Lec 3 9/5/19 Math 621

$$\sum_{i=1}^{r} X_i \sim \text{NegBin}(r,p) = \binom{t+r-1}{r-1}(1-p)^t p^r$$

$X \sim \text{Bin}(n,p)$. Let n be large and p be small s.t.

$$\lambda = np \Rightarrow p = \frac{\lambda}{n}, \quad \lambda \in (0,\infty)$$

param. space

$$\text{Bin}(n,p) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = p(x)$$

consider r.v. X s.t.

Let $n \to \infty$

$$\lim_{n\to\infty}\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty}\frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x\text{ terms}}}{\underbrace{n\cdot n\cdots n}_{x\text{ terms}}}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}_{\to 1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} = \text{Poisson}(\lambda) \qquad \text{Supp}(X) = \{0,1,\dots\}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$$T = X_1 + X_2 \sim \sum_{x\in \text{Supp}(X)} p(x)\,p(t-x) = \sum_{x=0}^{\infty}\frac{\lambda^x e^{-\lambda}}{x!}\frac{\lambda^{t-x}e^{-\lambda}}{(t-x)!}\mathbb{1}_{t-x\in\{0,1,\dots\}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x=0}^{\infty} \frac{1}{x!\,(t-x)!} \mathbb{1}_{x \le t} \left(\frac{t!}{t!}\right)$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x=0}^{\infty} \binom{t}{x} \mathbb{1}_{x\le t} = \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x=0}^{t} \binom{t}{x}}_{\text{another identity}}$$

Recall powersets. Consider set A s.t. $|A| = n$.

$$2^A := \{B : B \subseteq A\} = \{B : B \subseteq A, |B| = 0\} \cup \{B : B \subseteq A, |B| = 1\} \cup \vdots \; \{B : B \subseteq A : |B| = n\}$$

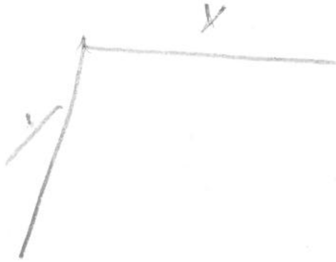
$$2^{|A|} = |2^A| = |\{B : B \subseteq A, |B| = 0\}| + |\{B : B \subseteq A : |B| = 1\}| + \cdots + |\{B : B \subseteq A : |B| = n\}|$$

$= 2^n$

$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$

You will solve the general case on the HW



Let $X, Y \overset{iid}{\sim} \text{Geom}(p)$

$$P(X>Y) = ? \overset{?}{=} \frac{1}{2}$$

Since iid $P(X>Y) = P(Y>X)$

$$P(X>Y) + P(Y>X) + \underbrace{P(X=Y)}_{\neq 0} = 1$$ Law of total prob.

$$\Rightarrow P(X>Y) < \frac{1}{2}$$

How to solve? Use indicator functions to select the areas

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p_{X,Y}(x,y)\, \mathbb{1}_{x>y} \overset{iid}{=} \sum \sum p(x)\, p(y)\, \mathbb{1}_{x>y}$$

$$= \sum_{x>y} \sum_{y\in\mathbb{R}} p(x)\, p(y)$$

$$= \sum_{x>y} \sum_{y\in\mathbb{R}} \left((1-p)^x p\, \mathbb{1}_{x\in\mathbb{N}_0}\right)\left((1-p)^y p\, \mathbb{1}_{y\in\mathbb{N}_0}\right)$$

$$= p^2 \sum_{x=y+1}^{\infty} \sum_{y=0}^{\infty} (1-p)^x (1-p)^y$$

$$= p^2 \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} (1-p)^x (1-p)^y = p^2 \sum_{y=0}^{\infty} \sum_{x'=0}^{\infty} (1-p)^{x'+(y+1)} (1-p)^y$$

reindexing trick let $x' = x-(y+1)$

$$\Rightarrow x = x' + y + 1$$

$$= p^2 \sum_{y=0}^{\infty} (1-p)^{2y+1} \sum_{x'=0}^{\infty} (1-p)^{x'}$$

$$= \cancel{p^2}(1-p) \sum_{y=0}^{\infty} (1-p)^{2y} \frac{1}{\cancel{p}}$$

$$= p(1-p) \sum_{y=0}^{\infty} \left((1-p)^2\right)^y$$

$$= p(1-p) \frac{1}{1-\underbrace{(1-p)^2}_{=1-2p+p^2}}$$

$$= \frac{p-p^2}{2p-p^2}$$

$$= \frac{1-p}{2-p}$$

easier way? Yes.... HW...

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this concept connects...

Recall $E[g(X)] = \sum_{x \in R} g(x) p(x)$

let $g(x) = \mathbb{1}_{x \in A}$

$$E[\mathbb{1}_{X \in A}] = \sum_{x \in R} \mathbb{1}_{x \in A}\, p(x) = \sum_{x \in A} p(x) = P(X \in A)$$

$$E[g(X,Y)] = \sum \sum g(x,y)\, p(x,y)$$

$$E[\mathbb{1}_{X>Y}] = \sum \sum \mathbb{1}_{x>y}\, p(x,y) = P(X>Y)$$

Vector r.v.'s

Basket of fruit: apples & bananas

p_1 = Prob of picking apple

p_2 = prob of picking banana

$p_1 + p_2 = 1$

draw with replacement n times,

How many apples?

$$X_1 \sim Bin(n, p_1) = \frac{n!}{x_1!(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1} \mathbb{1}_{x_1 \in \{0,\dots,n\}}$$

How many bananas?

$X_2 \sim Bin(n, p_2)$ but X_1, X_2 dependent! $X_2 = n - X_1$

JMF!

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim ? \quad p_{X_1,X_2}(x_1,x_2) = \frac{n!}{x_1!\,x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,\dots,n\}} \mathbb{1}_{x_2 \in \{0,\dots,n\}}$$

vector of r.v.'s

$$\frac{n!}{x_1!\,x_2!} = \binom{n}{x_1, x_2}$$ multinomial coefficient, "multi-choose notation"

Add cantaloupes to basket... p_3 = Prob of picking cantaloupe

$p_1 + p_2 + p_3 = 1$

$X_1 \sim Bin(n, p_1)$

$X_2 \sim Bin(n, p_2)$

$X_3 \sim Bin(n, p_3)$

JMF!

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim ? \quad p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{n!}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n} \mathbb{1}_{x_1 \in \mathbb{N}_0} \mathbb{1}_{x_2 \in \mathbb{N}_0} \mathbb{1}_{x_3 \in \mathbb{N}_0}$$

$$\frac{n!}{x_1!\,x_2!\,x_3!} = \binom{n}{x_1, x_2, x_3}$$