9/14/17 p148 X, Y are continuous r.v.'s with jdf (joint density function)

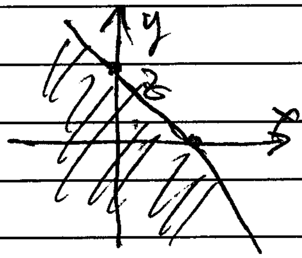
$f_{X,Y}(x,y)$ and $Z = g(X,Y)$

$$F_Z(z) = P(Z \le z) = P(g(X,Y) \le z)$$

$$= \iint_{\{(x,y):\, g(x,y) \le z\}} f_{X,Y}(x,y)\,dx\,dy = \cdots = \int_{-\infty}^{z} \underbrace{f_Z(t)}_{\text{must be the PDF of } Z}\,dt$$

$T = X + Y$

$$F_T(t) = \iint_{\{(x,y):\, x+y \le z\}} f_{X,Y}(x,y)\,dx\,dy \qquad y \le z - x \qquad = \int_R \left(\int_{\overset{-\infty}{y_l}}^{\overset{y_u}{z-x}} f_{X,Y}(x,y)\,dy \right) dx$$



Let $t = x + y$

$\Rightarrow y = t - x$

$y_l = -\infty = t_l - x \Rightarrow t_l = -\infty$

$y_u = z - x = t_u - x \Rightarrow t_u = z$

$$= \int \left(\int_{-\infty}^{z} f_{X,Y}(x, t-x)\,dt \right) dx$$

$$= \int_{-\infty}^{z} \underline{\left(\int_R f_{X,Y}(x, t-x)\,dx \right)}\,dt \quad \text{convolution}$$

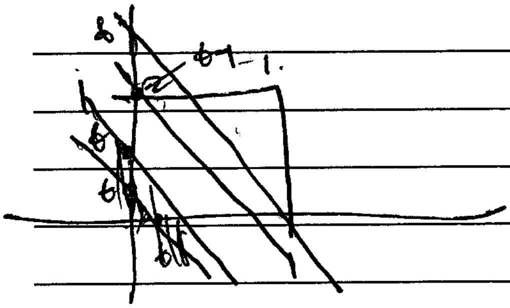
$$f_T(t) = f_X(x) * f_Y(x) = (f_X * f_Y)(x)$$

If X, Y are independent $\Rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$f_T(t) = f_X(x) * f_Y(x) = \int_{\mathbb{R}} f_X(x) f_Y(t-x)\, dx.$$

true of $\mathbb{1}$ for supports, inside the PDF is,

$$= \int_{\mathrm{supp}(X)} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in \mathrm{supp}(Y)}\, dx.$$



$X, Y \overset{iid}{\sim} U(0,1)$

$T = X + Y \sim f_T(t)$

$f_{(X,Y)}(x,y) = 1 \cdot \mathbb{1}_{x \in [0,1] \text{ and } y \in [0,1]}.$

$$F_T(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \in [0,1] \\ \frac{1}{2} + \left(\frac{1}{2} - \frac{(2-t)^2}{2}\right) & \text{if } t \in [1,2] \end{cases}$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in [1,2]. \end{cases}$$

$X_1, X_2 \overset{iid}{\sim} U(a,b) = \frac{1}{b-a}$

$T = X_1 + X_2$

$\mathrm{Supp}(X_2) = \mathrm{Supp}(X_1) = [a,b]$

$\mathrm{Supp}(T) = [2a, 2b]$

$$f_T(t) = \int f_{X_1}(x)\, f_{X_2}(t-x)\, \mathbb{1}_{t-x \in \operatorname{supp}[X_2]}\, dx$$

$$= \int_a^b \left(\frac{1}{b-a}\right)^2 \underbrace{\mathbb{1}_{t-x \in [a,b]}}_{\mathbb{1}_{x \in [t-b,\, t-a]}} dx$$

$$= \frac{1}{(b-a)^2} \int_{\max\{a,\, t-b\}}^{\min\{b,\, t-a\}} dx = \frac{1}{(b-a)^2}\left(\min\{b, t-a\} - \max\{a, t-b\}\right).$$

$$f_T(t) = \frac{\mathbb{1}_{t \in [2a, 2b]}}{(b-a)^2} \begin{cases} t - 2a & \text{if } t < a+b \\ 2b - t & \text{if } t \geq a+b \end{cases}$$

$X \sim \text{Geom}(p) = (1-p)^x p$

$F(x) = P(X \leq x) = 1 - P(X > x) = 1 - (1-p)^x \Rightarrow 1 - F(x) = (1-p)^x$

$\longrightarrow t$

I put n experiments in each time per

$X = nt$

$p(t) = (1-p)^{nt} p$ if $n \to \infty$, $p \to 0$ but $\lambda = np$

$\Rightarrow p(t) = \left(1 - \frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n}$

$\lim_{n\to\infty} p(t) = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt}\lim_{n\to\infty}\frac{\lambda}{n} = 0$ $\quad \forall t \; p(t) = 0$

$\Rightarrow$ PMF doesn't exist

$F_n(t) = 1-\left(1-\frac{\lambda}{n}\right)^{nt}$

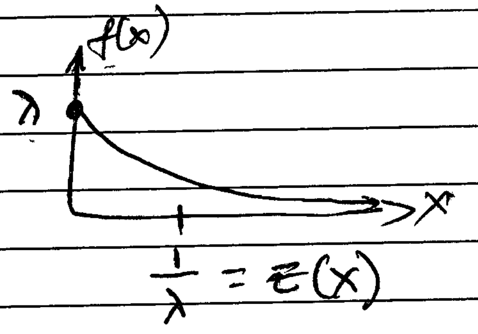
$F(t) = \lim\left(1-\left(1-\frac{\lambda}{n}\right)^{nt}\right) = 1-e^{-\lambda t} \Rightarrow 1-F(t) = e^{-\lambda t}$

$f(t) = F'(t) = \lambda e^{-\lambda t}$

$\boxed{X \sim Exp(\lambda) = \lambda e^{-\lambda x}}$ $\quad supp(X) = (0,\infty)$

$\lambda \in (0,\infty)$

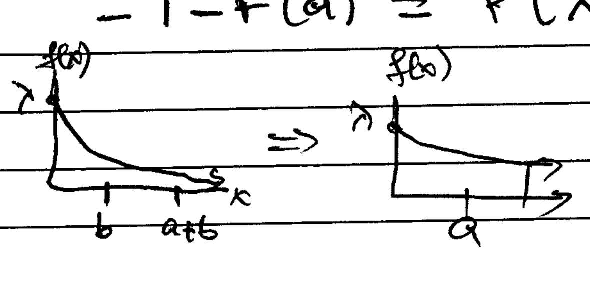
exponential r.v.'s.



$P(X>a+b \mid X>b) = \frac{P(X>a+b \,\&\, X>b)}{P(X>b)} = \frac{P(X>a+b)}{P(X>b)}$

$= \frac{1-F(a+b)}{1-F(b)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}\cdot \cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}} = e^{-\lambda a}$

$= 1-F(a) = P(X>a)$ "memorylessness"



$X_1, X_2, \dots \overset{iid}{\sim} Exp(\lambda)$

$$T_2 = X_1 + X_2 \sim f_T(t) = \int_{Supp(X_1)} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in (0,\infty)} dx$$

$Supp(T_2) = (0,\infty)$

$$= \int_0^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^{\infty} \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = \lambda^2 e^{-\lambda t} t$$

$$T_3 = X_3 + T_2 \sim \int_{Supp[X_3]} f_{X_3}(x) f_{T_2}(t-x) \mathbb{1}_{t-x \in Supp[T_2]} dx$$

$$= \int_0^{\infty} \lambda e^{-\lambda x} \lambda^2 e^{-\lambda(t-x)} (t-x) \mathbb{1}_{t-x \in (0,\infty)} dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t (t-x) dx = \lambda^3 e^{-\lambda t} \Big(t \underbrace{\int_0^t dx}_{t^2} - \underbrace{\int_0^t x\, dx}_{t^2/2} \Big)$$

$$= \lambda^3 e^{-\lambda t} \frac{t^2}{2}$$

$T_4 = X_4 + T_3$

$\sim \int_{supp(X_1)} f_{X_4}(x) \, f_{T_3}(t-x) \, \mathbb{1}_{t-x \in supp(T_3)} \, dx$

$= \int_0^\infty \lambda e^{-\lambda x} \lambda^3 e^{-\lambda(t-x)} \frac{(t-x)^2}{2} \mathbb{1}_{t-x \in (0,\infty)} \, dx$

$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t (t-x)^2 \, dx \to t^2 \int_0^t dx - 2t \int_0^t x \, dx + \int_0^t x^2 \, dx$

$\sim t^3 - \cancel{\frac{2t^3}{2}} + \int_0^t \frac{t^3}{3}$

$= \lambda^4 e^{-\lambda t} \frac{t^3}{3 \cdot 2}$

$T_k = X_1 + \cdots + X_k \sim \underbrace{\frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!}}_{f_T(t)} = Erlang(k, \lambda)$

$X \sim Erlang(k, \lambda) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}$

$supp(X) = (0, \infty)$

$\lambda \in (0, \infty)$

$k \in \mathbb{N}$.

$F_X(x) = \int_0^{x} \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} dy = \frac{1}{(k-1)!} \int_{0 \to y_L}^{x = y_u} \lambda (\lambda y)^{k-1} e^{-\lambda y} dy$

let $u = \lambda y \Rightarrow \frac{du}{dy} = \lambda \Rightarrow dy = \frac{du}{\lambda}$

$y_\ell = 0 \Rightarrow u_\ell = 0$

$y_u = x \Rightarrow u_x = \lambda x.$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \underbrace{\int_0^a t^{x-1} e^{-t} dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} dt}_{\Gamma(x,a)}$$

lower incomplete gamma function

upper incomplete gamma function