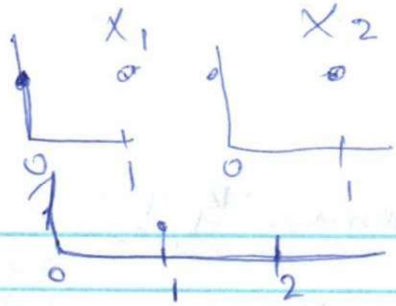

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(1/2)$

$T_2 = X_1 + X_2$

Convolution of 2 PMFs: $X_1 + X_2 \sim p_{X_1}(x) * p_{X_2}(x) = \sum_{x\,\text{Supp}[X_1]} p_{X_1}(x)\, p_{X_2}(t-x)$

let $X_1, X_2 \overset{iid}{\sim} \text{Bin}(n, p)$

$$Y = X_1 + X_2 \sim p_{X_1}(x) * p_{X_2}(x) = \sum_{x\,\text{Supp}[X_1]} p_{X_1}(x)\, p_{X_2}(y-x)$$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \underbrace{\mathbb{1}_{x\in\{0,1,\dots,n\}}}_{\text{not need}} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x} \underbrace{\mathbb{1}_{y-x\in\{0,1,\dots\}}}_{\text{not needed}}$$

$$= \sum_{x\in\{0,\dots,n\}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x}$$

$$= p^y (1-p)^{2n-y} \overbrace{\sum_{x\in\{0,\dots,n\}} \binom{n}{x}\binom{n}{y-x}}^{\binom{2n}{y}}$$

Vandermonde's Identity

$$= \text{Binom}(2n, p)$$

Consider $B_1, B_2 \dots \overset{iid}{\sim} \text{Bern}(p)$

let $X = \min_t \{B_t = 1\} - 1$

$X \sim \text{Geometric} = (1-p)^x p$, $\text{Supp}[X] = \{0,1,2\dots\} = \mathbb{N}_0$

$P(X=0) = p$

$P(X=1) = (1-p)p$

$P(X=2) = (1-p)^2 p$

$p(x) = P(X=x) = (1-p)^x p$

$T_2 = X_1 + X_2 \sim p(t) = p_{X_1}(x) * p_{X_2}(x)$

$$= \sum_{x \, \text{sup}[X_1]} p_{X_1} p_{X_2}(t-x) =$$

$$= \sum_{x \in [0,1\dots]} (1-p)^x p (1-p)^{t-x} p \, \mathbb{1}_{t-x \in [0,1,\dots]}$$

$$= (1-p)^t p^2 \overbrace{\sum_{x \in \{0,1\dots\}} \mathbb{1}_{x \le t}}^{\sum_{x=0}^{t} 1 = t+1}$$

$$= (t+1)(1-p)^t p^2$$

support of $[T_2] = \{0, 1, \dots\}$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p(t) = p_{X_3}(x) * p_{T_2}(x)$$

$$= \sum_{x \in \text{supp}[X_3]} p_{X_3}(x)\, p_{T_2}(t-x)$$

$$= \sum_{x \in \{0,\dots\}} (1-p)^x p\,(t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$= (1-p)^t p^3 \sum_{x \in \{0,\dots\}} (t-x+1)\, \mathbb{1}_{x \le t}$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \le t} - \sum_{x \in \{0,1,\dots\}} x\, \mathbb{1}_{x \le t} \right)$$

$$= (1-p)^t p^3 \left((t+1) \underbrace{\sum_{x=0}^{t} 1}_{t+1} - \underbrace{\sum_{x=0}^{t} x}_{\frac{t(t+1)}{2}} \right)$$

$$= (1-p)^t p^3 \left(\frac{t^2+3t+2}{2} \right) = \binom{t+2}{2} (1-p)^t p^3$$

T_3 # of failure until 3 successes

$$\underbrace{\underline{0}\ \underline{1}\ \_\ \_\ \_\ \dots\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}}_{t+2}\ \underset{t+3}{\underline{1}}$$

$$\binom{t+2}{2} = \frac{(t+2)!}{2!\,t!} = \frac{(t+2)(t+1)}{2} = \frac{t^2+3t+2}{2}$$

(5b)

$T_2 \sim \text{NegBin}(2, p)$ "Negative Binomial"

$T_3 \sim \text{NegBin}(3, p)$

$$P(T_3 = t) = \binom{t+2}{2}(1-p)^t p^3$$

$$X \sim \text{Negative}(k, p)$$

$$= \binom{x+k-1}{k-1}(1-p)^x p^k$$

$X \sim \text{Bin}(n, p)$, support$[X] = \{0, 1, \dots n\}$

what if n is really big?

" " p " " small? $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

What is the PMF if $n \longrightarrow \infty$?

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim \frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \underbrace{\lim \frac{n!}{(n-x)!\,x^n}}_{1} \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^{-x}}_{1} = \frac{\lambda^x}{x!}e^{-\lambda}$$

$$X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\text{support}[x] = \{0, 1, \ldots\}$$

$$\lambda \in (0, \infty)$$

let $X_1, X_2 \overset{iid}{\sim} \text{poisson}(\lambda)$

$$T = X_1 + X_2 = p_{X_1}(x) * p_{X_2}(x) = \sum_{x \in \text{sup}[X_1]} p_{X_1}(x)\, p_{X_2}(t-x)$$

$$= \sum_{x \in [0, \ldots]} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{\substack{t-x \in [0, \ldots] \\ x \le t}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in [0, \ldots]} \frac{1}{x!(t-x)!} \mathbb{1}_{x \le t} \cdot \frac{t!}{t!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in [0, \ldots]} \binom{t}{x} \mathbb{1}_{x \le t}$$

$$= \frac{\lambda^t e^{-2t}}{t!} \sum_{x=0}^{t} \binom{t}{x} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} =$$

Poisson(2λ)