11/30/17

Let X be a non-negative rv with finite expectation μ. Consider $a > 0$, a is constant. Consider the inequality.

$a\mathbb{1}_{X \geq a} \leq X$ Is this true? Yes

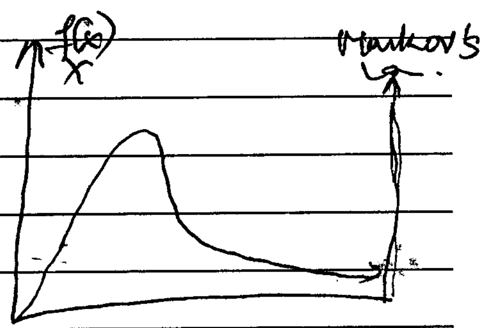
If $X \geq a$: $a(1) \leq X \Rightarrow X \geq a$ ✓
if $X < a$: $a(0) \leq X \Rightarrow X \geq 0$ true by assumption

$E[a\mathbb{1}_{X\geq a}] \leq \mu$

$\Rightarrow aE[\mathbb{1}_{X\geq a}] \leq \mu$

$\Rightarrow aP(X \geq a) \leq \mu$

$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$



tail bound

Markov's Inequality

$P(X \geq a) \leq 1$

Tons of Corollaries

* let $a = a'\mu \Rightarrow P(X \geq a'\mu) \leq \frac{1}{a'}$

* let h be a strictly monotonically increasing function

$h(a)\mathbb{1}_{h(X)\geq h(a)} \leq h(X) \Rightarrow P(h(X) \geq h(a)) \leq \frac{E(h(X))}{h(a)}$

$\Rightarrow P(X \geq a) \leq \frac{E(h(X))}{h(a)}$

$\cancel{* \; h(x) = |x|^p \quad \text{st. } p > 1}$

$$\cancel{P(X \geq a) \leq \frac{E(|x|^p)}{a^p}}$$

$* \; h(x) = x^p \quad \text{st } p > 1$

$$P(X \geq a) \leq \frac{E(x^p)}{a^p}$$

* Recall

$\text{Quantile}[X, p] = F_X^{-1}(p)$ if cont.

$P(X \geq a) \leq \frac{\mu}{a}$

$1 - P(X \leq a) \leq \frac{\mu}{a}$

$\Rightarrow 1 - F(a) \leq \frac{\mu}{a}$

let $a = F_X^{-1}(p)$

$\Rightarrow 1 - F(F^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$

$\Rightarrow 1 - p \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow \text{Quantile}[X, p] \leq \frac{\mu}{1-p}$

$\Rightarrow \text{med}[X] \leq 2\mu$

* Consider any r.v X. $|X|$ is non-negative

$$P(|X| \geq a) \leq \frac{E(|X|)}{a}$$

both fails

* let X be any r.v with finite μ, finite δ^2

let $Y = (X-\mu)^2$ Note: Y is non-negative.

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2} = \frac{E(X-\mu)^2}{a^2} = \frac{\delta^2}{a^2}$$

$$\Rightarrow P((X-\mu)^2 \geq a^2) \leq \frac{\delta^2}{a^2}$$

$$\Rightarrow P(|X-\mu| \geq a) \leq \frac{\delta^2}{a^2}$$ Chebyshev's Inequality.

* let X be any r.v

let $Y = e^{tX}$ Note: Y is non-negative.

$$P(Y \geq a) \leq \frac{E(Y)}{c}$$

$$P(e^{tX} \geq a) \leq \frac{E(e^{tX})}{c}$$

let $c = e^{ta}$

$$\Rightarrow P(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}} = \frac{M_X(t)}{e^{ta}}$$

Note: $M_X(t) = E(e^{tX}) \Rightarrow$ moment generating function

If $t > 0 \Rightarrow P(X \geq a) \leq e^{-ta} M_X(t)$

If $t < 0 \Rightarrow P(X \leq a) \leq e^{-ta} M_X(t)$

$\Rightarrow P(X \geq a) \leq \min_{t>0} \{ e^{-ta} M_X(t) \}$ Chernoff's Inequality

$\Rightarrow P(X \leq a) \leq \min_{t<0} \{ e^{-ta} M_X(t) \}$

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let $X \sim Bin(n, \frac{1}{4}) \Rightarrow \mu = \frac{1}{4}n, \ \sigma^2 = \frac{3}{16}n$

$P(X > \frac{3}{4}n)$ If n is large

$$X \simeq N\left(\frac{1}{4}n, \left(\sqrt{\frac{3}{16}n}\right)^2\right)$$

$$P(X \geq \frac{3}{4}n) = P\left(\frac{X - \frac{1}{4}n}{\sqrt{\frac{3}{16}n}} > \frac{\frac{3}{4}n - \frac{1}{4}n}{\sqrt{\frac{3}{16}n}}\right)$$

$$= P\left(Z > \frac{2}{\sqrt{3}}\sqrt{n}\right) = 0$$

Markov's $P(X \geq \frac{3}{4}n) \leq \dfrac{\frac{1}{4}n}{\frac{3}{4}n} = \frac{1}{3}$

Chebyshow's $P(X \geq \frac{3}{4}n)$

$$= P(X - \tfrac{1}{4}n \geq \tfrac{3}{4}n - \tfrac{1}{4}n)$$

$$\leq P(X - \tfrac{1}{4}n \geq \tfrac{1}{2}n) + P(\tfrac{1}{4}n - X \geq \tfrac{1}{2}n)$$

$$= P\left(X - \frac{1}{4}n \geq \frac{1}{2}n \text{ or } \frac{1}{4}n - X \geq \frac{1}{2}n\right)$$

$$= P\left(\left|X - \frac{1}{4}n\right| \geq \frac{1}{2}n\right) \leq \frac{\frac{3}{16}n}{\frac{1}{4}n^2} = \frac{3}{4n}$$

$X \sim Bin(n, p)$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \overset{\text{Bino theorem.}}{=} (1 - p + pe^t)^n$$

$$X \sim Bin\left(n, \frac{1}{4}\right) \Rightarrow M_X(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n$$

$$P\left(X \geq \frac{3}{4}n\right) \leq \min_{t \geq 0} \left\{ e^{-t\left(\frac{3}{4}n\right)} \left(\frac{3}{4} + \frac{1}{4}e^t\right)^n \right\}$$

$$= \min_{t > 0} \left\{ \left(\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t}\right)^n \right\}.$$

now $\frac{d}{dt}\left(\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t}\right)^n$

$$= n \cancel{\left(\frac{3}{4}e^{-\frac{3}{4}t} + \frac{1}{4}e^{\frac{1}{4}t}\right)^{n-1}} \left(\frac{-9}{16}e^{-\frac{3}{4}t} + \frac{1}{16}e^{\frac{1}{4}t}\right) = 0$$

$$\Rightarrow e^{\frac{1}{4}t} = 9e^{-\frac{3}{4}t} \quad \Rightarrow \frac{1}{4}t = \ln(9) - \frac{3}{4}t$$

$$\Rightarrow \underline{t_{min} = \ln(9)}$$

$$= \left(\frac{3}{4} e^{-\frac{3}{4}\ln(9)} + \frac{1}{4} e^{\frac{1}{4}\ln(9)} \right)^n$$

$$= \left(\frac{3}{4} 9^{-\frac{3}{4}} + \frac{1}{4} 9^{\frac{1}{4}} \right)^n$$

$$= \frac{\sqrt[4]{9}}{4^n} \left(\frac{3}{9^3} + 1 \right)^n = \sqrt[4]{9} \left(\frac{1.004}{4} \right)^n$$

$\rightarrow 0$ exponentially fast.

Consider any two r.v's X, Y with finite μ's, σ^2's.

Let $W := (X - cY)^2$ s.t. $c \in \mathbb{R}$. Note that W is non-nega

$$\Rightarrow E(W) \geq 0 \Rightarrow E(X - cY) \geq 0$$

$$\Rightarrow E(X^2 - 2cXY + c^2Y^2) \geq 0$$

$$\Rightarrow E(X^2) - 2cE(XY) + c^2 E(Y^2) \geq 0$$

Pick $c = \frac{E(XY)}{E(Y^2)}$

Then $\Rightarrow E(X^2) - 2\frac{E(XY)}{E(Y^2)} E(XY) + \frac{E(XY)^2}{E(Y^2)^2} E(Y^2) \geq 0$

$$\Rightarrow E(X^2)E(Y^2) - \cancel{2}E(XY)^2 + E(XY)^2 \geq 0$$

$$\Rightarrow E(XY)^2 \leq E(X^2)E(Y^2)$$

$\Rightarrow |E(XY)| \le \sqrt{E(X^2)E(Y^2)}$

~~Chain~~ Cauchy-Schwartz Inequality.

~~If~~ When equality if $X = cY$.

When is the correlation?

$$Corr(X,Y) = corr(cY, Y) = \frac{Corr(cY, Y)}{SE(cY)SE(Y)}$$

$$= \frac{c\,Corr(Y,Y)}{|c|\,SE(Y)^2}$$

$$= \frac{c}{|c|} = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$$