Lec. 21

Nov. 30

Let X - be non negative r.v. with finite expectat° is consider $a > 0$ a condition

Consider the inequality

$a \mathbb{1}_{X \geq a} \leq X$ True because

if $X \geq a$ $a(1) \leq X \Rightarrow X \geq a$

$X \leq a$ $P(0) \leq X \Rightarrow X \geq a$ by assumpt°

$E[\mathbb{1}_{X \geq a}] \leq \mu$

$\Rightarrow a E[\mathbb{1}_{X \geq a}] \leq \mu \Rightarrow P(X \geq a) \leq \mu$

$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$

Tail bound

Markov's inequality $P(X \geq a) \leq 1$

* Let $a = a'\mu \Rightarrow P(X \geq a'\mu) \leq \frac{1}{a'}$

$\rightarrow$ Let h be a strictly monotonically increasing funct°

$h(a) \mathbb{1}_{h(x) \geq h(a)} \leq h(x)$

$\Rightarrow P(h(x) \geq h(a)) \leq \frac{E[h(x)]}{h(a)}$

$\Rightarrow P(X \geq a) \leq \frac{E[h(x)]}{h(x)}$

$h(x) = X^p$ s.t. $p > 1$

$P(X \geq a) \leq \frac{E[X^p]}{a^p}$

* Recall Quantile$(X, p) = F_X^{-1}(p)$ inv. CDF

$P(X \geq a) \leq \frac{\mu}{a}$

$\Rightarrow 1 - P(X \leq a) \leq \frac{\mu}{a} \Rightarrow 1 - F(a) \leq \frac{\mu}{a}$

let $a = F_X^{-1}(p) \Rightarrow 1 - F(F^{-1}(p)) \leq \frac{\mu}{F^{-1}(p)}$

$\Rightarrow 1 - p \leq \frac{\mu}{F_X(p)} \Rightarrow$ Quantile$(X, p) \leq \frac{\mu}{p}$

$\Rightarrow$ Med$(X) = 2\mu$

* Consider any r.v. X. $|X|$ is non negative

$P(|X| \geq a) \leq \frac{E(|X|)}{a}$

both fails

* Let X be any r.v. with finite μ, finite σ^2

let $Y = (X-\mu)^2$ note Y is non-negative

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2} = \frac{E[(X-\mu)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

$$\Rightarrow P((X-\mu)^2 \geq a^2) \leq \frac{\sigma^2}{a^2}$$

$$\Rightarrow P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ Chebyshev's inequality

let X be any r.v.

let $Y = e^{tX}$ Y is non-negative

$$P(Y \geq c) \leq \frac{E(Y)}{c}$$

$$\Rightarrow P(e^{tX} \geq c) \leq \frac{E[e^{tX}]}{c}$$

Note

$M_X(t) = E[e^{tX}]$

Moment g- funct$^{\underline{c}}$

let $c = e^{ta}$

$$\Rightarrow P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = \frac{M_X(t)}{e^{ta}}$$

if $t > 0$ $P(X \geq a) \leq e^{-ta} M_X(t)$

$\Rightarrow P(X \geq a) \leq \min_{t>0} \{ e^{-ta} M_X(t) \}$

if $t < 0$

$P(X \leq a) \leq e^{-ta} M_X(t)$ Chernoff's inequality

$\Rightarrow P(X \leq a) \leq \min_{t<0} \{ e^{-ta} M_X(t) \}$

application for each inequality

$\rightarrow X \sim Bin($

$P(X \geq \frac{3}{4} n)$ if n is large

$X \simeq N(\frac{1}{4} n, (\sqrt{\frac{3}{16} n})^2)$

$P(X \geq \frac{3}{a} n) = P\left(\frac{X - \frac{1}{4} n}{\sqrt{\frac{3}{16} n}} \geq \frac{\frac{3}{4} n - \frac{1}{4} n}{\sqrt{\frac{3}{16} n}}\right)$

$= P(Z > \frac{2}{\sqrt{n}} \sqrt{n})$

1) Markov's $P(X \geq \frac{3}{4}n) \leq \frac{\frac{1}{4}n}{\frac{3}{4}n} = \frac{1}{3}$

Chebyshev's $P(X \geq \frac{3}{4})$

$= P(X - \frac{1}{4}n \geq \frac{3n}{4} - \frac{1}{a})$

$= P(X - \frac{1}{4}n \geq \frac{1}{2}n) + P(\frac{1}{4}n - X \geq \frac{1}{2}n)$

$= P(X - \frac{1}{4}n > \frac{1}{2}n \text{ OR } \frac{1}{4}n - X \geq \frac{1}{2}n)$

$= P(|X - \frac{1}{4}n| \geq \frac{1}{2}n) \leq \frac{\frac{3}{16}n}{\frac{1}{4}n^2} = \frac{3}{4n}$

$X \sim Bin(n, p)$

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} = (1 - p + pe^t)^n$

Bin. Theor.

$X \sim Bin(n, \frac{1}{4}) \Rightarrow M_X(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n$

$P\left(X \geq \frac{3}{4}n\right) \leq \min_{t>0} \left\{ e^{-t\left(\frac{3}{4}n\right)} \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n \right\}$

$= \min_{t>0} \left\{ \left(\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t}\right)^n \right\}$

Aside

$\Rightarrow \cancel{n\left(\quad\right)^{n-1}} \left(-\frac{9}{16}e^{-\frac{3}{4}t} + \frac{1}{16}e^{\frac{1}{4}t}\right) = 0$

$\rightarrow e^{\frac{1}{4}t} = 9e^{-\frac{3}{4}t}$

$\Rightarrow \frac{1}{4}t = \ln(9) - \frac{3}{4}t$

$\Rightarrow t_{min} = \ln(9)$

Now go back to min formula

$= \left(\frac{3}{4}e^{-\frac{3}{4}\ln(9)} + \frac{1}{4}e^{\frac{1}{4}\ln(9)}\right)^n$

$$\rightarrow = \left(\frac{3}{4} 9^{-\frac{3}{4}} + \frac{1}{4} 9^{\frac{1}{4}}\right)^n$$

$$= \frac{\sqrt[4]{9}}{4^n}\left(\frac{3}{9^3}+1\right)^n = \sqrt[4]{9}\left(\frac{1.004}{4}\right)^n \to 0$$

$\rightarrow$ Exponentially fast

Consider any 2 r.v. X, Y with finite μ_s, σ_s^2.

Let $W = (X - cY)^2$, $c \in \mathbb{R}$. W is non negative

$\Rightarrow E(W) \geq 0$

$\Rightarrow E(X - cY)^2 \geq 0 \Rightarrow E(X^2 - 2cXY + c^2Y^2) \geq 0$

$\Rightarrow E(X^2) - 2c\,E[XY] + c^2 E[Y^2] \geq 0$

Pick $c = \frac{E[XY]}{E[X^2]}$

$\Rightarrow E[W] = E[X^2] - 2\frac{E[XY]}{E[Y^2]}E[XY] + \frac{E[XY]^2}{E(Y^2)^2}E[Y^2]$

$\Rightarrow E(X^2)E(Y^2) - 2E[XY]^2 + E(XY)^2 \geq 0$

$\Rightarrow E[XY]^2 \leq E[X^2]E[Y^2]$

$\Rightarrow |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$

Cauchy - schwartz Inequality

What is the conclusion

$\text{Corr}[X,Y] = \text{Corr}[cY,Y] = \frac{\text{Corr}[cY,Y]}{SE(cY)SE(Y)}$

$= \frac{c\,\text{Corr}[Y,Y]}{|c|\,SE(X)^2} = \frac{c}{|c|} = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$