Lec 17 Math 621 11/18/19

CLT: $X_1, \ldots, X_n \overset{iid}{\sim}$ w/ $\mu, \sigma^2 \Rightarrow \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, $X = \sigma Z + \mu \sim N(\mu, \sigma^2)$

$Z_1, \ldots, Z_k \overset{iid}{\sim} N(0,1)$

$Z_1^2 + \ldots + Z_k^2 \sim \chi^2_k = \text{Gamma}(\frac{k}{2}, \frac{1}{2})$, $R = \frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) = \frac{1}{\pi} \frac{1}{r^2+1}$

$X_1 \sim \chi^2_{k_1}$ indep of $X_2 = \chi^2_{k_2} \Rightarrow X_1 + X_2 \sim \chi^2_{k_1+k_2} = \text{Gamma}\left(\frac{k_1+k_2}{2}, \frac{1}{2}\right)$

already proved when we did conv. of Gammas.

$X \sim \text{Gamma}(\alpha, \beta)$

$Y = cX \sim ?$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{x}{c}\right)^{\alpha-1} e^{-\beta\frac{x}{c}}$$

$$= \underbrace{\frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{c^{\alpha-1}}} e^{-\frac{\beta}{c}x}$$

$$= \frac{\left(\frac{\beta}{c}\right)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\beta}{c}x} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

If $X \sim \chi^2_k = \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

$Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

Let $X_1 \sim \chi^2_{k_1}$ ind. of $X_2 \sim \chi^2_{k_2}$ $\Rightarrow U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\frac{k_1}{2}, \frac{k_1}{2}\right)$, let $a := \frac{k_1}{2}$

$V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\frac{k_2}{2}, \frac{k_2}{2}\right)$, let $b := \frac{k_2}{2}$

let $Y = \frac{X_1/k_1}{X_2/k_2} = \frac{U}{V}$ We've done something similar before...

this formula came from a previous lecture

$$Y \sim \int_{\text{Supp}(U)} |t| f_U(rt) f_V(t)\, dt = \int_0^\infty t \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} \mathbb{1}_{t \in (0,\infty)} dt$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\,dt = \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \frac{\Gamma(a+b)}{(ar+b)^{a+b}} = \frac{a^a b^b}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} b^{-(a+b)} \left(1 + \frac{a}{b} r\right)^{-(a+b)} = \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)}$$

$$\Rightarrow \frac{X_1/k_1}{X_2/k_2} \sim \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\left(\frac{k_1+k_2}{2}\right)} \mathbb{1}_{r \geq 0} = F(k_1, k_2)$$

the "F-distribution" or "Fisher-Snedecor" distr.

<u>Param Space</u>

$k_1, k_2 \in \mathbb{N}$

this distr. comes up a lot, especially in linear regression.

Let $Z \sim N(0,1)$ indep of $X \sim \chi^2_k$. Let $W = \frac{Z}{\sqrt{\frac{X}{k}}} \sim \underbrace{T_k}_{\text{student's T-distr. with k d.f.}} = ?$

First note that $W^2 = \frac{Z^2/1}{\frac{X}{k}} \sim F(1,k)$ since $Z^2 \sim \chi^2_1$ indep of $X \sim \chi^2_k$

$$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$$

Take d/dw of both sides

$$2w\, f_{W^2}(w^2) = f_W(w) - - f_W(w) = 2 f_W(w) \Rightarrow f_W(w) = w\, f_{W^2}(w^2)$$

$$= w \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} \underbrace{(w^2)^{\frac{1}{2}-1}}_{w^{-1}} \left(1 + \frac{w^2}{k}\right)^{-\left(\frac{1+k}{2}\right)} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k}\, \underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} \Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{w^2}{k}\right)^{-\left(\frac{k+1}{2}\right)}$$

$$\Rightarrow f_W(w) = \boxed{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\, \Gamma\left(\frac{k}{2}\right)} \cdot \left(1 + \frac{w^2}{k}\right)^{-\left(\frac{k+1}{2}\right)} = T_k}$$

On the hw, you will prove this differently...

HW . . .   C3

$\lim_{k\to\infty} T_k = N(0,1)$ $Supp(R) = \mathbb{R}$

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |u|\, du$

$= \frac{1}{2\pi}\left(\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\, du + \int_{-\infty}^0 e^{-\left(\frac{1+r^2}{2}\right)u^2} u\, du\right) = \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\, du$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2}\frac{1}{u}dt$, $u = 0 \Rightarrow t = 0$, $u = \infty \Rightarrow t = \infty$

$= \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)t} \cancel{u}\frac{1}{2}\frac{1}{\cancel{u}}\, du = \frac{1}{2\pi}\frac{1}{\left(\frac{1+r^2}{2}\right)} = \frac{1}{\pi}\frac{1}{1+r^2} = \text{Cauchy}(0,1)$

s.t. $\sigma > 0$

$X = \sigma R + c \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi}\frac{1}{1+\left(\frac{r-c}{\sigma}\right)^2}$ HW

Note $\text{Cauchy}(0,1) = T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{(1)\pi}\,\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}}\left(1+\frac{r^2}{1}\right)^{-\frac{(1)+1}{2}} = \frac{1}{\pi}\frac{1}{1+r^2}$ why?

$\frac{Z_1}{Z_2} \overset{?}{=} \frac{Z_1}{\sqrt{\frac{Z_2^2}{1}}}$ modulo a negative sign but it doesn't matter....

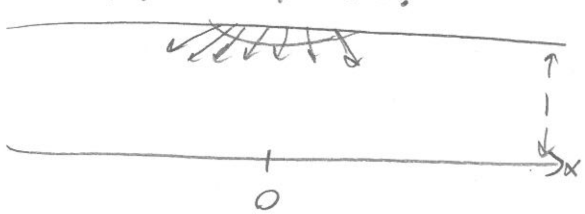
$E[R] = \int_{\mathbb{R}} r \frac{1}{\pi}\frac{1}{1+r^2}\, dr = \infty$.   No expectation!

$M_R(t) = \int_{\mathbb{R}} e^{tr} \cdot \frac{1}{\pi}\frac{1}{1+r^2}\, dr = \infty$ No MGF! (HW)

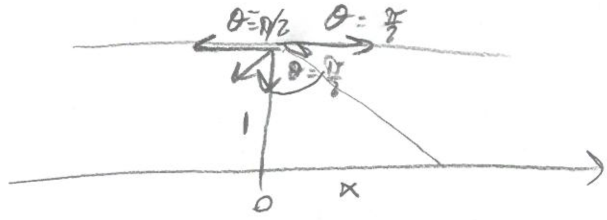
$\phi_R(t) = \int_{\mathbb{R}} e^{itr} \frac{1}{\pi}\frac{1}{1+r^2}\, dr = \underset{\text{complex analysis}}{\cdots\cdots} = e^{-|t|}$ $\phi'_R(t) = -\frac{t}{|t|} e^{-|t|}$

$\phi_R'(0)$ not defined $\Rightarrow E(R)$ d.n.e.

A physical interpretation:



Imagine a light at height = 1 shining uniformly $U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. What is the density of the brightness on the ground below (x) if the light is hung over $x = 0$?



$f_\Theta(\theta) = \frac{1}{\pi} \mathbb{1}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

$$\frac{x}{1} = \tan(\theta) \Rightarrow x = \tan(\theta) = g(\theta) \Rightarrow \theta = \arctan(x) = g^{-1}(x)$$

This is 1:1 if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which it is in our case here.

$$f_X(x) = f_\Theta(g^{-1}(x)) \left| \frac{d}{dx}\left[g^{-1}(x)\right] \right| = \frac{1}{\pi} \mathbb{1}_{\underbrace{\arctan(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{x \in \mathbb{R}}} \frac{1}{1+x^2} = \frac{1}{\pi} \frac{1}{1+x^2} = \text{Cauchy}(0,1)$$

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our project...

We have all the tools to prove everything in an intro stats course!

$X_1, \ldots, X_n \overset{iid}{\sim}$ something. $\bar{x}$ estimates μ since $E[\bar{X}] = \mu$, $S^2 := \frac{1}{n-1} \sum (X_i - \bar{x})^2$ estimates σ^2 as $E[S^2] = \sigma^2$

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow T_n \sim N(n\mu_x, n\sigma^2)$ proven using ch. f.'s

Applications to Statistics: Z, T, F, χ^2

$\Rightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ because you ~~scale~~ T_n ~~by~~ $\frac{1}{n}$ (also proven $\frac{1}{n}$ using ch. f.'s)

$$S_n^2 = \frac{1}{n-1}\left((X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2\right) \sim ?$$

Let's begin with something easier...

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$ $\quad \sum_{i=1}^n Z_i^2 \sim \chi^2_n$ $\quad$ let $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \Rightarrow \vec{Z}^T \vec{Z} \sim \chi^2_n$

Now $Z_i = \frac{X_i - \mu}{\sigma}$ $\quad \Rightarrow \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum (X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$

Now, $X_i - \mu = X_i - \bar{X} + \bar{X} - \mu \Rightarrow (X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2 = (X_i - \bar{X})^2 + 2\overbrace{(X_i - \bar{X})(\bar{X} - \mu)}^{X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu} + (\bar{X} - \mu)^2$

$$\sum (X_i - \mu)^2 = \sum (X_i - \bar{X})^2 + 2\left(\sum X_i \bar{X} + \sum \bar{X}^2 - \sum X_i \mu + \sum \bar{X}\mu\right) + \sum (\bar{X} - \mu)^2$$

$$= \underbrace{\sum (X_i - \bar{X})^2}_{(n-1)S^2} + 2\left(\cancel{n\bar{X}^2 - n\bar{X}^2 - n\bar{X}\mu + n\bar{X}\mu}\right) + n(\bar{X} - \mu)^2$$

$$\overbrace{\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}^{Z^2} \sim \chi^2_1$$

$$\Rightarrow \frac{\sum (X_i - \mu)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_n$$