(Lecture 4) September 9th, 2019

Bag of fruit

P_1 : Prob of Apple
P_2 : Prob of Banana
P_3 : Prob of Cantelope

$P_1 + P_2 + P_3 = 1$

(JMF)

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim P_{\vec{X}}(\vec{x}) = \frac{n!}{x_1! x_2! x_3!} P_1^{x_1} P_2^{x_2} P_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n} \mathbb{1}_{x_1 \in \{0,1,\cdots,n\}} \mathbb{1}_{x_2 \in \{0,1,\cdots,n\}} \mathbb{1}_{x_3 \in \{0,1,\cdots,n\}}$$

X_k : # of fruit k

$\underline{A}\ \underline{B}\ \underline{C}\ \underline{A}\ \underline{B}\ \underline{C}\ \underline{C}$

$$= \binom{n}{x_1, x_2, x_3} P_1^{x_1} P_2^{x_2} P_3^{x_3}$$

$$= \text{Multinomial}\left(n, \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}\right)$$

Generally, with K Categories

$$\vec{X} \sim \text{Multinomial}(n, \vec{p}) := \binom{n}{x_1, x_2, \cdots, x_k} P_1^{x_1} P_2^{x_2} \cdots P_k^{x_k}$$

$$\text{Supp}[\vec{X}] = \left\{ \vec{x} : \vec{x} \in \{0,1,\cdots,n\}^k, \vec{x} \cdot \vec{1} = n \right\}$$

Note: $\vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$n \in \mathbb{N}$, $\vec{p} \in \{\vec{v} : \vec{v} \in (0,1)^k, \vec{v} \cdot \vec{1} = 1\}$

$$\vec{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \text{Multinomial}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right)$$

$p_1 = p$
$p_2 = 1-p$

$$= \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

$$X_1 \sim \text{Bin}(n, p)$$
$$X_2 \sim \text{Bin}(n, 1-p)$$

Are $X_1, X_2 \overset{ind}{\sim}$?

If so then $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)\ \forall \vec{x} \in \text{Supp}[\vec{x}]$

$$P(X_1 = 1 \mid X_2 = n) \overset{?}{\neq} \underbrace{P(X_1 = 1) = np(1-p)^{n-1}}_{\text{pmf using binomial.}}$$

$= 0$

Therefore x_1, x_2 are dependent

$$p_{X_1|X_2}(x_1, x_2) = P(X_1 = x_1 \mid X_2 = x_2)$$

$$= \frac{p_{X_1, X_2}(x_1, x_2)}{p_{X_2}(x_2)} \quad \left(\text{by definition of conditional probability}\right)$$

We should get $\text{Deg}(n - x_2) = \mathbb{1}_{x_1 = n - x_2}$

Marginal PMF

$$p_{X_2}(x_2) = \sum_{x_1 \in \text{Supp}[x_1]} p_{X_1, X_2}(x_1, x_2)$$

$$= \sum_{x_1 \in \{0,1,\cdots,n\}} \frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \sum_{x_1 \in \{0,1,\cdots,n\}} \frac{1}{x_1!} p^{x_1} \mathbb{1}_{x_1 = n - x_2}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \frac{1}{(n-x_2)!} p^{n-x_2}$$

$\frac{n!}{x_2!(n-x_2)!}$ becomes $\binom{n}{x_2}$

$$= \binom{n}{x_2} (1-p)^{x_2} p^{n-x_2}$$

$$= \text{Bin}(n, 1-p)$$

$$\frac{P_{X_1 X_2}(x_1, x_2)}{P_{X_2}(x_2)} = \frac{\frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}}{\frac{n!}{x_2!(n-x_2)!} (1-p)^{x_2} p^{n-x_2}}$$

$$= \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$$= \begin{cases} \frac{x_1!}{x_1!} p^0 = 1 & \text{if } x_1 + x_2 = n \ (\text{or } x_1 = n - x_2) \\ 0 & \text{otherwise} \end{cases}$$

$$= \text{Deg}(n - x_2)$$

$$= \{ n - x_2 \ \text{ w.p. } 1$$

$$P_{\vec{x}_{-j}|x_j}(\vec{x}_{-j}, x_j)$$

let $\vec{x}_{-j} = \begin{bmatrix} x_1 \\ \vdots \\ x_{j-1} \\ x_{j+1} \\ \vdots \\ x_k \end{bmatrix}$

$$P_{\vec{x}_{-j}|x_j}(\vec{x}_{-j}, x_j) = \frac{\text{Multinom}(n, \vec{p})}{\text{Bin}(n, p_j)}$$

$$= \frac{\frac{n!}{x_1! \cdots x_{j-1}! x_{j+1}! \cdots x_k!}\left(p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_j^{x_j} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}\right)}{\frac{n!}{x_j!(n-x_j)!} p_j^{x_j}(1-p_j)^{n-x_j}}$$

let $n' := n - x_j$ $\quad = \dfrac{n'!}{x_1! \cdots x_{j-1}! x_{j+1}! \cdots x_k!} \cdot \dfrac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{n'}}$

Note: $n' = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k$

$$(1-p_j)^{n'} = (1-p_j)^{x_1} \cdots (1-p_j)^{x_{j-1}} (1-p_j)^{x_{j+1}} \cdots (1-p_j)^{x_k}$$

$= \text{Multinom}(n', \vec{p})$ where

$\vec{p} := \begin{bmatrix} \frac{p_1}{1-p_j} \\ \vdots \\ \frac{p_{j-1}}{1-p_j} \\ \frac{p_{j+1}}{1-p_j} \\ \vdots \\ \frac{p_k}{1-p_j} \end{bmatrix}$ such that $\dim[\vec{p}] = k-1$

$$\vec{X} := \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \qquad E(\vec{X}) = ? \qquad Var(\vec{X}) = ?$$

Let $X_1, \dots, X_1$ be r.v's

$$E[aX + c] = am + c \text{ where } m = E[X],\ a, c \in \mathbb{R} \text{ Constants}$$

$$E\left[\sum_{X_1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i) \quad (\text{true always})$$

$$\sigma^2 := Var[X] := E[(X-m)^2] = \dots = E(X^2) - m^2$$

$$\sigma := SE(X) := \sqrt{\sigma^2}$$

$$\text{If discrete} = \sum_{x \in \mathbb{R}} (x-m)^2 p(x)$$

$$\text{If Continuous} = \int_{\mathbb{R}} (x-m)^2 f(x)\,dx$$

$$Var[X_1 + X_2] = E\left[\left((X_1+X_2) - (m_1 + m_2)\right)^2\right]$$

$$= E\left[X_1^2 + X_2^2 + m_1^2 + m_2^2 - 2m_1X_1 + 2m_1X_2 + 2m_2X_2 + 2X_1X_2 + 2m_1m_2\right]$$

$$= E[X_1^2] + E[X_2^2] + m_1^2 + m_2^2 - 2m_1^2 - 2m_1m_2 - 2m_2m_1 - 2m_2^2$$

$$+ 2E[X_1X_2] + 2m_1m_2$$

$$= \sigma_1^2 + \sigma_2^2 + 2\left(E(X_1X_2) - m_1m_2\right)$$

$$= \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}$$

Covariance

$$\sigma_{1,2} := \mathrm{Cov}[X_1, X_2] := E[X_1 X_2] - m_1 m_2$$

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E(X_i) \text{ if } X_1 \cdots X_n \text{ ind.}$$

$$E[X_1, X_2] - m_1 m_2 = E\left[(X_1 - m_1)(X_2 - m_2)\right]$$