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Consider r.v X, Y with $m_X, m_Y, \sigma_X^2, \sigma_Y^2$

Let $W = (X - cY)^2$ for const $c \in \mathbb{R}$

W non-neg bcc. squared $\rightarrow E[W] \geq 0$

$\rightarrow E[(X-cY)^2] = E[X^2 - 2cXY + c^2Y^2]$

$= E[X^2] - 2cE[XY] + c^2E[Y^2] \geq 0$

Let $c = \dfrac{E[XY]}{E[Y^2]}$

$$\rightarrow E[X^2] - \overbrace{\frac{2E[XY]}{E[Y^2]}E[XY]}^{E[XY]^2} + \cancel{\frac{E[XY]^2}{E[Y^2]}E[Y^2]} \geq 0$$

$$\rightarrow E[X^2] - \frac{E[XY]^2}{E[Y^2]} \geq 0$$

$$\rightarrow E[X^2]E[Y^2] - E[XY]^2 \geq 0$$

$$E[XY]^2 \leq E[X^2]E[Y^2]$$

$$\rightarrow |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

Cauchy schwarz ineq

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we reasoned through $Cov[X,Y] = [-1,1]$

$$Z_x = \frac{X - \mu x}{\sigma x} \rightarrow X = \sigma_x Z_x + \mu x$$

$$Z_Y = \frac{Y - \mu Y}{\sigma Y} \rightarrow Y = \sigma_Y Z_Y + \mu Y$$

$E[Z_x] = E[Z_Y] = 0$

$SE[Z_x] = SE[Z_Y] = 1$

$E[Z_x^2] = E[Z_Y^2] = 1$

By Cau. Sc $|E[Z_x Z_Y]| \leq \sqrt{E[Z_x^2] E[Z_Y^2]} = 1$

$\rightarrow E[Z_x Z_Y] \in [-1,1]$

$$Cov[X,Y] = \frac{E[XY] - \mu_x \mu_Y}{\sigma_x \sigma_Y}$$

Corr?

$$= \frac{E[(\sigma_x Z_x + \mu x)(\sigma_Y Z_Y + \mu_Y)] - \mu_x \mu_Y}{\sigma_x \sigma_Y}$$

$$= \frac{\sigma_x \sigma_Y E[Z_x Z_Y] + \mu_x \mu_Y - \mu_x \mu_Y}{\sigma_x \sigma_Y} \in [-1,1]$$

* Correlation is bounded proof

12/4/19

Let g be convex fn on interval $I \in \mathbb{R}$

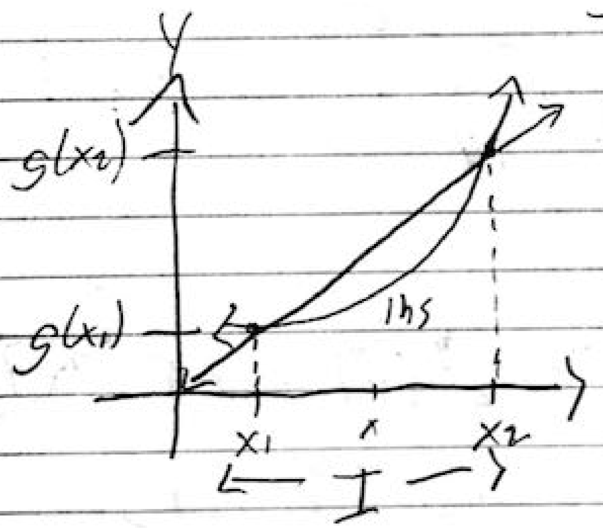
w are weights

Means $\forall \{x_1, x_2 \ldots\} \ \forall \{w_1, w_2 \ldots\}$ where $\sum w_i = 1, w_i \geq 0 \ \forall i$

weighted avg of x's

$$g(w_1 x_1 + w_2 x_2 + \ldots) \leq \underbrace{w_1 g(x_1) + w_2 g(x_2) + \ldots}_{\text{weight avg y's}}$$

$$\rightarrow g\left(\sum w_i x_i\right) \leq \sum w_i g(x_i)$$



Thm

$g''(x) \geq 0 \ \forall x \in I$

g concave on I

Let X be a discrete r.v with $supp[X] = \{x_1, x_2 \ldots\}$

Let $p(x_1), p(x_2), \ldots$ be the "weights"

If g is convex on I...

$$g\left(\sum_{x \in supp[X]} p(x)\,x\right) \leq \sum_{x \in supp[X]} p(x)\,g(x)$$

$$g(E[X]) \leq E[g(X)]$$

Jensen's ineq

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If g is "concave", $g(E[X]) \geq E[g(x)]$

Trivial Ineq.

$$g(x) = x^2 \longrightarrow \underbrace{E[X]^2}_{\mu^2} \leq \underbrace{E[X^2]}_{\sigma^2 + \mu^2}$$

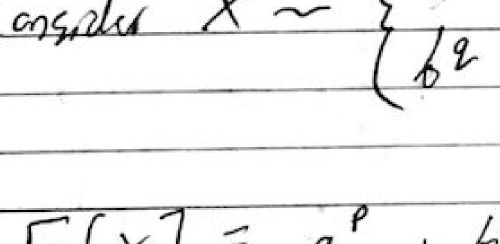
equal when $\sigma^2 = 0$ for const r.v's

1-1 Let $a, b \geq 0$, $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$

Consider $X \sim \begin{cases} a^p & \text{wp } \frac{1}{p} \\ b^q & \text{wp } \frac{1}{q} \end{cases}$ $\quad g(x) \sim \begin{cases} -p\ln(a) & \text{wp } \frac{1}{p} \\ -q\ln(b) & \text{wp } \frac{1}{q} \end{cases}$

$E[x] = \frac{a^p}{p} + \frac{b^q}{q}$, $E[g[x]] = -\ln(a) + -\ln(b)$
$= -\ln(ab)$

Consider $g(t) = -\ln(t)$ (convex) and 1-1



By Jensen's

$$-\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \leq -\ln(ab)$$

$$\rightarrow \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \ln(ab)$$

$$\rightarrow \ln(ab) \leq \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$ Young's Ineq

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Let $a = X$, $b = Y$ non-neg

$$E[XY] \le \frac{E[X^p]}{p} + \frac{E[Y^q]}{q}$$

Let $a = \frac{X}{E[X^p]^{1/p}}$, $b = \frac{Y}{E[Y^q]^{1/q}}$

$$\frac{E[XY]}{E[X^p]^{\frac{1}{p}} E[Y^q]^{1/q}} \le \frac{E\left[\left(\frac{X}{E[X^p]^{1/p}}\right)^p\right]}{p} + \frac{E\left[\left(\frac{Y}{E[Y^q]^{1/q}}\right)^q\right]}{q}$$

$$= \frac{\frac{E[X^p]}{E[X^p]}}{p} + \frac{\frac{E[Y^q]}{E[Y^q]}}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

$\rightarrow E[XY] \le E[X^p]^{1/p} E[Y^q]^{1/q}$ Holder ineq

Let $r, s > 0$ and $r < s$

Let $p = \frac{s}{r}$, $q = \frac{s}{s-r}$ $\rightarrow$ $p, q > 0$

$$\frac{1}{p} + \frac{1}{q} = \frac{r}{s} + \frac{s-r}{s} = 1$$

Let $X = |v|^r$, $Y = 1$ both non-neg

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By Holder's

$$E[|V|^r] \le E\left[(|V|^r)^{s/r}\right]^{r/s} = E[|V|^s]^{r/s}$$

If $E[|V|^s]$ if finite $\rightarrow$ $E[|V|^r]$ is finite

Convergence and Distribution

$X_n \xrightarrow{dist} X$ means CDF converges

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x$$

$$X_n \sim \begin{cases} \frac{1}{n+1} & wp\ \frac{1}{3} \\ 1-\frac{1}{n+1} & wp\ \frac{2}{3} \end{cases} \longrightarrow Bern\left(\frac{2}{3}\right)$$

$$\lim P_{X_n}(x) = \begin{cases} 0 & wp\ \frac{1}{3} \\ 1 & wp\ \frac{2}{3} \end{cases}$$

Thm $Supp[X_n] \subseteq \mathbb{Z}$, $Supp[X] \subseteq \mathbb{Z}$

then PMF conv. $\longleftrightarrow$ CDF conv

but does not apply because this ~~[illegible]~~ applies to integers

$$p_{X_n}(x) = F_{X_n}(x+\tfrac{1}{2}) - F_{X_n}(x-\tfrac{1}{2}) = P(X_n \in [x \pm \tfrac{1}{2}]) \quad \forall x \in \mathbb{Z}$$

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Pf $\leftarrow$

$$\lim p_{X_n}(x) = \lim F_{X_n}(x+\tfrac{1}{2}) - \lim F_{X_n}(x-\tfrac{1}{2})$$

$$= F_X(x+\tfrac{1}{2}) - F_X(x-\tfrac{1}{2}) = p_X(x)$$

Pf $\rightarrow$

$$\lim F_{X_n}(x) = \lim P(X_n \leq x) = \lim \sum_{y=-\infty}^{x} p_{X_n}(y)$$

$$= \sum_{y=-\infty}^{x} \lim p_{X_n}(y)$$

$$= \sum_{y=-\infty}^{x} p_X(y) = P(X \leq x) = F_X(x)$$

If $X_n \sim \text{Binom}\left(n, \frac{\lambda}{n}\right)$, $X \sim \text{Poisson}(\lambda)$

$\rightarrow X_n \xrightarrow{d} X$

$\rightarrow$ we've shown $\lim p_{X_n}(x) = p_X(x)$

Is PDF conv $\leftrightarrow$ CDF conv?

Only PMF conver. $\rightarrow$ CDF conver.

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For a counterexample, CDF conv $\nrightarrow$ PDF conv,

$$X_n \sim U[0, \tfrac{1}{n}) = n \mathbb{1}_{x \in [0, \frac{1}{n}]} = f_{X_n}(n)$$

$$\lim F_{X_n}(x) = \lim \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } x \in [0, \frac{1}{n}) \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases} = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} = \text{Deg}(0)$$

$$\lim f_{X_n}(x) = \lim n \mathbb{1}_{x [0, \frac{1}{n})} = \begin{cases} \infty & \text{if } x = 0 \\ 0 \end{cases}$$ has a PDF!

Convergence in Prob to a const

$X_n \xrightarrow{P} c$ means $\forall \epsilon > 0 \lim_{n \to \infty} P(|X_n - c| \geq \epsilon) = 0$

$\longleftrightarrow$

$$X_n \sim U(-\tfrac{1}{n}, \tfrac{1}{n}) = \frac{1}{2} \mathbb{1}_{x \in [-\frac{1}{n}, \frac{1}{n}]}$$