Complex numbers continued

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad \cos x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!}$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - i\frac{t^3x^3}{3!} + \frac{t^4x^4}{4!} + i\frac{t^5x^5}{5!} - \dots$$

$$i\sin(tx) = itx - i\frac{t^3x^3}{3!} + i\frac{t^3x^5}{5!} + \dots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \dots$$

$$\Rightarrow e^{itx} = \cos(tx) + i\sin(tx)$$

let $\theta = tx$

$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$ if $\theta = \pi$

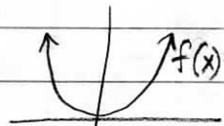
$\Rightarrow$ if $\theta = \pi$, then $e^{i\pi} = -1$

$\Rightarrow e^{i\pi} + 1 = 0$

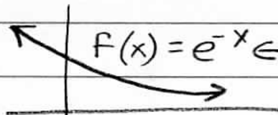
Define $L^1 := \{ f : \int_{\mathbb{R}} |f(t)|\, dt < \infty \}$

"ell-one" which is set of all "L^1 integrable functions" or all "absolutely integrable functions"

so



not L^1



is L^1

also, $f(x) = e^{-x} \mathbb{1}_{x>0} \in L^1$

all PDF's are L^1 (all PDFs $\in L^1$)

since $\int_{\mathbb{R}} |f(x)|\, dx = \int_{\mathbb{R}} f(x)\, dx < \infty$

If $f(t) \in L^1$ then there exists function $\hat{f}(\omega)$ which can be found using the "Fourier transform":

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-2\pi i \omega t} f(t)\, dt$$

Furthermore, if $\hat{f}(\omega) \in L^1$ then we can use the inverse Fourier transform to recover $f(t)$

$$f(t) = \int_{\mathbb{R}} e^{2\pi i \omega t} \hat{f}(\omega)\, d\omega$$

$\Rightarrow f, \hat{f} \in L^1$

$\Rightarrow f$ and $\hat{f}$ are one-to-one

$f(t)$ is known as the "true domain"

$\hat{f}(\omega)$ is known as the "frequency domain"

Characteristic Functions

Let X be a random variable

Define: $\phi_X(t) := E[e^{itX}] = \begin{cases} \int_{\mathbb{R}} e^{itx} f(x)\,dx & \text{if continuous} \\ \sum_{x \in \mathbb{R}} e^{itx} p(x) & \text{if discrete} \end{cases}$

Ch.f means characteristic function

which is called the characteristic function of X

We care about this because it gives us tools to solve problems and we can prove new theorems

properties/rules

(P0) $\phi_X(0) = E[e^{i(0)X}] = E[1] = 1$ for all X

(P1) $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$ (X and Y have same dist.)

(P2) If $Y = aX + b$,

then $\phi_Y(t) = E[e^{it(aX+b)}] =$

$= E[e^{itaX} e^{itb}]$

$= e^{itb} E[e^{i(at)X}]$

$= e^{itb} \phi_X(at)$

(P3) If X_1, X_2 are indep. and $T = X_1 + X_2$

then, $\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}]$

~~sir~~ since X_1, X_2 are indep, $= E[e^{itX_1}] E[e^{itX_2}] = \phi_{X_1}(t)\, \phi_{X_2}(t)$

(P3) If X_1, X_2 are indep. and $T = X_1 + X_2$

then, $\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}]$

since X_1, X_2 are indep., $= E[e^{itX_1}] E[e^{itX_2}]$

$= \phi_{X_1}(t)\, \phi_{X_2}(t)$

if X_1, X_2 are i.i.d, $= (\phi_X(t))^2$

(P4) "Moment Generator"

X is the r.v.
t is just a variable for function ϕ_X

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] \overset{\text{by Real Analysis}}{=} E\left[\frac{d}{dt}[e^{itX}]\right]$$

derivative

$$= E[iXe^{itX}]$$

$$\Rightarrow \phi_X'(0) = E[iX] \quad \Rightarrow E[X] = \frac{\phi'(0)}{i} =$$

$$\phi_X''(t) = \frac{d}{dt}[\phi_X'(t)] = E\left[\frac{d}{dt}[iXe^{itX}]\right]$$

$$= E[i^2X^2e^{itX}]$$

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 0$
etc.

$$\Rightarrow \phi_X''(0) = E[i^2X^2] \quad \Rightarrow E[X^2] = \frac{\phi''(0)}{i^2}$$

$\phi_X^{(n)}(t)$ means nth derivative of $\phi_X(t)$

$$\Rightarrow \text{In general, } E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}$$

$$\phi_X^{(n)}(t) = E[i^nX^ne^{itX}]$$

characteristic functions always exist (for any PDF)

(P5) Existence

$$|\phi_X(t)| = |E[e^{itX}]| = \begin{cases} \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| & \text{if X is continuous} \\ \left|\sum_{x\in\mathbb{R}} e^{itx} f(x)dx\right| & \text{if X is discrete} \end{cases}$$

r.v. random variable

$\forall$ for all

Triangle inequality $\left|\int_{\mathbb{R}} g(y)\right| dy \leq \int_{\mathbb{R}} |g(y)| dy$

characteristic function $\phi_x(t)$ exists for any probability distribution (of a r.v. X)

Triangle inequality $\left|\sum g(y)\right| \leq \sum |g(y)|$

$\Rightarrow$ then

(P5) Existence (of characteristic function for any density)

if X is continuous r.v. with PDF $f(x)$

(note $f(x) \geq 0 \ \forall x$)

$\int_{\mathbb{R}} f(x)\,dx = 1$

so $|f(x)| = f(x)$

$$
\begin{aligned}
|\phi_x(t)| &= \left|E[e^{itx}]\right| \\
&= \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| \\
&\leq \int_{\mathbb{R}} |e^{itx}| f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = 1
\end{aligned}
$$

because $|e^{itx}| = 1$

$$
\text{since } |e^{itx}| = |\cos(tx) + i\sin(tx)| = \sqrt{\cos^2(tx) + \sin^2(tx)} = \sqrt{1} = 1
$$

if X is discrete r.v. with PF (prob. function (PMF)) $p(x) = P(X = x)$

(note $p(x) \geq 0 \ \forall x$)

$\sum_{x \in \mathbb{R}} p(x) = 1$

so $|p(x)| = p(x)$

$$
\begin{aligned}
|\phi_x(t)| &= \left|E[e^{itx}]\right| \\
&= \left|\sum_{x \in \mathbb{R}} e^{itx} p(x)\right| dx \\
&\leq \sum_{x \in \mathbb{R}} |e^{itx}| p(x)\,dx = \sum_{x \in \mathbb{R}} p(x)\,dx = 1
\end{aligned}
$$

because $|e^{itx}| = 1$

so $|\phi_x(t)| \leq 1$ for any distribution (discrete or continuous)

so $\phi_{(x)} \in [-1, 1] \ \forall x, t$

if can't be integrated, then are dealing with a discrete r.v.

P6 Inversion:

if $\phi_X(t) \in L^1$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t)\, dt$$

$$t = -2\pi\omega$$

P7 Levy's CDF formula

$P(X \in [a,b])$ also written $P(a \le X \le b) = F(b) - F(a)$

F is CDF

For all characteristic functions $\phi_X(t)$'s

$$P(X \in [a,b]) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\, dt$$

Consider a sequence of r.v.'s $X_1, X_2, \ldots, X_n$

If $\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \;\; \forall x$

then we say "X_n converges in distribution to X"

and shorthand $X_n \xrightarrow{d} X$

P8 Levy's continuity Theorem

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)$$

$$\Rightarrow X_n \xrightarrow{d} X$$

Moment Generating Functions

The Moment Generating Functions (mgf) for r.v. X is

$$M_X(t) := E[e^{tX}]$$

analogous to "simpler" version of Characteristic function (but not as useful)

properties

(P0) $M_X(0) = 1$

(P1) $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$ (X and Y have same dist.)

(P2) $Y = aX + b \Rightarrow M_Y(t) = e^{tb} M_X(at)$

(P3) If X_1, X_2 are indep. and $T = X_1 + X_2$
then $M_T(t) = M_{X_1}(t) M_{X_2}(t) = (M_X(t))^2$

if X_1, X_2 are i.i.d.

m.g.f's may or may not exists (characteristic functions always do)

(P4) $E[X^n] = M_X^{(n)}(0)$

No (P5) ← mgf may or may not exist

$X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\beta - it)^\alpha} = \frac{\beta^\alpha}{(\beta - it)^\alpha}$$

$$= \left(\frac{\beta}{\beta - it}\right)^\alpha \leftarrow \text{Characteristic function for Gamma}(\alpha, \beta)$$

$X_1 \sim \text{Gamma}(\alpha, \beta)$, $X_2 \sim \text{Gamma}(\alpha, \beta)$ — Given X_1, X_2 are indep.

$$T = X_1 + X_2$$

$$\phi_T(t) = \phi_{X_1 + X_2}(t) \overset{P2}{=} \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right)^{\alpha_2}$$

$$= \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2}$$

$$\overset{P1}{\Longrightarrow} X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

$X \sim \text{Poisson}(\lambda)$ → find characteristic function

$$\phi_X(t) = \sum_{x \in \mathbb{R}} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,2,\dots\}}$$

Use: $e^{itx} = (e^{it})^x$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^{it}}$$

$$= e^{\lambda(e^{it}-1)}$$

← characteristic function for $X \sim \text{Poisson}(\lambda)$

$X_1 \sim \text{Poisson}(\lambda_1)$
$X_2 \sim \text{Poisson}(\lambda_2)$
> Given X_1, X_2 are indep.

$T = X_1 + X_2 \sim ?$

$$\phi_T(t) = \phi_{X_1+X_2}(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t)$$

$$= e^{\lambda_1(e^{it}-1)} e^{\lambda_2(e^{it}-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

would be $\phi_T(t)$ for $\text{Poisson}(\lambda_1+\lambda_2)$

$\Rightarrow T = X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

END OF MIDTERM 2 STUFF

Reviews for Test 2: Fri 10AM KY242
Tues RATHOUS 202 noon

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim}$ r.v.'s with finite expectation μ and finite variance σ^2

let $T_n = X_1 + X_2 + \ldots + X_n$ ← the sum r.v.

let $\overline{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n}$ ← the average r.v.

$\overline{X}_n = \frac{1}{n} X + 0$

From Math 241, $E[\overline{X}_n] = \mu$

$Var[\overline{X}_n] = \frac{\sigma^2}{n}$

so $SE[\overline{X}_n] = \frac{\sigma}{\sqrt{n}}$

$E[X] = \mu$
$Var[X] = \sigma^2$
$SE[X] = \sigma$

$T_n = \sum_{i=1}^{n} X_i$
so
$\phi_{T_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t)$
$\phi_{T_n}(t) = (\phi_X(t))^n$
since X's are i.i.d.

$\phi_{T_n}(t) \overset{P3}{=} (\phi_X(t))^n$

$\phi_{\overline{X}_n}(t) \overset{P2}{=} \left(\phi_X\left(\frac{t}{n}\right)\right)^n$

$\phi_Z(t) \overset{P2}{=} \left(e^{\frac{it\mu}{\sigma\sqrt{n}}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$

more next time...

"standardization"

$Z_n = \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$

$Z_n = \frac{\sqrt{n}}{\sigma} X_n - \frac{\sqrt{n}}{\sigma}\mu$

$E[Z_n] = 0$

$SE[Z_n] = 1$

$aX + b$

if $Y = aX + b$
$\phi_Y(t) = e^{itb}\phi_X(at)$
here, $a = \frac{\sqrt{n}}{\sigma}$
$b = \frac{\sqrt{n}}{\sigma}\mu$