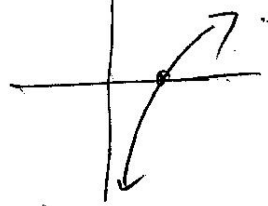
Lecture #9.

Let $X \sim U(0,1)$ $\quad Y = -\ln\left(\frac{1}{x} - 1\right) = g(x)$

$X \in (0,1) \quad \frac{1}{X} \in (1, \infty) \quad \frac{1}{X} - 1 \in (0, \infty) \quad \ln\left(\frac{1}{X}-1\right) \in \mathbb{R}$

$-\ln\left(\frac{1}{X}-1\right) \in \mathbb{R}$



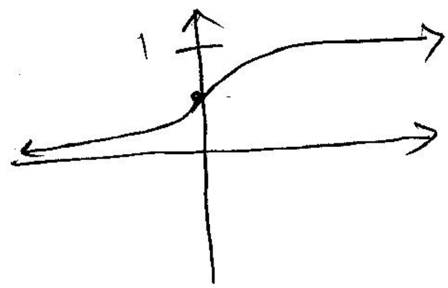
So $\text{Supp}[Y] = \mathbb{R}$

$y = -\ln\left(\frac{1}{x} - 1\right) \Rightarrow -y = \ln\left(\frac{1}{x} - 1\right)$

$\Rightarrow e^{-y} = \frac{1}{X} - 1 \Rightarrow 1 + e^{-y} = \frac{1}{x} \Rightarrow x = \frac{1}{1+e^{-y}} = g^{-1}(y)$

$f(x) = \frac{L}{1+e^{-k(x-x_0)}}$ Logistic function

L : max
k : steepness
x_0 : mid point



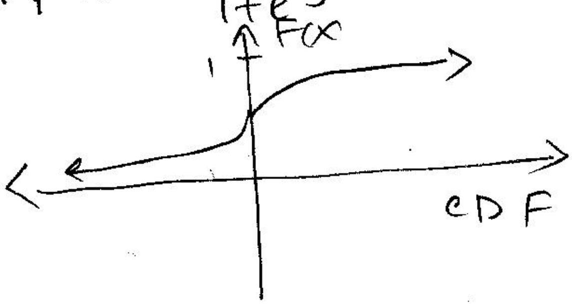
$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$ standard Logistic

$L = 1 \quad k = 1 \quad x_0 = 0$

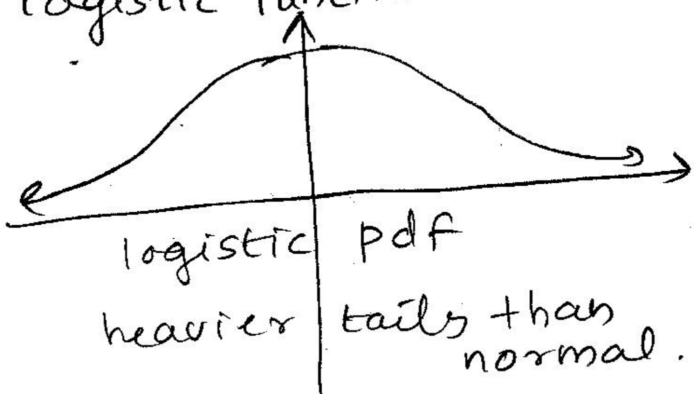
$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$

$= f_X\left(\frac{1}{1+e^{-y}}\right) \cdot \frac{e^{-y}}{(1+e^{-y})^2} = \frac{e^{-y}}{(1+e^{-y})^2}$ = Logistic (0,1) dist. (standard Logistic)

$F_Y(y) = \frac{1}{1+e^{-y}}$ standard logistic function.



CDF



logistic pdf

heavier tails than normal.

*Actually used remarkably Elo system chess ratings.

Let $X \sim Exp(\lambda)$ $\quad Y = k e^{X}$ s.t. $k \in (0, \infty)$

if $k = 1$ Supp$[Y]$?

Supp$[X] = (0, \infty)$

Supp$[Y] = (1, \infty)$

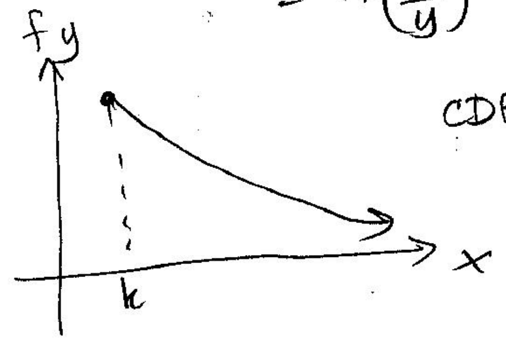
general k Supp$[Y] = (k, \infty)$

$$y = k e^{x} \Rightarrow \ln \frac{y}{k} = x \Rightarrow x = \ln\left(\frac{y}{k}\right) \Rightarrow g^{-1}(y) = \ln\left(\frac{y}{k}\right)$$

$$\Rightarrow \left| \frac{d}{dy} (g^{-1}(y)) \right| = \frac{1}{y/k} \cdot \frac{1}{k} = \frac{k}{y} \cdot \frac{1}{k} = \frac{1}{y} = y^{-1}$$

$$f_Y(y) = f_X\left(\ln\left(\frac{y}{k}\right)\right) y^{-1} = \lambda e^{-\lambda \ln\left(\frac{y}{k}\right)} \cdot y^{-1} = \lambda e^{\ln\left(\frac{y}{k}\right)^{\lambda}} \cdot y^{-1}$$

$$= \lambda \left(\frac{k}{y}\right)^{\lambda} \cdot \frac{1}{y} = \frac{\lambda k^{\lambda}}{y^{\lambda+1}} = \text{Pareto I}(k, \lambda)$$

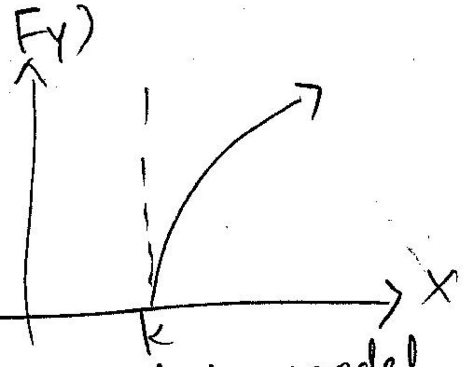


CDF
$$F_Y(y) = \int_k^y \frac{\lambda k^{\lambda}}{t^{\lambda+1}}\, dt$$

$$= \lambda k^{\lambda} \left[\frac{t^{-\lambda-1+1}}{-\lambda-1+1}\right]_k^{y}$$

$$= -k^{\lambda} \left[t^{-\lambda}\right]_k^y = -k(y^{-\lambda} - k^{-\lambda})$$

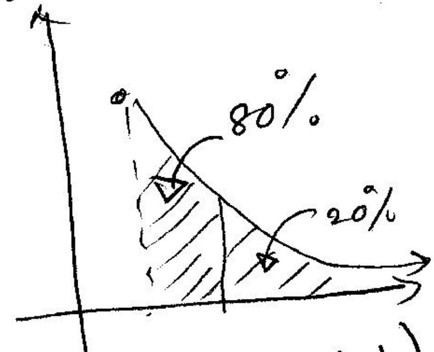
$$= k^{\lambda}(k^{-\lambda} - y^{-\lambda})$$



$$\boxed{F_Y(Y) = 1 - \left(\frac{k}{y}\right)^{\lambda}}$$

Used to model

- population spread towns/cities.
- survival, reliability, Hard drive failures
- sizes of sand particles
- file size / packet size in Internet traffic
- AND ... "Pareto Principle"
- 1896 80% of the land in Italy was owned by 20% of the population



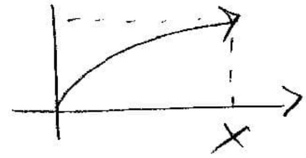
$X \sim$ Pareto $(1, \underbrace{\log_4(5)}_{1.16})$

$F_Y^{-1}(p) = ?$

Quantile $[X,p] = \inf_x \{F(x) \geq p\}$. which value of x has $p = P(X \leq x)$ what is 99% ile of the SAT ?

1590

if cont is $F_X^{-1}(p)$

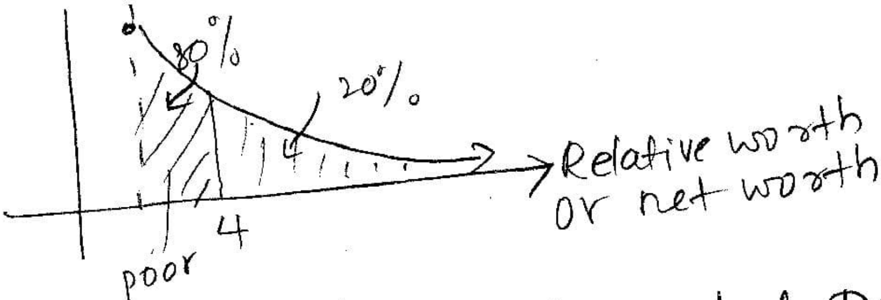


$p = F_Y(y) = 1-\left(\frac{k}{y}\right)^{\alpha}$

$\Rightarrow 1-p = \left(\frac{k}{y}\right)^{\alpha} \Rightarrow (1-p)^{\frac{1}{\alpha}} = \frac{k}{y} \Rightarrow y = k(1-p)^{-1/\alpha} = F_Y^{-1}(p)$

$F_X^{-1}(p) = (1-p)^{0.861}$ $\quad F_X^{-1}(0.8) = (1-0.8)^{-0.861} = 4$

$1-F_X(4) = 1-\left(\frac{1}{4}\right)^{1.16} = 0.8$



Let $X, Y \overset{iid}{\sim} Exp(1)$. Let $D = X-Y$. Let $Z = -Y$.

$F_Z(z) = f_Y(-z) = e^z$ $\quad$ Supp$[D] = \mathbb{R}$

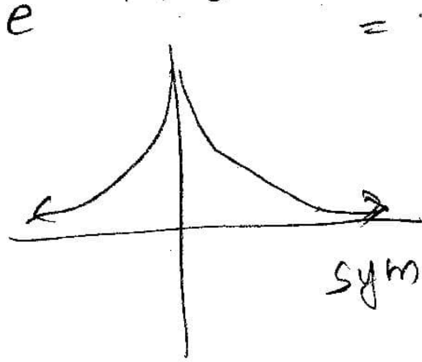
$D = X+Z \sim \int_{supp[X]} f_X(x) \cdot f_Z(d-x)\,dx = \int_0^\infty \underbrace{e^{-x} e^{d-x}}_{e^d \cdot e^{-x}} \underbrace{1_{d-x\in(-\infty,0)}}_{x \geq d}\,dx$

$= e^d \int_{\max\{0,d\}}^{\infty} e^{-2x}\,dx = e^d \left[-\frac{1}{2}e^{-2x}\right]_{\max\{0,d\}}^{\infty}$

$= \frac{1}{2} e^d e^{-2\max\{0,d\}} = \frac{1}{2}\begin{cases} e^d & \text{if } d \leq 0 \\ e^d & \text{if } d > 0 \end{cases} = \frac{1}{2}e^{-|d|}$

= Laplace (0,1)

"double exponential"



1774 "First Law of errors"

Imagine you are measuring some value V. Your measurement system is not perfect. So your measure $Y \neq V$ but "close".

$$Y = V + \underbrace{\varepsilon}_{\text{error}}$$

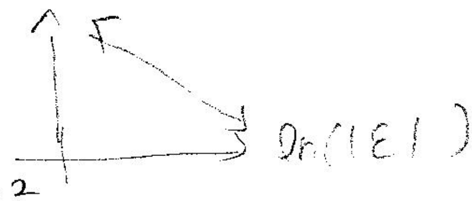
It seems reasonable that $E(\varepsilon) = 0 \Rightarrow E(Y) = V$

$med(\varepsilon) = 0 \Rightarrow med(Y) = V$

$f_\varepsilon(\varepsilon) = f_\varepsilon(-\varepsilon)$ over/underestimate of the same magnitude are equiprobable.

$f'(\varepsilon) < 0$ if $\varepsilon > 0$

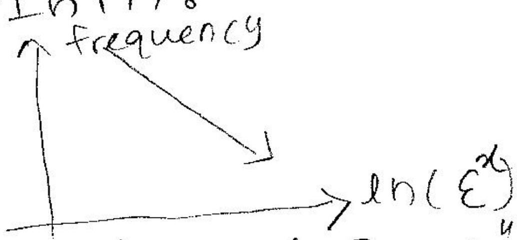
$f''(\varepsilon) = f'(\varepsilon) \Rightarrow f(\varepsilon) = c e^{-mx}$



$f(\varepsilon) \propto e^{-\varepsilon^2}$

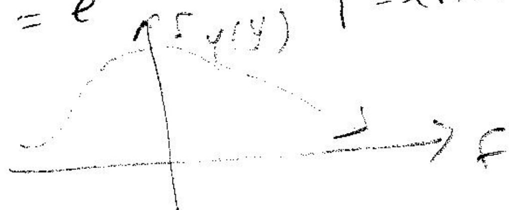
$=$ Normal

In 1778



"Second Law of Errors"

Let $X \sim Exp(1) = e^{-x}$ $\quad Y = \ln x = \ln\left(\frac{1}{x}\right)$ $\quad Supp[Y] = \mathbb{R}$



$e^y = \frac{1}{x} \Rightarrow x = \frac{1}{e^y} = e^{-y} = g^{-1}(y)$

$\left|\frac{d}{dy}(g^{-1}(y))\right| = e^{-y}$

$f_Y(y) = f_X(e^{-y}) = e^{-e^{-y}} \cdot e^{-y} = e^{-(y + e^{-y})}$ = Gumbel $(0,1)$ "standard Gumbel"

$X \sim$ Gumbel$(0,1)$ $\quad Y = \mu + \beta X \sim \frac{1}{|\beta|} f_X\left(\frac{y-\mu}{\beta}\right)$

$= \frac{1}{|\beta|} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$ = Gumbel (μ, β)

parameter space $\beta > 0$, $\mu \in \mathbb{R}$.