11/11 properties of ch.f.

where $Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$

$\phi_{Z_n}(t)$

P2

$$= e^{-it\frac{\sqrt{n}}{\sigma}\mu}\, \phi_X\left(\frac{\sqrt{n}}{\sigma}\cdot\frac{t}{n}\right)^n$$

$$= e^{-\frac{it\mu}{\sigma\sqrt{n}} n}\, \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n$$

$$= e^{-\frac{it\mu}{\sigma\sqrt{n}} n}\, e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{n\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}$$

$$= e^{\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)}$$

examine as $\lim_{n\to\infty}$ with the hope ~~that~~ converging so $Z_n \xrightarrow{d} Z$ using P8 and find the density using P6.

11/11 $\lim_{n\to\infty} \phi_{Z_n}(t) =$

Let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n \to \infty \Rightarrow u \to 0$

$$e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\phi_x(u)) - \cancel{\#} i u \mu \}}{u^2}}$$

L'Hopital rule $\Rightarrow$ $e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\frac{\phi_x'(u)}{\phi_x(u)} - i\mu}{2u}}$ we use L'Hop again

$$e^{\frac{t^2}{2\sigma^2} \lim_{n\to\infty} \frac{d}{du}\left[\frac{\phi_x'(u)}{\phi_x(u)}\right]}$$

$$e^{\frac{t^2}{2\sigma^2} \lim_{n\to\infty} \frac{\phi_x(u)\phi_x''(u) - \phi_x'(n)^2}{\phi_x(u)^2}} = e^{\frac{t^2}{2\sigma^2} \frac{\phi_x(0)\phi_x''(0) - \phi_x'(0)^2}{\phi_x(0)^2}}$$

(P0) $= e^{\frac{t^2}{2\sigma^2}(\phi_x''(0) - \phi_x'(0)^2)}$

(P4) $= e^{\frac{t^2}{2\sigma^2}(i^2 E[X^2] - i^2 E[X]^2)}$

$$= e^{-\frac{t^2}{2\sigma^2}(\underbrace{E[X^2] - E[X]^2}_{VAR[X] = \sigma^2})}$$

$= e^{-\frac{t^2}{2}} = \phi_Z(t)$ is this integrable?

is $\phi_Z(t) \in L^1$?

$$\int_{\mathbb{R}} \left|e^{-\frac{t^2}{2}}\right| dt = \underbrace{\int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt}_{} = \boxed{} < \infty$$

gaussian integral

11/11 Now we can use property 6 to find its pdf.
we will compute annoying integral now.

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(itz + \frac{t^2}{2})}\, dt$$

$ax^2 + bx + c \longleftrightarrow d(x+e)^2 + f$ plug back in

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 - \underbrace{\left(\frac{\sqrt{2}iz}{2}\right)^2}_{\frac{2i^2z^2}{4}} \rightarrow -\frac{z^2}{2} =$$

$$\frac{1}{2\pi}\int e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 + \frac{z^2}{2}}\, dt$$

$=$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} dt$$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}$

$$\frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dy = \frac{dt}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\, dy$$

$$\frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} \underbrace{e^{-y^2} dy}_{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = N(0,1)$$

$$\Rightarrow \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$ "Central Limit Theorem"

FIND Expectation and variance of this

$\phi_Z(t) = e^{-\frac{t^2}{2}}$ take derivative of $\phi_Z(t)$

$E[Z] = \phi_Z'(t) = -t e^{-\frac{t^2}{2}} \Big|_0 = 0 \quad E[Z]=0$

$VAR[Z] = E[Z^2] - E[Z]^2 = E[Z^2] \overset{p.4}{=} \frac{\phi_Z''(0)}{i^2}$

$\phi_Z''(t) = -\left(-t^2 e^{-t^2/2} + e^{-\frac{t^2}{2}}\right)$

$=$ evaluate at $t=0$ $= 0^2 e^0 + e^{-0} = 1$

cheat sheet

$X = \sigma Z + \mu \quad \sigma > 0 \quad \overset{?}{\sim} \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$

$$\boxed{\begin{array}{l} E[X] = \mu \\ VAR[X] = \sigma^2 \end{array}}$$

$$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

11/11 $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$

$T = X_1 + X_2 \sim = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(t-x-\mu_2)^2} dx$

using convolution first

use $\phi_X(t) \overset{P2}{=} e^{it\mu} \phi_Z(\sigma t)$

$= e^{it\mu} e^{\frac{-\sigma^2 t^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$\phi_T(t) = \phi_{X_1}(t) \phi_{X_2}(t) = \left(e^{it\mu_1} - e^{\frac{\sigma_1^2 t^2}{2}}\right)\left(e^{it\mu_2} - e^{\frac{\sigma_2^2 t^2}{2}}\right)$

$= e^{it\mu_1 + it\mu_2 - \left(\frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2 t^2}{2}\right)} = e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$

$\Rightarrow X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim ?$

$X = \ln(y) = g^{-1}(Y)$ $\quad \frac{\partial}{\partial y} \ln(y) = \frac{1}{y}$

$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial}{\partial y}[g^{-1}(y)] \right|$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln y - \mu)^2} \cdot \frac{1}{y}$

$\ln(y) \in (-\infty, \infty)$

$y \in (0, \infty)$

1/11

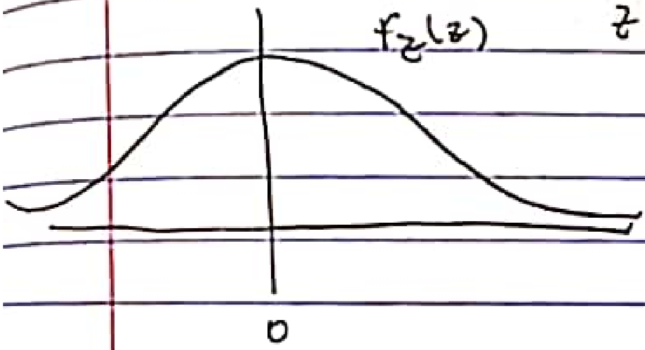
$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} \, e^{-\frac{1}{2\sigma^2}(\ln y - \mu)^2} \, \mathbb{1}_{y \ge 0} = \log N(\mu, \sigma^2)$$

"Log Normal"

$\sim ?^{2}$

$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$, g is not 1-1

$$F_Y(y) := P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}])$$



Z is symmetric

$$P(Z \in [-\sqrt{y}, \sqrt{y}]) = \int_{-\sqrt{y}}^{\sqrt{y}} f_Z(z)\,dz = 2\int_{0}^{\sqrt{y}} f_Z(z)\,dz$$

$$= 2\big(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{50\%}\big)$$

$\boxed{\text{Laplace, Standard t-system, even functions} \Rightarrow \text{symmetric}}$

$$2\big(F_Z(\sqrt{y}) - \tfrac{1}{2}\big)$$

$$2F_Z(\sqrt{y}) - 1$$ take derivative

$$\Rightarrow f_Y(y) = \cancel{2}\left(\frac{1}{\cancel{2}} y^{-\frac{1}{2}}\right) f_Z(\sqrt{y}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} \propto y^{\overbrace{\frac{1}{2}-1}^{\alpha}} e^{\overbrace{-\frac{1}{2}y}^{\beta}}$$

Gamma$(\frac{1}{2}, \frac{1}{2})$

$\propto$ ~~gamma~~

$$:= \chi^2$$

chi-squared distribution 1 degree of freedom

11/11 $Z_1, Z_2, \ldots, Z_k \overset{iid}{\sim} N(0,1)$

$X = Z_1^2 + Z_2^2 + \ldots + Z_k^2 \sim \chi_k^2 := \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

chi-squared with k degree of freedom.

$$= \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{1}{2}x} \mathbb{1}_{x \geq 0}$$

or $$\frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{1}{2}x} \mathbb{1}_{x \geq 0}$$

$$\chi_1^2 = \frac{1}{2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)} x^{-\frac{1}{2}} e^{-\frac{1}{2}x} = \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}.$$

$X \sim \chi_k^2$, $Y = \sqrt{X} \sim \chi_k$: ?? what is pdf.

$X = Y^2$ 1-1 as only concerned with $x \geq 0$.

$y^2 = g^{-1}(y)$ $\quad \frac{d}{dy} g^{-1}(y) = 2y$

$$f_Y(y) = f_X(g^{-1}(y))\,|2y| =$$

$$\frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}} 2y \, \mathbb{1}_{y \geq 0}$$

$$= \frac{1}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y \geq 0} = \chi_k$$

$Z_1, Z_2 \overset{iid}{\sim} N(0,1) \quad R = \frac{Z_1}{Z_2} \quad Supp[R] = \mathbb{R}$

$$\sim \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |u|\, du$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{e^{-\left(\frac{r^2+1}{2}\right)u^2}|u|}\, du$$

even function, symmetric around zero

$$= \frac{\cancel{2}\,1}{\cancel{2}\,\pi}\int_0^\infty e^{-\left(\frac{r^2+1}{2}\right)u^2} u\, du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u} dt$

$$\frac{1}{\pi}\int_0^\infty e^{-\left(\frac{r^2+1}{2}\right)t} u \frac{1}{2u\,dt} = \frac{1}{2\pi}\int_0^\infty e^{-at} dt = \frac{1}{\pi}\frac{1}{r^2+1}$$

$= \text{Cauchy}(0,1)$

"Standard Cauchy"