$X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$

$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$

$$Z_1 = \frac{X_1 - \mu}{\sigma}, \quad Z_2 = \frac{X_2 - \mu}{\sigma}, \quad \ldots, \quad Z_n = \frac{X_n - \mu}{\sigma}$$

same as dot product $\vec{Z} \cdot \vec{Z}$ $\rightarrow$

$$\vec{Z}^T \vec{Z} = \sum_{i=1}^{n} Z_i^2 = \sum \left(\frac{X_i - \mu}{\sigma} \right)^2 = \ldots \text{ algebra}$$

$$= \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_n$$

where $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

because

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} = \frac{(\bar{X} - \mu)^2}{\frac{\sigma^2}{n}} = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \right)^2 \sim \chi^2_1$$

assumed U_1, U_2 are indep. $\rightarrow$

Used:

if $U_1 \sim \chi^2_{k_1}$ and $U_2 \sim \chi^2_{k_2}$

then $U_1 + U_2 \sim \chi^2_{k_1 + k_2}$

uses Cochrain's Theorem $\rightarrow$

Conjecture

(1) $\dfrac{(n-1)S^2}{\sigma^2}$ is independent of $\dfrac{n(\bar{X} - \mu)^2}{\sigma^2}$

$$\Longleftrightarrow$$

S^2 is indep. of $\bar{X}$

(2) $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\vec{Z}^T \vec{Z} = \underbrace{\vec{Z}^T \overset{\text{identity}}{I_n} \vec{Z}}_{\text{quadratic form}}$$

Consider $$\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = Z_1^2 \sim \chi_1^2$$

$$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \\ & & & & \\ 0 & & & & 0 \end{bmatrix}}_{B_2} \vec{Z} = Z_2^2 \sim \chi_1^2$$

and so on...

$$\vec{Z}^T \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \vec{Z} = Z_n^2 \sim \chi_1^2$$

<u>Note</u>

1. All quadratic forms are independent
2. $B_1 + B_2 + \dots + B_n = I_n$
3. $\text{rank}[B_1] = \text{rank}[B_2] = \dots = \text{rank}[B_n] = 1$

   $\sum \text{rank}[B_i] = n$

$$\vec{Z}^T B_1 \vec{Z} + \dots + \vec{Z}^T B_n \vec{Z}$$
$$= \vec{Z}^T (B_1 \vec{Z} + \dots + B_n \vec{Z})$$
$$= \vec{Z}^T (B_1 + \dots + B_n) \vec{Z} = \vec{Z}^T I \vec{Z} = \vec{Z}^T \vec{Z}$$

$A\vec{x}$

$\vec{x} \in \{$ space with dim rank[K] $\}$

Cochran's Theorem (1934)

Given $Z_1, Z_2, \ldots, Z_n \overset{i.i.d.}{\sim} N(0,1)$

If (a) $B_1 + B_2 + \cdots + B_K = I_n$

and (b) $\sum_{j=1}^{K} \text{rank}[B_j] = n$

then:

(a) $\vec{Z}^T B_j \vec{Z} \sim \chi^2_{\text{rank}[B_j]}$

and

(b) $\vec{Z}^T B_{j_1} \vec{Z}$ is indep of $\vec{Z}^T B_{j_2} \vec{Z}$ $\forall j_1 \neq j_2$

$\vec{a}^T\vec{b}$ is $\vec{a} \bullet \vec{b}$ (Dot Product) "inner product"

$$\vec{Z}^T\vec{Z} = \sum z_i^2 = \sum((z_i - \bar{z}) + \bar{z})^2$$
$$= \sum \left((z_i - \bar{z})^2 + 2(z_i - \bar{z})\bar{z} + \bar{z}^2\right)$$
$$= \sum(z_i - \bar{z})^2 + 2\left(\sum z_i \bar{z} - \sum \bar{z}^2\right) + \sum \bar{z}^2$$
$$= \sum(z_i - \bar{z})^2 + 2(n z^2 - n z^2) + n\bar{z}^2$$
$$= \sum(z_i - \bar{z})^2 + 2(0) + n\bar{z}^2$$
$$= \sum(z_i - \bar{z})^2 + n\bar{z}^2$$

J = matrix of all 1's

$\vec{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \}n$

dim is n

Note: $\bar{z} = \frac{1}{n}(\sum z_i) = \frac{1}{n}\vec{1}_n^T\vec{Z} = \frac{1}{n}\vec{Z}^T\vec{1}_n$

$\sum_{i=1}^{n} c = nc$

constant

Define: $\vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ dim n

$J_n = \vec{1}_n \vec{1}_n^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$ ← $n \times n$ matrix all 1's

"outer Product"

$B_2 = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$ ← n by n matrix all $\frac{1}{n}$

$\text{rank}(B_2) = 1$

B_2 is $n \times n$ matrix of all $\frac{1}{n}$'s

$\text{rank}(B_2) = 1$

$$\Rightarrow n\bar{Z}^2 = n\left(\tfrac{1}{n}\vec{Z}^T \vec{1}_n\right)\left(\tfrac{1}{n}\vec{1}_n^T \vec{Z}\right)$$
$$= \vec{Z}^T\left(\tfrac{1}{n}\vec{1}_n \vec{1}_n^T\right)\vec{Z}$$
$$= \vec{Z}^T \underbrace{\left(\tfrac{1}{n} J_n\right)} \vec{Z}$$
$$= \vec{Z}^T B_2 \vec{Z}$$

$\text{rank}(B_2) = 1$

$$B_2 \vec{x} = \begin{bmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{bmatrix}$$

$$\sum (z_i - \bar{z})^2 = \sum z_i^2 - 2z_i\bar{z} + \bar{z}^2$$
$$= \sum z_i^2 - 2n\bar{z} + n\bar{z}^2$$
$$= \sum z_i^2 - n\bar{z}^2$$
$$= \vec{Z}^T I \vec{Z} - Z^T\left(\tfrac{1}{n} J_n\right)\vec{Z}$$
$$= \vec{Z}^T \underbrace{\left(I_n - \tfrac{1}{n} J_n\right)}_{B_1} \vec{Z}$$

$$B_1 = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & & \vdots \\ & & \ddots & -\frac{1}{n} \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & 1-\frac{1}{n} \end{bmatrix}$$

rank not obvious

$\text{rank}[B_1] + \text{rank}[B_2]$
$= 1 + (n-1) = n$ ✓

need $\text{rank}(B_1)$

trace is sum of entries on main diagonal of matrix

trace

$$tr[B_1] = n\left(1 - \tfrac{1}{n}\right)$$
$$= n - 1$$

$$B_1 + B_2 = \left(I_n - \tfrac{1}{n}J_n\right) + \tfrac{1}{n}J_n = I_n$$

identity matrix

Use this with Theorem...

Theorem If A is symmetric and idempotent (idempotent means $AA = A$) then $\operatorname{rank}[A] = \operatorname{trace}[A]$

trace is sum of entries on main diagonal

$$\left(I_n - \frac{1}{n}J_n\right)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J \Rightarrow \text{symmetric}$$

$$\left(I_n - \frac{1}{n}J_n\right)\left(I_n - \frac{1}{n}J_n\right) =$$
$$= I_n I_n - \frac{1}{n} I_n J_n - \frac{1}{n} I_n J_n + \frac{1}{n^2} J_n J_n$$

$J_n J_n = n J_n$

$$= I_n - 2\left(\frac{1}{n}J_n\right) + \frac{1}{n^2} \cdot n J_n$$
$$= I_n - 2\frac{1}{n}J_n + \frac{1}{n}J_n$$
$$= I_n - \frac{1}{n}J_n \Rightarrow \text{idenpotent}$$

---

since $\operatorname{rank}[B_1] + \operatorname{rank}[B_2] = n$ and $\Sigma B_i = 1$ can use Cochran's Theorem

By Cochran's Theorem:

$$\vec{Z}^T \vec{Z} = \underbrace{\sum_{i=1}^{n} (Z_i - \bar{Z})^2}_{\chi^2_{n-1}} + \underbrace{n\bar{Z}^2}_{\chi^2_1}$$

independent

$$\bar{Z} = \frac{Z_1 + Z_2 + \dots + Z_n}{n} = \frac{\frac{X_1 - \mu}{\sigma} + \frac{X_2 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}}{n}$$

$$\bar{Z} = \frac{X_1 + X_2 + \dots - n\mu}{n\sigma}$$

$$\bar{Z} = \frac{\bar{X} - \mu}{\sigma}$$

$$n\bar{Z}^2 = n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 = \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_1$$

$$n\bar{Z}^2 = n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 = \frac{n(\bar{X}-\mu)^2}{\sigma^2} = \chi_1^2$$

$$\sum(Z_i - \bar{Z})^2 = \sum\left(\frac{X_i-\mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \frac{\sum(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

independent

$\Rightarrow$ $S^2, \bar{X}$ are independent

Normal dist.

$Z = N(0,1)$

μ σ^2

Can also do "2 sample z test"

$Z_d = Z_2 - Z_1$

To test $\mu_1 = \mu_2$ would test $\mu_d = \mu_2 - \mu_1 = 0$

If σ is known and you wish to test null hypothesis $H_0: \mu = \text{value}$

$$\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

"one sample Z test"

You wish to test $H_0: \sigma^2 = \text{some value}$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

"one sample variance test"
"χ^2 test of variance"

You wish to test for two independent samples $H_0: \sigma_1^2 = \sigma_2^2$

$$\frac{\frac{(n_1-1)S_1^2}{\sigma_1^2} \Big/ n_1 - 1}{\frac{(n_2-1)S_2^2}{\sigma_2^2} \Big/ n_2 - 1} = \frac{S_1^2}{S_2^2} \sim F_{n_1, n_2}$$

"F test for equality of variances"

You want to test $H_0: \mu = \text{value}$
but you don't know σ

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}S^2}}$$

$$= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/n-1}} = \frac{Z}{\sqrt{\frac{U}{n-1}}} \sim T_{n-1}$$

"one - sample T test"

Note: $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = Z \sim N(0,1)$

indep.

$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

"one - sample T test"

can also compare means "2-sample T-test"

think of it as $T_d = T_2 - T_1$

can test $\mu_1 = \mu_2$ by testing $\mu_d = \mu_2 - \mu_1 = 0$

but have to find "pooled variance" and d.f.

Multivariate Normal

$$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} \quad \text{where } Z_1, Z_2, \ldots, Z_n \overset{i.i.d.}{\sim} N(0,1)$$

$$E[\vec{Z}] = \vec{0} \quad \text{since} \quad E[Z_1]=0,\ E[Z_2]=0, \ldots, E[Z_n]=0$$

$\Sigma \rightarrow$ the variance covariance matrix

$$Var[\vec{Z}] = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix} = I_n \quad \text{since } Var[Z_i] = 1$$

← identity matrix

and $Cov[Z_i, Z_j] = 0$ for $i \neq j$

since Z's are indep.

$$Var[\vec{Z}] = I_n$$

$$f_{\vec{Z}}(\vec{z}) = f_{Z_1 Z_2 \ldots Z_n}(z_1, z_2, \ldots, z_n)$$

$$= f_{Z_1}(z_1) f_{Z_2}(z_2) \ldots f_{Z_n}(z_n) = \prod_{i=1}^{n} f(z_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

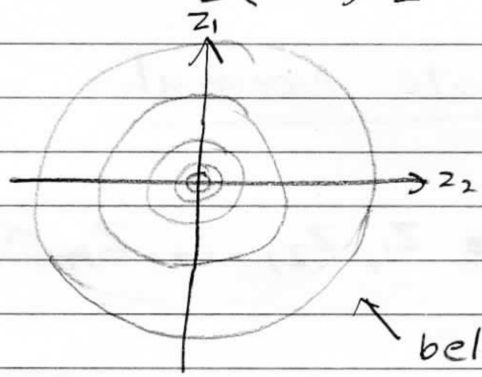
"Mu

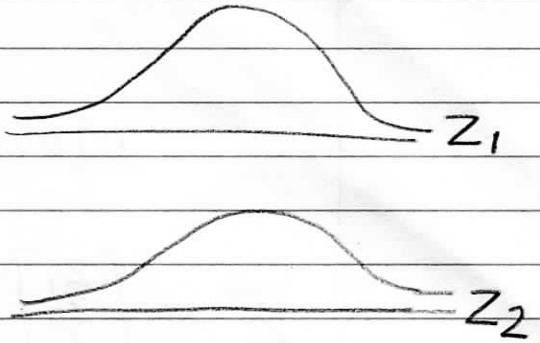
$$= N_n(\vec{0}, I_n)$$

"Multivariate Normal"

N: multivariate normal; n: dimensionality; $\vec{0}$: mean (expectation); I_n: variance expectations, variance-covariance matrix

$$\vec{Z} \sim N_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$





bell in 3-dimensions

① $\vec{\mu} \in \mathbb{R}^n$ where Z's are indep.

vector $\vec{\mu} = \begin{bmatrix}\mu_1\\\mu_2\\\vdots\\\mu_n\end{bmatrix}$

$$\vec{X} = \vec{Z} + \mu = \begin{bmatrix} Z_1+\mu_1 \\ Z_2+\mu_2 \\ \vdots \\ Z_n+\mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1, 1) \\ \sim N(\mu_2, 1) \\ \\ \sim N(\mu_n, 1) \end{matrix}$$

all are independent

$$= N_n(\vec{\mu}, I_n)$$

but what if Z's are not indep.?

② let $\vec{A} = \begin{bmatrix} 1&0&0&0&0&&0 \\ 1&1&0&0&0&&0 \\ 1&1&1&0&0&&0 \\ 1&1&1&1&0&&0 \\ &&&&&\ddots& \\ 1&1&1&1&1&&1 \end{bmatrix}$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1+Z_2 \\ Z_1+Z_2+Z_3 \\ \\ Z_1+Z_2+Z_3+\dots+Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \\ \sim N(0,n) \end{matrix}$$

but these are not indep

vector r.v.

Show X_1, X_2 are not indep.

$Cov[X_1, X_2]$
$= Cov[Z_1, Z_1+Z_2]$
$= Cov[Z_1, Z_2] + Cov[Z_1, Z_n]$
$= Var[Z_1] + 0$
$= 1 + 0 \neq 1 \Rightarrow X_1, X_2$ are dependent

$X_j = \sum_{i=1}^{j} Z_i \sim N(0, j)$

Want $E[\vec{X}] = E[A\vec{Z}] = ?$
Want $Var[\vec{X}] = Var[A\vec{Z}] = ?$