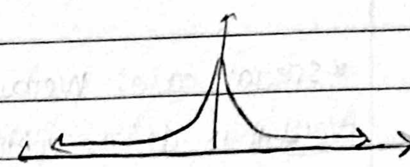
$$= e^d \begin{cases} \int_d^\infty e^{-2x}\,dx & \text{if } d>0 \\ \int_0^\infty e^{-2x}\,dx & \text{if } d\le 0 \end{cases}$$

$$= e^d \begin{cases} \left[-\frac{1}{2} e^{-2x}\right]_d^\infty & \text{if } d>0 \\ \left[-\frac{1}{2} e^{-2x}\right]_0^\infty & \text{if } d\le 0 \end{cases} = \frac{1}{2} e^d \begin{cases} e^{-2d} & \text{if } d>0 \\ 1 & \text{if } d\le 0 \end{cases}$$

$$= \frac{1}{2} \begin{cases} e^{-d} & \text{if } d>0 \quad \to d=|d| \\ e^{d} & \text{if } d\le 0 \quad \to -d=|d| \end{cases} = \frac{1}{2} e^{-|d|} = \underline{\text{Laplace}(0,1)}$$



$L = \mu + \sigma D \sim \frac{1}{2\sigma} e^{-\left|\frac{l-\mu}{\sigma}\right|}$

s.t. $\sigma > 0$ $\quad \mu \in \mathbb{R}$ $\quad \text{supp}[D] = \mathbb{R}$

10/23

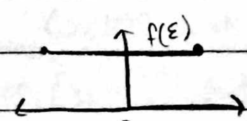
Laplace first published the distribution in 1774. He called it the "first law of error".

Imagine you want to measue a quantity v. But your measuring procedure has random additive error, $\mathcal{E}$(epsilon). so the measurement M is also random.

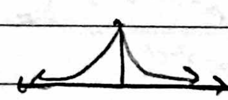
$$M = v + \mathcal{E}$$

It would make sense if:

- $E[\mathcal{E}] = 0 \rightarrow E[M] = v$ (AKA M is an "unbiased estimator")
- $\text{Med}[\mathcal{E}] = 0 \rightarrow$ 50% of the time you over estimate + 50% underestimate
- $f(\mathcal{E}) = f(-\mathcal{E})$



- $f'(\mathcal{E}) < 0$ if $\mathcal{E} > 0$ and $f'(\mathcal{E}) > 0$ if $\mathcal{E} < 0$



- $f''(\mathcal{E}) = f'(\mathcal{E})$
  - $\hookrightarrow f(\mathcal{E}) = ce^{-dx}$ $\quad \propto \text{Laplace}(0,1)$

- $X \sim Exp(1) = e^{-x} \mathbb{1}_{x \geq 0}$, $Y = \frac{1}{\lambda} X^{\frac{1}{k}}$ where $\lambda, k > 0$

$\lambda Y \quad X^{k \cdot 1/k} \rightarrow X = \lambda^k Y^k = g^{-1}(y)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = |\lambda^k k y^{k-1}| = k\lambda^k y^{k-1}$

$f_Y(y) = e^{-(\lambda y)^k} \underbrace{\mathbb{1}_{\lambda^k y^k \geq 0}}_{y^k \geq 0 \rightarrow y \geq 0} \; k\lambda^k y^{k-1}$

$= k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = \underline{Weibull(k, \lambda)}$

a poplar "survival" or "waiting time" method

* special case: $Weibull(k=1, \lambda) = \lambda e^{-\lambda y} \mathbb{1}_{y \geq 0} = Exp(\lambda)$

weibull is a generelization of the exponential

$F_Y(y) = \int_0^y k\lambda(\lambda t)^{k-1} e^{-(\lambda t)^k} dt$

- let $u = (\lambda t)^k = \lambda^k t^k$

$\frac{du}{dt} = k\lambda^k t^{k-1} \rightarrow dt = \frac{1}{k\lambda^k t^{k-1}} du$

$t = y \rightarrow u = (\lambda y)^k$

$t = 0 \rightarrow u = 0$

$= \int_0^{(\lambda y)^k} k\lambda^k t^{k-1} e^{-u} \frac{1}{k\lambda^k t^{k-1}} du = \int_0^{(\lambda y)^k} e^{-u} du = [-e^{-u}]_0^{(\lambda y)^k} = 1 - e^{-(\lambda y)^k} = F(y)$

$\rightarrow \bar{F}(y) = 1 - F(y)$ = survival function $= e^{-(\lambda y)^k}$

- let $c > 0$

consider: $P(Y \geq y+c \mid Y \geq c)$

$\frac{P(Y \geq y+c + P(Y \geq c))}{P(Y \geq c)} = \frac{P(Y \geq y+c)}{P(Y \geq c)} = \frac{\bar{F}(y+c)}{\bar{F}(c)}$

$= \frac{e^{-\lambda^k (y+c)^k}}{e^{-\lambda^k c^k}} = e^{\lambda^k (c^k - (y+c)^k)}$

$\rightarrow$ If $k=1$ $\quad P(Y \geq y+c \mid Y \geq c) = P(Y \geq y)$

$e^{\lambda(c-(y+c))} = e^{-\lambda y}$

$e^{-\lambda y} \stackrel{\checkmark}{=} e^{-\lambda y}$

Memoryless

more realistic model

→ If $k>1$ → $P(Y \geq y+c \mid Y \geq c) < P(Y \geq y)$ gets less probable as c increases

→ If $k<1$ → $P(Y \geq y+c \mid Y \geq c) > P(Y \geq y)$ gets more probable as c increases

→ If $k=2$ → $e^{\lambda^2}(c^2-(y+c)^2) < e^{-\lambda^2 y^2}$

$$\lambda^2(c^2-(y+c)^2) < -\lambda^2 y^2$$

$$c^2-(y+c)^2 < -y^2$$

$$c^2+y^2 < (y+c)^2 = y^2+2cy+c^2$$

$$0 < 2cy \checkmark$$

→ If $k=\frac{1}{2}$ → "

"

"

$$c^{1/2}+y^{1/2} > (c+y)^{1/2}$$

$$(c^{1/2}+y^{1/2})^2 > c+y$$

$$c+y+2\sqrt{cy} > c+y$$

$$2\sqrt{cy} > 0$$

~ Order Statistics p. 160-161

let $x_1, x_2, \ldots, x_n$ be a collection of continuous r.v.'s

Define $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ as follows:

$X_{(1)} := \min\{x_1, x_2, \ldots, x_n\}$ = first order statistic "minimum"

$X_{(n)} := \max\{x_1, x_2, \ldots, x_n\}$ = n^{th} order statistic "maximum"

$X_{(k)} := k^{th}$ largest of $\{x_1, x_2, \ldots, x_n\}$

$R := X_{(n)} - X_{(1)}$ "Range"

- $n=4$ realizations

$x_1 = 9 \quad x_2 = 2 \quad x_3 = 12 \quad x_4 = 7$

$X_{(1)} = 2 \quad X_{(2)} = 7 \quad X_{(3)} = 9 \quad X_{(4)} = 12 \qquad r = 12-2 = 10$

- Let's derive the PDF and CDF of $X_{(n)}$, the maximum:

$F_{X_{(n)}}(x) := P(X_{(n)} \leq x) = P(X_1 \leq x \;\&\; X_2 \leq x \;\&\; \dots \;\&\; X_n \leq x)$

If $X_1, \dots, X_n$ ind

$= \prod_{i=1}^{n} P(X_i \leq x) = \prod_{i=1}^{n} F_{X_i}(x)$

If $X_1, \dots, X_n$ iid

$= F(x)^n$

$f_{X_{(n)}}(x) = \frac{d}{dx}\left[F_{X_{(n)}}(x)\right] \overset{\text{iid}}{=} n f(x) F(x)^{n-1}$

- Let's derive the PDF & CDF of $X_{(1)}$, the minimum:

$F_{X_{(1)}}(x) := P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, X_2 > x, \dots, X_n > x)$

If $X_1, \dots, X_n$ ind

$= 1 - \prod_{i=1}^{n} P(X_i > x) = 1 - \prod_{i=1}^{n} (1 - F_{X_i}(x))$

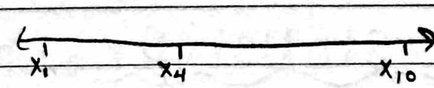
If $X_1, \dots, X_n$ iid

$= 1 - (1 - F(x))^n$

$f_{X_{(1)}}(x) = \frac{d}{dx}\left[F_{X_{(1)}}(x)\right] \overset{\text{iid}}{=} -(-f(x)) n (1 - F(x))^{n-1} = n f(x) (1 - F(x))^{n-1}$

- Let's get the PDF and CDF of $X_{(k)}$, the distribution of the k^{th} largest

consider $n = 10$ $k = 4$



consider $P(X_1 \leq x, \dots, X_4 \leq x, X_5 > x, \dots, X_{10} > x)$

$\overset{\text{ind}}{=} \prod_{i=1}^{4} P(X_i \leq x) \prod_{i=5}^{10} P(X_i > x)$

$= \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} (1 - F_{X_i}(x))$

$\overset{\text{iid}}{=} F(x)^4 (1 - F(x))^6$

- consider $P(\text{any } 4\ X_i\text{'s} < x \text{ and the other } 6 > x)$

$= \sum_{\text{over all subsets}} P(X_{s_1} \leq x, \dots, X_{s_4} \leq x, X_{s_5} > x, \dots, X_{s_{10}} > x)$

$\overset{\text{ind}}{=} \sum_{\text{subsets}} \prod_{i=1}^{4} F_{s_i}(x) \prod_{i=5}^{10} (1 - F_{s_i}(x))$

$\overset{\text{iid}}{=} \sum_{\text{subsets}} F(x)^4 (1 - F(x))^6 = \binom{10}{4} F(x)^4 (1 - F(x))^6$

$F_{X_{(4)}}(x) := P(X_{(4)} \le x) = P(\text{any 4 } x\text{'s are below } x \text{ and the other } 6 > x) \; +$

$P(\text{any 5 } x\text{'s are below } x \text{ and the other } 5 > x) \; +$

$\vdots$

$P(\text{all 10 } x\text{'s} < x)$

$\stackrel{iid}{=} \binom{10}{4} F(x)^4 (1-F(x))^6 + \binom{10}{5} F(x)^5 (1-F(x))^5 + \ldots + \binom{10}{10} F(x)^{10} (1-F(x))^{10-10}$

$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$

Generally.... with arbitrary n, k for $x_1, \ldots, x_n \stackrel{iid}{\sim}$

$F_{X_{(k)}} = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$

$F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \binom{n}{n} F(x)^n (1-F(x))^{n-n} = F(x)^n$

$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \underbrace{\left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right)}_{(F(x)+(1-F(x)))^n} - \overbrace{\binom{n}{0} F(x)^0 (1-F(x))^{n-0}}^{(1-F(x))^n}$

$= 1-(1-F(x))^n$ ✓