01-26/17

$Q \sim \text{Poisson}(\lambda)$

X Gamma Y Poisson(x)

$E(Q) = \lambda$

$SE(Q) = \lambda$

$Q \sim \text{NegBin}(k, p)$

$E(Q) = kp$

$SE(Q) \neq E(Q)$

$X \sim \text{Beta}(\alpha, \beta)$, $Y/X \sim \text{Bin}(n, X)$

n fixed.

X Beta(α,β) Y 0, 1, …, n Bin(n, x)

$$p_Y(y) = \int_{\text{supp}\{X\}} P_{Y|X}(y,x) f_X(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y}$$

$$= \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\,dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1}\,dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \text{Beta Binomial}(\alpha, \beta, n)$$

overdispersed binomial.

n fixed

$X \sim \text{Gamma}(\alpha, \beta)$, $Y/X \sim \text{Exp}(x)$

X Gamma(α,β) Y Exp(x)

$$f_Y(y) = \int_{\text{supp}\{x\}} f_{Y|X}(y,x) f_X(x)\,dx$$

$$= \int_0^\infty x \, e^{-xy} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

let $u = (\beta+y)x$

$\Rightarrow x = \frac{1}{\beta+y} u$

$\Rightarrow \frac{du}{dx} = \beta+y$

$\Rightarrow dx = \frac{1}{\beta+y} du$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1-1} e^{-(\beta+y)x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{u^{\alpha+1-1}}{(\beta+y)^\alpha} e^{-u} \frac{1}{\beta+y} du$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)(\beta+y)^{\alpha+1}} \int_0^\infty u^{\alpha+1-1} e^{-u} du = \frac{\cancel{\Gamma(\alpha+1)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\beta^{\alpha+1}}{\beta} \cdot \frac{1}{(\beta+y)^{\alpha+1}}$$

$$= \frac{\alpha}{\beta}\left(1+\frac{y}{\beta}\right)^{-(\alpha+1)} = \text{lomax}(\beta, \alpha)$$

Survival distn

$\text{supp}(y) = [0, \infty)$

$a, b \in \mathbb{R}$

$z = a + bi \in \mathbb{C}$

$i = \sqrt{-1} \Rightarrow i^2 = -1, \ i^3 = -i, \ i^4 = 1$

$Re(z) = a$

$Im(z) = b$

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{itx} = \sum_{k=0}^\infty \frac{(itx)^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

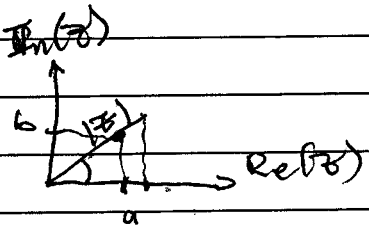
$$i\sin(tx) = itx - \frac{i\,t^3x^3}{3!} + \frac{i\,t^5x^5}{5!} - + \cdots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + \cdots$$

$$e^{itx} = \cos(tx) + i\sin(tx) \quad \text{if } \pi = tx$$

$$e^{i\pi} = -1 \implies e^{i\pi} + 1 = 0$$

Eulers Identity.



$$|z| = \sqrt{a^2+b^2} \in [0, \infty)$$

$$\operatorname{Arg}(z) = \theta = \arctan\left(\frac{b}{a}\right) \in (-\pi, \pi)$$

$$z = |z|e^{i\theta} = \sqrt{a^2+b^2}\Big(\underbrace{\cos\left(\arctan\left(\tfrac{b}{a}\right)\right)}_{\frac{a}{|z|}} + i\underbrace{\sin\left(\arctan\left(\tfrac{b}{a}\right)\right)}_{\frac{b}{|z|}}\Big)$$

$$= \cancel{|z|}\left(\frac{a}{\cancel{|z|}} + \frac{ib}{\cancel{|z|}}\right) = a + ib.$$

Define $L^1 = \left\{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \right\}$

absolute integral

All PDF's $\in L^1$ mean will be integral to 1

If $f \in L^1$ then $\mathcal{F} \hat{f}$ defined as

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i t x} f(x)\,dx$$

Know as the fourier transform of f
Not $\hat{f}$ doesn't necessarily $\in L^1$
$f(x)$ is called the "time domain."
$\hat{f}(t)$ is " " "frequency domain."

$f(x)$ can be as a sum of sines and cosines.

$Re\{\hat{f}(t)\} \to$ Amplitude of
$Arg\{\hat{f}(t)\} \to$ phase shift of

let $\phi(t) = \hat{f}\left(\frac{-t}{2\pi}\right) = \int_{\mathbb{R}} e^{itx} f(x)\,dx = E(e^{itx})$

If $f(x)$ is an PDF

Note: If $\hat{f} \in L^1$ then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i t x} \hat{f}(t)\,dt$$

Inverse fourier of transformation.

If $\phi(t) \in L^1$

let $u = -2\pi t$ $\quad \frac{du}{dt} = -2\pi$

$t = \frac{-u}{2\pi}$ $\quad dt = \frac{-1}{2\pi} du$

$t = \infty \Rightarrow u = -\infty$
$t = -\infty \Rightarrow u = \infty$

$$f(x) = \int_{\infty}^{-\infty} e^{2\pi i \left(\frac{-u}{2\pi}\right) x} \hat{f}\left(\frac{-u}{2\pi}\right) \left(\frac{-1}{2\pi}\right) du$$

PDF

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \hat{f}\left(\frac{-u}{2\pi}\right) du = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \underbrace{\phi(u)}_{\text{Ch. f.}} du$$

$\phi_X(t)$ ~~this is~~ the characteristic function of r.v. X. "Ch. f."

$$E(e^{itX}) = \begin{cases} \sum_{x \in \operatorname{supp}[x]} e^{itx} p(x) & \text{if } x \text{ discrete} \\ \int_{\operatorname{supp}(x)} e^{itx} f(x)\, dx & \text{if } x \text{ is contin} \end{cases}$$

① $\phi(0) = 1$

② If X_1, X_2 indep., $Y = X_1 + X_2$

$$\phi_Y(t) = \phi_{X_1+X_2}(t) = E\left[e^{it(X_1+X_2)}\right] = E\left[e^{itX_1} e^{itX_2}\right]$$

$$= E(e^{itX_1}) E(e^{itX_2}) = \phi_{X_1}(t) \phi_{X_2}(t)$$

③ If $Y = aX + b$, $a, b \in \mathbb{R}$.

$$\phi_Y(t) = E(e^{itY}) = E\left(e^{it(a\cancel{+}bX)}\right) = E\left(e^{itaX} e^{itb}\right)$$

$$= e^{itb} E(e^{itaX}) = e^{itb} \phi_X(at)$$

ME TO WE

(4) $\phi_X(t)$ always exists since

$$|\phi_X(t)| \leq 1 \quad \forall t.$$

$$|\phi_X(t)| = |E(e^{itX})| = \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| \leq \int_{\mathbb{R}} |e^{itx} f(x)|\,dx$$

$$= \int_{\mathbb{R}} |e^{itx}|\,|f(x)|\,dx = \int_{\mathbb{R}} |f(x)|\,dx = 1$$

$$|\cos(tx) + i\sin(tx)| = \sqrt{\cos^2(tx) + \sin^2(tx)} = \sqrt{1} = 1$$

$$M_X(t) = E(e^{tX})$$