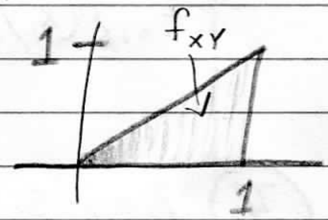
Conditional Densities

Conditional Densities example:

$X \sim \text{Uniform}(0,1)$

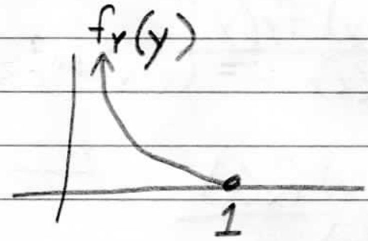
$\rightarrow \quad Y/X = x \sim \text{Uniform}(0, x)$



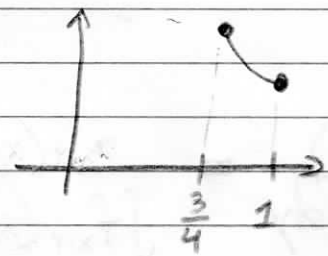
$f_{Y|X=x}(y) = \frac{1}{x} \mathbb{1}_{y \in (0,x)}$

Goal: find pdf's f_X, f_Y, $f_{X|Y}$

Guess:



Guess: $f_{X/Y}(x, \frac{3}{4})$



$$\boxed{\begin{aligned} P(X=x, Y=y) &= P(Y=y \mid X=x) P(X=x) \\ p_{XY}(x,y) &= p_{Y|X}(y) p_X(x) \\ f_{XY}(x,y) &= f_{Y|X}(y) f_X(x) \end{aligned}}$$

We'll use:

I Margining

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx$$

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy$$

for discrete

$$f_Y(y) = \sum_{y \in \mathbb{R}} f_{Y|X}(x,y)$$

II Def of Conditional Density

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(y)} \quad \text{for } f_X(y) \neq 0$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{for } f_Y(y) \neq 0$$

II Def. of Conditional density

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)} \quad \text{for } f_X(x) > 0 \Rightarrow f_{XY}(x,y) = f_{Y|X}(x,y) f_Y(y)$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{for } f_Y(y) > 0 \Rightarrow f_{XY}(x,y) = f_{X|Y}(x,y) f_X(x)$$

III Bayes Rule

Notation:

$f_{Y|X=x}(y)$ same as $f_{Y|X}(y)$

also same as $f_{Y|X}(x,y)$

"conditional density of Y given X"

$$f_{Y|X}(x,y) = \frac{f_{X|Y}(x,y) f_Y(y)}{f_X(x)}$$

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y) f_X(x)}{f_Y(y)}$$

combine I and II

IV Bayes Theorem

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{\int_{\mathbb{R}} f_{XY}(x,y)\,dy} = \frac{f_{X|Y}(x,y) f_Y(y)}{\int_{\mathbb{R}} f_{X|Y}(x,y) f_Y(y)}$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{\int_{\mathbb{R}} f_{XY}(x,y)\,dx} = \frac{f_{Y|X}(x,y) f_X(x)}{\int_{\mathbb{R}} f_{Y|X}(x,y) f_X(x)}$$

discrete version

$$p_{Y|X} = \frac{p_{XY}(x,y)}{\sum_{y \in \mathbb{R}} p_{XY}(x,y)} = \frac{p_{X|Y}(x,y) p_Y(y)}{\sum_{y \in \mathbb{R}} p_{Y|X}(x,y) p_Y(y)}$$

$X \sim U(0,1)$

$Y | X = x \sim U(0,1)$

$$f_X(x) = 1 \cdot \mathbb{1}_{x \in [0,1]}$$

$$f_{XY}(x) = \frac{1}{x} \mathbb{1}_{y \in [0,x]} \mathbb{1}_{x \in [0,1]}$$

$$= \frac{1}{x} \mathbb{1}_{0 \le y \le x \le 1}$$

$$= \frac{1}{x} \mathbb{1}_{y \in [0,1]} \mathbb{1}_{x \in [y,1]}$$

$$f_Y(y) = \int_{\mathbb{R}} \frac{1}{x} \mathbb{1}_{y \in [0,1]} \mathbb{1}_{x \in [y,1]}$$

$$= \mathbb{1}_{y \in [0,1]} \int_y^1 \frac{1}{x} \, dx$$

$$= \mathbb{1}_{y \in [0,1]} \left[\ln(x) \right]_y^1$$

$$= \mathbb{1}_{y \in [0,1]} (\ln 1 - \ln y)$$

$$= \mathbb{1}_{y \in [0,1]} (0 - \ln y)$$

$$= -\ln y \cdot \mathbb{1}_{y \in [0,1]}$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{x}\mathbb{1}_{y\in[0,1]}\mathbb{1}_{x\in[y,1]}}{-\ln(y)\mathbb{1}_{y\in[0,1]}}$$

$$= -\frac{1}{x\ln(y)}\mathbb{1}_{x\in[y,1]}$$

e.g. $f_{X|Y}(x, \frac{3}{4}) \approx \frac{3.5}{x}\mathbb{1}_{x\in[\frac{3}{4},1]}$

$f_{X|Y}(1, \frac{3}{4}) \approx 3.5$

$f_{X|Y}(\frac{3}{4}, \frac{3}{4}) \approx 4.6$

Mixture and Compound Distributions

e.g. If there is no internet traffic, downloads take $\sim Exp(\frac{1}{5})$ seconds.
If there is internet traffic, downloads take $\sim Exp(\frac{1}{20})$ seconds
Traffic is happening $\frac{1}{3}$ of the time.
How long do downloads take in general?
(Find the density)

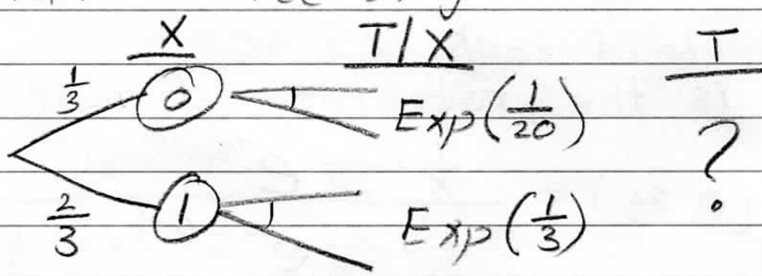
Let $X \sim Bern(\frac{2}{3})$
if $X=0 \Rightarrow$ traffic
$X=1 \Rightarrow$ no traffic

Let $T|X=1 \sim Exp(\frac{1}{5})$
be time for downloads with no traffic

Let $T|X=0 \sim Exp(\frac{1}{20})$
be time for downloads with traffic

Want density of T

Draw a tree diagram:



$T \leftarrow$ mixture model
hierarchical model
mixture dist.
multilevel dist.

$$f_T(t) = \sum_{x\in\mathbb{R}} f_{T|X}(x,t)\,p_X(x) = \sum f_{X,T}(x,t)$$

$$= \sum_{x\in\{0,1\}} \left(\frac{1}{5}e^{-\frac{1}{5}t}\mathbb{1}_{x=1} + \frac{1}{20}e^{-\frac{1}{20}t}\mathbb{1}_{x=0}\right)\cdot\left(\frac{2}{3}\right)^{x}\left(\frac{1}{3}\right)^{1-x}$$

$$= \frac{1}{3}\left(\frac{1}{20}e^{-\frac{1}{20}t}\right) + \frac{2}{3}\left(\frac{1}{5}e^{-\frac{1}{5}t}\right)$$

$$f_T(t) = \frac{1}{60}e^{-\frac{1}{20}t} + \frac{2}{15}e^{-\frac{1}{5}t}$$

this is dist. of time that dowloads take
(in general)

Given a download time,
what are the traffic probabilities?

$$p_{X|T}(x,t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{\left(\frac{1}{5}e^{-\frac{1}{5}t}\mathbb{1}_{x=1} + \frac{1}{20}e^{-\frac{1}{20}t}\mathbb{1}_{x=0}\right)\left(\frac{2}{3}\right)^{x}\left(\frac{1}{3}\right)^{1-x}\mathbb{1}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}$$

$$= \text{Bern}\left(\frac{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}\right)$$

since if $Q \sim \text{Bern}(p)$
then $P(Q=1) = p$

If download took 25 sec,
what is the probability of traffic?

$$p_{X|T}(0,25) = \frac{\frac{1}{x}\cdot\frac{1}{20}e^{-\frac{1}{20}(25)}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}(25)} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}(25)}}$$

≈ 0.84

$\approx 84\%$

e

$X \sim U(0,1)$
$Y | X=x \sim U(0,x)$

$$f_Y(y) = \int_{\mathbb{R}} \underbrace{f_{Y|X}(x,y)}_{\text{model}} \underbrace{f_X(x)}_{\text{mixing dist.}} dx$$

$Y \sim f_Y(y)$ mixture dist

X	$Y|X$	Y
$\prec U(0,1)$	$\prec U(0,x)$	?

* If the mixture dist. is continuous, then Y is called "compound"

Ex: $X \sim \text{Gamma}(\alpha, \beta)$
$Y|X=x \sim \text{Poisson}(x)$ here, $\lambda = x$

$$p_Y(y) = \int_{\mathbb{R}} p_{Y|X}(x,y) f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \left(\frac{x^y e^{-x}}{y!} \mathbb{1}_{y \in \{0,1,2,\dots\}}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}\right) \cancel{dx}$$

$$\underset{\substack{\text{proportional}\\ \text{to}}}{\propto} \frac{1}{y!} \mathbb{1}_{y \in \underbrace{\{0,1,2,\dots\}}_{\mathbb{N}_0}} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

$$= \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}}$$

$$\propto \text{ExtNegBin}\left(\alpha, \frac{\beta}{\beta+1}\right)$$

"Extended Negative Binomial"

Ex: n is Known

$X \sim Beta(\alpha, \beta)$

$Y | X = x, n \sim Bin(n, x)$

$$p_Y(y) = \int_{\mathbb{R}} p_{Y|X}(x,y)\, f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \left(\binom{n}{x} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,1,\dots,n\}} \right) \left(\frac{1}{\mathcal{B}(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in (0,1)} \right) dx$$

$$= \frac{\binom{n}{x}}{\mathcal{B}(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1} dx}_{\text{Beta function}}$$

$$= \frac{\binom{n}{x}}{\mathcal{B}(\alpha,\beta)} \mathcal{B}(y+\alpha, n-y+\beta) \mathbb{1}_{y \in \{0,1,\dots,n\}}$$

$$= BetaBinomial(\alpha, \beta, n)$$

$X \sim Gamma(\alpha, \beta)$

$Y | X = x \sim Exp(x)$

$\xrightarrow{\text{for HW}} Y \sim Lomax(\beta, \alpha)$

Next Tuesday 12PM
Office hours Mon 3:45PM

Complex numbers

$a, b \in \mathbb{R}$

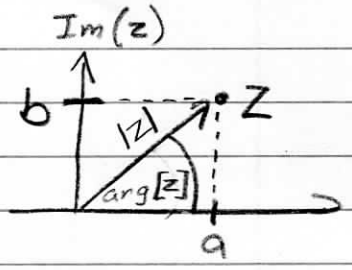
define complex number z

$z := a + bi \in \mathbb{C}$ ← $\mathbb{C}$ means the complex numbers

where $i = \sqrt{-1}$

real component $Re[z] = a$

imaginary component $Im[z] = b$



$$|z| = \sqrt{a^2 + b^2}$$

angle → $Arg[z] =^* \arctan\left(\frac{b}{a}\right)$ ← def. depends on quadrant

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
etc.