Continuous r.v's $|Supp[X]| = |\mathbb{R}| \Rightarrow p(x) = 0$ 09/18/2019.

$P(X = x) = 0$ $\quad f_X(x) = F_X'(x)$ → Probability Density Function

CDF when picking more prob than + ; prob = 0 as moving along line.

$$P(X \in [a,b]) = F(b) - F(a) \overset{\text{by FTC}}{=} \int_a^b \underbrace{f(x)}_{\text{PDF}}\, dx$$

$Supp[X] = \{x, f(x) > 0\}$

Properties of PDF

→ PDF

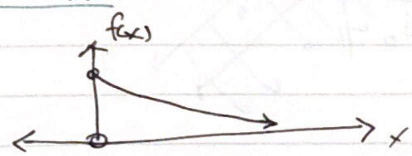
→ CDF how fast changing

① $\int_{\mathbb{R}} f(x)\, dx = 1 = \underbrace{F(\infty)}_{1} - \underbrace{F(-\infty)}_{0}$

② $f(x) \geq 0$ because F is monotonically increasing

$X \sim Exp(\lambda) = \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$

exponential r.v.



like time nonnegative.

$Supp[X] = [0, \infty)$

~~parameter space~~

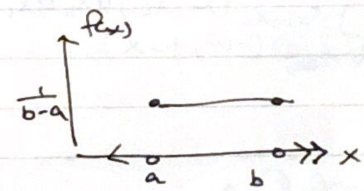
$\lambda \in (0, \infty)$

Another continuous r.v $X \sim U(a,b) = \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$

uniform r.v

$Supp[X] = [a, b]$

parameter space $a, b \in \mathbb{R}$ but $b > a$



$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$

standard uniform

$$\vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \quad \text{it has a joint density function (JDF)}$$

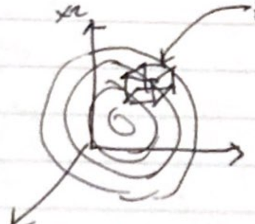
$$f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_k}(x_k) = f(x_1) \cdot \ldots \cdot f(x_k)$$

if iid

if $X_1, \ldots X_k$ ind

$$\int_{\mathbb{R}^k} \cdots \int f_{X_1, \ldots, X_k}(x_1, \ldots, x_n)\, dx_1 \ldots dx_k = 1$$

If $k = 2$



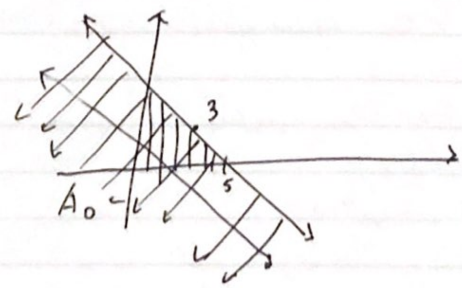
$$P(\vec{X} \in A) = \iint_A f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

Continuous Convolution.

Book Pg 143 $\rightarrow$

$$T = X_1 + X_2 \sim f_T(t) = ?$$
$$= f_{X_1}(x) * f_{X_2}(x)$$

$$F_T(t) = P(T \le t) = P(A_t)$$
$$A_t = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underbrace{x_1 + x_2 \le t}_{x_2 \le t - x_1} \right\}$$



$$\Rightarrow \iint_{A_t} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$$

$$= \iint_{x_1 \in \mathbb{R},\ x_2 \in (-\infty, t-x)} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

let $x_1 = x$

$x_2 = v - x \Rightarrow v = x_2 + x$

$dv_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$F_T(t) = \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1, X_2}(x, t-x)\, dv\, dx$$

$$f_T(t) = \frac{d}{dt}\Big(\quad\Big) = \underbrace{\int_{\mathbb{R}} f_{X_1 X_2}(x, t-x)\, dx}_{\text{general convolution formula}} = \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx$$

if X_1, X_2 ind

$$\int_{supp[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, \mathbb{1}_{t-x \in supp[X_2]}\, dx$$

$x \in \mathbb{R}$

PMF $\Rightarrow$ density

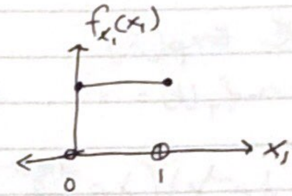
$\Sigma \Rightarrow \int$ intervals.

if iid

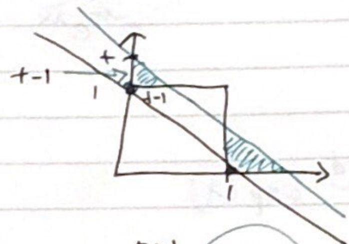
$$= \int_{\mathbb{R}} f(x) f(t-x)\, dx = \int_{Supp[X]} f(x) f(t-x) \mathbb{1}_{t-x \in Supp[X]}\, dx$$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$

$T = X_1 + X_2 \sim f_T(t) = ?$



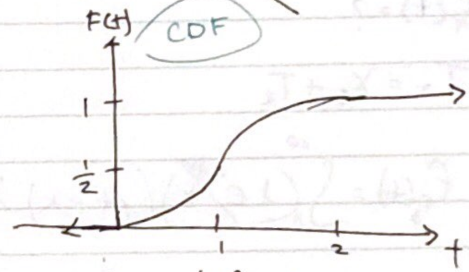
$$f_{X_1 X_2}(x_1, x_2) = f(x_1) f(x_2) = \mathbb{1}_{x_1 \in (0,1)} \mathbb{1}_{x_2 \in [0,1]}$$



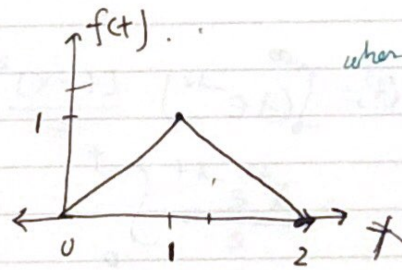
$Supp[T] = [0, 2]$

$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1}{2}t^2 & \text{if } t \in (0,1] \\ \frac{1}{2}t^2 + 2t - 1 & \text{if } t \in (1,2] \\ 1 & \text{if } t > 2 \end{cases}$$

$$\frac{1}{2}t^2 - 2\left(\frac{1}{2}(t-1)^2\right) = \frac{1}{2}t^2 - (t^2 - 2t + 1) = -\frac{1}{2}t^2 + 2t - 1$$



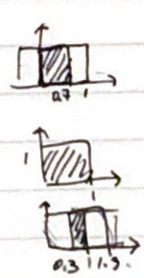
$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$



when overlaps.

$$f_T(t) = \int_0^1 \underbrace{f(x)}_{1} \underbrace{f(t-x)}_{1} \underbrace{\mathbb{1}_{t-x \in [0,1]}}_{\substack{x-t \in [-1,0] \\ \Downarrow \\ x \in [t-1,\, t]}} dx$$

$$\Rightarrow \int_0^1 \mathbb{1}_{x \in [t-1, t]}\, dx = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$



$1 - 0.3$

$1 - (t-1)$

$2 - t$

$t-x \Rightarrow x \leq t$

$$\int_{Supp[X]} f(x)\, f(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

$X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

$T = X_1 + X_2 \sim f_T(t) = ?$ $\quad Supp[T] = [0, \infty)$

$$f_T(t) = \int_0^\infty (\lambda e^{-\lambda x})\Big(\lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{t-x \in [0,\infty)}\Big)\, dx = \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \leq t}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx$$

$$= t\lambda^2 e^{-\lambda t}$$

$X_1, X_2 \ldots \overset{iid}{\sim} Exp(\lambda)$

$f_T(t) = ?$

$T_3 = X_3 + T_2$

$$f_{T_3}(t) = \int_0^\infty (\lambda \cancel{e^{-\lambda x}})\Big((t-x)\lambda^2 \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\Big) \mathbb{1}_{x \leq t}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t (t-x)\, dx = \frac{t^2}{2} \lambda^3 e^{-\lambda t}$$

$$f_{T_4}(t) = \int_0^\infty (\lambda e^{-\lambda x}) \frac{(t-x)^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{x \leq t}\, dx$$

$$= \frac{\lambda^4 e^{-\lambda t}}{2} \int_0^t$$