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$\vec{X}$ vector r.v of dim n

$\vec{\mu} := E[\vec{X}]$

$\Sigma := Var[\vec{X}] := E[\vec{X}\vec{X}^T] - \underbrace{E[\vec{X}]E[\vec{X}]^T}_{\vec{\mu}\vec{\mu}^T}$

$$= \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & \\ & Var[X_2] & & \\ & & \ddots & \\ & & & Var[X_n] \end{bmatrix}$$

Let $A \in \mathbb{R}^{m \times n}$ of constants

if $\vec{c} \subset \mathbb{R}^n$

$E[\vec{X} + \vec{c}] = \vec{\mu} + c$, $E[\vec{c}^T \vec{X}] = \vec{c}^T \vec{\mu}$

$Var[\vec{X} + \vec{c}] = Var[\vec{X}]$

$Var[\vec{c}^T \vec{X}] = \vec{c}^T \Sigma \vec{c}$

Let $A \in \mathbb{R}^{m \times n}$ of constants

$$E[A\vec{X}] = E\begin{bmatrix} a_{1n}\vec{X} \\ a_{2n}\vec{X} \\ a_{3n}\vec{X} \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} E[a_1\vec{X}] \\ E[a_2\vec{X}] \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{a}_1\vec{\mu} \\ \vec{a}_2\vec{\mu} \\ \vdots \\ \vec{a}_n\vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}\vec{\mu} = A\vec{\mu}$$

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$E[\vec{X}A^T] = \vec{\mu}A^T$

$Var[A\vec{X}] = E[(A\vec{X})(A\vec{X})^T] - E[A\vec{X}]E[A\vec{X}]^T$

$= E[AXX^TA^T] - A\vec{\mu}\vec{\mu}^TA^T$

$= AE[XX^T]A^T - A\vec{\mu}\vec{\mu}^TA^T$

$= A(E[XX^T]A^T - \vec{\mu}\vec{\mu}^TA^T)$

$= A(E[XX^T] - \vec{\mu}\vec{\mu}^T)A^T = A\Sigma A^T$

$U \sim \chi^2_k = Gamma(\frac{k}{2}, \frac{1}{2})$

$E[U] = E[Z_1^2] + \dots + E[Z_k^2] = kE[Z^2] = k \cdot 1 = k$

Let $Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$U = Z_1^2 + \dots + Z_k^2$

$\underbrace{Var[Z]}_{1} = E[Z^2] - \underbrace{E[Z]}_{0}{}^2$

$Z \sim N(0,1)$
$X = \sigma Z + \mu$
$\sim N(\mu, \sigma^2)$

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$ $\vec{z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$

$\vec{z} \sim f_Z(\vec{z}) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}_n, I_n)$

$\vec{X} = A\vec{z} + \vec{\mu} \sim f_{\vec{X}}(\vec{x}) = ?$
where $A \in \mathbb{R}^{n \times n}$, $\mu \in \mathbb{R}^n$

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$$E[\vec{X}] = E[A\vec{z} - \vec{m}] = E[A\vec{z}] + \vec{m}$$

$$= A\vec{0}_n + \vec{m} = \vec{m}$$

$$Var[\vec{X}] = Var[A\vec{z} + \vec{m}] = Var[A\vec{z}]$$

$$= A\,Var[\vec{z}]\,A^T = A I_n A^T = AA^T$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{z}}(h(\vec{x}))\,|J_n|$$

$$\vec{X} = A\vec{z} + \vec{m} \Rightarrow \vec{X} - \vec{m} = A\vec{z}$$

Assume A is invertible

$$\rightarrow \vec{z} = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{m}) = h(\vec{X})$$

$$J_n = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial h_n}{\partial x_1} & & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix}$$

$$\rightarrow = B\vec{X} - B\vec{m} = \begin{array}{c} \begin{bmatrix} \vec{b}_1\vec{X} - \vec{b}_1\vec{m} \\ \vec{b}_2\vec{X} - \vec{b}_2\vec{m} \\ \vdots \\ \vec{b}_n\vec{X} - \vec{b}_n\vec{m} \end{bmatrix} \end{array} \begin{array}{l} = h_1(\vec{x}) \\ = h_2(\vec{x}) \\ \\ = h_n(\vec{x}) \end{array} = \det \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & & & \\ b_{n1} & & & b_{nn} \end{bmatrix}$$

$$= \det[B] = \det[A^{-1}]$$

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$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(A^{-1}(\vec{x}-\vec{m}))\,|\det[A^{-1}]|$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-m))^T(A^{-1}(\vec{x}-\vec{m}))}\,|\det[A^{-1}]|$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-1/2(\vec{x}-\vec{m})^T(A^{-1})^T A^{-1}(\vec{x}-\vec{m})}\,|\det[A^{-1}]|$$

<u>Fact 1</u> $I = AA^{-1}$

$1 = \det[I] = \det[AA^{-1}] = \det[A]\det[A^{-1}]$

$\rightarrow \det[A^{-1}] = \frac{1}{\det[A]}$

<u>Fact 2</u> $\Sigma = AA^T$

$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T]$

$= \det[A]^2 \rightarrow \det[A] = \sqrt{\det[\Sigma]}$

<u>Fact 3</u> $I = AA^{-1} \rightarrow I^T = (AA^{-1})^T \rightarrow I = (A^{-1})^T A^T$

Note: $I = (A^T)^{-1}A^T$

$\rightarrow (A^{-1})^T = (A^T)^{-1}$

<u>Fact 4</u> $\Sigma = AA^T$

$\rightarrow \Sigma^{-1} = (AA^T)^{-1}$

$= (A^T)^{-1}A^{-1}$ $\quad (AB)^{-1} = B^{-1}A^{-1}$

$= (A^{-1})^T A^{-1}$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{m})\Sigma^{-1}(\vec{x}-\vec{m})} = N_n(\vec{m}, \Sigma)$$

general multivariate normal

If $\vec{m} = \vec{0}$, $\Sigma = I$

$$= \frac{1}{\sqrt{(2\pi)^n (1)}} e^{-\frac{1}{2}\vec{x}^T\vec{x}} = N_n(\vec{0}, I)$$

let $n=1 \rightarrow \vec{m} = m$, $\Sigma = [\sigma^2]$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\underbrace{-\frac{1}{2}(x-m)\frac{1}{\sigma^2}(x-m)}_{\frac{1}{2\sigma}(x-m)^2}} = N(m, \sigma^2)$$

$$\vec{X} = A\vec{Z} + \vec{m}$$

$A \in \mathbb{R}^{m \times n}$ s.t $m < n$, $\vec{m} \in \mathbb{R}^m$

$$\begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \vec{z} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \vec{m} \end{bmatrix}, \quad \begin{array}{cc} & n \\ m & \begin{bmatrix} \\ A \\ \\ \\ \end{bmatrix} \end{array} \begin{bmatrix} \\ \end{bmatrix}$$

its span is at most an n-dimen subspace of a larger space $\mathbb{R}^n$

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$$AA^T = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{array}{cc} & m \\ n & \begin{bmatrix} AA^T \end{bmatrix} \end{array}$$

· rank deficient, $n < m$
· $\Sigma = AA^T$ is not invertible

$\det[\Sigma] = 0$, $f_{\vec{X}}$ undefined

$$\vec{X} = A\vec{Z} + \vec{m} \sim f_{\vec{X}}(\vec{x}) = ?$$

Let $\tilde{\vec{X}} = \begin{bmatrix} \vec{X} \\ X_1 \\ \vdots \\ X_{n-m} \end{bmatrix}$, $\tilde{\vec{m}} = \begin{bmatrix} \vec{m} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} A \\ \vec{v}_1 \\ \vdots \\ \vec{v}_{n-m} \end{bmatrix}$

do not care

pick $\vec{v}_{1,\ldots}$ s.t. $\tilde{A}$ is full rank

$$\tilde{\vec{X}} = \tilde{A}\vec{Z} + \tilde{\vec{m}} \sim f_{\tilde{\vec{X}}}(\tilde{\vec{x}})$$

$$f_{\vec{X}}(\vec{x}) = \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n-m} N_n(\tilde{\vec{m}}, \tilde{\Sigma})\, dx_{1,\ldots,n-m}$$ annoying, use ch.f instead

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Let $\vec{X}$ be a vector r.v. Ch.f is:

$$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1+\dots+t_nX_n)}]$$

$$= E[e^{it_1X_1}e^{it_2X_2}\cdot\ldots\cdot e^{it_nX_n}]$$

ind, $= E[e^{it_1X_1}]\cdot\ldots\cdot E[e^{it_nX_n}] = \prod_{i=1}^{n}\phi_{X_i}(t_i)$

iid $= \prod_{i=1}^{n}\phi_X(t_i)$

(P0) $\phi_{\vec{X}}(\vec{0}_n) = E[e^{i\vec{0}_n^T\vec{X}}] = E[e^{i0}] = 1$ ✓

(P1) $\phi_{\vec{X}}(\vec{t}) = \phi_{\vec{Y}}(\vec{t}) \longleftrightarrow \vec{X} \overset{d}{=} \vec{Y}$

(P2) $\vec{Y} = A\vec{X} + \vec{b}$ where $A \in \mathbb{R}^{m\times n}$, $\vec{b} \in \mathbb{R}^m$ constants

$$\phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{i\vec{t}^TA\vec{X}}e^{i\vec{t}^T\vec{b}}]$$

$$= e^{i\vec{t}^T\vec{b}}E[e^{i\underbrace{(A^T\vec{t})^T\vec{X}}_{\vec{t}^TA\vec{X}}}]$$

$$= e^{i\vec{t}^T\vec{b}}\phi_{\vec{X}}(A^T\vec{t})$$

Let $\vec{Z} \sim N_n(\vec{0}_n, I_n)$

$$\left(\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n}\phi_Z(t_i) = \prod_{i=1}^{n}e^{-\frac{t_i^2}{2}}\right.$$

by ind of Z_i's

$$= e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

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Let $A \in \mathbb{R}^{m \times n}$, $\vec{\mu} \in \mathbb{R}^m$

$$\vec{X} = A\vec{z} + \vec{\mu}$$

$$\phi_X(\vec{t}) \underset{p2}{=} e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}(A^T\vec{t})^T A^T\vec{t}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T A A^T \vec{t}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T \Sigma \vec{t}}$$

Let $B \in \mathbb{R}^{l \times m}$, $\vec{c} \in \mathbb{R}^l$

$$\vec{Y} = B\vec{X} + \vec{c}$$

$$\phi_{\vec{Y}}(\vec{t}) \underset{p2}{=} e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(B^T\vec{t})$$

$$= e^{i\vec{t}^T\vec{c}} \left(e^{i\overbrace{(B^T\vec{t})^T\vec{\mu}}^{\vec{t}^T B\vec{\mu}} - \frac{1}{2}\overbrace{(B^T\vec{t})^T}^{\vec{t}^T B} \Sigma (B^T\vec{t})} \right)$$

$$= e^{i\vec{t}^T(\vec{c} + B\vec{\mu}) - \frac{1}{2}\vec{t}^T B \Sigma B^T \vec{t}}$$