Math 621 Lec 2 8/31/17

Review: Resources on course homepage

$X \sim Bern(p)$ $\quad Supp(X) = \{0,1\}$

$X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = p(x)$

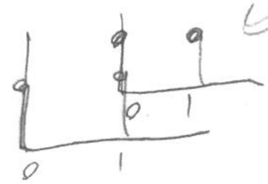
sample with replacement. Bag with 117 balls, 38 are black, the rest white

Pull out 51 balls. How many are black? white?

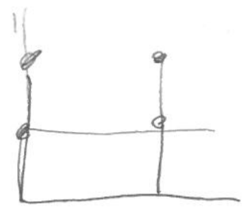
with replacement

$X_1, X_2 \overset{iid}{\sim} Bern\left(\frac{1}{2}\right)$, $T_2 = X_1 + X_2$ supports are aligned

pass this over



What's the overlap? $p_{X_1}(x) * p_{X_2}(x) =$



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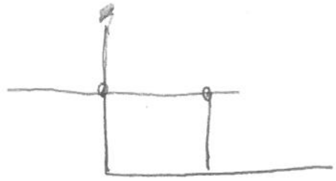

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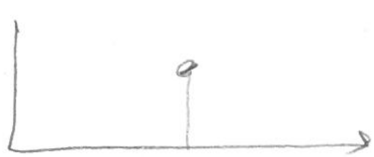
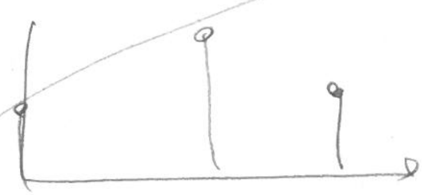


should be 0!

=



$= \sum_{x \in Supp(X)} p_{X_1}(x)\, p_{X_2}(t-x)$

without 1 function

$p(2) \overset{?}{=} \underbrace{P(X_1=0)}\ \underbrace{P(X_2=2)}\ \underbrace{P(X_1=1)}\ \underbrace{P(X_2=1)}$

$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x}\, p^{t-x} (1-p)^{1-t+x}\, \mathbb{1}_{t-x \in \{0,1\}}$

$= p^0 (1-p)^1\, p^{2-1} (1-p)^{1-2+0} + p^1 (1-p)^0\, p^1 (1-p)^0$

let $X_1, X_2 \overset{iid}{\sim} Bin(n, p)$

let $Y = X_1 + X_2 \overset{?}{\sim} p_{X_1}(x) * p_{X_2}(x) = \sum_{x \in Supp[X_1]} p_{X_1}(x)\, p_{X_2}(y-x)$

$$= \sum_{x \in \{0,\dots,n\}} \binom{n}{x} p^x (1-p)^{n-x} \underbrace{\mathbb{1}_{x \in \{0,\dots,n\}}} \binom{n}{y-x} p^{y-x} (1-p)^{n-y+x} \underbrace{\mathbb{1}_{y-x \in \{0,\dots,n\}}}$$

no need since and takes care of it

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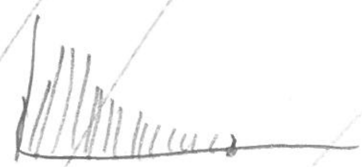
$$= p^y (1-p)^{n-y} \underbrace{\sum_{x \in \{0,\dots,n\}} \binom{n}{x}\binom{n}{y-x}} = Binom(2n, p)$$

why does this make sense?

$= \binom{2n}{y}$ by Vandermonde's identity

What if you don't believe me about the indicator function?

$$\sum_{x \in \{0,\dots,n\}} \binom{n}{x}\binom{n}{y-x} \mathbb{1}_{y-x \in \{0,\dots,n\}}$$



$y \in Supp[Y] = \{0, 1, \dots, 2n\}$
$x \in \{0, 1, \dots, n\}$

if $y = 2n \Rightarrow x \in \{n\}$ otherwise $\binom{n}{y-x}$ is illegal

lets say $x = n-1$ $\binom{n}{n+1}$! illegal $:= 0$

Vandermonde's Identity bakes this in...

You should also know the Geometric r.v.

Consider $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

$X := \min_t \{B_t = 1\} - 1$ i.e. the # of failures until the first success

$P(X=0) = p$
$P(X=1) = (1-p)p$
$P(X=2) = (1-p)^2 p$
$\vdots$
$p(x) = (1-p)^x p$

$X \sim \text{Geom}(p) := (1-p)^x p$ $\quad \text{Supp}(X) = \mathbb{N}_0 = \{0, 1, \ldots\}$

Param space? $p \in (0,1)$

Consider $X_1, X_2, \ldots \overset{iid}{\sim} \text{Geom}(p)$

$T_2 := X_1 + X_2$ # of failures until two successes

$$\sim p_{X_1}(x) * p_{X_2}(x) = \sum_{x \in \text{Supp}(X)} P(X_1 = x)\, P(X_2 = t - x) = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \underbrace{\mathbb{1}_{t-x \in \{0,1,\ldots\}}}_{\substack{t \in \{x, x+1, \ldots\} \\ t \geq x \\ x \leq t}} = (t+1)(1-p)^t p^2$$

$\sum_{x=0}^{t} 1 \} \; t+1$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p_{X_3}(x) * p_{T_2}(x) = \sum_{x \in \text{Supp}(X_1)} p_{X_1}(x)\, p_{T_2}(t-x) = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= (1-p)^t p^3 \sum_{x \in \{0,1,\ldots\}} (t-x+1)\mathbb{1}_{t-x \in \{0,1,\ldots\}} = (1-p)^t p^3 \sum_{x=0}^{t} (t-x+1) = (1-p)^t p^3 \underbrace{\left((t+1)\sum_{y=0}^{t} 1 - \sum_{x=0}^{t} x\right)}$$

$$(t+1)(t+1) - \frac{t(t-1)}{2}$$
$$= t^2 + 2t + 1 - \frac{t^2}{2} \cdot \frac{t}{2}$$
$$\frac{t^2 + 3t + 2}{2}$$

PAUSE: demonstrate PMF (next page)

Note $\binom{t+2}{2} = \frac{(t+2)!}{t!\,2!} = \frac{(t+2)(t+1)}{2} = \frac{t^2+3t+2}{2}$

$$\Rightarrow \binom{t+2}{2}(1-p)^t p^3$$

$\Rightarrow T_2 \sim NegBin(2, p) = \binom{t+1}{1}(1-p)^t p^2$

$\Rightarrow T_3 \sim NegBin(3, p) = \binom{t+2}{2}(1-p)^t p^3$

$$\underbrace{-\ -\ p\ -\ -\ -\ p\ -}_{t+2 \text{ spots}} \quad \underset{\substack{\uparrow \\ 3^{rd} \text{ success}}}{p}$$

$t+2$ spots

$\uparrow$ # failures $\uparrow$ # of "pre-successes" & which can go anywhere

$\binom{t+2}{2}$

$X \sim Bin(n, p)$ let's say n is large and p is small s.t.

$:= \binom{n}{x} p^x (1-p)^{n-x}$

$Supp(X) = \{0, 1, \ldots, n\}$

$$\lambda = np \quad \Rightarrow p = \frac{\lambda}{n}$$

$$= \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

What happens when $n \to \infty$?

$$\lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{n\cdot(n-1)\,(n-2)\cdot\ldots\cdot(n-x+1)}^{x \text{ terms}}}{\underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{x \text{ terms}}} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}^{\,1}$$

$$= \frac{\lambda^x}{x!}\,(1)\cdot(1)\cdot(1)\cdot\ldots\cdot(1)\,e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!} := Poisson(\lambda)$$

$X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!}$

$\text{Supp}(X) = ?$ since $n \to \infty$ $= \mathbb{N}_0$

$\lambda \in (0, \infty)$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \Rightarrow \text{Supp}[T] = \mathbb{N}_0$

$$p(t) = \sum_{x \in \text{Supp}(X_1)} P(X_1 = x)\, P(X_2 = t - x)$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}_0} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \mathbb{N}_0}$$

$$= e^{-2\lambda} \lambda^t \underbrace{\sum_{x=0}^{\infty} \frac{1}{x!(t-x)!} \mathbb{1}_{t \in \{x, x+1, \dots\}}}$$

$$\underbrace{\frac{t!}{t!} \sum_{x=0}^{t} \frac{1}{x!(t-x)!} \mathbb{1}_{t \geq x}}$$

$$\frac{1}{t!} \underbrace{\sum_{x=0}^{t} \binom{t}{x}}_{2^t} \mathbb{1}_{t \geq x}$$

can be ignored since def of $\binom{t}{x}$ has this condition built-in

Recall powersets. let A be a set s.t. $|A| = n$

$$2^A := \{B : B \subseteq A\} = \{B : B \subseteq A \,\&\, |B| = 0\} \cup \{B : B \subseteq A \,\&\, |B| = 1\} \cup \dots \cup \{B : B \subseteq A \,\&\, |B| = n\}$$

$$|2^A| = |\{B : B \subseteq A \,\&\, |B| = 0\}| + |\{B : B \subseteq A \,\&\, |B| = 1\}| + \dots + |\{B : B \subseteq A \,\&\, |B| = n\}|$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

$$\underset{\substack{1 \\ \text{on/off}}}{2} \cdot \underset{\substack{2 \\ \text{on/off}}}{2} \cdot \underset{3}{2} \cdot \dots \cdot \underset{\substack{n \\ \text{on/off}}}{2}$$

$$= \frac{e^{-2\lambda} (2\lambda)^t}{t!}$$

$\Rightarrow T_2 \sim \text{Poisson}(2\lambda)$