$$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \neq Bin(n, p)$$

Ind. prob. $\rightarrow$ $P_{X_1X_2}(x_1, x_2) = P_{X_1}(x) \cdot P_{X_2}(x)$

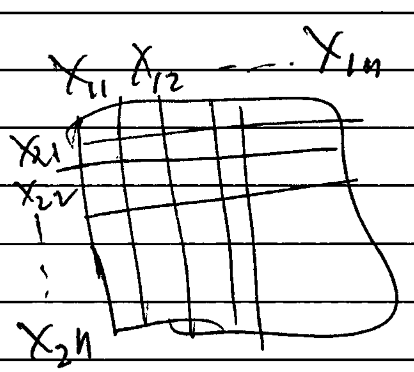
Bayes Rule

$$P_{X_1|X_2}(x_1|x_2) = \frac{P_{X_1X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

Cond. Prob.

Marginal distribution

$$P_{X_1|X_2}(x_1|x_2) = \frac{P_{X_1X_2}(x_1, x_2)}{P_{X_2}(x_2)} = \frac{\binom{n}{x_1, x_2} p^{x_1}(1-p)^{x_2}}{\binom{n}{x_2}(1-p)^{x_2}p^{n-x_2}}$$



$$P_{X_2}(x) = P(X_2 = x)$$

$$= \sum_{x_1 \in Supp(X)} p(x_1, x_2)$$

$$\sum_{x \in \{0, \dots, n\}} \binom{n}{x_1, x_2} p^{x_1}(1-p)^{x_2} = \sum_{x \in \{0, \dots, n\}} \frac{n!}{x_1!\, x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}$$

$$\frac{n!}{x_2!}(1-p)^{x_2} \sum_{x_1 \in \mathrm{supp}(X_1)} \frac{p^{x_1}}{x_1!} \underbrace{\mathbb{1}_{x_1+x_2=n}}_{\mathbb{1}_{x_1=n-x_2}}$$

$$\frac{n!}{x_2!}(1-p)^{x_2} \frac{p^{n-x_2}}{(n-x_2)!}$$

$$= \mathrm{Bin}(n, 1-p) \Longrightarrow X_1 \sim \mathrm{Bin}(n,p)$$

From (*) in the back

$$= \underline{\frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}}$$

$$\frac{n!}{x_1!\,(n-x_2)!}(1-p)^{x_2} p^{n-x_2}$$

$$= \frac{(n-x_2)}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$X \sim \text{Multinom}(n, p)$ $\text{Multinom}(n,p) = \text{JMF}$

$$P_{X_{-j} | X_j}(x_{-j} | x_j) = \frac{P_{X_1 \cdots X_n}(x_1, \cdots x_n)}{P_{X_j}(x_j)}$$

$\text{Bin}(n, p_j)$

$$= \frac{\frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}}{\frac{n!}{x_j!(n-x_j)!} p_j^{x_j}(1-p_j)^{n-x_j}}$$

$$= \frac{(n-x_j)!}{x_1! \cdots x_{j-1}! x_{j+1}! \cdots x_k!} \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{n-x_j}}$$

let $n' = n - x_j$

Recall $\sum_{i=1}^{k} x_i = n \Rightarrow x_1 + x_2 + \cdots + x_{j-1} + x_j + x_{j+1} + \cdots + x_n = n$

$$\Rightarrow n' = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k$$

$$= \binom{n'}{x_1, \cdots x_{j-1}, x_{j+1}, \cdots x_k} \frac{\left(p_1'(1-p_j)\right)^{x_1} \cdots \left(p_{j-1}'(1-p_j)\right)^{x_{j-1}} \left(p_{j+1}'(1-p_j)\right)^{x_{j+1}} \cdots \left(p_k'(1-p_j)\right)^{x_k}}{(1-p_j)^{n'}}$$

$E(X) = \mu$

$$E\left(g(X_1 \cdots X_n)\right) = \sum_{x_1 \in Supp(X_1)} \cdots \sum_{x_n \in Supp(X_n)} g(x_1 \cdots x_n)\, p(x_1 \cdots x_n)$$

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n\mu$$

$$E\left(\prod_{i=1}^{n} X_i\right) = \sum \cdots \sum x_1 x_2 \cdots x_n\, p(x_1, x_2 \cdots x_n)$$

$$= \sum \cdots \sum \left(x_1 p(x) \cdots\right) x_n p(x_n)$$

Variance = Var

$$Var(X) = E\left[(x - \mu)^2\right] = E(g(x))$$

$$= \sum g(x)\, p(x) = \sum (x - \mu)^2 p(x)$$

$$= \sum x^2 p(x) + \sum -2n\mu p(x) + \sum \mu^2 p(x)$$

$$= E(x^2) - 2n^2 + n^2$$

$$= E(x^2) - n^2$$

$Var(X+\mu) = Var(X)$ | $E(X+c) = E(X) + c$

$Var(cX) = c^2 Var(X)$ | $E(\mu X) = \mu E(X)$

$$Var(X_1 + X_2) = E\left[\left((X_1+X_2) - (\mu_1+\mu_2)\right)^2\right]$$

$$= Var(X_1) + Var(X_2) +$$

$$2\left(E(X_1 X_2) - \mu_1\mu_2\right) \quad \rho_{X_1X_2} = Cov(X_1, X_2)$$

$$= \frac{Cov(X_1, X_2)}{\Sigma E(X_1)\ \Sigma E(X_2)}$$

<u>Proprties</u>

$Cov(X, X) = Var(X)$.

$Cov(a_1X_1, a_2X_2) = a_1a_2\, Cov(X_1, X_2)$

$Cov(X_1 + c_1, X_2 + c_2) = Cov(X_1, X_2)$

$Cov(X_2, X_1) = Cov(X_1, X_2)$

$$\mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

$$\mathrm{Cov}\left(\sum_{i=1}^{n_i} X_i, \sum_{j=1}^{n_j} Y_j\right) = \sum_{i=1}^{n_i} \sum^{n_i} \mathrm{Cov}(X_i, Y_i)$$

$$\mathrm{Var}[X_1 + \dots + X_k] = \sum_{i=1}^{k} \sum_{j=1}^{K} \mathrm{Cov}(X_i, X_j)$$

if $\vec{X}$ in a vector

$$E(\vec{X}) = E\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_k) \end{bmatrix}$$

$$\mathrm{Var}(\vec{X}) = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \cdots \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \cdots \\ \vdots & & \ddots \\ & & \mathrm{Var}(X_k) \end{bmatrix}$$

Symetric $k \times k$ Matrix