Math 631 Lec 11 10/17/17

Waiting time for k $\exp(\lambda)$-timed events where k could be fractional

$X \sim \text{Gamma}(k_1, \lambda)$, $Y \sim \text{Gamma}(k_2, \lambda)$ X, Y ind.

WTS $X+Y \sim \text{Gamma}(k_1+k_2, \lambda)$ this is intuitive since we are now waiting for k_1+k_2 events

$$X+Y \sim f_X(x) * f_Y(y) = \dots = \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 t^{k_1-1} t^{k_2-1} u^{k_1-1} (1-u)^{k_2-1} t \, du$$

$=$ (next page)

$$= \frac{\lambda^{k_1+k_2} e^{\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du$$

PAUSE...

$X \sim Exp(\lambda) \overset{= f(x)}{:} \lambda e^{-\lambda x}$

$\int_{Supp(X)} f(x)\,dx = 1$ otherwise f is not a PDF

Note $f(x) = \lambda e^{-\lambda x} \underset{\text{direct proportional}}{\propto} \underbrace{e^{-\lambda x} = k(x)}_{\text{Kernel}}$

$k(x) = c\, f(x) \Rightarrow f(x) = \frac{1}{c} k(x)$

where C is not a func of x.

$1 = \int_{Supp(X)} f(x)\,dx = \int_{Supp(X)} \frac{1}{c} k(x)\,dx \Rightarrow c = \int_{Supp(X)} k(x)\,dx$

In this case, $\frac{1}{c} = \lambda \Rightarrow c = \frac{1}{\lambda}$ $\int e^{-\lambda x}dx = \frac{1}{\lambda}$

$k(x)$ can be restored to $f(x)$ by multiplying by $\frac{1}{c}$.

Who cares? $k(x)$ still identifies the PMF or PDF as the brand name r.v.

eg $X \sim Bin(n,p) = p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \propto \underbrace{(x!(n-x)!)^{-1} \left(\frac{p}{1-p}\right)^x}_{\text{Identifies the binomial!}} = k(x)$

$X \sim Weibull(k,\lambda) = f(x) = k\lambda (x\lambda)^{k-1} e^{-(x\lambda)^k} \propto \underbrace{x e^{-(x\lambda)^k}}_{\text{identifies a weibull}} = k(x)$

$X \sim Gamma(k,\lambda) = f(x) = \frac{\lambda^k e^{\lambda x} x^{k-1}}{\Gamma(k)} \propto e^{\lambda x} x^{k-1} = k(x)$

$$f_{X+Y}(t) = \frac{\lambda^{k_1+k_2} e^{\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\Gamma(k_2)} \underbrace{\int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du}_{h(k_1,k_2)} \propto e^{\lambda t} t^{k_1+k_2-1}$$

$h(k_1,k_2)$ some function ... but not of x!

$\propto \text{Gamma}(k_1+k_2, \lambda)$ ■ which completes the proof

As a corollary....

$$\Rightarrow f_{X+Y}(t) = \frac{\lambda^{k_1+k_2} e^{\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1+k_2)} \underset{\text{be}}{\overset{\text{which must}}{=}} \frac{\lambda^{k_1+k_2} e^{\lambda t} t^{k_1+k_2-1}}{\Gamma(k_1)\,\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du$$

$$\Rightarrow \frac{1}{\Gamma(k_1)\Gamma(k_2)} \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du = \frac{1}{\Gamma(k_1+k_2)}$$

$$\Rightarrow \int_0^1 u^{k_1-1}(1-u)^{k_2-1}\,du = \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)}$$

Note

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \Rightarrow B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\int_0^\infty t^{\alpha-1}e^{-t}\,dt \int_0^\infty t^{\beta-1}e^{-t}\,dt}{\int_0^\infty t^{\alpha+\beta-1}e^{-t}\,dt}$$

"Beta Function"

is a famous integral

just proved above

these two being the same is NOT obvious but prob. theory gave us this nice proof

Order statistics (p160)

$X_1, X_2, \dots, X_n$ are a sequence of cont. r.v.'s, then

$X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics

where $X_{(1)} := \min\{X_1, \dots, X_n\}$

$X_{(n)} := \max\{X_1, \dots, X_n\}$

$X_{(k)} := \{k^{th}$ largest of $X_1, \dots, X_n\}$

e.g.

$X_1 = 9 = X_{(3)}$

$X_2 = 2 = X_{(1)}$

$X_3 = 12 = X_{(4)}$

$X_4 = 7 = X_{(2)}$

let $R = X_{(n)} - X_{(1)}$ and R is called the "range" "max - min"

Under the assumption of iid of the $X_1, \dots, X_n$,

let's first derive the distn of the maximum

e.g.

$12 = \max\{2, 7, 9, 12\}$

$X_{(n)} = \max\{X_1, \dots, X_n\}$

↑

this means all X_i's are less than $X_{(1)}$!

$9 \le 12,\ 7 \le 12,\ 2 \le 12,\ 12 \le 12$

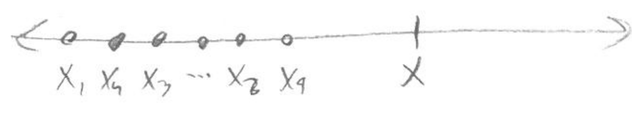
$$F_{X_{(n)}}(x) := P(X_{(n)} < x) = P(X_1 < x \ \&\ X_2 < x \ \&\ \dots \ \&\ X_n < x)$$

$$= \prod_{i=1}^{n} P(X_i < x) \quad \text{by indep.}$$

$$= P(X_1 < x)^n \quad \text{by iden. distr.}$$

$$= F(x)^n \quad \text{by def of CDF}$$

$$f_{X_{(n)}}(x) = F'_{X_{(n)}}(x) = n\, f(x)\, F(x)^{n-1}$$



$2 = \min\{2,7,9,12\}$

$X_{(1)} = \min\{X_1,\dots,X_n\}$

this means all X_i are greater than $X_{(1)}$! $\quad 7 \geq 2,\ 9 \geq 2,\ 12 \geq 2,\ 2 \geq 2$

$$F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - P(X_{(1)} \geq x)$$

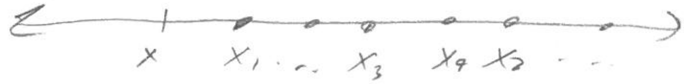
$$= 1 - P(X_1 \geq x \,\&\, X_2 \geq x \,\&\, \dots \,\&\, X_n \geq x)$$

$$= 1 - \prod_{i=1}^{n} P(X_i \geq x) \quad \text{by indep}$$

$$= 1 - P(X_i \geq x)^n \quad \text{by ident. distr}$$

$$= 1 - (1 - F(x))^n \quad \text{by def of CDF}$$

$$f_{X_{(1)}} = n(-f(x)) - (1-F(x))^{n-1} = n f(x)(1-F(x))^{n-1}$$



$X_{(k)}$: this is the k^{th} largest of $X_1, \dots X_n$. What's its distr? $F_{X_{(k)}}(x)$

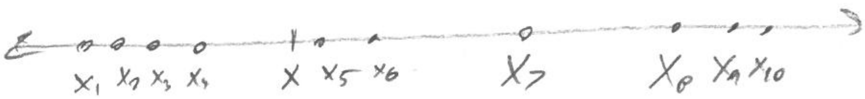
e.g 9 is the 3^{rd} largest of $\{2, 7, 9, 12\}$ $\quad X_{(3)} = 9$

this means it is larger than or equal to 3 and less than 1 obs.

Goal: $F_{X_{(k)}}(x)$, the CDF of the k^{th} largest r.v. of the $X_1, \ldots, X_n$

Consider $n = 10$

What is the $P\left(X_1, \ldots X_4 \in (-\infty, x] \text{ and } X_5, \ldots X_{10} \in (x, \infty)\right)$?



$$= P(X_1 \le x, \ldots, X_4 \le x, X_5 > x, \ldots, X_{10} > x)$$
$$= P(X_1 \le x) \cdot \ldots \cdot P(X_4 \le x) \cdot P(X_5 > x) \cdot \ldots \cdot P(X_{10} > x)$$
$$= F(x)^4 (1 - F(x))^6$$

What is the $P\left(\text{any } 4 \in (-\infty, x] \text{ AND the other } 6 \in (x, \infty)\right)$

$$P(\underbrace{X_1 \le x, \ldots, X_4 \le x}_{\text{these 4 below}}, \underbrace{X_5 > x, \ldots, X_{10} > x}_{\text{these 6 above}})$$

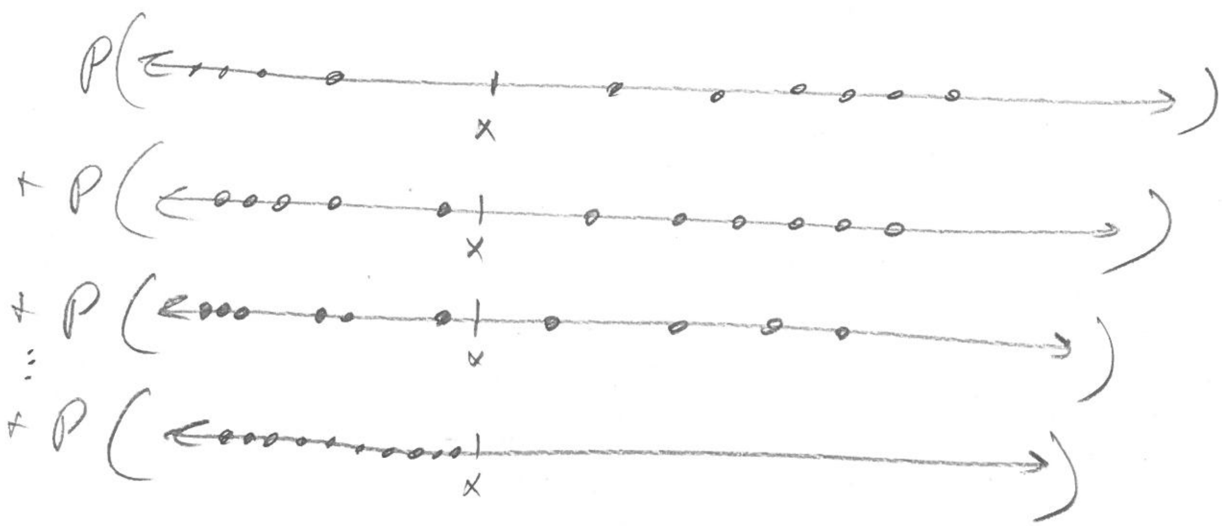
$$+ P(\underbrace{X_{10} \le x, X_7 \le x, X_2 \le x, X_9 \le x}_{\text{these 4 below}}, \underbrace{X_1 > x, X_3 > x, \ldots, X_8 > x}_{\text{these 6 above}})$$

$+$
$\vdots$ all other possibilities

$$= \binom{10}{4} F(x)^4 (1 - F(x))^6$$

looks like binomial where $n = 10$, $p = F(x)$

$F_{X_{(4)}}(x) = P(X_{(4)} \le x) =$



we just did this situation

$= \binom{10}{4}F(x)^4(1-F(x))^6 + \binom{10}{5}F(x)^5(1-F(x))^5 + \dots + \binom{10}{10}F(x)^{10}(1-F(x))^0$

$= \sum_{j=4}^{10}\binom{10}{j}F(x)^j(1-F(x))^{10-j}$

Generalizing this logic ... to arbitrary n, k:

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}$$

Verify this works for the max:

$$F_{X_{(n)}}(x) = \sum_{j=n}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j} = \cancel{\binom{n}{n}}F(x)^n\cancel{(1-F(x))^{n-n}} = F(x)^n$$

and for the min

trick...

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j} = \left(\sum_{j=0}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}\right) - \cancel{\binom{n}{0}}\cancel{F(x)^0}(1-F(x))^{n-0}$$

$$= \left(\overbrace{F(x)+(1-F(x))}^{1}\right)^n - (1-F(x))^n = 1-(1-F(x))^n$$

$$f_{X_{(k)}}(x) = F'_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \underbrace{\frac{d}{dx}\left[F(x)^j (1-F(x))^{n-j}\right]}$$

$\frac{d}{dx}(fg) = fg' + gf'$

$$F(x)^j (n-j)(1-F(x))^{n-j-1}(-f(x)) + (1-F(x))^{n-j}\, j\, F(x)^{j-1} f(x)$$

@ $n \Rightarrow 0$ so $\Rightarrow n-1$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \underbrace{\sum_{j=k}^{n} \frac{n!}{j!(n-j-1)!} f(x) F(x)^j (1-F(x))^{n-j-1}}$$

$$\sum_{j=k}^{n-1} \text{———}$$

reindex so that we sum from $k+1 \ldots n$

let $l = j+1 \Rightarrow j = l-1$

$$\Rightarrow \sum_{l=k+1}^{n} \frac{n!}{(l-1)!\underbrace{(n-(l-1)-1)!}_{(n-l)!}} f(x) F(x)^{l-1} (1-F_x)^{\overbrace{n-(l-1)-1}^{n-l}}$$

let $j = l$

$$\Rightarrow \sum_{j=k+1}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j}$$

$$= (a_k + a_{k+1} + \ldots + a_n) - (a_{k+1} + \ldots + a_n) = a_k$$

$$\boxed{f_{X_{(k)}} = \frac{n!}{\underbrace{(k-1)!(n-k)!}} f(x) F(x)^{k-1} (1-F(x))^{n-k}}$$