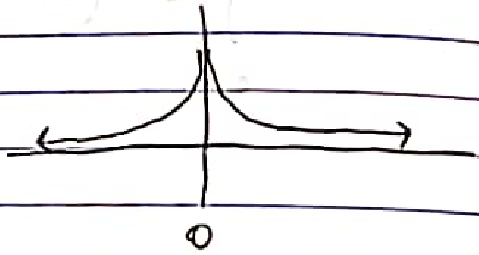
10/16 $L = \mu + \sigma D \sim \frac{1}{2\sigma} e^{-\frac{|L-\mu|}{\sigma}}$ = laplace(μ,σ)

$\mu \in \mathbb{R}$,

$\sigma > 0$

10/23 [next Friday HW Due.]

laplace $(0,1) = \frac{1}{2}e^{-|d|}$



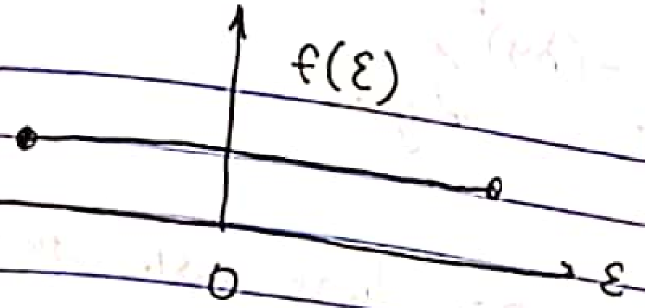
laplace first published this distribution in 1774 called it the "first law of errors."
He derived a different way.
Imagine you wanted to measure a quantity v.
But your measuring device has random additive error ε (epsilon).
So the measurement M is also random:

$$\underset{\text{random}}{M} = \underset{\text{fixed}}{V} + \underset{\text{random}}{\varepsilon}$$

It would make sense if

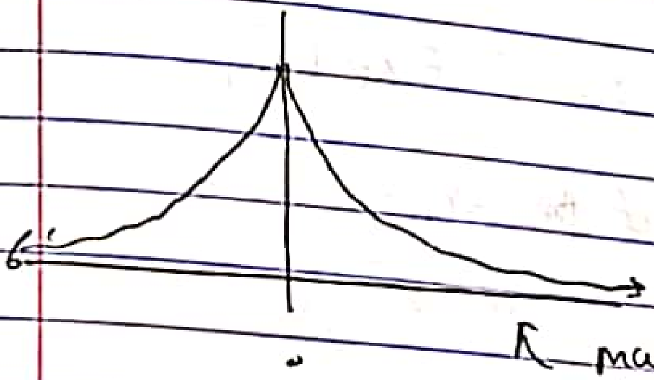
- $E[\varepsilon] = 0 \Rightarrow E[M] = v$ (AKA M is an unbiased estimator)
- $\text{Med}[\varepsilon] = 0 \Rightarrow$ 50% of the time you over estimate & 50% you under estimate.
- $f(\varepsilon) = f(-\varepsilon)$

10/23



← this is bad. likelyhood of each error is the same. you don't want this!

$f'(\varepsilon) < 0$ if $\varepsilon > 0$ and $f'(\varepsilon) > 0$ if $\varepsilon > 0$



↖ more likely to have smaller errors
less likely to have large errors.

Laplace says: $f''(\varepsilon) = f'(\varepsilon)$.

$\Rightarrow f(\varepsilon) = ce^{-dx} \sim \text{Laplace}(0,1)$

Next topic:

$X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x \geq 0} \quad Y = \frac{1}{\lambda} X^{\frac{1}{k}}$ where $\lambda, k > 0$

Find inverse function.

$\lambda Y = X^{\frac{1}{k}} \Rightarrow X = \lambda Y^k = \lambda^k Y^k = g^{-1}(y)$

$\frac{d}{dy}\left[g^{-1}(y)\right] = |\lambda^k k Y^{k-1}| = k\lambda^k Y^{k-1}$

$f_Y(y) = e^{-\lambda Y^k} \mathbb{1}_{\underbrace{\lambda^k Y^k > 0}_{Y^k > 0}} k\lambda^k Y^{k-1} =$

10/23 $f_Y(y) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}$

$= \text{Weibull}(k, \lambda)$

[survival, ~~...~~ or waiting time distribution.]

Special case: Weibull $(k=1, \lambda)$

$= \lambda e^{-\lambda y} \mathbb{1}_{y \geq 0} = \text{Exp}(\lambda)$

Weibull is a generalization of the exponential.

$$F_Y(y) = \int_0^y k\lambda(\lambda t)^{k-1} e^{-(\lambda t)^k} dt$$

let $u = (\lambda t)^k = \lambda^k t^k$

$\frac{du}{dt} = k\lambda^k t^{k-1}$

$dt = \frac{1}{k\lambda^k t^{k-1}} du$

$t = y \Rightarrow u = (\lambda y)^k$

$t = 0 \Rightarrow u = 0$

$$\int_0^{(\lambda y)^k} \cancel{k\lambda^k t^{k-1}} e^{-u} \frac{1}{\cancel{k\lambda^k t^{k-1}}} du$$

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10/23 $\displaystyle\int_0^{(\lambda y)^K} e^{-u} = \left[e^{-u}\right]_0^{(\lambda y)^K} = 1 - e^{-(\lambda y)^K} = F(y)$

$\bar{F}(y) = 1 - F(y) = e^{-(\lambda y)^K}$

↳ inverse CDF
↳ survival function $P(Y \geq y)$ probability light bulb
↳ last's longer than y years.

Consider: $P(Y \geq y+c \mid Y \geq c)$

$= \dfrac{P(Y \geq y+c \cap Y \geq c)}{P(Y \geq c)} \Rightarrow \dfrac{P(Y \geq y+c)}{P(Y \geq c)}$

$= \dfrac{\bar{F}(y+c)}{\bar{F}(c)} = \dfrac{e^{-\lambda^K (y+c)^K}}{e^{-\lambda^K c^K}} = e^{\lambda^K (c^K - (y+c)^K)}$

if $K = 1$, then $P(Y \geq y+c \mid Y \geq c) = P(Y \geq y)$

verify: $e^{\lambda(c-(y+c))} = e^{-\lambda y}$

$\Rightarrow e^{-\lambda y} = e^{-\lambda y}$ (memoryless) honda civic

10/23 If $k > 1 \Rightarrow P(Y \geq y+c \mid Y \geq c) < P(Y \geq y)$

gets less probable as c increases

If $k < 1 \Rightarrow P(Y \geq y+c \mid Y \geq c) > P(Y \geq y)$

gets more probable as c increases.

infant mortality

If $k = 2$

$$e^{\lambda^2(c^2-(y+c)^2)} < e^{-\lambda^2 y^2}$$

$$\lambda^2(c^2-(y+c)^2) < \lambda^2 y^2$$

$$(c^2-(y+c)^2) < -y^2$$

$$c^2+y^2 < -y^2$$

$$c^2+y^2 < (y+c)^2 = y^2+2cy+c^2$$

$$0 < 2cy \checkmark$$

If $k = \frac{1}{2}$

similar steps as above

$$c^{\frac{1}{2}}+y^{\frac{1}{2}} > (c+y)^{\frac{1}{2}}$$

$$(c^{\frac{1}{2}}+y^{\frac{1}{2}})^2 > c+y$$

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10/27

$c+y+2\sqrt{cy} > c+y$

$2\sqrt{cy} > 0$.

New Unit. Called order Statistics. pg (160 - 161)

Let $X_1, X_2, \ldots X_n$ be a collection of continuous random variables.

Define $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ as follows:

$X_{(1)} := \min\{X_1, X_2, \ldots X_n\}$ 1^{st} order statistic "minimum"

$X_{(n)} := \max\{X_1, X_2, \ldots, X_n\}$ n^{th} order statistic "maximum"

$X_{(k)} := k^{th}$ largest of $\{X_1, \ldots, X_n\}$

$R := X_{(n)} - X_{(1)}$ "Range"

$n=4$ realizations

$X_1 = 9$, $X_2 = 2$, $X_3 = 12$ $X_4 = 7$

$X_{(1)} = 2$, $X_{(2)} = 7$, $X_{(3)} = 9$ $X_{(4)} = 12$

$r = 12 - 2 = 10$

lets derive the pdf and CDF of $X_{(n)}$ the maximum.

10/23

$$F_{X_{(n)}}(x) := P(X_{(n)} \le x) = P(X_1 \le x \cap X_2 \le x \cap \ldots \cap X_n \le x)$$

first assumption: independence

IF $X_1, X_2, \ldots, X_n \overset{ind}{\sim}$

$$\prod_{i=1}^{n} P(X_i \le x) = \prod_{i=1}^{n} F_{X_i}(x) \overset{(\text{if iid})}{=} F(x)^n$$

$$\Rightarrow f_{X_{(n)}}(x) = \frac{d}{dx}\left[F_{X_{(n)}}(x)\right]$$ chain rule

(iid)

$$= n f(x) F(x)^{n-1}$$

Derive PDF, CDF for $X_{(1)}$ minimum:

$$F_{X_{(1)}}(x) := P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x \cap X_2 > x \cap \ldots \cap X_n > x)$$

assume independence. then we can re-write as

$$1 - \prod_{i=1}^{n} P(X_i > x) = 1 - \prod_{i=1}^{n}\left(1 - F_{X_i}(x)\right)$$

assume iid now

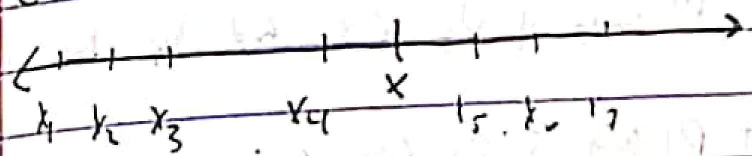
$$= 1 - (1 - F(x))^n$$

10/23 take derivative

$$-(-f(x))n(1-F(x))^{n-1} = nf(x)(1-F(x))^{n-1}$$

lets get the PDF and CDF of $X_{(k)}$, the distribution of the k^{th} largest.

Consider $n=10$, $k=4$.



Consider $P(X_1 \le x, \ldots, X_4 \le x, X_5 > x, \ldots X_{10} > x)$

need assumptions to make progress.

first: independence

$$= \prod_{i=1}^{4} P(X_i \le x) \prod_{i=5}^{10} P(X_i > x) = \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} (1-F_{X_i}(x))$$

now IID $= F(x)^4 (1-F(x))^6$

now consider P(any 4 X_i's $\le x$ and the other 6 $> x$)

$$= \sum_{\substack{\text{over all} \\ \text{subsets}}} P(X_{s_1} \le x, \ldots, X_{s_4} \le x, X_{s_5} > x, \ldots X_{s_{10}} > x)$$

If independent

$$\sum \prod_{i=1}^{4} F_{X_{s_i}}(x) \prod_{i=5}^{10} (1-F_{X_{s_i}}(x))$$

10/23 now iid $= \sum F(x)^4 (1-F(x))^6 = \binom{10}{4} F(x)^4 (1-F(x))^6$

$$F_{X_{(4)}}(x) := P(X_{(4)} \leq x)$$

consider $X_{(4)} \leq 3.7$

this only tells you the 4^{th} largest is less than 3.7.

$P(X_{(4)} \leq x) = P(\text{any } 4 \text{ } x\text{'s} < x \text{ and the other } 6 > x)$

$+ P(\text{any } 5 \text{ } x\text{'s} < x \text{ and the other } 5 \geq x)$

$+$

$\vdots$

$P(\text{all } 10 \text{ } x\text{'s} < x)$.

If iid

$$= \binom{10}{4} F(x)^4 (1-F(x))^6 + \binom{10}{5} F(x)^5 (1-F(x))^5 + \dots$$

$$+ \dots \binom{10}{10} F(x)^{10} (1-F(x))^{10-10}$$

$$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$$

Generally with arbitrary n, K cdf ~~of~~

$$F_{X_{(k)}} = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$$

10/27 $F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} = \underbrace{\binom{n}{n}}_{1} F(x)^n \underbrace{(1-F(x))}_{1}$

$= F(x)^n$

$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$

$(1-F(x))^n$

$= \underbrace{\left(\sum_{j=0}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right)}_{} - \underbrace{\binom{n}{0}}_{1} \underbrace{F(x)^0}_{1} \overbrace{(1-F(x))}^{n-0}$

changed index

$F(x) + (1-F(x))^n = 1-(1-F(x))^n$ ✓

$f_{X_{(k)}}(x) = \frac{\partial}{\partial x}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$