Lecture 5:

$\vec{x} \sim \text{multinomial}(n, \vec{p})$

$x_j \sim \text{Bin}(n, p_j)$

$$\vec{\mu} := E[\vec{x}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}$$

$$\hookrightarrow E[\vec{x}] = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$$m = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix} \Rightarrow E[m] = \begin{bmatrix} E(X_{11}) & \cdots & E(X_{1m}) \\ \vdots & & \\ E(X_{n1}) & \cdots & E(X_{nm}) \end{bmatrix}$$

$\sigma^2 = \text{Var}[X] = E[X^2] - \mu^2$

$\sigma_{12} := \text{Cov}[X_1, X_2] = E[X_1 X_2] - \mu_1\mu_2$

$= E[(X_1 - \mu_1)(X_2 - \mu_2)]$

$\text{Var}[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$

$X_1, \cancel{X_2}$ ind $\Rightarrow \sigma_{12} = 0$

- Rules for covariance:

① $\text{Cov}[X, X] = E(X^2) - \mu^2 = \text{Var}[X] = \sigma^2$

② $\text{cov}[X_1, X_2] = \text{cov}[X_2, X_1]$

③ $\text{Cov}[X_1 + X_2, X_3] = \text{Cov}[X_1, X_3] + \text{Cov}[X_2, X_3]$

④ $\text{Cov}[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$

a_1 & $a_2 \in$ constants

⑤ $\text{Var}[X_1 + \dots + X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}$

$\text{Var}[X_1 + X_2] = \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij} = \sigma_{11} + \sigma_{12} + \sigma_{21} + \sigma_{22}$

$= \sigma_1^2 + \underbrace{2\sigma_{12}}_{} + \sigma_2^2$

$$\bullet\ \Sigma := Var[\vec{X}] = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_k] \\ Cov[X_2, X_1] & Var[X_2] & & \\ \vdots & & \ddots & \\ Cov(X_k, X_1) & & & Var[X_k] \end{bmatrix} = \{cov[X_i, X_j]\}$$

$k \times k$ matrix symmetric

"Variance matrix"

"covariance matrix."

"variance - covariance matrix."

Note that the diagonals cannot be negative.
so, $Var[X_1]$ & $Var[X_2]$ & ... & $Var[X_k] \neq 0$.

$\rightarrow$ Why is it same as $E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T$

$\bullet\ X_1 \cdots X_k \overset{ind}{\sim}$ $\rightarrow$ independent

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \sigma_k^2 \end{bmatrix}$$

$\sigma_1^2\ \sigma_2^2 \cdots \sigma_k^2$

They are nonnegative

$\bullet\ X_1 \ldots X_k \overset{iid}{\sim}$

$$\Sigma = \begin{bmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{bmatrix}$$

$\rightarrow$ They are identical

$= \sigma^2 I_k$

Rules for expectation & variance of vectors of r.v's:

$$\bullet\ E[\vec{X} + \vec{a}] = \begin{bmatrix} E[X_1 + a_1] \\ \vdots \\ E[X_k + a_k] \end{bmatrix} = \begin{bmatrix} \mu_1 + a_1 \\ \vdots \\ \mu_k + a_k \end{bmatrix} = \vec{\mu} + \vec{a}$$

$\vec{a} \in \mathbb{R}^k$ constant

- $E[\vec{a}^T \vec{x}] = E[a_1 X_1 + \dots + a_k X_k] = a_1 \mu_1 + \dots + a_k \mu_k$
$$= \vec{a}^T \vec{\mu}$$

- $E[AX]$

$A \in \mathbb{R}^{L \times k}$
matrix of constraints.

$$= \begin{bmatrix} E[a_{11} X_1 + \dots + a_{1k} X_k] \\ E[a_{21} X_1 + \dots + a_{2k} X_k] \\ \vdots \\ E[a_{L1} X_1 + \dots + a_{Lk} X_k] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot}^T \vec{\mu} \\ \vdots \\ \vec{a}_{L\cdot}^T \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

- $Var[\vec{a}^T X] = Var[a_1 X_1 + \dots + a_k X_k]$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} Cov[a_i X_i, a_j X_j]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j \sigma_{ij} \implies \text{Hence, } \vec{a}^T \Sigma \vec{a}$$

This is not sum. → very useful

$Var[\vec{x}]$

- ~~Consider~~ $\vec{c} \in \mathbb{R}^k$
Let $V \in \mathbb{R}^{k \times k}$

Consider $\underbrace{\vec{c}^T V \vec{c}}_{1\times k \;\; k\times k \;\; k\times 1} = [c_1 c_2 \cdots c_k] \begin{bmatrix} c_1 V_{11} + \dots + c_k V_{1k} \\ c_1 V_{21} + \dots + c_k V_{2k} \\ \vdots \\ c_1 V_{k1} + \dots + c_k V_{kk} \end{bmatrix} =$

(Quadratic forms)

$$c_1 c_1 V_{11} + \dots + c_1 c_k V_{1k} +$$
$$c_2 c_1 V_{21} + \dots + c_2 c_k V_{2k} +$$
$$\vdots$$
$$c_k c_1 V_{k1} + \dots + c_k c_k V_{kk}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j V_{ij}$$

- Finance

let $X_1 \cdots X_k$ be r.v model for the yearly ~~term~~ (return) of assets $1, \dots, k$

let $\vec{x} = [X_1 \cdots X_k]^T$

let $\vec{w} = [w_1, \dots w_k]$ be weights of the K assets. s.t. $\vec{w}^T \vec{1} = 1$

let $F = w_1 X_1 + \dots + w_k X_k = \vec{w}^T \vec{X}$ be the yearly return of your total profile.

$E[F] = \mu_F$

I want $\mu_F = \mu_0$ with minimal variance

select $\vec{w}$ s.t. $Var[F]$ is minimal.

$$\vec{w}^* = \underset{\vec{w}^T \cdot \vec{1} = 1}{\arg\min} \left\{ \vec{w}^T \Sigma \vec{w} \right\}$$

Markowitz Optimal Portfolio Theorem.

Go back to multinomial:

$\vec{X} \sim \text{Multinorm}(n, \vec{p})$

$X_j \sim \text{bin}(n, p_j)$

$$Var[\vec{x}] = \begin{bmatrix} np_1(1-p_1) & & \sigma_{ij} \\ & np_2(1-p_2) & \\ \sigma_{ij} & & \ddots \\ & & np_k(1-p_k) \end{bmatrix}$$

$\sigma_{ij} < 0$

$$\sigma_{ij} = Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j = \underbrace{\left(\sum_{\substack{X_1 \in supp[X_1] \\ = \{0 \cdots n\}}} \sum_{\substack{X_2 \in supp[X_2] \\ \{0 \cdots n\}}} X_i X_j p_{X_i, X_j}(X_1, X_2) \right)}_{\text{I think this is impossible}} - n^2 p_i p_j$$

• Recall $X_i \sim bin(n, p_i)$
$X_j \sim bin(n, p_j)$

They are all independent

$$X_i = X_{1i} + X_{2i} + \dots + X_{ni} \quad \text{where } X_{1i}, X_{2i}, \dots X_{ni} \overset{iid}{\sim} Bern(p_i)$$

they are dependent

~~$X_j \sim bin(n$~~

$$X_j = X_{1j} + X_{2j} + \dots + X_{nj} \quad \text{where } X_{1j}, X_{2j} \dots X_{nj} \sim Bern(p_j)$$

$$\vec{X} = \vec{X_{1\cdot}} + \vec{X_{2\cdot}} + \dots + \vec{X_{n\cdot}} \quad \text{where } \vec{X_{1\cdot}}, \dots \vec{X_{n\cdot}} \sim multinomial(1, \vec{p})$$

How are they distributed?

$$\bullet\ Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$$
$$= \sum_{l=1}^{n}\sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

$l \neq m \quad Cov[X_{li}, X_{mj}] = 0$

$$\sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$$

$$Cov[X_{li}, X_{lj}] = \left(\sum_{x_1 \in [0,1]}\sum_{x_2 \in [0,1]} x_1 x_2 \, p_{X_{li}, X_{lj}}(x_1, x_2)\right) - p_i p_j = \cancel{p_{X_{li}, X_{lj}}(1,1)}$$

(1,1) is only that is not 0

Say · So, $\underbrace{p_{X_{li}, X_{lj}}(1,1)}_{0} - p_i p_j = -p_i p_j$

Impossible to get an apple & banana at the same time

$$\boxed{\text{Hence, } \sum_{l=1}^{n} Cov[X_{li}, X_{lj}] = \cancel{+np_ip_j} \sum_{l=1}^{n} -p_i p_j = -np_i p_j}$$