Math 368 / 621 Fall 2019
Final Examination

Solutions

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Full Name ____________________________________ Circle Section and Class: A B C 368 621

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signature date

Instructions

This exam is 120 minutes and closed-book. You are allowed three pages (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in choose, permutation, exponent, factorial or any other notation which could be resolved to a number with a computer. Questions marked "[Extra Credit]" are extra credit for both 368 and 621 students. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 For all problems below, let $Z_1, Z_2, \ldots \overset{iid}{\sim} \mathcal{N}(0,\, 1)$ and let the column random variable vector $\boldsymbol{Z} = [Z_1 \ \ldots \ Z_n]^\top$ where n is finite. Consider $a, b, c \in \mathbb{N}$ and three matrices B_1, B_2 and B_3 where

$$\begin{aligned} B_1 + B_2 + B_3 &= I_n, \\ \text{rank}\,[B_1] &= a, \\ \text{rank}\,[B_2] &= b, \\ \text{rank}\,[B_3] &= c, \\ a + b + c &= n. \end{aligned}$$

(a) [20 pt / 20 pts] Each one of the following expressions is distributed as a r.v. we learned about. Write explicitly the PDF or, more recommended is to use brand name notation e.g "$\sim$ Beta$(a,\, b)$". Make sure you make all parameters as clear as possible. Some are challenging — leave those blank until the end of the exam. The first one is done for you as an example.

i) $Z_{17}^2 + Z_{37}^2 + Z_{1984}^2 \sim \chi^2_3$

2 ii) $|Z_{17}| \sim \chi_1$

1 iii) $\boldsymbol{Z} \sim N_n(\vec{0}_n, I_n)$

1 iv) $\dfrac{Z_1}{Z_2} \sim$ Cauchy$(0, 1)$

2 v) $\boldsymbol{1}_n^\top \boldsymbol{Z} \sim N(0, n)$

1 vi) $\boldsymbol{Z}^\top (B_1 + B_2 + B_3) \boldsymbol{Z} \sim \chi^2_n$

1 vii) $\boldsymbol{Z}^\top B_2 \boldsymbol{Z} \sim \chi^2_b$

2 viii) $\boldsymbol{Z}^\top (I_n - B_2) \boldsymbol{Z} \sim \chi^2_{n-b}$
$= \chi^2_{a+c}$

2 ix) $\boldsymbol{Z}^\top B_1 \boldsymbol{Z}/a \sim$ Gamma $\left(\frac{a}{2}, \frac{a}{2}\right)$

1 x) $\dfrac{Z_{n+1}}{\sqrt{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}/a}} \sim T_a$

2 xi) $\dfrac{Z_{n+1}^2}{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}/a} \sim F_{1,a}$

1 xii) $\dfrac{\boldsymbol{Z}^\top B_3 \boldsymbol{Z}/c}{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}/a} \sim F_{c,a}$

2 xiii) $\dfrac{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}}{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}} \sim$ Deg(1)

2 xiv) $\dfrac{\boldsymbol{Z}^\top B_3 \boldsymbol{Z}}{\boldsymbol{Z}^\top B_1 \boldsymbol{Z}} \sim$ BetaPrime $\left(\frac{c}{2}, \frac{a}{2}\right)$

$$\underbrace{}^{A}\ \underbrace{}^{\vec{Z}}$$

(b) [5 pt / 25 pts] Let $\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. Find the PDF of $\boldsymbol{X}$ without using any vector or matrix notation (i.e. the PDF must be a function of x_1, x_2, numbers and fundamental constants) and simplify.

$$\vec{X} = A\vec{Z} \sim N_2(\vec{0}, \underbrace{AA^T}_{\Sigma}) = \frac{1}{\sqrt{(2\pi)^2 \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{0})^T \Sigma^{-1}(\vec{x}-\vec{0})}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}[x_1\ x_2]\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad = \frac{1}{2\pi} e^{-\frac{1}{2}[x_1\ x_2]\begin{bmatrix} x_1 - x_2 \\ 2x_2 - x_1 \end{bmatrix}}$$

$$\det[\Sigma] = 2\cdot 1 - 1\cdot 1 = 1 \qquad = \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2 - 2x_1x_2 + 2x_2^2)}$$

$$\Sigma^{-1} = \frac{1}{\det[\Sigma]}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_2 - x_1 \end{bmatrix}, \quad [x_1\ x_2]\begin{bmatrix} x_1 - x_2 \\ 2x_2 - x_1 \end{bmatrix} = (x_1^2 - x_1x_2) + (2x_2^2 - x_1x_2) = x_1^2 - 2x_1x_2 + 2x_2^2$$

Problem 2 Consider

$$X_1, X_2 \overset{iid}{\sim} \text{Bernoulli}(p)$$

(a) [6 pt / 31 pts] Prove that $T = X_1 + X_2 \sim \text{Binomial}(2, p)$ using the discrete convolution formula.

$$p_T(t) = \sum_{x \in Supp(X)} p(x)\, p(t-x)\, \mathbb{1}_{t-x \in Supp(X)}$$

$$= \sum_{x \in \{0,1\}} p^x(1-p)^{1-x}\, p^{t-x}(1-p)^{1-t+x}\, \underbrace{\mathbb{1}_{t-x \in \{0,1\}}}_{t \in \{x, x+1\}}$$

$$= p^t(1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x,x+1\}} = \begin{cases} 1 & \text{if } t=0 \\ 2 & \text{if } t=1 \\ 1 & \text{if } t=2 \\ 0 & \text{o/w} \end{cases} = \binom{2}{t}$$

$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}}\right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t} = \text{Binom}(2, p)$$

Problem 3 Consider

$$\boldsymbol{X} \sim \text{Multinomial}\left(n,\ [p_1\ p_2\ \dots\ p_K]^\top\right)$$

where both n and $[p_1\ p_2\ \dots\ p_K]^\top$ are in the parameter space for the multinomial and $K > n$. Its ch.f is

$$\phi_X([t_1\ t_2\ \dots\ t_K]^\top) = \left(p_1 e^{it_1} + p_2 e^{it_2} + \ldots + p_K e^{it_K}\right)^n.$$

(a) [6 pt / 37 pts] Write the PMF of $\boldsymbol{X}$ valid for all $\boldsymbol{x} \in \mathbb{R}^K$ using the gamma function.

$$p_{\vec{X}}(\vec{x}) = \binom{n}{x_1 \dots x_K} p_1^{x_1} \cdots p_K^{x_K}$$

$$= \frac{n!}{x_1! \cdots x_K!} p_1^{x_1} \cdots p_K^{x_K} \mathbb{1}_{\vec{x} \in \mathbb{N}^K} \mathbb{1}_{\vec{1}^\top \vec{x} = n}$$

$$= \frac{\Gamma(n+1)}{\Gamma(x_1+1) \cdots \Gamma(x_K+1)} p_1^{x_1} \cdots p_K^{x_K} \mathbb{1}_{\vec{x} \in \mathbb{N}^K} \mathbb{1}_{\vec{1}^\top \vec{x} = n}$$

(b) [3 pt / 40 pts] Find $\text{Cov}[X_2, X_3]$.

$-n p_2 p_3$

(c) [4 pt / 44 pts] Find $\mathbb{P}\left(\boldsymbol{X} = [0\ 0\ n\ 0\ 0\ \dots\ 0]^\top\right)$

p_3^n

(d) [4 pt / 48 pts] Find $\mathbb{P}\left(\boldsymbol{X} = [0\ K\ 0\ 0\ 0\ \dots\ 0]^\top\right)$

0 since $K > n$

(e) [6 pt / 54 pts] Prove that $X_2 \sim \text{Binomial}(n, p_2)$.

$$\phi_{X_2}(t) = \phi_{\vec{X}}\left([0\ t\ 0\ \cdots\ 0]^T\right) = \left(p_1 e^{i(0)} + p_2 e^{i(t)} + p_3 e^{i(0)} + \cdots + p_K e^{i(0)}\right)^n$$

$$= \left(p_2 e^{it} + p_1 + p_3 + \cdots + p_K\right)^n$$

Note: $p_1 + p_2 + p_3 + \cdots + p_K = 1$

$\Rightarrow p_1 + p_3 + \cdots + p_K = 1 - p_2$

$$= \left(1 - p_2 + p_2 e^{it}\right)^n \leftarrow \text{ch.f. of Binomial}$$

$$\overset{(P1)}{\Longrightarrow} X_2 \sim \text{Binom}(n, p_2)$$

Problem 4 This question is about indicator functions

(a) [3 pt / 57 pts] Expand and simplify as much as you can: $\sum_{x \in \mathbb{R}} x \mathbb{1}_{x \in \{-1,0,1\}}$.

$-1 + 0 + 1 = 0$

(b) [3 pt / 60 pts] Expand and simplify as much as you can: $\prod_{x \in \mathbb{R}} \mathbb{1}_{x \in \{-1,0,1\}}$.

0

Problem 5 Consider

$$Y \sim \text{Laplace}(0, b) \quad \text{and} \quad \phi_Y(t) = \frac{1}{1 + b^2 t^2}$$

(a) [3 pt / 63 pts] Write a complex integral expression that will recover the PDF of Y using its ch.f. Do not evaluate.

By (P6) $f_Y(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ity} \frac{1}{1 + b^2 t^2}\, dt$ since $\frac{1}{1+b^2t^2} \in L^1$

(b) [4 pt / 67 pts] Show that $\mathbb{V}\mathrm{ar}[Y] = 2b^2$.

$Var[Y] = E[Y^2] - E[Y]^2 = 2b^2 - 0^2 = 2b^2$

$E[Y] = \frac{\phi_Y'(0)}{i} = \frac{0}{i} = 0$

$\phi_Y'(t) = -(1+b^2t^2)^{-2}(2b^2t) \Rightarrow \phi_Y'(0) = -(1+b^2(0))^{-2}(2b^2(0)) = 0$

$E(Y^2) = \frac{\phi_Y''(0)}{i^2} = -\phi_Y''(0) = -(2b^2) = 2b^2$

$\phi_Y''(t) = -((1+b^2t^2)^{-2}(2b^2) + -2(1+b^2t^2)^{-3}(2b^2t)^2) \Rightarrow \phi_Y''(0) = -2b^2$

(c) [5 pt / 72 pts] Let $X_n \sim \text{Laplace}\left(0, \frac{1}{n}\right)$. Prove $X_n \xrightarrow{d} X$ where $X \sim \text{Deg}(0)$ *without* using the answer from the next question.

$\phi_{X_n}(t) = \frac{1}{1+\frac{t^2}{n^2}} = \frac{n^2}{n^2+t^2}$

Lévy's Cont. Thm

PB

$\lim_{n\to\infty} \phi_{X_n}(t) = 1 = \phi_X(t)$ where $X \sim Deg(0)$ $\Rightarrow X_n \to Deg(0)$

This can also be solved by showing CDF convergence, but it is messier.

(d) [4 pt / 76 pts] Let $X_n \sim \text{Laplace}\left(0, \frac{1}{n}\right)$. Prove $X_n \xrightarrow{\mathbb{L}^2} 0$.

$\lim E[|X_n - 0|^2] = \lim E[X_n^2] = \lim \frac{2}{n^2} = 0$ ✓

(b)

Problem 6

Consider

$$X_n \sim \text{ParetoI}(k, n)$$

(a) [4 pt / 80 pts] What is $f_{X_n}(x)$? Make sure the function is valid for all $x \in \mathbb{R}$.

$f_{X_n}(x) = \frac{nk^n}{x^{n+1}} \mathbb{1}_{x\in(k,\infty)}$

(b) [5 pt / 85 pts] Prove $X_n \xrightarrow{p} k$.

Note: $F_{X_n}(x) = 1 - \left(\frac{k}{x}\right)^n$ for $x > k$ $\Rightarrow 1 - F_{X_n}(x) = \left(\frac{k}{x}\right)^n$ for $x > k$

$\lim P(|X_n - k| \geq \varepsilon) = \lim P(X_n - k < -\varepsilon) + \lim P(X_n - k > \varepsilon)$

$= \lim \underbrace{P(X_n < k - \varepsilon)}_{0} + \lim P(X_n > k+\varepsilon) = \lim \left(1 - F_{X_n}(k+\varepsilon)\right) = \lim \left(\frac{k}{k+\varepsilon}\right)^n = 0$ ✓

Since $\text{supp}(X_n) = (k, \infty)$ and $\varepsilon > 0 \Rightarrow k - \varepsilon < k$

Since $\frac{k}{k+\varepsilon} \in (0,1)$

Since $\varepsilon > 0$, $k > 0$.

Problem 7 Consider $X \sim \text{Beta}(\alpha, \beta)$.

(a) [5 pt / 90 pts] Let $Y = \sqrt{X}$. Find the PDF of Y. Do not simplify.

$X = Y^2 = g^{-1}(Y)$, $\frac{d}{dy}[g^{-1}(y)] = 2y$

$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{B(\alpha,\beta)} (y^2)^{\alpha-1}(1-y^2)^{\beta-1} 2y \, \mathbb{1}_{y \in (0,1)}$

Problem 8 Below are some questions about inequalities.

(a) [4 pt / 94 pts] Prove Markov's inequality from first principles. Make sure you state all assumptions clearly.

Let X be a non-negative r.v. with finite mean μ. Let a be a positive constant.

The following inequality is true:

$a \mathbb{1}_{X \geq a} \leq X$

$\Rightarrow E[a \mathbb{1}_{X \geq a}] \leq E(X) = \mu$

$\Rightarrow a E[\mathbb{1}_{X \geq a}] \leq \mu \Rightarrow E[\mathbb{1}_{X \geq a}] \leq \frac{\mu}{a} \Rightarrow P(X \geq a) \leq \frac{\mu}{a}$

(b) [3 pt / 97 pts] Let X_1 be a r.v. with mean zero and variance σ_1^2 and X_2 be a r.v. with mean zero and variance σ_2^2. Show that $\mathbb{E}[|X_1 X_2|]$ cannot be more than $\sigma_1\sigma_2$. If you use any of the results from class, make sure you cite those results at the point in your proof where they are used.

$$E[|X_1 X_2|] \underset{\uparrow}{\le} \sqrt{E(X_1^2)E(X_2^2)} = \sqrt{(\sigma_1^2+\cancel{\mu_1^2})(\sigma_2^2+\cancel{\mu_2^2})} = \sqrt{\sigma_1^2\sigma_2^2} = \sigma_1\sigma_2$$

By Cauchy-Schwarz (or Holder's)

Problem 9 Consider

$$Y \mid X = x \sim \text{Exp}\left(\frac{1}{x}\right) \quad \text{and} \quad X \sim \text{Exp}(1).$$

(a) [3 pt / 100 pts] Find $\mathbb{E}[Y]$.

$$E(Y) = E_X\left[E_Y[Y \mid X=x]\right] = E_X\left[\frac{1}{\frac{1}{X}}\right] = E_X(X) = 1.$$

Problem 10 Below are extra credit exercises.

(a) [4 pt / 104 pts] [Extra Credit] Let r.v. X have a PMF or PDF which is even. Prove that its ch.f. is real. For example, the PDF of the Laplace with mean zero is even and its ch.f. is real. You must show all work clearly to get credit.

(b) [4 pt / 108 pts] [Extra Credit] Place three points inside a circle at random. Connect the three points by lines to form a triangle. What is the probability the triangle contains the center of the circle? You must show all work clearly to get credit.