1

Lec 4 Math 621 9/9/19

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cancel{2} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 210$$

$$\binom{7}{2\ 2\ 3} = \frac{7!}{2!2!3!}$$

Bag of fruits

p_1: prob of drawing apple

p_2: prob of drawing banana

p_3: prob of drawing cantaloupe

s.t. $p_1 + p_2 + p_3 = 1$

$\underline{A}\ \underline{A}\ \underline{B}\ \underline{B}\ \underline{C}\ \underline{C}\ \underline{C}$

Draw n with replacement. Let X_1, X_2, X_3 be ~~#~~ of apple, banana, can

$X_1 \sim \text{Bin}(n, p_1)$, $X_2 \sim \text{Bin}(n, p_2)$, $X_3 \sim \text{Bin}(n, p_3)$

$$\vec{X} := \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim p_{\vec{X}}(\vec{x}) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1 + x_2 + x_3 = n} \prod_{k=1}^{3} \mathbb{1}_{x_k \in \{0, \ldots, n\}}$$

$$= \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \text{Multinomial}(n, \vec{p})$$

In general, K categories of items n draws:

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} \sim p_{\vec{X}}(\vec{x}) = \binom{n}{x_1, x_2, \ldots, x_K} \prod_{k=1}^{K} p_k^{x_k} = \text{Multinom}(n, \vec{p})$$

In general... K categories of items
let $\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$
let $p_1 + p_2 + \ldots + p_K = 1$

$$p(\vec{x}) := P(\vec{X} = \vec{x}) = \binom{n}{x_1, x_2, \ldots, x_K} p_1^{x_1} \cdot p_2^{x_2} \cdot \ldots \cdot p_K^{x_K}$$

No indicator function necessary since multichoose takes care of it

$$\text{Supp}[\vec{X}] = \{ \vec{x} : \vec{x} \in \{0, 1, \ldots, n\}^K \ \& \ \vec{x} \cdot \vec{1} = n \}$$

Param space: $n \in \mathbb{N}$, $\vec{p} \in \{ \vec{v} : \vec{v} \in (0,1)^K \ \& \ \vec{v} \cdot \vec{1} = 1 \}$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \text{Multinomial}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

Intuitively, but not proven...

$$\Rightarrow X_1 \sim \text{Bin}(n, p), \quad X_2 \sim \text{Bin}(n, 1-p)$$

$$X_1 \overset{d}{\neq} X_2 \quad \text{but is } X_1, X_2 \text{ ind?}$$

In order for there to be independence...

$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1) \quad \forall x_1 \in \text{Supp}[X_1], \forall x_2 \in \text{Supp}[X_2]$$

Does this hold?

$$0 = P(X_1 = 1 \mid X_2 = n) = P(X_1 = 1) = n p (1-p)^{n-1} \Rightarrow \text{dependent}$$

Of course they're dependent since $X_1 = n - X_2$

Define a new r.v. $X_1 | X_2$

PMF $$p_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_2 = x_2)}$$

We need to derive $P(X_2 = x_2)$, the "marginal" distribution.

We know it will be $\text{Bin}(n, 1-p)$

$$p(x_2) = \sum_{x_1 \in \text{Supp}[X_1]} p(x_1, x_2) = \sum_{x_1=0}^{n} \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

Since n is assumed in param space... no need for this

$$= \sum_{x_1=0}^{n} \frac{n!}{x_1!\, x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1+x_2=n} \underbrace{\mathbb{1}_{x_1 \in \{0,\ldots,n\}}}_{x_1 \text{ assumed in support}} \underbrace{\mathbb{1}_{x_2 \in \{0,\ldots,n\}}}_{x_2 \text{ assumed in support}} \mathbb{1}_{n \in \mathbb{N}}$$

$$= \sum_{x_1=0}^{n} \frac{n!}{x_1!\, x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1 = n - x_2}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \sum_{x_1=0}^{n} \frac{1}{x_1!} p^{x_1} \mathbb{1}_{x_1 = n - x_2}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \frac{1}{(n-x_2)!} p^{n-x_2} = \binom{n}{x_2} (1-p)^{x_2} p^{n-x_2} \sim \text{Bin}(n, 1-p)$$

$$p_{X_1|X_2}(x_1, x_2) = \frac{\binom{n}{x_1, x_2} p^{x_1} \cancel{(1-p)^{x_2}}}{\binom{n}{x_2} \cancel{(1-p)^{x_2}} p^{n-x_2}} = \frac{\frac{\cancel{n!}}{x_1!\, \cancel{x_2!}} \mathbb{1}_{x_1+x_2=n}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!}} p^{x_1+x_2-n} = \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$$= \begin{cases} \frac{x_1!}{x_1!} p^{n-n} = 1 & \text{if } x_1 + x_2 = n \\ 0 & \text{o/t} \end{cases}$$

Degenerate!

$X_1 | X_2 \sim Deg(n - x_2)$

$= n - x_2$ w.p. 1

Now question...

$$P(X_{-j} | X_j) = \frac{P(X_1, \ldots, X_K)}{P(X_j)} = \frac{Multinom(n, \vec{p})}{Bin(n, p_j)}$$

all elements except the jth

$$= \frac{\frac{\cancel{n!}}{x_1! \cancel{x_j!} x_K!} \mathbb{1}_{x_1 + x_j + x_K = n} \; p_1^{x_1} \cdots \cancel{p_j^{x_j}} \cdots p_K^{x_K}}{\cancel{\frac{n!}{x_j! (n-x_j)!}} \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}}$$

$$= \frac{(n-x_j)!}{x_1! \cdots x_{j-1}! x_{j+1}! \cdots x_K!} \; \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_K^{x_K}}{(1-p_j)^{n-x_j}}$$

let $n' := n - x_j$ $\quad$ x_j is fixed and known in advance just like n!

let $p_1' := \frac{p_1}{1-p_j}$, $p_2' := \frac{p_2}{1-p_j}$, $p_{j-1}' = \frac{p_{j-1}}{1-p_j}$, $p_j' := \frac{p_{j+1}}{1-p_j}$, $\ldots p_{K-1}' := \frac{p_K}{1-p_j}$

$$= Multinom(n', \vec{p}') \quad \text{where} \quad \vec{p}' = \begin{bmatrix} p_1' \\ \vdots \\ p_{K-1}' \end{bmatrix}$$

Review of expectation. Let $X_1, \dots, X_n$ be r.v.'s

$$E[aX+c] = a\,E(X)+c$$

$$E\left[\sum X_i\right] = \sum E(X_i) \quad \text{always}$$

$$= n\mu \quad \text{if identically distr.} \qquad \mu := E(X)$$

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E(X_i) \quad \text{if independent. Pf. on HW.}$$

$$\sigma^2 := Var(X) := E\big(\underbrace{(X-\mu)^2}_{g(X)}\big) = \sum_{x\in\mathbb{R}} (x-\mu)^2 p(x) \quad \text{if discrete}$$

$$= \int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx \quad \text{if continuous}$$

$$= E(X^2) - \mu^2$$

$\sigma := SE(X) = \sqrt{Var(X)}$ same unit as X!!

$$Var[X_1+X_2] = E\left[\big((X_1+X_2)-(\mu_1+\mu_2)\big)^2\right]$$

$$= E\left(X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2\mu_1 X_1 - 2\mu_1 X_2 - 2\mu_2 X_1 - 2\mu_2 X_2 + 2X_1X_2 + 2\mu_1\mu_2\right)$$

$$= E(X_1^2) + E(X_2^2) + \cancel{\mu_1^2} + \cancel{\mu_2^2} - \cancel{2}\mu_1^2 - 2\mu_1\mu_2 - \cancel{2\mu_1\mu_2} - \cancel{2}\mu_2^2 + 2E[X_1X_2] + \cancel{2\mu_1\mu_2}$$

$$-\mu_1^2 \qquad -\mu_2^2$$

$$= \sigma_1^2 + \sigma_2^2 + 2\left(E[X_1X_2] - \mu_1\mu_2\right) = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

let $\sigma_{12} := Cov(X_1, X_2) := E(X_1X_2) - \mu_1\mu_2 = E\left[(X_1-\mu_1)(X_2-\mu_2)\right]$ HW problem