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• $Y = \dfrac{X_1}{X_1+X_2} \sim ?$ supp of Gamma $(0,\infty)$

$X_1 \sim \text{Gamma}(\alpha, \lambda)$ ind. of

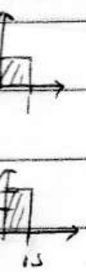
$X_2 \sim \text{Gamma}(k, \lambda)$

supp(Y) = [0, 1]

b/c if $X_1 \approx 0,\ X_2 \approx \infty \Rightarrow Y \approx 0$

$X_1 \approx \infty,\ X_2 \approx 0 \Rightarrow Y \approx 1$

$X_1 \approx X_2 \Rightarrow Y \approx \frac{1}{2}$



$$\text{So, } Y = \frac{X_1}{X_1+X_2} \sim \int_0^\infty f_{X_1}(y y_2)\, f_{X_2}(\underset{\underset{y_2(1-y)}{\shortparallel}}{y_2 - y y_2})\, \underset{\underset{|y_2|}{\shortparallel}}{y_2}\, dy_2$$

$$= \int_0^\infty \frac{\lambda^\alpha (y y_2)^{\alpha-1} e^{-\lambda y y_2}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta (y_2(1-y))^{\beta-1} e^{-\lambda y_2(1-y)}}{\Gamma(\beta)} \cdot y_2^1\, dy_2$$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_2^{\alpha+\beta-1} \underbrace{e^{-\lambda y y_2 - \lambda y_2(1-y)}}_{\substack{e^{-\lambda(y y_2 + y_2(1-y))} \\ e^{-\lambda y_2(y+1-y)} \\ e^{-\lambda y_2}}} dy_2$$

~~Looks simpler but there are a lot of things going on~~

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$$\int_0^\infty u^{\alpha+\beta-1} e^{-u}\, du = \Gamma(\alpha+\beta)$$

$u = \lambda y_2 \Rightarrow \frac{du}{dy_2} = \lambda$

$\Rightarrow y_2 = \frac{1}{\lambda} u \quad \Rightarrow dy_2 = \frac{1}{\lambda} dy$

$$= \frac{\lambda^{\alpha+\beta} y^{\alpha-1} (1-y)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^\infty \frac{u^{\alpha+\beta-1}}{\lambda^{\alpha+\beta-1}} e^{-u} \frac{1}{\lambda}\, du}_{\lambda^{\alpha+\beta}}$$

$$= \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\frac{1}{\beta(\alpha,\beta)}} y^{\alpha-1}(1-y)^{\beta-1} = Beta(\alpha,\beta)$$

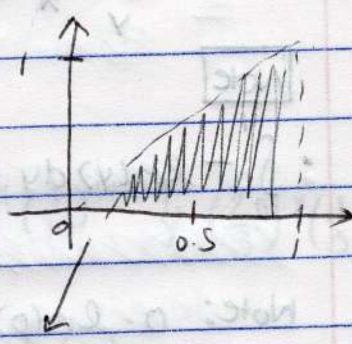
- Next unit (pg 155)

Conditional Densities

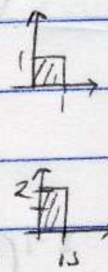
let $X \sim U(0,1)$

let $Y | X = x \sim U(0, x)$

$Y \sim ?$



$f_{X,Y}(x,y)$



slice

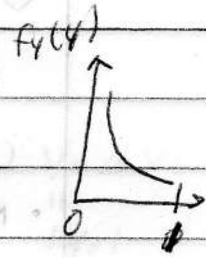
$supp[Y] = [0,1]$

· $f_Y(y) = \int_{supp(X)} f_{X,Y}(x,y)\,dx = \int_{supp(Y)} f_{Y|X}(y,x)\, f_X(x)\,dx$

· $f_Y(y) = \int_0^1 f_{Y|X}(y,x)\, f_X(x)\,dx$

$= \int_0^1 \frac{1}{x}\, \underbrace{\mathbb{1}_{y\in[0,x)}}_{\substack{0 \le y \le x \\ \Rightarrow y \le x}}\, 1\, \mathbb{1}_{x\in[0,1]}\, dx$

$= \int_y^1 \frac{1}{x}\,dx = \ln(x) \Big|_y^1 = -\ln(y)$



$\boxed{\text{Note}}$

· $\int_0^1 -\ln(y)\,dy = [y\ln(y) - y]_0^1 = y - y\ln(y) \Big]_0^1 = (1-0) - (0.0) = 1$ ✓

Note: $0 \cdot \ln(0) = 0$

• A download is either exponential with 10min w/ wo ~~network~~ network traffic, or exponential w/ 20min w/ network traffic.

How long does the download take?

Network traffic is $\frac{2}{3}$ of the time.

$$Y \sim \begin{cases} Exp(\frac{1}{10}) & wp\ \frac{1}{3} \\ Exp(\frac{1}{20}) & wp\ \frac{2}{3} \end{cases} \qquad X = \mathbb{1}_{\text{network traffic}} \sim Bern(\tfrac{2}{3}) = (\tfrac{2}{3})^x (\tfrac{1}{3})^{1-x}$$

$$Y|X \sim Exp\left((\tfrac{1}{20})^x (\tfrac{1}{10})^{1-x}\right)$$

$$= (\tfrac{1}{20})^x (\tfrac{1}{10})^{1-x} e^{-(\frac{1}{20})^x (\frac{1}{10})^{1-x} y}$$

Note: In prob, the sum & the integral are the same

$$f_Y(y) = \sum_{x \in supp(X)} f_{Y|X}(y,x)\, p_X(x)$$

$$= \sum_{x \in \{0,1\}} (\tfrac{1}{20})^x (\tfrac{1}{10})^{1-x} e^{-(\frac{1}{20})^x (\frac{1}{10})^{1-x} y} (\tfrac{2}{3})^x (\tfrac{1}{3})^{1-x}$$

$$= \frac{1}{10} e^{-\frac{1}{10}y} (\tfrac{1}{3}) + \frac{1}{20} e^{-\frac{1}{20}y} (\tfrac{2}{3})$$

$$= \frac{1}{3} Exp(\tfrac{1}{10}) + \frac{2}{3} Exp(\tfrac{1}{20})$$

• If its download week 25min,
What is its prob there was network traffic?

$$\frac{\text{Joint}}{\text{marginal}} \quad P_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y,x)\,P_X(x)}{f_Y(y)}$$

$$P_{X|Y}(1,25) = \frac{f_{Y|X}(25,1)\,P_X(1)}{f_Y(25)}$$

$$= \frac{\overbrace{\left(\frac{1}{20} e^{-\frac{1}{20}(25)} \cdot \frac{2}{3}\right)}^{\to\ 29.68}}{\underbrace{\frac{1}{3}\left(\frac{1}{10} e^{-\frac{1}{10}(25)}\right)}_{\hookrightarrow\ 1.22} + \underbrace{\frac{2}{3}\left(\frac{1}{20} e^{-\frac{1}{20}(25)}\right)}_{\hookrightarrow\ 21.68}}$$

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$$= 98\% \neq \frac{2}{3}$$

• Car Accidents are poison distribution ~~X~~ but λ their rate parameter is not the same for all drivers. λ is a gamma dist.

$\lambda \sim \text{Gamma}(\alpha, \beta)$.

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

$$Y \mid X = x \sim \text{Poisson}(x) = \frac{e^{-x} x^{y}}{y!}$$

here x is y.
& $\lambda \Rightarrow X$

X   Y

Gamma   Poisson   Poisson

X is cont so $\int$

What is the dist. of the accident?

$$P_Y(y) = \int_{supp(X)} p_{Y|X}(y,x) f_X(x)\,dx = \int_0^\infty \frac{e^{-x}x^y}{y!} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}\,dx$$

$supp(Y) = \{0,1,2,\dots\}$

$$= \frac{\beta^\alpha}{y!\,\Gamma(x)} \int_0^\infty x^{y-x+1} e^{-(\beta+1)x}\,dx$$

$\downarrow$ Gamma fn

$$\text{let } u = (\beta+1)x \Rightarrow x = \frac{1}{\beta+1}u$$

$$\frac{du}{dx} = \beta+1 \Rightarrow dx = \frac{1}{\beta+1}du$$

Compound Dist.

(Poisson + Gamma ⇒ NegBin)

$$\frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_0^\infty \frac{u^{y+\alpha-1}}{(\beta+1)^{y+\alpha-1}} e^{-u} \frac{1}{\beta+1}\,du$$

$(\beta+1)^{y+\alpha}$

$$= \frac{\beta^\alpha\,\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^{y+\alpha}} = \left(\frac{\beta}{\beta+1}\right)^\alpha \frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)(\beta+1)^y}$$

let $k=\alpha$, $p = \frac{\beta}{1+\beta} \Rightarrow 1-p = \frac{1}{1+\beta}$

$P_{Y|X}(y,x) = \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y-1)} p^k (1-p)^y$ = Exp NegBin(k,p)

if $y,k \in \mathbb{N}$, $\binom{y+k-1}{k} p^k (1-p)^y = Neg(k,p)$