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$\vec{X} \sim \text{Multin}(n, \vec{p}) \Rightarrow X_j \sim \text{Bin}(n, p_j)$

either cantaloupe or not

$$E[\vec{X}] = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} \Leftarrow \quad \text{let } \vec{\mu} = E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}$$

Let $M = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{1n} & \cdots & X_{nm} \end{bmatrix}$, $E(M) = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{nM}] \\ \vdots & & \vdots \\ E[X_{M1}] & \cdots & E[X_{nM}] \end{bmatrix}$

$\sigma^2 = Var[X] = E[X^2] - \mu^2$

$\sigma_{12} = Cov[X_1, X_2] = E[X_1, X_2] - \mu_1 \mu_2 = \ldots = E[(X_1 - \mu_1)(X_2 - \mu_2)]$

If X_1, X_2 iid $\Rightarrow \sigma_{12} = 0$

$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$

Rules for Covariance. measures linear type of relationship.

① $Cov[X_1, X_1] = Var[X_1] = \sigma_1^2 = \sigma_{11}$

② $Cov[X_1, X_2] = Cov[X_2, X_1]$

③ $Cov[X_1 + X_2, X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$

④ $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$

⑤ $Var[X_1 + \ldots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j]$

$$Var[X_1 + X_2] = \sum_{i=1}^{2} \sum_{j=1}^{2} Cov(X_i, X_j) = \sigma_1^2 + \underbrace{\sigma_{12} + \sigma_{21}}_{2\sigma_{12}} + \sigma_2^2$$

σ capital

$$\Sigma = Var[\vec{X}] = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_k] \\ Cov[X_2, X_1] & Var[X_2] & & \\ \vdots & & \ddots & \\ Cov[X_k, X_1] & \cdots & & Var[X_k] \end{bmatrix}$$

$\Rightarrow$ variance, covariance, variance covariance matrix.

$\Rightarrow$ diagonal

$\Rightarrow$ non negative

$\Rightarrow k \times k$

$= E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T$

$\Rightarrow$ If $X_1, \ldots, X_k$ ind

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_2^2 \end{bmatrix}$$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & & \\ & np_2(1-p_2) & & \\ & & \ddots & \\ & & & np_K(1-p_K) \end{bmatrix}$$

We need to compute σ_{ij} where $i \neq j$

we know $\sigma_{ij} < 0 \;\forall i \neq j$

$\sigma_{ij} = E[X_i X_j] - \mu_i \mu_j$

$= \sum_{x_1 \in Supp[X_1]} \sum_{x_2 \in Supp[X_2]} x_1 x_2 \underbrace{P_{X_i, X_j}(x_1, x_2)}_{E.C.} - n^2 p_i p_j$

Rules for vector expontion and variance

$E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a}$ where $\vec{a} \in \mathbb{R}^k$ constant.

$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + \dots + a_k X_k] = a_1 \mu_1 + \dots + a_k \mu_k = \vec{a}^T \mu$

$\underset{\uparrow \text{scalar}}{Var[\vec{a}^T \vec{X}]} = Var[a_1 X_1 + \dots + a_k X_k] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[a_i X_i, a_j X_j]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}$

$= \vec{a}^T \Sigma \vec{a}$

$$E[A\vec{X}] = \begin{bmatrix} E[a_1 X_1 + a_2 X_2 + \dots + a_{1k} X_k] \\ E[a_{21} X_1 + a_{22} X_2 + \dots + a_{2k} X_k] \\ \vdots \\ E[a_{L1} X_1 + a_{L2} X_2 + \dots + a_{Lk} X_k] \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T \cdot \vec{\mu} \\ \vec{a}_2^T \cdot \vec{\mu} \\ \vdots \\ \vec{a}_L^T \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

s.t $A \in \mathbb{R}^{L \times K}$ constant

Consider $\underset{1 \times k}{\vec{a}^T} \underset{k \times k}{V} \underset{k \times 1}{a} = [a_1 \dots a_k] \begin{bmatrix} a_1 V_{11} + \dots + a_k V_{1k} \\ a_1 V_{21} + \dots + a_k V_{2k} \\ \vdots \\ a_1 V_{k1} + \dots + a_k V_{kk} \end{bmatrix}$

where $V \in \mathbb{R}^{k \times k}$ "quadticform"

$\vec{a} \in \mathbb{R}^k$

$$\begin{aligned} = &a_1 a_1 V_{11} + a_1 a_2 V_{12} + \dots + a_1 a_k V_{1k} \\ &a_2 a_1 V_{21} + a_2 a_2 V_{22} + \dots a_2 a_k V_{2k} \\ &\vdots \\ &a_k a_1 V_{k1} + a_k a_2 V_{k2} \dots a_k a_k V_{kk} \end{aligned} = \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j V_{ij}$$

Recall:

Fig apples

$X_i \sim Bin(n, p_i)$ $X_i = X_{1i} + X_{2i} + \dots + X_{ni}$ where $X_{1i}, \dots X_{ni} \overset{iid}{\sim} Bern(p_i)$

↑dependen independen

$X_j \sim Bin(n, p_j)$ $X_j = X_{1j} + X_{2j} + \dots + X_{nj}$ where $X_{1j}, \dots X_{nj} \overset{iid}{\sim} Bern(p_j)$

$\vec{X} = \vec{X}_{1.} + \dots + \vec{X}_{n.}$ s.t $\vec{X}_{1.}, \vec{X}_{2.}, \dots, \vec{X}_{n.} \overset{iid}{\sim} Multin(1, \vec{p})$

$\sigma_{ij} = Cov[X_i, X_j] = Cov[X_{1i} + X_{2i} + \dots + X_{ni}, X_{1j} + X_{2j} + \dots + X_{nj}]$

independen covariance=0

$= \sum_{l=1}^{n} \sum_{m=1}^{n} cov[X_{li}, X_{mj}]$

$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$ because $l \neq m \Rightarrow$ covariance is zero due to independence

$\Rightarrow$

$$= \sum_{l=1}^{n} \left(E[X_{li} X_{lj}] - p_i p_j \right)$$

$$= \cancelto{0}{\left(\sum_{l=1}^{n} E[X_{li} X_{lj}] \right)} - n p_i p_j$$

$$= - n p_i p_j$$

Note:

$$E[X_{li} X_{lj}] = \sum_{x_1 \in \{0,1\}} \sum x_1 x_2 \, p_{X_{li} X_{lj}}(x_1, x_2)$$

$$= p_{X_{li}, X_{lj}}(1,1)$$

$$= P(X_{li} = 1 \,\&\, X_{lj} = 1)$$

$$= 0$$

what if $\vec{p} = \frac{1}{k} \vec{1}$

$\boxed{\text{Discrete r.v.}}$