9/4

Convolution operator

$T = X_1 + X_2 \sim p(t) = p_{X_1}(x) * p_{X_2}(x) = \sum p_{X_1}(x)\, p_{X_2}(t-x)$

$X_1 X_2 \overset{iid}{\sim} \text{Bern}(p)$

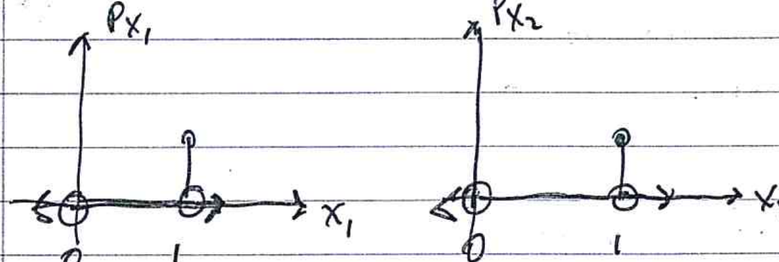
$$p(t) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right) p^{t-x} (1-p)^{1-t-x}\, \mathbb{1}_{t-x \in \{0,1\}}$$

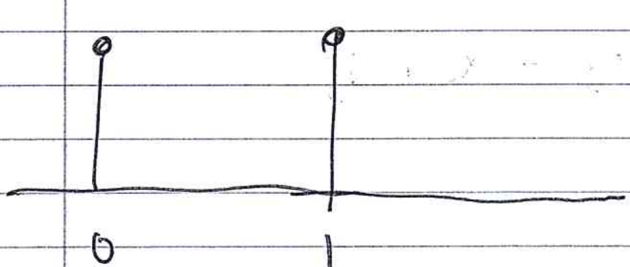
$$= p^t (1-p)^{2-t} \sum_{x\in\{0,1\}} \mathbb{1}_{t-x \in \{0,1\}} = \mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} = \begin{cases} 1 & t=0 \\ 2 & t=1 \\ 1 & t=2 \\ 0 & \text{o/t} \end{cases}$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

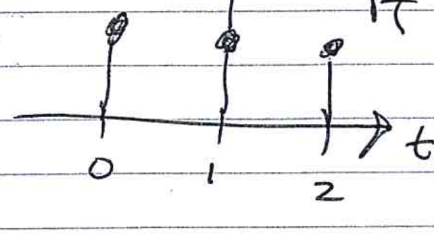
$= \text{Binomoial}(2,p)$

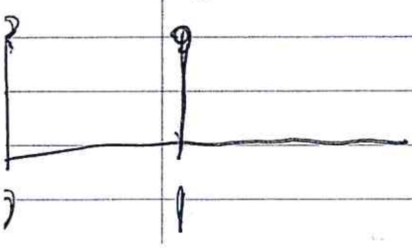
MAKE A PICTURE:





[convolution as a sum product.]





9/4 check wikipedia for convolution.

$$\text{Bern}(p) = \binom{1}{x} p^x (1-p)^{1-x}$$

$$\parallel$$

$$\mathbb{1}_{x \in \{0,1\}}$$

$$p(t) = \sum_{x=\{0,1\}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t-x} \cancel{\mathbb{1}_{t-x \in \{0,1\}}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x} = p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right)$$

$$= \binom{2}{t}$$

(by pascal)

$$\boxed{\text{Pascal's Identity} \quad \binom{N-1}{k} + \binom{N-1}{k-1} = \binom{N}{k}}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$

$$T = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_3$$

9/4

$$p(t) = \sum_{x \in \{0,1\}} \left(\binom{1}{x} p^{x} (1-p)^{1-x} \right) \left(\binom{2}{t-x} p^{t-x} (1-p)^{2+x-t} \cancel{\mathbb{1}_{t-x \in \{0,1,2\}}} \right)$$

$$= p^{t} (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{1}{x}\binom{2}{t-x} = \binom{2}{t} + \binom{2}{t-1} = \binom{3}{t}$$

$$= \binom{3}{t} p^{t} (1-p)^{3-t} = \text{Binomial}(3,p)$$

$$X_1, X_2 \overset{iid}{\sim} \text{Binom}(n,p) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\dots n\}} \left(\binom{n}{x} p^{x} (1-p)^{n-x} \right)\left(\binom{n}{t-x} p^{t-x} (1-p)^{n-(t-x)} \right)$$

$$= p^{t} (1-p)^{2n-t} \sum_{x \in \{0,1,\dots n\}} \binom{n}{x}\binom{n}{t-x}$$

$$\binom{2n}{t} \text{ by vandermonde's identity}$$

$$= \binom{2n}{t} p^{t} (1-p)^{2n-t}$$

$$= \text{Binomial}(2n,p)$$

9/4 geometric

$B_1, B_2, \dots \overset{iid}{\sim} \text{Bern}(p)$

X = # of zeroes realized before the first one = geometric(p)

$\text{Support}(X) = \{0, 1, 2, \dots\} = \mathbb{N}_0$

$p(0) = p$

$p(1) = (1-p)p$

$p(2) = (1-p)^2 p$

$\vdots$

$p(x) = (1-p)^x p$

$X_1, X_2 \overset{iid}{\sim} \text{Geo}(p)$

$$p(t) = \sum_{x \in \mathbb{N}_0} \left((1-p)^x p\right)\left((1-p)^{t-x} p\right) \mathbb{1}_{t-x \in \mathbb{N}_0}$$

$$= (1-p)^t p^2 \sum_{x \in \mathbb{N}_0} \mathbb{1}_{t-x \in \mathbb{N}_0}$$

$$= (t+1)(1-p)^t p^2$$

9/4 $\sum_{x\in\mathbb{N}_0} \mathbb{1}_{t \geq x} = \sum_{x\in\mathbb{N}_0} \mathbb{1}_{x\leq t} = \mathbb{1}_{0\leq t} + \mathbb{1}_{1\leq t} + \mathbb{1}_{2\leq t} + \mathbb{1}_{3\leq t} + \mathbb{1}_{4\leq t} \cdots$

# of ways of getting $T=4$.

$$p(4) = 5\underbrace{(1-p)^4}_{\text{prob 4 zero's}}\underbrace{p^2}_{\text{prob 2 1's}}$$

Experiment #	1	0	0	0	0	1	
↳	1	2	3	4	5	6	
	0	0	1	0	0	1	

$$T = X_1 + X_2 + X_3 = X_3 + T_2$$
$$T_2 = (X_1 + X_2)$$

$$p(t) = \sum_{x\in\mathbb{N}_0}\left(\left((1-p)^x p\right)\left(t-x+1\right)(1-p)^{t-x}p^2\,\mathbb{1}_{t-x\in\mathbb{N}_0}\right)$$

$$= (1-p)^t p^3 \sum (t-x+1)\,\mathbb{1}_{t-x\in\mathbb{N}_0}$$

$$= \sum_{x\in\mathbb{N}_0} (t+1)\,\mathbb{1}_{t-x\in\mathbb{N}_0} - \sum_{x\in\mathbb{N}_0} x\,\mathbb{1}_{t-x\in\mathbb{N}_0}$$

$$= (t+1)\underbrace{\sum_{x\in\mathbb{N}_0}\mathbb{1}_{x\leq t}}_{=t+1} - \sum_{x\in\mathbb{N}_0}$$

9/4 $\cancel{\frac{t(t+1)}{2}}$

$$= (t+1)^2 - \frac{t(t+1)}{2} = t^2 + 2t + 1 - \frac{t^2}{2} - \frac{t}{2}$$

$$= \frac{t^2 + 3t + 2}{2} = \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$X_1, X_2, \ldots X_r \overset{iid}{\sim} \text{Geo}(p)$

$$T = X_1 + X_2 + \ldots X_r \sim \binom{t+r-1}{r-1}(1-p)^t p^r$$

Neg Bin (r, p)

Negative Binomial r.V.