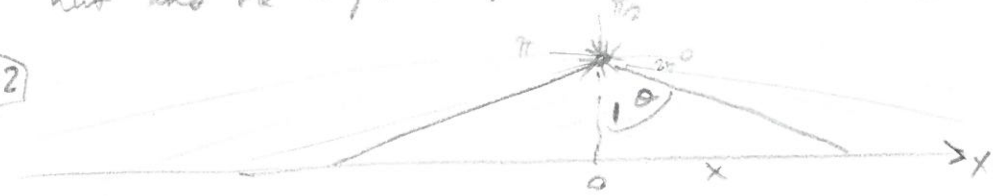
Lec 18 Math 621 11/9/17 (physical)

1

It is also known as the Lorentz distr. why? Imagine you have a source of light at $y=1$ above the origin and it shines light equally in all directions. What does the light density look like on the x-axis?

p122



obviously light extends to all $x \in \mathbb{R}$. If it shines equally ... for each θ.

So light shines $\theta \sim U(\pi, 2\pi) = \frac{1}{\pi}$

$\tan(\theta) = \frac{x}{1}$ $\quad x = \tan(\theta) = g(\theta) \quad \theta = \arctan(x) = g^{-1}(x) \quad \frac{d}{dx}\left[g^{-1}(x)\right] = \frac{1}{1+x^2}$

$$f_X(x) = f_\theta(g^{-1}(x)) \frac{d}{dx}\left[g^{-1}(x)\right] = \frac{1}{\pi} \frac{1}{1+x^2}$$

Proof of Cauchy using ratio $\qquad X_1 \sim N(0,1)$ ind. of $X_2 \sim N(0,1)$

$$R = \frac{X_1}{X_2} \sim \int_{Supp[X_2]} |x_2| f_{X_1}(x_2 r) f_{X_2}(x_2)\, dx_2$$

$$= \int_{\mathbb{R}} |x_2| \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2 r^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}} dx_2$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} |x_2| e^{-\frac{1}{2} x_2^2 (r^2+1)} dx_2 = \frac{1}{2\pi}\left(\int_{-\infty}^{0} -x_2\, e^{-\frac{1}{2}x_2^2(r^2+1)} dx + \int_0^\infty x_2\, e^{-\frac{1}{2}x_2^2(r^2+1)} dx\right)$$

$$= \frac{1}{2\pi} \int_0^\infty \text{---}\, dx - \int_{-\infty}^{0} \text{---}\, dx$$

let $u = -\frac{1}{2} x_2^2 (r^2+1)$

$\frac{du}{dx_2} = -x_2(r^2+1) \Rightarrow dx = -\frac{1}{x_2(r^2+1)}$

$$\int \cancel{x_2} e^u \, \frac{-1}{\cancel{x_2}(r^2+1)} du = -\frac{1}{r^2+1} \int e^u = -\frac{1}{r^2+1} e^u$$

$x_2 = 0 \Rightarrow u = 0$
$x_2 = \infty \Rightarrow u = -\infty$
$x_2 = -\infty \Rightarrow u = -\infty$

$$= \frac{1}{2\pi}\left(-\frac{1}{r^2+1}\right)\left(\left[e^u\right]_0^{-\infty} - \left[e^u\right]_{-\infty}^{0}\right) = -\frac{1}{2\pi}\frac{1}{r^2+1}(-2) = \frac{1}{\pi}\frac{1}{r^2+1}$$

$(0-1) - (1-0)$

MIDTERM 2

FINAL

Some Stat 633 Inference background.. $X_1, \dots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$ some distr.

$\bar{X}$ is the avg. random variable. It is often used as the "estimator" for μ.

It has nice properties e.g. $E[\bar{X}] = \mu$ (unbiased; on avg. it is spot on)

$\bar{x}$ is a realization from $\bar{X}$. $\bar{x}$ is an estimate of μ. This is why you use the sample avg to estimate the mean.

How to estimate σ^2? More rare, but definitely common.

$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$ the sample variance estimate

$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$, , , , , , , estimator. s^2 is a realization from S^2.

$E[S^2] = \sigma^2$ also unbiased

Assume $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ we know $X_1 + \dots + X_n \sim N(n\mu, n\sigma^2)$ we proved this with ch.f.'s

$$\bar{X} = \frac{X_1 + \dots + X_n}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$S^2 = \frac{1}{n-1}\left((X_1 - \bar{X})^2 + \dots + (X_n - \bar{X})^2\right) \sim ?$$ this is our project now..

$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$

Recall if $Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$ $\quad \vec{Z}^T \vec{Z} = \sum_{i=1}^n Z_i^2 \sim \chi^2_n$

$$\sum Z_i^2 = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$$

standardize

Note: $\sum (X_i - \mu)^2 = \sum (X_i - \bar{X} + \bar{X} - \mu)^2 = \sum \left((X_i - \bar{X}) + (\bar{X} - \mu)\right)^2 = \sum (X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2$

$= \sum (X_i - \bar{X})^2 + 2 \sum X_i \bar{X} - \bar{X}^2 - \mu X_i + \bar{X}\mu + n(\bar{X} - \mu)^2 = \sum (X_i - \bar{X})^2 + \underbrace{2\left(\cancel{n\bar{X}^2} - \cancel{n\bar{X}^2} - \cancel{\mu n \bar{X}} + \cancel{n\bar{X}\mu}\right)}_{0} + n(\bar{X} - \mu)^2$

$$\Rightarrow \underbrace{\frac{\sum (X_i - \bar{X})^2}{\sigma^2}}_{\frac{(n-1)S^2}{\sigma^2}} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_n \qquad \frac{n(\bar{X} - \mu)^2}{\sigma^2} = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 = Z^2 \sim \chi^2_1$$

$N(0,1)$

If $Y_1 \sim \chi^2_{k_1}$ ind. of $Y_2 \sim \chi^2_{k_2}$ $\Rightarrow Y_1 + Y_2 \sim \chi^2_{k_1+k_2}$

Wouldn't it be nice if $\dfrac{\sum(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$?

Then... $\chi^2_{n-1} + \chi^2_1 = \chi^2_n \ni$ WTS ① ~ χ^2_{n-1} ② Needs to be independent of $\bar{X}$

Turns out this is true! But it took until the 1930's to prove it.

Cochran's Thm. (1934)

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, Let $Q_1, \ldots, Q_K$ be scalar r.v.'s created via a quadratic form $Q_j = \vec{Z}^T B_j \vec{Z}$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

and $B_1, \ldots, B_K$ are positive semidefinite

$\sum Z_i^2 \sim \chi^2_n$

$$\vec{Z}^T \vec{Z} = \underbrace{\vec{Z}^T B_1 \vec{Z}}_{Q_1} + \underbrace{\vec{Z}^T B_2 \vec{Z}}_{Q_2} + \ldots + \underbrace{\vec{Z}^T B_K \vec{Z}}_{Q_K}$$

(a) $n = \sum_{j=1}^{K} \text{rank}(B_j)$ $\Leftrightarrow$ (b) All Q_j's are independent $\Leftrightarrow$ (c) $Q_j \sim \chi^2_{\text{rank}(B_j)}$

Proof: long, requires lots of linear algebra or adv. lin. alg + ch.f's! SKIP FOR NOW

Let's use the thm. $\sum Z_i^2 = \sum(Z_i - \bar{Z} + \bar{Z})^2 = \sum(Z_i - \bar{Z})^2 + 2\sum(Z_i - \bar{Z})\bar{Z} + \bar{Z}^2$

$$= \sum(Z_i - \bar{Z})^2 + 2\Big(\sum \underbrace{Z_i}_{n\bar{Z}} \bar{Z} - \sum \bar{Z}^2\Big) + n\bar{Z}^2$$

$$= \underbrace{\sum(Z_i - \bar{Z})^2}_{Q_1} + 2\big(\cancel{n\bar{Z}^2 - n\bar{Z}^2}\big) + \underbrace{n\bar{Z}^2}_{Q_2}$$

Even to use the thm., we gotta brush up on lin. alg!

Now... even to use Cochran's thm. we need linear algebra

$Q_2 = n\bar{Z}^2 = \vec{Z}^T \frac{1}{n} J_n \vec{Z}$ where $J_n = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ the matrix of all 1's

Proof

$$\frac{1}{n} J_n \vec{z} = \frac{1}{n}\begin{bmatrix} \sum z_i \\ \sum z_i \\ \vdots \\ \sum z_i \end{bmatrix} = \frac{1}{n}\begin{bmatrix} n\bar{z} \\ \vdots \\ n\bar{z} \end{bmatrix} = \bar{z}\vec{1}, \text{ then } \vec{z}^T\left(\frac{1}{n}J_n\vec{z}\right) = \vec{z}^T\bar{z}\vec{1} = \bar{z}\,\vec{z}^T\vec{1} = \bar{z}\sum z_i$$

$$= \bar{z}(n\bar{z}) = n\bar{z}^2$$

What does the first term look like?

$$Q_1 = \sum(z_i - \bar{z})^2 = \sum z_i^2 - 2z_i\bar{z} + \bar{z}^2 = \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2 = \sum z_i^2 - n\bar{z}^2$$

$$= \vec{z}^T\vec{z} \quad \frac{1}{n}\vec{z}^T J_n \vec{z}$$

$$= \vec{z}^T I_n \vec{z} - \vec{z}^T \frac{1}{n} J_n \vec{z}$$

$$= \vec{z}^T\left(I_n - \frac{1}{n}J_n\right)\vec{z}$$

$$\Rightarrow \sum z_i^2 = \overbrace{\vec{z}^T\left(I_n - \frac{1}{n}J_n\right)\vec{z}}^{Q_1} + \overbrace{\vec{z}^T\left(\frac{1}{n}J_n\right)\vec{z}}^{Q_2}$$

$$B_1 = I_n - \frac{1}{n}J_n, \quad B_2 = \frac{1}{n}J_n \qquad \text{Note: } B_1B_2 = \left(I_n - \frac{1}{n}J_n\right)\frac{1}{n}J_n = \frac{1}{n}J_n - \underbrace{\frac{1}{n}J_nJ_n}_{-\frac{1}{n}J_n} = 0$$

so this should

If Matrix A is both symmetric & idempotent $\Rightarrow tr(A) = rank(A)$

$$\frac{1}{n}J_n = \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$ clearly symmetric, idempotent? $AA = A$

$$\frac{1}{n}J_n\frac{1}{n}J_n = \frac{1}{n^2}J_nJ_n = \frac{1}{n^2}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \frac{1}{n^2}\begin{bmatrix} n & \cdots & n \\ \vdots & & \vdots \\ n & \cdots & n \end{bmatrix} = \frac{1}{n^2}nJ_n = \frac{1}{n}J_n \checkmark$$

$$rank\left(\frac{1}{n}J_n\right) = tr\left(\frac{1}{n}J_n\right) = \sum_{i=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$$

$$I_n - \frac{1}{n} J_n = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & & \ddots & \vdots \\ -\frac{1}{n} & \cdots & & 1-\frac{1}{n} \end{bmatrix}$$ clearly symmetric

idempotent?

$$\left(I_n - \tfrac{1}{n}J_n\right)\left(I_n - \tfrac{1}{n}J_n\right) = \underbrace{I_n I_n}_{I_n} - \underbrace{\tfrac{1}{n} J_n I_n}_{-\frac{1}{n}J_n} - \underbrace{\tfrac{1}{n} J_n I_n}_{\cancel{-\frac{1}{n}J_n}} + \underbrace{\tfrac{1}{n^2} J_n J_n}_{\cancel{\frac{1}{n}J_n}}$$ (we just did this)

$$= I_n - \tfrac{1}{n} J_n \checkmark$$

$$\Rightarrow \operatorname{rank}\left(I_n - \tfrac{1}{n}J_n\right) = \operatorname{tr}\left(I_n - \tfrac{1}{n}J_n\right) = \sum_{i=1}^{n} 1 - \frac{1}{n} = n\left(1-\frac{1}{n}\right) = n-1$$

we still need to prove B_1, B_2 are pos. semi def. Def. matrix A is pos. semi def. if

$\forall \vec{v} \neq \vec{0} \quad \vec{v}^T A \vec{v} \geq 0.$

Since $\quad \vec{z}^T B_2 \vec{z} = n\bar{z}^2 \geq 0 \checkmark$

$\quad\quad\quad \vec{z}^T B_1 \vec{z} = \sum (z_i - \bar{z})^2 \geq 0 \checkmark$

we can now apply Cochran's Thm!

(a) $\sum (z_i - \bar{z})^2 \sim \chi^2_{n-1}$ & $n\bar{z}^2 \sim \chi^2_1$ (b) $\sum(z_i - \bar{z})^2$ is indep. of $n\bar{z}^2$

$\Rightarrow X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\sum z_i^2 = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2} = \left(\frac{\vec{X}-\vec{\mu}}{\sigma}\right)^T \left(\frac{1}{n} J_n\right)\left(\frac{\vec{X}-\vec{\mu}}{\sigma}\right)$$

$\sim \chi^2_n$

$$= \left(\frac{\vec{X}-\vec{\mu}}{\sigma}\right)^T \left(I_n - \tfrac{1}{n}J_n\right)\left(\frac{\vec{X}-\vec{\mu}}{\sigma}\right)$$ HW!!

Using Cochran's Thm

(a) $\frac{\sum(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$ & $\frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2_1$ (b) $\frac{\sum(X_i-\bar{X})^2}{\sigma^2}$ and $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$ are independent!

Since $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum(X_i-\bar{X})^2}{\sigma^2} \Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$

$\Rightarrow \frac{\sqrt{n-1}}{\sigma} S \sim \chi_{n-1}$ (the chi distr) ALSO $\frac{(n-1)S^2}{\sigma^2}$ indep of $n\left(\frac{\bar{X}-\mu}{\sigma^2}\right)^2$

Since $n-1, n, \mu, \sigma^2$ are constants $\Rightarrow S^2, \bar{X}$ are indep!

First proved by Fisher, 1925

Geary, 1936 proved that the normal distr. is the only distr where $S^2, \bar{X}$ are independent.

Now... $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ Here's why all this is important...

this is why we can use the

Consider $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ "Z-test"

Consider $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim$ close to $N(0,1)$ since $S \approx \sigma$

Student, 1908... that was his intention

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)}{\sqrt{\frac{\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}}{n-1}}} \sim T_{n-1}$$

ind. of

both by Cochran's Thm

And this is why the t-test works!!