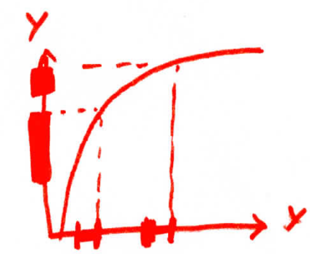
Lecture #18

- $X \sim \text{Beta}(\alpha, \beta) = \underbrace{\frac{1}{B(\alpha,\beta)}}_{C} x^{\alpha-1}(1-x)^{\beta-1} \underset{\text{kernel}}{\propto} x^{\alpha-1}(1-x)^{\beta-1}$

- New topic:
  X is a continuous random variable, g is a 1-1 function & $Y = g(X)$

$\Rightarrow$

$$f_Y(y) = f_X(g^{-1}(y)) \underbrace{\left| \frac{d}{dy}[g^{-1}(y)] \right|}_{\text{stretching/compressing}}$$



- Let $\vec{X}$ be a vector r.v. continuous with dimension n & $f_{\vec{X}}(\vec{x})$ known. Let $\vec{Y} = g(\vec{X})$ where $g: \mathbb{R}^n \to \mathbb{R}^n$ with an inverse, h (i.e. $\vec{X} = h(\vec{Y})$) and find $f_{\vec{Y}}(\vec{y})$

$$\begin{aligned} Y_1 &= g_1(X_1 \cdots X_n) \\ Y_2 &= g_2(X_1 \cdots X_n) \\ &\vdots \\ Y_n &= g_n(X_1 \cdots X_n) \end{aligned} \quad \text{and} \quad \begin{aligned} X_1 &= h_1(Y_1 \cdots Y_n) \\ X_2 &= h_2(Y_1 \cdots Y_n) \\ &\vdots \\ X_n &= h_n(Y_1 \cdots Y_n) \end{aligned}$$

Then, $f_Y(y) = f_X(h(y)) \left| \frac{d}{dy}[h(y)] \right|$ it becomes

$$\underbrace{f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y})) \, |J_n|}_{\text{change of variable formula}} \quad \text{where } J_n := \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

~~and the~~ $\Rightarrow$ "Jacobian Determinant"

- $T = X_1 + X_2 \sim f_T(t)$? $\Rightarrow$ Goal
  1. Find a "clever" g so that we can...
  2. Find h (the inverse)
  3. Compute Jacobian Determinant
  4. Substitute into change of variables formula
  5. Integrate our nuisana dimensions

① $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$

$Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - (-1 \cdot 0) = 1$

~~④ $f_Y(y)$~~

④ $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)\,|1|$

⑤ Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy \Rightarrow$ "margining"

$X_1, X_2 \overset{ind}{\sim}$ (Product Rule)

$\Rightarrow f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\,du = \int_{\mathbb{R}} f_{X_1}(t-u)\, f_{X_2}(u)\,du$

$= \int_{Supp[X_1]} f_{X_1}(t-u) f_{X_2}(u)\, \mathbb{1}_{u \in Supp[X_2]}\,du$

A formula for the ratio.

• $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

We need a clever g

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$

$Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 \cdot X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

Now, find inverses

③ $\det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2 \cdot 1 - (y_1 \cdot 0) = y_2$

④ $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2)\,|y_2|$

⑤ Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$

$\Rightarrow f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u)\,|u|\,du$

$X_1, X_2 \overset{ind}{\sim}$

$= \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\,|u|\,du$

$= \int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u)\, \mathbb{1}_{u \in Supp[X_2]}\,|u|\,du$.

• $R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ?$

A target

① $Y_1 = \frac{X_1}{X_1+X_2} = g_1(X_1, X_2)$

$Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1(X_1+X_2) = Y_1(Y_2) = h_1(Y_1, Y_2)$

$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix} = y_2(1-y_1) - (y_1 \cdot (-y_2))$

$= y_2 - y_1 y_2 + y_1 y_2$

$= y_2$

nonsense u.

④ $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2 - y_1 y_2)|y_2|$

⑤ $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u - ru) \, |u| \, du$

$\Rightarrow$ if X_1 & X_2 ~ ind $\int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u-ru) |u|$ €

$\Rightarrow \int_{supp(X_1)} f_{X_1}(ru) f_{X_1}(u-ru) \mathbb{1}_{[u-ru \in supp(X_2)]} |u| du$

• Use the formulas that we just derived:

$X_1 \sim$ Gamma (α_1, β) ind. of $X_2 \sim$ Gamma (α_2, β)

$R = \frac{X_1}{X_1+X_2} \sim \boxed{\int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u-ru)|u| du}$ → The formula that we'll be using

Note that both Gamma supp: $(0, \infty)$

or u is fine

* $Supp[R] = (0,1)$

* $\int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1-1}}_{r^{\alpha_1-1} u^{\alpha_1-1}} e^{-\beta ru} \right) \cdot \left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} \underbrace{(u-ru)^{\alpha_2-1}}_{u^{\alpha_2-1}(1-r)^{\alpha_2-1}} e^{-\beta(u-ru)} \mathbb{1}_{u-ru \in (0,\infty)} |u| du \right.$

$u(1-r) \in (0, \infty)$

$u \in (0, \infty)$

Note: $(1-r) = (0,1)$

So, here it becomes 1

$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta u} du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{\beta^{\alpha_1+\alpha_2}}}$

$$= \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}}_{\frac{1}{B(\alpha_1,\alpha_2)}} r^{\alpha_1 \ldots} (1-r) \ldots \quad = (\text{Beta}(\alpha_1, \alpha_2)$$

dist

b/c of supp

just to think about

- X_1 & X_2 same as above.

$$R = \frac{X_1}{X_2} \sim \quad \boxed{\int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\, |u|\, du} = \text{The other one.}$$

$\text{supp}[R] = (0,\infty)$

$$\int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1-1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u)^{\alpha_2-1} e^{-\beta u} \underbrace{\mathbb{1}_{u\in(0,\infty)}}_{\text{is } 1}\right) |u|\, du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u}\, du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{(\beta(r+1))^{\alpha_1+\alpha_2}}}$$

$$(\beta(r+1))^{\alpha_1+\alpha_2} = (\beta^{\alpha_1+\alpha_2})(r+1)^{\alpha_1+\alpha_2}$$

$$= \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r\in(0,\infty)}$$

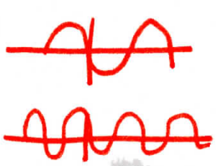
$= \text{BetaPrime}(\alpha_1, \alpha_2)$

- p155 ⇒ conditional density

Consider $X \sim U(0,1)$

$Y \mid X = x \sim U(0, x)$

* $f(x; a) = \sin(ax)$



b/c of a