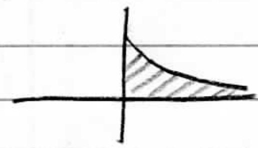
Let $X_1, X_2 \overset{i.i.d.}{\sim} Exp(1) = e^{-x} \mathbb{1}_{x \in (0,\infty)}$

X:



Double Exponential

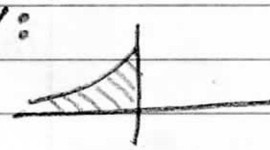
$D = X_1 - X_2 = X_1 + (-X_2) = X + Y$

where

$Y \sim e^{-(-y)} \mathbb{1}_{(-y) \in (0,\infty)}$

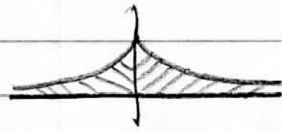
so $Y \sim e^{y} \mathbb{1}_{y \in (0,\infty)}$

Y:



$Supp[D] = \mathbb{R}$

D:



PDF of D

$$f_D(d) = \int_{Supp[X]} f_X(x) f_Y(d-x) \mathbb{1}_{d-x \in supp[Y]} dx$$

$$= \int_0^\infty (e^{-x})(e^{d-x} \mathbb{1}_{\underbrace{\underbrace{d-x \in (-\infty,0)}_{x-d \in (0,\infty)}}_{x \in (d,\infty)}}) dx$$

$$= \int_{max(0,d)}^\infty e^{d-2x} dx$$

$$= \begin{cases} e^d \int_d^\infty e^{-2x} dx & \text{if } d \geq 0 \\ e^d \int_0^\infty e^{-2x} dx & \text{if } d < 0 \end{cases}$$

$$= \begin{cases} e^d \left(-\frac{1}{2}\left[e^{-2x}\right]_d^\infty \right. & \text{if } d \geq 0 \\ e^d \left(-\frac{1}{2}\left[e^{-2x}\right]_0^\infty \right. & \text{if } d < 0 \end{cases}$$

$$= \begin{cases} -\frac{1}{2} e^d \left[0 - e^{2d}\right] & \text{if } d \geq 0 \\ -\frac{1}{2} e^d \left[0 - 1\right] & \text{if } d < 0 \end{cases}$$

$$f_D(d) = \begin{cases} -\frac{1}{2}e^{d}(-e^{2d}) & \text{if } d \geq 0 \\ -\frac{1}{2}e^{d}(-1) & \text{if } d < 0 \end{cases}$$

$$f_D(d) = \begin{cases} \frac{1}{2}e^{-d} & \text{if } d \geq 0 \\ \frac{1}{2}e^{d} & \text{if } d < 0 \end{cases} \xrightarrow{\text{notice}} \begin{array}{l} \text{if } d \geq 0 \Rightarrow d = |d| \\ \text{if } d < 0 \Rightarrow -d = |d| \end{array}$$

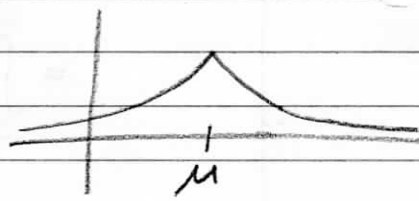
$$f_D(d) = \frac{1}{2}e^{-|d|}$$

This pdf is $\text{Laplace}\,(\underset{\mu}{\underset{\uparrow}{0}}, \underset{\sigma}{\underset{\uparrow}{1}})$

Laplace distribution:

$$L \sim \mu + \sigma D \sim \frac{1}{2\sigma} e^{-\frac{|l-\mu|}{l}} \quad \text{is Laplace}(\mu, \sigma)$$

where
$\mu \in \mathbb{R}$
$\sigma > 0$



$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$ is pdf for $Y = g(X)$

Step 1: find inverse function for g
Step 2: find the derivative

Ex: $X \sim Exp(1) = e^{-x}\mathbb{1}_{x\in(0,\infty)}$

$$Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{x}}{e^{x}}\right) = \ln(e^{x}-1) = g(x)$$

$$e^{Y} = e^{x} - 1$$
$$\Rightarrow e^{x} = e^{Y}+1$$
$$\Rightarrow X = \ln(e^{Y}+1) = g^{-1}(Y)$$

$$\frac{d}{dy}\left[\ln(e^{Y}+1)\right] = \frac{e^{Y}}{e^{Y}+1} > 0 \quad \forall y$$

so $f_Y(y) = e^{-\ln(e^{Y}+1)}\,\mathbb{1}_{\ln(e^{Y}+1)\in(0,\infty)} \cdot \frac{e^{Y}}{e^{Y+1}}$

$e^{Y}+1 \in (1,\infty)$
$e^{Y} \in (0,\infty)$
so $Y \in \mathbb{R}$

$$e^{-\ln(e^{Y}+1)} = e^{\ln\left(\frac{1}{e^{Y}+1}\right)} = \frac{1}{e^{Y}+1} = \frac{e^{-Y}}{1+e^{-Y}}$$

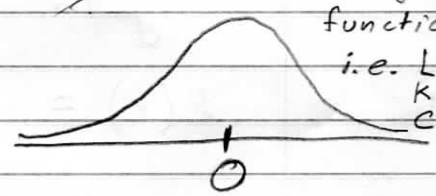
if multiply top and bottom by e^{Y}

$$= \frac{e^{Y}}{(e^{Y}+1)^2}$$

$$= \frac{e^{Y}}{(e^{Y}+1)^2} \cdot \frac{e^{-2y}}{(e^{-Y})^2}$$

are multiplying top and bottom by same thing

Standard Logistic function



i.e. $L=1$
$K=1$
$c=0$

its cdf is the logistic function

$$f_Y(y) = \frac{e^{-Y}}{(1+e^{-Y})^2}$$

this is pdf of
<u>Standard Logistic</u> dist. (like normal but more spread out)
or Logistic $(0,1)$
(used to model difference in scores, etc.)

general logistic:

$$L = \sigma Y + \mu \sim \frac{1}{\sigma} \frac{e^{-\left(\frac{\ell-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{\ell-\mu}{\sigma}\right)}\right)^2}$$

r.v. is L

pdf is $f(\ell)$

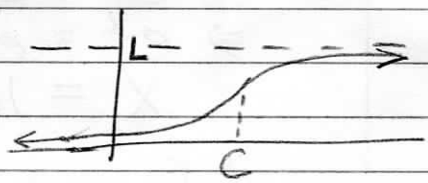
where $\sigma > 0$, $\mu \in \mathbb{R}$

ℓ is Logistic(μ, σ)

Note: $E[L] = \mu$

$$SE[L] = \sigma \frac{\pi}{\sqrt{3}}$$

The logistic function:



used to model population size over time, etc.

$$\ell(x) := \frac{L}{1+e^{-K(x-c)}}$$

where L is the max value

K is the steepness

c is the center

$e^x \cdot e^{-x} = 1$

If use $L=1$, $K=1$, $c=0$, get:

$$\ell(x) = \frac{1}{1+e^{-x}}$$ ← standard logistic function

$$\ell(x) = \frac{e^x}{e^x+1}$$ after multiply top & bottom by e^x

Show this is CDF for PDF $f_Y(y) = \frac{e^y}{(e^y+1)^2} = \frac{e^{-y}}{(e^{-y}+1)^2}$

$$F_Y(y) = \int_{-\infty}^{y} \frac{e^t}{(e^t+1)^2}\,dt$$

using CDF $= \int_{-\infty}^{y}$ PDF ← in support only

let $u = e^t + 1$ $\quad$ $u = e^{t+1}$

$e^t = u - 1$ $\quad$ $\frac{du}{dt} = e^t$

$t = \ln(u-1)$ $\quad$ $dt = \frac{1}{e^t}du$

$dt = \frac{1}{u-1}du$

$t = y \Rightarrow u = e^y + 1$

$t = -\infty \Rightarrow u = 1$

$$= \int_1^{e^y+1} \frac{u-1}{u^2}\frac{1}{u-1}\,du$$

$$= \int_1^{e^y+1} \frac{1}{u^2}\,du = \int^{e^y+1} u^{-2}\,du$$

$$= \left[-u^{-1}\right]_1^{e^y+1} = -\left(\frac{1}{e^y+1} - \frac{1}{1}\right)$$

$$= 1 - \frac{1}{e^y+1} = \frac{e^y+1}{e^y+1} - \frac{1}{e^y+1}$$

$$F_Y(y) = \frac{e^y}{e^y+1} = \frac{1}{1+e^{-y}}$$

Take derivative to get PDF

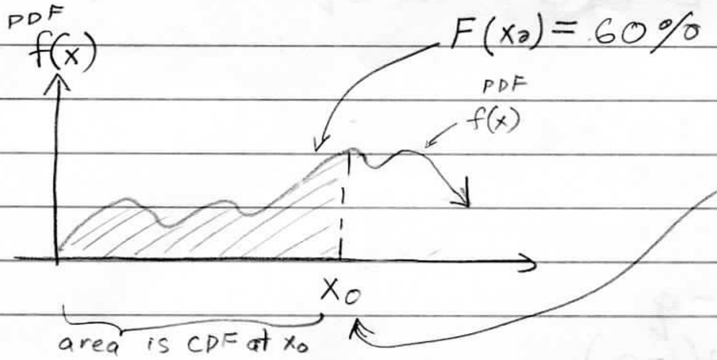
get:

$$f_Y(y) = \frac{e^y}{(e^y+1)^2} = \frac{e^{-y}}{(1+e^{-y})^2}$$

is PDF for Standard Logistic distribution

Logistic$(0,1)$

quantile is inverse of CDF (sort-of)



x_0 is 0.6 quantile or 60th percentile

$q \in (0,1]$

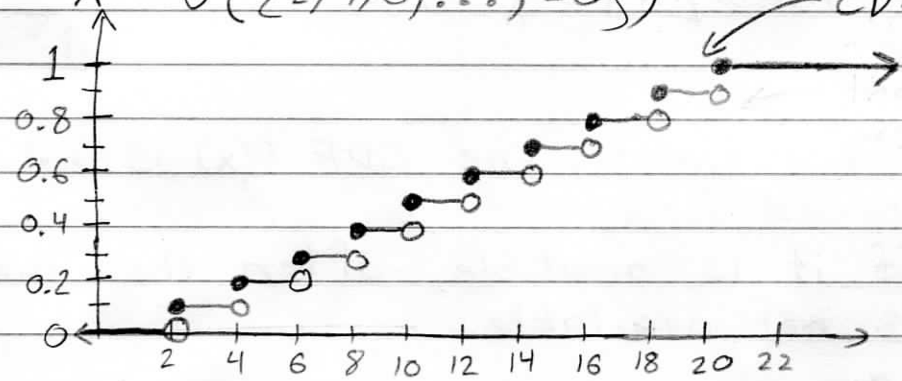
Find minimal x^* such that $P(X \leq x_0) \geq q$

This is called the <u>quantile operator</u> $Q[X,q]$ where q is the "quantile" and $100 \cdot q$ is the "percentile".

Ex: $X \sim U(\{2,4,6,\ldots,20\})$ ← Uniform

CDF drawn here

CDF of discrete dist. is step-wise function



0.1 quantile: $Q[X, 0.1] = 2$

quantile $Q[X, 1] = 20$

0.5 quantile $Q[X, 0.8] = 18$

0.85 quantile $Q[X, 0.85] = 18$

Define median of a r.v. $X := MED[X] = Q[y, \frac{1}{2}]$

If X is continuous and a strictly increasing CDF,
then $Q[X,q] = F_X^{-1}(q)$ "the quantile function"

often don't have closed form for quantile or CDF

Example:

$X \sim Exp(\lambda) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

so $q = 1 - e^{-\lambda x}$

$e^{-\lambda x} = 1 - q$

$-\lambda x = \ln(1-q)$

$\lambda x = \ln\left(\frac{1}{1-q}\right) \Rightarrow x = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right)$

$$F_X^{-1}(q) = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right)$$

our quantile function for Exp[X]

$$Med[X] = \frac{1}{\lambda}\ln(2)$$

Often times, the CDF $f(x)$ is not available in closed form.
If it is available, often the inverse is not available.

E.g.

$X \sim Erlang(k, \lambda) \rightarrow F(x) = P(k, \lambda x)$

"Computer, solve $q = P(k, \lambda x)$ for x as best you can"

$$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x\in(0,\infty)}$$

For $k>0$, let $Y = ke^X \Rightarrow g(X) = ke^x$

$$\Rightarrow \frac{Y}{K} = e^x$$

$$\Rightarrow x = \ln\left(\frac{Y}{K}\right)$$

$$x = \ln(Y) - \ln(K) \Rightarrow g^{-1}(Y) = \ln(Y) - \ln(K)$$

$$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{1}{y}$$

$$\mathbb{1}_{\ln(y)-\ln(k)\in(0,1)} \iff \ln(y) \in (\ln(k),\infty) \iff y \in (k,\infty)$$

$$\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right) = \ln\left(\frac{k^\lambda}{y^\lambda}\right)$$

PDF

$$f_Y(y) = \lambda e^{-\lambda \ln\left(\frac{y}{k}\right)} \mathbb{1} \cdot \frac{1}{y}$$

$$= \lambda e^{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)} \mathbb{1} \cdot \frac{1}{y}$$

$e^{\ln W} = W$

$$= \lambda e^{\ln\left(\frac{k^\lambda}{y^\lambda}\right)} \mathbb{1} \cdot \frac{1}{y}$$

$$= \lambda \frac{k^\lambda}{y^\lambda} \mathbb{1} \frac{1}{y}$$

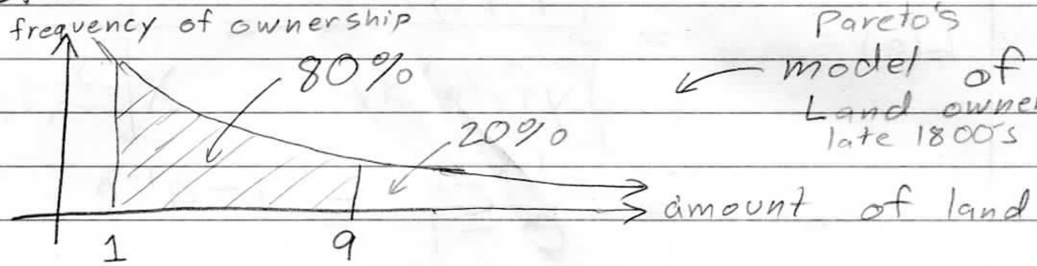
$$= \frac{\lambda}{y} \frac{k^\lambda}{y^\lambda} \mathbb{1}_{y\in(k,\infty)}$$

Pareto I used to model waiting times, and economic inequality

roman numeral I

difines <u>Pareto I(k, λ)</u> distribution

let $K = 1$ ← Pareto's assumption



Pareto's model of Land ownership late 1800's Italy

$Y \sim \text{Pareto I}(k, \lambda)$

has PDF

$$\boxed{f_Y(y) = \frac{\lambda k^\lambda}{y^{\lambda+1}} \mathbb{1}_{y \in (k,\infty)}}$$

CDF

$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}} dt = \lambda k^\lambda \left[-\frac{t^{-\lambda}}{\lambda} \right]_k^y$$

$$= k^\lambda \left(k^{-\lambda} - y^{-\lambda} \right)$$

$$= \left(1 - \left(\frac{k}{y}\right)^\lambda \right) \mathbb{1}_{y \in (k,\infty)}$$

let $q = 1 - \left(\frac{k}{y}\right)^\lambda$

$$\Rightarrow 1 - q = \frac{k^\lambda}{y^\lambda}$$

$$\Rightarrow y^\lambda = \frac{k^\lambda}{1-q}$$

$$\Rightarrow y = \left(\frac{k^\lambda}{1-q} \right)^{\frac{1}{\lambda}}$$

$$\Rightarrow y = \frac{k}{(1-q)^{\frac{1}{\lambda}}}$$

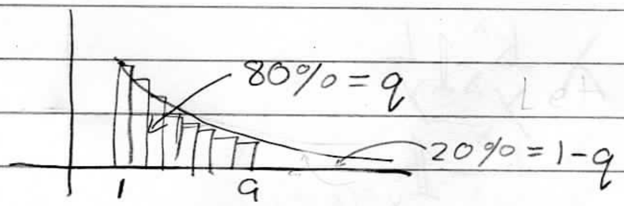
$$\Rightarrow y = k(1-q)^{-\frac{1}{\lambda}}$$

so

$$F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}$$

If $k = 1$ $\Rightarrow$ $f_Y(y) = \frac{\lambda}{y^{\lambda+1}} \mathbb{1}_{y \in (1,\infty)}$

$$F_Y^{-1}(q) = (1-q)^{-\frac{1}{\lambda}}$$



Let $L(a)$ be the proportion of people who own $\leq a$ (a is an amount of land)

$$L(a) = \frac{\int_1^a y f_Y(y) dy}{\int_1^\infty y f_Y(y) dy} = \frac{\lambda \left[\frac{y^{-\lambda+1}}{-\lambda+1} \right]_1^a}{\lambda \left[\frac{y^{-\lambda+1}}{-\lambda+1} \right]_1^\infty} = \frac{a^{1-\lambda} - 1}{0 - 1}$$

$$= \frac{a^{1-\lambda} - 1}{0 - 1} = 1 - a^{1-\lambda}$$

Set $a = F^{-1}(q)$

Let $L(q) = 1 - q = \bar{q}$

$$\Rightarrow \bar{q} = 1 - \left(\bar{q}^{-\frac{1}{\lambda}}\right)^{1-\lambda}$$

$$\Rightarrow q = \bar{q}^{\frac{\lambda-1}{\lambda}}$$

$$\Rightarrow q = \bar{q}^{1-\frac{1}{\lambda}}$$

$$\ln(q) = \left(1 - \frac{1}{\lambda}\right)\ln(\bar{q})$$

$$\Rightarrow \ln(q) = \ln(\bar{q}) - \frac{1}{\lambda}\ln(q)$$

$$\Rightarrow \ln(\bar{q}) - \ln(q) = \frac{1}{\lambda}\ln(q)$$

$$\Rightarrow \frac{1}{\lambda} = \frac{\ln(\bar{q}) - \ln(q)}{\ln(\bar{q})}$$

$$\Rightarrow \lambda = \frac{\ln(\bar{q})}{\ln(\bar{q}) - \ln(q)} = \frac{\ln(\bar{q})}{\ln\left(\frac{\bar{q}}{q}\right)} = \log_{\bar{q}/q}(\bar{q})$$

Pareto's 80-20 principle:
let $q = 0.8$ and $\bar{q} = 0.2$ ← if 80% of land owned by 20% of people

get $\lambda = \log_{0.2/0.8} \approx 1.161$ ← a measure of inequality

$Y \sim \text{Pareto I}(1, 1.161)$ you get the Pareto principle