what if $\vec{p} = \frac{1}{k}\vec{1}$

(only non-zero is when both are 1)

$$E[X_{\ell i}X_{\ell j}] = \sum_{x_1\in\{0,1\}} \sum_{x_2\in\{0,1\}} x_1 x_2 P_{\ell i,\ell j}(x_1, x_2) = P_{X_{\ell i}, X_{\ell j}}(1,1)$$

$$= P(X_{\ell i}=1 \ \& \ X_{\ell j}=1) = 0$$ (not possible for both to happen)

So $\left(\sum_{\ell=1}^{n} E[X_{\ell i}X_{\ell j}]\right) - n p_i p_j = -n p_i p_j$

(refer back to $Var[\vec{x}]$ matrix on p. 15)

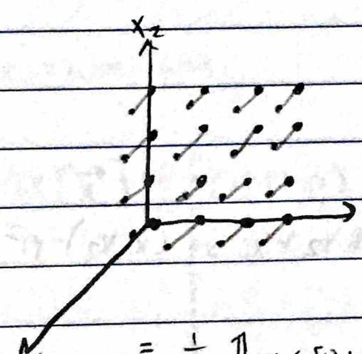
9/16 $X_1, X_2 \overset{iid}{\sim} U(\{0,1,2,3\}) := \begin{cases} 0 \text{ w.p. } 1/4 \\ 1 \text{ w.p. } 1/4 \\ 2 \text{ w.p. } 1/4 \\ 3 \text{ w.p. } 1/4 \end{cases} = \frac{1}{4}\mathbb{1}_{x\in\{0,1,2,3,4\}}$

uniform discrete

In general, $X \sim U(A) := \frac{1}{|A|}\mathbb{1}_{x\in A}$

parameter space: A is a finite set $\subset \mathbb{R}$

$supp[X] = A$



$P_{X_1,X_2}(x_1,x_2) = \frac{1}{16}\mathbb{1}_{x_1\in\{0,1,2,3\}}\mathbb{1}_{x_2\in\{0,1,2,3\}}$

$$p(t) = \sum_{x_1\in\mathbb{R}} \sum_{x_2\in\mathbb{R}} P_{X_1,X_2}(x_1,x_2)\mathbb{1}_{x_2 = t - x_1}$$

$$p(1) = \sum\sum p \, \mathbb{1}_{x_2 = 1 - x_1}$$

$p(.5) \to$ nothing there

$p(4) = 4/16$

- let $Y = \underbrace{-X}_{g(x)}$ $\Rightarrow X = -Y$

$X \sim U(\{0,1,2,3\}) \implies Y \sim U(\{0,-1,-2,-3\})$

$X=0 \to Y=0$ $\quad\quad$ $supp[Y] = -supp[X]$

$X=1 \to Y=-1$

$X=2 \to Y=-2$

$X=3 \to Y=-3$

we know $supp[X]$ and $p_X(x)$. we want $supp[Y]$ and $p_Y(y)$.

$p_Y(y) := P(Y=y) = P(-Y=-y) = P(X=-y) = p_X(-y)$ $\quad\quad$ let $z' = -z \Rightarrow z = -z'$

$supp[Y] := \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z' : p_X(z) > 0\}$

$= -\{z' : p_X(z') > 0\} = -supp[X]$

$$\left[X \sim Binom(n,p) \quad Y = -X \sim \binom{n}{-y} p^{-y} (1-p)^{n+y} = \binom{n}{x} p^{x} (1-p)^{n-x} \right]$$

- $\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1$

all integers and $c \in \mathbb{N}_0$

- $\sum_{x \in \{-d, d+1, \ldots, 0, \ldots, d-1, d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2d+1 & \text{if } d \le c \\ 2c+1 & \text{if } d > c \end{cases}$

where $d \in \mathbb{N}_0$

- $\int_{\mathbb{R}} \mathbb{1}_{x \in [-c,c]} \, dx = 2c$

- $\int_{-d}^{d} \mathbb{1}_{x \in [-c,c]} \, dx = \begin{cases} 2d & \text{if } d < c \\ 2c & \text{if } d \ge c \end{cases}$

- $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}$

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$$\underbrace{P_{X_1|T}}_{\neq \text{Poisson}(\lambda)}(x,t) := \frac{P_{x,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(X, t-x)}{P_T(t)} = \frac{p(x)\,p(t-x)}{P_T(x)}$$

$$= \frac{\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}\right)}{\left(\frac{e^{-2\lambda}(2\lambda)^t}{t!}\right)} = \underbrace{\binom{t}{x}\left(\tfrac{1}{2}\right)^t}_{\binom{t}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{t-x}} = \text{Bin}(t, \tfrac{1}{2})$$

- $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y}$ $\quad \text{supp}[D] = \mathbb{Z}$ ← all integers

$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad P_Y(y) = \frac{e^{-\lambda}\lambda^y}{y!}$

$$D \sim \sum_{x\in\{0,1,..\}} \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!}\right)\mathbb{1}_{d-x\in\{0,-1,\dots\}} = e^{-2\lambda}\sum_{x\in\{0,1,..\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x\geq d}$$

$$= e^{-2\lambda}\begin{cases} \sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \geq 0 \\ \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \end{cases}$$

let $d' = -d$ $\Rightarrow d' = |d|$ $\quad = \sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!}$

let $x' = x - d \Rightarrow x = x' + d \quad = \sum_{x'=0}^{\infty} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!(x')!}$

$$\hookrightarrow e^{-2\lambda}\underbrace{\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}}_{I_{|d|}(2\lambda)} = e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda)$$

$I_{|d|}(2\lambda)$ = Modified Bessel Function of the First Kind

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End Midterm 1 material

Continuous Random variables:

let $X_1 \sim Geom(p) = (1-p)^x p \mathbb{1}_{x \in \{0,1,\dots\}}$

$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = 1-(1-p)^x$

let n Bernoulli experiments occur between each time period of x scale.
let X_n be the waiting time

$P_{X_n}(x) = (1-p)^{nx} p \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}}$

$F_X(x) = 1-(1-p)^{nx}$

let $n \to \infty$, $p \to 0$ s.t. $\lambda = np \to p = \frac{\lambda}{n}$ similar to the Poisson construction

$= (1-\frac{\lambda}{n})^{nx} \frac{\lambda}{n}$

$= 1-(1-\frac{\lambda}{n})^{nx}$

$P_{X_\infty} := \lim_{n\to\infty} P_{X_n}(x) = \Big(\underbrace{\lim (1-\frac{\lambda}{n})^n}_{e^{-\lambda}}\Big)^x \quad \lim_{n\to\infty} \frac{\lambda}{n} = 0 \quad \forall x$

$F_{X_\infty} := \lim_{n\to\infty} F_{X_n}(x) = 1-(\lim(1-\frac{\lambda}{n}))^x = (1-e^{-\lambda x})\mathbb{1}_{x \ge 0}$

$Supp[X_\infty] = [0,\infty) \to$ continuous integral

$|Supp[X_\infty]| = |\mathbb{R}| \Rightarrow X_\infty$ a cont. r.v.

$\sum_{x \in supp[X_\infty]} P_{X_\infty}(x) = 0 \ne 1 \Rightarrow X_\infty$ has no PMF

Is $\lim_{x\to-\infty} F(x) = 0$ ✓

Is $\lim_{x\to\infty} F(x) = 1$

Is $F(x)$ monotonically increasing?

$\frac{d}{dx}[F(x)] = \lambda e^{-\lambda x} > 0 \quad \forall x \ge 0$ ✓

It is a CDF

$\Rightarrow X \sim Exp(\lambda)$ exponential r.v.