CDF:

$$F(x) = \int_0^x \frac{1}{\beta(\alpha,\beta)} t^{\alpha-1} (1-t)^{\beta-1} \, dt$$

$$= \frac{\int_0^x t^{\alpha-1} (1-t)^{\beta-1} \, dt}{\beta(\alpha,\beta)} = \frac{\beta(x,\alpha,\beta)}{\beta(\alpha,\beta)}$$

$$= I_x(\alpha,\beta).$$

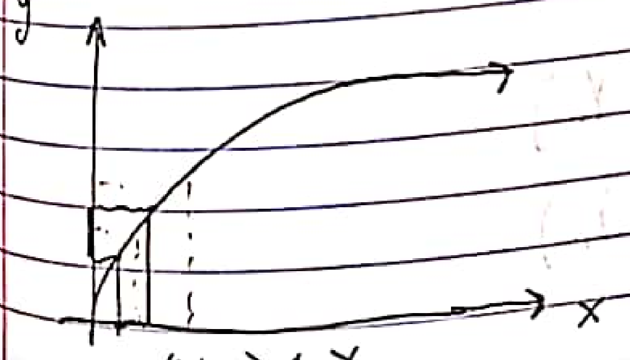
make up session (Review) Tues | Friday - one each day.

$$Y \sim Beta(\alpha,\beta) := \underbrace{\frac{1}{\beta(\alpha,\beta)}}_{c} x^{\alpha-1} (1-x)^{\beta-1} \quad \alpha \, x^{\alpha-1} (1-x)^{\beta-1}$$

New topic called

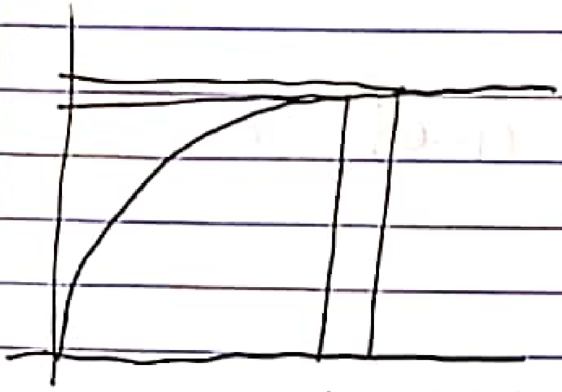
If X is a cont r.v. g a 1-1 function

$y = g(x) \Rightarrow f_y(y) = f_x(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$



$\Delta y > \Delta x$

10/30



Δy less meaningful per each Δx.

$\left|\frac{\partial}{\partial y} g^{-1}(y)\right|$ controls stretching & compressing

Let $\vec{X}$ be a vector r.v. continuous with dimension n and $f_{\vec{X}}(\vec{x})$ known. Let $\vec{Y} = g(\vec{X})$ where $g: \mathbb{R}^n \to \mathbb{R}^n$ with an inverse h (i.e. $\vec{X} = h(\vec{Y})$)

end goal: to find $f_{\vec{y}}(\vec{y})$

Think of $g_1(X_1, \ldots, X_n) = Y_1$

$g_2(X_1, \ldots, X_n) = Y_2$

$g_n(X_1, \ldots, X_n) = Y_n$

and because it is invertible:

$X_1 = h_1(Y_1, \ldots Y_n)$

$X_2 = h_2(Y_1, \ldots Y_n)$

$\vdots$

1/30

change of variables formula

$$\boxed{g^{-1}(y) = h}$$

$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\,|J_h|$ where J_h is defined to be: the determinant

$$J_h := \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

"Jacobian determinant"

handles the stretching and compressing within $\mathbb{R}^n$. multi-dimensional version of the derivative.

verify convolution formula for

$T = X_1 + X_2 \sim f_T(t)$: goal.

5 step process

1. Find a clever g s.t. we can...
2. find h
3. compute jacobian det
4. sub into change of var formula
5. Integrate out nuisance dimensions.

① $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$

$Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

10/30

(3) calculate J_h

$$= \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \vdots & \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} : \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$1 \cdot 1 - (-1 \cdot 0) = 1$$

(4) $$f_{Y_1, Y_2}(y_1 - y_2) = f_{X_1, X_2}(y_1 - y_2, y_2)\,|1|$$

(5) Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$: ~~marginny~~ marginning

T = var we care about.

u = ~~__~~ nuisance we want to go away.

$$f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\, du$$

If X_1, X_2 IND $= \int_{\mathbb{R}} f_{X_1}(t-u)\, f_{X_2}(u)\, du$

~~__~~ with indicator $\int_{\operatorname{supp} X_1} f_{X_1}(t-u)\, f_{X_2}(u)\, \mathbb{1}_{u \in \operatorname{supp} X_2}\, du$

10/30 $R = \frac{X_1}{X_2} \sim \cancel{f_R} \; f_R(r)$

$Y_1 = \frac{X_1}{X_2} = g_1(X_1, Y_2)$

$Y_2 = X_2 \sim g_2(X_1, X_2)$

$X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

$$\det \begin{bmatrix} Y_2 & Y_1 \\ 0 & 1 \end{bmatrix} = Y_2$$

$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(y_1 y_2, y_2) \, |y_2|$

$f_R(r) = \int_{\mathbb{R}} f_{X_1 X_2}(ru, u) \, |u| \, du$

if X_1 X_2 ind $= \int_{Supp(X_1)} f_{X_1}(ru) \, f_{X_2}(u) \, \mathbb{1}_{u \in Supp(X_2)} \, |u| \, du$

$\boxed{u = Y_2 = \text{nuisance} \quad Y_1 = \text{target}}$

10/31 $R = \dfrac{X_1}{X_1+X_2} \sim f_R(r) = ?$

$$Y_1 = \frac{X_1}{X_1+X_2} = g_1(X_1, X_2)$$

$$Y_2 = X_1 + X_2 = g_2(X_1, X_2)$$

$$X_1 = Y_1(X_1+X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$$

$$X_2 = Y_2 - Y_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$$

$$\det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix} = y_2$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|$$

$$f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$$

$$= f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u)\,du\,|u|$$

ind

$$\int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u - ru)\,|u| = \int_{\text{Supp}\,X_1} f_{X_1}(ru)\, f_{X_2}(u - ru)\,|u| \;!!$$

[illegible] Supp

$u - ru$

10/30 use formula's we just derived:

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ independent of $(0,\infty)$

$X_2 \sim \text{Gamma}(\alpha_2, \beta)$ $(0,\infty)$

$$R = \frac{X_1}{X_1+X_2} \sim \int_0^{\infty} \left(\frac{\beta^{\alpha_1}}{\Gamma\alpha_1}(ru)^{\alpha_1-1} e^{-\beta ru}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma\alpha_2} (u-ru)^{\alpha_2-1} e^{-\beta(u-ru)}\right) |u| \, \mathbb{1}_{u-ru \in (0,\infty)} \, du$$

Support$(R) = (0,1)$

$\boxed{\text{but } u \in (0,\infty) \quad |u| = u}$

remove indicator function first.

$u(1-r) \in (0,\infty)$

note $1-r \in (0,1) \Rightarrow u \in (0,\infty) \Rightarrow \mathbb{1}_{u-ru \in (0,\infty)} = 1$ always

simplify:

$$\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma\alpha_1 \Gamma\alpha_2} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^{\infty} u^{\alpha_1-1+\alpha_2} e^{-\beta u} \, du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{\beta^{\alpha_1+\alpha_2}}}$$

10/30 cancel $\beta^{\alpha_1+\alpha_2}$

$$= \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}}\, r^{\alpha_1-1}(1-r)^{\alpha_2-1} = \text{Beta}(\alpha_1, \alpha_2)$$

$$\parallel$$

$$\frac{1}{\beta(\alpha_1+\alpha_2)}$$

proportional waiting time of two gamma's.

X_1, X_2 same as above

$R = \frac{X_1}{X_2}$ Support(R) $= (0, \infty)$

$$R = \frac{X_1}{X_2} \sim \int_{= \text{Supp}\, X_1} f_{X_1}(ru)\, f_{X_2}(u)\, \mathbb{1}_{u \in \text{Supp}\, X_2}\, |u|\, du$$

$\mathbb{1} = 1$

$$\int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma\alpha_1} ru^{\alpha_1-1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma\alpha_2} u^{\alpha_2-1} e^{-\beta u}\right) u\, du$$

$$\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma\alpha_1\,\Gamma\alpha_2}\, r^{\alpha_1-1} \int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u}\, du$$

$$\rightarrow \frac{\Gamma(\alpha_1+\alpha_2)}{\beta(r+1)^{\alpha_1+\alpha_2}}$$

$$= \frac{1}{\beta(\alpha_1+\alpha_2)} \frac{r^{\alpha_1-1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r \in (0,\infty)}$$

10/30

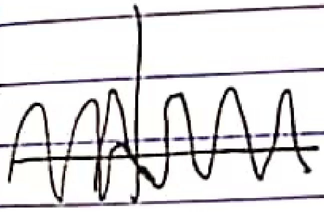
$$= \text{Beta Prime}(\alpha_1, \alpha_2)$$

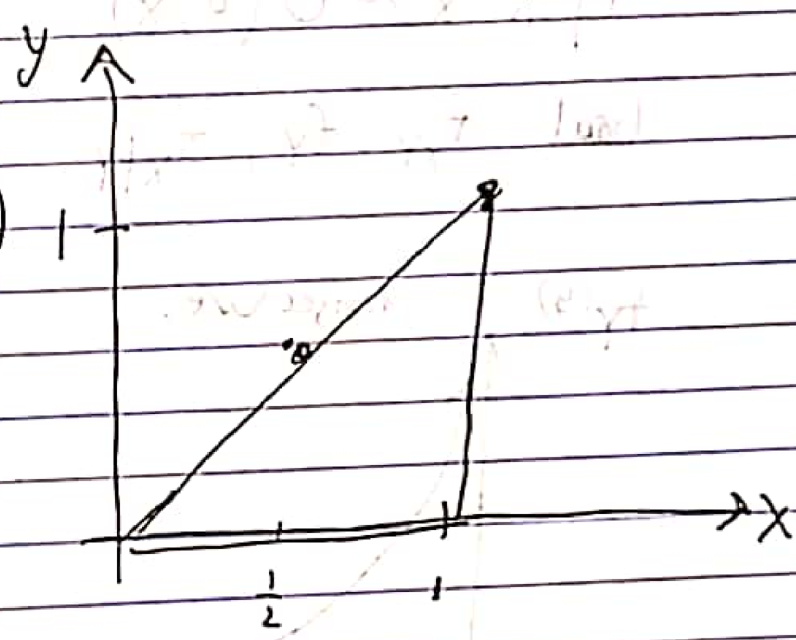
pg 105 conditional densities.

$X \sim U(0,1)$

$Y|X=x \sim U(0,x)$

$f(x,a) = \sin(ax)$





when $x = \frac{1}{2}$ what is y?

$f_{X,Y} = ??$ $\quad f_Y = ??$ $\quad f_{X|Y} = ?$

$f_{Y|X}(y,1) = 1$

$f_{Y|X}(y,.1) = 10$