(Review Tues - noon RA 202)

$$\phi_X(t) = E[e^{itX}]$$

Normal Distribution stuff

Properties of Characteristic Functions

(P0) $\phi_X(0) = 1$

(P2) $\phi_{aX+b}(t) = e^{itb}\phi_X(at)$

(P1) $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$

(P3) If X_1, X_2 are i.i.d then $\phi_{X_1+X_2}(t) = (\phi_X(t))^2$

(P4) $E[X^2] = \dfrac{\phi_X^{(n)}(0)}{i^n}$

(P6) If $\phi_X(t) \in L^1 \Rightarrow f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi_X(t)\,dt$

(P8) $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X$

$X_1, X_2, \ldots, X_n$ are i.i.d
with finite mean $E[X] = \mu$
with finite variance $Var[X] = \sigma^2$

let $T_n = X_1 + X_2 + \ldots + X_n \Rightarrow \phi_T(t) \overset{P3}{=} (\phi_X(t))^2$

let $\overline{X_n} = \dfrac{X_1 + X_2 + \ldots + X_n}{n} = \dfrac{T}{n}$

$$\Rightarrow \phi_{\overline{X_n}}(t) \overset{P2}{=} \left(\phi_X\left(\frac{t}{n}\right)\right)^n$$

let $Z_n = \dfrac{\overline{X_n} - \mu}{\sigma/\sqrt{n}} = \dfrac{\sqrt{n}}{\sigma}\overline{X_n} + \dfrac{-\sqrt{n}}{\sigma}\mu$

Note $E[\overline{X}] = \mu$ $\quad E[Z_n] = 0$
$Var[\overline{X}] = \dfrac{\sigma^2}{n}$ $\quad Var[Z_n] = 1$

$$\phi_{Z_n}(t) \overset{P2}{=} e^{-it\frac{\sqrt{n}}{\sigma}\mu}\left(\phi_X\left(\frac{\sqrt{n}}{\sigma}\cdot\frac{t}{n}\right)\right)^n$$

$\frac{n}{\sqrt{n}} = \sqrt{n}$

$$= e^{\frac{-it\mu}{\sigma\sqrt{n}}\cdot n}\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

$\ln(a^b) = b\ln(a)$

$\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$$= e^{\frac{-it\mu}{\sigma\sqrt{n}}n}\, e^{\ln\left(\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n\right)}$$

$$= e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{itn}{\sigma\sqrt{n}}\right)}$$

$$= e^{\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{itn}{\sigma\sqrt{n}}}{\frac{1}{n}}\cdot\frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$\exp(x) = e^x$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{itn}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

Examine $\lim_{n\to\infty} \phi_{Z_n}(t)$ with the hope

of it converging so we can get $Z_n \xrightarrow{d} Z$

Using P8 and find the densit with P6

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{itu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

let $u = \frac{t}{\sigma\sqrt{n}}$ so as $n\to\infty$, $u\to 0$

$$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_X(u)) - iu\mu}{u^2}}$$

The Guassian Integral

$\int e^{\frac{-t^2}{2}}\,dx = \sqrt{2\pi}$

by L'Hôpital's Rule:

$$= e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_X'(u)}{\phi_X(u)} - iu}{2u}}$$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n \to 0} \frac{\frac{\phi_x'(x)}{\phi_x(x)} - iu}{2u}}$$

by L'Hopital's Rule:

$$= e^{\frac{t^2}{\sigma^2} \lim_{u \to 0} \frac{\frac{d}{du}\left[\frac{\phi_x'(u)}{\phi_x(u)}\right]}{2}}$$

$\left[\frac{f}{g}\right]' = \frac{f'g + fg'}{g^2}$ Quotient Rule

$$= e^{\frac{t^2}{2\sigma^2} \lim_{u \to 0} \frac{\phi_x(u)\phi_x''(u) - (\phi_x'(u))^2}{(\phi_x(u))^2}}$$

Use $u = 0$ since doing $\lim_{u \to 0}$

$$= e^{\frac{t^2}{2\sigma^2}\left(\frac{\phi_x(0)\phi_x''(0) - (\phi_x'(0))^2}{(\phi_x(0))^2}\right)}$$

(PO) says $\phi_x(0) = 1$

by (PO)

$$= e^{\frac{t^2}{2\sigma^2}\left(\frac{(1)(\phi_x''(0)) - (\phi_x(0))^2}{1^2}\right)}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(\phi''(0) - (\phi_x(0))^2\right)}$$

(P4)

$$= e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - (i E[X])^2\right)}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - i^2 (E[X])^2\right)} = e^{\frac{t^2}{2\sigma^2} i^2 \overbrace{\left(E[X^2] - (E[X])^2\right)}^{Var[X]}}$$

$i^2 = -1$

$Var[X] = \sigma^2$

$$= e^{\frac{t^2}{2\sigma^2}(-1)\,Var[X]} = e^{-\frac{t^2}{2\sigma^2}\cdot\sigma^2} = e^{-\frac{t^2}{2}} = \phi_z(t)$$

$\lim_{n \to \infty} \phi_{z_n}(t) = \phi_z(t)$ so...

$$\phi_z(t) = e^{-\frac{t^2}{2}}$$

Is $\phi_z(t) \in L^1$?

since $\left|e^{-\frac{t^2}{2}}\right| = e^{-\frac{t^2}{2}}$ since $e^{-\frac{t^2}{2}} > 0$

$$\int_{\mathbb{R}} \left|e^{-\frac{t^2}{2}}\right| dt = \underbrace{\int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt}_{\text{Gamma}} = \sqrt{2\pi} < \infty$$

$$\int |\phi_z(t)|\, dt = \sqrt{2\pi} < \infty$$

so $\phi_z(t) \in L^1$

so can use P6 on $\phi_z(t) = e^{-\frac{t^2}{2}}$ to get density (PDF of Z)

$$\overset{P6}{\Longrightarrow} f_z(z) = \int_{\mathbb{R}} e^{-itz} e^{-\frac{t^2}{2}} dt$$

can use (P6) since $\phi_z(t) \in L^1$

(P6) $\Rightarrow f_z(z) = \frac{1}{\pi} \int_{\mathbb{R}} e^{-itz} e^{-\frac{t}{2}} dt$

$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-(itz + \frac{t^2}{2})} dt$

$\int_{\mathbb{R}} |e^{\ldots}| dt = \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt$ this is Gamma Function $= \sqrt{2\pi} < \infty$ so is $\in L^1$

completing the square:

$ax^2 + bx + c \iff d(x+e)^2 + f$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right) - \left(\frac{\sqrt{2}iz}{2}\right)^2$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right) - \left(\frac{2i^2z^2}{4}\right)$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right) - \left(-\frac{z^2}{2}\right)$$

$$f_z(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(\left(\frac{t}{\sqrt{2}}\right) + \frac{\sqrt{2}iz}{2}\right) - \frac{z^2}{2}} dt$$

$$= \frac{1}{2\pi} \int e^{-\left(\left(\frac{t}{\sqrt{2}}\right)^2 + \frac{\sqrt{2}iz}{2}\right)} e^{-\frac{z^2}{2}} dt$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}it}{2}\right)^2} dt$$

let $y = \frac{t}{\sqrt{n}} + \frac{\sqrt{2}iz}{2}$

$\frac{dy}{dt} = \frac{1}{\sqrt{2}}$

$dt = \sqrt{2}\, dy$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}i\, dy$$

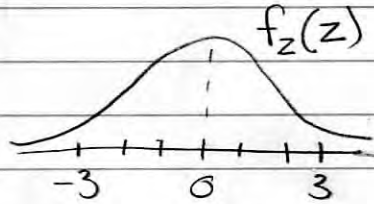
$$= \frac{1}{\sqrt{2}\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} dy = \frac{1}{\sqrt{2}\pi} e^{-\frac{z^2}{2}} \sqrt{\pi}$$

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = N(0,1)$$

$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = \underset{\text{Normal}}{N}(\overset{\mu}{0}, \overset{\sigma}{1}) \quad \sigma^2$$

"Standard Normal Distribution"

$$\boxed{\begin{array}{c}\text{Central Limit Theorem} \\ \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)\end{array}}$$



$\phi_z(t) = e^{-\frac{t^2}{2}}$

$$E[Z] \overset{P4}{=} \frac{\phi_z'(0)}{i} = 0$$

since $\phi_z'(t) = -t e^{-\frac{t^2}{2}}$

$$\phi_z'(0) = -0 e^{-\frac{0}{2}} = 0(1) = 0$$

$$Var[Z] = E[Z^2] - (E[Z])^2 = E[Z^2] - 0 = E[Z^2]$$

$$= E[Z^2] \overset{P4}{=} \frac{\phi_z''(0)}{i^2} = \frac{-0^2 e^{0^2/2} - e^{0^2/2}}{-1} = 1$$

$$Var[Z] = 1$$

since

$$\phi_z''(t) = -(-t^2 e^{-t^2/2} + e^{-t^2/2})$$

If $Z \sim N(0,1)$ then $\phi_z(t) = e^{-\frac{t^2}{2}}$

$$X = \sigma Z + \mu \quad \sigma > 0 \quad \sim \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$E[X] = \mu \qquad \text{PDF } f_X(x) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$Var[X] = \sigma^2 \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

"Normal Distribution"

pdf of "Normal r.v."

$$\phi_X(t) = e^{it\mu}\phi_Z(\sigma t)$$

$$= e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}}$$

remember:

$$\phi_{(Z)} = e^{-\frac{t^2}{2}}$$

$$\phi_X(t) = \phi_{\sigma Z+\mu}(t)$$

P2

$$= \phi_\mu(t)\phi_Z(\sigma t)$$

$$= e^{it\mu}\phi_Z(\sigma t)$$

P2

$$\phi_{aX+b}(t)$$

$$= e^{itb}\phi_X(at)$$

$$\phi_c = e^{itc}$$ constant

$$= e^{it\mu} e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}}$$

Characteristic function for $X \sim N(\mu, \sigma)$

$$\phi_X(t) = e^{itu - \frac{\sigma^2 t^2}{2}} = e^{t\left(u - \frac{\sigma^2 t}{2}\right)}$$

"Normal Density"

If $X \sim N(\mu, \sigma^2)$ then PDF is $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

μ is mean
σ^2 is variance σ is std. dev. (SE)

$X_1 \sim N(\mu_1, \sigma_1^2)$
$X_2 \sim N(\mu_2, \sigma_2^2)$ $\Big\rangle$ X_1, X_2 are indep.

$T = X_1 + X_2$ ← does not work well using convolution formulas, so use characteristic functions

$\phi_T(t) = \phi_{X_1+X_2}(t)$

$\phi_T(t) \overset{P2}{=} \phi_{X_1}(t) + \phi_{X_2}(t)$

$= e^{it\mu_1 - \frac{\sigma_1^2 t^2}{2}} \cdot e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$

$= e^{it\mu_1 + it\mu_2 - \left(\frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 t^2}{2}\right)}$

$= e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t}{2}}$ matches $\phi_{X_1}(t)$ for $X \sim N$ for $X \sim N(\mu = \mu_1+\mu_2, \sigma^2 = \sigma_1^2+\sigma_2^2)$

P1 → $T = X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim ?$ Find density (PDF)

$g(x) = e^x$

$g^{-1}(y) = \ln(y)$

$X = \ln(y) = g^{-1}$

$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{y}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}} \mathbb{1}_{\ln y \in (-\infty, \infty)} \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{y} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \mathbb{1}_{y \geq 0}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \mathbb{1}_{y \geq 0}$$

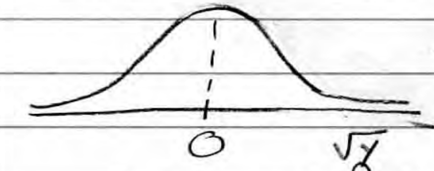
this density is called $\text{LogN}(\mu, \sigma^2)$ dist.

"Log-Normal"

$Z \sim N(0,1)$, and $Y = Z^2 = g(z)$

but now g is not one-to-one

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$



$$f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$ ← even function so symmetric around $z = 0$

$$\text{so } F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_z(z)\,dz = 2\int_0^{\sqrt{y}} f_z(z)\,dz$$

$$= 2\big(\underset{\text{cdf}}{F_z}(\sqrt{y}) - F_z(0)\big) = 2\big(F_z(\sqrt{y}) - 0.5\big)$$

$$F_Y(y) = 2F_Z(\sqrt{y}) - 1$$

$$\Rightarrow f_Y(y) = 2\left(\tfrac{1}{2} y^{-\frac{1}{2}}\right) F_Z(\sqrt{y})$$

$$= y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2}$$

$$\propto y^{\frac{1}{2}-1} e^{-\frac{1}{2}y} = y^{\alpha-1} e^{\beta y}$$

$$\propto \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \chi^2$$

χ^2 "Chi-Squared distribution with 1 degrees of freedom"

If

$X_1, X_2, \ldots, X_K$

are i.i.d

Gamma(α, β)

then

$T = X_1 + X_2 + \cdots + X_K$

is

Gamma$(K\alpha, \beta)$

$$Z_1, Z_2, \ldots, Z_K \overset{\text{i.i.d.}}{\sim} N(0,1)$$

$$X_K = Z_1^2 + Z_2^2 + \cdots + Z_K^2 \sim \chi_K^2$$

defines Chi-squared with K degrees of freedom χ_K^2

$$\chi^2 := \text{Gamma}\left(\frac{K}{2}, \frac{1}{2}\right)$$

pdf

$$f(x) = \frac{1}{2^{K/2}\Gamma(\frac{K}{2})} x^{\frac{K}{2}-1} e^{-\frac{1}{2}x} \mathbb{1}_{x \geq 0}$$

$$\chi_1^2 = \frac{1}{2^{1/2}\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$= \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

Test Wed

$X \sim \chi^2_K$

$Y = \sqrt{X} \sim \chi_K = ?$ find pdf

Y is a "Chi r.v." χ

inverse

$X = Y^2 = g^{-1}(Y)$

$f_Y(y) = f_X(y^2)\, 2y$

$$= \frac{1}{2^{K/2}\Gamma(K/2)} (y^2)^{\frac{K}{2}-1} e^{-\frac{y^2}{2}} (2y) \mathbb{1}_{y \geq 0}$$

$$= \frac{1}{2^{K/2}\Gamma(\frac{K}{2})} y^{K-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y \geq 0} = \chi_K$$

$Z \sim N(0,1)$ $\quad |Z| \sim \dfrac{1}{\underbrace{z^{-\frac{1}{2}}}_{\frac{1}{\sqrt{2}}} \underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}} e^{\frac{z^2}{2}} \mathbb{1}_{z \geq 0}$

$|z| = \sqrt{z^2} = \sqrt{\chi_1^2} = \chi_1$

$$= z\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathbb{1}_{z \geq 0}\right)$$

$$Z_1, Z_2 \overset{i.i.d.}{\sim} N(0,1)$$

$$R = \frac{Z_1}{Z_2} \sim f_R(r) = \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du$$

$Supp[R] = \mathbb{R}$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |u|\, du$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{e^{-\left(\frac{r^2+1}{2}\right)u^2} |u|\, du}_{\text{even function}}$$

symmetric around 0

$$= \frac{1}{2\pi} \cdot 2 \int_0^\infty e^{-\left(\frac{r^2+1}{2}\right)u^2} u\, du$$

Cauchy dist a.k.a. Lorenz dist.

Let $t = u^2$

$\Rightarrow \frac{dt}{du} = 2u$

$du = \frac{1}{2u} dt$

$$= \frac{1}{\pi} \int_0^\infty e^{-\left(\frac{r^2+1}{2}\right)t} u \cdot \frac{1}{2u} dt$$

$$= \frac{1}{2\pi} \int e^{-\frac{(r^2+1)}{2}} dt$$

Cauchy has no mean, no moments

let $a = \frac{r^2+1}{2}$

$$= \frac{1}{2\pi} \int_0^\infty e^{-at} dt = \frac{1}{2\pi} \cdot \frac{1}{a}$$

$E[X^n]$ is called nth moment

$$= \frac{1}{2\pi} \frac{1}{\frac{r^2+1}{2}} = \frac{1}{\pi} \cdot \frac{1}{r^2+1}$$

defines Cauchy $(0,1)$ "Standard Cauchy"