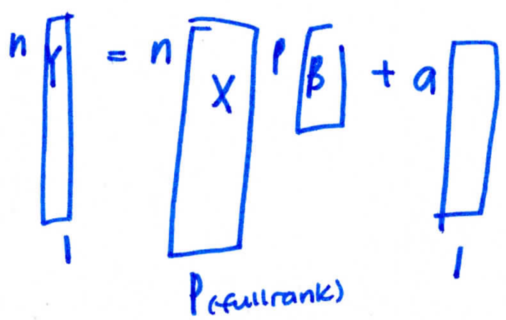
NOT on the test.

- $\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$



Standard Assumption:

$\vec{\varepsilon} \sim N_n(\vec{0}, \sigma^2 I)$

Independent, Normal, Homoskedastic (same variance)

- $\vec{\beta}, \sigma^2$ ~~unknown parameters~~,

$$\left\{ \begin{array}{l} * \frac{1}{\sigma}\vec{\varepsilon} \sim N_n(\vec{0}, I) = \vec{z} \\ \frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon} \sim \chi^2_n \end{array} \right.$$

With 30min of ~~algebra~~, minimum

we can find the least square ~~[illegible]~~ estimators,

$$\hat{\beta} = (X^TX)^{-1}X^T\vec{Y} = (X^TX)^{-1}X^T(X\vec{\beta} + \vec{\varepsilon})$$

$$= (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\vec{\varepsilon}$$

$$= \vec{\beta} + (X^TX)^{-1}X^T\vec{\varepsilon}$$

$$\hat{\vec{\beta}} \sim N_p(\vec{\beta}, \underbrace{(X^TX)^{-1}X^T \sigma^2 I\, (((X^TX)^{-1}X^T)^T)}_{\sigma^2 (X^TX)\cancel{^{-1}}(\cancel{(}X^TX)^T)^{-1}}) = N_p(\vec{\beta}, \sigma^2(X^TX)^{-1})$$

transpose

$\Rightarrow E[\hat{\vec{\beta}}] = \vec{\beta}$ unbiased estimate

margining

$$\Rightarrow \hat{\beta}_k \sim N(\beta_k, \sigma^2 (X^TX)^{-1}_{kk})$$

$$\Rightarrow \frac{\hat{\beta}_k - \beta_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}} \sim N(0,1)$$

Student's problem: σ is unknown

Use estimate instead.

similar $\to$ $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \approx \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$

- $$\underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon}}_{\sim \chi^2_n} = \frac{1}{\sigma^2}\vec{\varepsilon}^T P \vec{\varepsilon} + \frac{1}{\sigma^2}\vec{\varepsilon}^T_{(I-P)}\vec{\varepsilon}$$

Now, we need to worry about rank

$P := \underset{n \times n}{X(X^TX)^{-1}X^T}$ (orthogonal projection)

$\text{rank}[P] = p$

$I - P :=$ (orthogonal projection onto the map "missing" dimensions)

$P \cdot P = X(X^TX)^{-1}\cancel{X^TX(X^TX)^{-1}}X^T = P$

$\text{rank}[I-P] = n - p$

$(I-P)(I-P) = II - PI - IP + PP = I - P - P + P = I - P$

· Cochran's Thm

*Going back to $\underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon}}_{\chi^2_n} = \underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T P\vec{\varepsilon}}_{\chi^2_p} + \underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T(I-P)\vec{\varepsilon}}_{\chi^2_{n-p}}$

independent

$$\cdot\ \vec{\varepsilon}^T P\vec{\varepsilon} = \varepsilon^T PP\varepsilon$$

$$=(P\vec{\varepsilon})^T P\vec{\varepsilon}$$ ↲ symmetric $= (\hat{\vec{\beta}}-\vec{\beta})^T X^T X(\hat{\vec{\beta}}-\vec{\beta})$

$$\vec{\varepsilon}^T(I-P)\vec{\varepsilon} = \vec{\varepsilon}^T(I-P)(I-P)\vec{\varepsilon} = ((I-P)\vec{\varepsilon})^T(I-P)\vec{\varepsilon} = SSE$$

$$P\vec{\varepsilon} = P(\vec{Y}-X\vec{\beta}) = P\vec{Y} - PX\vec{\beta} = X\underbrace{(X^TX)^{-1}X^T X}_{\beta} - X\cancel{(X^TX)^{-1}X^TX}\beta$$

$$= X(\hat{\vec{\beta}}-\vec{\beta})$$

$$(I-P)\vec{\varepsilon} = (I-P)(\vec{Y}-X\vec{\beta}) = I\vec{Y} - P\vec{Y} - IX\vec{\beta} - PX\vec{\beta} = \vec{Y} - P\vec{Y} - \cancel{X\vec{\beta}} + \cancel{X\vec{\beta}}$$

$= \vec{Y} - X\hat{\vec{\beta}} = \vec{\hat{\varepsilon}}$ residuals

$$E\left[\frac{1}{\sigma^2} SSE\right] = n-p$$

$$E\left[\frac{1}{\sigma^2}\frac{SSE}{n-p}\right] = 1$$

$$E\left[\frac{SSE}{n-p}\right] = \sigma^2$$ → MSE

$$MSE \approx \sigma^2$$

$$\sqrt{MSE} = \sigma$$

⇒ Going back to the previous note,
you get $\dfrac{Z}{\sqrt{\dfrac{\chi^2_{n-p}}{n-p}}} \sim T_{n-p}$ justification for the T-test in a linear regression.

· F-test:

$$\frac{\frac{1}{\sigma^2}\vec{\varepsilon}^T P\vec{\varepsilon}/p}{\frac{1}{\sigma^2}\vec{\varepsilon}^T(I-P)\vec{\varepsilon}/n-p} \sim F_{p,n-p}$$