Standard Multivariate Normal

$$\vec{Z} \sim N_n(\vec{0}, I) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{z}^T z}$$

$$\phi_{\vec{z}}(t) = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

If $A \in \mathbb{R}^{n \times n}$ and A is invertible, $\vec{\mu} \in \mathbb{R}^n$

$$\vec{X} = A\vec{Z} + \mu \sim N_n(\vec{\mu}, \Sigma)$$

$\Sigma = AA^T$

Multivariate Normal

$N_n(\vec{\mu}, \Sigma)$ has pdf

$$f_{\vec{x}}(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

$$\phi_{\vec{x}}(t) = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T \Sigma \vec{t}}$$

Let $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^m$, $\vec{Y} = B\vec{X} + c$

$$\vec{Y} = \vec{B} + c \sim N_m(B\vec{\mu} + \vec{c}, B\Sigma B^T)$$

we used char. fun. to prove this

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let $A \in \mathbb{R}^{m \times n}$ and let A be full rank, $m \leq n$
let $c \in \mathbb{R}^m$

$$\vec{X} = A\vec{Z} + \vec{c} \sim N_m(A\vec{0} + \vec{c}, AIA^T)$$

$$= N_m(\vec{c}, \underbrace{AA^T}_{\Sigma})$$

$$= N_m(\vec{c}, \Sigma)$$

$$\vec{Z} \sim N_n(\vec{0}, I) \overset{\text{pdf}}{:=} \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$$\vec{X} = A\vec{Z} + \mu$$

$$\vec{X} \sim N_n(\vec{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

compare to

$$N_1(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})\frac{1}{\sigma^2}(\vec{x}-\vec{\mu})}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\left(\frac{\vec{x}-\mu}{\sigma}\right)^2}$$

z-score

Statistician from India

Mahalanobis Distance (1936)

multidim. analog to z-score

$$(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})$$

$$= (\vec{x}-\vec{\mu})(A A^T)^{-1}(\vec{x}-\vec{\mu})$$

$$= (\vec{x}-\vec{\mu})(A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})$$

$$= \left(A^{-1}(\vec{x}-\vec{\mu})\right)^T \left(A^{-1}(\vec{x}-\vec{\mu})\right)$$

$$A^{-1}(\vec{x}-\vec{\mu}) = \vec{Z}$$

$$= \vec{Z}^T \vec{Z} \sim \chi^2_n$$

like z-score functions to tell you how "far" you are from mean

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_5 \end{bmatrix} \qquad f_{X_2,X_4}(x_2,x_4) = \underset{\mathbb{R}\,\mathbb{R}\,\mathbb{R}}{\iiint} \underbrace{f_{X_1,X_2,X_3,X_4,X_5}}_{\text{Joint PDF (JMF)}}(x_1,x_2,\dots,x_5)\, \underbrace{dx_1\, dx_3\, dx_5}_{\text{all } x\text{'s except } x_2,\, x_4}$$

or use:

$$\phi_{\vec{X}}(\vec{t}) = E\left[e^{it\vec{X}}\right]$$

$$= E\left[e^{it\vec{X}_1} e^{itX_2} \cdots e^{it\vec{X}_5}\right]$$

$$\phi_{\vec{X}}\left(\begin{bmatrix} 0 \\ t_2 \\ 0 \\ t_4 \\ 0 \end{bmatrix}\right) = E\left[e^{it_2X_2} e^{it_4X_2}\right]$$

$$= E\left[e^{i[t_2\, t_4]\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}}\right]$$ same as dot product

$$= \phi_{X_2X_4}(t_2, t_4)$$

by P1 P6 get

$$\Longrightarrow f_{X_2X_4}(x_2, x_4)$$

So can find marginals using Characteristic functions instead of integrals

Multivariate Normal is composed of dependent normals

$$\phi_{\vec{X}_1}(t) = \phi_{\vec{X}}\left(\begin{bmatrix} t\\ 0\\ \vdots\\ 0\end{bmatrix}\right)$$

$$= e^{i[t\;0\ldots0]\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{bmatrix} - \frac{1}{2}[t\;0\ldots0]\Sigma\begin{bmatrix}t\\ 0\\ \vdots\\ 0\end{bmatrix}}$$

$$= e^{it[1\;0\ldots0]\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{bmatrix} - \frac{t^2}{2}[1\;0\ldots0]\underbrace{\Sigma\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}}_{\begin{bmatrix}\sigma_1^2 & \sigma_2 & \cdots\\ \sigma_{21} & \sigma_2^2 & \\ \vdots & & \\ \sigma_{n1} & & \text{etc.}\end{bmatrix}\begin{bmatrix}\sigma_1^2\\ \sigma_{21}\\ \vdots\\ \sigma_{n1}\end{bmatrix}}}$$

$$= e^{it\mu_1 - \frac{t^2\sigma_1^2}{2}}$$

this is ϕ for $N(\mu_1, \sigma_1^2)$, so by Forier Transform...

P1 $\Rightarrow X_1 \sim N(\mu_1, \sigma_1^2)$

In general,

$$X_j \sim N(\mu_j, \sigma_j^2)$$