Lecture 16

$$\phi_{T_n}(t) \underset{P3}{=} \phi_X(t)^n, \quad \phi_{\bar{X}_n}(t) \underset{P2}{=} \phi_X\left(\frac{t}{n}\right)^n$$

$$\phi_{Z_n} \underset{P2}{=} e^{-it\left(\frac{\sqrt{n}}{\sigma}\right)\mu} \phi_X\left(\frac{\sqrt{n}}{\sigma}\frac{t}{n}\right)^n = e^{\frac{-it n\mu}{\sigma\sqrt{n}}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n$$

$$= e^{\frac{-it\mu n}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$\phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}\right)}$$

Examine $\lim_{n\to\infty}$ in hopes of using P8, P6

take $\lim_{n\to\infty}$, move $e^{\frac{t^2}{\sigma^2}}$ outside

Let $u = \frac{t}{\sigma\sqrt{n}}$ $\quad n \to \infty \quad u \to 0$

$$= e^{\frac{t^2}{\sigma^2}} \lim_{n\to\infty} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}$$

L'hospital $$= e^{\frac{t^2}{\sigma^2}} \lim_{u\to 0} \frac{\frac{\phi_X(u)}{\phi_X(u)} - i\mu}{2u}$$

L'hospital again $$= e^{t^2/2\sigma^2} \lim_{u\to 0} \frac{d}{dt}\left[\frac{\phi'_X(u)}{\phi_X(u)}\right]$$

$$= e^{t^2/2\sigma^2} \lim_{n\to 0} \frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}$$

$$= e^{t^2/2\sigma^2} \frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi(0)^2}$$

$$P0 = e^{t^2/2\sigma^2}(\phi_X''(0) - \phi_X'(0)^2)$$

$$P4 = e^{t^2/2\sigma^2}(i^2 E[X^2] - i^2 E[X]^2)$$

$$= e^{-t^2/2\sigma^2}(E[X^2] - E[X]^2] = Var[X]$$

$$= e^{-t^2/2} = \phi_Z(t) \quad \text{Is } \phi_Z(t) \in L^1?$$

$$\int_{\mathbb{R}} |e^{-t^2/2}| dt = \int_{\mathbb{R}} e^{-t^2/2} = \sqrt{2\pi} < \infty \checkmark$$

$$P6 \quad f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz} e^{-t^2/2} dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(t^2/2 + itz)} dt$$

Note: $\frac{t^2}{2} - itz = \left(\frac{t}{\sqrt{2}} - \frac{\sqrt{2}iz}{2}\right)^2 - \underbrace{\left(\frac{\sqrt{2}iz}{2}\right)^2}_{= z^2/2}$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} - \frac{\sqrt{2}iz}{2}\right)^2} e^{-\frac{z^2}{2}} dt$$

$$= \frac{1}{2\pi} e^{-z^2/2} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} dt$$

Let $u = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}$

$$\frac{du}{dt} = \frac{1}{\sqrt{2}} \rightarrow dt = \sqrt{2}\, du$$

$$\frac{1}{2\pi} e^{-z^2/2} \int_{\mathbb{R}} e^{-v^2} \sqrt{2}\, du$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \int_{\mathbb{R}} e^{-v^2} du = \boxed{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}} = f_Z(t) = N(0,1)$$

$$Z \sim N(0,1) \rightarrow \phi_Z(t) = e^{-t^2/2}$$

$$\phi_Z'(t) = -te^{-t^2/2}$$

$$E[Z] = \frac{\phi_Z(0)}{i} = 0$$

$$Var[Z] = E[Z^2] - \cancel{E[Z]^2} = E[Z^2] = \frac{\phi_Z''(0)}{i^2} = \frac{-1}{-1} = 1$$

$$\phi_Z''(t) = -\left(-t^2 e^{-t^2/2} + e^{-t^2/2}\right)$$

trans of r.v

$$X = \sigma Z + \mu \sim \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

"normal r.v"

$$\rightarrow E[X] = \mu, \quad Var[X] = \sigma^2$$

$$\phi_X(t) \underset{P2}{=} e^{it\mu} \phi_Z(\sigma t) = e^{it\mu} e^{-\frac{\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

$$X_1 \sim N(\mu, \sigma^2) \text{ ind of } X_2 \sim N(\mu_2, \sigma_2^2)$$

$$T = X_1 + X_2$$

$$\sim \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\pi\sigma^2}(x-\mu)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\pi}((t-x)-\mu_2)^2} dx$$

(P3) $\Phi_T(t) = \Phi_{X_1}(t)\Phi_{X_2}(t)$

$$= e^{it\mu_1 - \frac{\sigma^2 t^2}{2}} e^{it\mu_2 - \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{it\mu_1 + it\mu_2 - \left(\frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 t^2}{2}\right)}$$

$$= e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

(P1) $T \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X$?

$X = \ln(y) = g^{-1}(y)$, $\frac{d}{dy}[g^{-1}(y)] = \frac{1}{Y}$

$f_Y(y) = f_X(g^{-1}(y))$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \mathbb{I}_{\substack{\ln(y)\in(-\infty,\infty) \\ y \geq 0}}$$

$$\frac{1}{Y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-m)^2} = \text{LogN}(m, \sigma^2)$$

"Log-Normal"

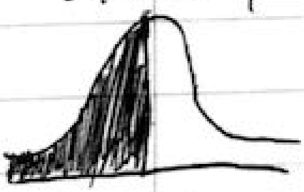
$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$? is <u>not</u> 1:1!

$$F_Y(y) = P(Y \leq y) = P(Z^2 \leq y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_Z(z)\,dz = 2\int_0^{\sqrt{y}} f_Z(z)\,dz \quad 2(F_Z(\sqrt{y}) - F_Z(0))$$

$$= F_Y(y) = 2(F_Z(\sqrt{y}) - F_Z(0))$$

symmetry



$$f_Y(y) = \cancel{2}\left(\frac{1}{\cancel{2}} y^{-1/2} f_Z(\sqrt{y})\right)$$

$$= y^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \propto y^{\overset{\alpha}{\frac{1}{2}}-1} e^{\overset{\beta}{-\frac{1}{2}}y}$$

$$\propto \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \chi^2$$

chi-square

$Z \sim N(0,1)$, $X = Z^2 \sim \chi^2$

$Z_1, \ldots, Z_K \overset{iid}{\sim} N(0,1)$

$\rightarrow Z_1 + Z_2 + \ldots + Z_K \sim \chi^2_K$ chi-squared with K deg of freedom

$$\chi^2_k = \text{Gamma}\left(\tfrac{k}{2}, \tfrac{1}{2}\right)$$

$$X \sim \chi^2_k, \quad Y \overset{1:1}{=} \sqrt{X} \sim \chi_k \to \text{"chi"}$$

$$X = Y^2 = g^{-1}(y), \quad \frac{d}{dy}\left[g^{-1}(y)\right] = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} (y^2)^{\frac{k}{2}-1} e^{\frac{1}{2}y^2} \mathbb{I}_{y^2 \in (0,\infty)} \quad (2y)$$

$$y \in (0,\infty)$$

$$= \frac{1}{2^{k/2}\Gamma(k/2)} y^{k-1} e^{\frac{1}{2}y^2} \mathbb{I}_{y \geq 0} \quad \checkmark$$

$$Z \sim N(0,1)$$

$$|Z| = \sqrt{Z^2} \sim \chi_1 = \frac{1}{\frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)} e^{-\frac{y^2}{2}} \mathbb{I}_{y \geq 0}$$

$$= 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbb{I}_{y \geq 0}\right)$$

$= 2 \cdot$ density of $N(0,1)$ for $z \geq 0$

$X_1, X_2 \underset{\sim}{iid} N(0,1)$

$$R = \frac{X_1}{X_2} \sim \int_{\mathbb{R}} f(ru) f(u) |u| \, du \qquad Supp[R] = \mathbb{R}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} |u| \, du$$

2 because even

$$= \frac{1}{2\pi}\left(2\int_0^{\infty} e^{-\left(\frac{r^2+1}{2}\right)u^2} |u| \, du\right) = \frac{1}{\pi}\int e^{-\left(\frac{r^2+1}{2}\right)u^2} u \, du$$

Let $t = u^2$, $\frac{dt}{du} = 2u \rightarrow du = \frac{1}{2}\frac{1}{u} dt$

$$= \frac{1}{\pi}\int_0^{\infty} e^{\frac{(-r^2+1)t}{2}} u \frac{1}{2}\frac{1}{u} \, dt$$

$$= \frac{1}{2\pi}\int_0^{\infty} e^{-at} dt = \frac{1}{\cancel{2}\pi} \cdot \frac{1}{\frac{r^2+1}{\cancel{2}}}$$

$$= \frac{1}{\pi}\frac{1}{r^2+1} = Cauchy(0,1)$$