Lecture 18

$$Z_1 := \frac{X_1 - \mu}{\sigma}, \ldots, Z_n := \frac{X_n - \mu}{\sigma} \overset{iid}{\sim} N(0,1)$$

$$\underbrace{\vec{Z}^T \vec{Z} = \sum z_i^2}_{\sim \chi^2_n} = \sum \left(\frac{X_i - \mu}{\sigma} \right)$$

$$= \frac{(n-1) S_n^2}{\sigma} + \underbrace{n \frac{(\bar{X}_n - \mu)^2}{\sigma}}_{= \left(\frac{\bar{X} - \mu}{\frac{\sigma^2}{n}}\right)^2} \sim \chi^2_n$$

Recall

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $\qquad = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \right)^2 \sim \chi^2_1$

$\rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

$$U_1 \sim \chi^2_{K_1} \text{ ind of } U_2 \sim \chi^2_{K_2}$$

$$\rightarrow U_1 + U_2 \sim \chi^2_{K_1 + K_2}$$

Conjecture

1) $\frac{(n-1) S^2}{\sigma^2}$ and $\frac{n(\bar{X} - \mu)^2}{\sigma^2}$ are ind $\longleftrightarrow S^2, \bar{X}^2$ ind

2) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Need to prove this. ^

$\vec{Z}^T \vec{Z} \sim \chi^2_n$

$= \underbrace{\vec{Z}^T I_n \vec{Z}}_{\text{quadratic}} \longrightarrow Z_1^2 + \ldots + Z_n^2$

$= Z^T B_1 \vec{Z}_1 + \vec{Z}^T B_2 \vec{Z} + \ldots + \vec{Z}^T B_n \vec{Z}$

Consider $\vec{Z}^T \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & \ddots & & & \\ 0 & & & & \\ & & & & 0 \end{bmatrix} \vec{Z} \longrightarrow$ only get back $Z_1^2 \sim \chi^2_1$

$\vec{Z}^T \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} \vec{Z} = Z_2^2 \sim \chi^2_1$

$\vec{Z}^T \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \vec{Z} = Z_n^2 \sim \chi^2_1$

$= \vec{Z}^T (B_1 \vec{Z} + B_2 \vec{Z} + \ldots + B_n \vec{Z})$

$= \vec{Z}^T (\underbrace{B_1 + B_2 + \ldots + B_n}_{I_n}) \vec{Z}$

Note: $B_1 + B_2 + \ldots + B_n = I_n$

$\text{Rank}[B_1] = \ldots = \text{Rank}[B_n] = 1$

$A\vec{x} \in \{$space of domain $\text{Rank}[A]\}$ $\quad \sum \text{Rank}[B_j] = n$

Cochran's Theorem (1934)

Let $z_1, \ldots, z_n \underset{\sim}{iid} N(0,1)$

a) $B_1 + \ldots + B_k = I_n$

b) $\sum_{j=1}^{k} rank[B_j] = n$

then

a) $\vec{z}^T B_j \vec{z} \sim \chi^2_{rank[B_j]}$

b) $\vec{z}^T B_{j_1} \vec{z}$ ind of $\vec{z}^T B_{j_2} \vec{z}$

$\forall \; j_1 \neq j_2$

$$Z^T I \vec{Z} = \sum z_i^2 = \sum ((Z_i - \bar{Z}) + (\bar{Z}))^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2\underbrace{(Z_i - \bar{Z})\bar{Z}}_{Z_i\bar{Z} - \bar{Z}^2} + \bar{Z}^2$$

$$= \sum (z_i - \bar{z})^2 + 2\cancel{(n\bar{Z}^2 - n\bar{Z}^2)}^{=0} + n\bar{Z}^2$$

Define: $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ dimension of n

$$\bar{Z} = \frac{1}{n}(Z_1 + \ldots + Z_n) = \frac{1}{n}\vec{1}^T \vec{Z} = \frac{1}{n}\vec{Z}^T \vec{1}$$

$$= n\bar{Z}^2 = n\left(\tfrac{1}{n}\vec{Z}^T \vec{1}\right)\left(\tfrac{1}{n}\vec{1}^T \vec{Z}\right)$$

$$= \vec{z}^T \left(\frac{1}{n} J_n\right) \vec{z}$$

$$\rightarrow B_2 = \frac{1}{n} J_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots \\ & \ddots & \\ & & \frac{1}{n} \end{bmatrix} \quad \text{rank}[B_2] = 1$$

$$\left(\frac{1}{n} J_n\right) \vec{x} = \begin{bmatrix} \bar{X} \\ \bar{X} \\ \vdots \\ \bar{X} \end{bmatrix}$$

$$\sum (z_i - \bar{z})^2 = \sum z_i^2 - 2 z_i \bar{z} + \bar{z}^2$$

$$= \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2 = \underbrace{\sum z_i^2 - n\bar{z}^2}$$

$$= \vec{z}^T I_n \vec{z} - \vec{z}^T \left(\frac{1}{n} J_n\right) \vec{z}$$

$$= \vec{z}^T \underbrace{\left(I_n - \frac{1}{n} J_n\right)}_{B_1} \vec{z}$$

$$B_1 + B_2 = I \qquad B_1 = \begin{bmatrix} 1-\frac{1}{n} & & & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & & \\ \vdots & & \ddots & \\ -\frac{1}{n} & & & 1-\frac{1}{n} \end{bmatrix}$$

* <u>Thrm</u>

If A is symmetric and idempotent, i.e $AA = A$, then $\text{rank}[A] = \text{trace}[A] = \sum_{i=1}^{n} A_{ii}$

Trace: sum of entries on main diagonal

Rank$[B_1]$

$(I_n - \frac{1}{n} J_n)^T = I_n^T - \frac{1}{n} J_1^T = I - \frac{1}{n} J$ ✓

$(I_n - \frac{1}{n} J_n)(I_n - \frac{1}{n} J_n) = I_n I_n - \frac{1}{n} J_n I_n + \frac{1}{n^2} J_n J_n$

$= I_n - 2(\frac{1}{n} J_n) + \frac{1}{n^2} n J_n$

$= I_n - \frac{1}{n} J_n$ (idempotent) ✓

$\sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$

ind of $n\bar{Z}^2 \sim \chi^2_1$

$$n\bar{Z}^2 = n\left(\frac{Z_1 + \dots + Z_n}{n}\right) = n\left(\frac{\frac{X_1 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}}{n}\right)$$

$$= n\left(\frac{X_1 + \dots + X_n - n\mu}{n\sigma}\right)^2$$

$$= n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 = \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_1$$

$$\sum (Z_i - \bar{Z})^2 = \sum \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma}\right)^2$$

$$= \sum \frac{(X_i - \bar{X})^2}{\sigma} = \frac{(n-1)S^2}{\sigma^2}$$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}, \quad Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$$

$$E[\vec{Z}] = \vec{0}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Var[\vec{Z}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & \vdots \\ & & \ddots & \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_n$$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = f_{Z_1, \ldots, Z_n}(z_1, \ldots, z_n)$$

$$= f(z_1) \cdot f(z_2) \cdot \ldots \cdot f(z_n)$$

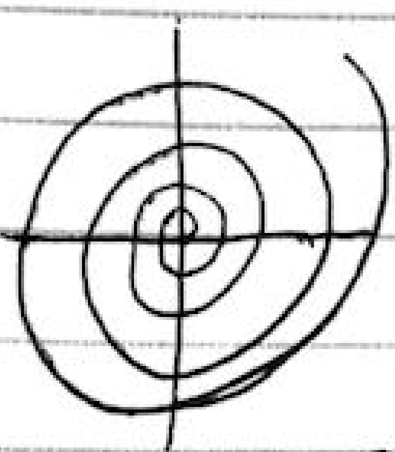
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \underbrace{N}_{\text{dimension}}{}_n(\overset{\text{expec}}{\vec{0}_n}, \overset{\text{var}}{I_n})$$

Standard Multivariate Normal

$$\vec{Z} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\vec{X} = \vec{Z} + \vec{\mu} = \begin{bmatrix} Z_1 + \mu_1 \\ \vdots \\ Z_n + \mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1, 1) \\ \text{ind} \\ \sim N(\mu_n, 1) \end{matrix}$$

$\vec{\mu} \in \mathbb{R}^n$



Multivariate Bell curve, 2D, 3D

Let $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 + \cdots \\ Z_1 + Z_2 + \cdots \\ Z_1 + Z_2 + Z_3 + \cdots \\ Z_1 + Z_2 + \cdots + Z_n \end{bmatrix} \left.\begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \sim N(0,n) \end{matrix}\right\} \text{dep.}$$

$$\begin{aligned} Cov[Z_1, Z_1 + Z_2] &= Cov[Z_1, Z_1] + Cov[Z_1, Z_2] \\ &= Var[Z_1] + 0 \\ &= 1 \neq 0 \rightarrow \text{dependent} \end{aligned}$$