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consider $\phi_X'(t) = \frac{d}{dt}\left(E(e^{itX})\right)$

$$= \frac{d}{dt}\left[\int_{\mathbb{R}} e^{itx} f(x)\,dx\right] \overset{?}{=} \int_{\mathbb{R}} f(x)\,\frac{d}{dt}\left(e^{itx}\right)dx$$

Does $\frac{d}{dt}\left(\int g(x,t)\,dx\right) \overset{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}\left[g(x,t)\right]dx$

conditions

a) $\exists\, t \in A$ s.t $\int_{\mathbb{R}} g(x,t)\,dx$ converges. $A = [a,b] \subseteq \mathbb{R}$

b) $g(x,t)$ continuous $\forall\, t \in A$

c) $g(x,t)$ continuous $\forall\, x \in \mathbb{R}$.

d) $\forall\, t \in A$ $\int_{\mathbb{R}} \frac{\partial}{\partial t} g(x,t)\,dt$ converges ~~contin~~ uniformly.

$$\phi_X'(t) = \int_{\mathbb{R}} f(x)\, ix\, e^{itx}\,dx$$

consider

$$\phi_X'(0) = \int_{\mathbb{R}} f(x)\, ix\,dx = i\int_{\mathbb{R}} x f(x)\,dx = iE(X)$$

$$\phi_X''(t) = \int_{\mathbb{R}} f(x)\, i^2x^2 e^{itx}\,dx$$

$$\phi_X''(0) = i^2\int_{\mathbb{R}} x^2 f(x)\,dx = i^2 E(X^2)$$

$$\phi_X'''(0) = i^3 E(X^3)$$

$$\Rightarrow \boxed{E(X^n) = \frac{\phi_X^{(n)}(0)}{i^n}}$$

Inversion the

⑥ $P(X \in (a,b)) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\,dt$

Motivation of $\phi_X \in L^1$

$\Rightarrow f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} \phi_X(t)\,dt$

$P(X \in (a,b)) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} \phi_X(t)\,dt\,dx.$

$= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_a^b e^{-itx}\,dx\right) \phi_X(t)\,dt$

$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\,dt.$

⑦ $\phi_X(t) = \phi_Y(t) \iff X \stackrel{d}{=} Y$

⑧ $\phi_{X_n}(t)$ is the ch. f for X_n

If $\forall t$ $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$

$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ also $\lim_{n\to\infty} X_n \stackrel{d}{=} X$

or $X_n \xrightarrow{d} X$ convergence in distribution/law.

$X \sim \text{Gamma}(k, \lambda)$ $\quad \phi_X(t) = \int_0^\infty e^{itx} \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)} dx$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{(it-\lambda)x} dx$$

~~let $u = (it-\lambda)x \Rightarrow x = \frac{1}{it-\lambda} t$~~

let $u = (\lambda - it)x \Rightarrow x = \frac{1}{\lambda - it} u \Rightarrow dx = \frac{1}{\lambda - it} du$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{u^{k-1}}{(\lambda - it)^{k-1}} e^{-u} \frac{1}{\lambda - it} du$$

$$= \frac{\lambda^k}{\Gamma(k)(\lambda - it)^k} \int_0^\infty u^{k-1} e^{-u} du = \frac{\lambda^k}{\cancel{\Gamma(k)}(\lambda - it)^k} \cdot \cancel{\Gamma(k)}$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^k = \left(1 - \frac{it}{\lambda}\right)^{-k}.$$

$X_1 \sim \text{Gamma}(k_1, \lambda)$ ind of $X_2 \sim \text{Gamma}(k_2, \lambda)$

$X_1 + X_2 \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k_1} \left(\frac{\lambda}{\lambda - it}\right)^{k_2}$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^{k_1 + k_2} \quad k_1+k_2 = k$$

$= \text{Gamma}(k_1 + k_2, \lambda)$.

~~PMF of Poisson~~ (λe^{it})

$X \sim$ Poisson (λ)

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x e^{-\lambda e^{it}}}{x!}$$

$$\frac{(e^{it})^x \lambda^x e^{-\lambda}}{x!} = \frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}} = e^{-\lambda + \lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$X_1 \sim$ Poisson (λ_1) ind of $X_2 \sim$ Poisson (λ_2)

$X_1 + X_2 \sim$ Poisson $(\lambda_1 + \lambda_2)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) = e^{\lambda_1(e^{it}-1)} \cdot e^{\lambda_2(e^{it}-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$X_1, \dots, X_n \overset{iid}{\sim}$ same distribution with finite mean μ & finite variance σ^2

$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad E(\bar{X}) = \mu, \quad Var(\bar{X}) = \frac{\sigma^2}{n}$

Define $\quad Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \qquad E(Z_n) = 0, \; Var(Z_n) = 1$

Standardization $\qquad\qquad\qquad\qquad EE(Z_n)$

$$\phi_{\bar{X}}(t) \overset{\#3}{=} \phi_{\sum X_i}\left(\frac{t}{n}\right) \overset{\#2}{=} \left(\phi_X\left(\frac{t}{n}\right)\right)^n$$

$$\phi_{Z_n}(t) \overset{\#3}{=} \phi_{\bar{X}_n}\left(\frac{t}{\frac{\sigma}{\sqrt{n}}}\right) e^{\imath t\left(\frac{-\mu}{\frac{\sigma}{\sqrt{n}}}\right)} = \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{\frac{-\imath t \mu \sqrt{n}}{\sigma} \cdot \frac{n}{n}}$$

$$= \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{\frac{-\imath t \mu n}{\sigma\sqrt{n}}} = \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n e^{\frac{-\imath t \mu n}{\sigma\sqrt{n}}}$$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \lim \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n e^{\frac{-\imath t \mu n}{\sigma\sqrt{n}}} = \cancel{\lim e^{\ln(\phi_X}}$$

$$= \lim e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n} e^{\frac{-\imath t \mu n}{\sigma\sqrt{n}}}$$

$$= \lim_{n\to\infty} e^{n \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{\imath t \mu n}{\sigma\sqrt{n}}}$$

$$= \lim e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{\imath t \mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{\imath t \mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{\imath t \mu}{\sigma\sqrt{n}}}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

let $u = \frac{t}{\sigma\sqrt{n}}$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{\imath t \mu}{\sigma\sqrt{n}}}{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}}$$

$n\to\infty \Rightarrow u \to 0$

$$= e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\phi_X(u)) - \imath \mu u}{u^2}}$$

$$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi'(u)}{\phi(u)} - i\mu}{u}} = e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right]}$$

Since $\lim_{u\to 0}\frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right] = \lim_{u\to 0}\frac{\phi''(u)\phi(u) - (\phi'(u))^2}{\phi(u)^2}$

$$= \frac{\phi''(0)\phi(0) - (\phi'(0))^2}{\phi(0)^2} = \frac{i^2 E(x^2)(1) - (i\mu)^2}{1}$$

$$= i^2\left(E(x^2) - \mu^2\right) = -\sigma^2$$

Now we have

$$\Rightarrow e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right]} = e^{\frac{t^2}{2\sigma^2}(-\sigma^2)} = e^{-\frac{t^2}{2}}$$

$$= \phi_Z(t)$$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-tx}\phi(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} e^{-\frac{t^2}{2}}\,dt.$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itx + \frac{t^2}{2}\right)}\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}ix}{2}\right)^2 + \frac{x^2}{2}\right)}\,dt$$

$$= \frac{1}{2\pi}e^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}ix}{2}\right)^2}\,dt = \frac{1}{2\pi}e^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-y^2}\sqrt{2}\,dy$$

$$= \frac{1}{\pi\sqrt{2}}e^{-\frac{x^2}{2}}\underbrace{\int_{\mathbb{R}} e^{-y^2}\,dy}_{\substack{\text{Gaussian Integral} \\ \| \\ \sqrt{\pi}}} = \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}_{\substack{\| \\ N(0,1) \\ \text{standard normal}}}$$

Let $\frac{\theta^2}{2} + \vartheta\theta x = \left(\frac{\theta}{\sqrt{2}} + \frac{\sqrt{2}\,\vartheta x}{2}\right)^2 + \frac{x^2}{2}$

$\frac{\cancel{2}\theta}{\sqrt{2}} \cdot \frac{\cancel{\sqrt{2}}\,\vartheta x}{\cancel{2}} = \vartheta\theta x \left(\frac{\sqrt{2}\,\vartheta x}{2}\right)^2 = -\frac{x^2}{2}$