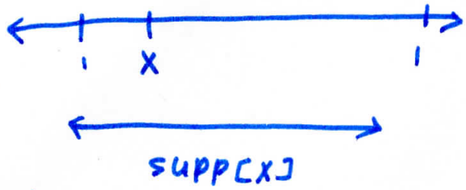
Lecture # 12

- $X_1 \dots X_n \overset{iid}{\sim} f(x)$ with CDF $F(x)$ — complete CDF binomial

$$\Rightarrow F_{X_{(n)}}(x) = \sum_{j=k}^{n} \underbrace{\binom{n}{j} F(x)^j (1-F(x))^{n-j}}_{\text{PDF binomial}}$$

- Let $L \sim Bin(n, p = F(x))$



How many X_i's ___ $\le x$ ___ ?

- Find $f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$

$$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx} \underbrace{F(x)^j}_{u} \underbrace{(1-F(x))^{n-j}}_{v}$$

$\vdots$ simplify

$$= f(x)\left(\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} j F(x)^{j-1}(1-F(x))^{n-j} - \underbrace{\sum_{j=k}^{n} \frac{n!}{j!(n-j)!}(n-j) F(x)^j (1-F(x))^{n-j-1}}\right)$$

$$\sum_{j=k}^{n-1} \frac{n!}{j!(n-(j+1))!} F(x)^j (1-F(x))^{n-(j+1)}$$

Let $\ell = j+1$

$\Rightarrow j = \ell - 1$

$$\sum_{\ell=k+1}^{n} \frac{n!}{(\ell-1)!(n-\ell)!} F(x)^{\ell-1}(1-F(x))^{n-\ell}$$

$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k}$$

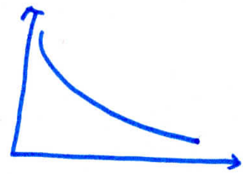
$$= f_{X_{(k)}}(x)$$ → density PDF

Minimum & maximum

$$f_{X_{(1)}}(x) = n f(x)(1-F(x))^{n-1}$$

$$f_{X_{(n)}}(x) = n f(x) F(x)^{n-1}$$

- $X_1 \dots X_n \overset{iid}{\sim} U(0,1) = 1, \ F(X) = X$



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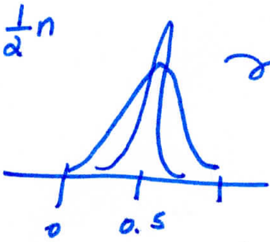
- $k \approx \frac{1}{2} n$



~ if n gets bigger

- $f_{X(k)}(X) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]}$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]}$$

$$= \text{Beta}(k, n-k+1)^{*}$$

- $X \sim \text{Gamma}(\alpha_1, \beta)$ independent of $Y \sim \text{Gamma}(\alpha_2, \beta)$

$$T = X + Y \overset{\text{conjecture}}{\sim} \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

$$\sim \int_{\text{supp}[X]} f_X(x)\, f_Y(t-x)\, \mathbb{1}_{t-x \in \text{supp}[X]}\, dx$$

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{\beta x}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} e^{\beta(t-x)}\, \underbrace{\mathbb{1}_{t-x \in (0,\infty)}}_{x \le t}\right) dx$$

check ⇒ $$= \underline{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha - 1}\, dx}$$

Use u-substitution:

$$\begin{cases} \text{let } u = \frac{x}{t} \Rightarrow x = nt \Rightarrow x = 0 \Rightarrow u = 0 \\ x = t \Rightarrow n = 1, \ \frac{dy}{dx} = \frac{1}{t} \Rightarrow dx = t\, du \end{cases}$$

$$\hookrightarrow \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\beta t} \int_0^1 (ut)^{\alpha_1 - 1} (t - ut)^{\alpha_2 - 1}\, t\, du$$

$$= \left(\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha-1}(1-u)^{\alpha_2-1}\,du\right) t^{\alpha_1+\alpha_2-1} e^{\beta x} \propto t^{\alpha_1+\alpha_2-1} e^{\beta t} \Rightarrow$$

$\propto$ Gamma $(\alpha_1+\alpha_2, \beta)$

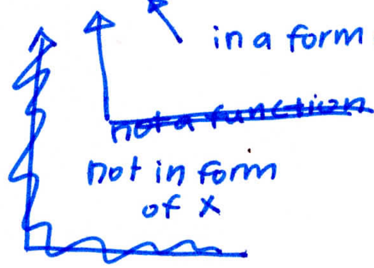
We can decompose $p(x) = c\,k(x) \propto k(x)$

strawberry

or water

$f(x) = c\,k(x) \propto k(x)$

in a form of x

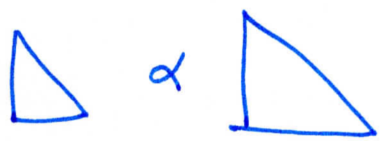


normalized compose

dehydrated strawberry → Expands

$$\sum p(x) = 1 \Rightarrow \sum c\,k(x) = 1 \Rightarrow c = \frac{1}{\sum k(x)} \Rightarrow$$

$k(x)$ specifies $p(x)$ or $f(x)$:=



$$\int f(x)\,dx = 1 \Rightarrow \int c f(x)\,dx = 1 \Rightarrow c = \frac{1}{\int f(x)\,dx}$$

- let $X \sim bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x}_{k(x)}$$

$$\propto \frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x \propto bin(n,p)$$

proportion

- let $X \sim Weibull(k, \lambda) = (k\lambda)(\lambda x)^{k-1} e^{-(\lambda x)^k} = \underbrace{k\lambda^k}_{c} \underbrace{x^{k-1} e^{-(\lambda x)^k}}_{k(x)}$

let $X \sim Gamma(\alpha, \beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{\beta x}}_{k(x)}$

$$\propto x^{\alpha-1} e^{\beta x} \propto Gamma(\alpha, \beta)$$

$$\Rightarrow \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}du \Rightarrow$$

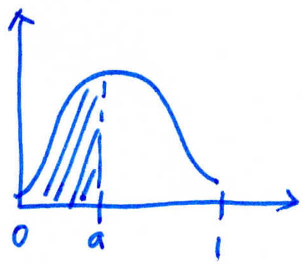
$\boxed{\text{Kernel}}$

$$B(\alpha_1,\alpha_2) = \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

Beta function

$$\int_0^\infty x^{\alpha_2-1}e^{-x}dx$$

"proportion"



$$B(a,\alpha_1,\alpha_2) = \int_0^a u^{\alpha_1-1}(1-u)^{\alpha_2-1}du$$

Incomplete Beta function

check

$$I_a(\alpha,a) = \frac{B(\beta,\alpha_1,\alpha_2)}{B(\alpha_1,\alpha_2)}$$

Regularized Incomplete Beta Function

- $X \sim \text{Beta}(\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{I}_{x\in(0,1)}$

$$\parallel$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\parallel$$

$\alpha,\beta\in\mathbb{N}$ $\quad \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!}$

CDF

$$F(x) := \int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}dt = \int_0^x \frac{t^{\alpha-1}(1-t)^{\beta-1}dt}{B(\alpha,\beta)} = \frac{B(x,a,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$