k number of categories to choose from

$$\vec{X} \sim \text{multinom}_k(n, \vec{p}) := \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

$\dim[X] = k$

No indicator function since multichoose is 0 unless

$$\sum_{i=1}^{k} x_i = n \text{ and } \forall_i \, x_i \in \mathbb{N}_0$$

$$\text{Supp}[\vec{X}] = \{\vec{x} : \vec{1} \cdot \vec{x} = n \text{ and } \vec{x} \in \mathbb{N}_0^k\}$$

parameter space $\vec{p} \in \{\vec{p} : \vec{p} \in (0,1)^k \text{ and } \vec{p} \cdot \vec{1} = 1\}$

Example: What is the probability of getting 3 apples, 2 bananas 5 cantiloupes if

$$p_A = \frac{1}{4}, \quad p_B = \frac{1}{8}, \quad p_C = \frac{5}{8}$$

$$p\left(\vec{X} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}\right) = \binom{10}{3, 2, 5} \left(\frac{1}{4}\right)^3 \left(\frac{1}{8}\right)^2 \left(\frac{5}{8}\right)^5$$

let $k = 2$, $\vec{p} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$

$$p(\vec{x}) = p(x_1, x_2) = \text{multinom}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) =$$

$$\binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$ (this is not a binomial $\text{Bin}(n,p)$)

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(10b)

(11b)

let $X \sim \text{Multinomial}(n, \vec{p})$ multi$(n, \vec{p})$

$$P_{X_{-j}|X_j}(x_{-j}|x_j) = \frac{\overbrace{P_{X_1,\dots X_k}(x_1,\dots x_k)}^{\text{multi}(n,\vec{p})}}{\underbrace{P_{X_j}(x_j)}_{\text{Bin}(n, p_j)}} =$$

X_{-j}: all x_i such that $i \neq j$

$$= \frac{\frac{n!}{x_1!\, x_2! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}}{\frac{n!}{x_j!\,(n-x_j)!} p_j^{x_j} (1-p_j)^{n-x_j}}$$

$$= \frac{(n-x_j)!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_k!} \cdot \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{n-x_j}}$$

let $n' = n - x_j$,

recall $\sum_{i=1}^{k} x_i = n \Rightarrow x_1 + x \cdots x_{j-1} + x_j + x_{j+1} \cdots x_n = n$

$\Rightarrow n' = x_1 + \cdots x_{j-1} + x_{j+1} \cdots x_k$

let $p_i' = \frac{p_i}{1-p_j} \Rightarrow p_i = p_i'(1-p_j)$

$$= \binom{n'}{x_1 \cdots x_{j-1}\, x_j \cdots x_k} \frac{(p_1'(1-p))^{x_1} \cdots (p_{j-1}'(1-p_j))^{x_{j-1}} \cdots (p_{j+1}'(1-p_j))^{x_{j+1}} \cdots p_k'(1-p_j)^{x_k}}{(1-p_i)^{n'}}$$

$$= \binom{n'}{x_{\dots}} \frac{(1-p_j)^{\overbrace{x_1 \cdots x_{j-1} + x_j \cdots x_k}^{n'}} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cdots p_k^{k}}{(1-p_j^n)'}$$

multinomial

$$E\left[g(x_1 \dots x_n)\right] = \sum_{x_1 \in \text{supp}[x_1]} \cdots \sum_{x_n \,\text{supp}[x_n]} g(x_1 \dots x_n)\, p(x_1 \dots x_2)$$

$$E[x_1 + \dots + x_n] = E[x_1] + E[x_2] \dots + E[x_n] = nN$$

linearity of Expectation — being identically distributed

$$E\left[\prod_{i=1}^{n} x_i\right] = \sum \dots \sum x_1 x_2 \dots x_k\, p(x_1 x_2 \dots x_2) =$$

$$= \sum \left(x_1 p(x_1) \dots \right) x_2\, p(x_2)$$

if $X_1 \dots X_n$ ind $= \prod_{i=1}^{n} E(x_i)$

$$Var[x] = E\left[(\underbrace{x - \mu)^2}_{g(x)}\right] = E[g(x)] =$$

$$\sum g(x) p(x) =$$

$$\sum (x-\mu)^2 p(x)$$

$$= \sum x^2 p(x) + \sum -2x\mu p(x) + \sum \mu^2 p(x)$$

$$= E[x^2] - 2\mu^2 + \mu^2 = E[x^2] - \mu^2$$

$$Var[x] = E[x^2] - \mu^2$$

$$Var[X+a] = Var[X]$$

$$Var[cX] = c^2\, Var[x]$$

$$E(x+c) = E(x) + c$$

$$E[aX] = a\,E(x)$$

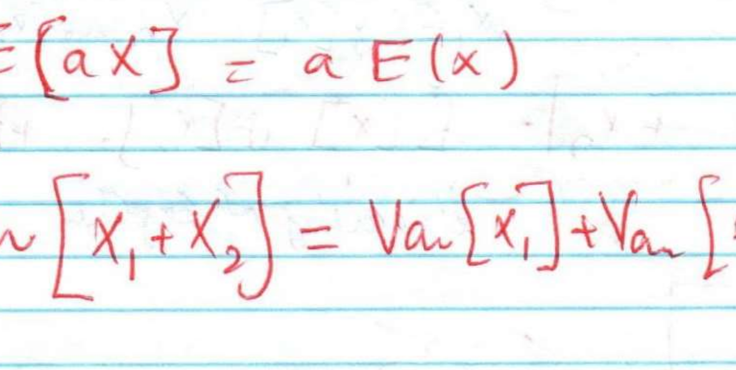
$$Var[X_1 + X_2] = Var[X_1] + Var[X_2] + 2\underbrace{\left(E[X_1X_2] - \mu_1\mu_2\right)}_{E[(X_1-\mu_1)(X_2-\mu_2)]}$$

$$\sigma_{X_1X_2} = Cov[X_1, X_2]$$

$$\rho_{X_1X_2} = \underset{\text{Correlation}}{Corr}[X_1, X_2] = \frac{Cov[X_1, X_2]}{\sigma E[X_1]\ \sigma E[X_2]}$$

Correlation $\in [-1, 1]$

$$\sigma E[x] = \sqrt{Var[x]}$$



$$\mathrm{Cov}[X, X] = \mathrm{Var}[X]$$

$$\mathrm{Cov}[a_1X_1, a_2X_2] = a_1 a_2 \mathrm{Cov}[X_1, X_2]$$

$$\mathrm{Cov}[X_1 + c_1, X_2 + c_2] = \mathrm{Cov}[X_1, X_2]$$

$$\mathrm{Cov}[X_2, X_1] = \mathrm{Cov}[X_1, X_2]$$

$$\mathrm{Cov}[X + Y, Z] = \mathrm{Cov}[X, Z] + \mathrm{Cov}[Y, Z]$$

$$\mathrm{Cov}\left[\sum_{i=1}^{n_i} X_i, \sum_{j=1}^{n_j} Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n_j} \mathrm{Cov}[X_i, Y_j]$$

$$\mathrm{Var}[X_1 + \cdots + X_k] = \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Cov}[X_i, X_j]$$

If $\vec{X}$ is a vector random variable

$$E[\vec{X}] := E\left[\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}\right] = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E[X_k] \end{bmatrix}$$

$$\mathrm{Var}[\vec{X}] = \begin{bmatrix} \overset{= \mathrm{Cov}[X_1, X_1]}{\mathrm{Var}[X_1]} & \mathrm{Cov}[X_1, X_2] & \cdots & \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & & \\ & & \ddots & \\ & & & \mathrm{Var}[X_k] \end{bmatrix}$$

Symmetric $k \times k$

$$\{\mathrm{Cov}[X_i, X_j]\} \quad \begin{matrix} i = 1 \ldots k \\ j = 1 \ldots k \end{matrix}$$