Lec 29 Math 621 11/20/18

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{Z}^T\vec{Z}}$$

Let $A \in \mathbb{R}^{n \times n}$ and invertible, $\vec{\mu} \in \mathbb{R}^n$, let $\Sigma = AA^T$

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma) := \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{X}-\vec{\mu})^T \Sigma^{-1} (\vec{X}-\vec{\mu})}$$

$$\phi_{\vec{Z}}(\vec{t}) = e^{-\frac{1}{2}\vec{t}^T\vec{t}}, \quad \phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$$= e^{i\vec{t}^T\vec{0} - \frac{1}{2}\vec{t}^T I \vec{t}}$$

Let $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^m$

$$\vec{Y} = B\vec{X} + \vec{c} \Rightarrow \phi_{\vec{Y}}(\vec{t}) = e^{-i\vec{t}^T(B\vec{\mu} + \vec{c}) - \frac{1}{2}\vec{t}^T(B\Sigma B^T)\vec{t}}$$

$$\overset{P1}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{\mu} + \vec{c}, B\Sigma B^T)$$

Let $A \in \mathbb{R}^{m \times n}$ where $m \le n$, A is full rank $\Rightarrow AA^T = \Sigma$ invertible

$$\vec{X} = A\vec{Z} + \vec{c} \sim N_m(A\vec{0}_n + \vec{c}, A I_n A^T) = N_m(\vec{c}, AA^T) = N_m(\vec{c}, \Sigma)$$

That's the general proof!

Review.

In general, if $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$ How is $(\vec{X}-\vec{\mu})^T \Sigma^{-1} (\vec{X}-\vec{\mu}) \sim ?$

$$= (\vec{X}-\vec{\mu})^T (A^{-1})^T A^{-1} (\vec{X}-\vec{\mu}) = (\underbrace{A^{-1}(\vec{X}-\vec{\mu})}_{\vec{Z}})^T (\underbrace{A^{-1}(\vec{X}-\vec{\mu})}_{\vec{Z}}) = \vec{Z}^T\vec{Z} \sim \chi^2_n$$

Mahalanobis Distance (1936)
analog of z-score $z = \frac{x-\mu}{\sigma}$ in 1 dim

$B \in \mathbb{R}^{m \times n}$

Now let $X \sim N_n(\vec{\mu}, \Sigma)$ and $\vec{Y} = B\vec{X} + \vec{c} \sim ?$

Use ch.f.'s!

$$\phi_{\vec{Y}}(\vec{t}) = e^{i\vec{t}^T\vec{c}}\,\phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}}\left(e^{i(B^T\vec{t})^T\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T\Sigma(B^T\vec{t})}\right)$$

we need to make sure $B\Sigma B^T$ still legit cov-matrix

$$= e^{i\vec{t}^T(B\vec{\mu}+\vec{c}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{P1}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{\mu}+\vec{c},\, B\Sigma B^T)$$

$\vec{Y} = a\vec{X} \sim N(a\vec{\mu}, a^2\Sigma)$

$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_m(\vec{\mu}, AA^T)$
$A \in \mathbb{R}^{m \times n}$, $\vec{\mu} \in \mathbb{R}^m$

Done!

Another cool fact about multivariate ch.f.'s

$$\phi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{X}}\right]$$

let $\vec{t} = \begin{bmatrix} t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{i[t\,0\dots0]\vec{X}}\right] = E\left[e^{itX_1}\right] = \phi_{X_1}(t) \overset{P1}{\Longrightarrow} X_1 \sim \dots$

this means you can find marginal distr's! No need for $\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f_{\vec{X}}(\vec{x})\,dx_1\,dx_2\dots$

e.g. $\vec{X} \sim N_m(\vec{\mu}, \Sigma)$ what is $X_1 \sim ?$

$$\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = e^{i[t\,0\dots0]\vec{\mu} - \frac{1}{2}[t\,0\dots0]\Sigma\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = e^{it\mu_1 - \frac{1}{2}\Sigma_{11}} \Rightarrow X_1 \sim N(\mu_1, \overset{\sigma_1^2\,=}{\Sigma_{11}})$$

HW: multiple dimension

We will now justify the T test and F tests from linear regression ECON 382, 387 using Math 390 material

NOT COVERED ON FINAL

this was a whole class in grad school called linear model theory

Let $\vec{Y} = X\vec{\beta} + \vec{\mathcal{E}}$ where $\vec{X} \in \mathbb{R}^{n \times p}$, $\vec{\beta} \in \mathbb{R}^p$ constants

$\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$

rank$(X) = p$ and $\vec{\mathcal{E}}_n \sim N_n(\vec{0}, \sigma I_n)$

Normal errors, Homoskedastic error assumption