Lec 10

$\boxed{\text{Oct. 10, 2017}}$

$X \sim \text{Exp}(1) \quad Y = -\ln(X) \sim \text{Gumbel}(0,1)$

$$= e^{-(y + e^{-y})}$$

$$= e^{-y} \cdot e^{-e^{-y}}$$

$Y \sim \text{Gumbel}(0,1), \quad X = e^{-Y} \sim \text{Exp}(1)$

cdf

$$F_Y(y) = P(Y \le x) = P(-Y \ge -x)$$

$$= P(e^{-Y} \ge e^{-x}) = P(X \ge e^{-y})$$

$$= 1 - F(e^{-y})$$

$$= e^{-e^{-y}}$$

$X \sim \text{Gumbel}(0,1) \quad Y = \mu + \beta X \sim \text{Gumbel}(\mu, \beta)$

$$= \frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)}$$

cdf

$$F_Y(y) = P(Y \le x) = P\left(\frac{Y-\mu}{\beta} \le \frac{y-\mu}{\beta}\right)$$

$$= P\left(X \le \frac{y-\mu}{\beta}\right) = F\left(\frac{y-\mu}{\beta}\right) = e^{-e^{-\left(\frac{y-\mu}{\beta}\right)}}$$

Valid for any linear transformation

$X \sim \text{Gumbel}(0,1) \quad Y = e^{-X} \sim \text{Exp}(1)$

$X \sim \text{Gumbel}(\mu, \beta) \quad Y = e^{-X}$?

$\text{Supp}(X) = \mathbb{R} \quad \text{Supp}(Y) = (0, \infty)$

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}[g^{-1}(y)]}$$

$X = -\ln(y) = g^{-1}(y)$

$\left|\frac{d}{dy}[-\ln(y)]\right| = \frac{1}{y} = y^{-1}$

$= f_X(-\ln(y))\, y^{-1} = \frac{1}{\beta} e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)} \cdot e^{-e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}} \cdot y^{-1}$

Note $-\left(\frac{-\ln(y)-\mu}{\beta}\right) = \frac{\ln(y)+\mu}{\beta}$

$= k(\ln(y)+\mu) = k[\ln(y) + \ln(\lambda)] = \ln(y\lambda)^k$

let $k = \frac{1}{\beta}$ and let $\mu = \ln(\lambda)$

$$e^{-\left(-\frac{\ln(y)-\mu}{\beta}\right)} = e^{\ln(\lambda y)} = (\lambda y)^k$$

$\beta \in$ Poisson $\Rightarrow k \in (0,\infty)$ $\quad \mu \in \mathbb{R}$

$\Rightarrow \lambda \in (0,\infty)$

$$f_Y(y) = k(\lambda y)^k e^{-(\lambda y)^k} y^{-1}$$

$$= \boxed{(k\lambda)(\lambda y)^{k-1} e^{-(\lambda y)^k} = \underline{\text{Weibull}(k,\lambda)}}$$

cdf $\quad F_Y(y) = P(Y \le y) = P(\ln(Y) \le \ln(y))$

$$= P(-\ln(Y) \ge -\ln(y))$$

$$= P(X \geq -\ln(y)) = 1 - [-(-\ln(y))]$$

$$= 1 - e^{\boxed{e^{-\left(\frac{-\ln(y)-\mu}{\beta}\right)}}} = 1 - e^{-(\lambda y)^k}$$

If $k = 1 \Rightarrow k = 1$

$$\text{Weibull}(1, \lambda) = \lambda e^{-\lambda y} = \text{Exp}(\lambda)$$ Memoryless

<u>Case 1</u> $k > 1$ exple $k = 2$

$$\text{Weibull}(2, \lambda) \Rightarrow F_X(x) = 1 - e^{-(\lambda x)^2}$$

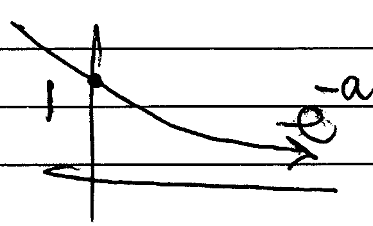
WTS $P(X \geq b) > P(X \geq a+b \mid X \geq a)$

Exple $P(X \geq 7) > P(X \geq 7+3 \mid X \geq 3)$

$$\Rightarrow 1 - F_X(x) = e^{-(\lambda x)^k}$$

$$e^{-(\lambda b)^2} = 1 - F_X(b) > \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{1 - F_X(a+b)}{1 - F_X(a)}$$

$$= \frac{e^{-(\lambda(a+b))^2}}{e^{-(\lambda a)^2}} = e^{-2\lambda^2 ab}\, e^{-(\lambda b)^2}$$



Case #2 $k < 1$ example $k = \frac{1}{2}$

WTS

$$P(X \geq b) < P(X \geq a+b \mid X \geq a)$$

$$e^{-(\lambda b)^{1/2}} < \frac{e^{-(\lambda(a+b))^{1/2}}}{e^{-(\lambda a)^{1/2}}} = e^{-(\lambda(a+b))^{1/2} + (\lambda a)^{1/2}}$$

$$= e^{-\lambda^{1/2}\left((a+b)^{1/2} - a^{1/2}\right)}$$

$$\Rightarrow \frac{-\lambda^{1/2} b^{1/2}}{-\lambda^{1/2}} < \frac{\lambda^{1/2}}{-\lambda^{1/2}}(\ ) \Rightarrow$$

$$b^{1/2} > (a+b)^{1/2} - a^{1/2} \Rightarrow$$

$$a^{1/2} + b^{1/2} > (a+b)^{1/2} \Rightarrow \left(a^{\frac{1}{2}} + b^{1/2}\right)^2 > a+b$$

$$\Rightarrow 2a^{1/2} > 0$$

$\text{Supp}(Y) = [0, \infty)$

$X \sim \text{Weibull}(k, \lambda) \qquad Y = \frac{1}{X}$

$g^{-1}(y) = \frac{1}{y} \qquad \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right| = \frac{1}{y^2}$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\frac{1}{y^2} = (k, \lambda)\left(\frac{\lambda}{y}\right)^{k-1} \cdot e^{-\left(\frac{\lambda}{y}\right)^k} \cdot \frac{1}{y^2}$$

$$= k\lambda^{k} \frac{1}{y^{k-1}} \frac{1}{y^2} e^{-\left(\frac{\lambda}{y}\right)^k}$$

$$= \frac{k}{\lambda} \lambda^{k+1} \frac{1}{y^{k+1}} e^{-\left(\frac{y}{\lambda}\right)^{-k}}$$

$$\boxed{= \frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{-(k+1)} e^{-\left(\frac{y}{\lambda}\right)^{-k}} = \text{Frechet}(k, \lambda, 0)}$$

{Gumbel, Weibull, Frechet} belong to a special familly called the generalized extreme value distributn

$$X \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\overbrace{\lambda^k x^{k-1} e^{-\lambda x}}^{\text{Gamma}(k,\lambda)}}{\Gamma(k)}$$

$$X \sim \text{NegBin}(k, p) = \binom{x+k-1}{k-1} p^k (1-p)^x$$

$$= \frac{\Gamma(x+k)}{\Gamma(x+1)\Gamma(k)} p^k (1-p)^x$$

Extend Negative Binomial

With $k \in \mathbb{N}$, $p \in (0,1)$ $\lambda \in (0, \infty)$

Waiting time
or survival time $\Rightarrow$ Erlang & NegBin

Recall $X \sim \text{Erlang}(k_1, \lambda)$ $\quad Y \sim \text{Erlang}(k_2, \lambda)$

Conv $X+Y = \text{Erlang}(k_1+k_2, \lambda)$

WTS if ~~$X \sim \text{Erlang}(k_1, \lambda)$~~

$X \sim \text{Gamma}(k_1, \lambda)$ $\quad Y \sim \text{Gamma}(k_2, \lambda)$

$X+Y = \text{Gamma}(k_1+k_2, \lambda)$

$$X+Y \sim \int_0^t f_X(x) f_Y(t-x)\,dx$$

$$= \int_0^t \frac{\lambda^{k_1} x^{k_1-1} e^{-\lambda x}}{\Gamma(k_1)} \cdot \frac{\lambda^{k_2} (t-x)^{k_2-1} e^{-\lambda(t-x)}}{\Gamma(k_2)}\,dx$$

$$= \frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\Gamma(k_2)} \int_0^t x^{k_1-1} (t-x)^{k_2-1}\,dx$$

~~Let $v = \frac{x}{t} \Rightarrow \frac{dv}{dx} = \frac{1}{t} \Rightarrow dx = t\,dv$~~

Let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

$x_l = 0 \Rightarrow u_l = 0$

$x_u = t \Rightarrow u_u = 1 \Rightarrow$

$$\frac{\lambda^{k_1+k_2} e^{-\lambda t}}{\Gamma(k_1)\,\Gamma(k_2)} \int_0^1 (ut)^{k-1} (t-ut)^{k_2-1}\, t\, du$$

$$= \frac{\quad}{\quad} \int_0^1 t^{k-1}\, t^{k_2-1}\, u^{k_2-1} (1-u)^{k_2-1}\, t\, du$$