(Lecture 3)

September 5^{th}, 2019

$X_1, X_2, \cdots, X_r \overset{iid}{\sim} Geom(p)$

$T = \sum_{i=1}^{r} X_i \sim NegBin(r,p) = \binom{t+r-1}{r-1}(1-p)^t p^r$

t

$\underline{0}\ \underline{1}\ \underline{0}\ \underline{1}\ \cdots\ \underset{t+r-1}{\underline{0}}\ \underset{t+r}{\underline{1}}$

$r-1$

$X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$

Let n be large and p be small such that

$\lambda := np \Rightarrow \frac{\lambda}{n}$, $n \in \mathbb{N}$, $p \in (0,1) \rightarrow \lambda \in (0,\infty)$

Let $n \to \infty$ Call it r.v. X

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda!}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!} \frac{1}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$n! : (n)(n-1)\cdots(n-x+1)$, there are x terms

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{(n)(n-1)\cdots(n-x+1)}{(n)(n)\cdots(n)} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\lim_{n\to\infty}\frac{n}{n} \lim_{n\to\infty}\frac{n-1}{n} \cdots \lim_{n\to\infty}\frac{n-x+1}{n} \qquad \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \qquad \text{Note: } e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

So

$$\frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^x e^{-\lambda}}{x!} = \text{Poisson}(\lambda) : \text{Poisson r.v.}$$

$$\text{Supp}[X] = \{0,1\ldots\} = \mathbb{N}_0$$

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$$

$$T = X_1 + X_2 \sim ?$$

Note: Convolution formula

$$p(t) = \sum_{x\in\text{Supp}(\cdot)} P_{OLD}(x)\, P_{OLD}(t-x)\, \mathbb{1}_{t-x\in\text{Supp}(x)}$$

$$T = X_1 + X_2 \sim \sum_{x\in\{0,1,\cdots\}}\left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\lambda^{t-x}e^{-\lambda}}{(t-x)!}\mathbb{1}_{t-x\in\{0,1,\cdots\}}\right)$$

$$= \lambda^t e^{-2\lambda}\sum_{x\in\{0,1\cdots n\}}\frac{1}{x!(t-x)!}\mathbb{1}_{x\le t}$$

$$= \lambda^t e^{-2\lambda}\sum_{x\in\{0,1,\cdots t\}}\frac{1}{x!(t-x)!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!}\sum_{x=0}^{t}\binom{t}{x}$$

Note. Powerset of A

$$2^A := \{B : B \subseteq A\} = B : B \subseteq A, |B| = 0 \cup$$
$$\{B : B \subseteq A, |B| = 1\} \cup$$
$$\{B : B \subseteq A, |B| = 2\} \cup$$
$$\vdots$$
$$\{B : B \subseteq A : |B| = n\}$$

$$= |2^A| = |\{B : B \subseteq A\}|$$

$$= |\{B : B \subseteq A, |B| = 0| + |\{B : B \subseteq A, |B| = 0| + |\{B : B \subseteq A, |B| = 1| + \cdots + |\{B : B \subseteq A, |B| = n\}$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

$$= \sum_{i=1}^{n} \binom{n}{i}$$

$$= 2^n$$

so $\sum_{x=0}^{t} \binom{t}{x} = 2^t$

so $\frac{\lambda^t e^{-2\lambda}}{t!} \binom{t}{x} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{poisson}(2\lambda)$

$X, Y \overset{iid}{\sim} \text{Geom}(p) = (1-p)^x p$

$P(X > Y)$

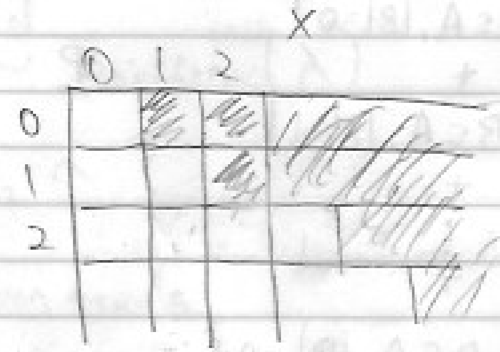
By law of total probability, $P(X>Y) + P(Y>X) + P(X=Y) = 1$

$P(X>Y)$ and $P(Y>X)$ are equal

$$P(X>Y) = \sum_{y\in R}\sum_{x\in R} P_{X,Y}(x,y)\,\mathbb{1}_{x>y}$$

$$= \sum_{y\in R}\sum_{\substack{x\in R \\ \text{and} \\ x>y}} P_{X,Y}(x,y)$$

$P_{X,Y}(x,y)$



$$\longrightarrow \; = \sum_{y\in R}\sum_{\substack{x\in R, \\ x>y}} \left((1-p)^x p\,\mathbb{1}_{x\in\{0,1,\dots\}}\right)\left((1-p)^y p\,\mathbb{1}_{y\in\{0,1,\dots\}}\right)$$

$$= p^2 \sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty} (1-p)^x(1-p)^y$$

$$= p^2 \sum_{y=0}^{\infty}(1-p)^y \sum_{x=y+1}^{\infty}(1-p)^x$$

Note: Reindexing trick

let $x' := x-(y-1) \rightarrow x = x' + (y+1)$

$$\sum_{x=y+1}(1-p)^x = \sum_{x'=0}^{\infty}(1-p)^{x'+y+1}$$

So $p^2 \sum_{y=0}^{\infty} (1-p)^y \sum_{x=y+1}^{\infty} (1-p)^x = p^2 \sum_{y=0}^{\infty} (1-p)^y \sum_{x'=0}^{\infty} (1-p)^{x'} (1-p)^{y+1}$

$$= p^2 \sum_{y=0}^{\infty} (1-p)^{2y+1} \underbrace{\sum_{x'=0}^{\infty} (1-p)^{x'}}_{\text{this is a geometric series}}$$

$$\left(\forall a \in (0,1) \sum_{x=0}^{\infty} a^x = \frac{1}{1-a} \right)$$

$$= p^2 \sum_{y=0}^{\infty} (1-p)^{2y+1} \frac{1}{p}$$

$$= p \sum_{y=0}^{\infty} (1-p)^{2y+1}$$

$$= p(1-p) \sum_{y=0}^{\infty} (1-p)^{2y}$$

$$= p(1-p) \sum_{y=0}^{\infty} \left((1-p)^2\right)^y$$

$$= \frac{p(1-p)}{1-(1-p)^2}$$

$$= \frac{p-p^2}{1-(1-2p+p^2)}$$

$$= \frac{p-p^2}{2p-p^2} = \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p}$$

Note: $E(X) = \sum_{X \in \mathbb{R}} X\,P(X)$

Recall for X discrete, $E(g(X)) = \sum_{X \in \mathbb{R}} g(X)P(X)$

Let $g(X) = \mathbb{1}_{X \in A}$

$$E[\mathbb{1}_{X \in A}] = \sum_{X \in \mathbb{R}} \mathbb{1}_{X \in A} P(X)$$

$$= \sum_{X \in A} P(X)$$

$$= P(X \in A)$$

Recall for X, Y discrete

$$E[g(X,Y)] = \sum_{X \in \mathbb{R}} \sum_{Y \in \mathbb{R}} g(X,Y)P(X,Y)$$

$$E[\mathbb{1}_{X>Y}] = \sum_{X \in \mathbb{R}} \sum_{Y \in \mathbb{R}} P(X,Y)\mathbb{1}_{X>Y}$$

Suppose you have a bag of apples and bananas

P_1 : Probability of picking apples

P_2 : Probability of picking bananas $\quad P_1 + P_2 = 1$

Let's pick one with replacement n times

let X_1 : # of apples

X_2 : # of bananas

$$X_1 \sim Bin(n, p_1) = \binom{n}{x_1} p_1^{x}(1-p_1)^{n-x_1}$$

$$X_2 \sim Bin(n, p_2) = \binom{n}{x_2} p_2^{x_2}(1-p_2)^{n-x_2}$$

Let $\vec{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{X_1 X_2}(x_1, x_2)$ (JMF)

$\dim[\vec{x}] = 2$

$$= \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1 + x_2 = n}$$

$$\mathbb{1}_{x_1 \in \{0, 1, \cdots n\}}$$

$$\mathbb{1}_{x_2 \in \{0, 1, \cdots, n\}}$$

$$\binom{n}{x_1, x_2} = \frac{n!}{x_1! x_2!} \mathbb{1} \ldots$$

so $\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$ = multinomial $(n, \vec{p})$ = vector r.v.