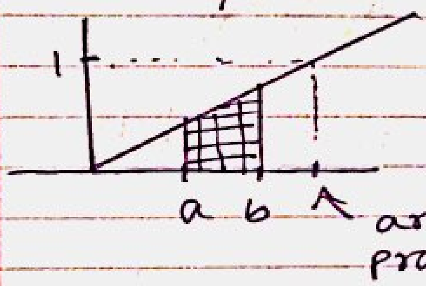
09|18|19 ①
Lecture# 7

Continuous r.v's X has $|supp[X]| = |\mathbb{R}|$
$\Rightarrow P(x) = P(X = x) = 0$
The derivative of the CDF is very important
$$f(x) = \frac{d}{dx}[F(x)] \text{ where } F(x) \text{ is the CDF}$$
which is called probability density function. (PDF). $f(x)$ is PDF.
$$P(X \in [a,b]) = F(b) - F(a) = \int_a^b f(x)\,dx$$
$|supp[X]| = |supp[X]|$

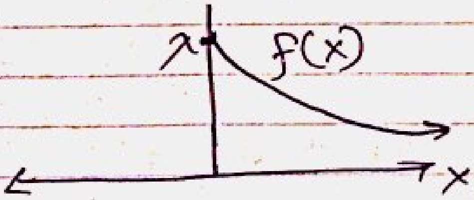
by Fundamental theorem of calculus.

$supp[X] = \{x : f(x) > 0\}$

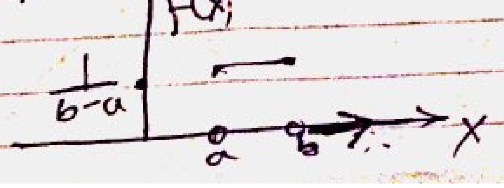


Properties of $f(x)$ = PDF
① $\int_{\mathbb{R}} f(x)\,dx = 1$ since $F(\infty) - F(-\infty) = 1 - 0 = 1$

⑪ $f(x) \geq 0$ for all x $supp[X] = \{x : f(x) > 0\}$
$$X \sim EXP(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$$
$supp[X] = [0, \infty]$



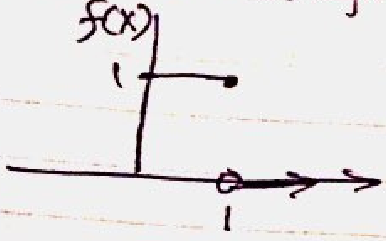
uniform continuous dist.
$$X \sim \underset{\text{uniform}}{U}(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in (a,b)}$$



$supp[X] = [a, b]$
$a, b \in \mathbb{R}$ but $b > a$

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$

standard uniform



$\mathrm{Exp}[1]$::

standard exponential

$\vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdots f_{X_k}(x_k)$; (Joint Density function)

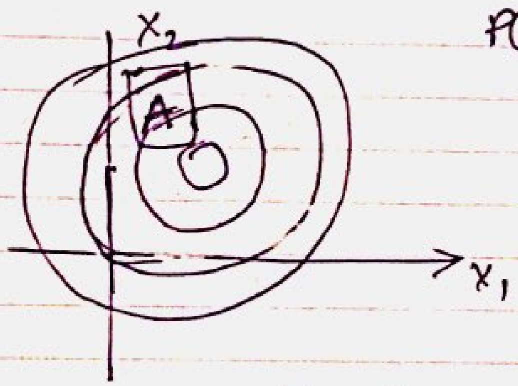
If $X_1 \cdots X_k$ are independent

If $X_1 \cdots X_k$ are $\overset{iid}{\sim}$ then $f(x_1) \cdots f(x_k)$ JDF

$\int \cdots \int_{\mathbb{R}^K} f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_2 = 1$

$K = 2$

$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$



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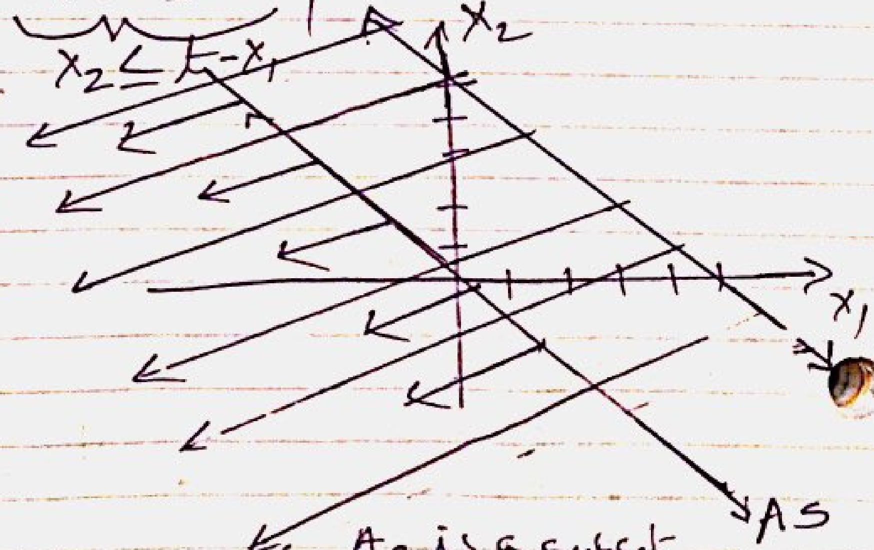
$T = X_1 + X_2 \sim f_T(t) = ?$

$F(t) = P(T \le t) = P(A_t)$

$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underbrace{x_1 + x_2 \le t}_{x_2 \le t - x_1} \right\} \subset \mathbb{R}^2$



If these approaches to ∞ then take $\int$ will be 1.

if $-\infty$, 0

A_0 is a subset of A_5

③

$$F(t) = P(A_t) = \int_{x_1 \in R} \int_{x_2 \in (-\infty, t-x_1)} f_{X_1} f_{X_2}(x_1, x_2)\, dx_1, dx_2$$

let $x = x_1 \Rightarrow dx = dx_1$

$v = x_2 + x_1 \Rightarrow x_2 = v - x_1$

$\Rightarrow dv = dx_2$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x_1 \Rightarrow v = t$

$$\int_{x \in R} \int_{-\infty}^{t} f_{X_1, X_2}(x, v-x)\, dv\, dx$$

$$F(t) = \int_{-\infty}^{t} \int_{R} f_{X_1 X_2}(x, v-x)\, dx\, dv$$

analysis

$$T = X_1 + X_2 \sim f_T(t) = ?$$

$$\underset{\text{PDF}}{f(t)} = \frac{d}{dt}[\underset{\text{CDF}}{F(t)}] = \int_{R} f_{X_1, X_2}(x, t-x)\, dx$$

General convolution formula

If X_1, X_2 are independent

$$f(t) = \int_{R} f_{X_1}(x) f_{X_2}(t-x)\, dx$$

If X_1, X_2 are iid $\rightarrow f(t) = \int_{R} f(x) f(t-x)\, dx$

$$= \int_{\text{supp}[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, \mathbb{1}_{t-x \in \text{supp}[X_2]}\, dx$$

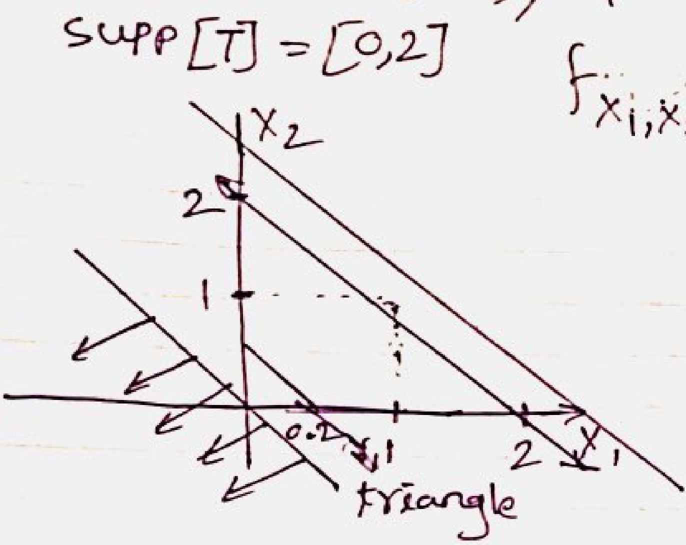
$$= \int_{\text{supp}[X]} f(x) f(t-x)\, \mathbb{1}_{t-x \in \text{supp}[X]}\, dx$$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f(t) = ?$

$supp[T] = [0,2]$

$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x) f_{X_2}(x_2)$

$= \mathbb{1}_{x_1 \in (0,1)} \cdot \mathbb{1}_{x_2 \in (0,1)}$



$$F(t) = \begin{cases} 0 \text{ if } t \le 0 \\ \frac{t^2}{2} \text{ if } t \in (0,1) \\ -\frac{t^2}{2} + 2t - 1 \text{ if } t \in (1,2) \\ 1 \text{ if } t > 2 \end{cases}$$

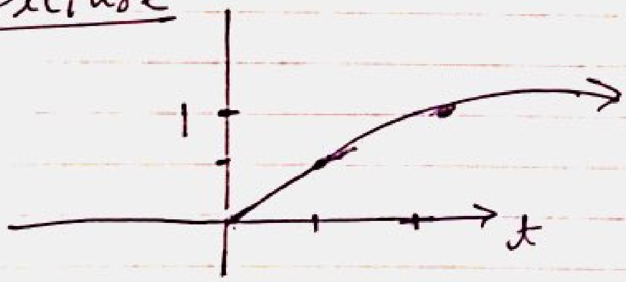
$A_{\square} = \frac{t^2}{2} - 2\left(\frac{(t-1)^2}{2}\right)$ → subtract the little triangle

$= \frac{t^2}{2} - t^2 + 2t - 1$

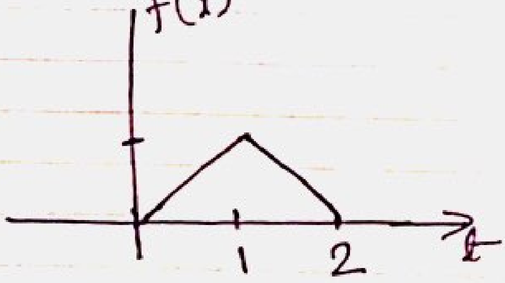
$= -\frac{t^2}{2} + 2t - 1$

need get ans

<u>picture</u>



$$\begin{cases} 0 \text{ if } t \le 0 \\ t \text{ if } t \in (0,1) \\ 2-t \text{ if } t \in (1,2) \\ 0 \text{ if } t > 2 \end{cases}$$



$f(t) = \int_0^1 (1)(1) \mathbb{1}_{t-x \in \{0,1\}} dx$

$= \int_0^1 \mathbb{1}_{x \in [t-1,t]} dx$

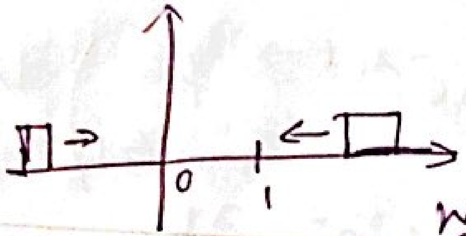
$= \mathbb{1}_{x-t \in (-1,0)}$

$\mathbb{1}_{x \in (t-1,t)}$

$t = -9$

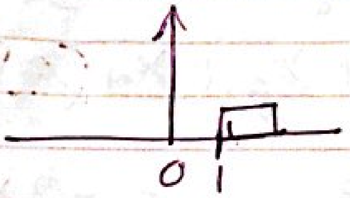
$\mathbb{1}_{x \in (-10,-9)}$

$$\begin{cases} 0 \text{ if } t \le 0 \\ t \text{ if } t \in (0,1) \\ 2-t = 1-(t-1) \text{ if } t \in [1,2] \\ 0 \text{ if } t > 2 \end{cases}$$

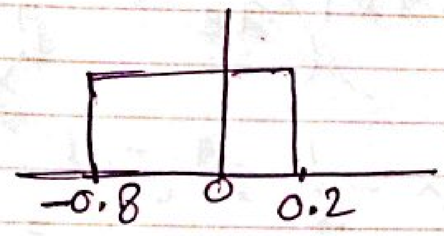


we can use either side to find the ans.

$t=10 \quad \mathbb{1}_{x\in(9,10)}$



$t=0.2$



$$= \begin{cases} 0 \text{ if } t \le 0 \\ t \text{ if } t\in(0,1) \\ 2-t \text{ if } t\in(1,2) \\ 0 \text{ if } t>2 \end{cases}$$

watch: video wikipedia

$T = X_1+X_2 \sim f_T(t) = ?\,,\quad X_1+X_2 \overset{iid}{\sim} Exp(\lambda)$

$$f_{T_2}(t) = \int_{supp[X_1]} f(x)\, f(t-x)\, \mathbb{1}_{t-x\in supp[X]}\, dx$$

$$f_{T_2}(t) = \int_0^\infty (\lambda e^{-\lambda x})\,(\underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t} e^{\lambda x}})\, \mathbb{1}_{t-x\in(0,\infty)}\, dx$$

$$= \lambda^2 e^{-\lambda t}\int_0^\infty \mathbb{1}_{x\le t}\, dx = \lambda^2 e^{-\lambda t}\int_0^t dx$$

$$= t\lambda^2 e^{-\lambda t} \ne Exp(\lambda)$$

$$supp[T_2] = (0,\infty)$$

using this :-

$$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = T_2 + X_3$$

$$f_{T_2}(t) = \int_0^\infty (\lambda e^{-\lambda x})\,(\underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t}\, \cancel{e^{\lambda x}}})\, \mathbb{1}_{t-x\in(0,\infty)}\, dx$$

⑥

$$f_{T_3}(t) = \int_0^{\infty} \left(x\lambda^2 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x \, dx = \frac{1}{2} t^2 \lambda^3 e^{-\lambda t}$$

$$T_4 = \underbrace{X_1 + X_2 + X_3}_{T_3} + X_4 = T_3 + X_4$$

$$f_{T_4}(t) = \int_0^{\infty} \left(\frac{1}{2} t^2 \lambda^3 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2 \, dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \frac{1}{3} t^3$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$$

$$f_{T_5}(t) = \int_0^{\infty} \left(\frac{1}{2 \cdot 3} x^3 \lambda^4 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{t \le t}\right) dx$$

$$= \frac{1}{2 \cdot 3} \lambda^5 e^{-\lambda t} \int_0^t x^3 \, dx = \frac{1}{2 \cdot 3 \cdot 4} \lambda^5 e^{-\lambda t} t^4$$

In General

$$f_{T_k}(t) = \frac{1}{(k-1)!} \lambda^k e^{-\lambda t} t^{k-1} = \text{Erlang}(k, \lambda)$$

$$\text{supp}[T_k] = (0, \infty)$$

$$\frac{1}{\lambda} \in \mathbb{N}, \lambda \in (0, \infty)$$

$$f_{T_2}(t) = \lambda^2 e^{-\lambda t} t$$

$$f_{T_3}(t) = \frac{1}{2} \lambda^3 e^{-\lambda t} t^2$$

$$f_{T_4}(t) = \frac{1}{2 \cdot 3} \lambda^4 e^{-\lambda t} t^3$$

$$f_{T_5}(t) = \frac{1}{2 \cdot 3 \cdot 4} \lambda^5 e^{-\lambda t} t^4$$