lecture

$X_1, X_2 \sim \text{Bern}(p)$

$$p(t) = \sum_{x\in\{0,1\}} (p^x(1-p)^{1-x})(p^{t-x}(1-p)^{1-(t-x)}\mathbb{1}_{t-x\in\{0,1\}}$$

$$= p^t(1-p)^{2-t} \underbrace{\sum_{x\in\{0,1\}} \mathbb{1}_{t-x\in\{0,1\}}}$$

$$\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}} = \begin{cases} 1 & \text{if } t=0 \\ 2 & t=1 \\ 1 & t=2 \\ 0 & \text{o/t} \end{cases}$$

$$= \binom{2}{t}$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

$$= \text{Bin}(2,p)$$

$$\text{Bern}(p) = \underbrace{\binom{1}{x}}_{\mathbb{1}_{x\in\{0,1\}}} p^x(1-p)^{1-x}$$

Using this, $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$$p(t) = \sum \left(\binom{1}{x} p^x (1-p)^{1-x}\right)\left(\binom{1}{t-x} p^{t-x}(1-p)^{1+x-t} \underbrace{\mathbb{1}_{t-x\in\{0,1\}}}_{\text{redundant}}\right)$$

$$= p^t(1-p)^{2-t} \sum_{x\in\{0,1\}} \underbrace{\binom{1}{x}}_{=1} \binom{1}{t-x}$$

$$= p^t(1-p)^{2-t}\left(\binom{1}{t} + \binom{1}{t-1}\right) = \underbrace{\binom{2}{t}}_{\text{2 choose } t} p^t (1-p)^{2-t}$$

* Pascal's identity:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$

$$T = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2$$

$$p(t) = \sum_{x\in\{0,1\}} \left(\binom{1}{x} p^x(1-p)^{1-x}\right)\binom{2}{t-x} p^{t-x}(1-p)^{2+x-t} \underbrace{\mathbb{1}_{t-x\in\{0,1,2\}}}_{\text{redundant}}$$

$$= p^t(1-p)^{3-t} \sum_{x\in\{0,1\}} \binom{1}{x}\binom{2}{t-x}$$

$$= \binom{3}{t} p^t (1-p)^{3-t} = \text{Bin}(3,p)$$

So, we get bin(3, p)
If we keep doing this, bin(n, p)

• $X_1, X_2 \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

$T = X_1 + X_2$?

$= Bin(2n, p)$

$$\sum_{x\in\{0,1,\dots n\}} \left(\binom{n}{x} p^x (1-p)^{n-x}\right)\binom{n}{t-x} p^{t-x}(1-p)^{n-t+x} \underbrace{\mathbb{1}_{t-x\in\{0,\dots,n\}}}_{\text{redundant}}$$

$$= p^t(1-p)^{2n-t} \sum_{x\in\{0,1\dots n\}} \binom{n}{x}\binom{n}{t-x}$$

$$= p^t(1-p)^{2n-t}\binom{2n}{t} = bin(2n,p)$$

• Geometric:

$B_1, B_2 \dots \overset{iid}{\sim} Bern(p)$

$X := \#$ of zero realized before the first one

$= Geo(p)$

$Supp(X) = \{0,1,2\dots\} = \mathbb{N}_0$

$p(0) = p$ → # of times that you don't succeed

$p(1) = (1-p)p$

$p(2) = (1-p)^2 p$

$\vdots$

$p(x) = (1-p)^x p$

How many times you fail

PMF: $X_1, X_2 \overset{iid}{\sim} Geom(p)$

$T = X_1 + X_2 \sim$?

$$p(t) = \sum_{x\in\{0,1,2\dots n\}} \left((1-p)^x p\right)\left((1-p)^{t-x} p\right) \mathbb{1}_{t-x\in\{0,1,\dots n\}}$$

$$= p^2(1-p)^t \underbrace{\sum_{x\in\{0,1\dots n\}} \mathbb{1}_{t-x\in\{0,1,\dots n\}}}_{\cancel{t\geq x}} = \sum_{x\in\{0,\dots,n\}} \mathbb{1}_{t\geq x} = \sum_{x\in\{0,\dots\}} \mathbb{1}_{x\leq t}$$

$= \cancel{\sum}$

$$= \mathbb{1}_{0\leq t} + \mathbb{1}_{1\leq t} + \mathbb{1}_{2\leq t} + \mathbb{1}_{3\leq t} + \dots$$

Ex) choose 3 then there are 4.

$= t+1$

$$= p^2(1-p)^t(t+1)$$

ⓧ $P(4) = \underbrace{5}_{\text{\# of ways of grouping}}\underbrace{(1-p)^4}_{\text{prob. of failures (zeros)}}\underbrace{p^2}_{\text{prob. of success}}$

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$
$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6$$

- $X_1, X_2 \,\&\, X_3 \sim \text{Geom}(p)$

$$T = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2$$

$$p(t) = \sum_{x \in \{0,1,\dots,n\}} \left((1-p)^x p\right)\left(p^2 (1-p)^{t-x}(t-x+1)\right) \mathbb{1}_{t-x \in \{0,1,2,\dots,n\}}$$

$$= \underbrace{p^3}_{\text{3 successes}} (1-p)^t \underbrace{\sum_{x \in \{0,1,\dots,n\}} (t-x+1)\, \mathbb{1}_{t-x \in \{0,1,\dots,n\}}}$$

$$\sum_{x \in \{0,1,\dots,n\}} (t+1)\, \mathbb{1}_{t-x \in \{0,1,\dots,n\}} - \sum_{x \in \{0,1,\dots,n\}} x\, \mathbb{1}_{t-x \in \{0,1,\dots,n\}}$$

$$= (t+1) - \sum_{x \in \{0,1,\dots,n\}} x\, \mathbb{1}_{t-x \in \{0,1,\dots,n\}}$$

$$= (t+1) \underbrace{\sum_{x \in \{0,\dots,n\}} \mathbb{1}_{x \le t}} - \sum_{x \in \{0,\dots,n\}} x\, \mathbb{1}_{t-x \in \{0,\dots,1,\dots,n\}}$$

$$= (t+1)\ (t+1\ )$$

$$= (t+1)^2 - \underbrace{\sum x\, \mathbb{1}_{t-x \in \{\dots,n\}}}$$

$$= 0 + 1\,\mathbb{1}_{1 \le t} + 2\,\mathbb{1}_{2 \le t} + 3\,\mathbb{1}_{3 \le t} + 4\,\mathbb{1}_{4 \le t} + \dots$$

$$= \frac{t(t+1)}{2}$$

⇒ ~~Altogether~~ $\cancel{p^3 (1-p)^t \left(\frac{t(t+1)}{2}(t+1)^2\right)}$

$$= (t+1)^2 - \frac{t(t+1)}{2} = \frac{(t+2)(t+1)}{2}$$