$D \sim \text{Laplace}(0,1) = \frac{1}{2}e^{-|d|}$

Laplace discoved this model in 1774 calling it the first "law of error"

Imagine you are measuring a quantity but your measurement device has some random additive error, ε.
So your measurement is $M = \underset{\uparrow \text{ actual value}}{v} + \varepsilon$.
Thus M is random

Certain properties of $\varepsilon \sim \overset{\text{pdf}}{f}(\varepsilon)$:

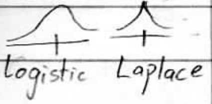
★ $E[\varepsilon] = 0 \Rightarrow E[M] = v$
(this means that M is an "unbiased estimator")

★ $\text{Med}[E] = 0 \Rightarrow$ 50% of the time you overestimate
50% of the time you underestimate

★ $f(\varepsilon) = f(-\varepsilon)$ symmetric

★ $f'(\varepsilon) < 0$ if $\varepsilon > 0$ and $f'(\varepsilon) > 0$ if $\varepsilon < 0$
(the further away from 0, the smaller the probabilities)

this is general concept of an error distribution (many different distributions fit this)



Consider $f''(\varepsilon) = f'(\varepsilon) \Rightarrow f(\varepsilon) = ce^{-b\varepsilon}$ = Laplace(0,1)

$X \sim \text{Exp}(1) = e^{-x}\mathbb{1}_{x \geq 0}$ Let $Y = \frac{1}{\lambda}X^{\frac{1}{k}} \sim ?$
where $\lambda k > 0$

$\Rightarrow \lambda Y = X^{\frac{1}{k}}$
$\Rightarrow X = (\lambda Y)^k = \lambda^k Y^k = g^{-1}(Y)$

$Y = X^k$
$Y = X^{\alpha}$

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \geq 0}$

Find pdf of
$Y = \frac{1}{\lambda} X^{\frac{1}{k}}$
$g(X) = \frac{1}{\lambda} X^{\frac{1}{k}}$

got $g^{-1}(Y) = \lambda^k Y^k$

$\left| \frac{d}{dy}\left[g^{'}(y) \right] \right| = \lambda^k k y^{k-1}$

Weibull is common survival model

$$f_Y(y) = e^{-(\lambda y)^k} \underbrace{\mathbb{1}_{\lambda^k y^k \geq 0}}_{\substack{\text{same as} \\ y^k \geq 0 \\ y \geq 0}} \lambda^k k y^{k-1}$$

$$= (ky)(\lambda y)^{k-1} \mathbb{1}_{y \geq 0}$$

$$= \text{Weibull}(k, \lambda)$$

Very functional waiting time/survival model

Weibull is a generalization of the Exponential

$$\text{Weibull}(k=1, \lambda) = \lambda e^{-\lambda y} = Exp(\lambda)$$

CDF

$$F(y) = \int_0^y (k\lambda)(\lambda t)^{k-1} e^{-(\lambda t)^k} dt =$$

let $u = (\lambda t)^k$

$\frac{du}{dt} = \lambda^k k t^{k-1}$

$dt = \frac{1}{\lambda^k k t^{k-1}} du$

$t = y \rightarrow u = (\lambda y)^k$

$t = 0 \rightarrow y = 0$

$$= \int_0^{(\lambda y)^k} k \cancel{\lambda^k} \cancel{t^{k-1}} e^{-u} \frac{1}{\cancel{\lambda^k} k \cancel{t^{k-1}}} du = \int_0^{(\lambda y)^k} e^{-u} du$$

$$= \left[-e^{-u} \right]_0^{(\lambda k)^y} = -\left(e^{-(\lambda k)^y} - e^0 \right) =$$

$$F(y) = 1 - e^{-(\lambda y)^k}$$ ← CDF for Weibull

$\overline{F}(y)$ survival function since represents $P(Y > y)$ (last longer than y)

$\overline{F}(y) = 1 - F(y)$
$\overline{F}(y) = e^{-(\lambda y)^k}$

survival function $\overline{F}(y) = 1 - F(y) = e^{-(\lambda y)^K}$

Is the Weibull memoryless?

memoryless means tha prob. that it takes c time units more is the same as prob. that takes c years

$$P(Y \geq y + c \mid Y \geq c) = \frac{P(Y \geq y+c,\ Y \geq c)}{P(Y \geq c)}$$

$$= \frac{P(Y \geq y+c)}{P(Y \geq c)}$$

$$= \frac{\overline{F}(y+c)}{\overline{F}(c)} = \frac{e^{-(\lambda(y+c))^K}}{e^{-(\lambda c)^K}}$$

↑ by weibull

↑ does not matter how much time passed already is "Memoryless"

$$= e^{-\lambda^K (y+c)^K} e^{\lambda^K c^K}$$

$$= e^{\lambda^K (c^K - (y+c)^K)}$$

If $K = 1$ (this covers Exponential)

$$P(Y \geq y + c \mid Y \geq c) = e^{\lambda(c-(y+c))}$$

$$= e^{-\lambda y} =$$

$$= P(Y \geq y)$$

Weibull is memoryless only for $K=1$ (Exponental)

this is Memoryless (meaning $P(Y \geq y + c \mid Y \geq c) = P(Y \geq y)$)

but if $K > 1$:

$$P(Y \geq y + c \mid Y \geq c) < P(Y \geq y)$$

this inequality gets more severe as we increase K

human lifespan → Ex: $P(Y \geq 100 \mid Y \geq 99) < P(Y \geq 1)$

prob. of lasting an additional year < prob. lasts a year inequality more severe → as decrease K

If $K < 1 \Rightarrow P(Y \geq y + c \mid Y \geq c) > P(Y \geq y)$

infant mortality, "burn-in" period for machines or electronics →

$P(Y \geq 15 \text{ years an } 1 \text{ day} \mid Y \geq 15 \text{ years}) \leq P(Y \geq 1 \text{ day})$

prob. of lasting an additional day
is less than prob. of lasting a day

Verify the inequality if $K = 2$

$$e^{\lambda^2(c^2-(y+c)^2)} \overset{?}{<} e^{-\lambda^2 y^2}$$

$$\lambda^2(c^2-(y-c)^2) < -\lambda^2 y^2$$

$$c^2 + y^2 < (y+c)^2$$

$$c^2 + y^2 < y^2 + 2yc + c^2$$

$$0 < 2yc$$

since $yc > 0$ ✓

Verify the inequality if $K = \frac{1}{2}$

"   "   "

$$c^{\frac{1}{2}} + y^{\frac{1}{2}} > (y+c)^{\frac{1}{2}}$$

$$\left[c^{\frac{1}{2}} + y^{\frac{1}{2}}\right]^2 > \left[(y+c)^{\frac{1}{2}}\right]^2$$ square both sides

$$c + 2c^{\frac{1}{2}}y^{\frac{1}{2}} + y > y + c$$

$$c + 2c^{\frac{1}{2}}y^{\frac{1}{2}} > c$$

$$2c^{\frac{1}{2}}y^{\frac{1}{2}} > 0$$ ✓

because $c > 0,\ y > 0$

so $c^{\frac{1}{2}} > 0,\ y^{\frac{1}{2}} > 0$

so $c^{\frac{1}{2}}y^{\frac{1}{2}} > 0$

so $2c^{\frac{1}{2}}y^{\frac{1}{2}} > 0$

Order Statistics (p. 160-161)

Consider continuous r.v.'s $X_1, X_2, \ldots, X_n$

Define $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ as the "order statistics" as follows:

$X_{(1)} := \min\{X_1, \ldots, X_n\}$ $\quad X_{(n)} := \max\{X_1, \ldots, X_n\}$

$X_{(k)}$ is the kth largest of $\{X_1, \ldots, X_n\}$ called "k^{th} order statistics"

$X_{(k)}$ is kth one if we put in order smallest to largest

Define "range" $R = X_{(n)} - X_{(1)}$

Example

For $n=4$ realizations

$x_1 = 9,\ x_2 = 2,\ x_3 = 7,\ x_4 = 14$

$x_{(1)} = 2,\ x_{(2)} = 7,\ x_{(3)} = 9,\ x_{(4)} = 14$

$r = 19 - 2 = 12$

Let's find the PDF and CDF of $X_{(n)}$, the max

$$F_{X_{(n)}}(x) := P(X_{(n)} \leq x)$$

$$= P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x)$$

if $X_1, X_2, \ldots, X_n$ are independent

$$= P(X_1 \leq x) P(X_2 \leq x) \ldots P(X_n \leq x)$$

if $X_1, X_2, \ldots X_n$ are i.i.d.

$$= \underbrace{P(X \leq x) P(X \leq x) \ldots P(X \leq x)}_{n \text{ of these}}$$

$$= [P(X \leq x)]^n$$

$$F_{X_{(n)}}(x) = [F(x)]^n$$

$$\overset{n}{}F'(x)$$

CDF $F_{X_{(n)}}(x) = [F(x)]^n$

to get PDF do derivative

since X's are i.i.d. $\boxed{F'(x) = f(x)}$

PDF $f_{X_{(n)}}(x) := \frac{d}{dx}\left[F_{X_{(n)}}(x)\right] = n f(x)[F(x)]^{n-1}$ by chain rule

---

Now, look at the minimum $X_{(1)}$

CDF

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

if are independent

$$= 1 - P(X_1 > x)P(X_2 > x)\ldots P(X_n > x)$$

if are i.i.d.

$$= 1 - \underbrace{P(X > x)P(X > x)\ldots P(X > x)}_{n \text{ of these}}$$

$$= 1 - [P(X > x)]^n$$

$$F_{X_{(1)}}(x) = 1 - [F(X)]^n$$

find PDF by taking the derivative

$$f_{X_{(1)}} = \frac{d}{dx}\left[F_{X_{(1)}}(x)\right] = -(-f(x))\, n\, (1 - F(x))^{n-1}$$

$$= n f(x) (1 - F(x))^{n-1}$$

Let's find CDF of $X_{(K)}$, the k^{th} ordered statistic

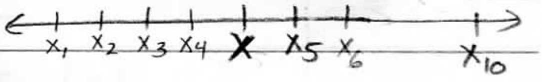
Consider $n = 10$, $k = 4$

$F_{X_{(4)}}(x) = P(X_{(4)} \le x)$ Assume X's are i.i.d.

$$= P(X_1 \le x, X_2 \le x, X_3 \le x, X_4 \le x, X_5 > x, X_6 > x, \ldots, X_{10} > x)$$

$$= \prod_{i=1}^{5} F_x(x) \prod_{i=6}^{10} (1 - F_x(x))$$

$$= F(x)^4 (1 - F(x))^6$$



this works if all of them are in order

note that left 5th one out

Consider P(4 of the 10 r.v.'s $\le x$
and the remaining 6 of the 10 r.v.'s $> x$)

actually $\sum_{\{S_1, S_2, S_3, S_4\} < \{1, 2, \dots 10\}}$

$$= \sum P(X_{S_1} \le x, \dots, X_{S_4} \le x, X_{S_5} > x, \dots X_{S_{10}} \ge x)$$

$$= \sum \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=5}^{10} (1 - F_{X_{S_i}}(x))$$

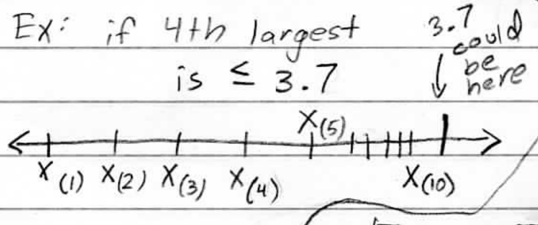
since are i.i.d.,

$$= \sum (F(x))^4 (1 - F(x))^6$$

$$= \binom{10}{4} (F(x))^4 (1 - F(x))^6 + \text{stuff}$$

$$F_{X_{(4)}} = P(X_{(4)} \le x) \underset{\substack{\uparrow \\ \text{if i.i.d.}}}{=} \binom{10}{4} (F(x))^4 (1 - F(x))^6 + \binom{10}{5} (F(x))^5 (1 - F(x))^5 + \binom{10}{6} (F(x))^6 (1 - F(x))^4 + \dots + \binom{10}{10} (F(x))^{10} (1 - F(x))^0$$

Ex: if 4th largest is ≤ 3.7



$$F_{X_{(4)}} = \sum_{j=4}^{10} \binom{10}{j} (F(x))^j (1 - F(x))^{10-j}$$

verify for $F_{X_{(n)}}, F_{X_{(1)}}$

here, had $k = 4$, $n = 10$

In general,

For arbitrary n, k, and r.v.'s $X_1, X_2, \dots, X_{10}$ which are i.i.d.

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1 - F(x))^{n-j}$$

use binomial CDF? $p = F(x)$

$$P(\max \leq x) = F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j}$$

CDF of max

$$= \binom{n}{n} (F(x))^n (F(x))^{n-n} \leftarrow (F(x))^0 = 1$$

$$= 1 \, (F(x))^n \, 1$$

$$F_{X(n)}(X) = (F(x))^n$$

$$P(\min \leq x) = F_{X(1)} = \sum_{j=1}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j}$$

CDF of min

using $\sum_{K=0}^{n} k(i) = k(0) + \sum_{K=1}^{n} k(i)$

or $\sum_{K=1}^{n} k(i) = \sum_{k=0}^{n} k(i) - k(0)$

$$= \left(\sum_{j=0}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j} \right) - \binom{n}{0} (F(x))^0 (1-F(x))^n$$

$$= 1 - 1 \cdot 1 \cdot (1-F(x))^n$$

$$F_{X(1)} = 1 - (1-F(x))^n$$

this is a sum of all the binomial probobabilites so must be 1

$$\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} = 1$$

for this summation, notice

$$(F(x) + (1-F(x)))^n$$

$$= (1)^n$$

$$= 1$$

so $F_{X(1)} = 1 - (1-F(x))^n$ ← CDF of min

In general,

For arbitrary n, K and i.i.d r.v.'s $X_1, X_2, \ldots, X_n$

CDF of k th largest one →

CDF

$$F_{X_{(k)}}(x) = \sum_{j=K}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-1}$$

PDF

$$f_{X(k)}(x) = \frac{d}{dx}\left[F_{X_{(K)}}(x)\right] = \frac{d}{dx}\left[\text{all of that stuff}\right] =$$

done in class next time