Lecture #6

multinom$(2n, p)$
$(n, \vec{p})$

$\boxed{\text{prob}}$

- Uniform discrete :

$$X_1 \,\&\, X_2 \overset{iid}{\sim} U(\{0,1,2,3\}) := \begin{cases} 0 & \text{w.p } \frac{1}{4} \\ 1 & \text{w.p } \frac{1}{4} \\ 2 & \text{w.p } \frac{1}{4} \\ 3 & \text{w.p } \frac{1}{4} \end{cases} = \frac{1}{4} \mathbb{1}_{v \in \{0,1,2,3\}}$$

Generally, $X \sim U(A) := \frac{1}{|A|} \mathbb{1}_{v \in A}$

$\text{supp}[X] = A$

Parameter Space: $A \subset \mathbb{R}$ s.t. A is finite.

- $P_{X_1,X_2}(x_1,x_2) = \frac{1}{16} \mathbb{1}_{x_1 \in \{0,1,2,3\}} \mathbb{1}_{x_2 \in \{0,1,2,3\}}$
- $T = X_1 + X_2$?

$$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2) \mathbb{1}_{x_2 = t - x_1}$$

$p(1) = \sum\sum p \, \mathbb{1}_{x_2 = 1 - x_1} = 2\left(\frac{1}{16}\right)$ $\longrightarrow$ pic taken

$p(0.5) = 0$

$p(3) = 4 \cdot \left(\frac{1}{16}\right)$

$p(6) = \frac{1}{16}$

$\longrightarrow$ how are they distributed?

- let $Y = \underbrace{-X}_{g(X)} \sim P_Y(y)$

$X \sim U(0,1,2,3)$

$X=0 \Rightarrow Y=0$ w.p $1/4$
$X=1 \quad Y=-1 \quad 1/4$
$X=2 \quad Y=-2 \quad 1/4$
$X=3 \quad Y=-3 \quad 1/4$

$\Rightarrow Y \sim U(\{0,-1,-2,-3\})$

It's a rule.

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

Conditional Probability

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$p_{X_1|T}(x,t) = P(X_1 = x_1 | T = t) \neq \text{poisson}(\lambda)$

$$p_{X_1|T}(x,t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} = \frac{p_{X_1}(x)\, p_{X_2}(t-x)}{p_2(t)}$$

$$= \frac{\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}\right)}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} = \frac{\lambda^t}{(2\lambda)^t}\binom{t}{x} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

$$= \text{Bin}(t, \tfrac{1}{2}).$$

Negative transformation will be included in the midterm.

$X_1 \,\&\, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} = \sum_{x \in \text{supp}[X]} p_X(x)\, p_Y(d-x)\, \mathbb{1}_{d-x \in \text{supp}[Y]}$

$\text{supp}[D] = \mathbb{Z}$ (integers)

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} = p_X(-y)$$

$$= \sum_{x \in \{0,1,2\cdots\}} \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!} \cdot \mathbb{1}_{d-x \in \{0,-1,-2,\cdots\}}$$

$$= e^{-2\lambda}\sum_{x \in \{0,1,2\cdots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x \geq d} = \begin{cases} e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d<0 \\ e^{-2\lambda}\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \geq 0 \end{cases}$$

$\Rightarrow$ pic taken

After that,

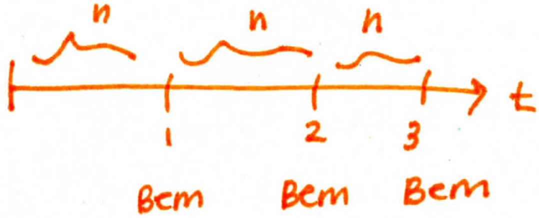
$$\Rightarrow e^{-2\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}} = e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda)$$

modified bessel function of the first kind

midterm 2 material:

Let $X_1 \sim Geom(p) = \underbrace{(1-p)^x p \, \mathbb{1}_{x\in\{0,1,2\ldots\}}}_{p(x)}$

$\boxed{\text{CDF}}$ $$F(x) = P(X \le x) = 1 - P(X > x) = 1 - P(X \ge x+1)$$
$$= \left(1-(1-p)^{x+1}\right)\mathbb{1}_{x\ge 0}$$



We run n trials within each "original" period.
X_n is the # 0's until a success using the new trial def.

$$\text{so, } X_n \sim \underbrace{(1-p)^{nx}}_{} p \, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\ldots\}}$$

Every time I take a minute, there are n in each side

Let $n\to\infty$, $p\to 0$, but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

Similar to the derivation of the poisson from the binomial.

$$F_{X_n}(x) = 1-(1-p)^{nx+1}$$

• $P_{X_\infty}(x) = \lim\limits_{x\to\infty} P_{X_n}(x) = \underbrace{\left(\lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda}} \underbrace{\lim\limits_{n\to\infty}\frac{\lambda}{n}}_{0} \underbrace{\lim\limits_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\ldots\}}}_{\mathbb{1}_{x\in[0,\infty)}} = 0$

$F_{X_\infty}(x) = \lim\limits_{n\to\infty} F_{X_n}(x) = 1-\left(\lim\left(1-\frac{\lambda}{n}\right)^n\right)^x \underbrace{\left(\lim 1-\frac{\lambda}{n}\right)}_{1} \mathbb{1}_{x\ge 0}$

$$= \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\ge 0}$$

$Supp[X_\infty] = [0,\infty)$

$|Supp[X_\infty]| = |\mathbb{R}| \Rightarrow X_\infty$ is a continuous r.v

Has <u>no</u> PMF

CDF is valid

I $\lim\limits_{x\to-\infty} F(x) = 0$

II $\lim\limits_{x\to\infty} F(x) = 1$

III $F(x)$ montonically increasing

$F'(x) = \lambda e^{-\lambda x} > 0$