Gamma Function

Gamma Function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt$$

a is a constant $a \in (0,\infty)$

$$= \underbrace{\int_0^{a} t^{x-1} e^{-t}\, dt}_{\substack{\gamma(x,a) \\ \text{lower incomplete} \\ \text{Gamma function}}} + \underbrace{\int_a^{\infty} t^{x-1} e^{-t}\, dt}_{\substack{\Gamma(x,a) \\ \text{upper incomplete} \\ \text{Gamma function}}}$$

Notice:

$$\Gamma(x) = \gamma(x,a) + \Gamma(x,a) \quad \text{for any } a$$

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \underbrace{\frac{\gamma(x,a)}{\Gamma(x)}}_{\substack{\text{Lower} \\ \text{regularized} \\ \text{Gamma} \\ \text{Function}}} + \underbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}_{\substack{\text{Upper} \\ \text{regularized} \\ \text{Gamma} \\ \text{Function}}}$$

$$P(x,a) = \frac{\gamma(x,a)}{\Gamma(x)} \qquad Q(x,a) = \frac{\Gamma(x,a)}{\Gamma(x)}$$

$$\lim_{a\to\infty} \gamma(x,a) = \Gamma(x) \qquad \lim_{a\to\infty} P(x,a) = 1$$

$$\lim_{a\to 0} \gamma(x,a) = 0 \qquad \lim_{a\to 0} P(x,a) = 1$$

$c \in \mathbb{R}$ is a constant

Useful reference stuff

$$\int_0^\infty t^{x-1} e^{-ct}\,dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du = \frac{1}{c^x} \int_0^\infty u^{x-1} e^{-u}\,du$$

use $u = ct \Rightarrow dt = \frac{1}{c}du$, $t = \frac{u}{c}$

$$= \frac{\Gamma(x)}{c^x}$$

So

$$\int_0^\infty t^{x-1} e^{-ct}\,dt = \frac{\Gamma(x)}{c^x}$$

$$\int_0^a t^{x-1} e^{-ct}\,dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du =$$

$$= \frac{1}{c^x} \int_0^{ac} u^{x-1} e^{-u}\,du$$

$$\int_0^a t^{x-1} e^{-ct}\,dt = \frac{\gamma(x, ac)}{c^x}$$

$\leftarrow$ lower partial Gamma

$$\int_a^\infty t^{x-1} e^{-ct}\,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, ac)}{c^x} = \frac{\Gamma(x, ac)}{c^x}$$

lower partial Gamma

Properties of Gamma Function

$\Gamma(1) = 0$ because

$$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t}\, dt = \left[-e^{-t}\right]_0^{\infty} = 1$$

$\Gamma(x+1) = x\Gamma(x)$ e.g. $\Gamma(2) = 1 \cdot \Gamma(1) = 1$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!$

$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$

so

$$\Gamma(n) = (n-1)!$$

$$\Gamma(n+1) = n!$$

also, $\boxed{n! = \Gamma(n+1)}$

$$3.5! = \Gamma(4.5)$$

$n \in \mathbb{N}$

$$\Gamma(n,a) = \int_a^{\infty} \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}\, dt}_{v} = \left[uv\right]_a^{\infty} - \int_a^{\infty} v\, du$$

integration by parts

$$\int u\, dv = uv - \int v\, du$$

$$= \left[t^{n-1} e^{-t}\right]_a^{\infty} + \int_a^{\infty} (e^{-t})(n-1)\, t^{n-2}\, dt$$

$$= a^{n-1} e^{-a} + (n-1) \int_a^{\infty} t^{(n-1)-1} t\, dt$$

$$= a^{n-1} e^{-a} + (n-1) \underbrace{\Gamma(n-1, a)}_{\text{incomplete Gamma}}$$

$$\Gamma(n,a) = a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{(n-1)-1}e^{-t}\,dt$$

$$= a^{n-1}e^{-a} + (n-1)\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\Gamma(n-2,a)\right)$$

$$= a^{n-1}e^{-a} + (n-1)a^{n-1} + (n-1)(n-2)\left(a^{n-3}e^{-a} + (n-3)\Gamma(n-3,a)\right)$$

pattern...

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + (n-1)(n-2)(n-3)a^{n-4} + \ldots + (n-1)!\,a^0\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \ldots + \frac{a^0}{0!}\right)$$

$$\Gamma(n,a) = e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

So,

$$\Gamma(n+1,a) = e^{-a}n!\sum_{i=0}^{n}\frac{a^i}{i!}$$

$$\Rightarrow \boxed{\underbrace{Q(n+1,a)}_{\text{upper regularized Gamma function}} = \frac{\Gamma(n+1,a)}{n!} = e^{-a}\sum_{i=0}^{n}\frac{a^i}{i!}}$$

$X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} Exp(\lambda)$

PMF

$T_K = X_1 + X_2 + \ldots + X_K \sim Exp(K, \lambda)$

$Exp(\lambda) = Erlang(1, \lambda)$

analogous to

$Geom(p) = NegBin(1, p)$

T_K has PMF $f(t) = \dfrac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \geq 0}$

find CDF of this (Erlang(k, λ))

CER is a constant

$$\underset{\text{CDF}}{F_T(t)} = P(T \leq t) = \int_0^t \frac{v^{k-1} \lambda^k e^{-\lambda v}}{(k-1)!} dv$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^t v^{k-1} e^{-\lambda v} dv$$

by our formulas earlier

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\gamma(k, \lambda t)}{\lambda^k}$$

$$= \frac{\gamma(k, \lambda t)}{\Gamma(k)}$$

$F_T(t) = P(k, \lambda t)$ regularized lower Gamma function

therefore,

$P(T > t) = 1 - F_T(t) = Q(k, \lambda t)$ upper regularized Gamma function

$$N \sim \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$\text{PMF} \quad f_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$\text{CDF} \quad F_N(n) = P(N \leq n) = \sum_{i=0}^{n} \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= e^{-\lambda}\sum \frac{\lambda^i}{i!} = e^{-\lambda}\sum \frac{\lambda^i}{\Gamma(i+1)}$$

$$= Q(n+1, \lambda)$$

lower regularized Gamma function

$X_1, X_2, \ldots \overset{i.i.d.}{\sim} \text{Exp}(\lambda)$ where
X is measured in seconds.

What is the probability you observe zero (less than one) event in the first second?

let $T_1 \sim \text{Erlang}(1, \lambda) = \text{Exp}(\lambda)$

$$P(T_1 > 1) = 1 - F_T(1) = Q(1, \lambda) = e^{-\lambda}$$

$$= F_N(0) = P(N \leq 0)$$

$N \sim \text{Poisson}(\lambda)$

What is the probability that you observe less than 2 events in the first second?

let $T_2 \sim \text{Erlang}(2, \lambda)$

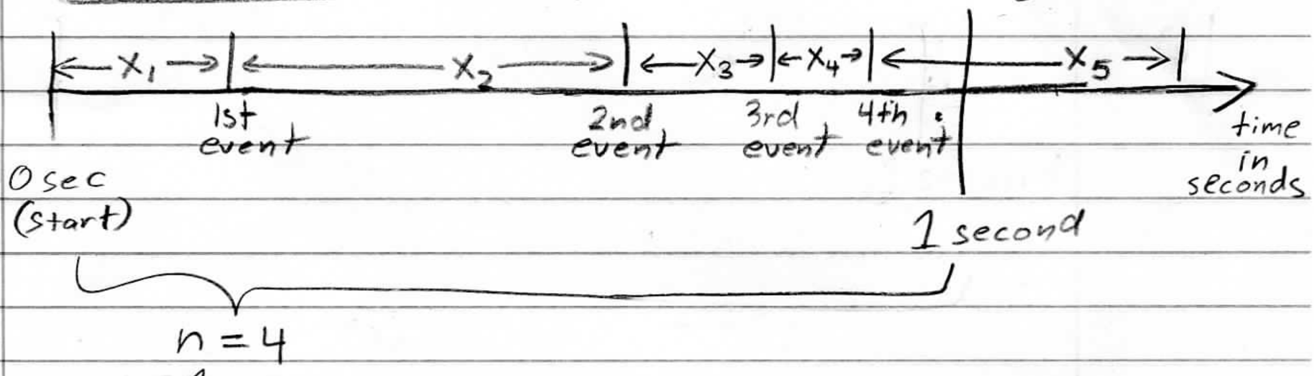
$$P(T_2 > 1) = 1 - F_T(1) = \underbrace{Q(2, \lambda)}_{e^{\lambda}(1+\lambda)} = F_N(1) = P(N \leq 1)$$

What is the probability that you observe less than K events in the first second?

Let $T_K \sim \text{Erlang}(K, \lambda)$ and $N \sim \text{Poisson}(\lambda)$

$$P(T_K > 1) = 1 - F_{T_K}(1) = \varphi(K, \lambda) = F_N(K-1)$$
$$= P(N \leq K-1)$$

<u>Poisson Process</u> $X_1, X_2, X_3, X_4, X_5, \ldots \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$



$N \sim \text{Poisson}(\lambda)$

$T_4 = X_1 + X_2 + X_3 + X_4$ is $\text{Erlang}(4, \lambda)$