Math 621 Fall 2017
Midterm Examination Two

Solutions

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Full Name ______________________________

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an *unauthorized* cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________ signature   ______________ date

Instructions

This exam is seventy five minutes and closed-book. You are allowed one 8.5" × 11" page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in choose, permutation, exponent, factorial or any other notation which could be resolved to a number with a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 Below are some theoretical questions.

(a) [10 pt / 10 pts] Let $X \sim \text{Beta}(\alpha, \beta)$ and $Y = 1 - X$. (i) Find $\text{Supp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$. (iv) If Y is a brand name r.v., indicate it with the notation used in class.

$Y = 1 - X = -X + 1$ $\quad$ $Supp(Y) = (0,1)$

$f_Y(y) = \frac{1}{|-1|} f_X\left(\frac{y-1}{-1}\right) = f_X(1-y) = \frac{1}{B(\alpha,\beta)}(1-y)^{\alpha-1}(1-(1-y))^{\beta-1} = \frac{1}{B(\beta,\alpha)} y^{\beta-1}(1-y)^{\alpha-1} = Beta(\beta,\alpha)$

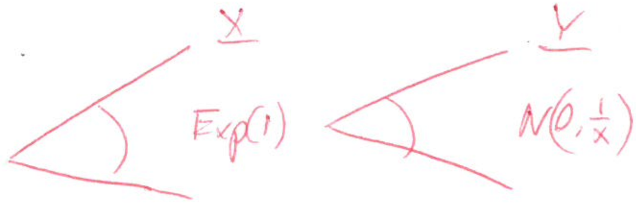
(b) [10 pt / 20 pts] Let $X \sim \text{Logistic}(0, 1)$ and $Y = e^X$ i.e. the log-logistic distribution analogous to the log-normal distribution. (i) Find $\text{Supp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$.

$Y = e^X = g(X)$. Note $g(X)$ is a 1:1 function $\quad g^{-1}(y) = \ln(y) = x \quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\ln(y)]\right| = \frac{1}{y}$

$f_Y(y) = f_X(\ln y)\frac{1}{y} = \frac{e^{\ln(y)}}{(e^{\ln(y)}+1)^2}\frac{1}{y} = \frac{\cancel{y}}{(y+1)^2 \cancel{y}} = \frac{1}{(y+1)^2}$ $\quad$ $Supp(Y) = (0,\infty)$

$X \sim Logistic(0,1) = \frac{e^x}{(e^x+1)^2}$

(c) [6 pt / 26 pts] If $Y \mid X = x \sim \mathcal{N}\left(0, \frac{1}{x}\right)$ and $X \sim \text{Exp}(1)$, draw a tree diagram for X, Y.



(d) [10 pt / 36 pts] As given previously, if $Y \mid X = x \sim \mathcal{N}\left(0, \frac{1}{x}\right)$ and $X \sim \text{Exp}(1)$. (i) Find $\text{Supp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$. (iv) If Y is a brand name r.v., indicate it with the notation used in class.

$\text{Supp}(Y) = \mathbb{R}$

$$f_Y(y) = \int_{\text{Supp}[X]} f_{Y|X}(y,x) f_X(x)\,dx = \int_0^\infty \frac{1}{\sqrt{2\pi\frac{1}{x}}} e^{-\frac{1}{2\frac{1}{x}}(y-0)^2} e^{-x}\,dx \propto \int_0^\infty x^{\frac{1}{2}} e^{-\frac{xy^2}{2}} e^{-x}\,dx$$

$$= \int_0^\infty x^{\frac{1}{2}} e^{-x\left(1+\frac{y^2}{2}\right)}\,dx = \int_0^\infty \frac{u^{\frac{1}{2}}}{\left(1+\frac{y^2}{2}\right)^{\frac{1}{2}}} e^{-u} \frac{1}{1+\frac{y^2}{2}}\,du = \frac{1}{\left(1+\frac{y^2}{2}\right)^{3/2}} \int_0^\infty u^{\frac{1}{2}} e^{-u}\,du \propto \left(1+\frac{y^2}{2}\right)^{-\frac{3}{2}} \propto T_2$$

let $u = x\left(1+\frac{y^2}{2}\right) \Rightarrow x = \frac{1}{1+\frac{y^2}{2}} u \Rightarrow dx = \frac{1}{1+\frac{y^2}{2}}\,du$

$$\Rightarrow f_Y(y) = \frac{\Gamma\left(\frac{2+1}{2}\right)}{\sqrt{2\pi}\,\Gamma\left(\frac{2}{2}\right)} \left(1+\frac{y^2}{2}\right)^{-\frac{3}{2}} = \frac{\frac{3}{2}\sqrt{\pi}}{\sqrt{2\pi}} = \frac{3}{2\sqrt{2}}\left(1+\frac{y^2}{2}\right)^{-3/2}$$

$\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$

(e) [10 pt / 46 pts] Let $X \sim \text{Laplace}(0, 1)$ and $Y = X\mathbb{1}_{X \geq 0}$. (i) Find $\text{Supp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$. (iv) If Y is a brand name r.v., indicate it with the notation used in class.

$X \sim \text{Laplace}(0,1) := \frac{1}{2}e^{-|x|}$

$\text{Supp}(Y) = [0, \infty)$

$Y = X\mathbb{1}_{X \geq 0} \sim \dfrac{\frac{1}{2}e^{-|y|}\mathbb{1}_{y \geq 0}}{P(X \geq 0)} = \dfrac{\frac{1}{2}e^{-|y|}\mathbb{1}_{y\geq 0}}{\frac{1}{2}} = e^{-y}\mathbb{1}_{y \geq 0} = \text{Exp}(1)$

Since X is symmetric about 0,
$P(X < 0) = P(X > 0) = \frac{1}{2}$

Since $y \geq 0 \Rightarrow |y| = y$

(f) [10 pt / 56 pts] Consider modeling human survival (with unit years) by $T \sim \text{Weibull}(\lambda, k)$. Make up values of k and λ that are appropriate for this modeling challenge. Note that $\mathbb{E}[T] = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{k}\right)$. Explain why you chose these k and λ values.

Survival should be getting less and less likely $\Rightarrow k > 1$. So let's say e.g. $\underline{k = 2}$. Average survival is ≈ 80yr $\Rightarrow$

$80 = E(T) = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{k}\right) = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{\lambda}\Gamma\left(\frac{3}{2}\right) = \frac{1}{\lambda}\frac{3}{2}\sqrt{\pi}$

$\Rightarrow \lambda = \dfrac{3\sqrt{\pi}}{2 \cdot 80} \approx \underline{0.033}$

(g) [4 pt / 60 pts] Why wouldn't $W \sim \text{Gumbel}(\mu, \beta)$ be appropriate for modeling human survival (with unit years)?

$\text{Supp}(W) = \mathbb{R}$ and survival is positive (# of yrs).

(h) [12 pt / 72 pts] If $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0, 1)$ and $Y = (Z_1/Z_2)^2$. (i) Find $\mathbb{S}\text{upp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$. (iv) If Y is a brand name r.v., indicate it with the notation used in class.

$$Y = \frac{Z_1^2}{Z_2^2} = \frac{\frac{Z_1^2}{1}}{\frac{Z_2^2}{1}} \sim F_{1,1} = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} y^{-\frac{1}{2}} (1+y)^{-1} = \frac{1}{\pi (y+1)\sqrt{y}}$$

$$\frac{1}{B(\frac{1}{2}, \frac{1}{2})} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} = \frac{1}{\pi}$$

$\text{Supp}(Y) = (0, \infty)$

(i) [12 pt / 84 pts] Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Gamma}(1, \beta)$ and $Y = \min\{X_1, \ldots, X_n\}$. (i) Find $\mathbb{S}\text{upp}[Y]$. (ii) Find $f_Y(y)$. (iii) Simplify $f_Y(y)$. (iv) If Y is a brand name r.v., indicate it with the notation used in class.

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Gamma}(1, \beta) = \text{Erlang}(1, \beta) = \text{Exp}(\beta) = \beta e^{-\beta x} \Rightarrow F(x) = 1 - e^{-\beta x}$$

$$Y \sim f_{X_{(1)}}(y) = n f_X(y) (1 - F_X(y))^{n-1} = n \beta e^{-\beta x} (e^{-\beta x})^{n-1} = n\beta e^{-n\beta x} = \text{Exp}(n\beta)$$

$\text{Supp}(Y) = (0, \infty)$

(j) [12 pt / 96 pts] Let $X \sim \text{U}(0, 1)$ independent of $Y \sim \text{U}(0, 1)$ and $R = \frac{X}{Y}$. (i) Find $\text{Supp}[R]$. (ii) Find $f_R(r)$. (iii) Simplify $f_R(r)$. (iv) If R is a brand name r.v., indicate it with the notation used in class.

$\text{Supp}(R) = (0, \infty)$

$$R \sim \int_{\text{Supp}(Y)} y\, f_X(ry)\, f_Y(y)\, dy = \int_0^1 y\,(1)\,\mathbb{1}_{ry \in [0,1]}\,(1)\,\mathbb{1}_{y \in [0,1]}\, dy$$

$$= \int_0^1 y\, \mathbb{1}_{y \in [0, \min\{1, \frac{1}{r}\}]}\, dy = \left[\frac{y^2}{2}\right]_0^{\min\{1, \frac{1}{r}\}}$$

$$= \begin{cases} \frac{1}{2} & \text{if } \frac{1}{r} \geq 1 \Rightarrow r \leq 1 \\ \frac{1}{2r^2} & \text{if } \frac{1}{r} < 1 \Rightarrow r > 1 \end{cases}$$

Problem 2 There are 5 class lectures left in Math 621. Write below about what kind of probability material *you* want to learn. [4 pt / 100 pts]