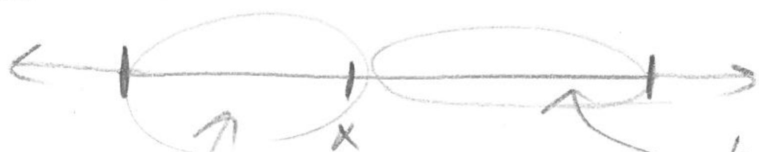
# Lec 12 Math 621 10/28/19

$X_1, \dots, X_n \overset{iid}{\sim}$ with known $f(x), F(x)$ then the distr of the $k^{th}$ order statistic is

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

Let $L \sim Bin(n, p = F(x))$ why?




landing here has prob $1-p$

landing here has prob $P(X \leq x) = F(x) = p$

$L$ is the # of landings less than $x$.

If $x$ is low, it is difficult to get many landings ($k$) which means $f_{X_{(k)}}(x)$ will be small there!

If $x \to$ largest value in support, then the prob of landing $\leq x$ becomes 1 so it is easy for the $k^{th}$ largest eg to be less than that.

Now let's find the density...

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right) - \binom{n}{0} F(x)^0 (1-F(x))^n$$

$$= \underbrace{(F(x) + (1-F(x)))^n}_{1} - (1-F(x))^n = 1-(1-F(x))^n \checkmark$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[F_{X_{(k)}}(x)\right] = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx}\left[F(x)^j (1-F(x))^{n-j}\right]$$

$$F(x)^j (n-j)(-f(x))(1-F(x))^{n-j-1} + j f(x) F(x)^{j-1} (1-F(x))^{n-j}$$

$$= j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^j (1-F(x))^{n-j-1}$$

$\frac{d}{dx}[uv] = uv' + u'v$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} j f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!(n-j)!} (n-j) f(x) F(x)^j (1-F(x))^{n-j-1}$$

$(j-1)!$ $\quad$ $(n-j-1)!$

if $j = n$ $\quad$ $\overset{!!}{0}$

let $l = j+1 \Rightarrow j = l-1$

$$= f(x)\left(\sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} F(x)^{j-1} (1-F(x))^{n-j} - \sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!} F(x)^{l-1} (1-F(x))^{n-l}\right)$$

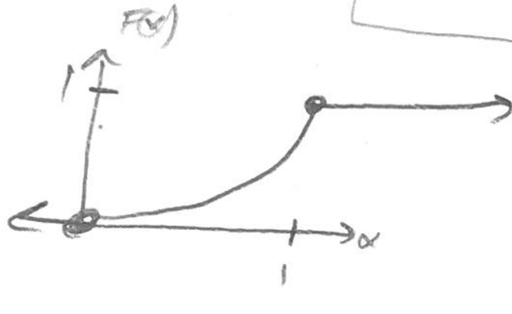
$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k}$$

No sum needed!!

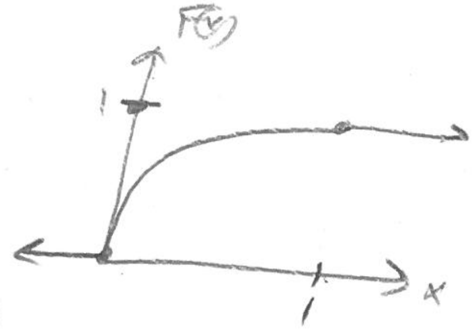
$f_{X_{(1)}}(x) = n f(x) (1-F(x))^{n-1}$
$f_{X_{(n)}}(x) = n f(x) F(x)^{n-1}$

e.g. $X_1, \dots, X_n \overset{iid}{\sim} U(0,1)$

$F_{X_{(n)}}(x) = F_X(x)^n = x^n$

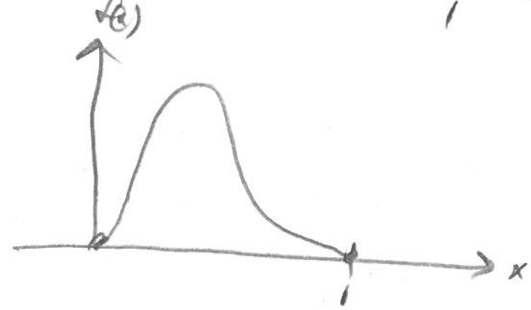


$$F_{X_{(1)}}(x) = 1 - \left(1 - F(x)\right)^n = 1 - (1-x)^n$$



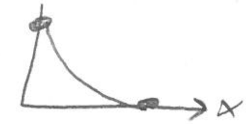

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in[0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in[0,1]}$$




$= \text{Beta}(k, n-k+1)$ normally arises as an order statistic of std. uniform

$f_{X_{(1)}} = \text{Beta}(1,n)$




$f_{X_{(n)}} = \text{Beta}(n,1)$


the general beta model is:

$$X \sim \text{Beta}(\alpha,\beta) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in(0,1)}$$

$$\int_{Supp(X)} f(x)\,dx = 1 \Rightarrow \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = 1 \Rightarrow \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$X \sim \text{Gamma}(\alpha_1, \beta)$, $Y \sim \text{Gamma}(\alpha_2, \beta) \Rightarrow X + Y \overset{?}{\sim} \text{Gamma}(\alpha_1+\alpha_2, \beta)$

this would make sense since $X \sim \text{Erlang}(k_1, \lambda)$, $Y \sim \text{Erlang}(k_2,\lambda) \Rightarrow X+Y \sim \text{Erlang}(k_1+k_2, \lambda)$

To prove this, we need a new concept called "kernels".

not a function of x

$p(x) = c\,k(x)$ or $f(x) = c\,k(x)$ we can say that $p(x) \propto k(x)$, $f(x) \propto k(x)$

since $\forall x$, we differ only by a constant multiple, $c$.

we can find $c$ from $k(x)$: function of $x$

$\sum_{x\in Supp(X)} p(x) = \sum c\,k(x) = 1 \Rightarrow \sum k(x) = 1$ | $\int f(x)\,dx = 1 \Rightarrow \int c\,k(x)\,dx = 1 \Rightarrow \int k(x) = \frac{1}{c}$

eg $X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \mathbb{1}_{x \in \{0,\dots,n\}}$

$$= \underbrace{\left(n!(1-p)^n\right)}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1}_{x\in\{0,\dots,n\}}}_{k(x)}$$

$X \sim Weibull(k,\lambda) := (k\lambda)(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{x \geq 0} = \underbrace{(k\lambda\lambda^{k-1})}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{x\geq 0}}_{k(x)}$

$X \sim Gamma(\alpha,\beta) := \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x}}_{k(x)}$

Sometimes, $k(x) = f(x)$

$X \sim Logistic(0,1) := \frac{e^{-x}}{(1+e^{-x})^2}$

Who cares?

If you're doing a calculation and you remove the constants, then "$=$" $\Rightarrow$ "$\propto$" and the "$\propto$" can then become a density. E.g. if I see

$x^a e^{-bx} \propto Gamma(a+1, \beta)$. I don't need to find $c$! Since using an integral/sum I know it can be found.

Let's add two gammas together:

$$f_{X+Y}(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t}\cancel{e^{+\beta x}}} \underbrace{\mathbb{1}_{t-x\in(0,\infty)}}_{x\leq t} dx$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx \propto e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx$$

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t (tx)^{\alpha_1-1} (t-tx)^{\alpha_2-1} dx = e^{-\beta t} t^{\alpha_1+\alpha_2-1} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du}_{f(\alpha_1,\alpha_2)}$$

Best u subst. I've ever seen!

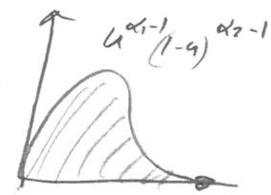
let $u = \frac{x}{t} \Rightarrow x=0 \Rightarrow u=0$, $x=t \Rightarrow u=1$
$du/dx = \frac{1}{t} \Rightarrow dx = t\,du$

but not $t$!!

$$\propto t^{\alpha_1+\alpha_2-1} e^{-\beta t} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} x^{\alpha_1+\alpha_2-1} e^{-\beta x}$$

$$\Rightarrow \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)}$$

$$\underbrace{B(\alpha_1,\alpha_2) := \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du}_{\text{Beta function}} = \underbrace{\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}}_{\text{expressed as gamma funcs}} =$$




for $\alpha_1, \alpha_2 > 0$

which is true in this setup because gamma param space.

$$X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in[0,1]}$$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = 1 \Rightarrow \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = B(\alpha,\beta)$$

by definition of beta!

$$B(q,\alpha,\beta) := \int_0^q u^{\alpha-1}(1-u)^{\beta-1}\,du < B(\alpha,\beta)$$

is the incomplete beta function

$$I_q(\alpha,\beta) := \frac{B(q,\alpha,\beta)}{B(\alpha,\beta)}$$ % of beta integral $\leq q$.

regularized incomplete beta function

$$F_X(x) = \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$