09/16/19
Lecture # 6

H/W

$$\sum_{x_j \in R^{k-1}} \binom{n}{x_1, x_2, \cdots, x_n} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

$$= \binom{n}{x_j} p_j^{x_j} \sum \binom{n - x_j}{x_1, x_2, \cdots x_{j-1}, x_{j+1}, \cdots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k} \cdot \frac{(1-p_j)^{n-x_j}}{(1-p_j)^{n-x_j}}$$

$$= \text{multinomial}\left(n - x_j, \frac{\vec{p}}{1-p_j}\right)$$

$$X_1, X_2 \overset{iid}{\sim} \underset{\text{uniform}}{U}(\{0,1,2,3\}) := \begin{cases} 0 \text{ wp } \frac{1}{4} \\ 1 \text{ wp } \frac{1}{4} \\ 2 \text{ wp } \frac{1}{4} \\ 3 \text{ wp } \frac{1}{4} \end{cases}$$

uniform discrete $= \frac{1}{4} \mathbb{1}_{x \in \{0,1,2,3\}}$

Generally, $X \sim U(A) := \frac{1}{|A|} \mathbb{1}_{x \in A}$ $\quad$ [$|A|$ means number of elements of set A]

$\text{Supp}[X] = A$

Parameter space $A \subset \mathbb{R}$ s.t. A is finite.

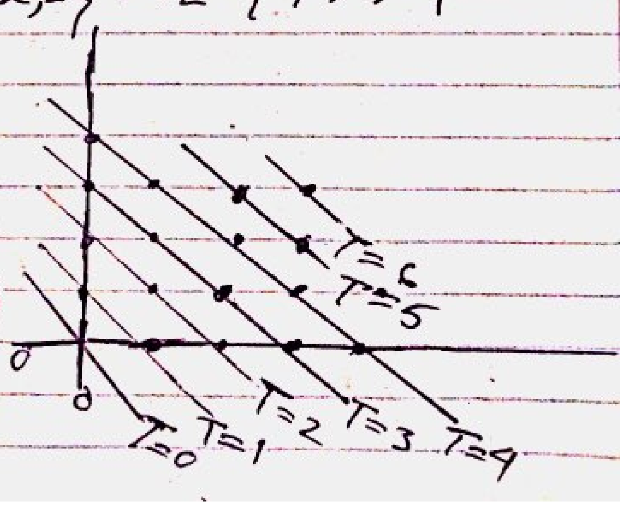
$$P_{X_1}, P_{X_2}(x_1, x_2) = \frac{1}{16} \mathbb{1}_{x_1 \in \{0,1,2,3\}} \mathbb{1}_{x_2 \in \{0,1,2,3\}}$$

$T = X_1 + X_2 \sim P_T(t)$

$$P(t) = \sum_{x \in R} \sum_{x_2 \in R} P_{X_1} P_{X_2}(x_1, x_2) \mathbb{1}_{x_2 = t - x_1}$$

$$P(1) = \sum\sum P \mathbb{1}_{x_2 = 1 - x_2} = 2\left(\frac{1}{16}\right)$$

$$P(0.5) = 0$$



$P(3) = 4\left(\frac{1}{6}\right)$

$P(6) = 1\left(\frac{1}{16}\right) = \frac{1}{16}$

let $\underbrace{y = -x}_{g(x)} \sim P_y(y)$

$X \sim \underset{\text{uniform}}{U}(\{0,1,2,3\})$

$$\left.\begin{aligned} X=0 &\Rightarrow y=0 \text{ wp } \tfrac{1}{4} \\ X=1 &\Rightarrow y=-1 \text{ wp } \tfrac{1}{4} \\ X=2 &\Rightarrow y=-2 \text{ wp } \tfrac{1}{4} \\ X=3 &\Rightarrow y=-3 \text{ wp } \tfrac{1}{4} \end{aligned}\right\} \Rightarrow y \sim U(\{0,-1,-2,-3\})$$

pmf of y

$P_y(y) := P(y=y) = P(-y=-y) = P(X=-y) = P_X(-y)$

$\Rightarrow \operatorname{supp}[y] = -\operatorname{supp}[X]$

e.g. $X \sim \operatorname{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

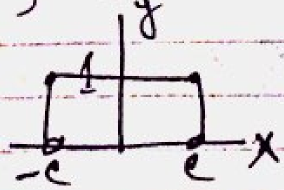
$y = -x \sim \binom{n}{-y} p^{-y} (1-p)^{n+y}$

Indicator functions :-

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1$$

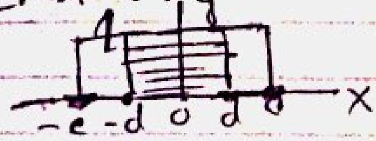
$[-c,c] = \{-c \cdots -2,-1,0,1,2 \cdots c\}$

$$\sum_{x \in \{-d,-d+1,\cdots,d-1,d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} (2d+1) & \text{if } c \geq d \\ 2c+1 & \text{if } c < d \end{cases} = 2\min(c,d)+1$$



$$\int_{\mathbb{R}} \mathbb{1}_{x \in \{-c,c\}}\, dx = 2c$$

$$\int_{-d}^{d} \mathbb{1}_{x \in \{-c,c\}}\, dx = \begin{cases} 2d & \text{if } c \geq d \\ 2c & \text{if } c < d \end{cases} = 2\min(c,d)$$



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$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$P_{X_1|T}(x,t) = P(X_1 = x_1 | T = t) \neq \text{Poisson}(\lambda)$ ↘ b/c limited by $T = t$

$$= \frac{P_{X_1, T}(x,t)}{P_T(t)}$$

$$= \frac{P_{X_1 X_2}(x, t-x)}{P_T(t)} = \frac{P_{X_1}(x)\, P_{X_2}(t-x)}{P_T(t)}$$

$$= \frac{\left(\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\right)\left(\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}\right)}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} = \binom{t}{x}\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

$$= \text{Bin}\left(t, \tfrac{1}{2}\right)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2 = X_1 + (-X_2)$, $\text{supp}[D] = \mathbb{Z}$

$\qquad\qquad\qquad \swarrow$ let X $\quad \swarrow$ let Y

$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $\quad P_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} = P_X(-y)$

Convolution formula

$$D = X_1 - X_2 = X_1 + (-X_2) = \sum_{x \in \text{supp}[X]} P_X(x)\, P_Y(d-x)\, \mathbb{1}_{d-x \in \text{supp}[Y]}$$

$$= \sum_{x \in \{0,1,2\cdots\}} \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!}\right)\mathbb{1}_{d-x \in \text{supp}[Y]}$$

$$= \sum \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{x-d}}{(x-d)!}\right)\mathbb{1}_{d-x \in \{0,-1,-2,\cdots\}}$$

$$= e^{-2\lambda}\sum_{x \in \{0,1,2,\cdots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x \geq d}$$

$$\begin{cases} = e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d<0 \\ e^{-2\lambda}\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d\geq 0 \end{cases}$$

let $x' = x-d \Rightarrow x = x'+d$

$$e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} \text{ if } d<0$$

$$e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'-d)!\,x!} \text{ if } d\geq 0$$

then $\displaystyle\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'-d)!\,x!}$

$$= \sum_{x'=0}^{\infty} \frac{\lambda^{2x'+d}}{x'!\,(x'-d)!}$$

use $d' = |d|$ if $d<0$, $d' = -d$

$\downarrow$

PMF of D
$$e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d)!} \text{ if } d<0$$

$$e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\lambda^{2x'+d}}{(x'-d)!\,x!} \text{ if } d\geq 0$$

PMF of D $\displaystyle = e^{-2\lambda}\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!} \rightarrow$ modified Bessel Function

$= e^{-2\lambda} I_{|d|}(2\lambda) \rightarrow$ of the 1st kind

$=$ Skellam (λ, λ) (1946)

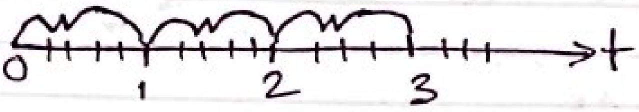
end of midterm I

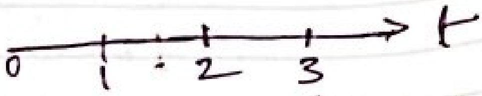
$X_1 \sim Geom(p) := \underbrace{(1-p)^x p\, \mathbb{1}_{x\in\{0,1,2\cdots\}}}_{p(x)}$

CDF is defined

$F(x) = P(X\le x) = 1-P(X>x) = 1-(1-p)^{x+1}$

ex: $F(10) = 1-P(X\ge 11) = 1-(1-p)^{12}$





1 trial each in period

We run n trials within each original period.

X_n is the # of trial & success being the next trial def.

$X_n \sim (1-p)^{nx}\, p\, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n}\cdots\}}$

let $n\to\infty$ $p\to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

similar to the derivation of the poisson from the Binomial.

$P_{X_\infty}(x) := \lim_{n\to\infty} P_{X_n}(x) = \Big(\overbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}^{e^{-\lambda}}\Big)^x \lim_{n\to\infty}\frac{\lambda}{n}\ \lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n}\cdots\}} = e^{-\lambda}\cdot 0\cdot 1$

PMF $P_{X_\infty}(x) = 0$

$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x)$

$= 1-\left(\lim\left(1-\frac{\lambda}{n}\right)^n\right)^x \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)\mathbb{1}_{x\ge 0}}_{supp[X_\infty] = [0,\infty]} = (1-e^{-\lambda x})\,\mathbb{1}_{x\ge 0}$

$|supp[X_\infty]| = |\mathbb{R}| \Rightarrow X_\infty$ is a continuous r.v

X_∞ has no PMF (does have PDF).

CDF is valid

(I) $\lim F(X) = 0$

(II) $\lim_{X \to \infty} F(X) = 1$

(III) $F(X)$ monotonically increasing

$F'(X) = \lambda e^{-\lambda x} > 0$