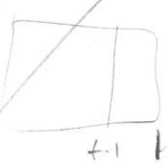
Lec 8 Math 621 10/3/19

$$f_T(t) = \int_{\underbrace{Supp(X)}_{(0,1)}} \underbrace{f(x)}_{1} \underbrace{f(t-x)}_{1} \underbrace{1\!\!1_{t-x \in \underbrace{Supp(X)}_{(0,1)}}}_{1\!\!1_{x-t \in [-1,0]}} dx$$

$$= \int_0^1 1\!\!1_{x \in [t-1,t]} dx = \begin{cases} 0 \text{ if } t<0 \\ t \text{ if } t \in [0,1) \\ 1-(t-1) = 2-t \text{ if } t \in [1,2) \\ 0 \text{ if } t \geq 2 \end{cases}$$



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$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda)$ $\quad T_2 = X_1 + X_2 \sim f_T(t) = ?$ $\quad Supp(T) = (0,\infty)$

$$f_{T_2}(t) = \int_0^\infty \left(\lambda e^{-\lambda x}\right) \lambda e^{-\lambda(t-x)} 1\!\!1_{t-x \in (0,\infty)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty 1\!\!1_{x \leq t} dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t}$$

$T = X_1 + X_2 + X_3 \sim ? = T_2 + X_3$

$$f_{T_3}(t) = \int_0^\infty \left(x \lambda^2 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} 1\!\!1_{x \leq t}\right) dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x\,dx = \frac{1}{2} t^2 \lambda^3 e^{-\lambda t}$$

$$T = X_1 + X_2 + X_3 + X_4 = T_3 + X_4$$

$$f_{T_4}(t) = \int_0^\infty \left(\frac{1}{2} x^2 \lambda^3 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\,dx$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$$

$$f_{T_5}(t) = \int_0^\infty \left(\frac{1}{2 \cdot 3} x^3 \lambda^4 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2 \cdot 3} \lambda^5 e^{-\lambda t} \int_0^t x^3\,dx$$

$$= \frac{1}{2 \cdot 3 \cdot 4} t^4 \lambda^5 e^{-\lambda t}$$

$$T = X_1 + \dots + X_k \sim \text{Erlang}(k, \lambda) := \frac{1}{(k-1)!} t^{k-1} \lambda^k e^{-\lambda t}$$

Param Space $k \in \mathbb{N}$, $\underbrace{\lambda \in (0, \infty)}_{\text{same as before}}$

$$\text{Erlang}(1, \lambda) = \text{Exp}(\lambda)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\text{NegBin}(1, p) = \text{Geom}(p)$$

Let's do some math definitions

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt = \underbrace{\int_0^a t^{x-1} e^{-t} dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} dt}_{\Gamma(x,a)}$$

"Gamma Function"

$\gamma(x,a)$ "lower incomplete gamma function"

$\Gamma(x,a)$ "upper incomplete gamma function"

$P(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1]$ % of $\Gamma(x)$ less than a

lower regularized gamma funct

$Q(x,a) := \Gamma(x,a) / \Gamma(x) \in [0,1]$ % of $\Gamma(x)$ above a

upper regularized gamma function

$\Rightarrow P(x,a) + Q(x,a) = 1$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -\left(e^{-t}\right)_0^\infty = -(0-1) = 1$$

$$\Gamma(x+1) \overset{thm}{=} x\,\Gamma(x)$$

$\Rightarrow \Gamma(2) = 1 \cdot 1, \quad \Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1, \quad \Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1, \quad \Gamma(5) = 4\,\Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1$

$\Rightarrow \Gamma(n+1) = n!$ or $\Gamma(n) = (n-1)!$

This is an "extension" of the factorial to all $x \in (0,\infty)$

$$X \sim \text{Erlang}(k,\lambda) := \overbrace{\frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}}^{f(x)} \mathbb{1}_{x \geq 0}$$

$$F(x) = P(X \leq x) = \int_0^x f(y)\,dy = \int_0^x \frac{\lambda^k e^{-\lambda y} y^{k-1}}{(k-1)!} dy = \frac{\lambda^k}{(k-1)!} \int_0^x e^{-\lambda y} y^{k-1} dy$$

see next page

$$= \frac{\cancel{\lambda^k}}{(k-1)!} \frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = \frac{\gamma(k,\lambda x)}{\Gamma(k)} = P(k,\lambda x)$$

$$\Rightarrow P(X > x) = 1 - F(x) = 1 - P(k,\lambda x) = Q(k,\lambda x)$$

Two useful integrals related to the gamma functions:

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c}du$

$$\int_0^\infty t^{x-1} e^{-ct}\,dt = \int_0^\infty \left(\frac{u}{c}\right)^{x-1} e^{-u}\frac{1}{c}\,\cancel{dt} = \frac{1}{c^x}\int_0^\infty u^{x-1}e^{-u}\,du = \frac{\Gamma(x)}{c^x}$$

$$\int_0^a t^{x-1} e^{-ct}\,dt = \int_0^{ac} \left(\frac{u}{c}\right)^{x-1} e^{-u}\frac{1}{c}\,dt = \frac{1}{c^x}\int_0^{ac} u^{x-1}e^{-u}\,du = \frac{\gamma(x, ac)}{c^x}$$

follows from the previous two

$$= \int_a^\infty t^{x-1}e^{-ct}\,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$$

$$\lim_{a\to\infty} \gamma(x,a) = \Gamma(x) \Rightarrow \lim_{a\to\infty} P(x,a) = 1$$

$$\lim_{a\to 0} \Gamma(x,a) = \Gamma(x) \Rightarrow \lim_{a\to 0} Q(x,a) = 1$$

Identity if $n \in \mathbb{N}$

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u}\,\underbrace{e^{-t}\,dt}_{dv} = \Big[uv\Big]_a^\infty - \int_a^\infty v\,du$$

$du = (n-1)t^{n-2}dt$, $v = -e^{-t}$

$$= \Big[-t^{n-1}e^{-t}\Big]_a^\infty - \int_a^\infty (-e^{-t})(n-1)t^{n-2}\,dt$$

$$= \underbrace{\Big[t^{n-1}e^{-t}\Big]_\infty^a} + (n-1)\underbrace{\int_a^\infty e^{-t}t^{n-2}\,dt}$$

$$= a^{n-1}e^{-a} + (n-1)\,\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\,a^{n-2}e^{-a} + (n-2)\,\Gamma(n-2,a)$$

$$= e^{-a}\left(a^{n-1} + (n-1)\,a^{n-2} + (n-2)\,a^{n-3} + (n-3)\,a^{n-4} + \dots + (1)\frac{\Gamma(1,a)}{e^{-a}}\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \dots + \frac{a^1}{1!} + \frac{a^0}{0!}\right)$$

$$e^{-a} = \int_a^\infty e^{-t}\,dt =$$

$$= e^{-a}(n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!} \quad \Rightarrow \quad \Gamma(n+1,a) = e^{-a}\, n! \sum_{i=0}^{n} \frac{a^i}{i!}$$

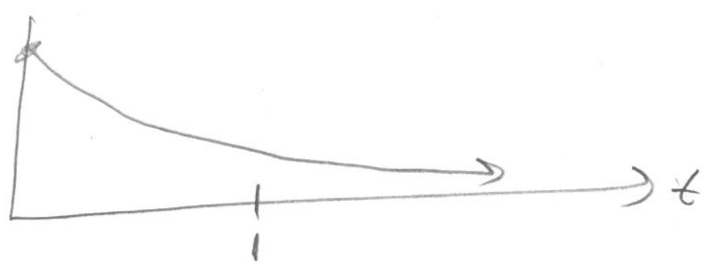
$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}$$

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{x} \frac{\lambda^i}{i!} = \frac{\Gamma(x+1,\lambda)}{x!} = \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

The CDF of the Poisson seems very similar to the CDF-complement of the Erlang! Let's see...

Let the rate of errors be λ. What is prob of 0 errors before 1 second?

Let $T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(\overset{k}{\overset{\shortparallel}{1}},\lambda)$ in seconds, $N \sim \text{Poisson}(\lambda)$



$$P(T_1 > 1) = 1 - F_{T_1}(1) = \overset{e^{-\lambda}}{\overset{\shortparallel}{Q(1,\lambda)}} = F_N(0)$$

What is prob of at most 1 error before 1 second?

Let $T_2 \sim \text{Erlang}(2,\lambda)$

$$P(T_2 > 1) = 1 - F_{T_2}(1) = \overset{e^{-\lambda}(1+\lambda)}{\overset{\shortparallel}{Q(2,\lambda)}} = F_N(1)$$

What is prob of at most 2 events before 1 second?

Let $T_3 \sim \text{Erlang}(3, \lambda)$

$$P(T_3 > 1) = Q(3, \lambda) = F_N(2)$$

$\vdots$

What is prob of at most k events before 1 second?

Let $T_k \sim \text{Erlang}(k, \lambda)$

$$P(T_k > 1) = Q(k, \lambda) = F_N(k-1)$$

"Poisson process". If there is an exponential waiting process, then the # of events that happen per unit time is Poisson-distr. $T_1, T_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$n = 5$
from Poisson(λ)