10/24/17 Lecture # 13

$Y = \frac{X_1}{X_1+X_2} \sim ?$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$ ind of $X_2 \sim \text{Gamma}(\beta, \lambda)$

$\text{Supp}[X_1] = (0,\infty)$
$\text{Supp}[X_2] = (0,\infty)$

$\text{Supp}[Y] = ?$

if $x_1 \approx 0$ $X_2 \approx \infty \Rightarrow Y \approx 0$
$x_1 \approx \infty$ $X_2 \approx 0 \Rightarrow Y \approx 1$
$x_1 \approx X_2 \Rightarrow Y \approx 1/2$

So $\text{Supp}[Y] = (0,1)$

$$Y = \frac{X_1}{X_1+X_2} \sim \int_0^\infty \underbrace{f_{X_1}(yy_2)}_{\text{Gamma}} \underbrace{f_{X_2}(\overset{y_2(1-y_2)}{y_2 - yy_2})}_{\text{Gamma}} \overset{=|y_2|}{y_2}\, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (yy_2)^{\alpha-1} e^{-\lambda y y_2}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta (y_2(1-y))^{\beta-1} \cdot e^{-\lambda y_2(1-y)}}{\Gamma(\beta)} \cdot y_2\, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} \underbrace{e^{-\lambda y y_2 - \lambda y_2(1-y)}}\, dy_2$$

$$e^{-\lambda(yy_2 + y_2(1-y))} = e^{-\lambda(\cancel{yy_2} + y_2 - \cancel{yy_2})} = e^{-\lambda y_2}$$

Let $u = \lambda y_2 \Rightarrow y_2 = \frac{u}{\lambda} = \frac{1}{\lambda} u$

$\frac{du}{dy_2} = \lambda \Rightarrow dy_2 = \frac{1}{\lambda} du$

$$Y = \frac{\lambda^{\alpha+\beta} y^{\alpha-1}(1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{u^{\alpha+\beta-1}}{\lambda^{\alpha+\beta-1}} \cdot e^{-u} \cdot \frac{1}{\lambda}\, du$$

$$= \frac{\cancel{\lambda^{\alpha+\beta}} y^{\alpha-1}(1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \frac{u^{\alpha+\beta-1}}{\cancel{\lambda^{\alpha+\beta}}} e^{-u}\, du$$

$$= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^\infty u^{\alpha+\beta-1} e^{-u}\, dy}_{\Gamma(\alpha+\beta)} = \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\frac{1}{B(\alpha,\beta)}} \cdot y^{\alpha-1}(1-y)^{\beta-1}$$

$= \text{Beta}(\alpha, \beta)$

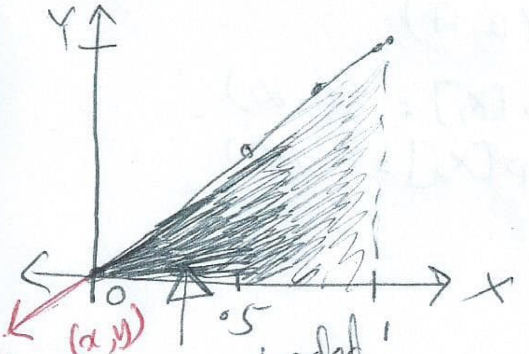
p.155
Ex: 8

# Conditional Densities.

Let $X \sim U(0,1)$. Let $Y|X = X \sim U(0, x)$ *Y be a uniformly distributed r.v. over $(0, X)$*

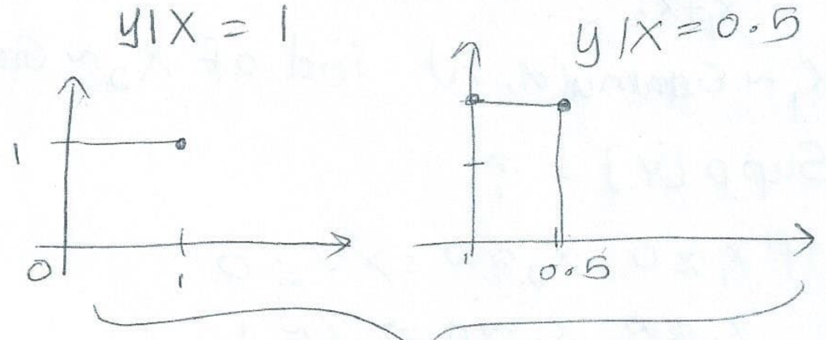
Find the joint density of $X$ & $Y$ and the marginal dens. of $Y$.

$Y \sim ?$




$f_{X,Y}(x,y)$ more shaded

Imagine this as the z axis.




conditional densities.

$$Supp[Y] = (0,1)$$

$$f_Y(y) = \int_{Supp[X]} f_{(X,Y)}(x,y)\,dx = \int_{Supp[X]} f_{Y|X}(y,x) f_X(x)\,dx$$

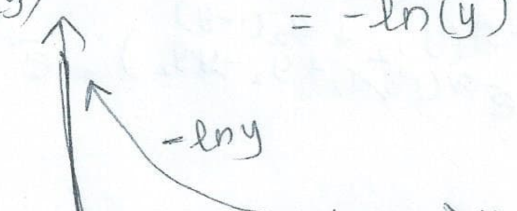
For the problem

$$f_Y(y) = \int_0^1 f_{Y|X}(y,x) \cdot f_X(x)\,dx = \int_0^1 \frac{1}{x} \underbrace{\mathbb{1}_{y\in(0,x)}}_{\substack{0 \le y \le x \\ x \ge y}} \cdot \mathbb{1}_{x\in(0,1)}\,dx$$

$$= \int_y^1 \frac{1}{x}\,dx = \ln(x)\Big]_y^1$$

$$= -\ln(y)$$

$f_Y(y)$




$$Y|X = 0.001 \sim U(0, 0.001)$$
$$= 1000\,\mathbb{1}_{x\in[0,.001]}$$

$* \; 0 \ln(0) = 0$

$$* \int_0^1 \ln(y)\,dy = [y\ln y - y]_0^1 = y - y\ln(y)\Big]_0^1 = (1-0)-(0-0) = 1$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad 0 < f_Y(y) < \infty$$

$$f_{Y|X}(y|x) = \frac{f(y,x)}{f_X(x)} \Rightarrow f(y,x) = f_{Y|X}(y|x) \cdot f_X(x)$$

Example: A download is either exponential with 10 min with/no network traffic or exponential with 20 min w/network traffic. How long does the download take? Network traffic is 2/3 of the time.

$$Y \sim \begin{cases} Exp\left(\frac{1}{10}\right) & w.p.\ 1/3 \\ Exp\left(\frac{1}{20}\right) & w.p.\ 2/3 \end{cases}$$

$$X = \mathbb{1}_{\text{Network traffic}} \sim Bern(2/3) = \left(\tfrac{2}{3}\right)^{x}\left(\tfrac{1}{3}\right)^{1-x}$$

$$Y|X \sim Exp\left(\left(\tfrac{1}{20}\right)^{x}\left(\tfrac{1}{10}\right)^{1-x}\right) = \left(\tfrac{1}{20}\right)^{x}\left(\tfrac{1}{10}\right)^{1-x} e^{-\left(\frac{1}{20}\right)^{x}\left(\frac{1}{10}\right)^{1-x}}.$$

$$f_Y(y) = \sum_{x \in supp[X]} f_{Y|X}(y,x) \cdot P_X(x) = \sum_{x \in \{0,1\}} \left(\tfrac{1}{20}\right)^{x}\left(\tfrac{1}{10}\right)^{1-x} \cdot e^{\left(\frac{1}{20}\right)^{x}\left(\frac{1}{10}\right)^{1-x}} \cdot *$$

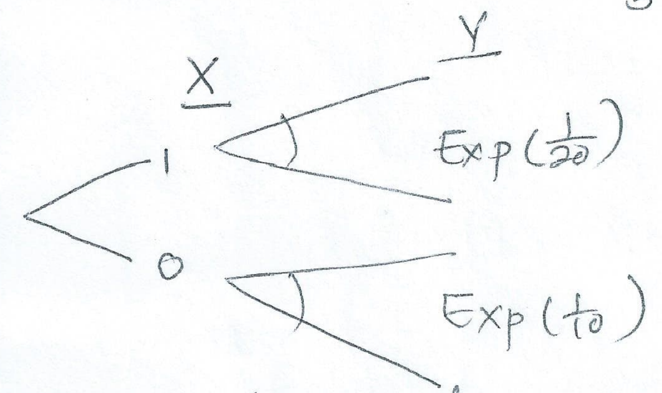
Marginal density of Y

Not $\int$ b/c it's a PMF

$$* = y\left(\tfrac{2}{3}\right)^{x}\left(\tfrac{1}{3}\right)^{1-x}$$

$$= \tfrac{1}{3}\left(\tfrac{1}{10} e^{-\frac{1}{10}y}\right) + \tfrac{2}{3}\left(\tfrac{1}{20} \cdot e^{-\frac{1}{20}y}\right)$$

$$= \tfrac{1}{3} Exp\left(\tfrac{1}{10}\right) + \tfrac{2}{3} Exp\left(\tfrac{1}{20}\right) \Rightarrow \text{Mixture Dist.}$$




If the download took 25 min's what is the probability there was network traffic?

$$P_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y,x) \cdot P_X(x)}{f_Y(y)}$$

$$P_{X|Y}(1,25) = \frac{\overset{Exp(\cdot)}{f_{Y|X}}(25,1) \cdot P_X(1)}{f_Y(25)} = \frac{\frac{1}{20} e^{-\frac{1}{20}(25)} \cdot \frac{2}{3}}{\frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}(25)}\right) + \frac{2}{3}\left(\frac{1}{20} \cdot e^{-\frac{1}{20}(25)}\right)}$$

from mixture distr.

$$= \frac{\frac{2}{3}(29\cdot 68)}{\frac{1}{3}(1\cdot 22)+\frac{2}{3}(22\cdot 60)} = 98\% \neq \frac{2}{3}$$

$X$ $\quad$ $Y$

$< U(0,1) < U(0,x)$

Example 9: p-157 in the book

car accidents are poisson dist. But their risk parameter $\lambda$ is not the same for all drivers. $\lambda$ is gamma dist. of accidents.

$$X \sim \text{Gamma}(\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \qquad \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$Y|X = x \sim \text{Poisson}(x = \frac{e^{-x}x^{y}}{y!}$$

← here $\lambda = x$, $x = Y$

$X$ $\quad$ $Y$

$< \text{Gamma}(\alpha,\beta) < \text{Poisson}(x)$

Compound Dist.
(Mix: continuous)

$P_Y(y)$ = marginal dist. of accidents.

$\text{Supp}[Y] = \{0,1,2,\dots\}$

$$P_Y(y) = \int P_{Y|X}(y,x) f_X(x)\,dx$$

$\int$ - b/e $x$ is continuous

$$= \int_0^{\infty} \frac{e^{-x}x^{y}}{y!}\cdot\frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}\,dx = \frac{\beta^{\alpha}}{y!\,\Gamma(\alpha)}\int_0^{\infty} x^{y+\alpha-1} e^{-(\beta+1)x}\,dx$$

← looks like Gamma

Let $u = (\beta+1)x \Rightarrow x = \frac{1}{\beta+1}\cdot u$

$\frac{du}{dx} = \beta+1 \Rightarrow dx = \frac{1}{\beta+1}\,du$

$$= \frac{\beta^{\alpha}}{y!\,\Gamma(\alpha)}\int_0^{\infty}\frac{u^{y+\alpha-1}}{(\beta+1)^{y+\alpha-1}}\cdot e^{-u}\cdot\frac{1}{\beta+1}\,du \Rightarrow * \int_0^{\infty}\frac{u^{y+\alpha-1}\cdot e^{-u}}{(\beta+1)^{y+\alpha}}$$

$$= \frac{\beta^{\alpha}\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}} = \left(\frac{\beta}{\beta+1}\right)^{\alpha}\frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y}}$$

Let $k = \alpha$ $\quad \beta = \frac{\beta}{1+\beta} \Rightarrow 1-p = \frac{1}{1+\beta}$

$$P_{Y|X}(y,x) = \frac{\Gamma(y+k}{\Gamma(k)\Gamma(y-1)}\cdot p^{k}(1-p)^{y} = \text{Ext. NegBin}(k,p).$$

if $y,k \in \mathbb{N}$, $\binom{y+k-1}{k}p^{k}(1-p)^{y} = \text{NegBin}(k,p)$