#10

- $Q[X, 0.1] = 2$ $\qquad$ $Q[X, 1] = 20$

$X \sim U(\{2, 4, 6, \ldots, 20\})$. $\qquad$ $Q[X, 0.9] = 18$

$Q[X, 0.85] = 18$

- Define median of r.v. X: $\text{med}(X) = Q[X, \frac{1}{2}]$
- If X is continuous & a strictly increasing CDF, then $Q[X, q] = F_X^{-1}(q)$, the "quantile function."

- $X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x} = Q \Rightarrow 1 - Q = e^{-\lambda x}$

$\ln(1-Q) = -\lambda x$ $\quad$ so $\quad x = \dfrac{\ln(1-Q)}{-\lambda}$

OR $e^{-\lambda x} = 1 - q$

$-\lambda x = \ln(1-q)$

$\lambda x = \ln\left(\frac{1}{1-q}\right)$

$x = \frac{1}{\lambda} \ln\left(\frac{1}{1-q}\right)$ which is $F_X^{-1}(z) = \frac{1}{\lambda} \ln\left(\frac{1}{1-z}\right)$

So, $\text{med}(x) = \frac{1}{\lambda} \ln(2)$

- Often times, the CDF $F(x)$ is not available in closed form,

* If it is available, often times, its mass is not available

E.g. $X \sim \text{Erlang}(k, \lambda) \Rightarrow F(x) = P(k, \lambda x)$

Computer solves $q = P(k, \lambda x)$ for x as best as you can.

- $X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)}$ $\quad$ $Y = ke^X \Rightarrow \frac{Y}{k} = e^X \Rightarrow \ln\left(\frac{Y}{k}\right) = X$

$= \ln(Y) - \ln(k) = g^{-1}(Y)$

$\frac{d}{dy}(g^{-1}(y)) = \frac{1}{y}$ $\qquad$ $f_Y(y) = \frac{1}{y} \cdot \lambda e^{-\lambda(\ln(y) - \ln(k))} = \frac{\lambda}{y} \cdot \frac{k^{\lambda}}{y^{\lambda}} = \frac{\lambda k^{\lambda}}{y^{\lambda+1}} \mathbb{1}_{y \in (k, \infty)}$

$\text{Supp}[Y] = (k, \infty)$ $\qquad$ $= \text{Pareto I}(k, \lambda)$

where

vice versa

$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}} dt = \lambda k^\lambda \left[\frac{-t^{-\lambda}}{\lambda} \right]_k^y = k^\lambda$

let $q = 1-\left(\frac{k}{y}\right)^\lambda \Rightarrow 1-q = \frac{k^\lambda}{y^\lambda} \Rightarrow y^\lambda = \frac{k^\lambda}{1-q} \Rightarrow y = \frac{k}{(1-q)^{1/\lambda}} = k(1-q)$ $= F_Y^{-1}(q)$

check

If $k=1 \Rightarrow f_Y(y) = \frac{\lambda}{y^{\lambda+1}} \mathbb{1}_{y\in(1,\infty)}$

$F_Y^{-1}(q) = (1-q)^{-1/\lambda}$

- Let $L(a)$ the proportion of land owned by all people who themselves own

$$L(a) = \frac{\int_1^a y f_Y(y)dy}{\int_1^\infty y f_Y(y)dy} = \frac{\lambda\left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]_1^a}{\lambda\left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]_1^\infty} = \frac{a^{1-\lambda}-1}{0-1} = 1-a^{1-\lambda}$$

- Set $a = F^{-1}(q)$

set $L(a) = 1-q = \bar{q}$

$\Rightarrow \bar{q} = 1-\left((1-q)^{-\frac{1}{\lambda}}\right)^{1-\lambda}$

$= 1-\left(\bar{q}^{-\frac{1}{\lambda}}\right)^{1-\lambda}$

$\Rightarrow q = \bar{q}^{\frac{-\lambda+1}{\lambda}} \Rightarrow q = \bar{q}^{1-\frac{1}{\lambda}} \Rightarrow \ln(q) = (1-\frac{1}{\lambda})\ln$

$\Rightarrow \ln(q) = \ln(\bar{q}) - \frac{1}{\lambda}\ln(\bar{q}) \Rightarrow \ln(q) = \ln(\bar{q}) = \frac{1}{\lambda}\ln(\bar{q}) \Rightarrow \frac{1}{\lambda} = \frac{\ln(\bar{q})-\ln(q)}{\ln(\bar{q})}$

$\Rightarrow \lambda = \frac{\ln(\bar{q})}{\ln(\bar{q})-\ln(q)} = \log_{\bar{q}/q}(\bar{q}) = \log_{0.25}(0.2) \approx 1.16$

pareto 80-20 principle
let $q = 0.8$ & $\bar{q} = 0.2$

$\Rightarrow Y_1 \sim$ Pareto I $(1, 1.161)$
You got the pareto principle

"Let $X_1, X_2 \overset{iid}{\sim} Exp(1) = e^{-x}$ [illegible]

$D = X_1 - X_2 = \underset{X}{\underset{\sim}{X_1}} + \underbrace{(-X_2)}_{Y} = X + Y$

$Y \sim e^{-(-y)} \mathbb{1}_{(-y) \in (0,\infty)} = e^{y} \mathbb{1}_{y \in (-\infty, 0)}$

$$f_D(d) = \int_{supp[X]} f_X(x) f_Y(d-x) \mathbb{1}_{d-x \in supp[Y]} \, dx$$

$$= \int_0^\infty (e^{-x})(e^{d-x} \mathbb{1}_{\underbrace{d-x \in (-\infty, 0)}_{\substack{x-d \in (0,\infty) \\ x \in (d,\infty)}}}) dx$$

$e^{d} e^{-2x}$

supp$[D] = \mathbb{R}$

$$= e^d \begin{cases} \int_d^\infty e^{-2x} dx & \text{if } d \geq 0 \\ \int_0^\infty e^{-2x} dx & \text{if } d < 0 \end{cases} = e^d \begin{cases} -\frac{1}{2} [e^{-2x}]_d^\infty & \text{if } d \geq 0 \\ -\frac{1}{2} [e^{-2x}]_0^\infty & \text{if } d < 0 \end{cases}$$

$$= \frac{1}{2} e^d \begin{cases} e^{-2d} & \text{if } d \geq 0 \\ 1 & \text{if } d < 0 \end{cases}$$

$$= \frac{1}{2} \begin{cases} e^{-d} & \text{if } d \geq 0 \\ e^{d} & \text{if } d < 0 \end{cases} = \frac{1}{2} e^{-|d|} = \text{Laplace}(0,1)$$

if $d \geq 0 \Rightarrow d = |d|$

$d < 0 \Rightarrow -d = |d| \Rightarrow d = -|d|$

$$L = \mu + \sigma D \sim \frac{1}{2\sigma} e^{-\frac{|\ell - \mu|}{\sigma}} = \text{Laplace}(\mu, \sigma)$$

where $\mu \in \mathbb{R}$

$\sigma > 0$