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$$\vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1} \qquad E(\vec{X}) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} = \vec{\mu}$$

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m} \qquad E(X) = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

$$\Sigma = Var[\vec{X}] = E[(\vec{X} - \vec{\mu})(\vec{X} - \mu)^T]$$

$$\delta^2 = Var(X) = E[(X-\mu)(X-\mu)]$$

Scalar.

$$= E[(X-\mu)^2]$$

$$= E\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots \\ (X_2-\mu_2)(X_1-\mu_1) & (X_2-\mu_2)^2 & \cdots \\ \vdots & & \vdots \\ & & (X_n-\mu_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \delta_1^2 & \delta_{1,2} & \cdots \\ \delta_{2,1} & \delta_2^2 & \cdots \\ \vdots & \cdots & \ddots \\ & & \delta_n^2 \end{bmatrix} = \Sigma \qquad (a+b)^T = a^T + b^T$$

Σ

Symmetric

$$= \Sigma = E[(X-\mu)(X-\mu)^T] = E[(X-\mu)(X^T-\mu^T)]$$

$$= E(XX^T - \underbrace{\mu X^T}_{\substack{n\times 1 \, 1\times n \\ n\times n}} - \underbrace{X\mu^T}_{n\times 1} + \mu\mu^T)$$

$$= E(XX^T) + E(-\mu X^T) + E(-X\mu^T) + E(\mu\mu^T)$$

$X \in \mathbb{R}^{n\times m}$ $\quad A \in \mathbb{R}^{p\times n}$

$$AX \in \mathbb{R}^{p\times m}, \quad E(AX) = E\left[\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{p1} & \cdots & a_{pn} \end{bmatrix}\begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}\right]$$

$$= E\begin{bmatrix} a_{11}X_{11} + a_{12}X_{21} + \cdots + a_{1n}X_{n1} & \cdots \\ \vdots & \ddots \end{bmatrix} = \begin{bmatrix} a_{1\cdot}\mu_{\cdot 1} & a_{1\cdot}\mu_{\cdot 2} & \cdots \\ a_{2\cdot}\mu_{\cdot 1} & a_{2\cdot}\mu_{\cdot 2} \\ \vdots & \vdots \end{bmatrix}$$

$$A = \begin{bmatrix} \leftarrow a_{1\cdot} \rightarrow \\ \leftarrow a_{2\cdot} \rightarrow \\ \vdots \\ \leftarrow a_{p\cdot} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_{\cdot 1} & a_{\cdot 2} & \cdots & a_{\cdot n} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$E(a_{1\cdot}X_{\cdot 1}) = E(a_{11}X_{11} + \cdots + a_{1n}X_{n1}) = a_{11}\mu_{11} + \cdots + a_{1n}\mu_{n1} = a_{1\cdot}\mu_{\cdot 1}$$

$$= \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ \vdots \\ a_{n\cdot} \end{bmatrix}\begin{bmatrix} \mu_{\cdot 1} & \mu_{\cdot 2} & \cdots & \mu_{\cdot n} \end{bmatrix} = AE(X)$$

$$XB, \quad E(XB) = E(X)B.$$

$$X + C = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix} + \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \\ c_{n1} & \cdots & c_{nm} \end{bmatrix} =$$

$$E(X + C) = E(X) + C$$

$$= E(XX^T - \mu X^T - X\mu^T + \mu\mu^T)$$

$$= E(XX^T) + E(-\mu X^T) + E(-X\mu^T) + E(\mu\mu^T)$$

$$\boxed{\Sigma = E(XX^T) - \mu\mu^T} \cancel{- \mu\mu^T + \mu\mu^T}$$

$$\sigma^2 = E(X^2) - \mu^2$$

Consider

$$Var(A^TX) = Cov(\underset{n\times p}{A^T}\underset{p\times m}{X}) = E[(A^TX)(A^TX)^T] - E(A^TX)E(A^TX)^T$$

$$= E(A^TXX^TA) \qquad A^TE(X)(A^TE(X))^T$$

$$= A^TE(XX^T)A \qquad A^T\mu\mu^TA$$

$$= A^T(E(XX^T) - \mu\mu^T)A = A^T\Sigma A$$

determining matrix

$$Var(\vec{a}^T\vec{X}) = \underset{1\times n}{\vec{a}^T}\,\underset{n\times n}{Var(X)}\,\underset{n\times 1}{\vec{a}} \quad \text{one quadratic form.}$$

$$Var[AX] = A\Sigma A^T$$

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$ $\quad E(\vec{Z}) = \vec{\mu}$ $\quad Var[\vec{Z}] = \Sigma$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim N_n(\vec{0}, I_n) = f_{Z_1 \dots Z_n}(z_1 \dots z_n)$$

$\mu = 0$

multivariate normal (MVN)

of dim n.

$$E(\vec{Z}) = n\vec{\mu} = n \cdot \vec{0} = 0$$

$$Var(\vec{Z}) = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & & \ddots & 1 \end{bmatrix} = I_n$$

$\vec{\mu} \in \mathbb{R}^n$, $\quad \Sigma \in \dots$ $\quad Supp(\vec{Z}) = \mathbb{R}^n$

$$f_{Z_1, \dots Z_n}(z_1, \dots, z_n) = f_{Z_1}(z_1) \cdots f_{Z_n}(z_n) = \prod_{i=1}^{n} f_Z(z_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T z}$$

multivariate normal density of dim n.

$f_{\vec{Z}}(\vec{z})$ $\quad \vec{X} = \vec{Z} + c$, $\quad E(\vec{X}) = \vec{0} + \vec{c}$,

$$Var(\vec{X}) = I_n \Rightarrow \vec{X} \sim N(\vec{c}, I_n)$$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-c_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum(x_i-c_i)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(\vec{x}-\vec{c})^T(\vec{x}-\vec{c})}$$

$$\vec{X} = A\vec{Z} \qquad \vec{Z} \in \mathbb{R}^{n\times 1},\ A \in \mathbb{R}^{m\times n}$$

$$E(\vec{X}) = A E(\vec{Z}) = A\vec{0}_n = 0_m$$

$$\Sigma = \mathrm{Var}(\vec{X}) = A\,\mathrm{Var}(\vec{Z})\,A^T = AIA^T = AA^T$$

$$\underset{\in\mathbb{R}^{n\times 1}}{\vec{X}} = A\vec{Z} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} a_{11}z_1 + a_{12}z_2 + \cdots a_{1n}z_n \\ a_{21}z_1 + a_{22}z_2 + \cdots a_{2n}z_n \\ \vdots \\ \vdots \end{bmatrix} \begin{matrix} \sim N(0, \Sigma a_{1i}^2) \\ \sim N(0, \Sigma a_{2i}^2) \\ \cdots \\ \cdots \end{matrix}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_{nm} \end{bmatrix}$$

$$\mathrm{Cov}[X_i, X_j] = \Sigma_{ji}$$

$$\mathrm{Cov}(X_1, X_2) = \Sigma_{12} = \delta_{12}$$

Assume $A \in \mathbb{R}^{n \times n}$ Also assume A is full rank.

$\vec{X} = A\vec{z} = g(\vec{z})$, $\vec{z} = h(\vec{X}) = A^{-1}\vec{X}$