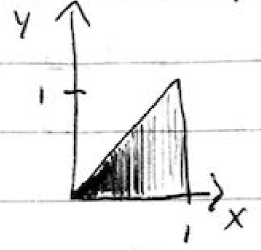
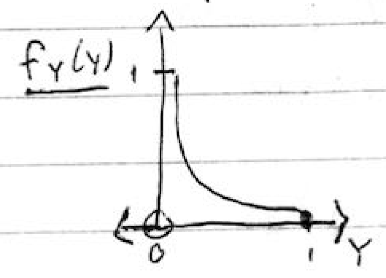
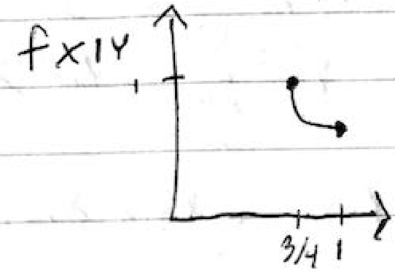
# Lecture 14

$X \sim U(0,1) \quad Y|X=x \sim U(0,x)$

Want: $f_{XY}, f_Y, f_{X|Y}$






$f_{X|Y}(x, 3/4)$

- need to be given a particular Y since X is condition on it

## (I) Marginalize

- $f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\, dy$ | $f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\, dx$

## (II) Definition of conditional probability

$f_Y(y) > 0$ ] $f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} \rightarrow f_{XY}(x,y) = \boxed{f_{X|Y}(x,y) f_Y(y)}$

$f_X(x) > 0$ ] $f_{Y|X}(x,y) = \dfrac{f_{XY}(x,y)}{f_X(x)} \rightarrow f_{XY}(x,y) = \boxed{f_{Y|X}(x,y) f_X(x)}$

## (III) Bayes Rule (consequence of II)

- $f_{X|Y}(x,y) = \dfrac{f_{Y|X}(x,y) f_X(x)}{f_Y(y)}$
- $f_{Y|X}(x,y) = \dfrac{f_{X|Y}(x,y) f_Y(y)}{f_X(x)}$

## IV Bayes Thrm I - III

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{\int_{\mathbb{R}} f_{XY}(x,y)\,dy} = \frac{f_{Y|X}(x,y) f_X(x)}{\int f_{Y|X}(x,y) f_X(x)\,dy}$$

This can be done on PMF's instead:

$$f_Y(y) = \sum_{x \in \mathbb{R}} f_{Y|X}(x,y) P_X(x)$$

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y) f_X(x)\,dx$$

---

Back to previous $U(0,1)$ example!

$$f_X(x) = \mathbb{I}_{x \in (0,1)}, \quad f_{Y|X}(x,y) = \frac{1}{x} \mathbb{I}_{y \in [0,x]}$$

from II $$f_{XY}(x,y) = \frac{1}{x} \mathbb{I}_{y \in [0,x]} \mathbb{I}_{x [0,1]}$$

$$= \frac{1}{x} \mathbb{I}_{0 \le y \le x \le 1} = \frac{1}{x} \mathbb{I}_{y \in [0,1]} \mathbb{I}_{x \in [y,1]}$$

from I $$f_Y(y) = \int_{\mathbb{R}} \frac{1}{x} \mathbb{I}_{y \in [0,1]} \mathbb{I}_{x \in [y,1]}\,dx$$

$$= \mathbb{I}_{y \in [0,1]} \int_y^1 \frac{1}{x}\,dx$$

$$= \mathbb{I}_{y \in [0,1]} \left[\ln(y)\right]_y^1 = -\ln(y)\,\mathbb{I}_{y \in [0,1]}$$

II $f_{X|Y}(x,y) = \dfrac{\frac{1}{x}\,\mathbb{I}_{y\in(0,1)}\,\mathbb{I}_{x\in(y,1)}}{-\ln(y)\,\mathbb{I}_{y\in[0,1]}}$

$= \dfrac{1}{x\ln(y)}\,\mathbb{I}_{y\in[0,1]}$ e.g. $y=\frac{3}{4}$, $f_{X|Y}(x,3/4) \approx \dfrac{3.5}{x}\,\mathbb{I}_{x\in[3/4,1]}$

$f_{X|Y}(3/4,3/4) \approx 4.6$ & $f_{X|Y}(1,3/4) \approx 3.5$

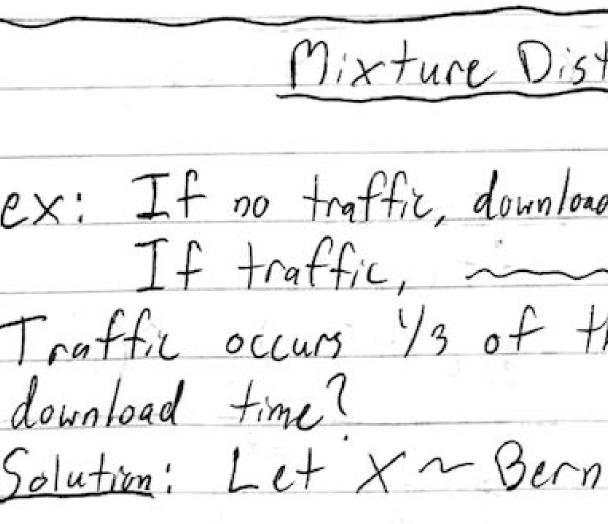
$f_{Y|X=1.7}$ is undefined, not even zero

---

## Mixture Distributions

ex: If no traffic, download speed $\text{Exp}(1/5)$
If traffic, $\text{Exp}(1/20)$
Traffic occurs $1/3$ of the time. What is overall download time?
Solution: Let $X \sim \text{Bern}(2/3)$ where $x=1$ no traffic
$x=0$ traffic

$\underline{X}$ $\underline{T|X}$ $\underline{T}$ ?




$$f_T(t) = \sum_{x\in\mathbb{R}} \underbrace{f_{T|X}(x,t)}_{\text{model}}\,\underbrace{f_X(x)}_{\text{mixing distr.}}$$

↓

$$= \sum_{x \in \mathbb{R}} \left( \frac{1}{5} e^{-1/5 t} \mathbb{I}_{x=1} + 1/20 e^{-1/20 t} \mathbb{I}_{x=0} \right) \left( \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x} \mathbb{I}_{x \in (0,1)} \right)$$

$$= \sum_{x \in [0,1]} \left( \frac{1}{5} e^{-1/5 t} \mathbb{I}_{x=1} + 1/20 e^{-1/20 t} \mathbb{I}_{x=0} \right) \left( \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x} \right)$$

$$f_T(t) = \frac{1}{3} \cdot \frac{1}{20} e^{-1/20 t} + \frac{2}{3} \cdot \frac{1}{5} e^{-1/5 t}$$

- T is called a "mixture model"
- Or "multi-level model"

---

ex: Traffic distribution if $t = 25$?

$$P_{X|T}(x,t) = \frac{f_{X,T}(x,t)}{f_T(t)}$$

$$= \frac{\left( \frac{1}{5} e^{-1/5 t} \mathbb{I}_{x=1} + \frac{1}{20} e^{-1/20 t} \mathbb{I}_{x=0} \right) \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}}{\left( \frac{1}{3} \cdot \frac{1}{5} e^{-1/5 t} + \frac{2}{3} \cdot \frac{1}{20} e^{-1/20 t} \right)}$$

$$= \frac{\frac{2}{3} \cdot \frac{1}{5} e^{-1/5 t} \mathbb{I}_{x=1} + \frac{1}{3} \cdot \frac{1}{20} e^{-1/20 t} \mathbb{I}_{x=0}}{\frac{2}{3} \cdot \frac{1}{5} e^{-1/5 t} + \frac{1}{3} \cdot \frac{1}{20} e^{-1/20 t}}$$

$$= \text{Bern}\left( \frac{\frac{2}{3} \cdot \frac{1}{5} e^{-1/5 t}}{\frac{2}{3} \cdot \frac{1}{5} e^{-1/5 t} + \frac{1}{3} \cdot \frac{1}{20} e^{-1/20 t}} \right)$$

If $t=25$; $X|T=25 \sim Bern(0.16)$

P of traffic $= 1-0.16 = 0.84$

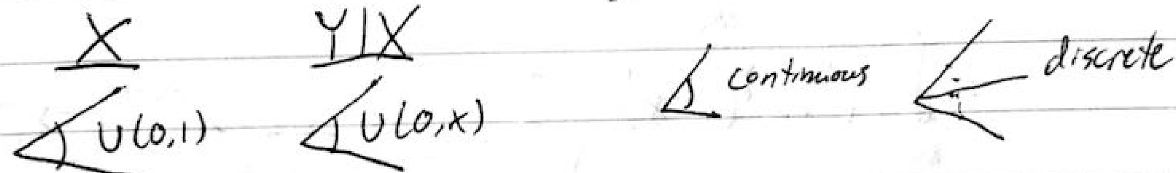
$P_{X|T}(0,25) = 0.84$

$$= \frac{\frac{1}{3}\cdot\frac{1}{20}e^{-1/20\, t}}{\frac{1}{3}\cdot\frac{1}{20}e^{-1/20\cdot 25}+\frac{2}{5}\frac{1}{5}e^{-1/5\, 25}}$$

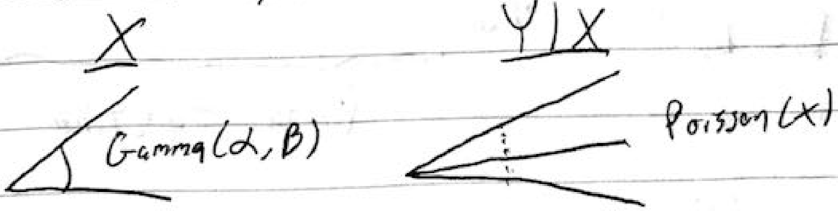
---

$X \sim U(0,1) \qquad Y|X=x \sim U(0,1)$

$$f_Y(y) = \int_{\mathbb{R}} \underbrace{f_{Y|X}(x,y)}_{\text{model}} \underbrace{f_X(x)}_{\text{mixing dist.}}\, dx$$




pg 156) $X \sim Gamma(\alpha,\beta)$, $Y|X=x \sim Poisson(x)$




$$p_Y(y) = \int p_{Y|X}(x,y)\, f_X(x)\,dx$$

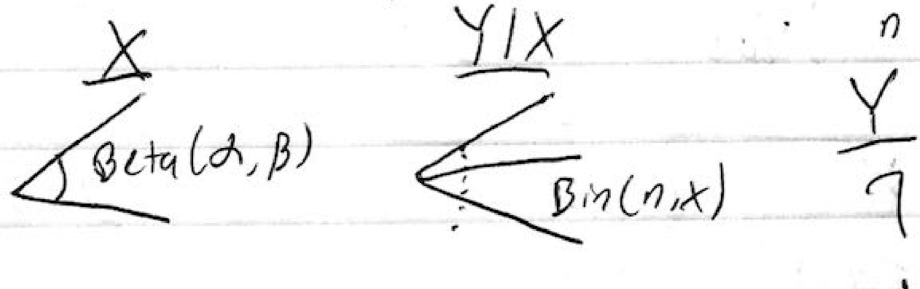
$$= \int_{\mathbb{R}} \left(\frac{x^y e^{-x}}{y!} \mathbb{I}_{y\in\mathbb{N}_0}\right)\cdot\left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, \mathbb{I}_{x\in(0,\infty)}\, dx\right)$$

$$\propto \frac{1}{y!}\int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

$$= \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(B+1)^{y+\alpha}} \mathbb{1}_{y \in \mathbb{N}_0} \propto \underset{\text{NegBin}}{\text{Extended}}\left(\alpha, \frac{B}{B+1}\right)$$

---

$X \sim \text{Beta}(\alpha, B)$, $Y|X=x, n \sim \text{Bin}(n, x)$   $n$ is known




$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y) f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \left(\binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,1,\dots,n\}}\right)\left(\frac{1}{B(\alpha,B)} x^{\alpha-1}(1-x)^{B-1} \mathbb{1}_{x \in (0,1)}\right) dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,B)} \mathbb{1}_{y \in \{0,1,\dots,n\}} \int_0^1 \underbrace{x^{y+\alpha-1}(1-x)^{n-y+\alpha-1}}_{\text{Beta function}}\, dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,B)} B(y+\alpha, n-y+B) \mathbb{1}_{y \in \{0,1,\dots,n\}}$$

$$= \text{BetaBinomial}(\alpha, B, n)$$

$X \sim Gamma(\alpha, \beta)$ $\quad Y|X=x \sim Exp(x)$

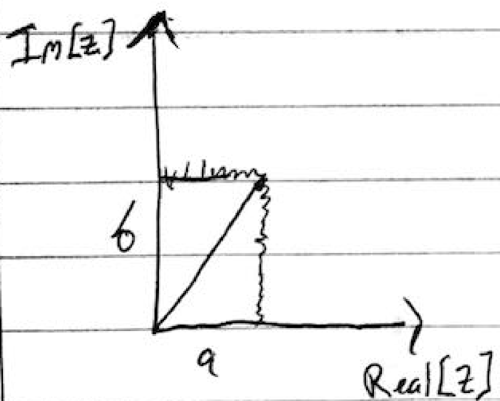
| $X$ | $Y\|X$ | $Y$ |
| --- | --- | --- |
| $Gamma(\alpha, \beta)$ | $Exp(x)$ | ? |

$Y \sim Lomax(\beta, \alpha)$

---

$a, b \in \mathbb{R}$ $\quad z := a + bi \in \mathbb{C}$ complex num

where $i := \sqrt{-1}$




$Real[z] = a$

$Im[z] = b$

$|z| := \sqrt{a^2 + b^2}$

$\theta = Ang[z] = \arctan\left(\frac{b}{a}\right)$