Let X, Y be continuous random variables with jdf $f_{X,Y}(x,y)$. Let $Z = g(X,Y)$ .Then

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(g(X,Y) \le z) = \int_{-\infty}^{z} f_Z(t)\,dt = \iint_{\left\{(x,y):g(x,y)\le z\right\}} f_{X,Y}(x,y)\,dxdy$$

where $f_Z(t)$ is the pdf of Z.
Let $T = X + Y$. Then

$$\begin{aligned}
F_Z(z) &= \iint_{\left\{(x,y):x+y\le z\right\}} f_{X,Y}(x,y)\,dxdy \\
&= \int_{\mathbb{R}} \Big(\int_{\left\{y:y\le z-x\right\}} f_{X,Y}(x,y)\,dy \Big)\,dx \\
&= \int_{\mathbb{R}} \Big(\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy \Big)\,dx \\
&= \int_{\mathbb{R}} \int_{-\infty}^{z} f_{X,Y}(x,t-x)\,dt\,dx \\
&= \int_{-\infty}^{z} \Big(\underbrace{\int_{\mathbb{R}} f_{X,Y}(x,t-x)\,dx}_{f_T(t)} \Big)\,dt
\end{aligned}$$

The convolution of $f_X(x) \times f_Y(y)$ is sometime notated as $(f_X \times f_Y)(x)$.

If $X, Y \overset{iid}{\sim}$, the definition of convolution for independent random variables is as follows

$$f_T(t) = \int_{\mathbb{R}} f_X(x) f_Y(t-x)\,dx = \int_{\mathrm{Supp}[X]} f_X(x) f_Y(t-x) \mathbb{1}_{t-x\in \mathrm{Supp}[Y]}\,dx$$

Note that the indicator functions are included in both $f_X(x)$ and $f_Y(t-x)$.

Let $X, Y \overset{iid}{\sim} U(0,1)$ and $T = X + Y$. What's $f_T(t)$?

$$\begin{aligned}
f_T(t) &= \int_{\mathrm{Supp}[X]} f_X(x) f_Y(t-x) \mathbb{1}_{t-x\in \mathrm{Supp}[Y]}\,dx \\
&= 1 \cdot \mathbb{1}_{x\in[0,1] \text{ and } y\in[0,1]} \\
F_T(t) &= \iint_{\left\{(x,y):x+y\le t\right\}} f_{X,Y}(x,y)\,dxdy \\
&= \begin{cases} \frac{1}{2}t^2 & \text{if } t \in [0,1] \\ \frac{1}{2} + (\frac{1}{2} - \frac{1}{2}(2-t)^2) & \text{if } t \in [1,2] \end{cases}
\end{aligned}$$

If we integrate this function to get $f_T(t)$,

$$f_T(t) = F_T'(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in [1,2] \end{cases}$$

Let $X_1, X_2 \overset{iid}{\sim} U(a,b)$ and $T_2 = X_1 + X_2$. Supp$[T] = [2a, 2b]$.

$$\begin{aligned}
f_{T_2}(t) &= \int_{\text{Supp}[X_1]} f_{X_1}(x) f_{X_2}(t-x)\,dx \\
&= \int_a^b \frac{1}{b-a}\frac{1}{b-a}\mathbb{1}_{t-x\in[a,b]\to x\in[t-b,t-a]}\,dx \\
&= \frac{1}{(b-a)^2}\int_{\max\{a,t-b\}}^{\min\{b,t-a\}} 1\,dx \\
&= \frac{1}{(b-a)^2}\Big(\min\{b,t-a\} - \max\{a,t-b\}\Big) \\
f_{T_2}(t) &= \begin{cases} \frac{t-2a}{(b-a)^2} & \text{if } t < a+b \\ \frac{2b-t}{(b-a)^2} & \text{if } t \geq a+b \end{cases} \mathbb{1}_{t\in[2a,2b}
\end{aligned}$$

Recall that if $X \sim \text{Geom}(p) = (1-p)^x p$, then $F(x) = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x) = 1-(1-p)^x$. If n many geometric realizations occur within each time period, then $x = tn$ and so $p(t) = (1-p)^{tn}p$. If $n \to \infty$ and $p \to 0$ but $\lambda = np$,

$$\begin{aligned}
p(t) &= \Big(1 - \frac{\lambda}{n}\Big)^{tn}\frac{\lambda}{n} \\
\lim_{n\to\infty} p(t) &= \underbrace{\Big(\lim_{n\to 0}(1-\frac{\lambda}{n})^n\Big)^t}_{e^{-\lambda t}} \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} = 0
\end{aligned}$$

Once the support is no longer discrete, the PMF vanishes. But recall that

$$\begin{aligned}
F(x) &= 1-(1-p)^x \\
F_n(t) &= 1-(1-p)^{nt} \\
F_n(t) &= 1-(1-\frac{\lambda}{n})^{nt} \\
\lim_{n\to\infty} F_n(t) &= 1 - \Big(\lim_{n\to\infty}(1-\frac{\lambda}{n})^n\Big)^t = 1-e^{-\lambda t} \\
\mathbb{P}(X > x) &= 1 - F(t) = e^{-\lambda t} \\
f_T(t) &= \frac{d}{dt}F_T(t) = \lambda e^{-\lambda t}
\end{aligned}$$

Let $X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x}$ where Supp$[X] = (0,\infty)$. Parameter space: $\lambda = np$ and $\lambda \in (0,\infty)$. This distribution can be used as a basic model for waiting time or failure time or survival.

If $a, b \in \mathbb{R}^+$,

$$\begin{aligned}
\mathbb{P}(x > a+b \mid x > b) &= \frac{\mathbb{P}(x > a+b \text{ and } x > b)}{\mathbb{P}(x > b)} \\
&= \frac{\mathbb{P}(x > a+b)}{\mathbb{P}(x > b)} \\
&= \frac{e^{-(a+b)x}}{e^{-bx}} \\
&= e^{-ax} \\
&= 1 - F(a) \\
&= \mathbb{P}(x > a)
\end{aligned}$$

For a continuous random variable X,

$$\mathrm{E}[X] = \int_{\mathrm{Supp}[X]} x f(x)\, dx$$

For the exponential distribution,

$$\int_0^\infty x\lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx = \cdots = \frac{1}{\lambda}$$

Let $X_1, X_2, \ldots \overset{iid}{\sim} \mathrm{Exp}(\lambda)$. What's $T_2 = X_1 + X_2 \sim ?$ $\mathrm{Supp}[T_2] = (0, \infty)$.

$$\begin{aligned}
f_{T_2}(t) &= \int_{\mathrm{Supp}[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, dx \\
&= \int_0^\infty \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)} \cdot \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in (0,\infty) \to x \in (-\infty, t)}\, dx \\
&= \lambda^2 \int_0^\infty e^{-\lambda t} \mathbb{1}_{x \in (-\infty, t)}\, dx \\
&= \lambda^2 e^{-\lambda t} \int_0^\infty dt \\
&= \lambda^2 t e^{-\lambda t}
\end{aligned}$$

Let $T_3 = X_1 + X_2 + X_3 = X_3 + T_2$.

$$\begin{aligned}
f_{T_3}(t) &= \int_{\text{Supp}[X_1]} f_{X_1}(x) f_{T_2}(t-x)\,dx \\
&= \int_0^\infty \lambda e^{-\lambda x} \cdot \lambda^2 (t-x) e^{-\lambda(t-x)} \mathbb{1}_{t-x\in(0,\infty)}\,dx \\
&= \lambda^3 e^{-\lambda t} \int_0^\infty (t-x)\mathbb{1}_{t-xin(0,\infty)}\,dx \\
&= \lambda^3 e^{-\lambda t}\Big(t\int_0^\infty \mathbb{1}_{t-x\in(0,\infty)}\,dx - \int_0^\infty x\mathbb{1}_{t-x\in(0,\infty)}\,dx\Big) \\
&= \lambda^3 e^{-\lambda t}\Big(t\int_0^t dx - \int_0^t x\,dx\Big) \\
&= \lambda^3 e^{-\lambda t}(t^2 - \frac{t^2}{2}) \\
&= \frac{\lambda^3 t^2}{2} e^{-\lambda t}
\end{aligned}$$

One more time

$$\begin{aligned}
f_{T_4}(t) &= f_{X_4}(x) f_{T_3}(t) \\
&= \int_0^\infty \lambda e^{-\lambda x} \frac{\lambda^3 (t-x)^2}{2} e^{-\lambda(t-x)} \mathbb{1}_{t-x\in(0,\infty)}\,dx \\
&= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t (t-x)^2\,dx \\
&= \lambda^4 e^{-\lambda t} \frac{1}{3\cdot 2} t^3
\end{aligned}$$

Following this pattern, we get

$$f_{T_k}(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \text{Erlang}(k,\lambda)$$

Its parameter space is as follows: $\lambda \in (0,\infty)$, $k \in \mathbb{N}$. $\text{Supp}[X] = (0,\infty)$.

What's F_{T_k} of the Erlang distribution?

$$\begin{aligned}
F_{T_k} &= \int_0^x \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}\,dy \\
&= \frac{1}{(k-1)!} \int_0^x \lambda (\lambda y)^{k-1} e^{-\lambda y}\,dy \\
&\text{Let } u = \lambda y \to \frac{du}{dy} = \lambda \to dy = \frac{du}{\lambda} \\
&= \frac{1}{(k-1)!} \int_0^{\lambda x} u^{k-1} e^{-u}\,du \\
&= \frac{\gamma(k,\lambda x)}{(k-1)!}
\end{aligned}$$

The Gamma function is as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \underbrace{\int_0^a t^{x-1}e^{-t}\,dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1}e^{-t}\,dt}_{\Gamma(x,a)}$$