$$\phi_X(t) = E[e^{itX}] = \begin{cases} \sum_{x \in supp(x)} e^{itx} p(x) & \text{if } x\text{-discrete} \\ \int_{supp(x)} e^{itx} f(x) & \text{if } x\text{-cont} \end{cases}$$

ch. f. of r.v. X.

① $\phi(0) = 1$

② $Y = X_1 + X_2$ and X_1, X_2 indep $\Rightarrow \phi_Y(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t)$.

③ $Y = aX + b \Rightarrow \phi_Y(t) = e^{itb}\phi_X(at)$

④ $|\phi_X(t)| \leq 1 \ \forall_X \forall_t \Rightarrow \phi_X$ always exists.

consider $\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] = \frac{d}{dt}\left[\int_{\mathbb{R}} e^{itx} f(x)dx\right] \overset{?}{=} \int_{\mathbb{R}} \frac{d}{dt}\left[e^{itx}\right]dx$

Does $\frac{d}{dt}\left[\int g(x)\,dx\right] \overset{?}{=} \int_{\mathbb{R}} \frac{\partial}{\partial t}[g(x,t)]\,dx$

conditions:

(a) $\exists\, t \in A$ s.t. $\int_{\mathbb{R}} g(x,t)dx$ converges $A\,[a,b] \subset \mathbb{R}$

(b) $g(x,t)$ cont. $\forall\, t \in A$

(c) $g(x,t)$ cont. $\forall\, x \in \mathbb{R}$.

(d) $\forall\, t \in A$, $\int_{\mathbb{R}} \frac{\partial}{\partial t} g(x,t)\,dt$ converges uniformly.

$\phi_X'(t) = \int_{\mathbb{R}} f(x)\, ix \cdot e^{itx}\,dx$

consider $\phi_X'(0) = \int_{\mathbb{R}} f(x)\, ix\,dx = i\int_{\mathbb{R}} x f(x)dx = iE[X]$

$\phi_X''(t) = \int_{\mathbb{R}} f(x)\, i^2 x^2 e^{itx}\,dx$

$\phi_X''(0) = i^2\int_{\mathbb{R}} x^2 f(x)\,dx = i^2\, E[X^2]$

So $\phi_X'''(0) = i^3 E[X^3]$

*Bern: PMF

Bern. doesn't have a pdf

$\Rightarrow$ 5 property $E[X^n] = \dfrac{\phi_X^{(n)}(0)}{i^n}$

Inversion Thm.

6 $\forall a<b$ $P(X\in(a,b)) = \dfrac{1}{2\pi}\displaystyle\int \frac{e^{-ita} - e^{-itb}}{it}\phi_X(t)dt$

Motivation:

If $\phi_X \in L'$ (if ϕ_X is integreble),

$\Rightarrow f(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{-itx}\phi_X(t)dt$

pdf

$$P(X\in(a,b)) = \int_a^b f(x)dx = \int_a^b \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi_X(t)dt \cdot dx$$

cdf

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_a^b e^{-tx}dx\right)\phi_X(t)dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}\cdot\phi_X(t)dt$$

7 $\Rightarrow \phi_X(t) = \phi_Y(t) \Leftrightarrow X \overset{d}{=} Y$ uniqueness property.

8 $\phi_{X_n}(t)$ is a ch. funct. for X_n.

if $\forall t$ $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow$

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ also .... $\lim_{n\to\infty} X_n \overset{d}{=} X$

or $X_n \xrightarrow{d} X$ convergence in distribution / Ian

Examples:

$X \sim \text{Gamma}(k, \lambda)$

$$\phi_X(t) = \int_0^\infty e^{itx}\cdot\frac{\lambda^k e^{-\lambda x}x^{k-1}}{\Gamma(k)}dx = \frac{\lambda^k}{\Gamma(k)}\int_0^\infty x^{k-1}e^{(it-\lambda)x}dx$$

Let $u = (\lambda - it)x$. $\Rightarrow x = \dfrac{1}{\lambda - it}u$.

$dx = \dfrac{1}{\lambda - it}\cdot du$

$$\phi_X(t) = \frac{\lambda^k}{\Gamma(k)} \cdot \int_0^\infty \frac{u^{k-1}}{(\lambda - it)^{k-1}} \cdot e^{-u} \cdot \frac{1}{\lambda - it} \cdot du$$

$$= \frac{\lambda^k}{\Gamma(k) \cdot (\lambda - it)^k} \underbrace{\int_0^\infty u^{k-1} e^{-u} du}_{= \Gamma(k)} = \frac{\lambda^k}{\cancel{\Gamma(k)} (\lambda - it)^k} \cdot \cancel{\Gamma}(k)$$

$$= \left(\frac{\lambda}{\lambda - it}\right)^k$$

⊛ $X_1 \sim \text{Gamma}(k_1, \lambda)$ ind of $X_2 \sim \text{Gamma}(k_2, \lambda)$.

$X_1 + X_2 \sim \text{Gamma}(k_1 + k_2, \lambda)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot Q_{X_2}(t) = \left(\frac{\lambda}{\lambda - it}\right)^{k_1} \left(\frac{\lambda}{\lambda - it}\right)^{k_2} = \left(\frac{\lambda}{\lambda - it}\right)^{k_1 + k_2}$$

$\Rightarrow \text{Gamma}(k_1 + k_2, \lambda)$

⊛ $X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = \sum_{x=0}^\infty e^{itx} \cdot \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^\infty \frac{(e^{it})^x \cdot \lambda^x e^{-\lambda}}{x!} = \sum \frac{(\lambda e^{it})^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}}$$

$$= \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \underbrace{\sum_{x=0}^\infty \frac{(\lambda e^{it})^x \cdot e^{-\lambda e^{it}}}{x!}}_{\text{PMF of Poisson}(\lambda e^{it})} = e^{-\lambda + \lambda e^{it}} = e^{\lambda(e^{it} - 1)}$$

⊛ $X_1 \sim \text{Poisson}(\lambda_1)$ ind of $X_2 \sim \text{Poisson}(\lambda_2)$.

$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) = e^{\lambda_1(e^{it}-1)} \cdot e^{\lambda_2(e^{it}-1)}$$

$$= e^{\underbrace{(\lambda_1 + \lambda_2)}_{\lambda'}(e^{it} - 1)}$$

Given (...)

* $(X_1, \ldots, X_n \overset{iid}{\sim}$ some dist. with finite mean μ and finite variance σ^2.) $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$E[\bar{X}] = \mu$

$Var[\bar{X}] = \frac{\sigma^2}{n}$

Define $Z_n := \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{\lim n\to\infty} N(0,1)$ Central Limit Theorem.

$E[Z_n] = 0 \qquad Var(Z_n) = 1$, $Var(Z_n) = SE(Z_n)$

Standardization

Rule #3, Rule #2

$$\phi_{\bar{X}}(t) \overset{\text{Rule \#3}}{=} \phi_{\sum X_i}\left(\frac{t}{n}\right) \overset{\text{Rule \#2}}{=} \left(\phi_X\left(\frac{t}{n}\right)\right)^n$$

$$\phi_{Z_n}(t) = \phi_{\bar{X}_n}\left(\frac{t}{\frac{\sigma}{\sqrt{n}}}\right) e^{it\left(\frac{-\mu}{\frac{\sigma}{\sqrt{n}}}\right)} = \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu\sqrt{n}}{\cancel{\sigma\sqrt{n}}}\frac{n}{n}}$$

$$= \phi_{\bar{X}_n}\left(\frac{t\sqrt{n}}{\sigma}\right) e^{-\frac{it\mu n}{\sigma\sqrt{n}}}$$

$$\phi_{Z_n}(t) = \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}$$ We want the limit of this thing.

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \lim_{n\to\infty} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}.$$

Let $y = e^{\ln(y)}$

$$= \lim_{n\to\infty} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n e^{-\frac{it\mu n}{\sigma\sqrt{n}}}\right)}$$

$$= \lim_{n\to\infty} e^{n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu n}{\sigma\sqrt{n}}}$$

$$= \lim_{n\to\infty} e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\lim \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{1/n}\cdot\frac{t^2/\sigma^2}{t^2/\sigma^2}}$$

$$= e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}}$$

Let $u = \frac{t}{\sigma\sqrt{n}}$ $\quad n \to \infty \Rightarrow u \to 0$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}} \lim_{u\to 0} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}$$

$$= e^{\frac{t^2}{\sigma^2}} \lim_{u\to 0} \frac{\frac{\phi'(u)}{\phi(u)} - i\mu}{2u}$$ ← using L'Hospital's Rule.

$$= e^{\frac{t^2}{2\sigma^2}} \lim_{u\to 0} \frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right] \Rightarrow$$ L'Hospital again.

$$\lim_{u\to 0}\frac{d}{du}\left[\frac{\phi'(u)}{\phi(u)}\right] = \lim_{u\to 0}\frac{\phi''(u)\,\phi(u) - (\phi'(u))^2}{\phi(u)^2} = \frac{\phi''(0)\phi(0) - (\phi'(0))^2}{\phi(0)^2}$$

$$= i^2\left[E[X^2] - \mu^2\right]$$

$$= -\sigma^2$$

$\phi(0) = 1$
$\phi'(0) = (i\mu)^2$
$\phi''(0) = i^2[E(X^2)]$

$$= e^{\frac{t^2}{2\sigma^2}(-\sigma^2)} = e^{-\frac{t^2}{2}} = \phi_Z(t)$$

Now get the ch. funct: & get the pdf.

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\cdot e^{-t^2/2}\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(itx + \frac{t^2}{2})}\,dt$$

piece of a quad. funct. (completing the square)

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2 - \frac{x^2}{2}\right)}dt$$

$$\frac{t^2}{2} + itx = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,it}{2}\right)^2 + \frac{x^2}{2}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}\right)^2} dt.$$

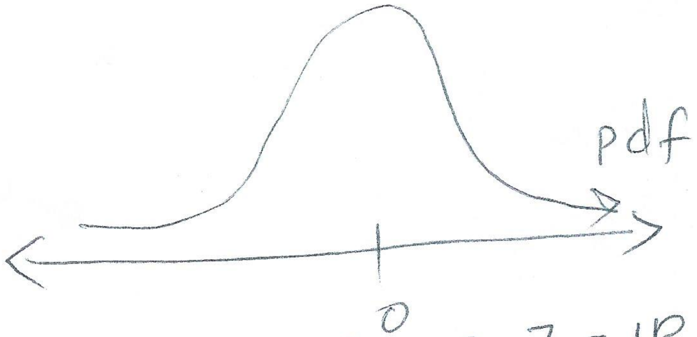
* Let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,ix}{2}$ $\quad \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,dy.$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-y^2}\sqrt{2}\,dy$$

$$= \frac{1}{\pi\sqrt{2}} e^{-\frac{x^2}{2}} \underbrace{\int_{\mathbb{R}} e^{-y^2} dy}_{\text{Gaussian Integral} = \sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$\Vert$

$N(0,1)$
Standard Normal.



$\operatorname{supp}[Z] = \mathbb{R}$.