Lecture 6 9/14/17

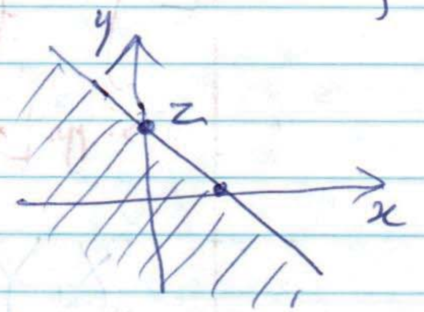
joint density funct

X, Y are continuous r'vs with JdF $f_{X,Y}(x,y)$
and $Z = g(X,Y)$

$$F_Z(x) = P(Z \le P(g(X,Y) \le z)$$

$$= \iint_{\{(x,y):\, g(x,y \le z\}} f_{X,Y}(x,y)\,dx\,dy \;\dots\; \int_{-\infty}^{z} \underbrace{f_Z}_{\text{PDF of } Z}(t)\,dt$$

$T = X + Y$



$$F_T(t) = \iint_{\{(x,y):\, x+y \le z\}} f_{X,Y}(x,y)\,dx\,dy \qquad y \le z - x$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty = y_l}^{z-x = y_u} f_{X,Y}(x,y)\,dy \right) dx$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{z} f_{X,Y}(x, t-x)\,dt \right) dx =$$

let $t = X + Y \Rightarrow dx = dy$
$\Rightarrow y = t - x$

$$= \int_{-\infty}^{z} \underbrace{\left(\int_{\mathbb{R}} f_{X,Y}(x, t-x)\,dx \right)}\,dt$$

$y_l \; -\infty \Rightarrow t_l = -\infty$
$y_u = z - x \Rightarrow t_u = z$

$$f_T(t) = f_X(x) * f_Y(x)$$
$$= (f_X * f_Y)(x)$$

(17b)

If X, Y are independent $\Rightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_T(t) = f_X(x) * f_Y(y) = \int_{\mathbb{R}} f_X(x) f_Y(x-z)\,dx$$

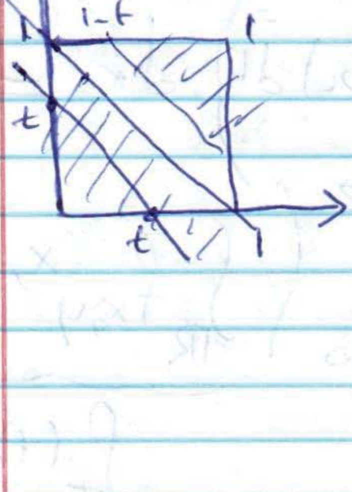
These if $\mathbb{1}$ for support is inside the PDF's

$$= \int_{Supp[X]} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in Supp[Y]}\,dx$$

$X, Y \underset{\sim}{iid}\ U(0,1)$

$T = X + Y \sim f_T(t)$

$$f_{X,Y}(x,y) = 1\ \mathbb{1}_{\substack{x \in [0,1] \\ y \in [0,1]}}$$



$$F_T(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \in [0,1] \\ \frac{1}{2} + \left(\frac{1}{2} - \frac{(2-t)^2}{2}\right) & \text{if } t \in [1,2] \end{cases}$$

$$f_T(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in [1,2] \end{cases}$$

2 standard uniform (0,1)

let $X_1, X_2 \overset{iid}{\sim} U(a,b) = \frac{1}{b-a}$

$T = X_1 + X_2$

$Supp[X_2] = Supp[X_1] = [a,b]$

$\Rightarrow Supp[T] = [2a, 2b]$

$$f_T(t) = \int_{Supp[X_1]} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]} dx$$

$$= \int_a^b \left(\frac{1}{b-a}\right)^2 \underbrace{\mathbb{1}_{t-x \in [a,b]}}_{\mathbb{1}_{x \in [t-b,\, t-a]}} dx$$

$$= \frac{1}{(b-a)^2} \int_{\max\{a, t-b)}^{\min(b, t-a)} dx$$

$$= \frac{1}{(b-a)^2} \left(\min[b, t-a] - \max[a, t-b]\right)$$

$$f_T(t) = \mathbb{1}_{t \in [2a, 2b]} \cdot \begin{cases} t - 2a & \text{if } t < a+b \\ 2b - t & \text{if } t \geq a+b \end{cases}$$

$$X \sim Geom(p) := \overbrace{(1-p)^{x} p}^{P(x)}$$

$$F(x) = P(X \le x) = 1 - P(X > x) = 1 - (1-p)^x \Rightarrow$$

$$1 - F(x) = (1-p)^x$$

$$\vdash\vdash\vdash\vdash\vdash \longrightarrow t$$

I put n experiment in each point

$x = tn$

$p(t) = (1-p)^{tn} p$ if $n \to \infty$, $p \to 0$ $\lambda = np$

$\Rightarrow p(t) = \left(1 - \frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n}$

$\lim_{n\to\infty} p(t) = 0 \quad \forall_t \; p(t) = 0 \Rightarrow$ PMF does not exist

$$F_n(t) = 1 - \left(1 - \frac{\lambda}{n}\right)^{nt}$$

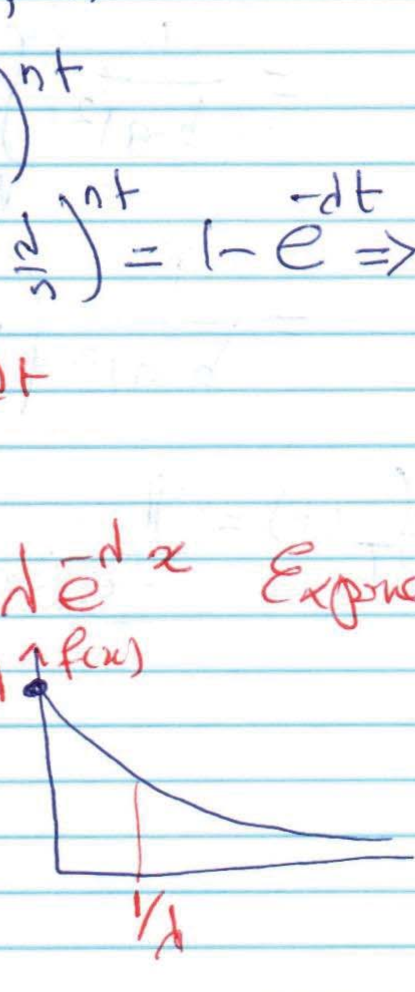
$$F(t) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nt} = 1 - e^{-\lambda t} \Rightarrow 1 - F(t) = e^{-\lambda t}$$

$$f(t) = F'(t) = \lambda e^{-\lambda t}$$

$$X \sim Exp(\lambda) := \lambda e^{-\lambda x} \quad \text{Exponential R.V's}$$

$Sup[X] = (0, \infty)$

$\lambda \in (0, \infty)$



$E[X] = \frac{1}{\lambda}$

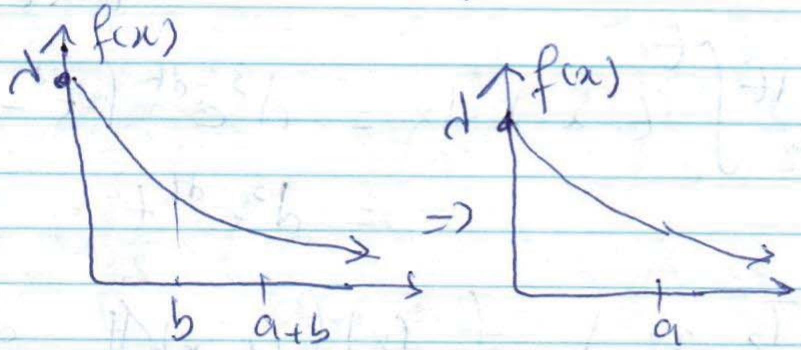
$$\boxed{P(X > a+b \mid X > b)} = \frac{P(X > a+b \text{ and } X > b)}{P(X > b)}$$

$$= \frac{P(X > a+b)}{P(X > b)} = \frac{1 - F(a+b)}{1 - F(b)} = \frac{e^{-d(a+b)}}{e^{-db}}$$

$$= e^{-da}$$

$$= 1 - F(a)$$

$$= P(X > a)$$



let $X_1, X_2 \ldots \overset{iid}{\sim} Exp(d)$

$$T_2 = X_1 + X_2 \sim f_T(t) = \int_{Sup[X_1]} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in (0,\infty)} \, dx$$

$Sup[T_2] = (0, \infty)$

$$= \int_0^\infty d e^{-dx} d e^{-d(t-x)} \mathbb{1}_{t-x \in (0,\infty)} \, dx$$

$$= d^2 e^{-dt} \int_0^\infty \mathbb{1}_{t-x \in (0,\infty)} \, dx = d^2 e^{-dt} \int_0^t dx$$

$$= d^2 e^{-dt} t$$

$$T_3 = X_3 + T_2 \sim \int_{Sup[X_3]} f_{X_3}(x) f_{T_2}(t-x) \mathbb{1}_{t-x \in Sup[T_2]} \, dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \lambda^2 e^{-\lambda(t-x)} (t-x) \mathbb{1}_{t-x \in (0,\infty)} \, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t (t-x) \, dx = \lambda^3 e^{-\lambda t} \left(t^2 - \frac{t^2}{2} \right)$$

$$= \lambda^3 e^{-\lambda t} \frac{t^2}{2}$$

$$T_4 = X_4 + T_3 \sim \int_{Sup[X_4]} f_{X_4}(x) f_{T_3}(t-x) \mathbb{1}_{t-x \, Sup[T_3]} \, dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \lambda^3 e^{-\lambda(t-x)} \frac{(t-x)^2}{2} \mathbb{1}_{t-x \in (0,\infty)} \, dx$$

$$T_k = X_1 + \cdots X_k \quad \boxed{\frac{\lambda^n e^{-\lambda t} t^{n-1}}{(n-1)!}} \quad \text{Erlang}(k, \lambda)$$

$$\boxed{T_k = X_1 + \cdots + X_k \sim \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!}} = f_T(t)$$

Earlang(k, λ)

$$\frac{\lambda^n e^{-\lambda t} t^{n-1}}{(n-1)!}$$

let $X \sim \text{Earlang}(k, \lambda) = \dfrac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}$

$\text{Supp}[X] = (0, \infty)$

$\lambda \in (0, \infty)$ let $u = \lambda y \Rightarrow dy = \frac{du}{\lambda}$

$k \in \mathbb{N}$

$$F_X(x) = \int_0^x \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} dy = \frac{1}{(k-1)!} \int_0^{x = u/\lambda} \lambda (\lambda y)^{k-1} e^{-\lambda y} dy$$

$$= \frac{1}{(k-1)!} \int_0^{\lambda x} u^{k-1} e^{-u} du$$

let define $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt =$

$$\underbrace{\int_0^a t^{x-1} e^{-t} dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} dt}_{\Gamma(x,a)}$$

lower incomplete gamma Function upper incomplete gamma Function