Lec 6 Math 241 9/18/17 — Derivation of convolution formula for cont. r.v.'s.

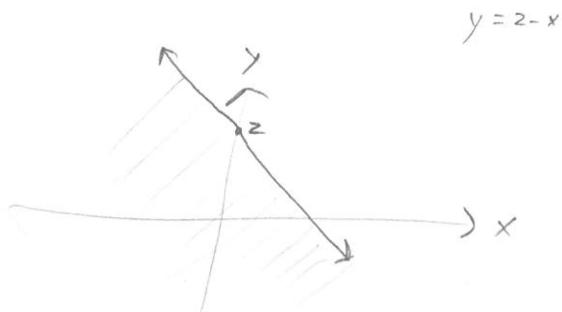
p145. X, Y are cont. r.v.'s w/ j.d.f. $f_{X,Y}(x,y)$

$$Z = g(X,Y)$$

$$F_Z(z) := P(Z \le z) = P(g(X,Y) \le z) = \iint_{\{(x,y):\, g(x,y) \le z\}} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{z} f_Z(t)\, dt$$

PDF of Z



Let $T = X + Y$

$$F_Z(z) = \iint_{\{(x,y):\, x+y \le z\}} f_{X,Y}(x,y)\, dx\, dy = \int_{\mathbb{R}} \left(\int_{\{y:\, y \le z-x\}} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{z-x} f_{X,Y}(x,y)\, dy \right) dx = \int_{\mathbb{R}} \int_{-\infty}^{z} f_{X,Y}(x, t-x)\, dt\, dx = \int_{-\infty}^{z} \underbrace{\left(\int_{\mathbb{R}} f_{X,Y}(x, t-x)\, dx \right)}_{f_T(t)} dt$$

let $t = x + y$

$\Rightarrow y = t - x$

$t_\ell - x \quad y_\ell = -\infty \Rightarrow t_\ell = -\infty$

$y_u = z - x = t_u - x \Rightarrow t_u = z$

$f_T(t)$ is the PDF of the sum

sometimes notated as $(f_X * f_Y)(x)$

The def of conv $f_X(x) * f_Y(y)$

If $X, Y \overset{\text{ind}}{\sim}$, the def of conv for indep. r.v.'s

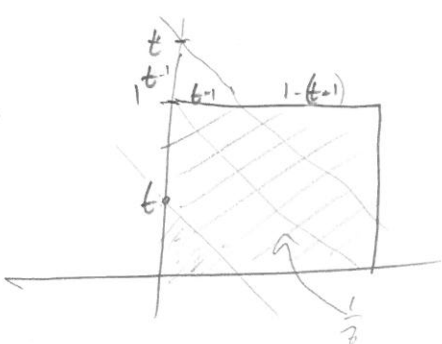
$$f_T(t) = \int_{\mathbb{R}} f_X(x)\, f_Y(t-x)\, dx$$

Note: indicator functions are included in the f. If not...

$$= \int_{Supp(X)} f_X(x)\, f_Y(t-x)\, \mathbb{1}_{t-x \in Supp(Y)}\, dx \qquad *$$

let $X, Y \overset{iid}{\sim} U(0,1)$ $T = X+Y \sim f_T(t) = ?$ We solve this using $*$

$$f_{X,Y}(x,y) = 1\, \mathbb{1}_{x\in[0,1]\ \&\ y\in[0,1]}$$



$$F_T(t) = \iint_{\{(x,y)\,:\, x+y \le t\}} f_{X,Y}(x,y)\, dx\, dy = \begin{cases} \frac{1}{2} t^2 & \text{if } t\in(0,1) \\ \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}\underbrace{(2-t)^2}_{u}\right) & \text{if } t\in(1,2] \end{cases}$$

$u \frac{du}{dt} = (2-t)(-1) = 2-t$

$$\Rightarrow f_T(t) = F_T'(t) = \begin{cases} t & \text{if } t\in[0,1] \\ 2-t & \text{if } t\in(1,2] \end{cases}$$

$X_1, X_2 \overset{iid}{\sim} U(a,b)$, $T_2 = X_1 + X_2$ $\quad$ $Supp(T) = [2a, 2b]$

$$f_{T_2}(t) = \int_{Supp(X_1)} f_{X_1}(x)\, f_{X_2}(t-x)\, dx$$

$$= \int_a^b \left(\frac{1}{b-a}\right)\left(\frac{1}{b-a}\right) \mathbb{1}_{t-x \in [a,b]}\, dx$$

$x - t \in [-b, -a]$

$x \in [t-b, t-a]$

$$= \frac{1}{(b-a)^2} \int_{\max\{a, t-b\}}^{\min\{b, t-a\}} (1)\, dx$$

$$= \frac{1}{(b-a)^2} \left(\min\{b, t-a\} - \max\{a, t-b\}\right)$$

$$f_{T_2}(t) = \begin{cases} \text{if } t < a+b \Rightarrow \dfrac{t-2a}{(b-a)^2} \\ \text{if } t \geq a+b \Rightarrow \dfrac{2b-t}{(b-a)^2} \end{cases} \quad \mathbb{1}_{t \in [2a, 2b]}$$

Recall $X \sim Geom(p) := (1-p)^x p$, $F(x) = P(X \le x) = 1 - P(X > x)$
$= 1-(1-p)^x$

if n many geometric realizations occur within each time period,

$x = tn \Rightarrow p(t) = (1-p)^{tn} p$

if $n \to \infty$, $p \to 0$ but $\lambda = np$

$$p(t) = \left(1-\frac{\lambda}{n}\right)^{tn} \frac{\lambda}{n} \quad \text{and} \quad \lim_{n\to\infty} p(t) = \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^t}_{e^{-\lambda t}} \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} = 0$$

$\lim_{n\to\infty}$ effectively $|Supp(X)| = |\mathbb{N}| \Rightarrow |Supp(X)| = |\mathbb{R}|$

once the support is no longer discrete, the PMF vanishes.

But... recall $F(x) = 1-(1-p)^x$

$$\Rightarrow F_n(t) = 1-(1-p)^{nt}$$

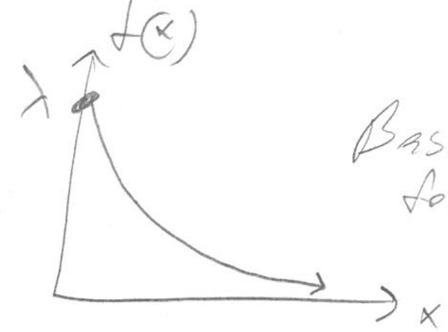
$$\Rightarrow F_n(t) = 1-\left(1-\frac{\lambda}{n}\right)^{nt}$$

$$F(t) \quad \lim_{n\to\infty} F_n(t) = 1 - \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^t = 1-e^{-\lambda t}$$

$P(X>t)$
$\Rightarrow 1-F(t)$
$= e^{-\lambda t}$

$$\Rightarrow f(t) = \frac{d}{dt}F(t) = \lambda e^{-\lambda t}$$

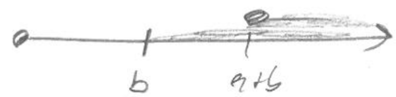
$$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$$



Basic model for waiting time / failure time / survival

$$Supp(X) = (0,\infty)$$

Parm space $\lambda = \underset{(+)}{u}\,\underset{(+)}{p}$ $\quad \lambda \in (0,\infty)$



if $a,b \in \mathbb{R}^+$,

$$P(X > a+b \mid X > b) = \frac{P(X > a+b \;\&\; X > b)}{P(X>b)} = \frac{P(X>a+b)}{P(X>b)}$$

$$= \frac{e^{-(a+b)x}}{e^{-bx}} = e^{-ax} = 1-F(a) = P(X>a)$$

Memorylessness

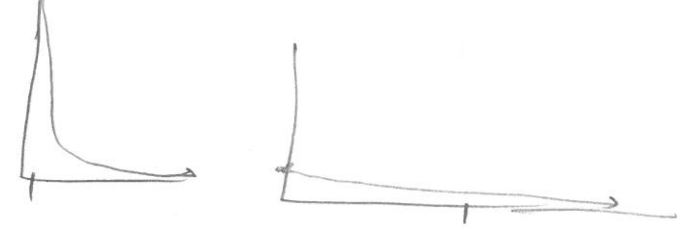
For a cont r.v. X,

$$E(X) = \int_{Supp(X)} x\, f(x)\, dx$$

$X \sim Exp(\lambda)$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx = \dots = \frac{1}{\lambda}$$

expected waiting time is inverse λ.



$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda)$, $T_2 = X_1 + X_2 \sim ?$ $\quad Supp(T_2) = (0, \infty)$

$$f_{T_2}(t) = \int_{Supp(X_1)} f_{X_1}(x) f_{X_2}(t-x)\, dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)} \lambda e^{\overbrace{-\lambda(t-x)}^{-\lambda t + \lambda x}} \underbrace{\mathbb{1}_{t-x \in (0,\infty)}}\, dx$$

$$\underbrace{\mathbb{1}_{x-t \in (-\infty, 0)}}$$

$$\mathbb{1}_{x \in (-\infty, t)}$$

$$= \lambda^2 \int_0^\infty e^{-\lambda t} \mathbb{1}_{x \in (-\infty, t)}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = \lambda^2 t e^{-\lambda t}$$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2$$

$$f_{T_3}(t) = \int_{Supp(X_1)} f_{X_1}(x) f_{T_2}(t-x)\, dx = \int_0^\infty \lambda e^{-\lambda x} \lambda^2 (t-x) e^{-\lambda(t-x)} \mathbb{1}_{t-x \in (0,\infty)}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty (t-x) \mathbb{1}_{t-x \in (0,\infty)}\, dx$$

$$= \lambda^3 e^{-\lambda t} \left(t \int_0^\infty \mathbb{1}_{t-x \in (0,\infty)}\, dx - \int_0^\infty x \mathbb{1}_{t-x \in (0,\infty)}\, dx \right) = \lambda^3 e^{-\lambda t} \left(t \int_0^t dx - \int_0^t x\, dx \right)$$

$$= \lambda^3 e^{-\lambda t}\left(t^2 - \frac{t^2}{2}\right) = \frac{\lambda^3 t^2}{2} e^{-\lambda t}$$

Again...

$$f_{T_4}(t) = f_{X_4}(x) * f_{T_3}(t) = \int_0^\infty \lambda e^{-\lambda x} \frac{\lambda^3 (t-x)^2}{2} e^{-\lambda(t-x)} \mathbb{1}_{t-x\in(0,\infty)}\, dx$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t (t-x)^2 dx$$

$$\parallel$$

$$t^2\int_0^t dx - 2t\int_0^t x\,dx + \int_0^t x^2 dx$$

$$\parallel$$

$$\cancel{t^3} - \cancel{2t\frac{t^2}{2}} + \frac{t^3}{3}$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{3\cdot 2} t^3$$

$$\vdots$$

$$f_{T_K}(x) = \frac{\lambda^K x^{K-1} e^{-\lambda x}}{(K-1)!}$$

Param space: $\lambda \in (0,\infty)$ as before

$K \in \mathbb{N}$

$Supp(X) = (0,\infty)$ as before

$$\text{Erlang}(k,\lambda) :=$$

let $u = \lambda y \Rightarrow \frac{du}{dy} = \lambda \Rightarrow dy = \frac{du}{\lambda}$

$\Rightarrow y = 0 \Rightarrow u = 0$

$\Rightarrow y = x \Rightarrow u = \lambda x$

$$F_{T_K}(x) = \int_0^x \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} dy = \frac{1}{(k-1)!}\int_0^x \lambda(\lambda y)^{k-1} e^{-\lambda y} dy$$

$$= \frac{1}{(k-1)!}\int_0^{\lambda x} u^{k-1} e^{-u} du = \frac{\gamma(k,\lambda x)}{(k-1)!}$$

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,\, a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,\, a)}$$

gamma function

lower incomplete gamma function

upper incomplete gamma function

Lec 7

The gamma function is known as the extension of the factorial function to all real #'s.

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = -e^{-t}\Big]_0^\infty = -(0-1) = 1$$

$$\Gamma(x+1) = x\,\Gamma(x) \quad (HW)$$

$$\Rightarrow \Gamma(2) = 1 \cdot 1$$

$$\Gamma(3) = 2\,\Gamma(2) = 2 \cdot 1$$

$$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1$$

$$\vdots$$

$$\Gamma(n) = (n-1)!$$

$$\Rightarrow F_{T_k}(x) = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

Sometimes called the "standard" gamma functions

$$1 - F_{T_k}(x) = 1 - \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \frac{\Gamma(k, \lambda x)}{\Gamma(k)}$$