lecture #20 11/28/17

Standard MVN

$\vec{z} \sim N_n(\vec{0}_n, I_n)$

$E[\vec{z}] = \vec{0}_n$

$Var[\vec{z}] = I_n$

If $\vec{x} = \vec{z} + \vec{\mu} \sim N(\vec{\mu}, I_n)$

If $\vec{x} = A\vec{z}$, $E[\vec{x}] = A E[\vec{z}] = A\vec{0}_n = \vec{0}_m$

$\Sigma = Var[\vec{x}] = AA^T \in \mathbb{R}^{m \times n}$

Is Σ symmetric?

$\Sigma \overset{?}{=} \Sigma^T$

$\Sigma^T = (AA^T)^T = A^{T^T} \cdot A^T = AA^T$ ✓

We want $f_{\vec{x}}(\vec{x})$?
(jdf)

$\vec{x} = g(\vec{z}) = A\vec{z}$. Assume $A \in \mathbb{R}^{n \times n}$ for now.

we can say $\vec{z} = h(x)$ where h is the inverse function.

$\vec{z} = h(x) = A^{-1}\vec{x}$ g is only 1:1 if A is full rank.

We need $m = n$ i.e A is square and A is full rank

Let's do multivariate change of variable like before.

$$A^{-1}\vec{x} = \begin{bmatrix} h_1(\vec{x}) \\ h_2(\vec{x}) \\ \vdots \\ h_n(\vec{x}) \end{bmatrix}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\ |J_n(\vec{x})| \quad (*)$$

generalization of univariate

Let $B = A^{-1}$

canatical notation.

$$B = A^{-1} = \begin{bmatrix} \leftarrow b_{1\cdot} \rightarrow \\ \leftarrow b_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow b_{n\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b_{\cdot 1} & b_{\cdot 2} & \cdots & b_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{n \ldots} \end{bmatrix}$$

$$\vec{Z} = \vec{B}\vec{X} = \begin{bmatrix} h_1(\vec{x}) = \vec{b}_{1\cdot}\vec{x} = b_{11}x_1 + \cdots + b_{1n}x_n \\ h_2(\vec{x}) = \vec{b}_{2\cdot}\vec{x} = b_{21}x_1 + \cdots + b_{2n}x_n \\ \vdots \\ h_n(\vec{x}) = \vec{b}_{n\cdot}\vec{x} = b_{n1}x_1 + \cdots + b_{nn}x_n \end{bmatrix}$$

$$\vec{B}\vec{X} = h(\vec{x})$$

$$J_n(\vec{x}) = \det\left(\begin{bmatrix} \frac{\partial}{\partial x_1}[h_1(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_1(\vec{x})] \\ \vdots & & \\ \frac{\partial}{\partial x_1}[h_n(\vec{x})] & \cdots & \frac{\partial}{\partial x_n}[h_n(\vec{x})] \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right) = \det(B) = \det(A^{-1})$$

$$\frac{\partial}{\partial \vec{x}}[c\vec{x}] =$$

$$(*)\ f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(\overset{h(\vec{x})}{B\vec{x}})\,|J_n(\vec{x})| = f_{\vec{Z}}(A^{-1}\vec{x})\,|\det(A^{-1})|$$

Side no…

Recall $\det(A^{-1}) = \frac{1}{\det(A)}$

Proof: Recall $AA^{-1} = I$

$\det(AA^{-1}) = \det(I)$

$= 1$ (product of the diagonal)

Recall $\det(AB) = \det(A).\det(B)$

So; $\det(A).\det(A^{-1}) = 1$.

$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$

Now

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}\vec{x})^T(A^{-1}\vec{x})} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T(A^{-1})^T \cdot A^{-1} \cdot \vec{x}} \cdot \frac{1}{|\det(A)|}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{x}^T \cdot \underbrace{(A^T)^{-1} \cdot A^{-1}} \cdot \vec{x}} \cdot \frac{1}{\det(A)}$$

$$= \frac{1}{(2\pi)^{n/2}} \cdot e^{-\frac{1}{2}\vec{x}^T \mathcal{E}^{-1}\vec{x}} \cdot \frac{1}{\det(A)}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\mathcal{E})}} e^{-\frac{1}{2}\vec{x}^T \mathcal{E}^{-1}\vec{x}}$$

$(A^T)^{-1} \overset{?}{=} (A^{-1})^T$

$AA^{-1} = I$

$(AA^{-1})^T = I^T = I$

$(A^{-1})^T \cdot A^T = I$

$A^T(A^T)^{-1} = I$ equal

$(A^T)^{-1}A^T = I$

So $(A^T)^{-1} = (A^{-1})^T$.

$$f_{\vec{X}}(\vec{x}) = N_n(\vec{O}_n, \mathcal{E})$$

centered at zero $\quad = AA^T$

$\circledast\ \mathcal{E} = AA^T$

$\mathcal{E}^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$

$(AB)^{-1}(AB) = I$

It has to be

$$\underbrace{B^{-1}\underbrace{A^{-1} \cdot A}_{I}B}_{I} = I$$

So $(AB)^{-1} = B^{-1}A^{-1}$

$\circledast\ \vec{X} = A\vec{Z} + M \sim N_n(\vec{M}, \mathcal{E})$ general MVN

$\cdot\ Y = \vec{X} - M = A\vec{Z}$

$$N_n(\vec{M}, \mathcal{E}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathcal{E})}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \mathcal{E}^{-1}(\vec{x}-\vec{\mu})}$$

$\det(\mathcal{E}) = \det(A).\det(A^T)$

$= \det(A)^2$

$\Rightarrow \det A = \sqrt{\det \mathcal{E}}$

$\det A = \det(A^T)$

Characteristic function:

Recall: $\phi_X(t) := E\left[e^{itX}\right]$ generalizes to $\phi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}\right.$

Rules: Uni dimensional rules are valid for multidim.

$$\phi_{\vec{X}_1+\vec{X}_2}(\vec{t}) = E\left[e^{i\vec{t}^T(\vec{X}_1+\vec{X}_2)}\right] = E\left[e^{i\vec{t}^T\vec{X}_1} \cdot e^{i\vec{t}^T\vec{X}_2}\right]$$

$$= E\left[e^{i\vec{t}^T\vec{X}_1}\right] E\left[e^{i\vec{t}^T\vec{X}_2}\right] = \phi_{\vec{X}_1}(\vec{t}) \cdot \phi_{\vec{X}_2}(\vec{t})$$

characteristic function

<u>Recall</u>: $\phi_X(t) := E[e^{itX}]$ generalizes to

$$\phi_{\vec{X}}(\vec{t}) = E[e^{i\vec{t}^T\vec{X}}]$$

<u>Rules</u>: uni dimensional rules are valid for ~~...~~ multi dime

$$\phi_{\vec{X}_1+\vec{X}_2}(\vec{t}) = E\left[e^{i\vec{t}^T(\vec{X}_1+\vec{X}_2)}\right] = E\left[e^{i\vec{t}^T\vec{X}_1} \cdot e^{i\vec{t}^T\vec{X}_2}\right]$$

$$= E[e^{i\vec{t}^T\vec{X}_1}]\, E[e^{i\vec{t}^TX_2}] = \phi_{\vec{X}_1}(\vec{t}) \cdot \phi_{\vec{X}_2}(\vec{t})$$

Let $\vec{Y} = A\vec{X} + \vec{c}$

$A \in \mathbb{R}^{m\times n}$, $\vec{c} \in \mathbb{R}^m$

$\vec{X} \in \mathbb{R}^n$, $\vec{Y} \in \mathbb{R}^m$

$$\phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T\vec{Y}}] = E[e^{i\vec{t}^T(A\vec{x}+\vec{c})}]$$

$$= E[e^{i\vec{t}^TA\vec{X}} \cdot e^{i\vec{t}^T\vec{c}}] = e^{i\vec{t}^T\vec{c}}\, E[e^{i\vec{t}^TA\vec{x}}]$$

Let $t'^T = t^TA$, $t' = (t^TA)^T = A^T\vec{t}$

$$= e^{i\vec{t}^T\vec{c}}\, E[e^{i\vec{t}'^T\vec{x}}] = e^{i\vec{t}\vec{c}}\, \phi_{\vec{X}}(\vec{t}')$$

$$= e^{i\vec{t}^T\vec{c}}\, \phi_{\vec{X}}(A^T\vec{t})$$

Need to get characteristic function of Z.

$$\vec{z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{Z}}(\vec{t}) = E[e^{i\vec{t}^T\vec{Z}}]$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i\vec{t}^T\vec{z}} f_{\vec{Z}}(\vec{z})\, d\vec{z}.$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{i(t_1z_1+\cdots+t_nz_n)} \cdot \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(z_1^2+\cdots z_n^2)}\, dz_1 \cdots$$

$\Sigma = AA^T$

$\Sigma^{-1} = (AA^T)^{-1}$

$= (A^T)^{-1} A^{-1}$

$(AB)^{-1}(AB) = I$

It has to be $\underbrace{B^{-1}\underbrace{A^{-1}A}_{I}B}_{I} = I$

So $(AB)^{-1} = B^{-1}A^{-1}$

So $\Rightarrow \quad = \frac{1}{(2\pi)^{n/2}} e^{-1/2 \vec{x}^T \Sigma^{-1} \vec{x}} \frac{1}{\det(A)}$

$\det(\Sigma) = \det(A)\det(A^T) = \det(A)^2$

$\Rightarrow \det(A) = \sqrt{\det(\Sigma)}$

$\det(A) = \det(A^T)$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}}$$

$$= N_n(\vec{0}_n, \Sigma)$$

centered at zero

$\Sigma = AA^T$

$$\vec{X} = A\vec{z} + \mu \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

general MVN

- $Y = \vec{x} - \mu = A\vec{z}$