Poisson Process

$T_K \sim Erlang(k, \lambda)$

$N \sim Poisson(\lambda)$

$P(T_K > 1) = P(N \leq k-1)$

$\Rightarrow 1 - F_{T_K}(1) = F_N(k-1) = Q(k, \lambda)$

r.v.'s for counting and waiting

	Fixed time, count #	Fixed # measure time
Discrete	Bernoulli/Binomial	Geometric/Neg Bin
Continuous	Poisson	Exponential/Erlang

Ex

Experiments occur in discrete time. What is the probability of zero successes by time $t = 50$ if the probability of success $= 0.1$?

N = # of successes $\quad N \sim Binom(50, 0.1)$

T = amount of time $\quad T \sim NegBin(1, 0.1)$

$$P(N=0) = P(T > 49)$$

What is the probability of k or fewer successes ($\le k$) by time t if probability of success is p?

$N \sim Binom(t, p)$

$T \sim NegBin(k+1, p)$

up to $k+1$ failures

$$P(N \le k) = P(T > t-k-1)$$
$$= P(T \ge t-k)$$
$$F_N(k) = 1 - F_T(t-k-1)$$

$$F_N(k) = 1 - F_T(t-k-1)$$
$$\sum_{i=0}^{k} \binom{t}{i} p^i (1-p)^{t-i} = 1 - \sum_{i=0}^{t-k-1} \binom{k+i}{k} (1-p)^i p^k$$

Poisson Process

$$T \sim Erlang(k, \lambda) := \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$$
$$= \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$$

this is an example of a Gamma dist.

$$T \sim NegBin(k, p) := \binom{k+t-1}{k-1} (1-p)^t p^t \mathbb{1}_{t \in \underbrace{\mathbb{N} \cup \{0\}}_{\mathbb{N}_0}}$$
$$= \frac{(k+t-1)!}{(k-1)!\, t!} (1-p)^t p^t \mathbb{1}_{t \in \mathbb{N}_0}$$
$$= \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^t \mathbb{1}_{t \in \mathbb{N}_0}$$

$X \sim Gamma$

$$:= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Gamma dist.

r.v.
$X \sim \text{Gamma}(\alpha, \beta)$ defined by PDF

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$T \sim \text{Erlang}(k, \lambda)$ defined by PDF

$$f(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

$$= \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

This matches Gamma where $\alpha = k, \beta = \lambda$

$T \sim \text{Gamma}(\alpha = k, \beta = \lambda)$

so Erlang is a case of the Gamma distribution

Transformations of discrete r.v.'s

$X \sim Bern(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

$\underbrace{Y = X+3}_{Y = g(X)} \sim \left\{ \begin{array}{ll} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{array} \right\}$ Support is $y \in \{3,4\}$

$= p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}}$

$= p^{y-3}(1-p)^{1-y+3} \mathbb{1}_{y \in \{3,4\}}$

check:

$X = g^{-1}(Y) = Y - 3$

If g is invertible (g has an inverse function)

$$\boxed{p_Y(y) = P(Y=y) = P(g(x)=y)} = P(X = g^{-1}(y)) = p_X(g^{-1}(y))$$

$X \sim Uniform(\{1,2,\dots,10\}) := 0.1 \mathbb{1}_{x \in \{1,2,\dots,10\}}$

define r.v. $Y = g(X) = \min\{X, 3\}$

$p(x)$	x	y	$p(y)$
0.1	1	1	0.1
0.1	2	2	0.1
0.1	3	3	0.8
0.1	4	3	
⋮	⋮	⋮	
0.1	10	3	

General formula

$$\boxed{p_Y(y) = \sum_{\{x : y = g(x)\}} p_X(x)}$$

sum up all $p_X(x)$ where x matches $g(x)$

if g is invertible

$$p_Y(y) = \sum_{\{x:\, y=g(x)\}} p_X(x) = p_X(g^{-1}(y))$$

$\{x : y = g(x)\}$

$\downarrow$

$\{x : x = g^{-1}(y)\}$ if g is invertible

$\downarrow$

$\{g^{-1}(y)\}$ ← only have 1 thing to add so don't need summation

Example:

$$X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$$

$$Y = X^3 \sim \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y} \in \{0,1,2,\dots,n\}}$$

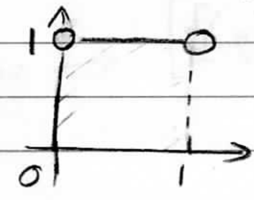
$y = g(x) = x^3$

$x = g^{-1}(y) = \sqrt[3]{y}$

$$\binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{y \in \{0,1,8,\dots,n^3\}}$$

Transformations of continuous r.v.'s

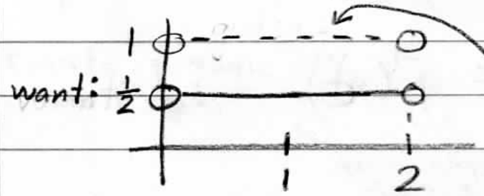
$X \sim \text{Uniform}(0,1)$ PDF $f(x) = 1 \cdot \mathbb{1}_{x \in (0,1)}$



$Y = 2X = g(X)$ Is this the PDF?

$\Rightarrow X = \frac{Y}{2} = g^{-1}(Y)$ $f_Y(y) \stackrel{?}{=} f_Y(g^{-1}(y)) = \mathbb{1}_{\frac{y}{2} \in (0,1)}$

$= \mathbb{1}_{y \in (0,2)}$



NO! if try to integrate this: $\int_0^2 f_Y(y)\,dy \neq 1$

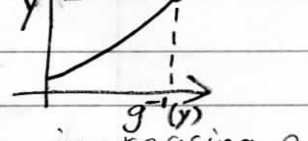
would be

*Use the CDF, not PDF for continuous r.v.'s

g is one-to-one $\Longleftrightarrow$ $\begin{cases} g \text{ is strictly increasing} \\ \text{OR} \\ g \text{ is strictly decreasing} \end{cases}$

$\Longleftrightarrow$ if and only if

case (A) strictly increasing



$F_Y(y) = P(Y \leq y) = P(g(X) \leq Y)$

$= \underbrace{P(X \leq g^{-1}(y))}_{\text{this is CDF}}$ for increasing g

$= F_X(g^{-1}(y))$

want PDF, f_Y

$$\boxed{\begin{array}{l}\text{Chain Rule}\\ \frac{d}{dt}[m(n(t))]\\ \quad = m'(n(t))n'(t)\end{array}}$$

$$f_Y(y) = \frac{d}{dy}\left[F_X(g^{-1}(y))\right]$$

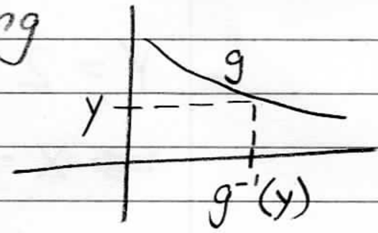
$$= F_X'(g^{-1}(y))\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{this is positive since } g \text{ is increasing } (g' \text{ is increasing})}$$

deriv. of CDF is PDF

$$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

← since is positive, is equal to its absolute value

case (B) g is strictly decreasing



$$F_Y(y) = P(Y \le y)$$

$$= P(g(X) \le y)$$

$$= \underbrace{P(X \ge g^{-1}(y))}_{\text{complement of CDF}}$$

since g strictly decreasing

$$= 1 - F_Y(g^{-1}(y))$$

want PDF, f_Y $\qquad$ (Know $f_Y(y) = \frac{d}{dy}[F_Y(y)]$)

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y)\right]$$

$$= F_X'(g^{-1}(y))\left(-\frac{d}{dy}[g^{-1}(y)]\right)$$

$\frac{d}{dy}[g^{-1}(y)]$ is negative since g is decreasing

so $-\frac{d}{dy}[g'(y)]$ is positive

$$-\frac{d}{dy}[g'(y)] = \left|\frac{d}{dy}[g'(y)]\right|$$

$$f_Y(y) = F_X'(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$\boxed{\text{In general, } f_Y(y) = F_X'(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| \text{ if } g \text{ is one-to-one}}$$

Ex:

$$Y = g(X) = aX + c \quad \text{where } a, c \in \mathbb{R}$$

(linear transformation/ shifts and/or scales)

linear transformation of X

$$X = g^{-1}(Y) = \frac{Y-c}{a}$$

$$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{1}{a}$$

$$\text{so } f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$$

PDF of Y

specific cases
$Y = X + c \sim f_X(y-c)$
$Y = -X \sim f_X(-y)$
$Y = aX \sim f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$

Ex:

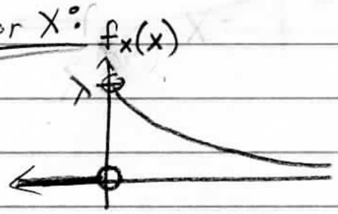
$X \sim \text{Exp}(\lambda)$

$$Y = -X \sim f_X(-y) = \lambda e^{-\lambda(-y)} \mathbb{1}_{-y \in (0,\infty)}$$

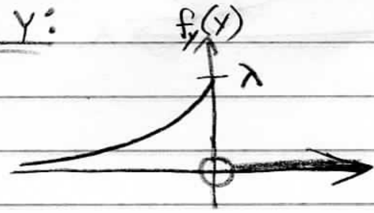
$$= \lambda e^{\lambda y} \mathbb{1}_{y \in (-\infty, 0)}$$

for

for X:



for Y:



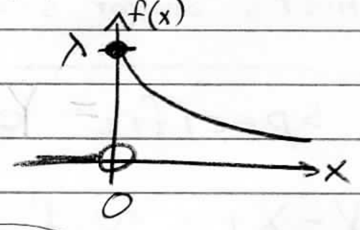
$X \sim Exp(\lambda)$

$$Y = X + c \sim f_X(y-c) = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y-c\in(0,\infty)}$$

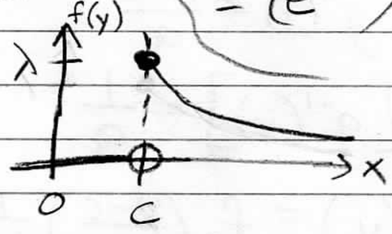
$$= \lambda e^{-\lambda y} e^{\lambda c} \mathbb{1}_{y\in(c,\infty)}$$

$$= (e^{\lambda c}) \lambda e^{-\lambda y} \mathbb{1}_{y\in(c,\infty)}$$

for X:



for Y:



Ex:

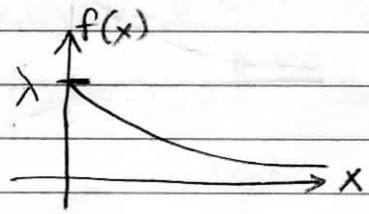
$X \sim Exp(\lambda)$ assume $a > 0$

$$Y = aX \sim f_X\left(\frac{y}{a}\right) \frac{1}{a} \mathbb{1}_{\frac{y}{a}\in(0,\infty)} =$$

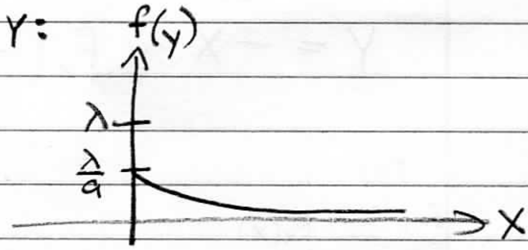
$$= \frac{\lambda}{a} e^{-\frac{\lambda}{a} y} \mathbb{1}_{y\in(0,\infty)}$$

$$= Exp\left(\frac{\lambda}{a}\right)$$

for X:



for Y:



$$X \sim \text{Uniform}(0,1) = \overset{\text{PDF}}{\mathbb{1}}_{x \in [0,1]} \sim f_x(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

find PDF of Y for

$$Y = -\ln(X)$$

$$X = e^{-Y}$$

$$\frac{d}{dy}[g^{-1}(y)] = -e^{-Y}$$

$$= \mathbb{1}_{e^{-Y} \in [0,1]}$$

$$= e^{-Y} \mathbb{1}_{y \in (0,\infty)}$$

$$= \text{Exp}(1)$$