lecture3 9/5/17 (6b)

$X_1, X_2 \overset{iid}{\sim} Bern(p)$

$T = X_1 + X_2 \sim P_{X_1}(x) * P_{X_2}(x) = \sum_{x \in [0,1]} P_{X_1}(x) P_{X_2}(t-x)$

$P(2) = \underbrace{P_{X_1}(0) P_{X_2}(2)}_{\text{Not legal}} + P_{X_1}(1) P_{X_2}(1)$

$2^n \overset{?}{=} \sum_{i=0}^{n} \binom{n}{i}$

set A st $|A| = n \Rightarrow A = \{w_1, w_2 \cdots w_n\}$

$2^A := \{B ; B \subseteq A\}$

$= \{B : B \subseteq A \text{ and } |B| = 0\}$

$\cup \{B : B \subseteq A \text{ and } |B| = 1\}$

$\vdots$

$\cup \{B : B \subseteq A \text{ and } |B| = n\}$

$2^A = \bigcup_{i=0}^{n} \{B : B \subseteq A \text{ and } |B| = i\}$

$2^{|A|} = |2^A| = \sum_{i=0}^{n} |\{B : B \subseteq A \text{ and } |B| = i\}| = \sum \binom{n}{i}$

Recall: $E(x) = \sum_{x \in sup} x\, p(x)$ for discrete X

$$E[g(x)] = \sum_{x \in sup[x]} g(x)\, p(x)$$

Some Expectation does not Exist

let $Y = g(X) = \ln(X)$ such that $X = \text{Poisson}(\lambda)$

$$E[Y] = \sum_{x \in \{0,1..\}} \ln(x) \frac{e^{-\lambda}\lambda^x}{x!} = \underbrace{\ln(0)\, 1}_{\text{Does not exist}} + \cdots \sum_{x \in [1,..]} \ln(x) \frac{e^{-\lambda}\lambda^x}{x!}$$

let $Z = \mathbb{1}_A \sim \text{Ber}(p(A))$

$E[Z] = E[\mathbb{1}_A] = P(A)$

Geometric: $X, Y \overset{iid}{\sim} \text{Geom}(p) := (1-p)^x p$

$$F_x(x) = P(X \leq x) = 1 - p(X \geq x)$$
$$= 1 - p(X \geq x+1)$$
$$= 1 - (1-p)^{x+1}$$

$$1 - F_x(x) = P(X > x) = (1-p)^{x+1}$$

let $Z := \mathbb{1}_{X>Y}$

$P(X>Y) = E[Z]$

$$E[Z] = E\left[g(X,Y)\right] = \sum_{y \in \text{Sup}[Y]} \sum_{x \in \text{Sup}[X]} g(x,y)\, \underbrace{p_{X,Y}(x,y)}_{\text{JMF}}$$

$$E[Z] = \sum_{\cancel{x} \in \text{Sup}[\cancel{X}]} \sum_{\cancel{y} \in \text{Sup}[\cancel{Y}]} \mathbb{1}_{x>y}\, p_{X,Y}(x,y)$$

$$= \sum_{x \in [0,1\ldots]} \sum_{y \in [0,1\ldots]} \mathbb{1}_{x>y}\, p(1-p)^x p(1-p)^y$$

Note: $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$ due to independence

$$= p^2 \sum_{y \in [0,\ldots]} (1-p)^y \sum_{x \in \{0,1\ldots\}} (1-p)^x\, \mathbb{1}_{x>y}$$

$$= p^2 \sum_{y \in [0,1\ldots]} (1-p)^y \sum_{x=y+1}^{\infty} (1-p)^x.$$

let $x' = x-(y+1) = x-y-1$
$\Rightarrow x = x'+y+1$

$$= p^2 \sum_{y \in [0,\ldots]} (1-p)^y \sum_{x'=0}^{\infty} (1-p)^{x'+y+1}$$

Recall: if $r \in (0,1)$; $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ geometric fun

$$= p^2 \sum_{Y\in[0,1,\dots\}} (1-p)^{2y+1} \frac{1}{1-(1-p)} = p^2(1-p) \sum_{y\in[0,\dots]} (1-p)^{2y} =$$

$$= p(1-p) \sum_{y=0}^{\infty} \left((1-p)^2\right)^y$$

$$= p(1-p) \frac{1}{1-(1-p)^2} = \frac{1-p}{2-p}, \quad \lim_{p\to 0} \frac{1-p}{2-p} = \frac{1}{2}.$$

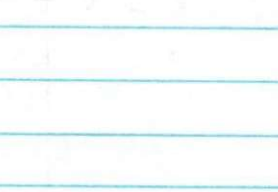
<u>Other method</u>

$$E(Z) = \sum_{X\in\{0,\dots\}} \sum_{Y\in[0,1,\dots]} \mathbb{1}_{X>Y}\, p(1-p)^X p(1-p)^Y$$

$$= \sum_{Y\in[0,1,\dots]} p(1-p)^Y \sum_{X\in[0,1,\dots\}} p(1-p)^X \mathbb{1}_{X>y}$$

$$= \sum_{Y\in[0,1,\dots]} p(1-p)^X \sum_{X=y+1}^{\infty} p(1-p)^X$$

$$= \sum_{Y\in[0,1,\dots]} P(Y=y)\, P(X\geq y+1)$$



$$Z := \mathbb{1}_{X=Y}$$

$$P(X=Y) = E[Z] = \sum_{x \in \mathrm{Sup}[X]} \sum_{y \in \mathrm{Sup}[Y]} \mathbb{1}_{x=y} P_{X,Y}(x,y)$$

$$= \sum_{y \in [0,1 \ldots]} \sum_{x \in [0,1 \ldots]} p(1-p)^x p(1-p)^y \mathbb{1}_{x=y}$$

$$= p^2 \sum_{y \in [0,1,\ldots]} (1-p)^y \sum_{x \in [0,1 \ldots]} (1-p)^x \mathbb{1}_{x=y}$$

$$= p^2 \sum_{y \in [0,1,\ldots]} (1-p)^y \sum_{x \in [0,1,\ldots]} (1-p)^x \mathbb{1}_{x=y}$$

$$= p^2 \sum_{y \in [0,1 \ldots]} \left((1-p)^2\right)^y = \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}$$

$$1 = P(X>Y) + P(X<Y) + P(X=Y)$$

$$= \frac{1-p}{2-p} + \frac{1-p}{2-p} + \frac{p}{2-p} = 1 \checkmark$$

$X, Y \overset{iid}{\sim} Bin(n, p)$

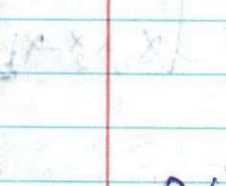
$$P(X>Y) = \sum_{Y \in [0,\dots,n]} P(Y=y)\left(1-F_X(y)\right)$$

↑ no closed form

$$P(X=Y) = \sum_{Y\in[0,\dots]} \sum_{X\in[0,\dots]} \mathbb{1}_{x=y} \underbrace{\binom{n}{x}p^x(1-p)^{n-x}}_{P_X(x)} \underbrace{\binom{n}{y}p^y(1-p)^{n-y}}_{P_Y(y)}$$

$$= \sum_{y=0}^{n} \binom{n}{y}^2 p^{2y}(1-p)^{2(n-y)}$$

$$P(X>y) = \frac{1-P(X=Y)}{2}$$



Basket of n apple and ban and Cantilopes

P_1 = proportion of apples

P_2 = proportion of Bannanas

P_3 is the proportion of cantilopes

in this cases $P_2 = 1-P_1$

X_1 be the number of apple drawn with replacement n times

$\cancel{X_1 \sim Bin(n, p_1)}$

X_2 = # Bannas drawn

X_3 = # of Cantilopes drawn

let $\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$$P(\vec{X} = \vec{x}) = \overbrace{P(X_1 = x_1, X_2 = x_2, X_3 = x_3)}^{\text{JMF}}$$

Note: $P(X_1=x_1) = \binom{n}{x_1} p^{x}(1-p)^{n-x_1} \mathbb{1}_{x\in\{0,\dots\}}$ $\binom{n}{x_1} = \frac{n!}{x_1!(n-x_1)!}$

$$P(X_1=x_1, X_2=\cancel{x}_2) = \frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2}$$

$$P(\vec{X}=\vec{x}) = \underbrace{\frac{n!}{x_1!\,x_2!\,x_3!}}_{\text{multichoose}} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n}$$

$\vec{X} \sim \text{Multinomial}(n, \vec{p}) = p(\vec{x}) = \binom{n}{x_1, x_2 \dots x_k} p_1^{x_1} \cdots p_k^{x_k}$ of dimension k

$$= p(\vec{x}) = \binom{n}{x_1, x_2 \dots x_k} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

$$\text{Support}[\vec{X}] = \{\vec{x} : \vec{1}\cdot\vec{x} = n, \vec{x} \in \mathbb{N}^k\}$$

$$= \{\vec{x} : \sum_{i=1}^{k} x_i = n, x_1, x_2, \dots x_k \in \mathbb{N}_0\}$$

Parameter Space

$$n \in \mathbb{N}, \vec{p} \in \{\vec{p} : \vec{p}\cdot\vec{1} = 1, \vec{p} \in (0,1)^k\}$$