Let $X \sim \text{Gamma}(k_1, \lambda)$ and $Y \sim \text{Gamma}(k_2, \lambda)$ $(X, Y \overset{iid}{\sim})$. The Gamma distribution describes waiting time for k Exponential(λ) timed events where k could be fractional. Then

$$X + Y \sim \text{Gamma}(k_1 + k_2, \lambda)$$

$$\begin{aligned} X + Y &\sim f_X(x) \times f_Y(y) \\ &= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1} (1-u)^{k_2-1} t \, du \\ &= \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1} (1-u)^{k_2-1} \, du \end{aligned}$$

Recall that $X \sim \text{Exp}(\lambda) = f(x) = \lambda e^{-\lambda x}$ and $\int_{\text{Supp}[X]} f(x) \, dx = 1$. Note

$$f(x) = \lambda e^{-\lambda x} \propto e^{-\lambda x} = k(x)$$

Here, $k(x)$ is called the kernel of the Exponential distribution and is proportional to $f(x)$.

$$k(x) = cf(x) \to f(x) = \frac{1}{c} k(x) \text{ where } c \text{ is not a function of } x$$

$$1 = \int_{\text{Supp}[X]} f(x) \, dx = \int_{\text{Supp}[X]} \frac{1}{c} k(x) \, dx \to c = \int_{\text{Supp}[X]} k(x) \, dx$$

In this case, $\frac{1}{c} = \lambda$ and so $c = \frac{1}{\lambda}$

$$\int e^{-\lambda x} \, dx = \frac{1}{\lambda}$$

$k(x)$ can be restored to $f(x)$ by multiplying it by $\frac{1}{c}$.

Let $X \sim \text{Binom}(n, p) = p(x) = \binom{n}{x} p^x (1-p)^{n-x}$. Then

$$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \propto \underbrace{(x!(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x}_{\text{identifies the Binomial}} = k(x)$$

Let $X \sim \text{Weibull}(k, \lambda) = f(x) = k\lambda(x\lambda)^{k-1} e^{-(x\lambda)^k}$. Then

$$p(x) \propto \underbrace{xe^{-(x\lambda)^k}}_{identifiestheWeibull} = k(x)$$

Let $X \sim \text{Gamma}(k, \lambda) = f(x) = \frac{\lambda^k e^{\lambda x} x^{k-1}}{\Gamma(k)}$. Then

$$p(x) \propto e^{\lambda x} x^{k-1} = k(x)$$

Therefore,

$$f_{X+Y}(t) = \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1} \, du \propto e^{-\lambda t} t^{k_1+k_2-1} \propto \text{Gamma}(k_1 + k_2, \lambda)$$

As a corollary,

$$\begin{aligned}
f_{X+Y}(t) &= \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1+k_2)} \\
&= \frac{\lambda^{k_1+k_2} e^{-\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du \\
\frac{1}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du &= \frac{1}{\Gamma(k_1+k_2)} \\
\int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du &= \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)}
\end{aligned}$$

Let $B(\alpha,\beta)$ be the beta function. Then

$$\begin{aligned}
B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&= \frac{\int_0^\infty t^{\alpha-1}e^{-t}\,dt \; \int_0^\infty t^{\beta-1}e^{-t}\,dt}{\int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dt}
\end{aligned}$$

Let $X_1, X_2, \ldots, X_n$ be a sequence of continuous random variables. Then $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics where

$$\begin{aligned}
X_{(1)} &= \min\Big\{X_1, \ldots, X_n\Big\} \\
X_{(n)} &= \max\Big\{X_1, \ldots, X_n\Big\} \\
X_{(k)} &= \Big\{k^{\text{th}} \text{ largest of } X_1, \ldots, X_n\Big\}
\end{aligned}$$

For example,

$$\begin{aligned}
X_1 &= 9 = X_{(3)} \\
X_2 &= 2 = X_{(1)} \\
X_3 &= 12 = X_{(4)} \\
X_4 &= 7 = X_{(2)}
\end{aligned}$$

Let $R = X_{(n)} - X_{(1)}$ be the range of the set under the assumption of $\overset{iid}{\sim}$ of $X_1, \ldots X_n$. Let's first derive the distribution of the maximum.

$$\begin{aligned}
12 &= \max\{2, 7, 9, 12\} \\
X_{(n)} &= \max\{X_1, \ldots, X_n\}
\end{aligned}$$

This means that all X_i's are less than $X_{(n)}$.

$$\begin{aligned}
F_{X_{(n)}}(x) &= \mathbb{P}(X_{(n)} < x) \\
&= \mathbb{P}(X_1 < x, X_2 < x, \ldots, X_n < x) \\
&= \prod_{i+1}^{n} \mathbb{P}(X_i < x) \\
&= \mathbb{P}(X_1 < x)^n \\
&= F(x)^n
\end{aligned}$$

Then

$$f_{X_{(n)}}(x) = F'_{X_{(n)}}(x) = nf(x)F(x)^{n-1}$$

On the other side,

$$\begin{aligned}
2 &= \min\{2, 7, 9, 12\} \\
X_{(1)} &= \min\{X_1, \ldots, X_n\}
\end{aligned}$$

This means that all X_i's are greater than $X_{(1)}$.

$$\begin{aligned}
F_{X_{(1)}}(x) &= \mathbb{P}(X_{(1)} \leq x) \\
&= 1 - \mathbb{P}(X_{(1)} \geq x) \\
&= 1 - \mathbb{P}(X_1 \geq x, X_2 \geq, \ldots, X_n \geq n) \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}(X_i \geq n) \\
&= 1 - \mathbb{P}(X_i \geq x)^n \\
&= 1 - (1 - F(x))^n
\end{aligned}$$

Then

$$f_{X_{(1)}} = n(-f(x))(-1)(1 - F(x))^{n-1} = nf(x)(1 - F(x))^{n-1}$$

What about $X_{(k)}$, the k^{th} largest of $X_1, \ldots, X_n$? In our example, 9 is the third largest of $\Big\{2, 7, 9, 12\Big\}$, and so $X_{(3)} = 9$.

Goal: $F_{X_{(k)}}(x)$, the CDF of the k^{th} largest random variable of $X_1, \ldots, X_n$.
Consider $n = 10$. What is the $\mathbb{P}(X_1, \ldots, X_4 \in (-\infty, x)$ and $X_5, \ldots, X_{10} \in (x, \infty))$? It is

$$\begin{aligned}
&\mathbb{P}(X_1 \leq x, \ldots, X_4 \leq x, X_5 > x, \ldots, X_{10} > x) \\
&\mathbb{P}(X_1 \leq x) \ldots \mathbb{P}(X_4 \leq x)\mathbb{P}(X_5 > x) \ldots \mathbb{P}(X_{10} > x) \\
&F(x)^4(1 - F(x))^6
\end{aligned}$$

More generally, what is the $\mathbb{P}(\text{any } 4 \in (-\infty, x) \text{ and the other } 6 \in (x, \infty))$?

$$
\begin{aligned}
&\mathbb{P}(\underbrace{X_1 \leq x, \dots, X_4 \leq x}_{\text{these 4 below}}, \underbrace{X_5 > x, \dots, X_{10} > x}_{\text{these 6 above}}) \\
&+ \mathbb{P}(\underbrace{X_{10} \leq x, X_7 \leq x, X_3 \leq x, X_9 \leq x}_{\text{these 4 below}}, \underbrace{X_1 > x, X_3 > x, \dots, X_8 > x}_{\text{these 6 above}}) \\
&+ \text{ all other possibilities} \\
&= \binom{10}{4} F(x)^4 (1 - F(x))^6
\end{aligned}
$$

This looks like the binomial where $n = 10$ and $p = F(x)$. Then

$$
\begin{aligned}
F_{X_{(4)}}(x) &= \mathbb{P}(X_{(4)} \leq x) \\
&= \binom{10}{4} F(x)^4 (1 - F(x))^6 + \binom{10}{5} F(x)^5 (1 - F(x))^5 + \cdots + \binom{10}{10} F(x)^{10} (1 - F(x))^0 \\
&= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1 - F(x))^{10-j}
\end{aligned}
$$

Generalizing this to arbitrary n and k:

$$
F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}
$$

Verify that this works for the max and min:

$$
\begin{aligned}
F_{X_{(n)}}(x) &= \sum_{j=n}^{n} F(x)^j (1 - F(x))^{n-j} = \binom{n}{n} F(x)^n (1 - F(x))^{n-n} = F(x)^n \\
F_{X_{(1)}}(x) &= \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \\
&= \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \right) - \binom{n}{0} F(x)^0 (1 - F(x))^{n-0} \\
&= \Big(F(x) + (1 - F(x)) \Big)^n - (1 - F(x))^n \\
&= 1 - (1 - F(x))^n
\end{aligned}
$$

Note that

$$
\begin{aligned}
f_{X_{(k)}}(x) &= F'_{X_{(k)}}(x) \\
&= \frac{d}{dt}\Big[\sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}\Big] \\
&= \sum_{j=k}^{n}\frac{n!}{j!(n-j)!}\underbrace{\frac{d}{dx}[F(x)^j(1-F(x))^{n-j}]}_{F(x)^j(n-j)(1-F(x))^{n-j-1}(-f(x))+(1-F(x))^{n-j}jF(x)^{j-1}f(x)} \\
&= \sum_{j=k}^{n}\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
&- \sum_{j=k}^{n}\frac{n!}{j!(n-j-1)!}f(x)F(x)^{j}(1-F(x))^{n-j-1}
\end{aligned}
$$

We can reindex this to end at $n-1$ since at n it is 0

$$
\begin{aligned}
&= \sum_{j=k}^{n}\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
&- \sum_{j=k}^{n-1}\frac{n!}{j!(n-j-1)!}f(x)F(x)^{j}(1-F(x))^{n-j-1}
\end{aligned}
$$

Reindex this again so that it sums from $k+1$ to n. Let $l=k+1$ so that $j=l-1$

$$
\begin{aligned}
&= \sum_{j=k}^{n}\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
&- \sum_{l=k+1}^{n}\frac{n!}{(l-1)!\,\underbrace{(n-(l-1)-1)!}_{(n-l)!}}f(x)F(x)^{l-1}(1-F(x)^{\overbrace{n-(l-1)-1}^{n-l}}
\end{aligned}
$$

Let $j=l$

$$
\begin{aligned}
&= \sum_{j=k}^{n}\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
&- \sum_{j=k+1}^{n}\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
&= (a_k + a_{k+1} + \cdots + a_n) - (a_{k+1} + \cdots + a_n) \\
&= a_k
\end{aligned}
$$

$$
f_{X_{(k)}} = \frac{n!}{(k-1)!(n-k)!}f(x)F(x)^{k-1}(1-F(x))^{n-k}
$$