Let $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$ and $T = X_1 + X_2$. Then

$$p(t) = \mathbb{P}_{X_1}(x) \cdot \mathbb{P}_{X_2}(x) = \sum_{x \in \text{Supp}[X]} \mathbb{P}_{X_1}(x) \cdot \mathbb{P}_{X_2}(t-x) \overset{?}{=} 2p^t(1-p)^{2-t}$$

$$\begin{aligned} p(2) &\overset{?}{=} \mathbb{P}_{X_1}(0)\mathbb{P}_{X_2}(2-0) + \mathbb{P}_{X_1}(1)\mathbb{P}_{X_2}(2-1) \\ &= \mathbb{P}_{X_1}(0)\mathbb{P}_{X_2}(2) + \mathbb{P}_{X_1}(1)\mathbb{P}_{X_2}(2-1) \\ &= p^{-}(1-p)^2 p^2 (1-p)^0 + p^1(1-p)^1 \cdot p^1(1-p)^1 \\ &= 2p^2(1-p)^2 \end{aligned}$$

Let $A = \Big\{w_1, w_2, \ldots, w_n\Big\}$ where $|A| = n$. Let

$$\begin{aligned} 2^A &= \Big\{B : B \subseteq A\Big\} \\ &= \Big\{B : B \subseteq A \text{ and } |A| = 0\Big\} \bigcup \\ &\Big\{B : B \subseteq A \text{ and } |A| = 1\Big\} \bigcup \\ &\Big\{B : B \subseteq A \text{ and } |A| = 2\Big\} \bigcup \\ &\ldots \\ &\bigcup \Big\{B : B \subseteq A \text{ and } |A| = n\Big\} \\ 2^n &= |2^A| \\ &= \sum_{i=1}^{n} |\Big\{B : B \subseteq A \text{ and } |A| = i\Big\}| \\ &= \sum_{i=0}^{n} \binom{n}{i} \end{aligned}$$

This proves that

$$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Recall $\text{E}[X] = \sum_{x \in \text{Supp}[X]} xp(x)$ for discrete random variables. Consider a function of a random variable g. Then $\text{E}[g(x)] = \sum_{x \in \text{Supp}[X]} g(x)p(x)$

Let $z = \mathbb{1}_A$. Then $z \sim \text{Bern}(P(A))$. Hence $\text{E}[z] = P(A)$.

If $z = g(x, y)$, a function of two random variables,

$$\text{E}[z] = \text{E}[g(x, y)] = \sum_{x \in \text{Supp}[X]} \sum_{y \in \text{Supp}[Y]} g(x, y)\mathbb{P}_{X,Y}(x, y)$$

where $\mathbb{P}_{X,Y}(x, y)$ is a jmf.

Let $X, Y \overset{iid}{\sim} \text{Geom}(p) = (1-p)^x p$. Then

$$\text{E}[X] = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1 - (1-p)^{x+1}$$

What is $\mathbb{P}(X > Y)$? Let $z = \mathbb{1}_{x>y} = g(x, y)$. Then

$$\begin{aligned}
\mathbb{P}(X > Y) &= \mathrm{E}[z] \\
&= \sum_{y\in\mathbb{N}_0}\sum_{x\in\mathbb{N}_0} \mathbb{1}_{x>y}\mathbb{P}_{X,Y}(x,y) \\
&= p^2 \sum_{y\in\mathbb{N}_0}(1-p)^y \sum_{x\in\mathbb{N}_0}(1-p)^x \mathbb{1}_{x>y} \\
&\text{since } X, Y \overset{iid}{\sim},\ \mathbb{P}_{X,Y}(x,y) = \mathbb{P}_X(x)\mathbb{P}_Y(y) = p(1-p)^x p(1-p)^y \\
&= p^2 \sum_{y\in\mathbb{N}_0}(1-p)^y \sum_{x=y+1}^{\infty}(1-p)^x \\
&\text{Let } x' = x-(y+1) = x-y-1 \to x = x'+y+1 \\
&= p^2 \sum_{y\in\mathbb{N}_0}(1-p)^y \sum x' \in \mathbb{N}_0 (1-p)^{x'+y+1} \\
&= p^2 \sum_{x\in\mathbb{N}_0}(1-p)^{2y+1} \sum_{x'\in\mathbb{N}_0}(1-p)^{x'} \\
&= p^2(1-p) \underbrace{\sum_{y\in\mathbb{N}_0}\left((1-p)^2\right)^y}_{\underbrace{\frac{1}{1-(1-p)^2}}_{\frac{1}{p(2-p)}}} \underbrace{\sum_{x'\in\mathbb{N}_0}(1-p)^{x'}}_{\underbrace{\frac{1}{1-(1-p)}}_{\frac{1}{p}}} \\
&= \frac{1-p}{2-p}
\end{aligned}$$

In fact,

$$\lim_{p\to 0}\mathbb{P}(X > Y) = \frac{1}{2}$$

What is $\mathbb{P}(X = Y)$? Let $z = \mathbb{1}_{x=y}$. Then

$$\begin{aligned}
\mathbb{P}(X = Y) &= \mathrm{E}[z] \\
&= \sum\sum \mathbb{1}_{x=y}\mathbb{P}_{X,Y}(x,y) \\
&= \sum_{y\in\mathbb{N}_0} p(1-p)^y \underbrace{\sum_{x=y}^{y} p(1-p)^x}_{\text{one element}} \\
&= p^2 \sum_{y\in\mathbb{N}_0}(1-p)^{2y} \\
&= p^2 \frac{1}{p(2-p)} \\
&= \frac{p}{2-p}
\end{aligned}$$

Let $X, Y \overset{iid}{\sim} \text{Binom}(n, p)$.Then

$$\mathbb{P}(X > Y) = \sum_{y \in \mathbb{N}_0} \mathbb{P}(Y = y)(1 - F_X(y))$$

But $F_X(y)$ has no closed form.

A basket has apples and bananas. Let p_1 = probability of getting apples and p_2 = probability of getting bananas. It is true that $p_2 = 1 - p_1$. Furthermore, $p_1 \in (0, 1)$. Represent apples as x_1. Then bananas can be represented as $x_2 = n - x_1$ where n is the total number of fruits in the basket. A vector can be created that represents this:

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let's add cantaloupes to the basket. p_3 = probability of getting cantaloupes. Now, the parameter space is such that $p_1 + p_2 + p_3 = 1$ and $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

What's $\mathbb{P}(\vec{X} = \vec{x})$?

$$\mathbb{P}_{\vec{X}}(x_1, x_2, x_3) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1 + x_2 + x_3 = n}$$

where the factorials term can be simplified to $\binom{n}{x_1, x_2, x_3}$.

In general,

$$\vec{X} \sim \text{Multinorm}(n, \vec{p}) := \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdot \cdots \cdot p_k^{x_k} \mathbb{1}_{\sum x_i = n}$$

such that $\binom{n}{x_1, x_2, \ldots, x_k} = \frac{n!}{x_1! x_2! \ldots x_k!}$. Note that $\vec{X}$ is a multidimensional random variable of dim K and $\vec{p}$ is a multidimensional parameter of dim $K$where $n, x_i \in \mathbb{N}$ and $\sum x_1 \leq n$. This is the multidimensional generalization of the binomial distribution. Instead of two categories (successes and failures), there are k categories.

Let's go back to the basket problem. If $k = 3$, $n = 10$ and $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{8}$, $p_3 = \frac{5}{8}$, how many mays are there to have 3 apples, 3 bananas and 4 cantaloupes?

$$\mathbb{P}(\vec{X} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}) = \binom{10}{3, 3, 4} \left(\frac{1}{4}\right)^3 \left(\frac{1}{8}\right)^3 \left(\frac{5}{8}\right)^4$$

What are the parameter space of the multinormal distribution? $n \in \mathbb{N}$. $p \in (0, 1)^k$ or sets of all k-tuples such that $\vec{p} \cdot \vec{1} = 1$ where $\sum p_k = 1$.