Lec 5 Math 621 9/11/19

$\vec{X} \sim \text{Multin}(n, \vec{p}) \Rightarrow X_j \sim \text{Bin}(n, p_j)$

let $\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_k) \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$

element-wise operation

If $M = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$, a matrix of r.v.'s, define $E(M) := \begin{bmatrix} E(X_{11}) & \cdots & E(X_{1m}) \\ \vdots & & \\ E(X_{n1}) & \cdots & E(X_{nm}) \end{bmatrix}$

$\sigma^2 := Var[X] = E[X^2] - \mu^2$

HW

$\sigma_{12} := Cov[X_1, X_2] := E[X_1 X_2] - \mu_1 \mu_2 = E[(X_1 - \mu_1)(X_2 - \mu_2)]$

$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$

Rules for covariance

If X_1, X_2 iid $\Rightarrow \sigma_{12} = 0$

1. $Cov[X, X] = \sigma^2$
2. $Cov[X_1, X_2] = Cov[X_2, X_1]$ HW
3. $Cov[X_1 + X_2, X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$
4. $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$

HW

5. $Var[X_1 + \ldots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j]$

$Var[X_1 + X_2] = \sum_{i=1}^{2} \sum_{j=1}^{2} Cov[X_i, X_j] = \sigma_1^2 + \sigma_2^2 + \overbrace{\sigma_{12} + \sigma_{21}}^{2\sigma_{12}}$

"outer product"

the

$$\mathrm{Var}[\vec{X}] := \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_k] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & & \\ \vdots & & \ddots & \\ \mathrm{Cov}[X_1,X_k] & \cdots & & \mathrm{Var}[X_k] \end{bmatrix} = E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T$$

$$= \{\mathrm{Cov}[X_i,X_j]\}^{k \times k \text{ matrix}}$$

since diagonal is $\mathrm{Cov}[X_i,X_i] = \mathrm{Var}[X_i]$

symmetric, diagonal ≥ 0

Why not just a vector? You can capture interactions within the elements of $\vec{X}$

if $X_1, \ldots, X_k$ iid $\Rightarrow \mathrm{Cov}[X_i,X_j] = 0 \quad \forall i \neq j$

$$\Rightarrow \mathrm{Var}[\vec{X}] = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_k^2 \end{bmatrix}$$

i.e. a "diagonal" matrix

Rules for vector-r.v. expectation & variance col. vector of all ones

~~let~~ $\cancel{T = X_1 + X_2 + \dots + X_K = \vec{1}_K^T \vec{X} = [1 \dots 1] \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix}}$

$$\cancel{E[T] = \mu_1 + \dots + \mu_K = \vec{1}^T \vec{\mu} = E[\vec{1}^T \vec{X}]}$$

$$\cancel{Var[T] = \sum_{i=1}^{K} \sum_{j=1}^{K} Cov[X_i, X_j] = Var[\vec{1}^T \vec{X}]}$$

Rules:

$$E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a} \quad \text{(vector)}$$

$\vec{a} \in \mathbb{R}^K$ const

$$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + a_2 X_2 + \dots + a_K X_K] = \sum a_i \mu_i = \vec{a}^T \vec{\mu} \quad \text{(scalar)}$$

$$E[A \vec{X}] = \begin{bmatrix} E[a_{11} X_1 + a_{12} X_2 + \dots + a_{1K} X_K] \\ E[a_{21} X_1 + a_{22} X_2 + \dots + a_{2K} X_K] \\ \vdots \\ E[a_{L1} X_1 + a_{L2} X_2 + \dots + a_{LK} X_K] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot} \vec{\mu} \\ \vec{a}_{2\cdot} \vec{\mu} \\ \vdots \\ \vec{a}_{L\cdot} \vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot} \\ \vec{a}_{2\cdot} \\ \vdots \\ \vec{a}_{L\cdot} \end{bmatrix} \vec{\mu} = A \vec{\mu} \in \mathbb{R}^L$$

where $A \in \mathbb{R}^{L \times K}$

$$Var[\vec{a}^T \vec{X}] = Var[a_1 X_1 + a_2 X_2 + \dots + a_K X_K]$$

s.t. $\vec{a} \in \mathbb{R}^K$ const.

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} Cov[a_i X_i, a_j X_j]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j Cov[X_i, X_j] = a^T \Sigma a$$

$Var[\vec{X}]$

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let $V \in \mathbb{R}^{K \times K}$

$$\underset{1 \times K}{\vec{a}^T} \underset{K \times K}{V} \underset{K \times 1}{\vec{a}} = [a_1 \dots a_K] \begin{bmatrix} a_1 V_{11} + \dots + a_K V_{1K} \\ a_1 V_{21} + \dots + a_K V_{2K} \\ \vdots \\ a_1 V_{K1} + \dots + a_K V_{KK} \end{bmatrix} =$$

scalar

$$a_1 a_1 V_{11} + a_1 a_2 V_{12} + \dots + a_1 a_K V_{1K} +$$
$$a_2 a_1 V_{21} + a_2 a_2 V_{22} + \dots + a_2 a_K V_{2K} +$$
$$a_K a_1 V_{K1} + a_K a_2 V_{K2} + \dots + a_K a_K V_{KK} = \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j V_{ij}$$

Quadratic Form

A little bit of finance..

let $X_1, \ldots X_K$ be assets. Each has mean return μ_i.

Assets are almost always dependent. Σ is the var cov matrix.

let $\vec{a}$ be a set of weights s.t. $\vec{1}^T\vec{a} = 1$. where you put your money and $\vec{a} \in (0,1)^K$

A portfolio $F = \vec{w}^T \vec{X}$ it has μ_F mean return $\vec{a}^T\vec{\mu}$

and variance $\sigma_F^2 := \vec{a}^T \Sigma \vec{a}$. Goal $\min_{\substack{\vec{a} \in (0,1)^K \\ \text{and} \\ \vec{1}^T\vec{a} = 1}} \sigma_F^2$ subject to $\mu_F = \mu_0$

$\vec{X} \sim \text{Multinom}(n, \vec{p})$, $E(\vec{X}) = n\vec{p}$ why?

$X_j \sim \text{Bin}(n, p_j) \Rightarrow E(X_j) = np_j$

$\text{Var}(\vec{X}) = ?$ Diagonal is easy... since $\text{Var}(X_j) = np_j(1-p_j)$ (Math 241)

Off-diagonal .. HARD. $\text{Cov}(X_i, X_j)$ $i \neq j$.

$$\text{Cov}(X_i, X_j) := E(X_iX_j) - \mu_i\mu_j = \sum_{x_1 \in \{0,..,n\}} \sum_{x_2 \in \{0,1,..,n\}} x_1 x_2 \underbrace{P_{X_i,X_j}(x_1,x_2)}_{\text{JMF of arbitrary 2-dim subset } \vec{X}} - (np_i)(np_j)$$

TOO hard!

E.C. on HW!

Recall $X_i \sim Bin(n, p_i)$

$X_j \sim Bin(n, p_j)$

represent whether the i^{th} category or j^{th} category drawn

So $X_i = X_{1i} + \ldots + X_{ni}$ where $X_{1i}, \ldots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + \ldots + X_{nj}$ where $X_{1j}, \ldots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

$\Rightarrow \vec{X} = \vec{X}_{1\cdot} + \ldots + \vec{X}_{n\cdot}$ where $\vec{X}_{1\cdot}, \ldots, \vec{X}_{n\cdot} \overset{iid}{\sim} Multi(1, \vec{p})$

$$Cov[X_i, X_j] = Cov[X_{1i} + \ldots + X_{ni}, X_{1j} + \ldots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} \left(E[X_{li}, X_{mj}] - \underset{p_i}{\mu_{li}} \, \underset{p_j}{\mu_{mj}} \right)$$

if $l \neq m$ it is a different draw, ind. so $E[X_{li}, X_{mj}] = p_i p_j$

$$= \sum_{l=1}^{n} E[X_{li}, X_{lj}] - p_i p_j$$

$$E[X_{li}, X_{mj}] = \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \, p_{X_{li}, X_{lj}}(x_{li}, x_{lj})$$

$$= p_{X_{li}, X_{lj}}(1, 1)$$

What is the prob of picking <u>both</u> an apple & rambutan on the l^{th} pick? Zero

$$\sum_{l=1}^{n} - p_i p_j = -n p_i p_j$$

Why negative? If $X_i \uparrow \Rightarrow X_j \downarrow$ in prob.

$$\Sigma = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_K \\ -np_2p_1 & np_2(1-p_2) & \cdots & -np_2p_K \\ \vdots & & & \vdots \\ -np_Kp_1 & \cdots & & np_K(1-p_K) \end{bmatrix}$$