Lecture #7 : Continuous r.v's:

X has $|Supp(X)| = |\mathbb{R}|$

$\Rightarrow p(x) = P(X=x) = 0$

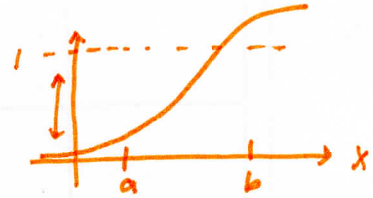
The derivative of CDF is very important.

$$f(x) = \frac{d}{dx}[F(x)]$$
$$= F'(x)$$

Probability density function (PDF)

CDF

$F(x) = P(X \leq x)$



$$P(X \in [a,b]) = F(b) - F(a)$$
$$= \int_a^b f(x)\,dx$$

(fundamental Thm. calculus)

~~Proba~~ Properties of $f(x)$

(I) $\int_{\mathbb{R}} f(x)\,dx = 1$

$= F(\infty) - F(-\infty)$

$= 1 - 0 = 1$

There is no more PMF

(II) $f(x) \geq 0$

(III) $Supp[X] = \{x : f(x) > 0\}$

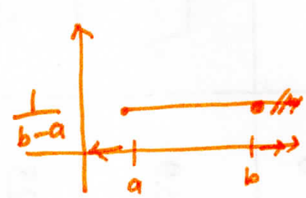
- $X \sim Exp(\lambda) = \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$

  You need to put the indicator function here since this is dealing with waiting time

  $Supp[X] = [0, \infty)$

- $X \sim U(a,b) = \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$

  (uniform)



$Supp[X] = [a,b]$

$a, b \in \mathbb{R}$

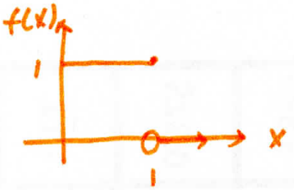
but $b > a$

$X \sim U(0,1) \quad = \mathbb{1}_{x \in [0,1]}$

standard uniform

$Exp(1)$

standard exponential



$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \quad f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdots f_{X_k}(x_k)$

joint density function (JDF)

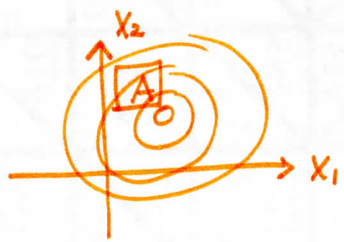
If $X_1 \ldots X_k$ are independent

If " $\overset{iid}{\sim}$, then $f(x_1) \cdots f(x_k)$

$\int \cdots \int_{\mathbb{R}^k} f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_k = 1$

$k = 2$

$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$



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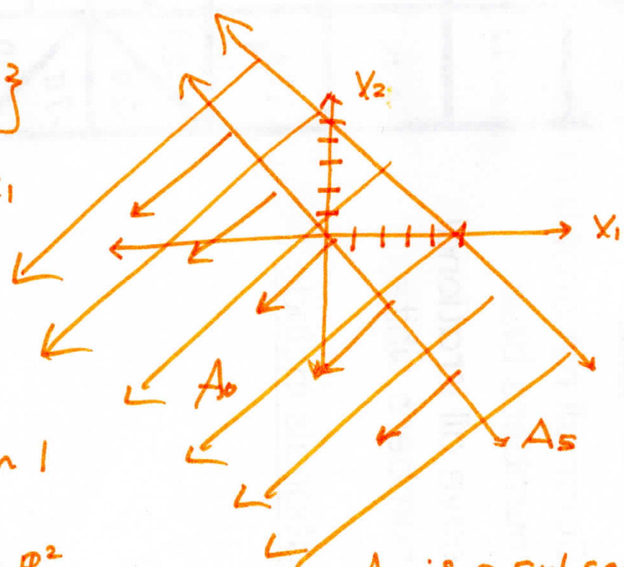
$T = X_1 + X_2 \sim f_T(t) = ?$

$F(t) = P(T \leq t) = P(A_t)$

$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underbrace{x_1 + x_2 \leq t}_{x_2 \leq t - x_1} \right\}$



If you take all of these to ∞, then take $\int$ then 1 ← b/c $\mathbb{R}^2$

if $-\infty$, 0.

A_0 is a subset of A_5.

$P(A_t) = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty, t-x_1)} f_{X_1,X_2}(x_1,x_2)\, dx_2\, dx_1$ $= \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\, dv\, dx$

$= \int_{-\infty}^{t} \int_{x \in \mathbb{R}} f_{X_1,X_2}(x, v-x)\, dx\, dv$

$= F(t)$

Let $x = x_1 \Rightarrow dx = dv_1$

$v = x_2 + x \Rightarrow x_2 = v - x$

$\Rightarrow dv = dx_2$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$\cancel{\frac{d}{dx}\int_{-\infty}^{a} f(x)\,dx} =$

general convolution formula

$* \quad f(t) = \frac{d}{dt}[\quad] = \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\, dx$

$= \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx \quad = \int_{\mathbb{R}} f(x) f(t-x)\, dx$

if X_1 & $X_2 \overset{ind}{\sim}$ if iid

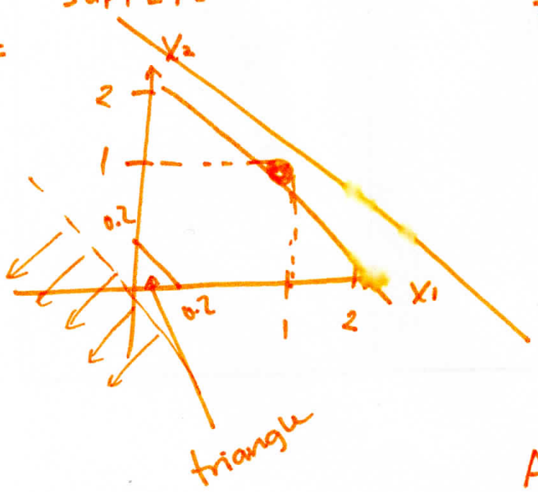
$= \int_{supp[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, \mathbb{1}_{t-x \in supp[X_2]}\, dx$

$= \int_{supp[X]} f(x) f(t-x)\, \mathbb{1}_{t-x \in supp[X]}\, dx$

- $X_1, X_2 \overset{iid}{\sim} U(0,1)$. $T = X_1 + X_2 \sim f(t)$?

$supp[T] = [0,2]$

JDF



$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

$= 1 \cdot \mathbb{1}_{x_1 \in [0,1]} \cdot 1 \cdot \mathbb{1}_{x_2 \in [0,1]}$

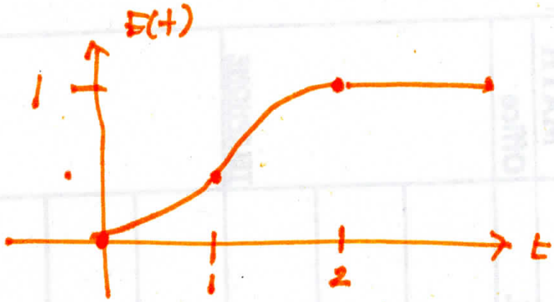
$F(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1}{2}t^2 & \text{if } t \in [0,1] \\ 1 & \text{if } t > 2 \\ & \text{if } t \in [1,2] \end{cases}$

$A_{\square} = \frac{t^2}{2} - 2\underbrace{\left(\frac{(t-1)^2}{2}\right)}$ $\rightarrow$ ~~Now,~~ subtract the little triangle

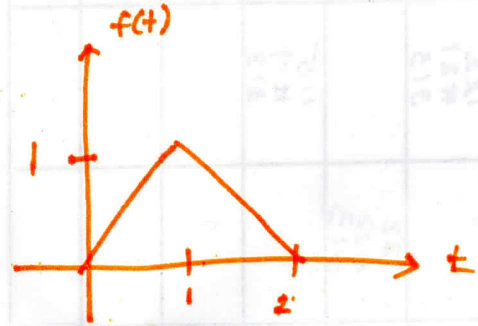
$= \frac{t^2}{2} - t^2 + 2t - 1$

$= -\frac{t^2}{2} + 2t - 1$

Now, draw a picture:



$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & t \in (0,1] \\ 2-t & t \in (1,2] \\ 0 & t > 2 \end{cases}$$

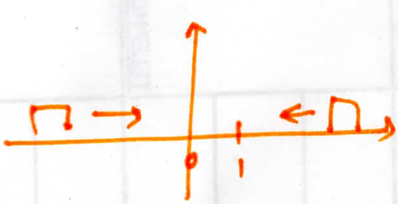


Let's use a formula:

$$f(t) = \int_0^1 1 \cdot 1 \cdot \underbrace{\underbrace{\mathbb{1}_{t-x \in [0,1]}}_{\mathbb{1}_{x-t \in [-1,0]}}}_{\mathbb{1}_{x \in [t-1,t]}} \, dx = \int_0^1 \mathbb{1}_{x \in [t-1,t]} \, dx$$

$$= \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in (0,1] \\ 2-t = 1-(t-1) & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$

Play with numbers



You can use either way to find the answer to this.

$T = X_1 + X_2 \sim f_T(t)$

$X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

$$f_T(t) = \int_0^\infty (\lambda e^{-\lambda x})(\lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t}e^{\lambda x}}) \mathbb{1}_{t-x \in (0,\infty)} \, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \le t} \, dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t\lambda^2 e^{-\lambda t} \neq Exp(\lambda)$$

$Supp[T_2] = (0,\infty)$

$$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 = X_3 + T_2$$

$$f_{T_3}(t) = \int_0^\infty (x\lambda^2 e^{-\lambda x})(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}) \, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x \, dx = \frac{1}{2} t^2 \lambda^3 e^{-\lambda t}$$

Find $T_4 \Rightarrow f_{T_4} = \int_0^\infty \frac{1}{2} x^2 \lambda^3 e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t} \, dx = \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$

Find $T_5 \Rightarrow f_{T_5} = \int_0^\infty \frac{1}{6} x^3 \lambda^4 e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t} \, dx = \frac{1}{2 \cdot 3 \cdot 4} \lambda^5 e^{-\lambda t} t^4$

$\vdots$

$$f_{T_k}(t) = \frac{1}{t!} \lambda^k e^{-\lambda t} t^{k-1} = Erlang(k, \lambda)$$

↑

sums of waiting time

exp ⇒ waiting time

- $Supp[T_k] = (0,\infty)$
  $k \in \mathbb{N}$
  $\lambda \in (0,\infty)$

Lecture #6

X_1 & X_2 iid $U(\{0,1,2$

Generally $X \sim U(A) := \frac{1}{|A|} \mathbb{1}_{x \in}$

$Supp[A] = A$ where

$\boxed{P_{X_1,X_2}(X_1,X_2)} = \frac{1}{4} \cdot \frac{1}{4} \cdot$

put them together

$P(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x$

Let's visualize this.

$P(1) = 2$

$P(0.5) = 0$

$P(3) = 4$

$P(6) = 1$

$P(7) = 0$

Add them together