(lecture 12)

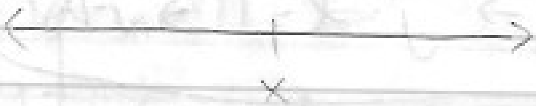
October 28th, 2019

$X_1, \dots, X_n \overset{iid}{\sim} f(x)$ & CDF $F(x)$

$$\Rightarrow F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

CDF of the k^{th} order statistic

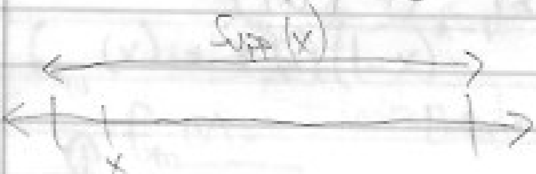
$L \sim Bin(n, p = F(x))$



n trials of landing on this number line

prob of landing $\le x$ = $F(x)$

L = # of landings $\le x$



$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[F_{X_{(k)}}(x)\right]$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx}\left[\underbrace{F(x)^j}_{u}\underbrace{(1-F(x))^{n-j}}_{v}\right]$$

$$= \sum_{j=k}^{n} \binom{n}{j} F(x)^j (-f(x))(n-j)(1-F(x))^{n-j-1} + (1-F(x))^{n-j} F(x)^{j-1} j f(x)$$

$$= j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$$= f(x) \left(\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} j F(x)^{j-1} (1-F(x))^{n-j} - \underbrace{\sum_{j=k}^{n-1} \frac{n!}{j!\underbrace{(n-j)!}_{\to (n-j-1)!}} (n-j) F(x)^{j} (1-F(x))^{n-j-1}}_{n-(j+1)} \right)$$

if $j = n \to$ term $= 0$
$\to j \in \{k, \cdots, n-1\}$

Reindex and let $l = j+1 \to j = l-1 \to j-1 = \cdots$

$$\to \sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!} F(x)^{l-1} (1-F(x))^{n-l}$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k}$$

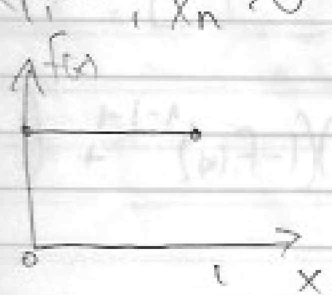
Density for minimum

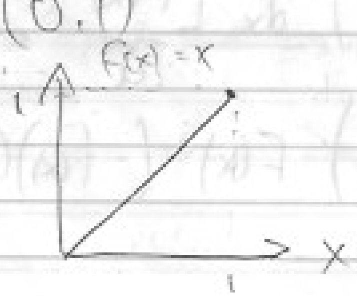
$$f_{X_{(1)}}(x) = n f(x) (1-F(x))^{n-1}$$

Density for maximum

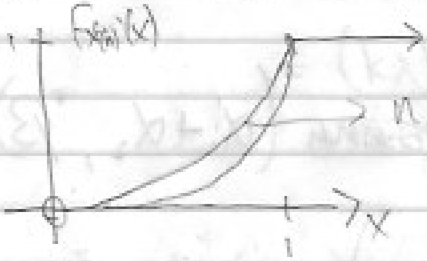
$$f_{X_{(n)}}(x) = n f(x) F(x)^{n-1}$$

$X_1, \cdots, X_n \overset{iid}{\sim} U(0,1)$

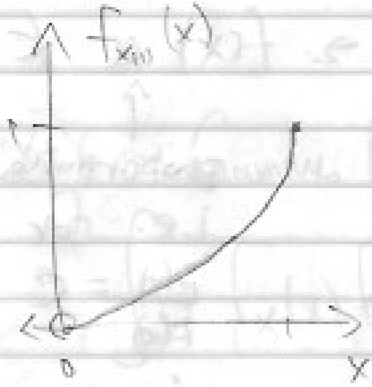




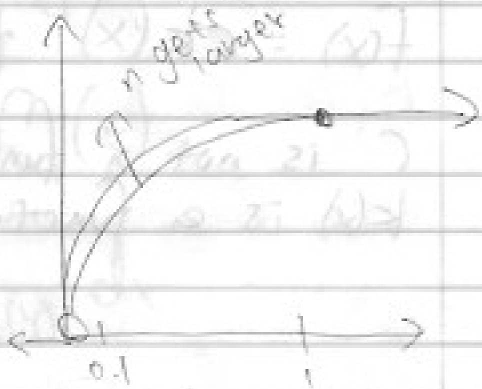
$F_{X_{(n)}}(x) = F(x)^n = x^n$



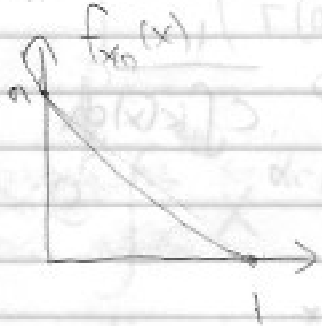
$f_{X_{(n)}}(x) = n(1)\,x^{n-1} = \text{Beta}(1,n)$



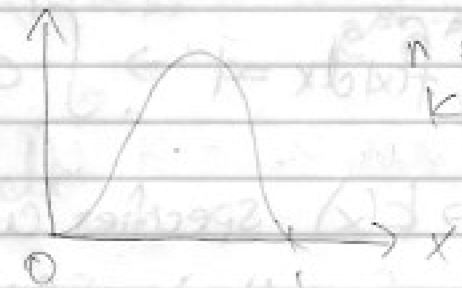
$F_{X_{(1)}}(x) = 1-(1-F(x))^n = 1-(1-x)^n$



$f_{X_{(1)}}(x) = n(1)(1-x)^{n-1}$
$= \text{Beta}(n,1)$



$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}$



n is large
$k = \frac{1}{2}n$

$$f_{X_{(n)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k-1)} x^{k-1}(1-x)^{n-k+1-1}$$

$= \text{Beta}(k, n-k+1) \rightarrow$ order statistics for iid $U(0,1)$ are beta

$X \sim Gamma(\alpha_1, \beta)$, $X, Y \sim$ independent
$Y \sim Gamma(\alpha_2, \beta)$

we would expect $X+Y \sim Gamma(\alpha_1+\alpha_2, \beta)$

kernel:

$p(x) = c\,k(x)$ for any PMF $\rightarrow p(x) \propto k(x)$

$f(x) = c\,k(x)$ for any PDF $\rightarrow f(x) \propto k(x)$

proportional to
i.e. for any x, $\frac{p(x)}{k(x)} = c$

c is not a function of x
$k(x)$ is a function of x

$$\sum_{Supp(x)} p(x) = 1 \rightarrow \sum_{Supp(x)} c\,k(x) = 1 \rightarrow c = \frac{1}{\sum k(x)}$$

$$\int f(x)dx = 1 \rightarrow \int c\,k(x)dx = 1 \rightarrow \frac{1}{c\int k(x)dx}$$

$\rightarrow k(x)$ specifies a r.v.

$$X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$
$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x}_{k(x)}$$
$$\propto \frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x$$

$X \sim \text{Weibull} := (k\lambda)(\lambda x)^{k-1} e^{-(\lambda x)^k}$

$= \underbrace{(k\lambda)\lambda^{k-1}}_{C} \underbrace{x^{k-1} e^{-(\lambda x)^k}}_{k(x)} \propto x^{k-1} e^{-(\lambda x)^k}$

$X \sim \text{Gamma}(\alpha, \beta) := \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{C} \underbrace{x^{\alpha-1} e^{-\beta x}}_{k(x)} \propto x^{\alpha-1} e^{-\beta x}$

$X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta) = f_{X+Y}(t)$

$= \int_{\text{Supp}(X)} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in \text{Supp}(Y)} \, dx$

$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-\beta x}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \underbrace{e^{\beta(t-x)}}_{e^{\beta t} e^{-\beta x}} \underbrace{\mathbb{1}_{t-x \in (0,\infty)}}_{x \le t}\right) dx$

$\propto e^{\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} e^{-\beta t} \, dx$

Let $u = \frac{x}{t} \rightarrow x=0, u=0$, if $x=t$, $u=1$

$x = ut$ $\frac{du}{dx} = \frac{1}{t} \rightarrow dx = t\,du$

$= e^{\beta t} \int_0^1 (tu)^{\alpha_1-1} (t-tu)^{\alpha_2-1} t \, du$

$$= \underbrace{t^{\alpha_1 - 1 + \alpha_2 - 1 + 1}}_{\alpha_1 + \alpha_2 - 1} e^{\beta t} \overbrace{\int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du}^{h(\alpha_1, \alpha_2) \neq h(t)}$$

$$\propto \underbrace{t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t}}_{\text{Kernel for Gamma}} \propto \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\underbrace{\Gamma(\alpha_1 + \alpha_2)}_{c}} t^{\alpha_1 + \alpha_2 - 1} e^{\beta t}$$

$$B(\alpha_1, \alpha_2) := \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du \qquad \int_0^\infty u^{\alpha_1 - 1} e^{-u}\, du$$

"Beta function"

$$\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2)} = \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

$$B(\alpha_1, \alpha_2) = \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du,$$

$$B(a, \alpha_1, \alpha_2) = \int_0^a u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du \rightarrow \text{"Incomplete Beta Function"}$$

$$I_a(\alpha_1, \alpha_2) := \frac{B(a, \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \rightarrow \text{Regularised Incomplete Beta Function}$$

$$X \sim \text{Beta}(\alpha, \beta) = \frac{1}{\underset{\displaystyle \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}}{\overset{\shortparallel}{B(\alpha, \beta)}}} x^{\alpha - 1}(1-x)^{\beta - 1}$$

$\mathrm{Supp}(X) = [0,1]$, $\alpha, \beta > 0$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)}$$

$$= I_x(\alpha,\beta)$$