9/9

Bag of Fruit

p_1 - prob of apple
p_2 - prob of banana
p_3 - prob of cantalope

$$p_1 + p_2 + p_3 = 1$$

Draw n w/ replacement

Let X_1 : # of apples
X_2 : # of bananas
X_3 : # of cantalope

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim P_{\vec{X}}(\vec{x}) = \cancel{\frac{n!}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}}$$

$$= \frac{n!}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n} \mathbb{1}_{x_1 \in \{0,\dots,n\}} \mathbb{1}_{x_2 \in \{0,\dots,n\}} \mathbb{1}_{x_3 \in}$$

$$= \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \text{Multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right)$$

- **Generally w/ K elements**

$$\text{Multinomial}\left(n, \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}\right) = \binom{n}{p_1, \dots, p_K} p_1^{x_1} \cdots p_K^{x_K}$$

- $Supp[\vec{X}] = \{ \vec{x} \mid \vec{x} \in \mathbb{N}^k, \vec{x} \cdot \vec{1} = n \}$

$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $\quad \vec{x} \in \{0, 1, \ldots, n\}^k$

$\vec{p} \in \{ \vec{p} : (0,1)^k, \vec{p} \cdot \vec{1} = 1 \}$

$n \in \mathbb{N}$

Back to 2-dimensional case

$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim Multi(n, \begin{bmatrix} p \\ 1-p \end{bmatrix})$

$p_1 = p$ $\qquad X_1 \sim Binom(n, p)$
$p_2 = 1-p$ $\qquad X_2 \sim Binom(n, 1-p)$

Is $X_1 \stackrel{d}{=} X_2$? NO
Is X_1, X_2 ind? No

For two indep. r.v's, $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$ $\forall x_1 \in Supp[X_1]$ & $x_2 \in Supp[X_2]$

$0 = P(X_1 = 1 \mid X_2 = n) \neq P(X_1 = 1) = \binom{n}{1} p (1-p)^{n-1} = np(1-p)^{n-1}$

* When $K = Dim[\vec{x}] = 2$ & $n = 1$, multinomial is a Bernoulli distribution.
* When $K = 2$ & $n > 1$, multinomial is a Binomial distribution.

Conditional PMF / JMF

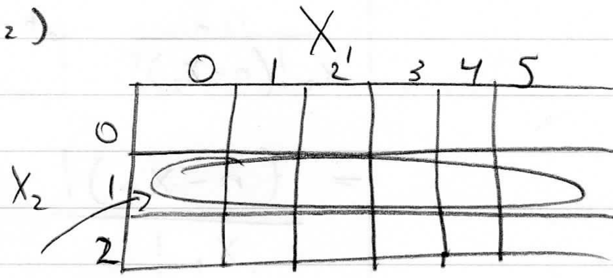
$$P_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \frac{P_{X_1, X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

Def of cond. prob.

Marginalization

$$P_{X_2}(x_2) = \sum_{x_1 \in supp[X_1]} P_{X_1, X_2}(x_1, x_2)$$

Marginal PMF



$$= \cancel{\sum_{x \in \{0,\dots,n\}} \frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}}$$

$$= \sum_{x_1 \in \{0,\dots,n\}} \frac{n!}{x_1! \, x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_2 \in \{0,\dots,n\}}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \mathbb{1}_{x_2 \in \{0,\dots,n\}} \underbrace{\sum_{x_1 \in \{0,\dots,n\}} \frac{1}{x_1!} p^{x_1} \mathbb{1}_{x_1 = n - x_2}}_{= \frac{1}{(n-x_2)!} p^{n-x_2}}$$

$$= \binom{n}{x_2} (1-p)^{x_2} p^{n-x_2} = Bin(n, 1-p)$$

Back to Cond. prob.

$$P_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 | X_2 = x_2) = \frac{P_{X_1,X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

$$= \frac{\frac{n!}{x_1!\, x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}}{\frac{n!}{x_2!(n-x_2)!} p^{n-x_2}(1-p)^{x_2}}$$

$$= \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$$= \begin{cases} \frac{x_1!}{x_1!} p^0 = 1 & \text{if } x_1+x_2=n \\ 0 & \text{if } x_1+x_2 \neq n \end{cases}$$

$$= \text{Deg}(n-x_2) = \{ n - x_2 \text{ w.p. } 1$$

Degenerate

$$P_{\vec{X}|x_j}(\vec{x}, x_j) = \frac{\dfrac{\cancel{n!}}{x_1! \cdots x_{j-1}! \cancel{x_j!} x_{j+1}! \cdots x_k!} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cancel{p_j^{x_j}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{\cancel{\dfrac{n!}{x_j!(n-x_j)!} p_j^{x_j}} (1-p_j)^{n-x_j}}$$

$$\left(\vec{X} = \begin{bmatrix} x_1 \\ x_{j+1} \\ x_{j-1} \\ \vdots \\ x_j \end{bmatrix} \right)$$

Let $n' := n - x_j$ (# of things left over in sample)

$$= \frac{n'!}{x_1! \cdots x_{j-1}! x_{j+1}! \cdots x_k!} \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{n'}}$$

$$\left(\begin{array}{l} \text{Recall: } n = x_1 + \dots + x_{j-1} + x_j + x_{j+1} + \dots + x_k \\ \Rightarrow n' = x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k \end{array} \right)$$

$$= \binom{n'}{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k} \left[\frac{p_1}{(1-p_1)}\right]^{x_1} \cdots \left[\frac{p_{j-1}}{(1-p_{j-1})}\right]^{x_{j-1}} \left[\frac{p_{j+1}}{(1-p_{j+1})}\right]^{x_{j+1}} \cdots \left[\frac{p_k}{(1-p_k)}\right]^{x_k}$$

$$= \text{Multin}(n', \vec{p}\,')$$

$$\vec{p}\,' = \begin{bmatrix} \frac{p_1}{1-p_1} \\ \vdots \\ \frac{p_{j-1}}{1-p_{j-1}} \\ \frac{p_{j+1}}{1-p_{j+1}} \\ \vdots \\ \frac{p_k}{1-p_k} \end{bmatrix} \qquad \dim[\vec{p}\,'] = k-1$$

- $E[\vec{X}] = ?$
- $Var[\vec{X}] = ?$
- $\mu := E[X] \overset{disc}{=} \sum_{x \in \mathbb{R}} x p(x)$

  $\overset{cont}{=} \int_{\mathbb{R}} x f(x)\, dx$

- $E[aX+b] = a\mu + b$
- $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$ always true
- $E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$

  if $X_1, \dots, X_n$ ind.

- $\sigma^2 := Var[X] := E\left[(X-\mu)^2\right] = E(X^2) - \mu^2$

  $\sigma := E[X] := \sqrt{\sigma^2}$

  Standard error

$$
\begin{aligned}
Var[X_1 + X_2] &= E\left[\left((X_1+X_2) - (\mu_1 + \mu_2)\right)^2\right] \\
&= E\big[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 \\
&\quad - 2X_1\mu_1 - 2X_2\mu_1 - 2X_1\mu_2 - 2X_2\mu_2 \\
&\quad + 2X_1X_2 + 2\mu_1 + 2\mu_2 \\
&= E(X_1^2) + E(X_2^2) + \mu_1^2 + \mu_2^2 \\
&\quad - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_2^2 \\
&\quad + 2E[X_1X_2] + 2\mu_1\mu_2 \\
&= \sigma_1^2 + \sigma_2^2 + 2\left(E[X_1X_2] - \mu_1\mu_2\right) \\
&= \boxed{\sigma_1^2 + \sigma_2^2}
\end{aligned}
$$

if X_1, X_2 ind

"Covariance"

$$\sigma_{12} := Cov[X_1, X_2] := E[X_1 X_2] - \mu_1 \mu_2$$

$$= E[(X_1 - \mu_1)(X_2 - \mu_2)]$$

Cov (+) means two r.v's are close

Cov (-) means two r.v's are far apart

(lots of apples, not a lot of bananas)