Lec 17 Nov. 07, 2017

$z_1, z_2, \dots, z_k \overset{iid}{\sim} N(0,1)$

$\sum_{i=1}^{k} z_i^2 \sim \chi_k^2 \rightarrow \sqrt{\sum_{i=1}^{k} z_i^2} = \chi$

$Y = |z| \qquad Y^2 = z^2 = \chi_1^2$

$|z| = \sqrt{z^2} = \chi_1 = \sqrt{\frac{2}{\pi}} e^{-\frac{z^2}{2}}$

$= 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)$

$X \sim \chi_k^2 \qquad Y = \frac{X}{K} \sim ?$

$\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

$X \sim \text{Gamma}(\alpha, \beta), \quad Y = cX \quad c > 0$

$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)}}{\Gamma(\alpha)}$

$$= \frac{\left(\frac{\beta}{c}\right)^{\alpha} y^{\alpha-1} e^{-\frac{\beta}{c} y}}{\Gamma(\alpha)} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$X_1 \sim \chi^2_{k_1}$ ind of $X_2 \sim \chi^2_{k_2}$

$$R = \frac{\frac{X_1}{k_1}}{\frac{X_2}{k_2}} = \frac{V_1}{V_2} \sim \int_{\text{Supp}[V_2]} t f_{V_1}(rt) f_{V_2}(t)\, dt$$

Supp$[R] = (0, \infty)$

$$= \int_0^\infty t \frac{a^a (rt)^{a-1} e^{-art}}{\Gamma(a)} \frac{b^b t^{b-1} e^{-bt}}{\Gamma(b)} dt$$

let $a = \frac{k_1}{2}$ $b = \frac{k_2}{2}$

$V_1 \sim \frac{a^a x^{a-1} e^{-ax}}{\Gamma(a)}$ $V_2 \sim \frac{b^b x^{b-1} e^{-bx}}{\Gamma(b)}$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} dt$$

let $u = (ar+b)t$

$dt = \frac{du}{(ar+b)}$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)(ar+b)^{a+b}} \underbrace{\int_0^\infty u^{a+b-1} e^{-u} du}_{\Gamma(a+b)}$$

$$= \frac{a^{a} b^{b} r^{a-1}}{B(a,b)(ar+b)^{a+b}} = \frac{a^{a} b^{b}}{B(a,b)\, b^{a+b}} r^{a-1} \left(\frac{a}{b} r + 1\right)^{-(a+b)}$$

$$= \frac{\left(\frac{a}{b}\right)^{a}}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)} = \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\frac{k_1+k_2}{2}}$$

$$= F_{k_1, k_2}$$

F distribution

"Fisher Snedecor" F is named after Fisher

$Z \sim N(0,1)$ ind. of $V \sim \chi^2_k$

let $Y = \frac{Z}{\sqrt{\frac{V}{k}}}$? $X = \frac{Z^2}{\frac{V}{k}} = \frac{\frac{U}{1}}{\frac{V}{k}}$

$$Y \sim F_{1,k} = \frac{\left(\frac{1}{k}\right)^{1/2}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} y^{-\frac{1}{2}} \left(1 + \frac{1}{k} y\right)^{-\frac{k+1}{2}}$$

$W = \pm\sqrt{Y}$ W is symetric

$$\cancel{F_y(y) = P(Y \le y) = P(Y^2 \le y^2)}$$

$$y = x^{1/2} = \sqrt{x}$$

$$f_Y(y) = f_x(g^{-1}(y)) \left| \left[\frac{d}{dy}(g^{-1}(y))\right] \right|$$

$$f_Y(y) = \frac{1}{2} f_X(y^2) 2y = \frac{1}{\sqrt{k} B\left(\frac{1}{2}, \frac{k}{2}\right)} (y^2)^{-\frac{1}{2}} \left(1 + \frac{y^2}{k}\right)^{-\frac{k+1}{2}} y$$

$$= \frac{1}{\sqrt{k} B\left(\frac{1}{2}, \frac{k}{2}\right)} \left(1 + \frac{y^2}{k}\right)^{-\frac{k+1}{2}} = T_k \xrightarrow{\lim} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Student T distribution with k degree of freedom

$$Y = \frac{Z}{\sqrt{\frac{V}{k}}} \longrightarrow Z \qquad \overset{k \to \infty}{T_k \to Z}$$

$E(\chi^2_k) = k$. $Var[\chi^2] = 2k$

$$\frac{V}{k} = \frac{\sum_{i=1}^{k} Z_i^2}{k} \qquad E\left[\frac{V}{k}\right] = 1 \qquad Var\left[\frac{V}{k}\right] = \frac{2}{k}$$

$$\simeq N\left(1, \frac{2}{k}\right) \xrightarrow[k \to \infty]{\lim} Deg(1)$$

$X_1 \sim N(0,1)$ ind. of $X_2 \sim N(0,1)$

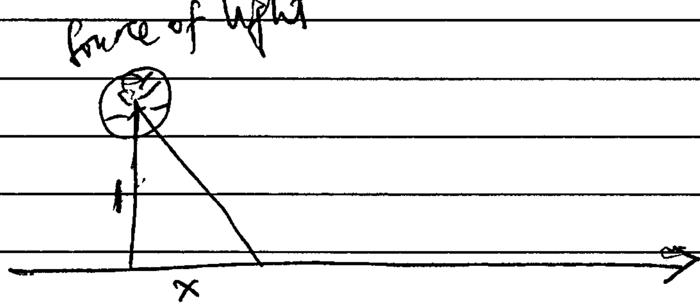
$$R = \frac{X_1}{X_2} = \frac{X_1}{\sqrt{\frac{X_2^2}{1}}} \sim T_1 = \frac{1}{\underbrace{B\left(\frac{1}{2}, \frac{1}{2}\right)}_{= \pi}} (1 + x^2)^{-1}$$

$$\frac{1}{\pi(1+x^2)} = \text{Cauchy}(0,1)$$

(standard)

AKA

the Lorentz distribution



$U \sim U(\pi, 2\pi) = \frac{1}{\pi}$

$X = \tan(U) \longrightarrow U = \arctan(x) = g^{-1}(x)$

$\left| \frac{d}{dx}\left(g^{-1}(x)\right) \right| = \frac{1}{1+x^2}$

$f_X(x) = f(g^{-1}(x)) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ Cauchy

$E(X) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{x}{1+x^2} dx = \frac{1}{\pi} \left[\frac{1}{2} \ln(x^2+1) \right]_{-\infty}^{\infty} = \infty$ DNE

$Var(X) = \infty$ DNE

$R = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} |x_2| f_{X_1}(x_2 r) f_{X_2}(x_2) dx$