11/2/17 Standard normal

$$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Leftrightarrow \phi_X(t) = e^{-\frac{t^2}{2}}$$

PDF

$X_1, \dots X_n \overset{iid}{\sim} ?$

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} N(0,1)$$

CLT

$E(X) = 0$

$SE(X) = 1$

$E(Y) = \mu + \sigma E(X) = \mu$

$SE(Y) = \sigma SE(X) = \sigma$

$Y = \mu + \sigma X, \ \sigma \in (0, \infty)$

$$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

$= N(\mu, \sigma^2)$ "Normal"

$X_1, \dots X_n \overset{iid}{\sim} ?$ μ (mean), σ^2 (variance)

$$T_n = X_1 + \dots + X_n \overset{d}{\approx} N(n\mu, n\sigma^2)$$

$$\bar{X}_n = \frac{X_1 + \dots X_n}{n} \overset{d}{\approx} N\left(\mu, \frac{\sigma^2}{n}\right)$$

"if n is "large enough""

$$\phi_Y(t) = e^{it\mu}\phi_X(\sigma t) = e^{it\mu}e^{-\frac{\delta^2 t^2}{2}} = e^{it\mu - \frac{\delta^2 t^2}{2}}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ and of $X_2 \sim N(\mu_2, \delta_1^2)$.

$$T = X_1 + X_2 \sim \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx$$

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta_1^2}} e^{-\frac{1}{2\delta_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\delta_2^2}} e^{-\frac{1}{2\delta_2^2}(t-x-\mu_2)^2} dx = \ldots \text{ bad}$$

No indicator function

$$\phi_Y(t) = \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 - \frac{\delta_1^2 t^2}{2}} e^{it\mu_2 - \frac{\delta_2^2 t^2}{2}}$$

$$= e^{it\overbrace{(\mu_1+\mu_2)}^{\mu'} - \frac{\overbrace{(\delta_1^2+\delta_2^2)}^{\delta^{2\prime}} t^2}{2}} \Rightarrow Y \sim N(\mu_1+\mu_2, \delta_1^2+\delta_2^2)$$

$$X \sim N(\mu, \delta^2),\ \underset{\text{supp}\{y\}=(0,\infty)}{Y} = e^X = g(x) \quad x = g^{-1}(y) = \ln y$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{y}$$

$$Y \sim f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$= \frac{1}{\sqrt{2\pi\delta^2}} \cdot \frac{1}{y} e^{-\frac{1}{2\delta^2}(\ln(y)-\mu)^2} = \text{LogN}(\mu, \delta^2)$$

"log normal"

you begin with Y_0 \$ s. You must return of for t time periods. R_t is the rate of return (varies)

$$Y_1 = Y_0(1+R_1)$$

$$Y_2 = Y_1(1+R_2) = Y_0(1+R_1)(1+R_2)$$

$$\vdots$$

$$Y_t = Y_0 \prod_{i=1}^{t}(1+R_i) = Y_0 e^{\ln\left(\prod_{i=1}^{t}(1+R_i)\right)}$$

$$= Y_0 e^{\sum_{i=1}^{t}\ln(1+R_i)} = Y_0 e^{\sum_{i=1}^{t} x_i}$$

$$\sum_{i=1}^{t} x_i \overset{d}{\approx} N(t\mu_x, t\delta_x^2)$$

Note: $e^{\sum x_i} \approx \log N(t\mu_x, t\delta_x^2)$

$X \sim \log N(\mu, \delta^2)$, $Y = aX$ $\quad a \in (0, \infty)$

$$Y \sim f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right) = \frac{1}{a}\frac{1}{\sqrt{2\pi\delta^2}} \cdot \frac{1}{\frac{y}{a}} e^{-\frac{1}{2\delta^2}\left(\ln\left(\frac{y}{a}\right)-\mu\right)^2}$$

$$= \log N(\mu + \ln(a), \delta^2)$$

$X_i = \ln(1+R_i)$ $\qquad R = 0.03 = 3\%$

μ_R, δ_R^2 $\qquad X = \ln(1+0.03) = .0296$

$R = -0.05 = -5\%$

$X = \ln(1-0.05) = -0.051$

$\Rightarrow X \approx R$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \approx x$

if x is small

$\Rightarrow Y_t = Y_0 e^{\sum x_i} \approx \text{LogN}(t\mu_R + \ln(Y_0), t\sigma_R^2)$

So:

$R \overset{iid}{\sim} N(10\%, 10\%^2)$

start with \$1000

In 5 years what is the prob you have more than \$1650?

$P(Y_t > 1650) = 1 - F_{Y_t}(1650)$

with $Y_t \sim \text{LogN}\left(.5 + \overset{(\ln(1000))}{6.91}, \frac{5 \cdot 10^2}{.05}\right)$

$P(Y_t > 1650) = 1 - \text{plnorm}(1650, 7.41, \sqrt{.05})$

If a quantity experiences normal percentage / proportional changes, then the result quantity is LogN.

$Z \sim N(0,1)$, $Y = Z^2$, $supp(Y) = (0, \infty)$
$\uparrow$ not 1-1

$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$

$= 2P(Z \in [0, \sqrt{y}])$

$= 2(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{1/2}) = 2F_Z(\sqrt{y}) - 1$

$f_Y(y) = F_Y'(y) = \cancel{2} f_Z(\sqrt{y}) \cdot \frac{y^{-1/2}}{\cancel{2}} = f_Z(\sqrt{y}) \cdot y^{-1/2}$

$= \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2} = \chi_1^2$

Chi-Squared with degrees of freedom = 1 Parameter

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt$

let $u = \sqrt{t}$, $\frac{du}{dt} = \frac{1}{2} \cdot \frac{1}{\sqrt{t}} \Rightarrow dt = 2\sqrt{t}\, du = 2u\, du$

$\Rightarrow t = u^2$

$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty \frac{1}{\cancel{u}} e^{-u^2} 2\cancel{u}\, du = 2\underbrace{\int_0^\infty e^{-u^2} du}_{\frac{\sqrt{\pi}}{2}} = \sqrt{\pi}$

$= \frac{(\frac{1}{2})^{1/2} y^{-1/2} e^{-\frac{y}{2}}}{\Gamma(\frac{1}{2})} = Gamma(\frac{1}{2}, \frac{1}{2}) = \chi_1^2$

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$$Z_1^2 + Z_2^2 \sim \text{conv of } \underset{\underset{\exp(\frac{1}{2})}{\parallel}}{\text{Gamma}\left(1, \tfrac{1}{2}\right)} = \chi_2^2.$$

convolution χ_1^2 with χ_1^2

$= \text{con of Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) \text{ with Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$$\sum_{i=1}^{k} Z_i^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi_k^2$$

$$= \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{k}{2}\right)}$$

$X \sim \chi_k^2, \quad Y = \sqrt{X}, \quad g^{-1}(y) = y^2 \Rightarrow \left|\frac{d}{dy}[g^{-1}(y)]\right| = 2y$

$\text{supp}[Y] = (0, \infty)$

$$f_Y(y) = f_X(y^2)\, 2y = \frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} \cdot (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}} \cdot 2y$$

$$= \frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} \left((y^2)^{1/2}\right)^{k-2} e^{-\frac{y^2}{2}}$$

$$= \frac{1}{2^{\frac{k}{2}-1} \Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}} \sim \chi_k$$

Chi-distribution