(Lecture 17) November 15th, 2019

$X_1, \dots, X_n \overset{iid}{\sim} \text{w}/\ \mu, \sigma^2$

$\rightarrow \dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$X = \sigma Z + \mu \sim N(\mu, \sigma^2) := \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$Z_i^2 \sim \chi_1^2 := \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$\underbrace{Z_1^2 + \dots + Z_m^2}_{\chi_m^2} + \underbrace{Z_{m+1}^2 + \dots + Z_k^2}_{\chi_{k+m}^2} \sim \chi_k^2 := \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

$X \sim \chi_k^2$

$Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

$X_1 \sim \chi_{k_1}^2$ indep of $X_2 \sim \chi_{k_2}^2$

$R = \dfrac{X_1/k_1}{X_2/k_2} \sim$

$U = \dfrac{Y_1}{k_1} \sim \text{Gamma}\left(\frac{k_1}{2}, \frac{k_1}{2}\right)$

$V = \dfrac{X_2}{k_2} \sim \text{Gamma}\left(\frac{k_2}{2}, \frac{k_2}{2}\right)$

$$R \sim \int_{Supp[U]} f_U(rt) f_V(t) \mathbb{1}_{t \in Supp[V]} |t| \, dt$$

$$= \int_0^\infty \left(\frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \right) \left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} \right) t \, dt$$

Note: $\frac{k_1}{2} = a$, $\frac{k_2}{2} = b$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} \, dt$$

$$= \frac{a^a b^b r^{a-1}}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+b)}{(ar+b)^{a+b}}$$

$Supp[R] = (0, \infty)$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)}$$

$\rightarrow (ar+b)^{-(a+b)} = \left(b\left(1+\frac{a}{b}r\right)\right)^{-(a+b)}$

$$= \frac{a^a b^b}{B(a,b)} \frac{1}{b^a b^b} r^{a-1} \left(1+\frac{a}{b}\right)^{-(a+b)}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1+\frac{k_1}{k_2}r\right)^{-\left(\frac{k_1+k_2}{2}\right)} \mathbb{1}_{r>0}$$

$= F_{k_1,k_2}$, the F-distribution

"Fisher-Snecedor" distribution

Parameter Space: $k_1, k_2 \in \mathbb{N}$

$Z \sim N(0,1)$ indep of $X \sim \chi^2_k$

$W = \frac{Z}{\sqrt{\frac{X}{k}}} \sim ?$ $\quad$ $Supp[W] = \mathbb{R}$

χ^2_1

$W^2 = \frac{Z^2}{\frac{X}{k}} = \frac{\frac{Z^2}{1}}{\frac{X}{k}} \sim F_{1,k}$

χ^2_k

$F_{W^2}(w^2) := P(W^2 \le w) = P(W \in [-w, w])$

$= F_W(w) - F_W(-w)$

$\frac{d}{dw}$ both sides $\rightarrow 2w f_{W^2}(w^2) = f_W(w) - -f_W(-w) = 2f_W(w)$

Assuming $f_W(w) = f_W(-w)$

$2f_W(w) \Rightarrow f_W(w) = 2f_{W^2}(w^2)$

$f_{W^2}(r) = \frac{\frac{k_1}{k_2}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\left(\frac{k_1+k_2}{2}\right)} \mathbb{1}_{r>0}$

$w f_{W^2}(w^2) = w \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} (w^2)^{\frac{1}{2}-1} \left(1 + \frac{1}{k} w^2\right)^{-\left(\frac{1+k}{2}\right)}$

$-\frac{1}{2}$

$\sqrt{2\pi} \rightarrow \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k}\,\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{w^2}{k}\right)^{-\left(\frac{k+1}{2}\right)} = T_k$$

Student's T distribution with k degrees of freedom

$$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$$

$$\frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) = \frac{1}{\pi}\frac{1}{1+r^2}$$

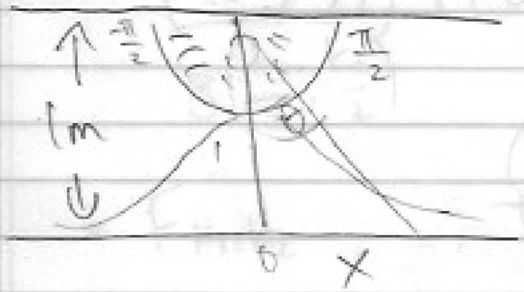
$$X \sim T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}\,\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}}\left(1+r^2\right)^{-1} = \frac{1}{\pi}\frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

$$X = \sigma R + c \sim \text{Cauchy}(c,\sigma) = \frac{1}{\sigma\pi}\frac{1}{1+\left(\frac{r-c}{\sigma}\right)^2}$$

$$\phi_R(t) = E\left[e^{itR}\right] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{itr}}{1+r^2}\,dr = \underset{\uparrow\ \text{Complex Analysis}}{\cdots} = e^{-|t|}$$

$$\phi_R'(t) = \frac{t}{|t|}e^{-|t|} = \begin{cases} e^{-|t|} & \text{if } t>0 \\ -e^{|t|} & \text{if } t<0 \\ \text{undefined} & \text{if } t=0 \end{cases}$$

Assume light from light source uniform



ceiling

floor

What is the distribution of light on the floor?

$$\tan(\theta) = \frac{x}{1} \qquad \theta = \arctan(x) = g^{-1}(x)$$

$$\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

$$f_X(x) = f_\theta\left(g^{-1}(x)\right) \left| \frac{d}{dx}\left[g^{-1}(x)\right] \right|$$

$$= \frac{1}{\pi} \underbrace{\mathbb{1}_{\arctan(\theta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}}_{x \in (-\infty, \infty)} \left| \frac{1}{1+x^2} \right|$$

$\uparrow$ always > 0

$$= \frac{1}{\pi} \frac{1}{1+x^2}$$

$$= \text{Cauchy } (0,1)$$

Statistics Application

$$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2), \quad X_1 + \dots + X_n \sim N(n\mu, n\sigma^2)$$

$$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$\uparrow$

estimator for μ

$$S^2 := \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

Big $X \to$ r.v.

Small $x \to$ realizations

$\uparrow$

estimator for σ^2

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \overline{x}\right)^2$$

$\uparrow$

estimate

$$S^2 = \frac{1}{n-1}\left((x_1-\bar{x}) + (x_2-\bar{x}) + \dots + (x_n-\bar{x})\right) \sim ?$$

$$Z_1, \dots Z_n \overset{iid}{\sim} N(0,1)$$

$$\to \sum Z_i^2 \sim \chi_n^2$$

let $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \to \vec{Z}^T \vec{Z} \sim \chi_n^2$

$$Z_i = \frac{x_i-\mu}{\sigma} \sim N(0,1) \to \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim \chi_n^2$$

$$x_i - \mu = x_i - \bar{x} + \bar{x} - \mu$$

$$(x_i-\mu)^2 = \left((x_i-\bar{x}) + (\bar{x}-\mu)\right)^2$$

$$= (x_i-\bar{x})^2 + \underbrace{2(x_i-x)(\bar{x}-\mu)}_{2(x_i\bar{x} - \bar{x}^2 - x_i\mu + \bar{x}\mu)} + (\bar{x}-\mu)^2$$

$$\sum\left(\frac{x_i-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 + 2\sum(x_i\bar{x} - \underset{n x^2}{x^2} - \overset{n\bar{x}\mu}{x_i\mu} + \overset{n\bar{x}\mu}{\bar{x}_n}) + \sum(x-\mu)^2$$

$$= \frac{1}{\sigma^2}\left(\sum(x_i-\bar{x})^2 + n(x-\mu)^2\right)$$

$$= \frac{(n-1)S^2}{\sigma^2} + \frac{n(x-\mu)^2}{\sigma^2}$$