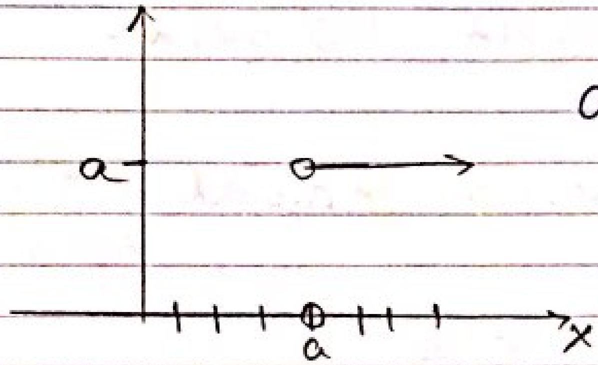
Lecture 21 ①
12/02/19

Let X be a non-negative r.v (ie - $supp[X] \geq 0$) and has finite expectation μ. Let $a > 0$ (a is constant) and consider the following inequality

$$a\mathbb{1}_{X \geq a} \leq X$$



verify the inequality:

If $X \geq a$ then $a(1) \leq X \Rightarrow X \geq a$

if $X < a$ then $a(0) \leq X \Rightarrow X \geq 0$ (this because X is non-negative)

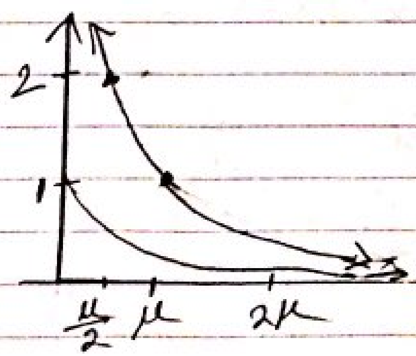
Let's take $E[X]$ on both sides

$$E[a\mathbb{1}_{X \geq a}] \leq E[X]$$

$$\Rightarrow a E[a\mathbb{1}_{X \geq a}] \leq \mu \Rightarrow E[a\mathbb{1}_{X \geq a}] \leq \frac{\mu}{a}$$

$$\Rightarrow P(X \geq a) \leq \frac{\mu}{a} \Rightarrow 1 - F(a) \leq \frac{\mu}{a}$$

CDF



Tail Markov's Inequality

"Crude Bound"

$X \sim Exp(1) \Rightarrow \mu = 1, \sigma^2 = 1 \Rightarrow P(X \geq a) = e^{-a}$

a	$P(X \geq a)$	markov	chebyshev	chernoff
2	0.1353	0.5	1	0.73526
5	0.0067	0.2	0.0635	0.09158
10	0.0004	0.1	0.0123	0.00123

Corollaries of markov's Inequality:

* $b = a\mu$

Note $b > 0$, $P(X \geq b) \leq \frac{\mu}{b}$

$\Rightarrow P(X \geq a\mu) \leq \frac{1}{a}$

* let $h(X)$ be 1:1 increasing

$P(h(X) \geq h(a)) \leq \frac{E[h(X)]}{h(a)}$

$\Rightarrow P(X \geq a) \leq \frac{E[h(X)]}{h(a)}$ If X is continuous with connected support

* Let $a = \text{Quantile}[X, P] = F_X^{-1}(P)$

$P(X \geq a) \leq \frac{\mu}{a} \Rightarrow 1 - F_X(a) \leq \frac{\mu}{a}$

$\Rightarrow 1 - F_X[F_X^{-1}(P)] \leq \frac{\mu}{F^{-1}(P)}$

$\Rightarrow 1 - P \leq \frac{\mu}{F_X^{-1}(P)} = F_X^{-1}(P) \leq \frac{\mu}{1-P}$

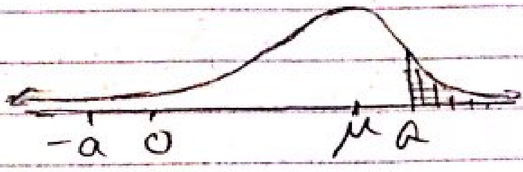
e.g.:- $P = \frac{1}{2} \Rightarrow \text{Med}[X] = F_X^{-1}(\frac{1}{2}) \leq 2\mu$

* let X be any r.v

Note $|X|$ is non-neg

$P(|X| \geq a) \leq \frac{E[|X|]}{a}$ Note $E[|X|] < \infty$

right and left tail



* Let X be any r.v mean μ, variance σ^2

let $Y = (X-\mu)^2$ Note Y is non-negative

$P(Y \geq a^2) \leq \frac{E[Y]}{a^2}$

$P((X-\mu)^2 \geq a^2) \leq \frac{E(X-\mu)^2}{a^2}$

$\Rightarrow P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$

Chebyshev's Inequality

Assume X is non-negative

$\Rightarrow P((X-\mu) \geq a \text{ or } (X-\mu) \leq -a)$

$= P(X-\mu \geq a) + P((X-\mu) - a)$

$= P(X \geq \mu + a) + P(X \leq \mu - a)$

let $a > \mu \Rightarrow P(X \geq \mu + a) + P(\cancel{X \leq 0})^{0}$

let $b = \mu + a \Rightarrow b > 2\mu$

$\Rightarrow P(X \geq b) \leq \frac{\sigma^2}{(b-\mu)^2}$

corollaries of Markov's Inequality:

* let X be any r.v

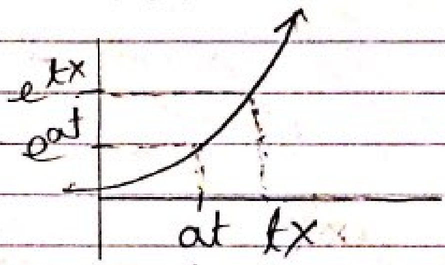
$y = e^{tx}$ where $t \neq 0$ Note y is non-neg $\forall t$

$P(y \geq c) \leq \frac{E[y]}{c}$ let $c = e^{at} \geq 0$

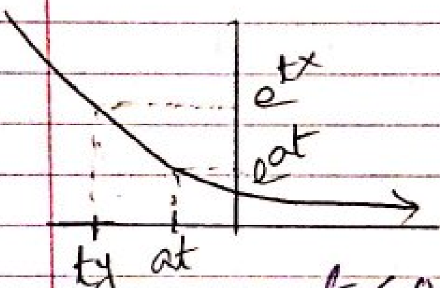
$\Rightarrow P(e^{tx} \geq e^{at}) \leq \frac{E(t^{tx})}{e^{at}} = e^{-at} M_X(t)$

moment generating function

$P(e^{tx} \geq e^{at}) \leq m_x(t)$



$t < 0$



$t < 0$, $P(X \leq a) < e^{-at} m_X(t)$

Inequality for both right and left tails

Note valid for all $t \neq 0$

If $t > 0$ $P(X \geq a) \leq \min_{t>0} \{ e^{-at} M_X(t) \}$

$t < 0$ $P(X \leq a) = \min_{t<0} \{ e^{-at} M_X(t) \}$

ex $X \sim Exp(\lambda)$

$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$

$$= \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty \overset{\text{if } t-\lambda<0}{\Longrightarrow} \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda-t}$$

if $\lambda=1$ (i.e. $X \sim EXP(1)$) then, $m_X(t) = \frac{1}{1-t}$

let $g(t) = \frac{e^{-at}}{1-t}$

$$g'(t) = \frac{(1-t)(-a)e^{-at} - e^{-at}(-1)}{(1-t)^2}$$

$$= e^{-at}(a(t-1)+1) = 0$$

$$at - a + 1 = 0 \Rightarrow at = a-1 \Rightarrow t = \frac{a-1}{a}$$

$\Rightarrow t = 1 - \frac{1}{a} \in (0,1)$ if $a>1$

$$P(X \leq a) \leq e^{-a(1-\frac{1}{a})} \frac{1}{1-(1-\frac{1}{a})}$$

$$= a e^{-a+1} = \frac{e \cdot a}{e^a}$$

X, Y are r.v's with means m_X, m_Y and variance σ_X^2, σ_Y^2

$cov[X,Y] := E[XY] - m_X m_Y$

measure of "linear dependence" in the units of X and Y

$$corr[X,Y] = \frac{cov[X,Y]}{SE[X]\,SE[Y]}$$

"correlation" which is unitless and bounded $\in (-1,1)$

let $y = cx$ where $c \neq 0 \rightarrow$ constant

$$\text{corr}[x,y] = \frac{\text{cov}[x, cx]}{SE[x]|c|SE[cx]} = \frac{c\,\text{Cov}[x,x]}{SE[x]\,|c|\,SE[x]}$$

$$= \frac{c}{|c|} \frac{\sigma^2 x}{\sigma_x \sigma_x} = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$$