Probability Theory — MATH 621 — CLASS 09/09/2019

Bag of Fruit

p_1 : probability of apple
p_2 : probability of banana $\qquad p_1 + p_2 + p_3 = 1$
p_3 : probability of cantalope

Draw n with replacment.

Let X_1 : # of apples
X_2 : # of bananas
X_3 : # of cantalopes

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim P_{\vec{X}}(\vec{x})$$

$$\sim P_{\vec{X}}(\vec{x}) = \underbrace{\frac{n!}{x_1!\,x_2!\,x_3!}\, p_1^{x_1} p_2^{x_2} p_3^{x_3} \cdot \mathbb{1}_{x_1+x_2+x_3=n} \cdot \mathbb{1}_{x_1\in\{0,1,\dots,n\}} \cdot \mathbb{1}_{x_2\in\{0,\dots,n\}} \cdot \mathbb{1}_{x_3\in\{0,\dots,n\}}}$$

$$\binom{n}{x_1, x_2, x_3}$$

so,

$$\sim P_{\vec{X}}(\vec{x}) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

$$= \text{Multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right)$$

Generally w/ k types of objects:

$$\vec{X} \sim \text{Multinomial}(n, \vec{p}) := \binom{n}{x_1, \ldots, x_k} p_1^{x_1} \cdots p_k^{x_k}$$

$$\text{Supp}[\vec{X}] = \left\{ \vec{x} : \vec{x} \in \{0, 1, \ldots, n\}^k \right\}$$

$$\vec{p} \in \left\{ \vec{p} : (0,1)^k, \ \vec{p} \cdot \vec{1} = \vec{1} \right\} \qquad \vec{1} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \text{Multi}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) \qquad \begin{array}{l} p_1 = p \\ p_2 = 1-p \end{array}$$

$$X_1 \sim \text{Bin}(n, p) \qquad X_2 \sim \text{Bin}(n, 1-p)$$

Is $X_1 \stackrel{d}{=} X_2$? NO! Is $X_1, X_2 \stackrel{ind}{\sim}$? NO!

☆ Remark: Any element inside a multinomial will be dependent ☆

For two independent r.v's.:

$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$$

$$P(X_1 = 1 \mid X_2 = n) = np(1-p)^{n-1}$$

$$\forall_{x_1} \in \text{Supp}[X_1] \text{ and } \forall_{x_2} \in \text{Supp}[X_2]$$

$$P_{X_1|X_2}(X_1, X_2) := P(X_1 = x_1 \mid X_2 = x_2)$$

$$= \frac{P_{X_1,X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

conditional PMF/JMF

definition of conditional probability

$$P_{X_2}(x_2) = \sum_{x_1 \in supp[X_1]} P_{X_1,X_2}(x_1, x_2)$$

$$= \sum_{x_1 \in \{0,1,\dots,n\}} \frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_2\in\{0,1,\dots,n\}}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2} \mathbb{1}_{x_2\in\{0,1,\dots,n\}} \sum_{x_1\in\{0,1,\dots,n\}} \frac{1}{x_1!} p^{x_1} \mathbb{1}_{x_1=n-x_2}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2} \mathbb{1}_{x_2\in\{0,1,\dots,n\}} \cdot \frac{1}{(n-x_2)!} p^{n-x_2}$$

$$= \frac{n!}{x_2!\,(n-x_2)!}(1-p)^{x_2} p^{n-x_2} \mathbb{1}_{x_2\in\{0,1,\dots,n\}}$$

$$= \binom{n}{x_2}(1-p)^{x_2} p^{n-x_2}$$

$$= \text{Binomial}(n, 1-p)$$

marginal PMF

so,

$$P_{X_1|X_2}(X_1, X_2) = \frac{P_{X_1,X_2}(X_1, X_2)}{P_{X_2}(x_2)}$$

$$= \frac{\cancel{n!}/x_1!\,x_2!\; p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}}{\cancel{n!}/x_2!\,(n-x_2)!\; p^{n-x_2}(1-p)^{x_2}}$$

$$= \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$$= \begin{cases} x_1!/x_1!\; p^0 = 1 & \text{if } x_1+x_2=n \\ 0 & \text{if } x_1+x_2\neq n \end{cases} = \text{Deg}(n-x_2) = \{\, n-x_2 \text{ w.p } 1$$

$$\vec{X}_{-j} = \begin{bmatrix} x_1 \\ \vdots \\ x_{j-1} \\ x_{j+1} \\ \vdots \\ x_k \end{bmatrix}$$

$$P(X_{-j} \mid X_j) = \frac{\left(\cancel{n!} / x_1! \cdots x_{j-1}! \, \cancel{x_j!} \, x_{j+1}! \cdots x_k! \right) p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cancel{p_j^{x_j}} \, p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{\cancel{n!} / \cancel{x_j!} \, (n-x_j)! \; \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}}$$

let $n' = n - x_j$

$$= \frac{n'!}{x_1! \cdots x_{j-1}! \, x_{j+1}! \cdots x_k!} \cdot \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \, p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{n'}}$$

Recall: $n = x_1 + \cdots + x_{j-1} + x_j + x_{j+1} + \cdots + x_k$

$\Rightarrow n = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k$

so, $$P(X_{-j} \mid X_j) = \binom{n}{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k} \frac{p_1^{x_1}}{(1-p_j)^{x_1}} \cdots \frac{p_{j-1}^{x_{j-1}}}{(1-p_j)^{x_{j-1}}} \cdot \frac{p_{j+1}^{x_{j+1}}}{(1-p_j)^{x_{j+1}}} \cdots \frac{p_k^{x_k}}{(1-p)^{x_k}}$$

$$= \binom{n}{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k} \left(\frac{p_1}{(1-p_j)}\right)^{x_1} \cdots \left(\frac{p_{j-1}}{(1-p_j)}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{(1-p_j)}\right)^{x_{j+1}} \cdots \left(\frac{p_k}{(1-p_j)}\right)^{x_k}$$

$$\vec{P} = \begin{bmatrix} p_1 / 1-p_j \\ \vdots \\ p_{j-1} / 1-p_j \\ p_{j+1} / 1-p_j \\ \vdots \\ p_k / 1-p_j \end{bmatrix} \qquad \dim[\vec{P}] = k-1$$

$E[\vec{X}] = ?$ $\quad$ $Var[\vec{X}] = ?$

discrete

$$\mu := E[X] = \sum_{x \in \mathbb{R}} x\, p(x)$$

$$\mu := E[X] = \int_{\mathbb{R}} x\, f(x)\, dx$$

continuous

$E[aX + c] = a\mu + c$ where a, c are constants

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$ always true

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

if $X_1, \ldots, X_n \overset{ind}{\sim}$

$$\underline{\sigma^2} := Var[X] := E\left[(X - \mu)^2\right]$$

Variance

$$\underline{\sigma} := E[X] := \sqrt{\sigma^2}$$

Standard error.