}(h_1(Y_1 \ldots Y_n) \ldots h_n(Y_1 \ldots Y_n)) \left| J_n(Y_1 \ldots Y_n) \right|$$

$$f_Y(y) = f_X(g^{-1}(x)) \left| \frac{d}{dy}[g^{-1}(x)] \right|$$

$$J_n : \left| \det \left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & & \frac{\partial h_n}{\partial y_n} \end{bmatrix} \right) \right|$$

$$J_n = \det\left(\left[\frac{\partial g^{-1}(y)}{\partial y} \right] \right) = \frac{\partial g^{-1}(y)}{\partial y}$$

- Given X_1, X_2

$$Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2) \qquad X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$Y_2 = Y_1 \quad g_2(X_1, X_2) \qquad X_2 = Y_2 = h_2(Y_1, Y_2)$$

$$J_n = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} \right)$$
$$= y_2 \cdot 1 - y_1 \cdot 0 = y_2$$

$$\frac{\partial h_1}{\partial y_1} = y_2 \qquad \frac{\partial h_1}{\partial y_2} = y_1$$

$$\frac{\partial h_2}{\partial y_1} = 0 \qquad \frac{\partial h_2}{\partial y_2} = 1$$

know where the formulas came from

· General formula:

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1 X_2}(y_1 y_2, y_2)\,|y_2|$$

$$\Rightarrow f_{Y_1}(y_2) = \int_{\substack{supp(Y_2)\\ \shortparallel \\ supp(X_2)}} f_{X_1 X_2}(y_1 y_2, y_2)\,|y_2|\,dy_2$$

$$\star\ f_Y(y) = \int_{supp(X_2)} x_2\, f_{X_1}(y_1, x_2)\, f_{X_2}(x_2)\,dx_2$$

if X_1 & X_2 are integrable & positive.

multi variable

$*$ Given X_1, X_2

$$Y_1 = \frac{X_1}{X_1 + X_2} = g(X_1, X_2) \qquad X_1 = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$Y_2 = X_1 + X_2 = g_2(X_1, X_2) \qquad X_2 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$$

$$\frac{\partial h_1}{\partial y_1} = y_2 \quad \frac{\partial h_1}{\partial y_2} = y_1$$

$$J_h = \det\left(\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix}\right)$$

$$\frac{\partial h_2}{\partial y_1} = -y_2 \quad \frac{\partial h_2}{\partial y_2} = 1 - y_1$$

$$= y_2(1-y_1) - (y_1)(-y_2)$$

$$= y_2 - y_1 y_2 + y_1 y_2 = y_2$$

• If X_1 X_2 are integrable & positive,

$$f_Y(y) = \int_{supp(y_2)} y_2 \; f_{X_1}(y_1 \, y_2) \; f_{X_2}(y_2(1-y)) \, y_2$$

• $X_1 \sim$ Gamma (α, λ) ind. of $X_2 \sim$ Gamma (β, λ)

$$Y_1 = \frac{X_1}{X_1 + X_2} \sim ?$$

proportion

$Supp[Y_1] = (0,1)$ $\qquad\qquad$ $Supp[Y_2] = (0, \infty)$

$$\int_0^\infty$$