Lec 16 Math 621 11/11/19

$X_1, \dots, X_n \overset{iid}{\sim}$ r.v.'s with finite expectation μ and finite variance σ^2

$T_n = X_1 + \dots + X_n$, the r.v. of the sum

$\bar{X}_n = \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$, the r.v. of the average

Poisson Process Review

$X_1, X_2, \dots \overset{iid}{\sim} Exp(\lambda)$ each represent a waiting time

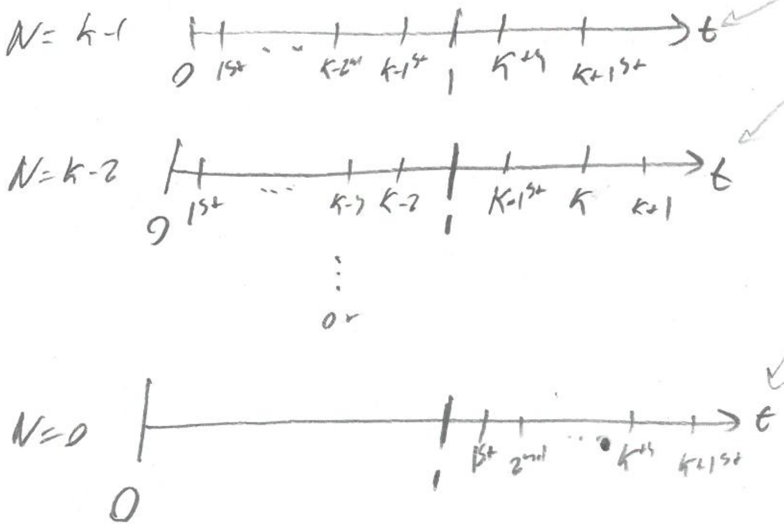
$T_k = X_1 + X_2 + \dots + X_k$ represents waiting for k of them

$T_k \sim Erlang(k, \lambda)$. Let N be # arrivals in $t \in [0,1]$

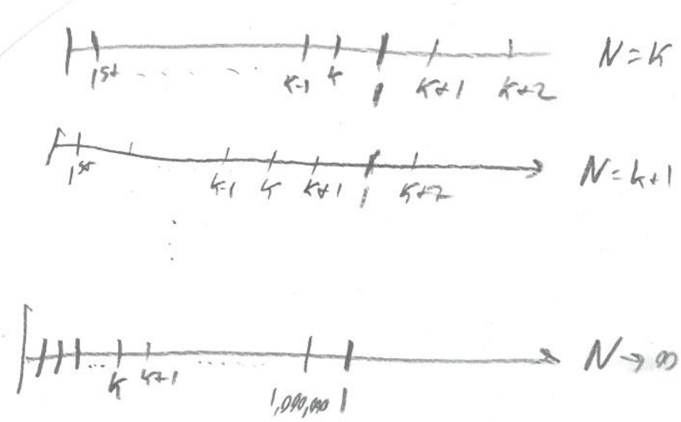
$N \sim Poisson(\lambda)$

The event $\{T_k > 1\}$ means either

the event $\{N \leq k-1\}$ means either



$\{T_k < 1\}$ means $\{N \geq k\}$



Note: $E(\overline{X}_n) = \mu$, $Var(\overline{X}_n) = \frac{\sigma^2}{n}$ (from Math 2#1)

let $Z_n := \dfrac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$, the standardized avg. $E(Z_n) = 0$, $Var(Z_n) = 1$ $= SE(Z_n)$

$$= \frac{\sqrt{n}}{\sigma}\overline{X}_n + -\frac{\sqrt{n}}{\sigma}\mu$$

$$\phi_{T_n}(t) \underset{(P3)}{=} \left(\phi_X(t)\right)^n$$

$$\phi_{\overline{X}_n}(t) \underset{(P2)}{=} \phi_T\left(\tfrac{1}{n}t\right) = \left(\phi_X\left(\tfrac{t}{n}\right)\right)^n$$

$$\phi_{Z_n}(t) \underset{(P2)}{=} e^{-it(\;)}\left(\phi_X\left((\;)\frac{t}{n}\right)\right)^n = e^{\frac{-it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}}\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = e^{-\frac{it\mu n}{\sigma\sqrt{n}}}\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

$$= e^{\frac{-it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n\right)}$$

$$= e^{\frac{-it\mu n}{\sigma\sqrt{n}} + n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}$$

$$= e^{n\left(-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\right)}$$

$$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}}\cdot\frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n\to\infty \Rightarrow u\to 0$

Examine; in hopes to use (P8)

$$\lim_{n\to\infty}\phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)} = e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

L'Hopital's Rule

$$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}} \overset{\text{L'Hopital's}}{=} e^{\frac{t^2}{2\sigma^2}\frac{d}{du}\left[\frac{\phi_X'(u)}{\phi_X(u)}\right]}$$

$$= e^{\frac{t^2}{2\sigma^2}} \lim_{u \to 0} \frac{\phi_X(u)\,\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2} = e^{\frac{t^2}{2\sigma^2}} \left(\frac{\phi_X(0)\,\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2} \right)$$

$$\underset{(P0)}{=} e^{\frac{t^2}{2\sigma^2}} \left(\phi_X''(0) - \phi_X'(0)^2 \right) \underset{(P7)}{=} e^{\frac{t^2}{2\sigma^2}} \left(i^2 E[X^2] - (iE[X])^2 \right) = e^{\frac{t^2}{2}} \overset{-1}{i^2} \underbrace{\left(E(X^2) - E(X)^2 \right)}_{\sigma^2}$$

$$= e^{-\frac{t^2}{2}}$$

$\phi_{Z_n}(t) \longrightarrow e^{-\frac{t^2}{2}} \overset{(P8)}{\Longrightarrow} Z_n \xrightarrow{d} Z$ some r.v. with ch.f. $e^{-\frac{t^2}{2}}$.

What's the density of this r.v.? Use (P6) Is $\phi_{Z_n}(t) \in L^1$? so...

Yes $\int_{\mathbb{R}} e^{-t^2/2}\,dt = \sqrt{2\pi} < \infty$ (Gaussian integral)

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \phi_Z(t)\,dt = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} e^{-\frac{t^2}{2}}\,dt = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)}\,dt$$

Note $\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2} \right)^2 - \left(\frac{\sqrt{2}\,iz}{2} \right)^2$

$$= \frac{t^2}{\sqrt{2}} + 2\,\frac{\sqrt{2}}{2}\,i\,\frac{t}{\sqrt{2}} + \underbrace{\frac{i^2 z^2}{2}}_{-\frac{z^2}{2}} - \underbrace{\frac{i^2 z^2}{2}}_{-\frac{z^2}{2}} \quad ✓$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\left(-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2} \right)^2 + \left(\frac{\sqrt{2}\,iz}{2} \right)^2 \right)} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2} \right)^2} e^{-\frac{z^2}{2}}\,dt$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2} \right)^2}\,dt =$$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}$ $\quad \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,dy$

$t=\infty \Rightarrow y=\infty$, $t=-\infty \Rightarrow y=-\infty$

Gaussian Integral $= \sqrt{\pi}$

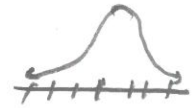
$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\,dy = \frac{1}{\sqrt{2}\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2}\,dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$= N(0,1)$$

Central Limit Thm: If $X_1, X_2, \ldots \overset{iid}{\sim}$ with mean μ, variance σ^2, then

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

we will see applications of this later



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where $\mu \in \mathbb{R}, \sigma > 0$

$Z \sim N(0,1)$, let $X = \mu + \sigma Z \sim \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$

"standard normal" $\quad$ "the normal"

$E[Z] = \phi_Z'(0) = 0e^{-\frac{0^2}{2}} = 0$

$\phi_Z'(t) = te^{-\frac{t^2}{2}}$

$Var(Z) = E(Z^2) - E(Z)^2 = E(Z^2) = 1$

$E(Z^2) = \phi_Z''(0) = 0^2 e^{-\frac{0^2}{2}} + e^{-\frac{0^2}{2}} = 1$

$\phi''(t) = t^2 e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}$

$\Rightarrow$

$E(X) \overset{E[\mu+\sigma Z]}{=} \mu$, $Var(X) \overset{Var[\mu+\sigma Z]}{=} \sigma^2$

$\phi_X(t) \overset{P2}{=} e^{it\mu} e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep of

$X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 + X_2 \sim ?$

$$\phi_{X_1+X_2}(t) \overset{P3}{=} \left(e^{it\mu_1 + \frac{\sigma_1^2 t^2}{2}}\right)\left(e^{it\mu_2 + \frac{\sigma_2^2 t^2}{2}}\right)$$

$$= e^{it(\mu_1+\mu_2) + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

P1

$\Rightarrow X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Convolution HARD!!

$Z \sim N(0,1)$ $\quad E(Z) = \frac{\phi'(0)}{i} = \frac{0}{i} = 0$

$\phi_Z'(t) = -t e^{-t^2/2}$ $\quad E(Z^2) = \frac{\phi''(0)}{i^2} = --1 = 1$ $\quad \Rightarrow Var(Z) = E(Z^2) - E(Z)^2 = 1^2 - 0^2 = 1$

$\phi_Z''(t) = -\left(t^2 e^{-t^2/2} + e^{-t^2/2}\right)$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

$X \sim N(\mu, \sigma^2) \sim ?$ $\quad \frac{1}{|\sigma|} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$X \sim N(\mu, \sigma^2)$, $\quad Y = e^X \sim f_X(\ln(y)) \left|\frac{1}{y}\right| = \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}$

$= LogN(\mu, \sigma^2)$

"log-normal"

Because its log is normally distr.

$\Downarrow$

$X = \ln(Y) = g^{-1}(Y)$

$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{y}$

there are finance applications

LogN is cool... but we are running behind...

$Z \sim N(0,1)$, $\quad Y = Z^2 = g(Z)$ $\quad$ not 1:1

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 2\left(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{=\frac{1}{2} \text{ due to symmetry}}\right) = 2\left(F_Z(\sqrt{y}) - \tfrac{1}{2}\right) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = 2\left(\tfrac{1}{2} y^{-\frac{1}{2}}\right) f_Z(\sqrt{y}) = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \propto y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}_{y>0}$$

$\cdot\, \mathbb{1}_{\sqrt{y} \in (-\infty, \infty)}$ $\quad \propto Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

$\parallel$

$\mathbb{1}_{y \in (0,\infty)}$

$Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$$X = Z_1^2 + Z_2^2 + \dots + Z_k^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$$

This distr has one parameter, k

$$X \sim \chi^2_k := \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x \geq 0}$$

"Chi-squared with k degrees of freedom"

we will discuss this later

$$= \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x \geq 0}$$

$$\chi^2_1 = \frac{1}{2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} e^{-\frac{x}{2}} \mathbb{1}_{x \geq 0} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}} \mathbb{1}_{x \geq 0}$$

Note: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

If $X \sim \chi^2_1$, $Y = \sqrt{X} \sim ?$

$\Rightarrow X = Y^2 = g^{-1}(y)$ 1:1 for $Supp(X)$

$\frac{d}{dy}[g^{-1}(y)] = 2y$

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}} |2y| \mathbb{1}_{y \geq 0} = \frac{1}{2^{k/2-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-y^2/2} \mathbb{1}_{y \geq 0}$$

$$= \chi_k$$

Chi with k degrees of freedom

$X \sim N(0,1)$

$$|X| \sim ?\ |X| = \sqrt{X^2} \sim \chi_1 = \frac{1}{2^{1/2-1}\Gamma\left(\frac{1}{2}\right)} x^{1-1} e^{-x^2/2} \mathbb{1}_{x \geq 0}$$

$$= 2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbb{1}_{x \geq 0}$$ which is $2\times$ density of $N(0,1)$.

Makes sense!