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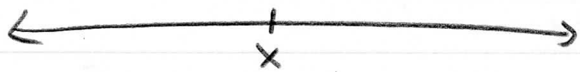
- $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ & CDF $F(x)$

$$\Rightarrow F_{X^{(n)}}(x) = \sum_{j=K}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

"CDF of the $K^{th}$ order Statistic"

$L \sim Bin(n, p = F(x))$




n trials of landing on this # line.
prob of landing $\leq x = F(x)$
$L =$ # of landings $\leq x$

$$f_{X_{(K)}}(x) = \frac{d}{dx}\left[F_{X_{(K)}}(x)\right]$$

$$= \frac{d}{dx}\left[\sum_{j=K}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=K}^{n} \binom{n}{j} \frac{d}{dx}\left[F(x)^j (1-F(x))^{n-j}\right]$$

Product Rule $= F(x)^j(-f(x))(n-j)(1-F(x))^{n-j-1}$

$$+ (1-F(x))^{n-j} f(x) \cdot j \cdot f(x)^{j-1}$$

$$= j f(x) F(x)^{j-1}(1-F(x))^{n-j}$$

$$- (n-j) f(x) F(x)^j (1-F(x))^{n-j-1}$$

$$= f(x)\left[ \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \, j \, F(x)^{j-1}(1-F(x))^{n-j} \right.$$

$$\left. - \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} (n-j) F(x)^{j} (1-F(x))^{n-j-1} \right]$$

( Note: For $2^{nd}$ term when $j=n$, term $=0$, so $j \in \{k, \dots, n-1\}$ )

Reindex: Let $\ell = j+1$

$\Rightarrow j = \ell - 1$, so $n-(j+1) = n-\ell$

$2^{nd}$ term: $\displaystyle\sum_{\ell=k+1}^{n} \frac{n!}{(\ell-1)!(n-\ell)!} F(x)^{\ell-1}(1-F(x))^{n-\ell}$

$$\boxed{f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1}(1-F(x))^{n-k}}$$

$f_{X_{(1)}}(x) = n f(x)(1-F(x))^{n-1}$ (density for minimum)

$f_{X_{(n)}}(x) = n f(x) F(x)^{n-1}$ (density for maximum)

" Beta $(n, 1)$ "

- $X \sim Gamma(\alpha_1, \beta)$
  $Y \sim Gamma(\alpha_2, \beta)$

We would express $X + Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$

– "Kernels"

$p(x) = c\,K(x)$ for any PMF $\Rightarrow p(x) \propto K(x)$

$f(x) = c\,k(x)$ for any PDF $\Rightarrow f(x) \propto K(x)$

not a function of f

is a function of f

proportional

for any $x$, $\frac{p(x)}{K(x)} = c$

(just like proportional triangles)

- $\sum_{Supp[X]} p(x) = 1 \Rightarrow \sum_{Supp[X]} c\,K(x) = 1 \Rightarrow c = \frac{1}{\sum K(x)}$

- $\int f(x)\,dx = 1 \Rightarrow \int c\,K(x)\,dx = 1 \Rightarrow c = \frac{1}{\int K(x)\,dx}$

$K(x)$ specifies a r.v

- $X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \underbrace{n!\,(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x}_{K(x)} \propto \frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x$$

- $X \sim \text{Weibull}(\kappa, \lambda) := \underbrace{(\kappa\lambda)(\lambda x)^{\kappa-1}}_{c} \underbrace{e^{-(\lambda x)^{\kappa}}}_{\kappa(x)} \propto x^{\kappa-1} e^{-(\lambda x)^{\kappa}}$

- $X \sim \text{Gamma}(\alpha, \beta) := \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x}}_{\kappa(x)} \propto x^{\alpha-1} e^{-\beta x}$

- $X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta) \underset{\cdot}{=} f_{X+Y}(t)$

$$= \int_{\text{Supp}[X]} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in \text{supp}[X]}$$

$$= \int_0^{\infty} \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{\beta x}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \underbrace{e^{\beta(t-x)}}_{e^{\beta t} e^{-\beta x}}\right) \underbrace{\mathbb{1}_{t-x \in (0,\infty)}}_{x \le t} dx$$

$$= \propto \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} e^{\beta t} \, dx$$

$$= \propto e^{\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} \, dx$$

$$= e^{\beta t} \int_0^1 (tu)^{\alpha_1 - 1} (t - tu)^{\alpha_2 - 1} t \, du$$

$$= t^{\overbrace{\alpha_1 - 1 + \alpha_2 - 1 + 1}^{\alpha_1 + \alpha_2 - 1}} e^{\beta t} \overbrace{\int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du}^{h(\alpha_1, \alpha_2) \neq h(t)}$$

$$\propto \underbrace{t^{\alpha_1 + \alpha_2 - 1} e^{\beta t}}_{\text{Kernel for the gamma}} \propto \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

$$= \underbrace{\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2)}}_{c} t^{\alpha_1 + \alpha_2 - 1} e^{\beta t}$$

u-sub

Let $u = \frac{x}{t}$

$\Rightarrow x = 0 \Rightarrow u = 0$

$x = t \Rightarrow u = 1$

$\Rightarrow x = ut$

$\Rightarrow \frac{du}{dx} = \frac{1}{t}$

$\Rightarrow dx = t \, du$

- $B(\alpha_1, \alpha_2) := \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du$
"Beta Function"

$$\frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)} = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1, \alpha_2)$$

$$\Rightarrow B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

"Beta-Gamma Identity"

- $B(a, \alpha_1, \alpha_2) := \int_0^a u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\, du$

"Incomplete Beta Function"

- $I_a(\alpha_1, \alpha_2) := \dfrac{B(a, \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)}$

"Regularized Incomplete Beta Function"

- $X \sim \mathrm{Beta}(\alpha, \beta) = \underbrace{\frac{1}{B(\alpha, \beta)}}_{\displaystyle = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}} x^{\alpha-1}(1-x)^{\beta-1}$

$\mathrm{Supp}[X] = [0, 1]$
$\alpha, \beta > 0$

$$F(x) = \int_0^x \frac{1}{B(\alpha, \beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta)$$