# Math 368/621 Lec 1 8/28/19

A "discrete" r.v. $X$ has prob. mass function (PMF)

$$p(x) := P(X = x)$$ and notation $X \sim p(x)$ and

cumul. distr. function (CDF) $F(x) := P(X \leq x)$. The r.v has

support $Supp(X) := \{ x : p(x) > 0, x \in \mathbb{R} \}$ and $|Supp(X)| \leq |\mathbb{N}|$

i.e. finite or cntbly infinite. It is called discrete because its

support is discrete. The support and PMF are related via $\sum_{x \in Supp(X)} p(x) = 1$.

The "most fundamental" r.v. (IMHO) is the Bernoulli r.v.

$$X \sim Bern(p) := \overbrace{p^x (1-p)^{1-x}}^{p(x)} \quad \text{and} \quad Supp(X) = \{0, 1\}$$

By def. if $x \notin \{0,1\}$, $p(x) = 0$. It is annoying to have to provide both $p(x)$ & $Supp(X)$. We combine the two together. First, we need to define a special math. function:

Indicator Function

$$\mathbb{1}_A = \begin{cases} 1 \text{ if } A \\ 0 \text{ if } A^c \text{ (not } A) \end{cases}$$

Can we use this to rewrite the PMF of the Bernoulli in a way s.t. we do not need to provide the PMF?

$$X \sim \text{Bern}(p) := \underbrace{p^x (1-p)^{1-x} \, \mathbb{1}_{x \in \{0,1\}}}_{p(x)}$$

$\mathbb{1}_{x \in \{0,1\}}$ ← Supp(X)

Some books refer to this as the PMF (with the indicator for the support).
We will be using this convention for the remainder. Now... legal PMFs feature:

$$\sum_{x \in \mathbb{R}} p(x) = 1$$

(degenerate)

Previously, $X \sim \text{Deg}(c) = \underset{\text{wp } 1}{c} \Rightarrow p(x) = 1$

Now $X \sim \text{Deg}(c) := \mathbb{1}_{x=c}$ very clean!

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If we have two or more r.v.'s $X_1, X_2, \ldots, X_n$

$p_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$ is the joint mass function (JMF)

If $X_1, X_2, \ldots, X_n \overset{ind}{\sim}$ "are independent", then..

$$p_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = p_{X_1}(x_1)\, p_{X_2}(x_2) \cdots p_{X_n}(x_n) = \prod_{i=1}^{n} p_{X_i}(x_i) \quad \forall\, x_1, x_2, \ldots, x_n \in \mathbb{R}$$

If $X_1 \overset{d}{=} X_2 \overset{d}{=} \ldots \overset{d}{=} X_n$ equal in distr., "identically distr."

$$p_{X_1}(x_1) = p_{X_2}(x_2) = \ldots = p_{X_n}(x_n) \quad \forall\, x_1, \ldots x_n \in \mathbb{R}$$

The special case of independent and identically distr. (iid) is denoted

$$X_1, \ldots, X_n \overset{iid}{\sim} p(x)$$

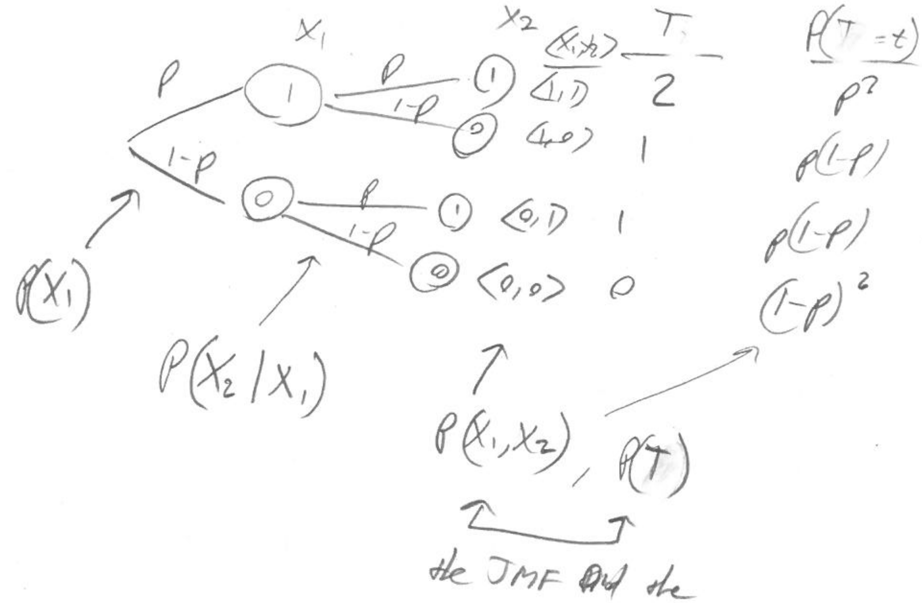
$$\text{JMF} = \prod_{i=1}^{n} p(x_i)$$

↑ no subscript

Consider $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$T := X_1 + X_2$, adding two r.v.'s together

Let's find the PMF for $T$. In 241... we drew a tree




$$\Rightarrow T \sim \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 2 & \text{w.p. } p^2 \end{cases}$$

$$\sum_{t \in supp(T)} p(t) = 1$$

$$= \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$

$$P(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_1+x_2=t} = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1, t-x_1)\, \mathbb{1}_{x_2 = t-x_1}$$

Select only the elements in the JMF where you get the sum you are looking for

independence

$$= \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x)$$

← the answer! Remember is convenient to use

if we demand $t$ and know $x_1$, $x_2$ is fixed

indep identical

$$= \sum_{x \in \mathbb{R}} \cancel{\sum_{x_2 \in \mathbb{R}}} p(x)\, p(t-x)\, \cancel{\mathbb{1}_{x_2 = t - x_1}}$$

Only one element in this sum is nonzero

$$= \sum_{x \in \mathbb{R}} p(x)\, p(t-x)$$

$$= \sum_{x \in \mathbb{R}} p^{x} (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}\, p^{t-x} (1-p)^{1-(t-x)} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= \sum_{x \in \{0,1\}} \underbrace{p^{t} (1-p)^{2-t}}_{\text{const}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^{t}(1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^{t}(1-p)^{2-t} \Big( \underbrace{\mathbb{1}_{t \in \{0,1\}} + \overset{\mathbb{1}_{t-1 \in \{0,1\}} \,=}{\mathbb{1}_{t \in \{1,2\}}}}_{c} \Big)$$

Correct

$\Downarrow$ if $t=0 \Rightarrow c=1$

$t=1 \Rightarrow c=2$

$t=2 \Rightarrow c=1$ $\Rightarrow c = \binom{2}{t}$

$t \notin \{0,1,2\} \Rightarrow c=0$

Otherwise correct notation only

$$p_{T_2}(t) = \binom{2}{t} p^{t}(1-p)^{2-t} = \text{Binomial}(2, p)$$

$$\text{Supp}(T) = \text{Supp}(X_1) + \text{Supp}(X_2) = \{0, 1, 2\}$$

$$A + B = \{a+b;\ a \in A,\ b \in B\}$$