Lecture 3

Proof by induction HW

$X_1 \ldots X_r \overset{iid}{\sim} \text{Geom}(p)$

$T = \sum_{i=1}^{r} X_i \sim \text{Neg Bin}(r, p) = \binom{t+r-1}{r-1} (1-p)^t p^r$ (t 0's, r 1's) → pmf

$\underset{1}{\_}\ \underset{2}{1}\ \underset{3}{1}\ \ldots\ 0\ 1\ \_\ \_\ 1$

t - Scores

$X \sim \text{Bin}(n, p) := \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$

Let n be large and p be small. S.t n, p are revelant via. | $P_{\lambda}(x)$ $\lim_{n\to\infty} p(x) \Rightarrow$

$\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$P_{\lambda}(x) = \lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)\left(1-\frac{\lambda}{n}\right)^{-x} = \frac{\lambda^x}{x!}$

factor out $\Rightarrow \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!} \frac{1}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \overset{\text{dist lim}}{=} \frac{\lambda^x}{x!} \lim \frac{n}{n} \lim \frac{n-1}{n} \ldots$

by fac def $\frac{\overbrace{n(n-1)(n-2)\ldots(n-x+1)}^{x = 4 \text{ terms}}}{n(n)(n)\ldots(n)}$

$\ldots \lim \frac{n-x+1}{n} \cdot \lim \left(1-\frac{\lambda}{n}\right)^n \cdot \lim \left(1-\frac{\lambda}{n}\right)^{-x} = \boxed{\frac{\lambda^x e^{-\lambda}}{x!}} = \text{Poisson}(\lambda)$

THM $\to e^{-\lambda}$

Recall
Simulation formula $P(t) = \sum_{x \in Supp[X]} P_{LO}(x) P_{LO}(t-x) \mathbb{1}_{t-x \in Supp[X]}$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$

$$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\dots\}} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!}\right) \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x \le t} = \lambda^t e^{-2\lambda} \sum_{x=0}^{t} \underbrace{\frac{1}{x!(t-x)!} \cdot \frac{t!}{t!}}_{\binom{t}{x}}$$

$$= \frac{\lambda^t e^{-\lambda}}{\lambda!} \underbrace{\sum_{x=0}^{t} \binom{t}{x}}_{2^t} = \boxed{\frac{(2\lambda)^t e^{-2\lambda}}{t!} = Poisson(2\lambda)}$$

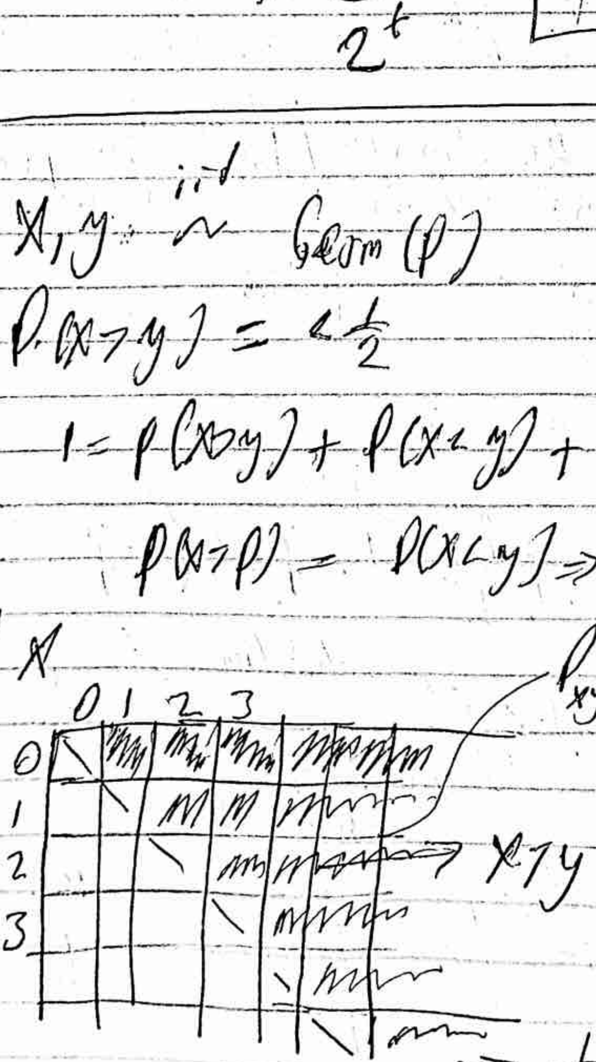
Ex:
$X, Y \overset{iid}{\sim} Geom(p)$ $\qquad E[\mathbb{1}_{X>Y}] = P(X>Y)$

$P(X>Y) = < \frac{1}{2}$

$1 = P(X>Y) + P(X<Y) + P(X=Y) = 2P(X>Y) + \underbrace{P(X=Y)}_{>0}$

$P(X>Y) = P(X<Y) \Rightarrow$ b/c they are arbitrary

Idea



$\Rightarrow \frac{1}{1-(1-p)^2}$ --- Increasing

Recall Geometric Series

$p_X(x) \underset{\shortparallel}{\ } p_Y(x)$ for $q \in (0,1) = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q}$

General Idea

$$P(X>Y) = \sum_{y \in \mathbb{R}} \sum_{x \in \mathbb{R}} p_{X,Y}(x,y) \mathbb{1}_{x>y} = \sum_{y<\mathbb{R}} \sum_{x \in (y,\infty)} \left((1-p)^x p\right) \mathbb{1}_{x \in \{0,1 \ldots\}} \Rightarrow$$

$$\left((1-p)^y p \, \mathbb{1}_{y \in \{0,1,\ldots\}}\right) = p^2 \sum_{y \in \{0,1..\}}^{\infty} (1-p)^y \sum_{x \in \{y+1, y+2 ..\}} (1-p)^x$$

let's re-index $x' = x-(y+1) \Rightarrow x = x' + y + 1$

$$\Rightarrow p^2 \sum_{y \in \{0,1..\}} (1-p)^y \sum_{x' \in \{0,1,\ldots\}} (1-p)^{x'+y+1} \Rightarrow (1-p)^{x'} (1-p)^y (1-p)$$

$$\Rightarrow p^2 (1-p) \sum_{y \in \{0,1,..\}} (1-p)^{2y} \sum_{x' \in \{0,1,..\}} (1-p)^{x'} = p(1-p) \sum_{y \in \{0,1..\}} (1-p)^{2y}$$

$$\underset{\frac{1}{1-(1-p)}}{\shortparallel} \Rightarrow \text{Geometric Formula}$$

Recall $q^{bc} = (q^b)^c \neq q^{(b^c)}$

Rewrite

$$\Rightarrow p(1-p) \sum_{y \in \{0,1..\}} \left((1-p)^2\right)^y$$

$$\Rightarrow \frac{1}{1-(1-p)^2} \Rightarrow \frac{1}{1-(1-2p+p^2)} \Rightarrow \frac{1}{2p-p^2} \Rightarrow \frac{1}{p(2-p)}$$

$$\Rightarrow \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \boxed{\frac{1-p}{2-p}} \quad p \Rightarrow 0$$

Note $\sum_{x\in\mathbb{R}}$ Can also be Represented as $\sum_{n=1}^{\infty}$

Expectation:

X, Y discrete $E[X] = \sum_{x\in\mathbb{R}} x\,p(x)$

Prob weighted avg.

$E[g(x)] = \sum_{x\in\mathbb{R}} g(x)\,p(x)$

$E[g(x,y)] = \sum_{x\in\mathbb{R}}\sum_{y\in\mathbb{R}} g(x,y)\,p(x,y) \Rightarrow$

Consider $E[\mathbb{1}_{x\in A}] \Rightarrow$

$\mathbb{1}_{x\in A} = g(x)$

$$\boxed{\Rightarrow \sum_{x\in\mathbb{R}} \mathbb{1}_{x\in A}\, p(x) = \sum_{x\in A} p(x) = P(X\in A)}$$

Multinomial Coefficient.

#6 R.V

P_1 = Prob of Apple on one Draw from bag

P_2 = " " banana " "

$\Rightarrow P_1+P_2=1$

Draw n fruit w/ replacement

X_1 = # Apple

X_2 = # Bananas

binomial

$X_1 \sim Bin(n, P_1) \Rightarrow \binom{n}{x_1} p_1^{x_1}(1-p)^{n-x_1}$

$X_2 \sim Bin(n, P_2) = \binom{n}{x_1 x_2} p^{x_1} p^{x_2}$

Note $X_1+X_2 = n$

Let $\vec{X} := \begin{bmatrix}X_1\\X_2\end{bmatrix} \sim P_{X_1,X_2}(x_1,x_2) = \underbrace{p_1^{x_1}}_{\text{Apple}} \underbrace{p_2^{x_2}}_{\text{Bananas}} \frac{n!}{x_1!\,x_2!}\, \mathbb{1}_{x_1+x_2=n}\, \mathbb{1}_{x_1\in\{0,1,..,n\}}\, \mathbb{1}_{x_2\in\{0,1,..,n\}}$

$\binom{n}{x_1} = \frac{n!}{x_1!\,(n-x_1)!}$

$n = 8$ fruits

$x_1 = 5$ Apple

$x_2 = 3$ Bananas

Next Class Exam

$$= \frac{n!}{x_1! \, x_2!}$$

Let us define multinomial coefficient.

$$\binom{n}{x_1, x_2} := \frac{n!}{x_1! \, x_2!} \quad \Uparrow_{x_1 + x_2 = n} \quad \Uparrow_{x_1 \in \{0, 1 \ldots n\}} \quad \Uparrow_{x_2 \in \{0, 1, \ldots n\}}$$

Rewriting the PMF we get

$$= \binom{n}{x_1, x_2} p^{x_1} p^{x_2} = \text{Multinomial}(n, \vec{p})$$