(Lecture 9) October 7^{th}, 2019

Poisson Process

$T_k \sim$ Erlang (k, λ)
$N \sim$ Poisson (λ)

$P(T_k > 1) = P(N \leq k-1) \Rightarrow 1 - F_T(1) = F(k-1) = Q(k, \lambda)$

"Counting and Waiting r.v.'s"

	Fixed time, Counting #	Fixed #, measure time
Discrete	Bern / Binomial	Geom / Neg. binomial
Continuous	Poisson	Exp / Erlang

What is the probability there are 0 successes by 50 trials each w.p. 0.1 of success?

$N \sim Bin(50, 0.1)$ $T \sim NegBin(1, 0.1)$

$P(N=0) = P(T > 49)$

$F_N(0) = 1 - F_T(49)$

What is the probability $\leq k$ events by experiment # t if probability of success is p?

$N \sim Binom(t, p)$, $T \sim NegBin(k+1, p)$

$P(N \leq t) = P(T > t-k-1) \Rightarrow F_N(k) = 1 - F_T(t-k-1)$

$$\Rightarrow \sum_{i=0}^{k} \binom{t}{i} p^i (1-p)^{t-i} = 1 - \sum_{i=0}^{t-k-1} \binom{k+i}{k} (1-p)^i p^k$$

$$T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0}$$

$$= \frac{\lambda^k}{\Gamma(k)} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0} \rightarrow \text{Gamma Distribution}$$

Parameter space: $k \in \mathbb{N}$, $\lambda \in (0, \infty)$

$$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

$$= \underbrace{\frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}}_{\text{Extended Negative Binomial}}$$

$k \in \mathbb{N}$, $p \in (0, 1)$

Can $k \in (0, \infty)$?

Gamma Distribution: $X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$

$\alpha, \beta > 0$

Transformation of Discrete Random Variables

$$X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = p_X(x)$$

$$Y = X + 3 \sim \begin{cases} 3 & \text{w.p } 1-p \\ 4 & \text{w.p } p \end{cases}$$

$$= p^{y-3}(1-p)^{1-(y-3)}\mathbb{1}_{y\in\{3,4\}}$$

$$Y = g(x) = x+3 \Rightarrow X = g^{-1}(Y) = Y-3$$

$$\left.\begin{aligned} P_Y(y) &= P(Y = y) \\ &= P(g(X) = y) \\ &= P(X = g^{-1}(y)) \\ &= P_X(g^{-1}(y)) \end{aligned}\right\} \text{Assuming } \exists g^{-1}$$

$$X \sim U(\{1,2,\dots,10\}) = \frac{1}{10}\mathbb{1}_{x\in\{1,2,\dots,10\}}$$

$$Y = g(X) = \min\{X, 3\}$$

	Y	$P_Y(y)$
	1	1/10
	2	1/10
	3	8/10

$$P_Y(y) = \sum_{\{x:g(x)=y\}} P_X(x) \underset{\text{if } \exists g^{-1}}{\uparrow}{=} P_X(g^{-1}(y))$$

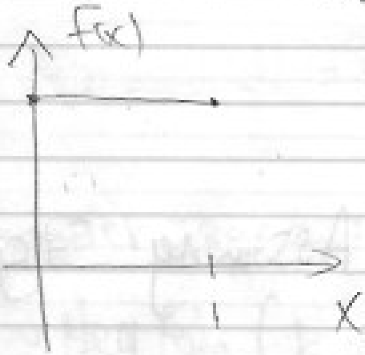
$$X \sim Bin(n,p)$$
$$Y = X^3 \rightarrow X = \sqrt[3]{Y}$$

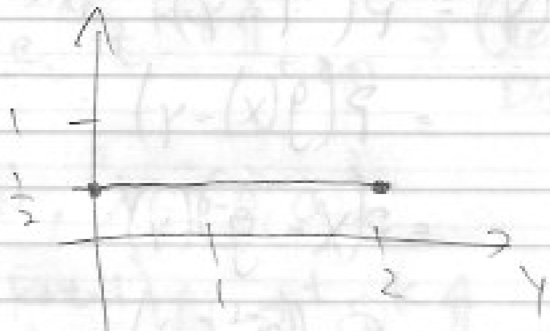
$$P_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}}(1-p)^{n-\sqrt[3]{y}}\mathbb{1}_{y\in\{0,1,8,\dots,n^3\}}$$

Transformation of Continuous Random Variables

$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ $\qquad g^{-1}(Y) = \frac{Y}{2} = x$

$X \sim (0,1) = \mathbb{1}_{x\in[0,1]}$ $\qquad Y = 2X \sim \frac{1}{2}\mathbb{1}_{y\in[0,2]}$



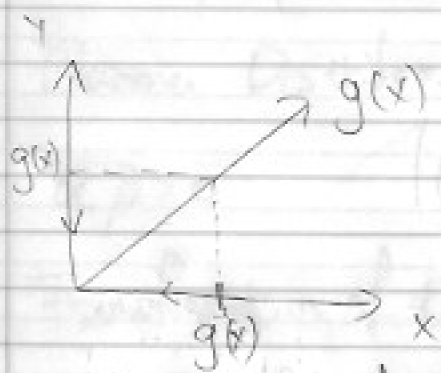


$f_Y(y) \overset{?}{=} f_X\left(\frac{y}{2}\right) = \mathbb{1}_{y\in[0,2]}$ ← Incorrect

Assume $\exists g^{-1}$

First consider g strictly increasing

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$



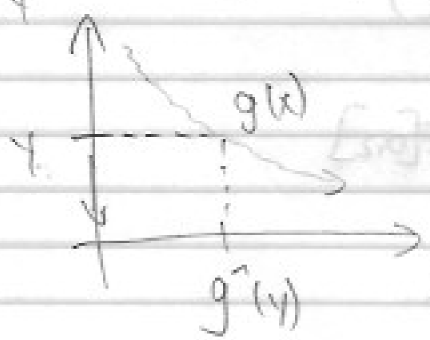
$$\Rightarrow f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right] = \frac{d}{dy}\left[F_X(g^{-1}(y))\right]$$
$$= F_X'(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$
$$= f_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$$

Assume g is strictly decreasing

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$



$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -F_X'(g^{-1}(y)) \frac{d}{dy}\left[g^{-1}(y)\right]$$

< 0

$$= F'(x)(g^{-1}(y))\left(-\frac{d}{dy}\left[g^{-1}(y)\right]\right)$$

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$$

If g 1:1, $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$

$Y = g(X) = aX + c$, linear transformation (shift and scale)

$$\rightarrow X = g^{-1}(Y) = \frac{Y - c}{a}$$

$$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|} = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}\mathbb{1}_{\frac{y-c}{a} \in Supp[X]} = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}\mathbb{1}_{y \in Supp[X]+c}$$

$$Y = -X \to a = -1,\ c = 0$$

$$\to f_Y(y) = f_X(-y)$$

$$Y = X + c \to f_Y(y) = f_X(y-c)$$

$$Y = 2X \sim f_Y\left(\frac{y}{2}\right)\frac{1}{2} = \frac{1}{2}\mathbb{1}_{y \in [0,2]}$$

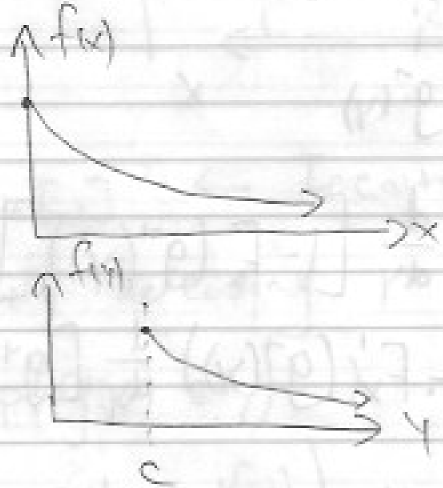
$X \sim Exp(\lambda)$

$Y \sim X + c$

$\sim \lambda e^{-\lambda(y-c)}\mathbb{1}_{y-c \in (0,\infty)}$

$= \lambda e^{-\lambda y + \lambda c}\mathbb{1}_{y \in (0,\infty)}$

$= \underbrace{e^{\lambda c}}_{\text{Scaling}}\lambda e^{-\lambda y}\mathbb{1}_{y \in (0,\infty)}$



$$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$$

$$Y = -\log(x) = g(x)$$

$$\to X = e^{-y} = g^{-1}(y)$$

$$\left|\frac{d}{dy}\left[e^{-y}\right]\right| = e^{-y}$$

$$f_Y(y) = f_X(e^{-y})e^{-y} = \mathbb{1}_{e^{-y} \in [0,1]}\, e^{-y}$$

$e^{-y} \in [0,1] \to -y \in (-\infty, 0)$

$\Downarrow$

$y \in (0,\infty)$

$$= e^{-y}\mathbb{1}_{y \in (0,\infty)} = Exp(1)$$