$p(4) = 5(1-p)^4 p^2$ 4 0's 1 1's

$\overbrace{0 \; 1 \; 0 \; 0 \; 0}^{\text{4 0's 1 1's}} \; 1$

- $X_1, X_2, X_3 \overset{iid}{\sim} Geom(p)$

$T = X_1 + X_2 + X_3$

$= X_3 + T_2 = p(t) = \sum_{x \in \{0,1,\dots\}} ((1-p)^x p)(t-x+1(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \{0,1,\dots\}})$

$= p^3 (1-p)^t \sum_{x \in \{0,1,\dots\}} (t-x+1) \underbrace{\mathbb{1}_{t-x \in \{0,1,\dots\}}}_{x \le t}$

$\hookrightarrow \sum_{x \in \{0,1,\dots\}} (t+1) \mathbb{1}_{x \le t} - \sum_{x \in \{0,1,\dots\}} x \mathbb{1}_{x \le t}$

$\hookrightarrow (t+1) \underbrace{\sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \le t}}_{t+1} - (1+2+\dots+t)$

$\hookrightarrow (t+1)^2 - \frac{t(t+1)}{2} = t^2+2t+1 - \frac{t^2+t}{2} = \frac{t^2+3t+2}{2}$

$= \frac{(t+2)(t+1)}{2} = \frac{(t+1)!}{t!\,2!} = \binom{t+2}{2}$

$p(t) = \binom{t+2}{2}(1-p)^t p^3 = NegBin(3,p)$

4 0's 3 1's $\binom{6}{2}$ ways

- $p(4) = 15(1-p)^4 p^3$ $\quad 0 \; 0 \; 1 \; 0 \; 1 \; 0 \; 1$

~ $X_1 + \dots + X_r \sim NegBin(r,p) = \binom{t+r-1}{r-1}(1-p)^t p^r = T = \sum_{i=1}^r X_i$

9/5 $X \sim Binom(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

let n be large and p be be small

s.t. $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ let $n \to \infty$

$n \in \mathbb{N}$ $\quad p \in (0,1) \Rightarrow \lambda \in (0,\infty)$

$\hookrightarrow = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$ call it r.v. Y

$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$$\lim_{x\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!} \frac{1}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

there are x terms $\overbrace{(n)(n-1)\ldots(n-x+1)}$

$$= \frac{\lambda^x}{x!} \underbrace{\lim_{n\to\infty} \frac{(n)(n-1)\ldots(n-x+1)}{(n)(n)\ldots(n)}} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \underbrace{\cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x}}_{1}$$

$$\underbrace{\cancel{\lim_{n\to\infty} \frac{n}{n}}}_{1} \; \underbrace{\cancel{\lim_{n\to\infty} \frac{n-1}{n}}}_{1} \ldots \underbrace{\cancel{\lim_{n\to\infty} \frac{n-x+1}{n}}}_{1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} = \text{Poisson}(\lambda)$$ poisson r.v.

$\text{Supp}[X] = \{0, 1, \ldots\} = \mathbb{N}_0$

~ $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$T = X_1 + X_2$

* iid convolution formula:

$$p(t) = \sum_{x \in \text{supp}[X]} P_{old}(x)\, P_{old}(t-x)\, \mathbb{1}_{t-x \in \text{supp}[X]}$$

$$T = X_1 + X_2 \sim \sum_{x\in\{0,1,\ldots\}} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \cdot \mathbb{1}_{t-x \in \{0,1,\ldots\}}\right)$$

$\hookrightarrow \lambda^t e^{-2\lambda} \sum_{x\in\{0,1,\ldots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x \le t}$

$\hookrightarrow \lambda^t e^{-2\lambda} \sum_{x\in\{0,1,\ldots,t\}} \frac{1}{x!(t-x)!} = \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x=0}^{t} \binom{t}{x}$

all subsets of A

$\Rightarrow$ (power set of A): 2^A: $\{B : B \subseteq A\}$

$|2^A| = |\{B : B \subseteq A\}|$

$\rightarrow \{B : B \subseteq A\} = \{B : B \subseteq A, |B| = 0\} \cup \{B : B \subseteq A, |B| = 1\}$

$\cup \{B : B \subseteq A, |B| = 2\} \cup \ldots \cup \{B : B \subseteq A, |B| = n\}$

$\hookrightarrow$ add them all together

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}=\sum_{i=1}^{n}\binom{n}{i}=2^n$$

$$\frac{\lambda^t e^{-2\lambda}}{t!}\sum_{x=0}^{t}\binom{t}{x}=\frac{(2\lambda)^t e^{-2t}}{t!}=\text{Poisson}(2t)$$

~ $X, Y \overset{iid}{\sim} \text{Geom}(p)=(1-p)^x p$

$P(X>Y)$

By Law of Total probability, $\underbrace{P(X>Y)+P(X<Y)}_{\text{equal}}+P(X=Y)=1$

nonzero

$$P(X>Y)=\sum_{y\in\mathbb{R}}\sum_{x\in\mathbb{R}}P_{X,Y}(x,y)\mathbb{1}_{x>y}=\sum_{y\in\mathbb{R}}\sum_{\substack{x\in\mathbb{R}\\x>y}}P_{X,Y}(x,y)=\sum_{y\in\mathbb{R}}\sum_{x\in\mathbb{R}}P(x)P(y)$$

$$=\sum_{y\in\mathbb{R}}\sum_{\substack{x\in\mathbb{R}\\x>y}}\left((1-p)^x p\,\mathbb{1}_{x\in\{0,1,\dots\}}\right)\left((1-p)^y p\,\mathbb{1}_{y\in\{0,1,\dots\}}\right)$$

$$=p^2\sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty}(1-p)^x(1-p)^y=p^2\sum_{y=0}^{\infty}(1-p)^y\sum_{x=y+1}^{\infty}(1-p)^x$$

*reindex trick: let $x'=x-(y+1)\Rightarrow x=x'+(y+1)$

$$\sum_{x=y+1}^{\infty}(1-p)^x=\sum_{x'=0}^{\infty}(1-p)^{x'+y+1}$$

$$=p^2\sum_{y=0}^{\infty}(1-p)^y\sum_{x'=0}^{\infty}(1-p)^{x'}(1-p)^{y+1}=p^2\sum_{y=0}^{\infty}(1-p)^{2y+1}\sum_{x'=0}^{\infty}(1-p)^{x'}$$

* $\forall\,\alpha\in(0,1)\ \sum_{x=0}^{\infty}\alpha^x=\frac{1}{1-\alpha}$ Geometric series

$$=p^2\sum_{y=0}^{\infty}(1-p)^{2y+1}\frac{1}{p}$$

$$=p(1-p)\sum_{y=0}^{\infty}(1-p)^{2y}=p(1-p)\underbrace{\sum_{y=0}^{\infty}\left((1-p)^2\right)^y}_{\frac{1}{1-(1-p)^2}}$$

$$=\frac{p(1-p)}{1-(1-p)^2}=\frac{p-p^2}{1-(1-2p+p^2)}$$

8 $$=\frac{p-p^2}{2p-p^2}=\frac{\cancel{p}(1-p)}{\cancel{p}(2-p)}=\boxed{\frac{1-p}{2-p}}$$

~ Recall $\overset{\text{expectation}}{E}[g(x)] = \sum_{x \in \mathbb{R}} g(x)\, p(x)$ for x discrete

$E[x] = \sum_{x \in \mathbb{R}} x\, p(x)$

let $g(x) = \mathbb{1}_{x \in A}$

$E[g(x)] = \sum_{x \in \mathbb{R}} \mathbb{1}_{x \in A}\, p(x) = \sum_{x \in A} p(x) = P(x \in A)$

~ for x, y discrete, $E[g(x,y)] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} g(x,y)\, p(x,y)$

$E[\mathbb{1}_{x>y}] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p(x,y)\, \mathbb{1}_{x>y}$

- p_1 = prob. of picking apple

p_2 = prob of picking banana

Let's pick one with replacement n times.

let x_1 = # of apples x_2 = # of bananas

$X_1 \sim Bin(n, p_1)$ $X_2 \sim Bin(n, p_2)$

$p_2 = 1 - p_1$

let $\vec{X} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim p_{x_1, x_2}(x_1, x_2)$ $\dim[\vec{x}] = 2$

$p_{x_1,x_2}(x_1, x_2) = \frac{n!}{x_1!\, x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$

$\binom{n}{x_1, x_2} = \frac{n!}{x_1!\, x_2!}$

$\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} = \text{Multinomial}(n, \vec{p})$ (vector r.v.)