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$X_1, \dots, X_r \overset{iid}{\sim} Geo(p)$

$$T = \sum_{i=1}^{r} X_i \sim NegBin(r, p) := \binom{t+r-1}{r-1}(1-p)^t p^r$$

$t + r$ total experiments.

$$\underline{0}\ \underline{1}\ \underline{0}\ \underline{1}\ \dots\ \underset{t+r-1}{\underline{1}}\ \underset{t+r}{\underline{1}}$$

$r-1$

$X \sim Bin(n, p) = \binom{n}{x} p^x (1-p)^{n-x} = p(x)$

let n get large and p get small but peg

$n \in \mathbb{N}$ $\quad \lambda = np \Rightarrow p = \frac{\lambda}{n}$

$p \in (0,1)$

$\lambda \in (0, \infty)$

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} =$$

(take out any constants.)

$$\frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\, n^x} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\downarrow$$

(A) $\dfrac{n(n-1)(n-2)\cdots\cdots(n-x+1)}{(n)(n)\cdots\cdots(n)}$] Next page.

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$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x} = \text{goes to } 1$$

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \boxed{\lim \frac{n}{n} \lim \frac{n-1}{n} \cdots \lim \frac{n-x+1}{n}}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$1 \quad 1 \quad 1$$

$$= X \sim \text{poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\text{Support}(X) = \{0, 1, \ldots\} = \mathbb{N}_0$$

9/5 $$p(t) = \sum_{x \in Supp(X)} P_{0,\lambda}(x)\, P_{0,\lambda}(t-x)\, \mathbb{1}_{t-x \in Supp(X)}$$

$$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\dots\}} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!}\, \mathbb{1}_{t-x \in Supp(X)}$$

$\in \mathbb{N}_0$

when $t-x \in \mathbb{N}_0$ you can convert to this.

$$= e^{-2\lambda} \lambda^t \sum \frac{1}{x!\,(t-x)!}\, \mathbb{1}_{x \le t}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\dots,t\}} \left[\frac{1}{x!\,(t-x)!} \cdot \frac{t!}{t!}\right]$$ nice trick

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,1,\dots t\}} \binom{t}{x} = \frac{2\lambda^t e^{-2\lambda}}{t!} = \text{poisson}(2\lambda)$$

combinatorial aside:

$A = \{a_1, a_2 \dots a_n\}$ $|A| = n$

$2^A :\doteq \{B : B \subseteq A\} = \{B : B \subseteq A, |B| = 0\} \cup$

power set of A

partition into disjoint subsets by size.

$\{B : B \subseteq A, |B| = 1\} \cup$

$\vdots$

$\{B : B \subseteq A, |B| = n\}$

$2^n = |2^A| = |\{B : B \subseteq A, |B| = 1\} + \{B : B \subseteq A, |B| = 2\}$

$+ \{B : B \subseteq A, |B| = 3\} + \dots$

$\{B : B \subseteq A, |B| = n\}$

$\binom{n}{0}$ $\binom{n}{1}$

$= 1 + n + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n$

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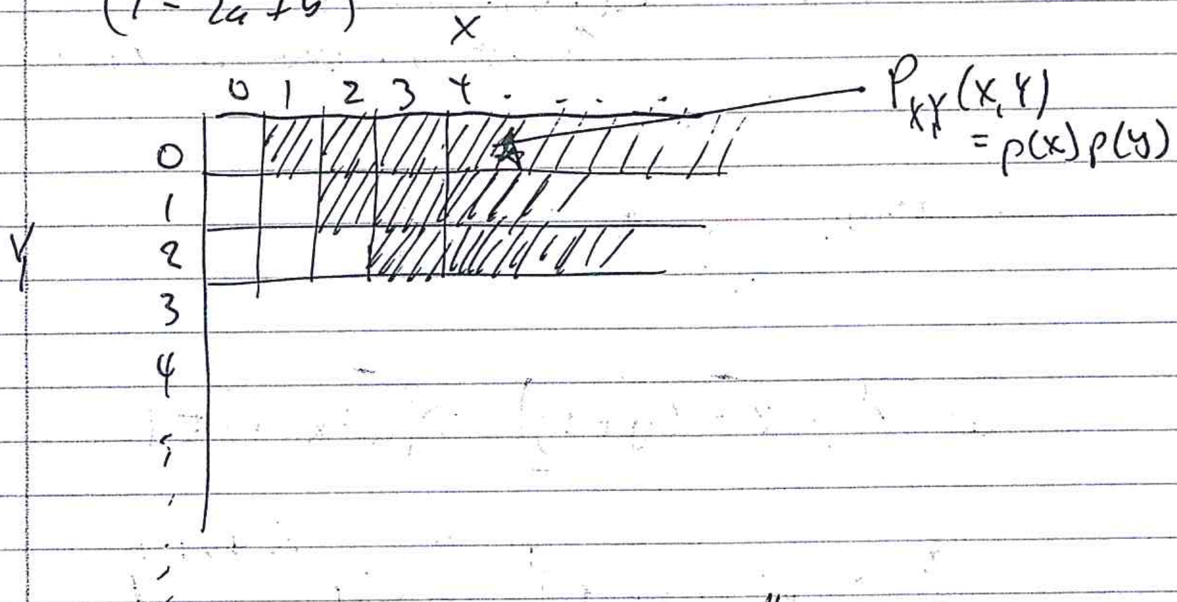
$X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X>Y) = P(Y>X) < \frac{1}{2}$ why?

$$1 = \underbrace{P(X>Y) + P(Y>X)}_{\text{these are equal}} + \underbrace{P(X=Y)}_{>0}$$

$(1 = 2a + b)$



$$P(X>Y) = \sum_{y\in\mathbb{R}} \sum_{x\in\mathbb{R}} p_{X,Y}(x,y)\, \mathbb{1}(x>y)$$

use indicator to indicate the sum and limit whats going on.

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$$= \sum_{y \in \mathbb{R}} \sum_{x \in (y,\infty)} p(1-p)^x \, p(1-p)^y \, \mathbb{1}_{x \in \mathbb{N}_0} \mathbb{1}_{y \in \mathbb{N}_0}$$

pull out constants.

$$p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \sum_{x \in \{y+1, y+2, \dots\}} (1-p)^x$$

(first y > itself is $y+1$)

$$\boxed{\text{if } a \in (0,1) \quad \sum_{i=0}^{\infty} a^i = \frac{1}{1-a}}$$

$\boxed{\text{Let } x' = x-(y+1)} \Rightarrow \boxed{x = x' + y + 1}$

$$= p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \sum_{x' \in \mathbb{N}_0} (1-p)^{x'+y+1}$$

[Re-indexing]

$$(1-p)^{x'}(1-p)^y(1-p)$$

$$p^2(1-p) \sum_{y \in \mathbb{N}_0} (1-p)^{2y} \sum_{x' \in \mathbb{N}_0} (1-p)^{x'} \Rightarrow \frac{1}{1-(1-p)} = \boxed{\frac{1}{p}}$$

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$$p(1-p)\sum_{y\in\mathbb{N}_0}(1-p)^{2y} = p(1-p)\underbrace{\sum_{y\in\mathbb{N}_0}\left((1-p)^2\right)^y}_{\text{use geometric series}} =$$

$$\boxed{\frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2} = \frac{1}{p(2-p)}}$$

$$= \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \frac{1-p}{2-p}$$

As p gets small answer tends to $\frac{1}{2}$.

X is a discrete r.v.

$$E(X) = \sum_{x\in\mathbb{R}} x\,p(x)$$

$$E[g(X)] = \sum_{x\in\mathbb{R}} g(x)\,p(x)$$

Now X, Y are discrete r.v.

$$E[g(X,Y)] = \sum_{x\in\mathbb{R}}\sum_{y\in\mathbb{R}} g(x,y)\,p(x,y)$$

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consider a simple function.

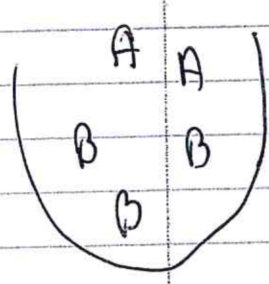
$$E\left[\mathbb{1}_{X\in A}\right] = \sum_{x\in\mathbb{R}} \mathbb{1}_{x\in A}\, p(x) = \sum_{x\in A} p(x) = p(X\in A)$$

$$E\left[\mathbb{1}_{X>Y}\right] = p(X>Y)$$

$$\parallel$$

$$\sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p(x,y)\, \mathbb{1}_{x>y}$$

Begin multinomial:



$p_1 :=$ probability of drawing an apple

$p_2 :=$ " " " " Banana

$$p_1 + p_2 = 1$$

Draw $\underline{n}$ with replacement

$X_1 :=$ # of apples drawn

$X_2 :=$ # of bananas drawn

$$X_1 + X_2 = \underline{n}$$

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$X_1 \sim \text{Binomial}(n, p_1)$

$X_2 \sim \text{Binomial}(n, p_2)$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{X_1 X_2}(x_1, x_2) = p_1^{x_1} p_2^{x_2}$$

$$\binom{n}{x_1} = \frac{n!}{x_1!(n-x_1)!} = \boxed{\frac{n!}{x_1! \, x_2!}}$$

Example:
$n = 8$
$x_1 = 5$
$x_2 = 3$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim p_{X_1 X_2 X_3}(x_1, x_2, x_3) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

expanding out.

→ add-in indicator functions

$$\frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\dots n\}} \mathbb{1}_{x_2 \in \{1,2 \dots n\}}$$

$$\binom{n}{x_1 \; x_2} := \frac{n!}{x_1! x_2!} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\dots n\}} \mathbb{1}_{x_2 \in \{0,1,\dots n\}}$$

multichoose
or multinomial
co-effecient

$$X \sim \text{multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) \sim \binom{n}{x_1 \; x_2} p_1^{x_1} p_2^{x_2}$$