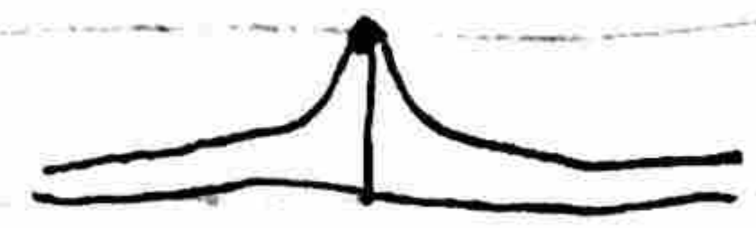
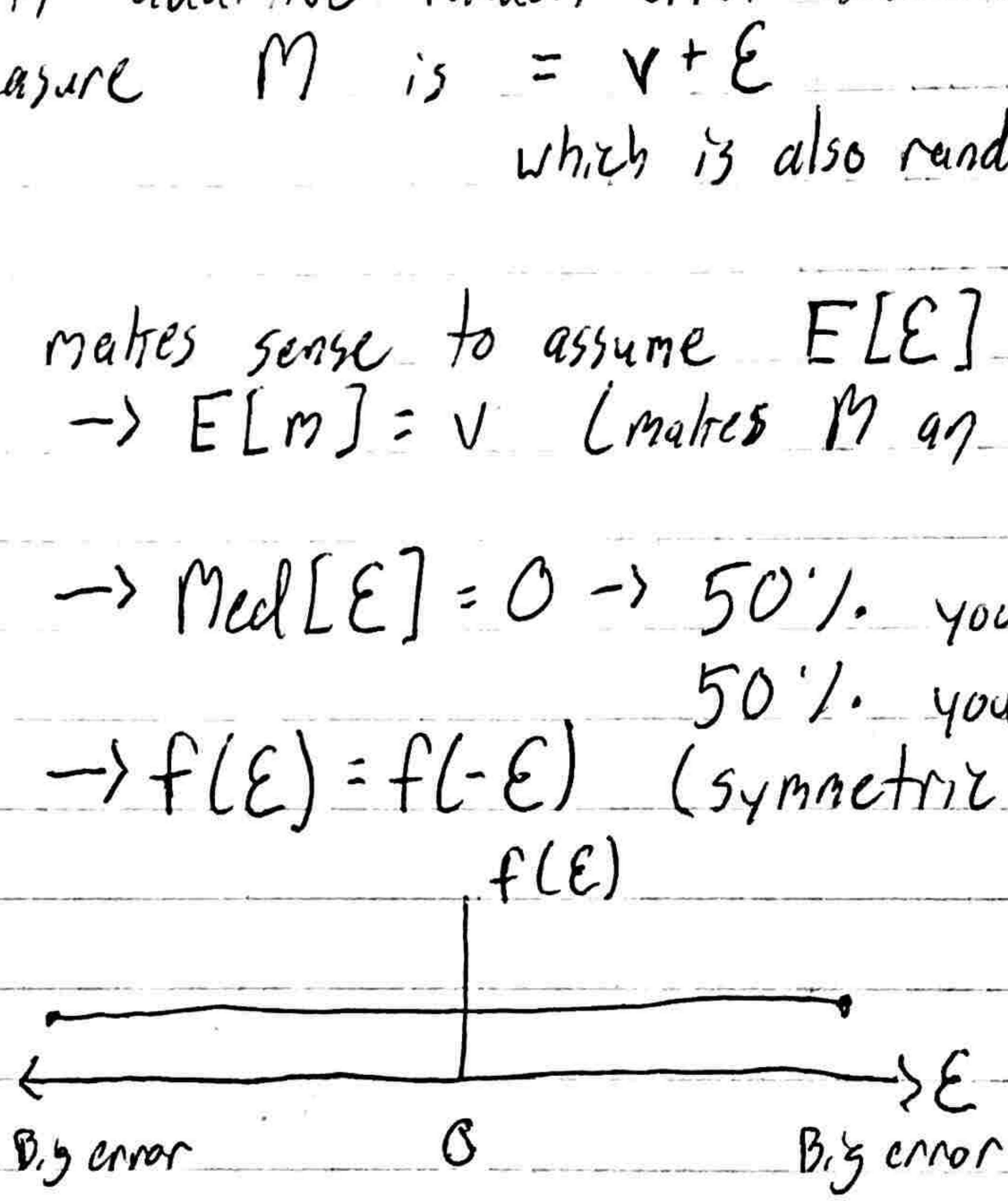
(1)

$D \sim$ Leplace$(0,1) = \frac{1}{2} e^{-|d|}$



He published this in 1774, called it the "first law of errors".
Imagine you're measuring a quantity "v" with additive random error denoted ε. So the measure M is $= v + \varepsilon$
which is also random

It makes sense to assume $E[\varepsilon] = 0$
$\to E[m] = v$ (makes M an "unbiased estimate")
$\to Med[\varepsilon] = 0 \to$ 50% you overestimate
50% you underestimate
$\to f(\varepsilon) = f(-\varepsilon)$ (symmetric around 0)



Problem: Big error probability = small error probability

$\to f'(\varepsilon) < 0$ if $\varepsilon > 0$
$\to f'(\varepsilon) > 0$ if $\varepsilon < 0$

Why not let $f''(\varepsilon) = f'(c) \to f(\varepsilon) = ce^{-b\varepsilon} \propto$ ~~Laplace(0,1)~~
$\propto$ Leplace(0,1)

Logistic also good for error.

$$X \sim \mathrm{Exp}(1) = e^{-x}\,\mathbb{1}_{x \geq 0}$$

$$Y = \frac{1}{\lambda} X^{1/k} \sim ? \qquad \text{where } k, \lambda > 0$$

- find inverse
- find derivative
- plug in

$$\lambda Y = X^{1/k} \rightarrow X = (\lambda k)^k$$
$$= \lambda^k k^k = g^{-1}(y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \lambda^k k y^{k-1}$$

$$\boxed{f_Y(y) = e^{-(\lambda y)^k}\,\mathbb{1}_{\substack{\lambda^k y^k \geq 0 \\ y^k \geq 0 \\ y \geq 0}}}$$

rewritten $\underbrace{\lambda^k}_{\lambda \cdot \lambda^{k-1}} k y^{k-1}$

$$= (k\lambda)(\lambda y)^{k-1} e^{-(\lambda y)^k}\,\mathbb{1}_{y \geq 0}$$

$$= \mathrm{Weiball}(k, \lambda)$$

→ famous waiting time / survival model

$$F(y) = \int_0^y (\kappa\lambda)(\lambda t)^{\kappa-1} e^{-(\lambda t)^\kappa}\, dt$$

let $u = (\lambda t)^\kappa = \lambda^\kappa t^\kappa$ $\quad t=0 \quad u=0 \quad t=y \rightarrow u=(\lambda y)^\kappa$

$$\frac{du}{dt} = \lambda^\kappa \kappa t^{\kappa-1} \rightarrow dt = \frac{1}{\lambda^\kappa \kappa t^{\kappa-1}}\, du$$

$$= \int_0^{(\lambda y)^\kappa} \kappa\lambda t^{\kappa-1} e^{u} \frac{1}{\kappa\lambda^\kappa t^{\kappa-1}}\, du$$

$$= \left[-e^{-u}\right]_0^{(\lambda y)^\kappa} = 1 - e^{-(\lambda y)^\kappa} \rightarrow P(Y \geq y)$$

$$\bar{F}(y) = 1 - F(y) = e^{-(\lambda y)^\kappa}$$

complement

$$\text{Weiball}(1, \lambda) = \lambda e^{-\lambda y} \mathbb{I}_{y \geq 0} = \text{Exp}(\lambda)$$

$y=1,\ c=15 \quad P(Y \geq y+c \mid Y \geq c) = \dfrac{P(Y \geq y+c \cap Y \geq c)}{P(Y \geq c)}$

$$= \frac{\bar{F}(y+c)}{\bar{F}(c)}$$

$Y \sim$ Weiball

$$\hookrightarrow \frac{e^{-(\lambda(y+c))^\kappa}}{e^{-(\lambda c)^\kappa}} = e^{-\lambda^\kappa (y+c)^\kappa} e^{\lambda^\kappa c^\kappa}$$

$$= e^{\lambda^\kappa (c^\kappa - (y+c)^\kappa)}$$

If $\boxed{K=1} \rightarrow e^{\lambda(c-(y+c))} = e^{-\lambda y} = \bar{F}(y)$
$= P(Y \geq y)$

$K > 1 \rightarrow P(Y \geq y+c \mid Y \geq c) < P(Y \geq y)$
with inequality getting greater if c gets larger.

$K < 1 \rightarrow P(Y \geq y+c \mid Y \geq c) > P(Y \geq y)$

Ex] $P(Y \geq 98_{yrs} \mid Y \geq 97_{yrs}) < P(Y \geq 1_{yr})$

"P that someone who is 97 lives to be 98 is less than the P that an infant who lives to be 1 yr old lives longer than 1 year"

Ex] $P(Y \geq 250{,}000 \text{ mi} \mid Y \geq 249{,}000 \text{ mi}) < P(Y \geq 1000)$

"P that a car makes it to 250,000 miles given it made it to 249,000 miles is less than the P that it makes it to at least 1000 miles."

$K = 1/2$ $\quad c^{1/2} + y^{1/2} > (y+c)^{1/2}$

$\rightarrow c + y + 2c^{1/2}y^{1/2} > y + c$

$\rightarrow 2c^{1/2}y^{1/2} > 0$

$K=2$ $\quad e^{\lambda^2(c^2-(y+c)^2)} < e^{\lambda^2 y^2}$

$\rightarrow \lambda^2(c^2-(y+c)^2) < -\cancel{\lambda^2} y^2$

$= \quad c^2+y^2 < (y+c)^2$

$= \quad c^2+y^2 \leq y^2+c^2+2cy$

$= \quad 0 < 2cy$ ✓ $\quad c, y > 0$

Order Statistics p 160-161

Consider continuous r.v's $X_1, \ldots, X_n$
Let the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be defined as

$X_{(1)} := \min\{X_1, \ldots, X_n\}$ called minimum
$X_{(n)} := \max\{X_1, \ldots, X_n\}$ called maximum
$X_{(k)} := k^{th}$ largest of $\{X_1, \ldots, X_r\}$ called the k^{th} largest

Let $R := X_{(n)} - X_{(1)}$ called range
As ex, $n=4$ for realizations

$\boxed{r = 12-2 = 10}$

$x_1 = 9 \quad x_2 = 2 \quad x_3 = 7 \quad x_4 = 12$

$x_{(1)} = 2 \quad x_{(2)} = 7 \quad x_{(3)} = 9 \quad x_{(4)} = 12$

Let's find the PDF and CDF of $X_{(n)}$, the max.

$$F_{X_n}(x) := P(X_{(n)} \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x)$$

$$X_1, \ldots, X_n \text{ iid} = \prod_{i=1}^{n} P(X_i \leq x)$$

$$= \prod_{i=1}^{n} F_{X_i}(x)$$

$$\text{iid} = F(x)^n$$

P x_1 less than max, x_2 less than max etc

$$f_{X_n}(x) = \frac{d}{dx}\left[F_{(x)_n}(x)\right] = n f(x) F_x^{\,n-1}$$

Let's get PDF and CDF for the minimum, $X_{(1)}$

$$F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - (P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

if iid $\rightarrow$ $$= 1 - \prod_{i=1}^{n} (P(X_i > x))$$

$$= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(x))$$

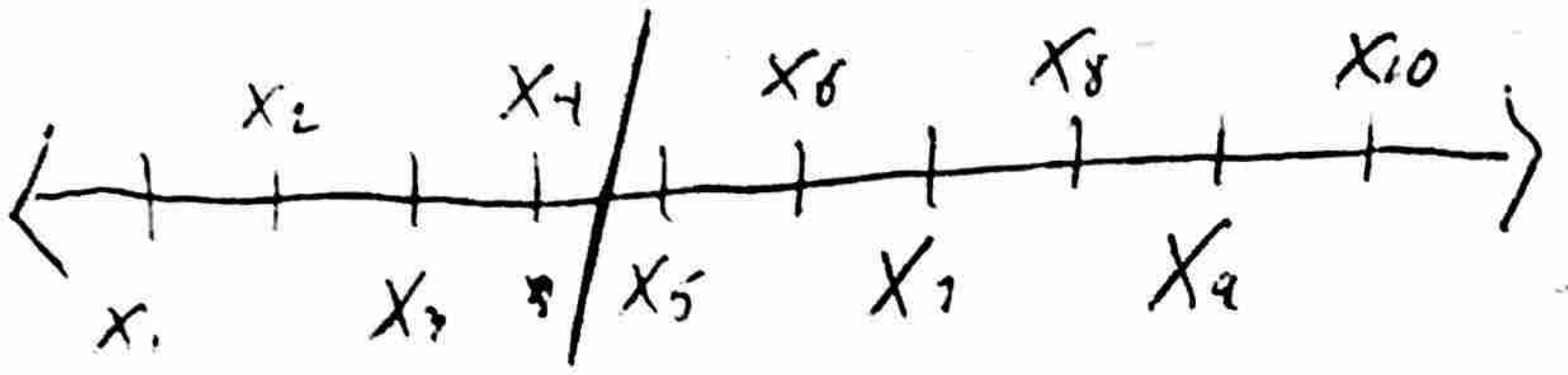
$$= 1 - (1 - F(x))^n$$

$$x := \frac{d}{dx}\left[F_{X_{(1)}}(x)\right] = -(-f(x))\,n(1 - F(x))^{n-1}$$

$$= n f(x)\left(1 - F(x)\right)^{n-1}$$

Find PDF and CDF of ~~8~~ $X_{(k)}$, the k^{th} order stat (difficult)

Let $n=10$, $k=4$



Consider $P(X_1 \le x, X_2 \le x, X_3 \le x, X_4 \le x,$
$X_5 > x, \dots, X_{10} > x)$

ind → $= \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} (1 - F_{X_i}(x))$

iid → $= F(x)^4 (1 - F(x))^6$

Now consider P that 4 of the 10 $\le x$, the other 6 of the 10 $> x$

$$= \sum_{\substack{\text{all subsets} \\ \{s_1, s_2, s_3, s_4\} \\ \subset \\ \{1,2,\dots,10\}}} P(X_{s_1} \le x, \dots, X_{s_4} \le x, X_{s_5} > x, \dots X_{s_{10}} > x)$$

$$\underline{\text{ind}} = \sum_{\text{all subsets}} \prod_{i=1}^{4} F_{X_{s_i}}(x) \prod_{i=5}^{10} (1 - F_{X_{s_i}}(x))$$

$$\underline{\text{iid}} = \sum_{\text{all subsets}} F(x)^4 (1 - F(x))^6 = \binom{10}{4} F(x)^4 (1 - F(x))^6$$

$$P(4 \text{ is less than } x)$$

$$F_{X_{(k)}}(x) := P(X_{(k)} \leq x) \overset{iid}{=} \binom{10}{4} F(x)^4 (1-F(x))^6 + \binom{10}{5} F(x)^5 (1-F(x))^5 + \cdots + \binom{10}{10} F(x)^{10} (1-F(x))^{10-10}$$

$$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$$

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j}$$

$$= \binom{n}{n} F(x)^n (1-F(x))^{n-n} = F(x)^n$$

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$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

$$= \underbrace{\left(\sum_{j=0}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j} \right)}_{=1-(1-F(x))^n} - \underbrace{\binom{n}{0}}_{1} \underbrace{F(x)^0}_{1} \underbrace{(1-F(x))^{n-0}}_{(1-F(x))^n}$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[F_{X_{(k)}}(x)\right] =$$

next lecture