$$X_1,\dots,X_n \overset{i.i.d.}{\sim} f(x) \quad \text{with CDF } F(x)$$

CDF of kth r.v. if put r.v.'s in order

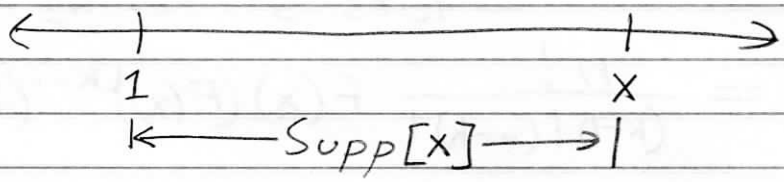
$$F_{X_{(k)}}(x) = \underbrace{\sum_{j=k}^{n} \binom{n}{j}(F(x))^j(1-F(x))^{n-j}}_{\text{this is compliment of CDF of binomial}}$$

$\binom{n}{j}(F(x))^j(1-F(x))^{n-j}$ is PMF for binomial

ex: $n=10$

$F_{X_{(7)}}(x)$

let $L \sim Bin(n, p=F(x))$



this is how many X's land $\leq x$ out of n

want PDF

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n}\binom{n}{j}(F(x))^j(1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n}\binom{n}{j}\frac{d}{dx}\left[(F(x))^j(1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n}\binom{n}{j}\left[(F(x))^j(-f(x))(n-j(1-F(x))^{n-j+1} + (1-F(x))^{n-j}f(x)(F(x))^{j-1}\right]$$

$$= \sum_{j=k}^{n}\binom{n}{j} f(x)\left(j\,F(x)^{j-1}(1-F(x))^{n-j} - (n-j)(F(x))^j(1-F(x))^{n-j-1}\right)$$

$$= f(x)\left(\sum \frac{n!}{j!(n-j)!} j\,(F(x))^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n} \frac{n!}{j!(n-j)!}(n-j)(F(x))^{j}(1-F(x))^{n-(j+1)} \right)$$

last sum is same as

$$= \sum_{j=k}^{n-1} \frac{n!}{j!(n-(j+1))!}(F(x))^{j}(1-F(x))^{n-(j+1)}$$

let $l = j+1, \Rightarrow j = l-1$

$$\sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!}(F(x))^{l-1}(1-F(x))^{n-l}$$

lots of terms cancel

PDF

$$f_{X(k)} = \frac{n!}{(k-1)!(n-k)!} f(x)(F(x))^{k-1}(1-F(x))^{n-k}$$

---

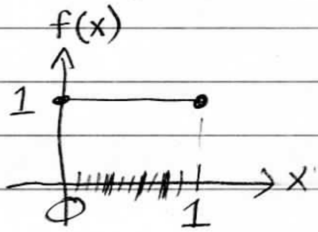
for the min:

$$f_{X_{(1)}}(x) = nf(x)(1-F(x))^{n-1}$$

for the max:

$$f_{X_{(n)}}(x) = nf(x)\,F(x)^{n-1}$$

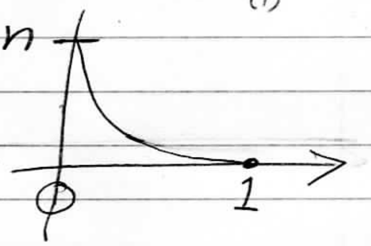
Given: $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \overset{\text{Uniform}}{U(0,1)}$ PDF $f(x) = 1$ for $0<x<1$
CDF $F(x) = x$ for $0<x<1$



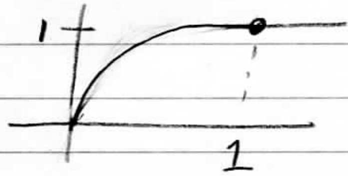
order statistics have the same support: $(0,1)$

get the min using the formula $f_{X_{(1)}}(x) = nf(x)\left(1-F(x)\right)^{n-1}$

PDF $f_{X_{(1)}}(x) = n(1-x)^{n-1}$



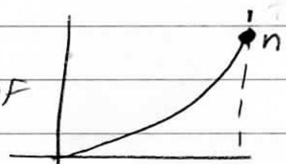
CDF $F_{X_{(1)}}(x) = 1-(1-x)^{n-1}$



get the max using the formula $f_{X_{(n)}} = nf(x)\left(F(x)\right)^{n-1}$

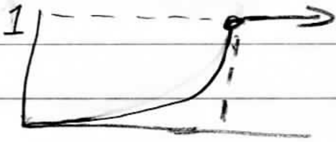
PDF $f_{X_{(n)}}(x) = nx^{n-1}$

PDF

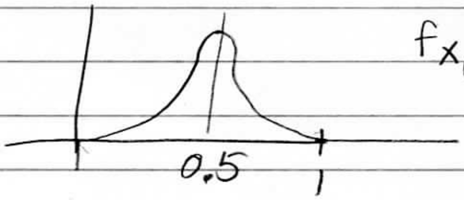


CDF $F_{X_{(n)}}(x) = x^n$

CDF



median $k \approx \frac{1}{2}n$



$$f_{X_{(k)}} = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in(0,1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in(0,1)}$$

$$= \mathrm{Beta}(k, n-k-1)$$

(which is generalization of Exponential)

notice that the order statistics for set of n i.i.d. r.v.'s that are $U(0,1)$ is Beta

$$X_{(K)} \sim Beta(K, n-k+1)$$

$$X_{(1)} \sim Beta(1, n)$$

$$X_{(n)} \sim Beta(n, 1)$$

We can decompose $p(x) = c\,k(x) \propto k(x)$

"proportional to"

or

$f(x) = c\,k(x) \propto k(x)$

not a function of X (normalization constant)

is a function of X (kernal)

for a probability dist.

$$\sum_{Supp[X]} p(x) = 1 \Rightarrow \sum c\,k(x) = 1 \Rightarrow c = \frac{1}{\sum_{Supp[X]} k(x)}$$

also,

$$\int_{Supp[X]} f(x)\,dx = 1 \Rightarrow \int c\,f(x)\,dx = 1 \Rightarrow c = \frac{1}{\int_{Supp[X]} f(x)\,dx}$$

— Let's find the "kernal"

$$X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$n!\,(1-p)^n \underbrace{\frac{1}{x!\,(n-x)!}\left(\frac{p}{1-p}\right)^x}_{\text{kernal}}$$

$$X \sim Weibull(k,\lambda) = (k\lambda)(\lambda x)^{k-1} e^{-(\lambda x)^k}$$
$$= \underbrace{k\lambda^k}_{c}\ \underbrace{x^{n-1} e^{-(\lambda x)^k}}_{\substack{k(x) \\ \text{kernal}}}$$

$$X \sim \text{Gamma}(\alpha, \beta) = \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{c} \underbrace{X^{\alpha-1} e^{\beta X}}_{\substack{k(x) \\ \text{kernal}}}$$

so $X^{\alpha-1} e^{\beta X} \propto \text{Gamma}(\alpha, \beta)$

proportional to

$X \sim Gamma(\alpha_1, \beta)$ $\quad Y \sim Gamma(\alpha_2, \beta)$
and X, Y are independent

conjecture: $T = X + Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$

Proof:

$$T = X + Y \sim \int_{Supp[X]} f_X(x) f_Y(t-x) \mathbb{1}_{Supp[Y]} \, dx$$

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{\beta x}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} e^{\beta(t-x)}\right)\mathbb{1}$$

$$e^{\beta(t-x)} = e^{\beta t} e^{-\beta x}$$

$t - x \in (0, \infty)$
same as $x \leq t$
so change integral to end at t

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} \, dx$$

let $u = \frac{x}{t} \Rightarrow x = ut$ $\quad x = 0 \Rightarrow u = 0$
$\quad x = t \Rightarrow u = 1$

$$\frac{du}{dx} = \frac{1}{t}$$

$$dx = t\,du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\beta t} \int_0^1 (ut)^{\alpha_1 - 1} (t - ut)^{\alpha_2 - 1} t \, du$$

factor out $t^{\alpha_2 - 1}$

$$= \left(\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du\right) t^{\alpha_1 + \alpha_2 - 1} e^{\beta x}$$

$$= \left(\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \right) \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} dx \Big) t^{\alpha_1 + \alpha_2 - 1} e^{\beta x}$$

$$\underset{\substack{\uparrow \\ \text{proportional} \\ \text{to}}}{\propto} t^{\alpha_1 + \alpha_2 - 1} e^{\beta t} \propto \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

this means that

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} = \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} du$$

$$\Rightarrow \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

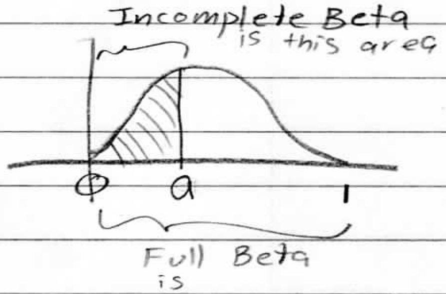
This is called the <u>Beta function</u>

Beta Function

$$\mathcal{B}(\alpha_1, \alpha_2) := \int_0^1 u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

There is no closed form solution for this

(Lower)
Incomplete Beta function:



$$\mathcal{B}(a, \alpha_1, \alpha_2) := \int_0^a u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1} du$$

Regularized Beta function

$$I_a(\alpha_1, \alpha_2) = \frac{\mathcal{B}(a, \alpha_1, \alpha_2)}{\mathcal{B}(\alpha_1, \alpha_2)}$$

represents the proportion of full area for Beta covered up to a

$$X \sim \text{Beta}(\alpha,\beta) \overset{\text{PDF}}{:=} \frac{1}{\mathcal{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in(0,1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in(0,1)}$$

$$f(x) = \frac{(\alpha+\beta+1)!}{(\alpha-1)!(\beta-1)!} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in(0,1)}$$

Find CDF

$$F(x) = \int_0^x \frac{1}{\mathcal{B}(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt}{\mathcal{B}(\alpha,\beta)}$$

$$= \frac{\mathcal{B}(x,\alpha,\beta)}{\mathcal{B}(\alpha,\beta)} = I_x(\alpha,\beta)$$

CDF is regularized beta (up to x)