Lecture #18

$X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$Z_1 = \frac{X_1 - \mu}{\sigma}, \dots\dots, Z_n = \frac{X_n - \mu}{\sigma} \overset{iid}{\sim} N(0,1)$

$\vec{Z}^T \vec{Z} = \sum_{i=1}^{n} Z_i^2 = \Sigma\left(\frac{X_i - \mu}{\sigma}\right)^2 =$ algebra

$\frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2_n$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right) \sim N(0,1)$

$\frac{n(\bar{X}-\mu)^2}{\sigma^2} = \frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}} = \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi_1^2$

$U \sim \chi^2_{k_1}$ and $U_2 = \chi^2_{k_2} \Rightarrow U_1 + U_2 \sim \chi^2_{k_1} + \chi^2_{k_2}$

conjecture

(1) $\frac{(n-1)S^2}{\sigma^2}$ is independent of $\frac{n(\bar{X}-\mu)^2}{\sigma^2}$

(2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$\vec{Z}^T\vec{Z} = \vec{Z}^T \underbrace{I_n}_{\text{Identity}} \vec{Z}$

quadratic formula

consider $\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = Z_1^2 \sim \chi_1^2$

$$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & \cdots & \ddots & 0 \end{bmatrix}}_{B_2} \vec{Z} = Z_2^2 \sim \chi_1^2$$

②

$$\vec{Z}^T \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \\ & & 0 & 0 \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \vec{Z} = Z_n^2 \sim \chi_1^2$$

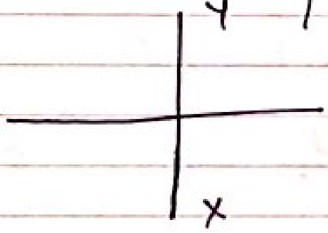
Note: ① All quadratic forms are independent

② $B_1 + B_2 + \cdots + B_n = I_n$

③ $\text{rank}[B_1] = \text{rank}[B_2] = \cdots = \text{rank}[B_n] = 1$

$$\Sigma\, \text{rank}[B_i] = n$$

matrix $A \cdot \bar{X} \in \{$space all dimension rank$\}$



Cochran's Theorem (1934)

$Z_1, \cdots, Z_n \overset{iid}{\sim} N(0,1)$ If

ⓐ $B_1, \cdots, B_K = I_K = I_n$ and

ⓑ $\sum_{j=1}^{k} \text{rank}[B_j] = n$ then

ⓐ $\vec{Z}^T B_j \vec{Z} \sim \chi^2_{\text{rank}[B_j]}$

ⓑ $\vec{Z}^T B_{j_1} \vec{Z}$ is indep of $\vec{Z} B_{j_2} \vec{Z} \; \forall j_1 = j_2$

$$\vec{Z}^T B_1 \vec{Z} + \cdots + \vec{Z}^T B_n \vec{Z}$$
$$= \vec{Z}^T (B_1 \vec{Z} + \cdots + B_n \vec{Z}) = \vec{Z}^T (B_1 + \cdots + B_n) \vec{Z} = \vec{Z}^T \vec{Z}$$

3

$$\vec{Z}^T\vec{Z} = \sum Z_i^2 = \sum\left((Z_i-\bar{Z}) + \bar{Z}\right)^2$$
$$= \sum\left((Z_i-\bar{Z})^2 + 2(Z_i-\bar{Z})\bar{Z} + (\bar{Z})^2\right)$$
$$= \sum(Z_i-\bar{Z})^2 + 2\sum Z_i\bar{Z} - \sum\bar{Z}^2) + \sum\bar{Z}^2$$
$$= \sum(Z_i-\bar{Z})^2 + n\bar{Z}^2$$

Note $\bar{Z} = \frac{1}{n}(Z_1 + \dots + Z_n) = \frac{1}{n}\vec{I_n}^T\vec{Z}$

$$= \frac{1}{n}\vec{Z}^T\vec{I_n} \Rightarrow n\bar{Z}^2 = n\left(\frac{1}{n}\vec{Z}^T I_n\right)\left(\frac{1}{n}I_n^T\vec{Z}\right)$$
$$= \vec{Z}^T\left(\frac{1}{n}\vec{I_n}I_n^T\right)\vec{Z} = \vec{Z}^T\left(\frac{1}{n}J_n\right)\vec{Z}$$
$$= \vec{Z}^T B_2\vec{Z}$$

Define $\vec{I_n} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$J_n = \vec{I_n}I_n^T = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}$

$B_2 = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \rightarrow \text{rank}(B_2) = 1$

$B_2\vec{X} = \begin{bmatrix} \bar{X} \\ \bar{X} \\ \vdots \\ \bar{X} \end{bmatrix}$

$$\sum(Z_i-\bar{Z})^2 = \sum Z_i^2 - 2Z_i\bar{Z} + \bar{Z}^2$$
$$= \sum Z_i^2 - 2n\bar{Z}^2 + n\bar{Z}^2 = \sum Z_i^2 - n\bar{Z}^2$$
$$= \vec{Z}^T I\vec{Z} - Z^T\left(\frac{1}{n}J_n\right)\vec{Z} = \vec{Z}^T\underbrace{\left(I_n - \frac{1}{n}J_n\right)}_{B_1}\vec{Z}$$

$$B_1 + B_2 = \left(I_n - \frac{1}{n}J_n\right) + \frac{1}{n}J_n = I_n$$

$$\text{rank}(B_1) + \text{rank}(B_2) = 1 + (n-1) = n$$

$tr(B_1) = n(1-\frac{1}{n}) = n-1$

trace

$$B_1 = \begin{bmatrix} (1-\frac{1}{n}) & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & (1-\frac{1}{n}) & \cdots & \\ \vdots & & \ddots & \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & (1-\frac{1}{n}) \end{bmatrix}$$

Theorem: If A is symmetric and independent (ie $AA=A$) then rank $[A]$ = trace $[A]$

$(I_n - \frac{1}{n}J_n)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J \Rightarrow$ symmetric

$(I_n - \frac{1}{n}J_n)(I_n - \frac{1}{n}J_n)$

$= I_n - 2(\frac{1}{n}J_n) + \frac{1}{n^2} \cdot n \cdot J_n = I_n - 2\frac{1}{n}J_n + \frac{1}{n}J_n$

$= I_n - \frac{1}{n}J_n \Rightarrow$ independent.

By Cochran's Theorem: $\rightarrow \chi^2_{n-1}$ $\rightarrow \chi^2_1$

$$\vec{Z}^T\vec{Z} = \sum_{i=1}^{n}(Z_i - \bar{Z})^2 + n\bar{Z}^2$$

independent

$\boxed{n\bar{Z}^2 = n}$

$$\bar{Z} = \frac{Z_1 + Z_2 + \cdots + Z_n}{n} = \frac{\frac{X_1-\mu}{\sigma} + \frac{X_2-\mu}{\sigma} + \cdots + \frac{X_n-\mu}{\sigma}}{n}$$

$$\bar{Z} = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{n\sigma}$$ By Cochran's theorem

$$n\bar{Z}^2 = n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 = \frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2_1 \rightarrow \text{ind}$$

$$\sum(Z_i - \bar{Z})^2 = \sum\left(\frac{X_i-\mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2$$

5

$$= \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \rightarrow \text{independent}$$

$\Rightarrow S^2_1 \bar{x}$ are independent

If σ is known and you wish to test Null hypothesis $H_0 : \mu = \text{value}$

$$\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \rightarrow \text{"one sample z test"}$$

You wish to test $H_0 : \sigma^2 = \text{same value}$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$ "One sample variance test" χ^2 test of variance

You wish to test for two independent samples

$H_0 : \sigma_1^2 = \sigma_2^2$

$$\frac{\frac{(n_1-1)S_1^2}{\sigma_1^2} \big/ n-1}{\frac{(n_2-1)S_2^2}{\sigma_2^2} \big/ n_2-1} = \frac{S_1^2}{S_2^2} \sim F_{n_1, n_2}$$

"F test for equality of variance".

You want to test $H_0: \mu = $ value but you don't know σ

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}S^2}}$$

$$= \frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} = \frac{Z}{\sqrt{\frac{U}{n-1}}} \sim T_{n-1}$$

"one same T test"

Note: $\frac{\bar{X}-\mu}{\sigma} = Z \sim N(0,1)$ independent

$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Multivariate Normal

$$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} \text{ where } Z_1, Z_2, \dots, Z_n \overset{iid}{\sim} N(0,1)$$

$$E[Z] = \vec{0}$$

$$\text{var}[\vec{Z}] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ 0 & & \ddots & & 0 \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix} = I_n \qquad \text{var}(\vec{Z}) = I_n$$

$$f_{\vec{Z}}(\vec{z}) = f_{Z_1 Z_2 \cdots Z_n}(z_1, z_2, z_3 \cdots z_n)$$
$$= f_{Z_1}(z_1)\, f_{Z_2}(z_2) \cdots f_{Z_n}(z_n)$$
$$= \prod_{i=1}^{n} f(z_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{\frac{-z_i^2}{2}}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$$= N_n(\vec{0}_n, I_n)$$

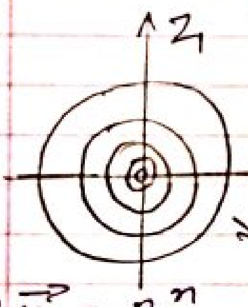
multivariate normal — dimension — mean — variance

"Multivariate Normal"

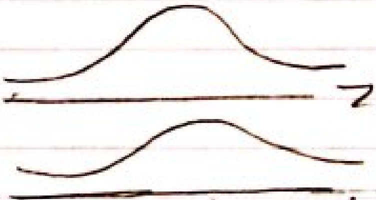
expetation

$$\vec{z} \sim N_2\left(\begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

variance-covariance matrix



ball in 3 dimensions



① $\vec{\mu} = R^n$

$$\vec{x} = \vec{z} + \mu = \begin{bmatrix} z_1+\mu_1 \\ z_2+\mu_2 \\ z_n+\mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1, 1) \\ \sim N(\mu_2, 1) \\ \sim N(\mu_n, 1) \end{matrix} \to \text{Independent} = N_n(\vec{\mu}, I_n)$$

② Let $A = \begin{bmatrix} 1&0&0&0&0&\cdots&0 \\ 1&1&0&0&0&\cdots&0 \\ 1&1&1&0&0&\cdots&0 \\ \vdots&&&&&\ddots& \\ 1&1&1&1&1&1&1 \end{bmatrix}$ and $\vec{x} = A\vec{z} = \begin{bmatrix} z_1 \\ z_1+z_2 \\ z_1+z_2+z_3 \\ \vdots \\ z_1+z_2+z_3+\dots+z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \vdots \\ \sim N(0,n) \end{matrix}$ these are not indep

$$x_j = \sum_{i=1}^{j} z_i \sim N(0, j)$$

$\text{cov}[x_1, x_2]$

$= \text{cov}[z_1, z_2, z_3)$

$= \text{cov}[z_1, z_2) + \text{cov}(z_1, z_n)$

$= \text{var}(z_1) + 0$

$= 1 + 0 \neq 1 \Rightarrow x_1, x_2$ are dependent

$E[X] = E[A\vec{x}] = ?$ $\quad \text{var}[\vec{x}] = \text{var}[A\vec{x}] = ?$