11/20

$$\mathrm{Cov}[X_1, X_2]$$
$$= \mathrm{Cov}[Z_1, Z_1 + Z_2]$$
$$= \mathrm{Cov}[Z_1, Z_1] + \mathrm{Cov}[Z_1, Z_2]$$
$$= \mathrm{Var}[Z_1] + 0$$
$$= 1 \neq 0 \Rightarrow X_1\ X_2 \text{ dependent.}$$

11/25 let $\vec{X}$ be a vector r.v of dim n.

$\vec{c} \in \mathbb{R}^n$ $\quad \vec{\mu} := E[\vec{X}]$, $E[\vec{X} + \vec{c}] = \vec{\mu} + \vec{c}$

$$E[\vec{c}^T \vec{X}] = \vec{c}^T \vec{\mu}$$

$$\Sigma := \mathrm{Var}[\vec{X}] := E[\vec{X}\vec{X}^T] - \underbrace{E[\vec{X}]E[\vec{X}]^T}_{\vec{\mu}\vec{\mu}^T} =$$

$$\begin{bmatrix} \mathrm{VAR}[X_1] & \mathrm{Cov}[X_1, X_2] & & \\ & \mathrm{Var}[X_2] & & \\ & & \ddots & \\ & & & \mathrm{Var}[X_n] \end{bmatrix}$$

$E[\vec{X}B] = \vec{\mu}B$

11/25 $VAR[\vec{X} + \vec{c}] = Var[\vec{X}]$

$Var[\vec{c}^T\vec{X}] = \vec{c}^T \Sigma \vec{c}$ } first time we saw quadratic forms from mid 1

Let $A \in \mathbb{R}^{m \times n}$ constants $E[A\vec{X}] = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vec{X} \end{bmatrix}$

$$= E\begin{bmatrix} \vec{a}_1 \cdot \vec{X} \\ \vec{a}_2 \cdot \vec{X} \\ \\ \vec{a}_n \cdot \vec{X} \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \vec{X}] \\ E[\vec{a}_2 \vec{X}] \\ \vdots \\ E[\vec{a}_n \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_n \cdot \vec{\mu} \end{bmatrix} = \vec{\mu} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = A\vec{\mu}$$

$$VAR[A\vec{X}] = E[(A\vec{X})(A\vec{X})^T] - E[A\vec{X}]E[A\vec{X}]^T$$

$$= E[A\vec{X}\vec{X}^TA^T] - A\vec{\mu}\vec{\mu}^TA^T$$

$$A\,E[\vec{X}\vec{X}^TA^T] - A\vec{\mu}\vec{\mu}^TA^T$$

$$AE[\vec{X}\vec{X}^T]A^T - A\vec{\mu}\vec{\mu}^TA^T$$

$$AE[\vec{X}\vec{X}^T]A^T - \vec{\mu}\vec{\mu}A^T$$

$$= A(E[\vec{X}\vec{X}^T - \vec{\mu}\vec{\mu}^T)A^T$$

$$= A\Sigma A^T$$

general case

Let $A = \vec{c}^T$

This is how you recover special case from

11/25 Let $U \sim \chi^2_K$ $\quad Z_1, Z_2, \ldots, Z_K \overset{iid}{\sim} N(0,1)$

$$U = Z_1^2 + Z_2^2 + \ldots + Z_K^2$$

$$E[U] = E[Z_1^2] + E[Z_2^2] + \ldots + E[Z_K^2]$$

$$\underbrace{1 + 1 + 1 \ldots + 1}_{K \text{ times}} = K.$$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{bmatrix} \sim N_n(\vec{0}_n, I_n) \quad Z_1, Z_2 \ldots Z_n \overset{iid}{\sim} N(0,1)$$

$$\frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{Z}^T\vec{Z}}$$

Let $A \in \mathbb{R}^{n \times n}$ constants $\vec{\mu} \in \mathbb{R}^n$ constants

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim f_{\vec{X}}(\vec{x}) = ?$$

$$X = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

$$E[\vec{X}] = \cancel{E[A\vec{Z}] + \mu A E[\vec{Z}]}$$

$$E[A\vec{Z} + \vec{\mu}] = A E[\vec{Z}] + \vec{\mu} = A\vec{0} + \vec{\mu} = \vec{\mu}$$

$$VAR[\vec{X}] = Var[A\vec{Z} + \vec{\mu}] = Var[A\vec{Z}] = A\, Var[\vec{Z}] A^T$$

$$= AIA^T = AA^T = \Sigma$$

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11/25 do change of variables to find the JDF.

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_h|$$

$$\vec{X} = AZ + \vec{\mu}$$

$$\vec{X} - \vec{\mu} = AZ \qquad \text{Assume } A \text{ is invertible}$$

$$A^{-1}(\vec{X} - \vec{\mu}) = A^{-1}A\vec{Z} \qquad \text{Let } B = A^{-1}$$

$$\vec{Z} = A^{-1}(\vec{X} - \vec{\mu}) = B(\vec{X} - \vec{\mu}) = B\vec{X} - B\vec{\mu} = h(\vec{x})$$

$$J_h = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_n}{x_1} & \cdots & \cdots & \frac{\partial h_n}{x_n} \end{bmatrix} = \det \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ \vdots & & & \\ b_{n1} & \cdots & \cdots & b_{nn} \end{bmatrix} = \det[B] = \det[A^{-1}]$$

$$h(\vec{x}) = \begin{bmatrix} \vec{b}_{1\cdot}\vec{X} - \vec{b}_{1\cdot}\vec{\mu} \\ b_{2\cdot}\vec{X} - b_{2\cdot}\vec{\mu} \\ \vdots \\ \\ \\ \end{bmatrix} \begin{matrix} = h_1(\vec{x}) = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n \\ = h_2(\vec{x}) \\ \\ \\ = h_n(\vec{x}) \end{matrix}$$

$$\underbrace{b_{11}\mu_1 - b_{12}\mu_2 - \cdots - b_{1n}\mu_n}_{\text{constants drop on taking derivative.}}$$

$$= f_{\vec{Z}}\left(A^{-1}(\vec{x} - \vec{\mu})\right)\left|\det[A^{-1}]\right| = \quad \rightarrow$$

11/25 $= \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}(A^{-1}(\bar{x}-\mu))^T A^{-1}(\bar{x}-\bar{\mu})} \left| \det[A^{-1}] \right|$

$= \frac{|\det[A^{-1}]|}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\bar{x}-\bar{\mu})}$

Fact #1

$I = AA^{-1}$ $\quad \det[I] = \det[AA^{-1}]$

$1 = \det[A]\det[A^{-1}]$

$\det[A^{-1}] = \frac{1}{\det[A]}$

Fact #2

$\Sigma = AA^T$ $\quad \det[\Sigma] = \det[A]\det[A^T]$

$\det[\Sigma] = \det[A]^2$

$\Rightarrow \det[A] = \sqrt{\det[\Sigma]}$

← always be positive for ...

Can use facts to simplify formula.

$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{[\text{-----}]}$

Fact 3:

$I = AA^{-1}$

$I^T = (AA^{-1})^T = (A^{-1})^T A^T$

$I = (A^T)^{-1} A^T \quad \Rightarrow (A^T)^{-1} = (A^{-1})^T$

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11/25 FACT 4:

$$\Sigma = AA^T, \quad \& \quad \Sigma^{-1} = (AA^T)^{-1}$$

$$\Sigma^{-1} = (A^T)^{-1} A^{-1}$$

$$= (A^{-1})^T A^{-1}$$ simplify $f_{\vec{Z}}$ even more...

$$N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

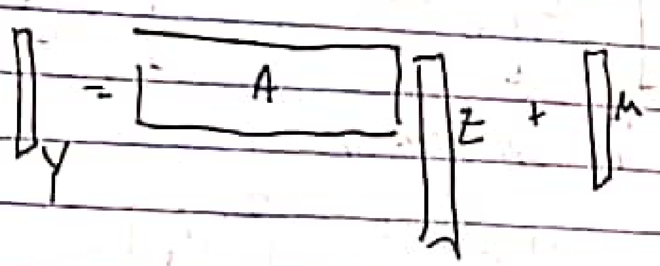
"general multivariate normal" JDF

$\vec{\mu} = \vec{0}, \Sigma = I \rightarrow$ standard normal

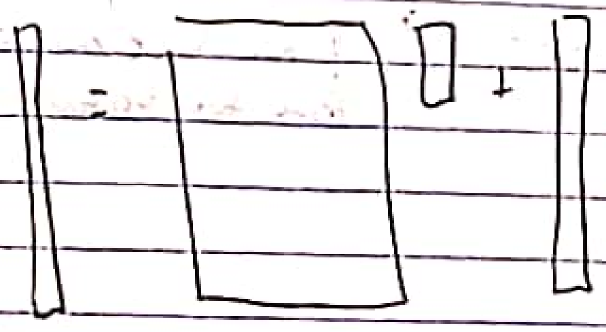
would be nice if A was non-square matrix. need a new formula because A will not have an inverse $\Rightarrow h(\vec{x})$ useless.

Let $A \in \mathbb{R}^{n \times m}$ $\vec{\mu} \in \mathbb{R}^m$, $m < n$ A has rank n (Full rank)

let $\vec{X} = A\vec{Z} + \vec{\mu}$



consider $m > n \Rightarrow A$ has at most rank n



$\Sigma = AA^T$ at most rank $n < m$

$m \times n$ $n \times m$

11/25

$\Rightarrow \Sigma$ not full rank $\Rightarrow \Sigma$ not invertible $\Rightarrow \det[\Sigma] = 0$

Thus m can never be greater than n.

Let $\tilde{A} = \begin{bmatrix} A \\ \vec{v}_1 \\ \vdots \\ \vec{v}_{n-m} \end{bmatrix}$, $\tilde{\vec{\mu}} = \begin{bmatrix} \vec{\mu} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \tilde{\vec{X}} = \tilde{A}\tilde{Z} + \tilde{\vec{\mu}}$

$\vec{v}_1 \ldots \vec{v}_{n-m}$ we chose so that $\tilde{A}$ is full rank $\Rightarrow \tilde{A}^{-1}$ exists.

we know this already

$\sim N(\tilde{\vec{\mu}}, \tilde{A}\tilde{A}^T)$

but this is density of $\tilde{\vec{X}}$ not $\vec{X}$, has extra dimensions.

$$F_{\vec{X}}(\vec{x}) = \underbrace{\int \cdots \int}_{(n-m)} \frac{1}{\sqrt{(2\pi)^n \det[\tilde{A}\tilde{A}^T]}} e^{-\frac{1}{2}(\tilde{\vec{x}}-\tilde{\vec{\mu}})^T(\tilde{A}\tilde{A}^T)^{-1}(\tilde{\vec{x}}-\tilde{\vec{\mu}})} \, d\ldots$$

We won't do this. It is possible.

There is another way using c.f.'s.

Multi-variate Characteristic Functions:

$$\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right] = E\left[e^{i(t_1X_1 + \cdots + t_nX_n)}\right] = E\left[e^{it_1X_1} e^{\ldots}\right]$$

If $X_1, \ldots, X_n \overset{ind}{\sim}$ you can split this up

$$E\left[e^{it_1X_1}\right] \cdot E\left[e^{it_2X_2}\right] \cdots E\left[e^{it_nX_n}\right] = \prod_{i=1}^{n} \phi_{X_i}(t_i)$$

t can take on different values from the vector

11/25 verify property 0.

(P-0) $\phi_{\vec{X}}(\vec{0}) = E[e^{i\vec{0}_n^T\vec{X}}] = E[e^{i0}] = 1$ ✓

(P1) $\phi_{\vec{X}}(\vec{t}) = \phi_{\vec{Y}}(\vec{t}) \longleftrightarrow \vec{X} \overset{d}{=} \vec{Y}$ no proof beyond course

(P2)

$\vec{Y} = A\vec{X} + \vec{b}$ where $A \in \mathbb{R}^{n\times n}$ $\vec{b} \in \mathbb{R}^n$ $\dim[\vec{X}] = n$

$\Rightarrow \phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}]$ $\quad i\vec{t}^TA\vec{X} + i\vec{t}^T\vec{b}$

$= E[e^{i\vec{t}^TA\vec{X}} e^{i\vec{t}^T\vec{b}}] = e^{i\vec{t}^T\vec{b}}$ constant

$\vec{t}^TA\} = (A^T\vec{t})^T$ re write $E[e^{i(A^T\vec{t})^T\vec{X}}] = e^{i\vec{t}^T\vec{b}}\phi_{\vec{X}}(A^T\vec{t})$

$\vec{Z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n}\phi_{Z_i}(t_i) = \prod_{i=1}^{n} e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2}$

$= e^{-\frac{1}{2}\vec{t}^T\vec{t}}$

Let $A \in \mathbb{R}^{n\times n}$ invertible, $\vec{\mu} \in \mathbb{R}^n$

$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma)$ we just proved this. lets get c.f.

$\phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu}}\phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}(A^T\vec{t})^T(A^T\vec{t})}$

$\phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^TAA^T\vec{t}} = \boxed{e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}} = $ c.f.

12/11 last day of class.
[Review]

11/25 **Prove one more FACT:**

Let $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^m$.

$\vec{Y} = B\vec{X} + \vec{c}$. lets find c.f. of Y

$\vec{t}^T B$ $\quad \vec{t}^T B$

$$\phi_{\vec{Y}}(\vec{t}) = e^{i\vec{t}^T\vec{c}}\,\phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}}\left(e^{i(B^T\vec{t})^T\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T \Sigma (B^T\vec{t})}\right)$$

$$= e^{i\vec{t}^T(\vec{c}+B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}}$$

by propert 1

$$\vec{Y} \sim N_m(\vec{c}+B\vec{\mu},\ B\Sigma B^T)$$