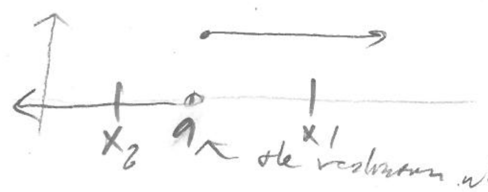
Lecture 21 12/6/19

New Material

Let X be a r.v. with non-neg support i.e. $Supp(X) \geq 0$ and finite expectation. Let $a \geq 0$, a constant. Consider:

$$a\mathbb{1}_{X \geq a}$$



the restriction which will be random

Is $a\mathbb{1}_{X\geq a} \leq X$?

If $X \geq a \Rightarrow a(1) \leq X \Rightarrow a \leq X \Rightarrow X \geq a$ i.e. same as antecedent

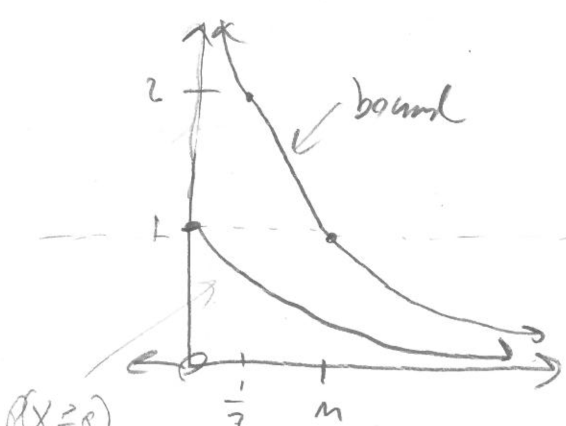
If $X < a \Rightarrow a(0) \leq X \Rightarrow 0 \leq X \Rightarrow X \geq 0$ true by premise

$\Rightarrow$ Yes

Now let's take expectation of both sides

"Markov's Inequality"

$$E[a\mathbb{1}_{X\geq a}] \leq E[X] \Rightarrow a\,E[\mathbb{1}_{X\geq a}] \leq \mu \Rightarrow \underbrace{P(X\geq a)}_{1-F_X(a)} \leq \frac{\mu}{a}$$



this bound is "crude" means seldom informative e.g. $X \sim Exp(1) \Rightarrow P(X \geq a) = e^{-a}$

a	$P(X \geq a)$	Markov Bound	Chebyshev's Bound	Chernoff
2	0.1353	0.5	1	0.73526
5	0.0067	0.2	0.0635	0.09158
10	0.00007	0.1	0.0.123	0.00123
⋮				

Tons of Corollaries

* let $b = a\mu$ $\quad P(X \geq b) \leq \frac{\mu}{b} \Rightarrow P(X \geq a\mu) \leq \frac{1}{a}$

* let $h(x)$ be a monotonically increasing function (thus 1:1)

$$P(h(X) \geq h(a)) \leq \frac{E[h(X)]}{h(a)}$$

$$\Rightarrow P(X \geq a) \leq \frac{E[h(X)]}{h(a)}$$

if X is continuous

* let $a = \text{Quantile}(X, p) = F_X^{-1}(p)$

e.g. $p = \frac{1}{2}$

$\text{Med}(X) \leq 2\mu$

$P(X \geq a) \leq \frac{\mu}{a}$

$1 - F_X(a) \leq \frac{\mu}{a}$

relates exp. to quantile!

$1 - F_X(F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow 1 - p \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow F_X^{-1}(p) \leq \frac{\mu}{1-p}$

* Let X be any r.v. (i.e. with possible negative support)

$\Rightarrow |X|$ has non-negative support — both tails now

$$\Rightarrow P(|X| \geq a) \leq \frac{E(|X|)}{a}$$ All above apply as well

Note $|X| = X\mathbb{1}_{X \geq 0} - X\mathbb{1}_{X<0} \Rightarrow E[|X|] = E[X\mathbb{1}_{X\geq 0}] - E[X\mathbb{1}_{X<0}]$

* Let X be any r.v. with finite mean and variance

let $Y := (X-\mu)^2$

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2}$$ by Markov's Inequality

$$= P\left((X-\mu)^2 \geq a^2\right) \leq \frac{E\left((X-\mu)^2\right)}{a^2}$$

$$= P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ Chebyshev's Inequality

difference between realization and mean is bounded by its variance. This is tighter! How to use this for one tail of a pos r.v.?

Assume $supp(X) \geq 0$

def of abs. value — addition for disjoint events

$$P(|X-\mu| \geq a) = P\left(X-\mu \geq a \;\cup\; -(X-\mu) \geq a\right) = P(X-\mu \geq a) + P(X-\mu \leq -a)$$

$$= P(X \geq \mu+a) + P(X \leq \mu-a) \xRightarrow{\text{if } a \geq \mu} P(X \geq \mu+a) \leq \frac{\sigma^2}{a^2}$$

let $b = \mu+a \Rightarrow a = b-\mu \Rightarrow b \geq 2\mu$ $\Rightarrow P(X \geq b) \leq \frac{\sigma^2}{(b-\mu)^2}$

e.g. $X \sim Exp(1)$

Markov $P(X \ge a) \le \frac{1}{a}$

Chebyshev $P(X \ge b) \le \frac{1}{(b-1)^2}$ if $b \ge 2$

Let X be any r.v. $Y = e^{tX}$ which is positive

Markov

$\Rightarrow P(Y \ge c) \le \frac{E(Y)}{c} \Rightarrow P(e^{tX} \ge c) \le \frac{E(e^{tX})}{c}$ let $c = e^{ta}$ $\Rightarrow P(e^{tX} \ge e^{ta}) \le \frac{\overset{M_X(t)}{\Downarrow}{E(e^{tX})}}{e^{ta}}$

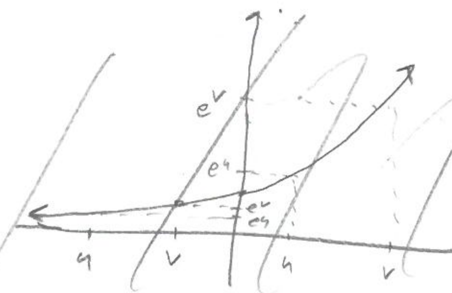
~~Precalculus~~

~~Let $u, v > 0$~~

~~$e^v \ge e^u \Rightarrow v > u$~~

~~Let $u, v < 0$~~

~~$e^v \ge e^u \Rightarrow$~~



$\Rightarrow P(tX \ge ta) \le e^{-ta} M_X(t)$

if $t > 0$

$\Rightarrow P(X \ge a) \le e^{-ta} M_X(t) \Rightarrow P(X \ge a) \le \min_{t>0} \{e^{-ta} M_X(t)\}$ ← valid for all t

if $t < 0$

$\Rightarrow P(X \le a) \le e^{-ta} M_X(t) \Rightarrow P(X \le a) \le \min_{t<0} \{e^{-ta} M_X(t)\}$

Chernoff's Inequality

e.g. $X \sim Exp(\lambda)$ $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{x(t-\lambda)} dx = \frac{\lambda}{t-\lambda} \left[e^{x(t-\lambda)} \right]_0^\infty$

For what λ does this work?

$= \frac{\lambda}{t-\lambda} \left(\begin{Bmatrix} \infty \text{ if } t \ge \lambda \\ 0 \text{ if } t < \lambda \end{Bmatrix} - 1 \right) = \frac{\lambda}{\lambda - t}$ only defined if $t < \lambda$ ← here a problem with ch.f.'s!