$$\vec{X} \sim Multin(n, \vec{p}) \Rightarrow X_j \sim Binom(n, p_j) \ \forall j$$

$$\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix}$$

If $\vec{X} \sim Multin(n, \vec{p}) \Rightarrow \vec{\mu} = \begin{bmatrix} np_1 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$

Matrix of r.v's

$$M = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$$

$$E[M] = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1m}] \\ \vdots & & \\ E[X_{n1}] & \cdots & E[X_{nm}] \end{bmatrix}$$

$$\sigma^2 := Var[X] = E[X^2] - \mu^2$$

$$\sigma_{ij} := Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j$$

$$= E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

If $X_1, X_2 \overset{ind}{\sim}$ $\sigma_{12} = 0$, i.e $Cov[X_1, X_2] = 0$.

Rules for Covariance

1. $Cov[X, X] = Var[X]$ $(\sigma_{11} = \sigma_1^2)$ Note $= \sigma^2$

2. $Cov[X_i, X_j] = Cov[X_j, X_i]$ (commutativity)

3. $Cov[X_1 + X_2 + X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$
(distributivity)

4. $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$, for $a_1, a_2 \in \mathbb{R}$ constant

5. $Var[X_1 + \dots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j]$

<u>Pf</u>: $Var[X_1 + X_2] \overset{?}{=} \sum_{i=1}^{2} \sum_{i=2}^{2} Cov[X_i, X_j]$

$$= \underset{\Downarrow}{\sigma_1^2} + \underbrace{\sigma_{12} + \sigma_{21}}_{2\sigma_{12}} + \underset{\Downarrow}{\sigma_2^2}$$

Rest by induction

$\sigma_{11} = \sigma_1^2$ $\qquad$ $\sigma_{22} = \sigma_2^2$

sigma also used for notation of Variance

$$\Sigma := Var[\vec{X}] := \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_k] \\ \vdots & Var[X_2] & & \\ & & \ddots & \\ Cov[X_k, X_1] & & & Var[X_k] \end{bmatrix}$$

"Variance Matrix"
"Covariance Matrix"
"Variance-Covariance Matrix".

$$:= E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T$$

- <u>Properties</u>

1. Symmetric (since Covariance is commutative)
2. Diagonal is non-negative

~~Independence ⇒ Cov = 0~~

Independence $\Rightarrow Cov = 0$

- If $X_1, \ldots, X_k \overset{ind}{\sim}$,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_k^2 \end{bmatrix} = I \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_k^2 \end{bmatrix}$$

"diagonal matrix"

- If $X_1, \ldots, X_k \overset{iid}{\sim}$

$$\Sigma = \begin{bmatrix} \sigma^2 & & & 0 \\ & \sigma^2 & & \\ & & \ddots & \\ 0 & & & \sigma^2 \end{bmatrix} = \sigma^2 I_k$$

- Rules for Expectation & Variance of r.v. vectors

1. $$E[\vec{X}+\vec{a}] = \begin{bmatrix} E[X_1]+a_1 \\ \vdots \\ E[X_K]+a_K \end{bmatrix} = \vec{\mu}+\vec{a}$$

where $\vec{a} \in \mathbb{R}^K$ constants

2. $$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + \dots + a_K X_K] = a_1 \mu_1 + \dots + a_K \mu_K = \vec{a}^T \vec{\mu}$$

$$E[A\vec{X}] = \begin{bmatrix} E[a_{11}X_1+\dots+a_{1K}X_K] \\ E[a_{21}X_1+\dots+a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1+\dots+a_{LK}X_K] \end{bmatrix}$$

$A \in \mathbb{R}^{L \times K}$ of constants

$$= \begin{bmatrix} \vec{a}_1^T \vec{\mu} \\ \vec{a}_2^T \vec{\mu} \\ \vdots \\ \vec{a}_L^T \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

3. $$Var[\vec{a}^T \vec{X}] = Var[a_1 X_1 + \dots + a_K X_K]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} Cov[a_i X_i, a_j X_j]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \sigma_{ij}$$

Now we want a nice linear algebra formula →

Let $z \in \mathbb{R}^K$, $V \in \mathbb{R}^{K \times K}$ (symmetric)

Consider the quantity $\underbrace{\vec{c}^{\,T}}_{1 \times K} \underbrace{V}_{K \times K} \underbrace{\vec{c}}_{K \times 1}$ called "quadratic form in c w/ determining matrix V"

this is a scalar quantity

$$\vec{c}^{\,T} V \vec{c} = [c_1, \dots, c_K] \begin{bmatrix} c_1 V_{11} + \dots + c_K V_{1K} \\ c_1 V_{21} + \dots + c_K V_{2K} \\ \vdots \\ c_1 V_{K1} + \dots + c_K V_{KK} \end{bmatrix}$$

Note: this multi out to 1 matrix Scalar qu

$$\begin{aligned} = \; & c_1 c_1 V_{11} + \dots + c_1 c_K V_{1K} \\ & + c_2 c_1 V_{21} + \dots + c_2 c_K V_{2K} \\ & + \\ & \vdots \\ & + c_K c_1 V_{K1} + \dots + c_K c_K V_{KK} \end{aligned}$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} c_i c_j V_{ij}$$

Thus $$\sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \sigma_{ij} = \vec{a}^{\,T} \underbrace{\Sigma}_{Var[\vec{X}]} \vec{a}$$

Markowitz Portfolio Theory

Let $\vec{X} = [X_1, \ldots, X_K]$ be the yearly terms of assets $1, \ldots, k$.
Let $\vec{w} = [w_1, \ldots, w_K]$ be a vector of weights s.t $\vec{w}^T \vec{1} = 1$
Let $F = \vec{w}^T \vec{X}$ be your total portfolio yearly return
I want $\mu_F = \mu_0$ & then select $\vec{w}$ s.t $Var[F]$ is minim

$$\vec{w} = \underbrace{\underset{\vec{w}^T \vec{1} = 1}{\operatorname{argmin}} \left\{ \vec{w}^T \Sigma \vec{w} \right\}}_{\text{Used in optimization theory}} \quad Var[F]$$

Back to Multinomial

$$\Sigma := Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & & \\ & np_2(1-p_2) & \sigma_{ij} & \\ \sigma_{ij} & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

We know $\sigma_{ij} < 0$

$$\sigma_{ij} = Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j$$

$$= \underbrace{\left(\sum_{x_1 \in Supp[X_1]} \sum_{x_2 \in Supp[X_2]} x_1 x_2 \, p_{X_i X_j}(x_1, x_2) \right) - \mu^2 p_i p_j}_{\text{FAIL}}$$

$\{0, \ldots, n\}$

Try again ☺

Recall: $X_i \sim Bin(n, p_i)$
$X_j \sim Bin(n, p_j)$

$X_i = X_{1i} + X_{2i} + \ldots + X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \ldots + X_{nj} \overset{iid}{\sim} Bern(p_j)$

$\Rightarrow X_{\ell i}$ & $X_{\ell j}$ are dependent $\forall \ell$

$\Rightarrow X_{\ell j}$ & X_{mj} are independent $\forall \ell \neq m$

$\Rightarrow \vec{X}_{1\cdot}, \vec{X}_{2\cdot}, \ldots, \vec{X}_{n\cdot} \overset{iid}{\sim} Multin(1, \vec{p})$

These dots mean choose an element from one of the X_i's (A vector of 0's & 1's)

* $X_{\ell i}$ & $X_{\ell j}$ are dependent b/c $X_{\ell i}$ is heads, $X_{\ell j}$ is tails. If you know what happened in $X_{\ell i}$, it automatically tells you the value of $X_{\ell j}$

$$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1i} + \dots + X_{kj}]$$

$$= \sum_{l=1}^{k} \sum_{m=1}^{k} Cov[X_{li}, X_{mj}]$$

$$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$$

• $l \neq m \Rightarrow Cov[X_{li}, X_{mj}] = 0$

if $l \neq m$ because it is signifying a different draw

$$= \sum_{l=1}^{n} \left(E[X_{li}, X_{lj}] - p_i p_j \right)$$

$$= \left(\sum_{l=1}^{n} E[X_{li}, X_{lj}] - np_i p_j \right) = \boxed{-np_i p_j}$$

Note: $E[X_{li}, X_{lj}] = \sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} p_{x_{li}, x_{lj}}(x_1, x_2)$

$$= \underbrace{p_{x_{li}, x_{lj}}(1,1)}_{\text{Probability of picking both an apple \& banana (clearly zero)}} = 0$$

only nonzero term

Result: $\sum_{l=1}^{n} Cov[X_{li}, X_{lj}] = \sum_{l=1}^{n} -p_i p_j = -np_i p_j$

• If $\vec{X} \sim Multin(n, \vec{p})$, $\vec{p} = \frac{1}{K}\vec{1}$, $k \to \infty$