(Lecture 16) November 11^{th}, 2019

$X_1, \ldots, X_n \overset{iid}{\sim}$
with finite $E(x) = \mu$ and finite variance$[x] = \sigma^2$

Let $T_n := X_1 + \cdots + X_n$
$\bar{X}_n := \frac{X_1 + \cdots + X_n}{n} = \frac{T_n}{n}$ Note: $E[\bar{x}_n] = \mu$, $Var[\bar{x}_n] = \frac{\sigma^2}{n}$

$Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ Note: $E[Z_n] = 0$, $Var[Z_n] = 1$

$\phi_{T_n}(t) \overset{by (?)}{=} \phi_x(t)^n$ $\phi_{Z_n}(t) = e^{-it\frac{\sqrt{n}}{\sigma}\mu}$

$\phi_{\bar{x}_n}(t) = \phi_x\left(\frac{t}{n}\right)^n$

$Z_n = \frac{\sqrt{n}}{\sigma}\bar{X}_n + \frac{-\sqrt{n}}{\sigma}\mu$

$$\phi_x\left(\frac{\sqrt{n}}{\sigma}\frac{t}{n}\right)^n = e^{-\frac{it\mu\sqrt{n}}{\sigma}} \phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)^n$$

$$= e^{-\frac{it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{n\left(-\frac{it\mu}{\sigma\sqrt{n}} + \ln\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}$$

$$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

Examine $\lim_{n\to\infty} \phi_{Z_n}(t)$ with hopes of using P5 and P6

$$= e^{\frac{t^2}{\sigma^2}\left(\lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

let $u = \frac{t}{\sigma\sqrt{n}} \to n\to\infty \Rightarrow u \to 0$

$$= e^{\frac{t^2}{\sigma^2}\left(\lim_{n\to\infty} \frac{\ln(\phi_X(u)) - it\mu \cdot u}{n^2}\right)}$$

L'Hospital

$$= e^{\frac{t^2}{\sigma^2}\left(\lim_{n\to\infty} \frac{\frac{\phi_X'(n)}{\phi_X(n)} - itm}{2n}\right)}$$

L'Hospital #2

$$= e^{\frac{t^2}{2\sigma^2}\left(\lim_{n\to\infty} \frac{d}{dn}\left[\frac{\phi'(u)}{\phi(u)}\right]\right)}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(\lim_{n\to\infty} \frac{\phi_X(n)\phi''(n) - \phi'(n)^2}{\phi_X(n)^2}\right)}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(\frac{\phi_X(0)\phi''(0) - \phi'(0)^2}{\phi_X(0)^2}\right)}$$

$$\overset{P6}{=} e^{\frac{t^2}{2\sigma^2}\left(\phi_X''(0) - \phi_X'(0)^2\right)}$$

$$\overset{P5}{=} e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - i^2 E[X]^2\right)}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(E[X^2] - E[X]^2\right)}$$

$$= e^{\frac{t^2}{2}} = \phi_Z(t)$$

$\lim_{n\to\infty} \phi_{Z_n}(t) = \phi_Z(t) = e^{t^2/2}$

Is $\phi_Z(t) \in L^1$ $\int_{\mathbb{R}} |e^{t^2/2}| dt = \int_{\mathbb{R}} e^{t^2/2} dt = \sqrt{2\pi} < \infty$ ✓

$$f(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)} dt$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2 - \left(\frac{\sqrt{2}\, iz}{2}\right)^2$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)} dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2} e^{-\frac{z^2}{2}} dt$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2} dt$$

Let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}$ $\quad \frac{dy}{dt} = \frac{1}{\sqrt{2}} \to dt = \sqrt{2}\, dy$

$t = -\infty \to y = -\infty$
$t = \infty \to y = \infty$

$$\frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} \underbrace{e^{-y^2} dy}_{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = f(z) = N(0,1)$$

Standard Normal

Central Limit theorem

$$E[Z] = \frac{\phi'_Z(0)}{i} = \frac{te^{-\frac{t^2}{2}}\Big]_0}{i} = 0$$

$$Var[Z] = E[Z^2] - E[Z]^2 = E[Z^2]$$

$$= \frac{\phi''_Z(0)}{i^2} = -\cdot-(-0+1) = 1$$

$$\phi_Z''(t) = -\left(t^2 e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}\right)$$

Parameter space

$$X = \sigma Z + \mu \quad (\leftarrow \sigma > 0) \sim N(\mu, \sigma^2), \ \mu \in R, \ \sigma^2 > 0$$

$Supp[Z] = R$

$$E[X] = \mu, \ Var[X] = \sigma^2$$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

$$\phi_X(t) \overset{(2)}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu} \cdot e^{-\frac{\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

$$X_1 \sim N(\mu_1, \sigma_1^2) \text{ indep. of } X_2 \sim N(\mu_2, \sigma_2^2)$$

$X_1 + X_2 \sim ?$ Convolution

$$\phi_T(t) = e^{it\mu_1 + \frac{\sigma_1^2 t^2}{2}} e^{it\mu_2 + \frac{\sigma_2^2 t^2}{2}}$$

$\phi_{X_1}(t)\phi_{X_2}(t)$

$$= e^{it\mu_1 + it\mu_2 + \frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{it(\mu_1+\mu_2) + \frac{(\sigma_1^2+\sigma_2^2)}{2}t^2}$$

$$= X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

$$X \sim N(\mu, \sigma^2), \ Y = e^X \rightarrow X = \ln(Y) = g^{-1}(Y) \quad \frac{d}{dy}[g^{-1}(y)] = \frac{1}{y}$$

$$Y \sim f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right| = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \underbrace{\mathbb{1}_{\ln(y)\in(-\infty,\infty)}}_{y\in(0,\infty)} \left|\frac{1}{y}\right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \mathbb{1}_{y\geq 0} = \text{LogNormal}(\mu, \sigma^2)$$

$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$, not 1:1!

$$F_Y(y) = P(Y \leq y) = P(Z^2 \leq y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Note: $f(z)$ is an even function i.e. $f(z) = f(-z)$

$$= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 2(F_Z(\sqrt{y}) - F_Z(0)) = F_Y(y)$$

$$f_Y(y) = 2\left(\frac{1}{2} y^{-\frac{1}{2}} f_Z(\sqrt{y}) - 0\right) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \propto y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}_{y\in(0,\infty)}$$

$$\propto \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$Z \sim N(0,1)$, $Z^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$= \chi^2_1$ (chi-squared with degrees of freedom 1)

$Z_1, Z_2, \ldots, Z_k \stackrel{iid}{\sim} N(0,1)$

$\rightarrow Z_1^2 + Z_2^2 + \cdots + Z_k^2 \sim \chi^2_k = \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

$Z \sim N(0,1)$, $|Z| = \sqrt{Z^2} \sim \chi_1 = ?$

$X \sim \chi^2_{k}$, $Y = \sqrt{X}$, $X = Y^2 = g^{-1}(Y)$ $\quad \frac{d}{dy}[g^{-1}(y)] = 2y$

$$Y \sim \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \underbrace{(y^2)^{\frac{k}{2}-1}}_{y^{k-2}} e^{-y^2/2} \mathbb{1}_{y^2 \geq 0} (2y)$$

$$= \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y \geq 0} = \chi_k$$

$$\text{So } \chi_1 = \frac{1}{2^{-\frac{1}{2}} \underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}} e^{-\frac{z^2}{2}} \mathbb{1}_{z \geq 0} = \sqrt{\frac{2}{\pi}} e^{-z^2/2} = 2 f_Z(z)$$

$X_1, X_2 \overset{iid}{\sim} N(0,1)$,

$$R = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} f(ru) f(u) |u| \, du$$

$\text{Supp}[R] = \mathbb{R}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2 u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |u| \, du$$

$$= \frac{1}{2\pi} \left(\int_0^{\infty} e^{-\left(\frac{1+r^2}{2}\right)u^2} u \, du + \int_{-\infty}^{0} e^{-\left(\frac{1+r^2}{2}\right)u^2} u \, du \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\left(\frac{1+r^2}{2}\right)u^2} u \, du$$

Let $t = u^2 \to \frac{dt}{du} = 2u \to du = \frac{1}{2} \cdot \frac{1}{u} dt$

$u = \infty \to t = \infty$, $u = 0 \to t = 0$

$$= \frac{1}{\pi} \int_0^\infty e^{-\left(\frac{1+k^2}{2}\right)t} u \cdot \frac{1}{2} \cdot \frac{1}{u}\, dt$$

$$= \frac{1}{2\pi} \int_0^\infty e^{-\left(\frac{1+k^2}{2}\right)t}\, dt$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\left(\frac{1+k^2}{2}\right)} = \frac{1}{\pi} \frac{1}{1+k^2} = \text{Cauchy}(0,1)$$