8/28/2019 Lecture 1

A discrete random variable (r.v.) $X$ has a probability mass function (PMF) $\Rightarrow p(x) := P(X=x)$ and cumulative distribution function (CDF) $\Rightarrow F(x) = P(X \le x)$

Support and PMF are related via $\sum_{x \in Supp[X]} p(x) = 1$

Also, $|Supp(X)| \le |\mathbb{N}| \Leftarrow X$ is discrete.

The most fundamental discrete r.v. is the $\boxed{\text{Bernoulli}}$:

$$X \sim Bern(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases} = p^x (1-p)^{1-x}$$

$$Supp[X] = \{0, 1\}$$

* $p$ is its parameter and belongs to the "parameter space." $p \in (0,1)$

if $p=1$, $X \sim Bern(1) := \{1 \text{ w.p. } 1 = Deg(1) = \mathbb{1}_{x=1}$

if $p=0$, $X \sim Bern(0) := \{0 \text{ w.p. } 1$ (degenerate r.v.)

$= Deg(0) = \mathbb{1}_{x=0}$

*** $X \sim Deg(c) = \{c$ with probability $1 = \boxed{\mathbb{1}_{x=c}}$

Ex) $p(x=3.7) = p(3.7) = p^{3.7}(1-p)^{-2.7} = 0.5$ if $p = \frac{1}{2} \Rightarrow 3.7$ is not in the support

$\quad = 0$

Let $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$ $\Downarrow$ Indicator function

$\forall x \in \mathbb{R}$

$p(x) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \underbrace{\{0,1\}}_{Supp[X]}}$

Ex) $f(x) = \mathbb{1}_{x<1}$

$f(19) = 0$ ; $f(-173) = 1$

$X_1, X_2, \dots, X_n$ are discrete r.v.'s which have a joint mass function (JMF): $P_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)$

$$= P(X_1 = x_1 \,\&\, X_2 = x_2, \,\&\, \dots \,\&\, X_n = x_n)$$

If $X_1 \overset{d}{=} X_2 \overset{d}{=} \dots \overset{d}{=} X_n \Rightarrow P_{X_1}(x) = P_{X_2}(x) = \dots P_{X_n}(x)$

(identically distributed)

* They share the same PMF.

$X_1, X_2$ iid $\Rightarrow$ independent + identically distributed

$$= \prod_{i=1}^{n} P(x_i)$$

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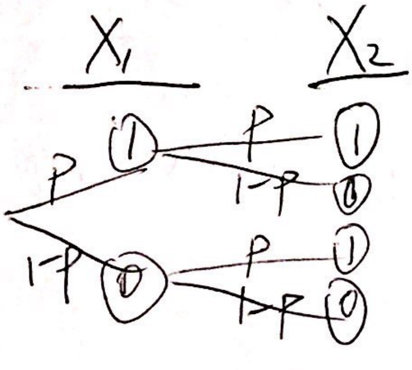
Let $X_1, X_2 \overset{iid}{\sim} Bern(p)$ and $T = X_1 + X_2 \Rightarrow P_T(t) = ?$

$Supp[T] = Supp[X_1] + Supp[X_2] = \{0, 1, 2\}$

$A + B = \{a + b \mid a \in A, b \in B\}$



| ~~$\langle x_1, x_2 \rangle$~~ | $P_{X_1,X_2}(x_1, x_2)$ | $T$ | $P_T(t)$ |
|---|---|---|---|
| $\langle 1, 1 \rangle$ | $p^2$ | 2 | $p^2$ |
| $\langle 1, 0 \rangle$ | $p(1-p)$ | 1 | $2p(1-p)$ |
| $\langle 0, 1 \rangle$ | $(1-p)p$ | 1 | |
| $\langle 0, 0 \rangle$ | $(1-p)^2$ | 0 | $(1-p)^2$ |

~~$\Rightarrow P_T(t)$~~ $P_T(t) = \begin{cases} 0 & w.p. \ (1-p)^2 \\ 1 & w.p. \ 2p(1-p) \\ 2 & w.p. \ p^2 \end{cases}$

$$\sum_{t \in Supp[T]} P_T(t) = (1-p)^2 + 2p(1-p) + p^2 = 1 - \cancel{2p} + \cancel{p^2} + \cancel{2p} - \cancel{2p^2} + \cancel{p^2}$$

$$= 1$$

$$P(T=t) = p_T(t) = \sum_{x\in\mathbb{R}}\sum_{x\in\mathbb{R}} p_{X_1,X_2}(x_1,x_2)\,\mathbb{1}_{t=x_1+x_2}$$

$$= \sum_{x\in\mathbb{R}}\sum_{x\in\mathbb{R}} p_{X_1,X_2}(x_1,t-x_2)\cdot\mathbb{1}_{x_2=t-x_1}$$

$X_1, X_2$ independent $\Rightarrow$

$$= \sum_{x\in\mathbb{R}} p_{X_1,X_2}(x_1,t-x_1)$$

$$= \sum_{x\in\mathbb{R}} p_{X_1}(x)\,p_{X_2}(t-x)$$

$X_1 = X_2 \Rightarrow$

$$= \sum_{x\in\mathbb{R}} p(x)\,p(t-x) \Rightarrow$$

General Discrete convolution Formula

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$$p_T(t) = \sum_{x\in\mathbb{R}} \left(p^x(1-p)^{1-x}\,\mathbb{1}_{x\in\{0,1\}}\right)\left(p^{t-x}(1-p)^{\overbrace{1-(t-x)}^{=1-t+x}}\,\mathbb{1}_{t-x\in\{0,1\}}\right)$$

$$= \sum_{x\in\{0,1\}} p^x(1-p)^{1-x}\,p^{t-x}\,(1-p)^{1-t+x}\,\mathbb{1}_{t-x\in\{0,1\}}$$

$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t-x\in\{0,1\}}\right)$$

$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}}\right)$$

$$= \begin{cases} (1-p)^2 & \text{if } t=0 \\ 2p(1-p) & \text{if } t=1 \\ p^2 & \text{if } t=2 \\ 0 & \text{otherwise} \end{cases} = \boxed{\binom{2}{t} p^t(1-p)^{2-t}}$$

$\Downarrow$

Binom$(2,p)$

$$\binom{2}{t} = \begin{cases} 1 & \text{if } t=0 \\ 2 & \text{if } t=1 \\ 1 & \text{if } t=2 \\ 0 & \text{o.w.} \end{cases}$$