(Lecture 10)

October 16^{th}, 2019

$$f_Y(y) = f_X((g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(x)\right]\right|$$

X

$X \sim Exp(1) = e^{-x}\,\mathbb{1}_{x\in(0,\infty)}$

let $Y = -\ln\left(\dfrac{e^{-x}}{1-e^{-x}}\right)$

$= \ln\left(\dfrac{e^{x}}{1-e^{-x}}\right)$

$= \ln(e^{x}-1)$

$= g(x)$

$y = \ln(e^{x}-1)$

$e^{y} = e^{x}-1$

$e^{y}+1 = e^{x}$

$x = \ln(e^{y}+1) = g^{-1}(y)$

$\dfrac{d}{dy}\left[\ln(e^{y}+1)\right] = \dfrac{e^{y}}{e^{y}+1} > 0 \quad \forall y$

$$f_Y(y) = e^{-\ln(e^{y}+1)}\,\frac{e^{y}}{e^{y}+1}\,\mathbb{1}_{\underbrace{\ln(e^{y}+1)\in(0,\infty)}}$$

$e^{y}+1 \in (1,\infty)$

$e^{y} \in (0,\infty)$

$y \in \mathbb{R}$

$$= \left(\frac{1}{e^{y}+1}\right)\frac{e^{y}}{(e^{y}+1)}$$

$$= \frac{e^{y}}{(e^{y}+1)^2}\left(\frac{e^{-2y}}{e^{-2y}}\right) = \frac{e^{-y}}{(1+e^{-y})^2} = \text{"Standard" Logistic}(0,1)$$

Let $\mu \in \mathbb{R}, \sigma > 0$

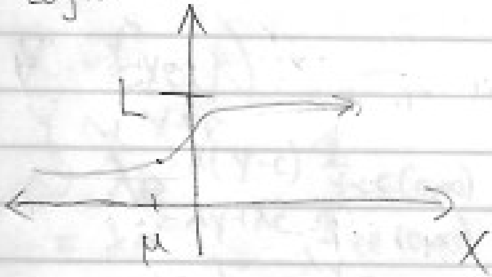
$$L = \sigma Y + \mu \sim \frac{1}{\sigma} \frac{e^{\frac{l-\mu}{\sigma}}}{\left(e^{\frac{l-\mu}{\sigma}}+1\right)^2} \sim \text{Logistic}(\mu, \sigma)$$

$$E[L] = \mu, \quad SE[L] = \sigma \frac{\pi}{\sqrt{3}}$$

Let $l(x) := \dfrac{L}{1+e^{-k(x-\mu)}}$

L : max value
k : Steepness
μ : Center

Logistic function



$$F_Y(y) = \int_{-\infty}^{y} \frac{e^t}{(1+e^t)^2} \, dt$$

Let $u = 1+e^t$, $du = e^t dt$, $dt = e^{-t} du$

$\Downarrow$

$u-1 = e^t$ $\qquad t = -\infty \rightarrow u = 1$

$\Downarrow$

$t = \ln(u-1)$ $\qquad t = y \rightarrow u = 1+e^y$

$\Downarrow$

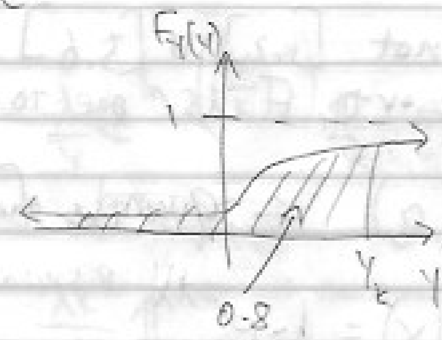
$dt = \frac{1}{u-1} du$

$$\int_{-\infty}^{y} \frac{e^t}{(1+e^t)^2} \, dt = \int_{1}^{1+e^y} \frac{u-1}{u^2} \frac{1}{u-1} \, du = \left[-u^{-1}\right]_1^{1+e^y}$$

$$= 1 - \frac{1}{1+e^y}$$

$= \frac{e^y}{1+e^y}$ ← Standard Logistic Function, i.e. $L=1, \mu=0, k=1$

$= \frac{1}{1+e^{-y}}$



New Question

Solve for minimum x such that $q \geq P(X \geq x) = F(x)$

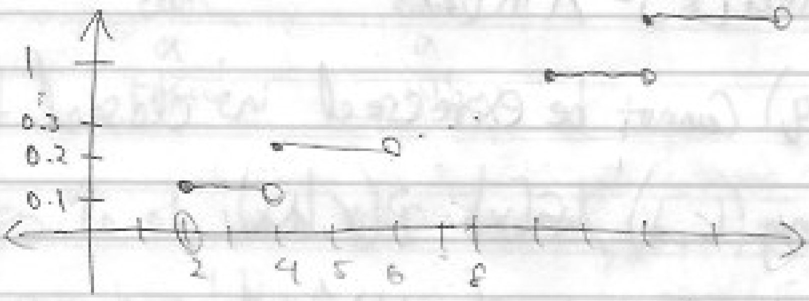
Ex: $q = 80\%$

This X_* is called the q^{th} quantile or $100q$ percentile

$X^* = Q[X, q]$

← quantile operator

Let $X \sim U(\{2, 4, \cdots, 20\})$



30^{th} Percentile

$Q[X, 0.3] = 6$ $\quad Q[X, 0.9] = 18$

$Q[X, 1] = 20$ $\quad Q[X, 0.85] = 18$

CDF's are step function

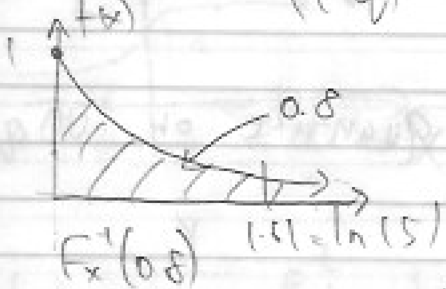
Continuous r.v.'s with continuous support

Ex: $[0,10]$, $[0,\infty)$, but not $[1,2] \cup [5,6]$, the $F(x)$ is strictly increasing on the support, F is one-to-one

$\Rightarrow X_x = Q[X,q] = F_x^{-1}(q) \leftarrow$ quantile function

$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \rightarrow F_x(x) = 1 - e^{-\lambda x}$

$F^{-1}(q) = \lambda \ln\left(\frac{1}{1-q}\right)$



Define medium, $\text{med}[X] = Q[X, 0.5] \overset{\text{if continuous}}{=} F_x^{-1}(0.5)$

$X \sim Exp(\lambda) \Rightarrow \text{Med}[X] = \lambda \ln(2)$

Often times $F_x^{-1}(q)$ cannot be expressed in closed form

Example: $X \sim Erlang(k, \lambda)$, $F(x) = P(x, \lambda x)$

to solve $Q[X,q]$, set $q = P(x, \lambda x)$ and use the computer to find x

$X \sim Exp(\lambda)$, let $Y = ke^{X} \sim ?$

$g^{-1}(y) = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k)$

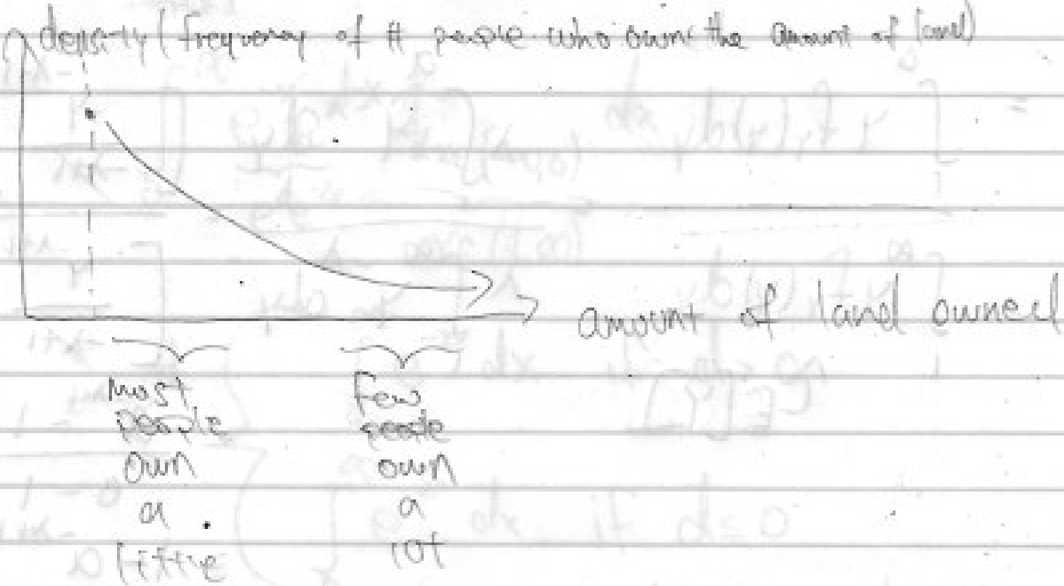
$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{y} > 0 \; \forall y > 0$

$$f_Y(y) = \lambda e^{-\lambda \ln\left(\frac{y}{k}\right)} \left(\frac{1}{y}\right) \mathbb{1}_{\underbrace{\ln(y)-\ln(k)\in(0,\infty)}_{\substack{\ln(y)\in(\ln(k),\infty) \\ y\in(k,\infty)}}}$$

$$= \frac{\lambda}{y} \underbrace{e^{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)}}_{\left(\frac{y}{k}\right)^{-\lambda}} \mathbb{1}_{y\in(k,\infty)} \qquad \text{Note: } a\ln(b) = \ln(b^a)$$

$$= \frac{\lambda k^\lambda}{y^{\lambda+1}} \mathbb{1}_{y\in(k,\infty)}$$

$$= \text{Pareto I}(k;\lambda) \rightarrow \text{"Pareto Distribution"}$$



50% of land was owned by 20% of the people.

$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}}\,dt = \lambda k^\lambda \left[\frac{-t^{-\lambda}}{\lambda}\right]_k^y$$

$$= k^\lambda\left(k^{-\lambda} - y^{-\lambda}\right)$$

$$= \left(1-\left(\frac{k}{y}\right)^\lambda\right)\mathbb{1}_{y\in(k,\infty)}$$

$F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}$

Let $k=1$, $Y \sim \text{Pareto}(1,\lambda) = \frac{\lambda}{y^{\lambda+1}} \mathbb{1}_{y\in(1,\infty)}$

$F_Y(y) = \left(1 - \frac{1}{y^\lambda}\right) \mathbb{1}_{y\in(1,\infty)}$

$F_Y^{-1}(q) = (1-q)^{-\frac{1}{\lambda}}$

Let $L(a)$ be % of land owned if land is owned by people who own $\le a$

$$L(a) = \frac{\text{amount owned if} \le a}{\text{Total amount owned}}$$

$$= \frac{\int_1^a y f_Y(y)\,dy}{\underbrace{\int_1^\infty y f_Y(y)\,dy}_{E[Y]}} = \frac{\lambda\int_1^a y^{-\lambda}\,dy}{\lambda\int_1^\infty y^{-\lambda}\,dy} = \frac{\left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]_1^a}{\left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]_1^\infty}$$

$$= \frac{a^{-\lambda+1}-1}{0-1}$$

$$= 1-a^{-\lambda+1}$$

80% of land was owned by 20% of the people

$$1-q = L(F_Y^{-1}(q)) = 1-\left((1-q)^{-\frac{1}{\lambda}}\right)^{-\lambda+1}$$

$$= 1-(1-q)^{\frac{\lambda+1}{\lambda}}$$

$q = (1-q)^{\frac{\lambda+1}{\lambda}} \to \ln(q) = \left(1+\frac{1}{\lambda}\right)\ln(1-q)$

$\frac{\ln(q)}{\ln(1-q)} = 1+\frac{1}{\lambda} \to \log_{1-q}(q) = 1+\frac{1}{\lambda} \to \lambda = \log_4(5) = 1.161$

$X \sim \text{Pareto}(1, 1.161)$ is the "Pareto Rule"

$X, Y \overset{iid}{\sim} \text{Exp}(1) = e^{-x} \mathbb{1}_{x \in (0,\infty)}$

Let $D = X - Y = X + \underbrace{(-Y)}_{Z} \sim ?$

$Z \sim e^{z} \mathbb{1}_{z \in (-\infty, 0)}$

$$f_D(d) = \int_{\text{Supp}[X]} f_X(x) f_Z(d-x) \mathbb{1}_{d-x \in \text{Supp}[Z]} \, dx$$

$$= \int_0^\infty \underbrace{e^{-x} e^{d-x}}_{e^d e^{-2x}} \underbrace{\mathbb{1}_{d-x \in (-\infty, 0)}}_{x \in (d, \infty)} \, dx$$

$$= e^d \begin{cases} \int_d^\infty e^{-2x} \, dx & \text{if } d > 0 \\ \int_0^\infty e^{-2x} \, dx & \text{if } d \leq 0 \end{cases}$$

$$= e^d \begin{cases} \left[-\frac{1}{2} e^{-2x}\right]_d^\infty & \text{if } d > 0 \\ \left[-\frac{1}{2} e^{-2x}\right]_0^\infty & \text{if } d \leq 0 \end{cases}$$

$d > 0 \Rightarrow d = |d|$
$d \leq 0 \Rightarrow -d = |d|$

$$= \frac{1}{2} e^d \begin{cases} e^{-2d} & \text{if } d > 0 \\ 1 & \text{if } d \leq 0 \end{cases} = \frac{1}{2} \begin{cases} e^{-d} & \text{if } d > 0 \\ e^{d} & \text{if } d \leq 0 \end{cases} = \frac{1}{2} e^{-|d|}$$

$= \text{Laplace}(0, 1)$

$\text{Supp}[D] = \mathbb{R}$

$L = \mu + \sigma D \sim \frac{1}{2\sigma} e^{-\frac{|\ell - \mu|}{\sigma}}$

Such that $\sigma > 0$

$\mu > \mathbb{R}$