Lecture 2

$\mathbb{1}_{x \in X} = x \mathbb{1}\ x = c$ Idea

$\mathbb{1}$ = is a notation we use to express 1 True 0 False | Like Dummy Variables

Convolution operator

X_1, X_2 are discrete r.v.'s Convolution operator → you multiply the together

$T = X_1 + X_2 \sim p(t) = ? \Rightarrow P_{X_1}(x) * P_{X_2}(x)$

$$P(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1 X_2}(x_1, x_2)\ \mathbb{1}_{x_1 + x_2 = t}$$

$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1, X_2}(x_1, x_2)\ \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x \in \mathbb{R}} P_{X_1 X_2}(x, t-x) \overset{\text{if independent}}{=} \sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x)$$

General Formula Convultion formula

$$\overset{\text{if } X_1 \overset{d}{=} X_2}{=} \boxed{\sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x)} \overset{\text{if } X_1 \overset{d}{=} X_2}{=} \sum_{x \in \mathbb{R}} P(x) P(t-x)$$

$$= \sum_{x \in \mathbb{R}} P_{oID}(x)\ \mathbb{1}_{x \in Supp(X_1)} P_{oID}(t-x)\ \mathbb{1}_{t-x \in Supp(X_2)}$$

$$= \boxed{\sum_{x \in Supp[X]} P_{oID}(x) P_{oID}(t-x)\ \mathbb{1}_{t-x \in Supp[X]}} \Rightarrow$$

You see this formula on book [formula]

$X_1, X_2 \overset{iid}{\sim} Bern(p)$

$T = X_1 + X_2$

$$p(t) = \sum_{x \in \{0,1\}} (p^x (1-p)^{1-x})(p^{t-x} (1-p)^{1-(t-x)}\ \mathbb{1}_{t-x \in \{0,1\}})$$

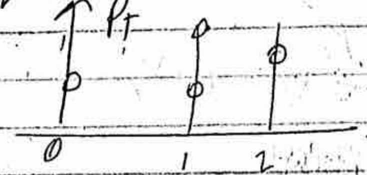
$1-(t-x) = 1+x-t$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}} = p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{0,1\}}\right) \Rightarrow$$

Formula $\Rightarrow$ $\left[\sum_{x \in Supp[X]} P_{oID}(x) P_{oID}(t-x)\ \mathbb{1}_{t-x \in Supp[X]}\right]$

$$\boxed{\sum_{x \in Supp[x]} \underbrace{P_{old}(x)}_{\sim P_{X_1}} \underbrace{P_{old}(t-x)}_{\sim P_{X_2}} \mathbb{1}_{t-x \in Supp[x]}}$$

Convolution Operator: you add two PMF together & multiply them together (Prob of)



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 etc

Pascal Identity

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$$

Proof #2

$$\text{Bern}(p) = \binom{1}{x} p^x (1-p)^{1-x}$$

$$= p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$$

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$$T = X_1 + X_2$$

$$P(t) = \sum_{x \in \{0,1\}} \left(\binom{1}{x} p^x (1-p)^{1-x}\right) \binom{1}{t-x} p^{t-x} (1-p)^{1-(t-x)} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{x} \binom{1}{t-x}$$

$$\sum_{x \in \{0,1\}} = \binom{1}{t} + \binom{1}{t-1} = \binom{2}{t}$$

This is another way to prove Binomial THM

Note Convolution formula work for $\boxed{X_1 + X_2 = T_2}$ ← Identity

Use ⇒ Proof sid or Pascal Identity Test

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$ WTF $P(t)$

$$T = \underbrace{X_1\ X_2}_{T_2} + X_3$$

$$T = X_3 + T_2 \sim \sum_{x \in [0,1]} \binom{1}{x} p^x (1-p)^{1-x} \Big(\binom{2}{t-x} p^{t-x} (1-p)^{2-(t-x)} \underbrace{\mathbb{1}_{t-x \in [0,2]}}_{\text{Not needed}} \Big) = p^t (1-p)^{3-t} \sum_{x \in [0,1]} \binom{1}{x}\binom{2}{t-x}$$

$$= \binom{3}{t} p^t (1-p)^{3-t} = Binom(3, p)$$

Ex: $X_1, X_2 \overset{iid}{\sim} Bin(n, p)$

$T = X_1 + X_2$

Def of Binomial

$$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\dots n\}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-(t-x)} \mathbb{1}_{t-x \in \{0,1,\dots n\}}$$

$n+x-t$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x \in \{0,1,\dots n\}} \binom{n}{x}\binom{n}{t-x}}_{\text{Vandermonde Identity}} = \binom{2n}{t} p^t (1-p)^{2n-t}$$

$$= Bin(2n, p)$$

$\mathbb{N}_0$ all natural # with 0 — All formally $\sum_{x \in Supp[x]} p_{Geo}(x)\, p_{Geo}(t-x)\, \mathbb{1}_{t-x}$

Geometric r.v (p)

Geometric Identity.

$B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$

$X := \#$ 0's before I see the first 1

$Supp[X] = \{0, 1, 2, \ldots\} = \mathbb{N}_0$

$p(0) = p$

$p(1) = (1-p)p$

$p(2) = (1-p)^2 p$

$\vdots$

$p(x) = (1-p)^x p \Rightarrow$ PMF of Geometric

Ex: $X_1, X_2 \overset{iid}{\sim} Geom(p)$

$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\ldots\}} \underbrace{(1-p)^x p}_{\text{Pmf Geometric}} \left((1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}\right) =$

$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{t-x \in \{0,1,\ldots\}}$

$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \le t} = (t+1)(1-p)^t p^2$

$p(t) = (t+1)(1-p)^t p^2$

by $\mathbb{1} = 1$

$\mathbb{1}_{0 \le t} + \mathbb{1}_{1 \le t} + \mathbb{1}_{2 \le t} \cdots \mathbb{1}_{4 \le t} + \cdots +$

$\frac{1}{4}\ \mathbb{1} = 0$

Note

$\mathbb{1}_{t-x \in \{0,1,\ldots\}}$

$= \mathbb{1}_{t \ge x}$

or

$\mathbb{1}_{x \le t}$

$\text{Geometric}(p) \Rightarrow P(t) = \underbrace{(t+1)}_{4}(1-p)^{\underbrace{t}_{2}} p^2$

Ex: $P(4) = \underbrace{5(1-p)^4}_{\text{Prob 4 zeros}} \, p^2$ ← Prob of 2 1's

Experiment Number #	0	1	0	0	0	1
	1	2	3	4	5	6
Prob						

Note adding the exponent to probabilities

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$ $\qquad P(t) = (t+1)(1-p)^t p^2$

Pmf Geometric $\qquad \text{Supp}(T) = \{0,1,\dots\} = \mathbb{N}_0$

$T = \underbrace{X_1 + X_2}_{} + X_3 \sim$

By Def $X_1 + X_2 = T_2$

$= X_3 + T_2 \quad \sum_{x \in \{0,1,\dots\}} \left((1-p)^x p\right)\left((t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \{0,\dots\}}\right)$

$= (1-p)^t p^3 \sum_{x \in \{0,1,\dots\}} (t-x+1) \mathbb{1}_{x \le t} = (1-p)^t p^3 \Rightarrow$

$\left(\sum_{x \in \{0,1,\dots\}} (t+1) \mathbb{1}_{x \le t} - \sum x \mathbb{1}_{x \le t}\right)$

$= (1-p)^t p^3 \left((t+1) \underbrace{\sum_{x \in \{0,1,\dots\}}}_{t+1} - \sum_{x \in \{0,\dots\}} x \mathbb{1}_{x \le t}\right) =$

↳ general formula $(1+2+\dots+t)$

↳ $\frac{t(t+1)}{2}$

$0 + 1 \cdot \mathbb{1}_{1 \le t} + 2 \cdot \mathbb{1}_{2 \le t} + 3 \cdot \mathbb{1}_{3 \le t} + 4 \cdot \mathbb{1}_{4 \le t} + 5 \cdot \mathbb{1}_{5 \le t}$ at $t = 3$

$= 1 + 2 + 3$

$\Rightarrow (1-p)^t p^3 \left((t+1)^2 - \frac{t(t+1)}{2}\right) = \underbrace{t^2 - 2t + 1 - \frac{t-t}{2}}_{\frac{t^2+3t+1}{2}} = \frac{(t+2)(t+1)}{2} = \binom{t+2}{2}$

Next class ⇒ poisson
⇒

waiting until you get r' successes

Negative Binomial Random Variable
↳ Sum of Geometrics

$= \binom{t-2}{2}(1-p)^t p^3$

Ex: $p(4) = \binom{6}{2}(1-p)^4 p^3$ ← note adding exponent = #
$\overset{\parallel}{15}$

✓ 4 zeros
3 ones

Events # any order $\underline{1}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}$ = Neg Bin(3, p)

For H.W
$X_1 \ldots X_2 \overset{iid}{\sim}$ Geom(p)

$T = X_1 + \ldots + X_r \sim$ Neg Bin$(r, p) = \binom{t+r-1}{r-1}(1-p)^t p^r$