Poisson Distribution

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$$X \sim Binom(n,p) = \binom{n}{x} p^x (1-p)^{1-x} = p(x)$$

Let n get large, let p get small

Fix $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$n \in \mathbb{N}$
$p \in (0,1)$
$\lambda \in (0,\infty)$

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{n^x(n-x)!} \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}^{1}$$

$$\downarrow$$

$$\left(\frac{n(n-1)(n-2)\cdots(n-x+1)}{n(n)\cdots\cdots(n)}\right)$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \lim_{n\to\infty} \cancel{\frac{n}{n}}^{1} \lim \cancel{\frac{n-1}{n}}^{1} \cdots \lim \cancel{\frac{n-x+1}{n}}^{1}$$

$$= \boxed{\frac{\lambda^x e^{-\lambda}}{x!}} = p(x)$$

$$Supp[x] = \{0, 1, \ldots\} = \mathbb{N}_0$$

Convolution of two r.v's

$X_1, X_2 \overset{iid}{\sim}$ Poisson (λ)

$$T = X_1 + X_2 \sim \sum_{x \in \mathbb{N}_0} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!}\right) \mathbb{1}_{t-x \in \mathbb{N}_0}$$

$$p(t) = \sum_{x \in Supp[X]} P_{old}(x) P_{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

$$= \lambda^t e^{-2\lambda} \sum \frac{1}{x!(t-x)!} \mathbb{1}_{x \le t}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,\dots,t\}} \frac{1}{x!(t-x)!} \cdot \frac{t!}{t!}$$

Because we want it to look like the combination formula

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,\dots,t\}} \binom{t}{x}$$

Side Note: we want to prove $\sum_{x \in \{0,\dots,t\}} \binom{t}{x} = 2^t$.

Proof

$A = \{a_1, \dots, a_n\}$, $|A| = n$

$$2^A := \{B \mid B \subset A\}$$

$$= \{B \mid B \subset A, |B| = 0\} \cup \{B \mid B \subset A, |B| = 1\} \cup$$

$$\vdots$$

$$\{B \mid B \subset A, |B| = n\}$$

$$2^n = |2^A| = |\{B \mid B \subset A, |B| = 0\}| + \dots + |\{B \mid B \subset A, |B| = n\}|$$

$\rightarrow$

$$= 1 + n + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n.$$

$\square$

(Back to distribution)

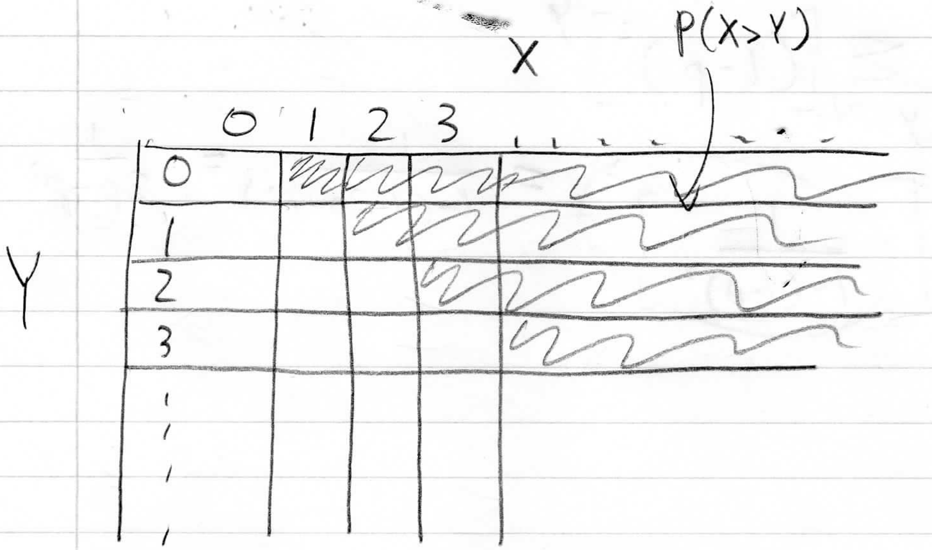
$$= \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda).$$

Note: Poisson(λ) + Poisson(λ) = Poisson(2λ).

$X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X > Y) = P(Y > X) < \frac{1}{2}$

$1 = P(X > Y) + P(Y > X) + P(X = Y)$



$P_{X,Y}(x,y) = P(x)P(y)$

$$P(X > Y) = \sum_{y \in \mathbb{R}} \sum_{x \in \mathbb{R}} P_{X,Y}(x,y) \mathbb{1}_{x>y}$$

$$= \sum_{y \in \mathbb{R}} \sum_{x \in (y, \infty)} p(1-p)^x \mathbb{1}_{x \in \mathbb{N}_0} \, p(1-p)^y \mathbb{1}_{y \in \mathbb{N}_0}$$

$$= p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \sum_{x \in \{y+1, \dots\}} (1-p)^x$$

Let $x' = x - (y+1) \Rightarrow x = x' + y + 1$

$$= p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \sum_{x' \in \mathbb{N}_0} (1-p)^{x'+y+1}$$

Note: Geometric series: If $q \in (0,1)$, $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$

$\longrightarrow$

$$= p^2(1-p) \sum_{y \in \mathbb{N}_0} (1-p)^{2y} \sum_{x' \in \mathbb{N}_0} (1-p)^{x'} = \frac{1}{p}$$

$$= p(1-p) \sum_{y \in \mathbb{N}_0} (1-p)^{2y}$$

$$= p(1-p) \sum_{y \in \mathbb{N}_0} \left[(1-p)^2\right]^y$$

$$= p(1-p) \left[\frac{1}{1-(1-p)^2} \right] = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2} = \frac{1}{p(2-p)}$$

$$= \boxed{\frac{1-p}{2-p}}$$

- X is a discrete r.v's

$$E(X) = \sum_{x \in \mathbb{R}} x\, p(x)$$ (Expectation)

$$E[g(X)] = \sum_{x \in \mathbb{R}} g(x)\, p(x)$$

- X, Y are discrete r.v's

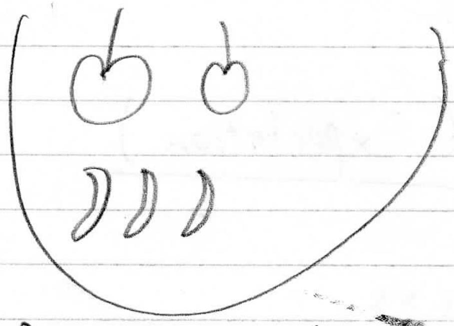
$$E[g(X,Y)] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} g(x,y)\, p(x,y)$$

- $$E[\mathbb{1}_{X \in A}] = \sum_{x \in \mathbb{R}} \mathbb{1}_{x \in A}\, p(x)$$

$$= \sum_{x \in A} p(x) = p(X \in A)$$

Note: $E[\mathbb{1}_{X > Y}] = P(X > Y) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p(x,y)\, \mathbb{1}_{x > y}$

Multi-nomial Distribution



$p_1 := $ prob of drawing an apple
$p_2 := $ prob of drawing a banana

$p_1 + p_2 = 1$

Draw n w/ replacement

$X_1 := $ # of apples drawn
$X_2 := $ # of bananas drawn

$$X_1 + X_2 = n$$

$X_1 \sim Binom(n, p_1)$
$X_2 \sim Binom(n, p_2)$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim P_{X_1, X_2}(x_1, x_2)$$

(Note: $\binom{n}{x_1} = \frac{n!}{x_1!(n-x_1)!} = \frac{n!}{x_1!x_2!}$)

$$= \frac{n!}{x_1!x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,\dots,n\}} \mathbb{1}_{x_2 \in \{0,\dots,n\}}$$

(Notation: $\binom{n}{x_1!, x_2!} := \frac{n!}{x_1!x_2!} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,\dots,n\}} \mathbb{1}_{x_2 \in \{0,\dots,n\}}$

Multichoose or Multinomial coefficient)

$$X \sim Multinom\left(n, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) p_1^{x_1} p_2^{x_2})$$