Lecture 4

$\frac{n!}{x_1!x_2!x_3!}$ = Multinomial

Bag of fruit

p_1 = Prob Apple

p_2 = Prob Banana

p_3 = Prob Cantaloupe

$$\vec{X} := \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim P_{\vec{X}}(\vec{x}) = \mathbb{1}_{x_1 x_2 x_3 = n} \; p_1^{x_1} p_2^{x_2} p_3^{x_3}$$ JMF

= Multinomial $\left(n, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right)$ $\vec{p}$

$p_1 + p_2 + p_3 = 1$

w/ Replacement

C A B ~~C~~ A C B

$x_1 = 2$

$x_2 = 3$

$x_3 = 3$

$$\mathbb{1}_{x_1 \in \{0,1,\dots n\}} \mathbb{1}_{x_2 \in \{0,1,\dots n\}} \mathbb{1}_{x_3 \in \{0,1,\dots n\}}$$

$$= \binom{n}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \text{Multinomial}(n, \vec{p})$$

~~General Multinomial.~~

Let's be K the # of Item

general Multinomial

$$\vec{X} \sim \text{Multinomial}(n, \vec{p}) := \binom{n}{x_1 x_2 \dots x_K} p_1^{x_1} p_2^{x_2} \dots p_K^{x_K}$$

$$\text{Supp}(\vec{X}) = \{\vec{x} : \vec{x} \in \{0,1,\dots n\}^K, \vec{x} \cdot \vec{1} = n\} \quad \vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$n \in \mathbb{N}, \; p \in \{\vec{p} \in (0,1)^K, \vec{p} \cdot \vec{1} = 1\}$$

Ex: Two Dimensions.

$\vec{X} \sim \text{Multinomial}\left(n, \begin{pmatrix} p \\ 1-p \end{pmatrix}\right) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$

let $p_1 = p,\ p_2 = 1-p$

$X_1 \sim Bin(n, p)$

$X_2 \sim Bin(n, 1-p)$

~~So we notice~~

$X_1 \overset{d}{\neq} X_2$

Independence Def

$A_r = X_1 X_2 \overset{iid}{\sim} ?$ no

Def of Independence

$P(X_1 = x \mid X_2 = x_2) = P(X_1 = x)$

$\forall \vec{x} \in Supp(\vec{X})$

Proof by Contradiction

TEST

$0 = P(X_1 = 1 \mid X_2 = n) \overset{?}{\neq} P(X_1 = 1) = np(1-p)^{n-1}$

$\Rightarrow$ Replaces

Proof of Conditional Pmf

$P_{X_1/X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \dfrac{P_{x_1 x_2}(x_1, x_2)}{P_{X_2}(x_2)}$

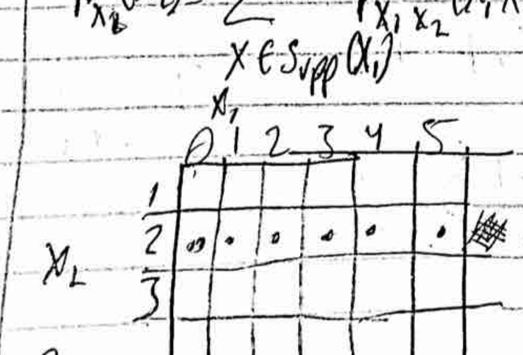
Conditional Pmf

by def Conditional Probability

marginal Prob

we need to use marginalization formula, $\Rightarrow$

$P_{X_2}(x_2) = \sum_{x \in Supp(x_1)} P_{X_1 X_2}(x_1, x_2)^2 = \sum_{x_1 \in \{0, 1, \ldots n\}} \frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1 + x_2 = n}$



HW. ms

take out constants

$$= \frac{n!}{X_2!} \sum_{X_1 \in \{0,1 \dots n\}} \frac{1}{X_1!} p^{X_1} \mathbb{1}_{X_1 = n - X_2} = \frac{n!}{X_2!}(1-p)^{X_2} \frac{1}{(n-X_2)!} p^{n-X_2}$$

$$= \binom{n}{X_2} (1-p)^{X_2} p^{n-X_2} = Bin(n, 1-p)$$

So, $f_{X_1/X_2} := P(X_1 = x / X_2 = x_2) = \frac{f_{X_1 X_2}(X_1 X_2)}{f_{X_2}(X_2)} = \frac{\frac{n!}{X_1! X_2!} p^{X_1} \cancel{(1-p)^{X_2}} \mathbb{1}_{X_1 + X_2 = n}}{\frac{n!}{X_2!(n-X_2)!} \cancel{(1-p)^{X_2}} p^{n-X_2}}$

$$= \frac{(n-X_2)!}{X_1!} p^{X_1 + X_2 - n} \mathbb{1}_{X_1 + X_2 = n} = \begin{cases} \frac{X_1!}{X_1!} p^0 = 1 & \text{if } X_1 + X_2 = n \\ 0 & \text{if } X_1 + X_2 \neq n \end{cases} = Deg(n - X_2) =$$

$\{ n - X_2$ w.p 1

Ex: $f_{\vec{X}_{-j}/X_j}(\vec{x}_{-j}, x_j) := P(\vec{X}_{-j} = \vec{x}_{-j} / X_j = x_j) = \frac{f_{X_1 \dots X_K}(X_1 \dots X_K)}{f_{X_j}(X_j)} =$

$$\vec{X}_j := \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{j-1} \\ X_{j+1} \\ \vdots \\ X_K \end{bmatrix} \qquad \frac{Multi(n, \vec{p})}{Bin(n, p_j)} = \frac{\frac{\cancel{n!}}{X_1! \dots X_{j-1}! \cancel{X_j!} X_{j+1}! \dots X_K!} p_1^{X_1} \dots p_{j-1}^{X_{j-1}} \cancel{p_j^{X_j}} p_{j+1}^{X_{j+1}} \dots p_K}{\frac{\cancel{n!}}{\cancel{X_j!}(n-X_j)!} \cancel{p_j^{X_j}} (1-p)^{n-X_j}}$$

$$= \frac{(n-X_j)!}{X_1! \dots X_{j-1}! X_{j+1}! \dots X_K!} \cdot \frac{p_1^{X_1} \dots p_{j-1}^{X_{j-1}} p_{j+1}^{X_{j+1}} \dots p_K^{X_K}}{(1-p)^{n-X_j}} = \binom{n'}{X_1 \dots X_{j-1} X_{j+1} \dots X_K}$$

n'

$n = X_1 + \dots + X_{j-1} + X_j + X_{j+1} \dots X_n$

$n' = X_1 + \dots + X_{j-1} + X_{j+1} \dots X_K$

2)

$$\binom{n'}{x_1 \cdots x_{j-1} x_{j+1} \cdots x_k} \frac{p_1^{x_1}}{(1-p_j)^{x_1}} \cdots \frac{p_{j-1}^{x_{j-1}}}{(1-p_j)^{x_{j-1}}} \cdot \frac{p_{j+1}^{x_{j+1}}}{(1-p_j)^{x_{j+1}}} \cdots \frac{p_k^{x_k}}{(1-p_j)^{x_k}}$$

$$= \text{Multi}(n', \vec{p}\,') \quad \text{where } \vec{p}\,' = \begin{bmatrix} \frac{p_1}{1-p_j} \\ \vdots \\ \frac{p_{j-1}}{1-p_j} \\ \frac{p_{j+1}}{1-p_j} \\ \vdots \\ \frac{p_k}{1-p_j} \end{bmatrix}$$ and $\text{dimension}(\vec{p}\,') = K-1$

Expectation of Multinomial & Variance of $Y(\vec{X})$?

$E(\vec{X})$?

$$\mu = E[X] = \sum_{x \in R} x\,p(x) \quad \text{discrete}$$
$$= \int_R x\,f(x)\,dx \quad \text{continuous}$$

$$\sigma^2 = Var(X) = \sum_{x \in R} (x-\mu)^2 p(x) \quad \text{discrete}$$
$$\int_R (x-\mu)^2 f(x)\,dx \quad \text{continuous}$$

note
$\sigma^2 = Var(X) = E[(X-\mu)^2]$
or
$= E[X^2] - \mu^2$

$\sigma = SE[X] = \sqrt{\sigma^2}$

↳ standard of error

$E[aX+c] = a\mu + c$ where $a, c \in \mathbb{R}$ constants

$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E(X_i)$ always there

$E[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} E(X_i)$ if ... $X_1 ... X_2 \overset{iid}{\sim}$ Sometimes true

$$Var[X_1 + X_2] = E[(X_1+X_2) - (\mu_1+\mu_2)^2]$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_2 + 2X_1X_2 + 2\mu_1\mu_2]$$

$$= E[X_1^2] + E[X^2] + \mu_1^2 + \mu_2^2 - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_2\mu_1 - 2\mu_2^2 + 2E[X_1X_2] + 2\mu_1\mu_2$$

Cancel & Combine

$$= \sigma_1^2 + \sigma_2^2 + 2(E[X_1X_2] - \mu_1\mu_2)$$

How

Covariance $\sigma_{12} = Cov[X_1, X_2] = E[X_1X_2] - \mu_1\mu_2 = E[(X_1-\mu_1)(X_2-\mu_2)]$

If $X_1 X_2 \overset{iid}{\sim} \Rightarrow E(X_1X_2) = \mu_1\mu_2 \Rightarrow \sigma_{11} = 0$

If independent $\Rightarrow$ Covariance Variate

Cov Regions