11/2/17 Lecture #16

Standard Normal:

$X \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Longleftrightarrow \phi_X(t) = e^{-\frac{t^2}{2}}$

$E(X) = 0$
$SE(X) = 1$
standard error

$X_1, \dots, X_n \overset{iid}{\sim} ?$

$Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$

CLT
n gets really big, standarized average gets normal

- Let $Y = \mu + \sigma X$, $\sigma \in (0, \infty)$

$f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}}$

cheat sheet

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$

$= N(\mu, \sigma^2)$
"Normal"

$E(Y) = \mu + \sigma E[X] = \mu$
$SE[Y] = \sigma SE[X] = \sigma$

- $X_1, \dots, X_n \overset{iid}{\sim} ?$ mean variance μ, σ

$X_1 + \dots + X_n \overset{d}{\approx} N(\underbrace{n\mu}_{\underbrace{\mu+\mu+\dots\mu}_{n\text{ times}}}, \underbrace{n\sigma^2}_{\underbrace{\sigma^2+\dots+\sigma^2}_{n\text{ times}}})$

$\bar{X}_n = \frac{X_1 + \dots + X_n}{n} \overset{d}{\approx} N\left(\mu, \frac{\sigma^2}{n}\right)$
if n is "large enough"

$\boxed{\text{Support of Normal} = \mathbb{R}}$

$\phi_Y(t) = e^{it\mu}\phi_X(\sigma t) = e^{it\mu}e^{-\frac{\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ ind of $X_2 \sim N(\mu_2, \sigma_2^2)$

$T = X_1 + X_2 \sim \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx$ $\quad$ continuous $\therefore \int$

$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(\cancel{t}-x-\mu_2)^2}\,dx$

supports are all real #s.

No need of indicator functions.

$= \text{----}$ BAD.

Rule # 2

$\phi_Y(t) = \phi_{X_1}(t)\phi_{X_2}(t) = e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} \cdot e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$

$= e^{it\underbrace{(\mu_1+\mu_2)}_{\mu'} - \underbrace{\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right)}_{\sigma^{2\prime}} t^2}$

$\Rightarrow Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

- $X \sim N(\mu, \sigma_1^2)$ $\quad Y = e^X = g(x)$; $x = g^{-1}(y) = \ln y$

$Y \sim f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right| = \frac{1}{\sqrt{2\pi\sigma_1^2}} \cdot e^{-\frac{1}{2\sigma_1^2}(\ln y - \mu)^2} \cdot \frac{1}{y}$

$= \log N(\mu, \sigma^2)$

"log Normal"

μ_R, σ_R^2 (iid)

You begin with Y_0 dollars. You invest it for t time period. R_t is the rate of return (variance). How much money you have in time period Y_1?

$Y_1 = Y_0(1+R_1)$

$Y_2 = Y_1(1+R_2) = Y_0(1+R_1)(1+R_2)$ - compound interest -

$\vdots$

$Y_t = Y_0 \prod_{i=1}^{t}(1+R_i) = Y_0\, e^{\ln\left(\prod_{i=1}^{t}(1+R_i)\right)}$

Define $X_i := \ln(1+R_i)$ $\quad X_i$'s iid.

Now

$Y_t = Y_0\, e^{\sum_{i=1}^{t} X_i}$

$$\sum_{i=1}^{t} X_i \overset{d}{\approx} N(t\mu_x, t\sigma_x^2)$$

Note $e^{\sum X_i} \approx \text{LogN}(t\mu_x, t\sigma_x^2)$

- Let $X \sim \text{LogN}(\mu, \sigma^2)$

$Y = aX \qquad a \in (0, \infty)$

$x = \frac{y}{a}$

$g^{-1}(y) = \frac{y}{a} \qquad (g^{-1}(y))' = \frac{1}{a}$

$$Y \sim f_Y(y) = \frac{1}{a} f_x\left(\frac{y}{a}\right)$$

$\text{LogN}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{1}{2\sigma^2}(\ln y - \mu)^2} \cdot \frac{1}{y}$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{1}{2\sigma^2}(\ln \frac{y}{a} - \mu)^2} \cdot \frac{1}{\frac{y}{a}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{1}{2\sigma^2}(\ln y - \ln a - \mu)^2} \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{1}{2\sigma^2}(\ln y - \underbrace{(\mu + \ln a)}_{\mu'})^2} \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{1}{2\sigma^2}(\ln y - \mu')} \cdot \frac{1}{y}$$

$$= \text{LogN}(\mu + \ln(a), \sigma^2)$$

$X = \ln(1 + R_i) \quad \mu_R, \sigma_R^2$

$- R = .003 = 3\%$

$X = \ln(1 + 0.3) = 0.0296$

$- R = -.05 = -5\%$

$X = \ln(1 + (-0.5)) = -0.051$

$\Rightarrow X \approx R$

If R is small this works!

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - + - \cdots$$

(*) This means $Y_t = Y_0 e^{\sum X_i} \approx \text{LogN}(t\mu_R + \ln(Y_0), t\sigma_R^2)$

$R_i \overset{iid}{\sim} N(10\%, 10\%^2)$

start with \$1000

In 5 yrs, what is the prob. you have more than \$1650 ?

- $P(Y_t > 1650) = 1 - F_{Y_t}(1650) = 1 - \text{plnorm}(1650, 7.41, \sqrt{.05})$?

- $Y_t \sim \text{Log}N(.5 + 6.91, \underbrace{5 \cdot 10^2}_{.05})$ $\ln(1000) = 6.91$

If a quantity experiences normal percentage/ proportional changes, then the resulting quantity $\sim$ LogN

- Let $Z \sim N(0,1)$ $\quad Y = Z^2$ $\quad \text{supp}[Y] = (0, \infty)$

Not 1-1
can't use our formula!

cdf of $Y \Rightarrow$

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$
$$= 2P(Z \in [0, \sqrt{y}]) = 2(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{=\frac{1}{2}})$$
$$= 2F_Z(\sqrt{y}) - 1$$

pdf $\Rightarrow$ $f_Y(y) = F_Y'(y) = 2F_Z(\sqrt{y}) \cdot \frac{y^{-\frac{1}{2}}}{2} = F_Z(\sqrt{y})\, y^{-\frac{1}{2}}$

$N(0,1) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$= \chi^2_{\text{①-parameter}} = \chi^2(1)$$ ← should look like this

(say) Chi-squared with degree of freedom $= 1$

degree of freedom = parameter.

support of chi squared $= [0, \infty)$