Gamma Function

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$= \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)} \quad \text{Some } a \in \mathbb{R}^+$$

$\gamma(x,a)$ Lower Incomplete Gamma Function

$\Gamma(x,a)$ Upper Incomplete Gamma Function

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\gamma(x,a)}{\Gamma(x)} + \boxed{\frac{\Gamma(x,a)}{\Gamma(x)}} = P(x,a) + \boxed{Q(x,a)}$$

$P(x,a)$ is "Lower Regularized Gamma Function"
$Q(x,a)$ is "Upper Regularized Gamma Function"

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = -\left[e^{-t}\right]\Big|_0^\infty = 1$$

Hw: Show $\Gamma(x+1) = x\,\Gamma(x)$

ex: $\Gamma(2) = 1\,\Gamma(1)$

$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1$

$\Gamma(4) = 3\,\Gamma(3) = 3\cdot 2\cdot 1$

$\vdots$

$\Gamma(n) = (n-1)!$

"$2.5!$" $= \Gamma(3.5)$

Gamma Function gives us values for the continuous factorial function rather than just discrete.

$$X \sim \text{Erlang}(k, \lambda) := \underbrace{\frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}}_{f(x)} \mathbb{1}_{x \geq 0}$$

$$P(X \leq x) = \overset{\text{CDF}}{F(x)} = \int_0^x f(y)\,dy = \int_0^x \frac{\lambda^k e^{-\lambda y} y^{k-1}}{(k-1)!}\,dy$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^x e^{-\lambda y} y^{k-1}\,dy$$

$$= \frac{\lambda^k}{(k-1)!} \frac{\gamma(k, \lambda x)}{\lambda^k} = \frac{\gamma(k, \lambda x)}{\Gamma(k)} = \boxed{P(k, \lambda x)}$$

$$P(X > x) = 1 - F(x) = 1 - P(k, \lambda x) = \boxed{Q(k, \lambda x)}$$

"CDF complement"

Integrals

1. $\int_0^\infty t^{x-1} e^{-ct}\, dt \quad , \quad c \in \mathbb{R}$

Let $u = ct \Rightarrow t = u/c \Rightarrow dt = \frac{1}{c} du$

$$= \int_0^\infty \left(\frac{u}{c}\right)^{x-1} e^{-u} \left(\frac{1}{c} du\right)$$

$$= \frac{1}{c^x} \underbrace{\int_0^\infty u^{x-1} e^{-u}\, du}_{\Gamma(x)}$$

$$= \boxed{\frac{\Gamma(x)}{c^x}}$$

2. $\int_0^a t^{x-1} e^{-ct}\, dt$ (use same u-sub)

$$= \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \left(\frac{1}{c} du\right)$$

$$= \frac{1}{c^x} \underbrace{\int_0^{ac} u^{x-1} e^{-u}\, du}_{\gamma(x,a)}$$

$$= \boxed{\frac{\gamma(x, ac)}{c^x}}$$

3. $$\int_q^\infty t^{x-1} e^{-ct}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, qc)}{c^x}$$

$$= \boxed{\frac{\Gamma(x, qc)}{c^x}}$$

- $\lim_{q \to \infty} \gamma(x, q) = \Gamma(x)$

- $\lim_{q \to 0} \gamma(x, q) = \Gamma(x)$

- $\lim_{q \to \infty} P(x, q) = 1$

- $\lim_{q \to 0} Q(x, q) = 1$

If $n \in \mathbb{N}$, $\Gamma(n,a) = \int_a^\infty t^{n-1} e^{-t}\, dt$

Let $u = t^{n-1}$ $\quad v = -e^{-t}$

$dv = e^{-t}\, dt$ $\quad \frac{du}{dt} = (n-1)t^{n-2}$

$\Rightarrow du = (n-1)t^{n-2}\, dt$

$= [uv]\Big|_a^\infty - \int_a^\infty v\, du$ (Integration by Parts)

$= \left[t^{n-1}e^{-t}\right]\Big|_\infty^a + \int_a^\infty (e^{-t})(n-1)\, t^{n-2}\, dt$

$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{(n-1)-1} e^{-t}\, dt$

$= a^{n-1}e^{-a} + (n-1)\,\Gamma(n-1,a)$

$= a^{n-1}e^{-a} + (n-1)\left[a^{n-2}e^{-a} + (n-2)\,\Gamma(n-2,a)\right]$

$= e^{-a}\left[a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \ldots + (n-1)!\,\frac{\cancel{\Gamma(1,a)}^{\,1}}{\cancel{e^{-a}}}\right]$

Note: $\Gamma(1,a) = \int_a^\infty t^{1-1}e^{-t}\, dt = e^{-a}$

$= e^{-a}(n-1)!\left[\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \ldots + \frac{1}{0!}\right]$

$= e^{-a}(n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$

$\Rightarrow \Gamma(n+1,a) = e^{-a}\, n! \sum_{i=0}^{n} \frac{a^i}{i!}$

$\Rightarrow \frac{\Gamma(n+1,a)}{n!} = e^{-a} \sum_{i=0}^{n} \frac{a^i}{i!}$

$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}$$

<u>CDF</u>

$$P(X \leq x) = F(x) = \sum_{i=0}^{x} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{x}\frac{\lambda^i}{i!} = \frac{\Gamma(x+1, \lambda)}{x!}$$

$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = \boxed{Q(x+1, \lambda)}$$

Poisson Process

- λ is rate of events. $X_1, X_2, \dots \overset{iid}{\sim} Exp(\lambda)$

Time in seconds.

Q: What is probability of zero events before 1 second?

$$T_1 \sim Erlang(1, \lambda) = Exp(\lambda)$$

$$P(T_1 > 1) = 1 - F_{T_1}(1) = Q(1, \lambda) = \frac{\Gamma(1, \lambda)}{\Gamma(1)} = \frac{\int_1^\infty t^{1-1} e^{-\lambda t}\, dt}{\int_0^\infty t^{1-1} e^{-t}\, dt} = \frac{e^{-\lambda}}{1} = e^{-\lambda}$$

$$= \boxed{F_N(0)}$$

Q: What is probability at most one event occurs by 1 second?

$$T_2 \sim Erlang(2, \lambda)$$

$$P(T_2 > 1) = 1 - F_{T_2}(1) = Q(2, \lambda) = e^{-\lambda}(1 + \lambda) = \boxed{F_N(1)}$$

Q: What is probability at most K events occurs by 1 second?

$$T_K \sim Erlang(K, \lambda)$$

$$P(T_K > 1) = 1 - F_{T_K}(1) = Q(K, \lambda) = \boxed{F_N(K)}$$

- Poisson Process: If exponential waiting times, then the # of events that happen per unit of time is Poisson distributed.