Let X, Y be r.v.'s with finite means μ_X, μ_Y
finite s.d.'s σ_X, σ_Y

let $W = (X - cY)^2$ where $c \in \mathbb{R}$, c is a constant

Note: W is non-negative so $E[W] \geq 0$

$$E[(X-cY)^2] = E[X^2 - 2cXY + c^2Y^2]$$
$$= E[X^2] - 2c\,E[XY] + c^2 E[Y^2] \geq 0$$

let $c = \dfrac{E[XY]}{E[Y^2]}$

$$E[(X-cY)^2] = E[X^2] - 2\frac{E[XY]}{E[Y^2]} + \frac{E[XY]^2}{E[Y]} E[Y^2] \geq 0$$

$$\Rightarrow \quad E[X^2] - \frac{E[XY]^2}{E[Y^2]} \geq 0$$

$$\Rightarrow \quad E[X^2]E[Y^2] - E[XY]^2 \geq 0$$

$$\Rightarrow \quad E[X^2]\,E[Y^2] \geq E[XY]^2$$

$$\Rightarrow \quad E[XY]^2 \leq E[X^2]E[Y^2]$$

$$\Rightarrow \quad |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

This is the Cauchy-Schwartz Inequality

WTS
want to show

Want to show:

$$\mathrm{Corr}[X,Y] \in [-1,1]$$

let $Z_X = \frac{X-\mu_X}{\sigma_X} \Rightarrow X = \sigma_X Z_X + \mu_X$

$Z_Y = \frac{Y-\mu_Y}{\sigma_Y} \Rightarrow Y = \sigma_Y Z_Y + \mu_Y$

$E[Z_X]=0 \quad E[Z_Y]=0$

$SE[Z_X]=1 \quad SE[Z_Y]=1$

$E[Z_X^2]=1 \quad E[Z_Y^2]=1$

$\mu_X = E[X]$
$\sigma_X = SE[X]$
$\sigma_X^2 = Var[X]$
and so on...

Using Cauchy-Schwartz Inequality

$$|E[Z_X Z_Y]| \leq \sqrt{E[Z_X^2]E[Z_Y^2]} =$$

$$\leq \sqrt{(1)(1)} = 1$$

$$|E[Z_X Z_Y]| \leq 1$$

$\mathrm{Corr}[X,Y] = \frac{\mathrm{Cov}[XY]}{SE[X]SE[Y]}$

$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

denoted ρ_{XY} →

$$\mathrm{Corr}[X,Y] = \frac{E[XY]-\mu_X\mu_Y}{\sigma_X\sigma_Y}$$

$$= \frac{E[(\sigma_X Z_X+\mu_X)(\sigma_Y Z_Y+\mu_Y)]-\mu_X\mu_Y}{\sigma_X\sigma_Y}$$

since
$X=\sigma_X Z_X+\mu_X$
$Y=\sigma_Y Z_Y+\mu_Y$

$$= \frac{E[\sigma_X\sigma_Y Z_X Z_Y+\mu_X\sigma_Y Z_Y+\mu_Y\sigma_X Z_X+\mu_X\mu_Y]-\mu_X\mu_Y}{\sigma_X\sigma_Y}$$

$$= \frac{\sigma_X\sigma_Y E[Z_X Z_Y]+0+0+\mu_X\mu_Y-\mu_X\mu_Y}{\sigma_X\sigma_Y}$$

are proving $\mathrm{corr}[X,Y] \in [-1,1]$

$$= \frac{\sigma_X\sigma_Y E[Z_X Z_Y]}{\sigma_X\sigma_Y}$$

$\mathrm{Corr}[X,Y] = E[Z_X Z_Y]$ and had $|E[Z_X Z_Y]| \leq 1$

so $|\mathrm{Corr}[X,Y]| \leq 1$ so $\mathrm{Corr}[X,Y] \in [-1,1]$

$\forall$ "for all"

w's are the weights

lhs left-hand side

rhs right-hand side

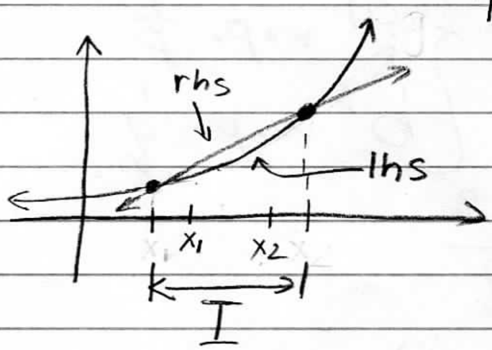
A function g is "convex" on interval $I \subseteq \mathbb{R}$ if $\forall\ x_1, x_2, \ldots \in I$ and $\forall\ w_1, w_2, \ldots$ such that $\sum w_i = 1$, $w_i > 0$ $\forall i$

we have

$$g(w_1 x_1 + w_2 x_2 + \ldots) \leq w_1 g(x_1) + w_2 g(x_2) + \ldots$$

OR $$g\left(\sum_{\text{all } i} w_i x_i\right) \leq \sum_{\text{all } i} w_i g(x_i)$$

"lhs" "rhs"



$w_1 x_1 + w_2 x_2 = x^*$

Let X be a discrete r.v. such that $Supp[X] = \{x_1, x_2, \ldots\} = I$ and let $p(x_1), p(x_2), \ldots$ be considered the "weights" (these probabalities are the w's)

For convex g, we know by definition,

$$g\left(\sum_{x \in Supp[X]} p(x)\, x\right) \leq \sum p(x) g(x)$$

$$g(E[X]) \leq E[g(X)] \text{ for convex functions}$$

this is <u>Jensen's Inequality</u>

(convex is concave up)

$g(t) = t^2$

by Jensen's Inequality

$E[X^2] \le E[X^2]$

$\mu^2 \le \mu^2 + \sigma^2$

(only equal in degenerate case: $\sigma = 0$)

Theorem from Calculus

if $g''(x) \ge 0$ for $x \in I$ then g is convex in I

Let $a, b > 0$ and $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$

w.p. with probability

consider $X \sim \begin{cases} a^p & \text{w.p. } \frac{1}{p} \\ b^q & \text{w.p. } \frac{1}{q} \end{cases}$

$$E[X] = \frac{a^p}{p} + \frac{b^q}{q}$$

let $g(t) = -\ln(t)$ which is one-to-one

$$g(X) \sim \begin{cases} -p\ln(a) & \text{w.p. } \frac{1}{p} \\ -q\ln(q) & \text{w.p. } \frac{1}{q} \end{cases}$$

$$E[g(X)] = -\ln(a) + -\ln(b) = -\ln(ab)$$

Use $g(E[X]) \le E[g(X)]$ ← Jensen's Inequality

$$-\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \le -\ln(a, b)$$

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \ge \ln(a, b)$$ since $f(u) = \ln u$ is increasing function

Young's Inequality $\frac{a^p}{p} + \frac{b^q}{q} \ge ab \Rightarrow ab \le \frac{a^p}{p} + \frac{b^q}{q}$

for $a, b > 0$, $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$

Young's Inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Let X, Y be non-negative r.v.'s

Let $a = X$ and $b = Y$ as above and take expectations of both sides

$$E[X,Y] \leq \frac{E[X^p]}{p} + \frac{E[Y^q]}{q}$$

Let X, Y be non-negative r.v.'s

Let $a = \frac{X}{(E[X^p])^{1/p}}$, $b = \frac{Y}{(E[Y^q])^{1/q}}$

as in Young's Inequality above and take expectation of both sides

using: $E[E[X]] = E[X]$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$$\frac{X}{(E[X^p])^{1/p}} \cdot \frac{Y}{(E[Y^q])^{1/q}} \leq \frac{\left(\frac{X}{(E[X^p])^{1/p}}\right)^p}{p} + \frac{\left(\frac{Y}{(E[Y^q])^{1/q}}\right)^q}{q}$$

$$\frac{E[XY]}{(E[X^p])^{1/p}(E[Y^q])^{1/q}} \leq \frac{E\left[\left(\frac{X}{(E[X^p])^{1/p}}\right)^p\right]}{p} + \frac{E\left[\left(\frac{Y}{(E[Y^q])^{1/q}}\right)^q\right]}{q}$$

$$\leq \frac{\frac{E[X^p]}{E[X^p]}}{p} + \frac{\frac{E[Y^p]}{E[Y^p]}}{q}$$

$$\frac{E[XY]}{\left(E[X^p]\right)^{1/p}\left(E[Y^q]\right)^{1/q}} \leq \frac{\frac{E[X^p]}{E[X^p]}}{p} + \frac{\frac{E[Y^q]}{E[Y^q]}}{q}$$

$$\leq \frac{1}{p} + \frac{1}{q}$$

$$\frac{E[XY]}{\left(E[X^p]\right)^{1/p}\left(E[Y^q]\right)^{1/q}} \leq 1$$

$$\Rightarrow E[XY] \leq \left(E[X^p]\right)^{1/p} E[Y^q]^{1/q}$$

This is <u>Holder's Inequality</u>

can also write

$$E[XY] \leq \sqrt[p]{E[X^p]} \cdot \sqrt[q]{E[Y^q]}$$

Let $r, s > 0$ and $s > r$

Let $p = \frac{s}{r}$ and $q = \frac{s}{s-r}$

$$\frac{1}{p} + \frac{1}{q} = \frac{r}{s} + \frac{s-r}{s} = 1$$

Let $X = |V|^r$ (X is non-negative)

Let $Y = 1$ (degenerate)

Via Holder's Inequality

$$E[|V|^r] \leq E[(|V|^r)^{s/r}]^{r/s} = E[|V|^s]^{r/s}$$

$$E[|V|^r] \leq E[|V|^s]$$

so...

*If $E[|V|^s]$ is finite

then $E[|V|^r]$ is finite (for $r < s$ and $r, s > 0$)

If s^{th} moment is finite and $r < s$ then r^{th} moment is finite

Convergence in Distribution

X_n is sequence of r.v.'s

$X_n \xrightarrow{d} X$ means the CDF of X_n converges to CDF of X

$$\left(\text{i.e. } \lim_{n\to\infty} F_{X_n}(x) = F_X(x)\right)$$

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w.p. with probability

$$X_n \sim \begin{cases} \frac{1}{n+1} & \text{with prob. } \frac{1}{3} \\ 1-\frac{1}{n+1} & \text{with prob. } \frac{2}{3} \end{cases}$$

e.g. $X_3 \sim \begin{cases} \frac{1}{4} & \text{w.p. } \frac{1}{3} \\ \frac{3}{4} & \text{w.p. } \frac{2}{3} \end{cases}$ and $X_{100} \sim \begin{cases} \frac{1}{100} & \text{w.p. } \frac{1}{3} \\ \frac{99}{100} & \text{w.p. } \frac{2}{3} \end{cases}$

here $X_n \xrightarrow{d} \text{Bern}\left(\frac{2}{3}\right) = \begin{cases} 1 & \text{w.p. } \frac{1}{3} \\ 0 & \text{w.p. } \frac{2}{3} \end{cases}$

Is PMF convergence equivalent to CDF convergence?

$\mathbb{Z}$ set of integers

Theorem If $\text{Supp}[X_n] \subseteq \mathbb{Z}$ and $\text{Supp}[X] \subseteq \mathbb{Z}$ then PMF convergence $\Leftrightarrow$ CDF convergence

lim means $\lim_{n\to\infty}$

$\Longleftrightarrow$ if and only if

Proof of $\Leftarrow$

First $P_{X_n}(x) = F_{X_n}(x-\frac{1}{2}) - F_{X_n}(x+\frac{1}{2}) = P(X_n \in [x \pm \frac{1}{2}])$

$$\lim p_X(x) = \lim F_{X_n}(x+\tfrac{1}{2}) - \lim F_{X_n}(x-\tfrac{1}{2})$$
$$= F_X(x+\tfrac{1}{2}) - F_X(x+\tfrac{1}{2})$$
$$= P_X(x) \checkmark$$

Proof of $\Rightarrow$

$$\lim F_{X_n}(x) = \lim_{n\to\infty} P(X_n \le x)$$

$$= \lim_{n\to\infty} \sum_{y=-\infty}^{x} p_{X_n}(y)$$

$$= \sum_{y=-\infty}^{x} \lim_{n\to\infty} p_{X_n}(y)$$

by dominating convergence theorem

$$= \sum_{y=-\infty}^{x} p_X(y) = P(X \le x) =$$

$$= F_X(x)$$

If $X_n \sim Binom(n, \frac{\lambda}{n})$ and $X \sim Poisson(\lambda)$

we showed $\lim_{n\to\infty} p_{X_n}(x) = p_X(x)$

$$\Longrightarrow X_n \xrightarrow{d} X$$

Is PDF convergence equivalent to CDF convergence?

No, only: PDF convergence $\Rightarrow$ CDF convergence
(not converse: $\Leftarrow$)

For a counterexample for the converse, consider $X \sim U(0, \frac{1}{n}) = n\mathbb{1}_{x\in[0,\frac{1}{n}]} = f_{X_n}(x)$

$$\lim_{n\to\infty} f_{X_n}(x) = \begin{cases} \infty & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x<0 \\ nx & \text{if } x\in[0,\frac{1}{2}] \\ 1 & \text{if } x\geq\frac{1}{2} \end{cases}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x\geq 0 \end{cases}$$

$= \text{CDF of } \deg(0)$
degenerate

so $X_n \xrightarrow{d} 0$ here

(but, $\lim_{n\to\infty} f_X(x) = \lim_{n\to\infty} n\mathbb{1}_{x\in[0,\frac{1}{n}]} = \infty$

so PDF does not converge here)

Convergence in Probability

① to a constant (let c be the constant) $c \in \mathbb{R}$

$$X_n \xrightarrow{P} c \text{ if } \forall \varepsilon \to 0, \underbrace{\lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0}_{\text{same as}}$$

$$\lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1$$

WTS
want to
Show

Let $X_n \sim \overset{\text{uniform}}{U}\left(-\frac{1}{n}, \frac{1}{n}\right) \overset{\text{PDF}}{=} \frac{n}{2} \mathbb{1}_{x \in [-\frac{1}{n}, \frac{1}{n}]}$

Want to Show $X_n \xrightarrow{P} 0$