Day #1 8/28/19

- A discrete r.v. X has probability mass function (PMF)

  $p(x) := P(X=x)$ notated $X \sim p(x)$ &

- A cumulative distribution function $F(x) := P(X \le x)$
- r.v. X has "support" $Supp[X] := \{x : p(x) > 0,\ x \in \mathbb{R}\}$

  ↳ Something that can happen.

- Support & PMF are related as $\sum_{x \in supp(x)} p(x) = 1$

- $|supp(x)| \le |\mathbb{N}|$ i.e. the # of possible different realizations

  (finite or discrete)

with probability

- Bernoulli: $X \sim Bern(p) := \begin{cases} 1 & w.p\ p \\ 0 & w.p\ 1-p \end{cases} = p^x(1-p)^{1-x}$

  $supp(x) = \{0,1\}$

  where p is its parameter (tuning knob) & belong to the parameter space"

  $p \in (0,1)$

If $p=1$, $X \sim Bern(1) = \{1 \ w.p\ 1 = Deg(1)$ degreate r.v $= \mathbb{1}_{x=1}$

If $p=0$, $X \sim Bern(0) = \{0\ w.p\ 1 = Deg(0)$ $= \mathbb{1}_{x=0}$

$X \sim Deg(c) = \{c\ w.p\ 1$ $= \mathbb{1}_{x=c}$

- Note: $P(X=3.7) = p(3.7) = p^{3.7}(1-p)^{-2.7} = 0.5$ if $p = \frac{1}{2}$

  $\| $

  0

doesn't work b/c of the support of the bernoulli

Let $\mathbb{1}_A := \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$

indicator function

$\forall x \in \mathbb{R}$, $p(x) = p^x(1-p)^{1-x}\,\mathbb{1}_{x \in \{0,1\}}$

- $X_1, X_2, \dots X_n$ are discrete r.v's which has a joint mass fun

(JMF) $p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots x_n) =$

$$P(X_1 = x_1 \;\&\; X_2 = x_2 \;\&\; \dots \;\&\; X_n = x_n)$$

If $X_1, X_2, X_3, \dots X_n \overset{iid}{\Rightarrow} p_{X_1 \dots X_n}(x_1 \dots x_n) = \prod_{i=1}^{n} p_{X_i}(x_i), \; \forall \vec{x} \in \mathbb{R}^n$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(independent) → product

If $X_1 \overset{d}{=} X_2 \overset{d}{=} \dots \overset{d}{=} X_n \Rightarrow p_{X_1}(x) = p_{X_2}(x) = \dots p_{X_n}(x)$

(identically distributed)

This means they share the same PMF.

$$\Rightarrow \prod_{i=1}^{n} p(x_i)$$

- $X_1, \dots X_n \overset{iid}{\sim}$

⇒ independent & identically distributed

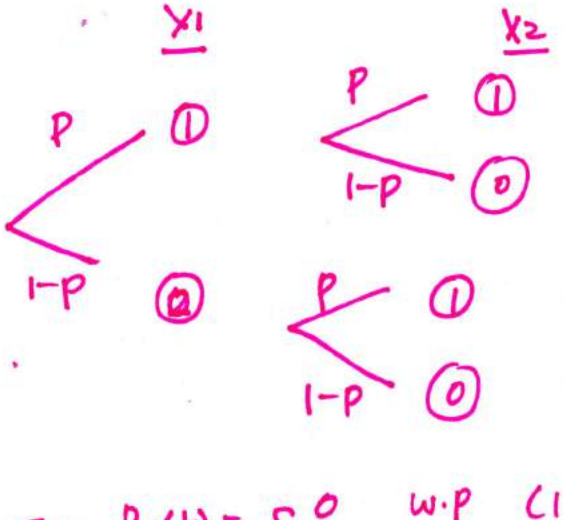
- Let $X_1 X_2 \overset{iid}{\sim} \text{Bern}(p)$

Let $T := X_1 + X_2 \sim p_T(t) = ?$

$\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2] = \{0, 1, 2\}$

$A + B := \{a + b : a \in A, b \in B\}$

Visualization:



$\langle X_1, X_2 \rangle$	$p_{X_1, X_2}(x_1, x_2)$	T	$p_T(t)$
$\langle 1, 1 \rangle$	p^2	2	p^2
$\langle 1, 0 \rangle$	$p(1-p)$	1	$2p(1-p)$
$\langle 0, 1 \rangle$	$(1-p)p$	1	
$\langle 0, 0 \rangle$	$(1-p)^2$	0	$(1-p)^2$

$\Rightarrow T = \{2, 1, 0\}$

So, $p_T(t) = \begin{cases} 0 & \text{w.p } (1-p)^2 \\ 1 & 2p(1-p) \\ 2 & p^2 \end{cases}$

$$\sum_{t \in supp(T)} p_T(t) = (1-p)^2 + 2p(1-p) + p^2 = 1$$

It's correct, but not practical

- $$P(T=t) = P_T(t) = \sum_{x \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1, X_2}(x_1, x_2) \mathbb{1}_{t = x_1 + x_2}$$

$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1, X_2}(x_1, t-x_1) \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x_1 \in \mathbb{R}} P_{X_1, X_2}(x_1, t-x_1)$$

General discrete convolution formula

$$= \sum_{x \in \mathbb{R}} P_{X_1, X_2}(x, t-x)$$

$$= \sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x) \qquad = \sum_{x \in supp[X]} P(x) P(t-x) \mathbb{1}_{t-x \in supp[X]}$$

if X_1, X_2 independent

if $X_1 = X_2$

$$= \sum_{x \in \mathbb{R}} P(x) p(t-x) = \sum_{x \in \mathbb{R}} P(x) \mathbb{1}_{x \in supp[X]} P(t-x) \mathbb{1}_{t-x \in supp[X]}$$

- $$P_T(t) = \sum_{x \in \mathbb{R}} \left(p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} \right) \left(p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}} \right)$$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= \sum_{x \in \{0,1\}} p^t (1-p)^{2-t} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t-1 \in \{0,1\}} \right)$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right) = \begin{cases} (1-p)^2 & \text{if } t=0 \\ 2p(1-p) & \text{if } t=1 \\ p^2 & \text{if } t=2 \\ 0 & \text{if otherwise} \end{cases}$$