Math 621 — Aug-28

Lecture-1

A discrete random variable X has probability mass function (PMF).
$P(x) := P(X = x)$ notation $X \sim P(X)$ and cumulative distinction factor.
$F(X) := P(X \le x)$ random variable X has "support" $supp(X) := \{x : P(x) > 0, x \in \mathbb{R}\}$
support and PMF are related via $\sum_{x \in supp[X]} P(X) = 1$

Also, (the size) $|supp(X)| \le |\mathbb{N}|$ i,e the number of possible different realization is (finite or discrete)

IMHO, the most fundamental random variable is the Bernoulli :

$$X \sim Bern(p) := \begin{cases} 1, & \text{w.p } p \\ 0 & \text{w.p } 1-p \end{cases} = p^x (1-p)^{1-x}$$

with probability

$supp[X] = \{0, 1\}$
where p is the parameter (turning knob) & belong to the parameter space" $p \in \{0, 1\}$

If $p = 1$, $X \sim Bern(1) = \{1 \text{ w.p } 1 = Deg\, 1 = \mathbb{1}_{x=1}$ (degenerate r.v)

If $p = 1$, $X \sim Bern(0) = \{0, \text{ w.p } = Deg(0) = \mathbb{1}_{x=0}$
$X \sim Bern(c) = \{c, \text{ w.p } = \mathbb{1}_{x=c}$

Note: $P(X = 3.7) = P($
$P(X = 3.7) = P(3.7) = p^{3.7}(1-p)^{-2.7} = 0.5$
$\parallel$
0
if $p = \frac{1}{2}$

does not work b/c of the supp of bernoulli

Let $\mathbb{1}_A := \begin{cases} 1 & \text{if } A \\ 0 & \text{if } A^c \end{cases}$

Scanned with CamScanner

$\forall \in \mathbb{R}, p(x) = p^x(1-p)^{1-x} \mathbb{1}_{x=\{0,1\}} \longrightarrow \text{supp}[x]$

$$\boxed{\begin{array}{lll} f(x) = \mathbb{1}_{x<17} & \mathbb{1}(A) & f(19)=0 \\ & I_A & f(-173)=1 \\ & 1(A) & f(-173)=1 \end{array}}$$

$P_{OLD}(X)\, \mathbb{1}_{x \in \text{Supp}[x]}$

* $X_1, X_2, X_3 \dots X_n$ are discrete random variables which are have a joint mass function (JMF) $\rightarrow$ $P_{\cancel{X_1}, \cancel{P X_2}} - P_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots x_n)$.

$$= P(X_1 = x_1 \ \& \ X_2 = x_2 \ \& \dots \& \ X_n = x_n)$$

If $X_1, X_2, X_3 \dots X_n \overset{iid}{\Rightarrow} P_{X_1 \dots X_n}(x_1 \dots x_n)$

$$= \prod_{i=1} P_{X_i}(x_i) \ \forall \vec{x} \in \mathbb{R}^n$$

(independent) $\quad \vec{x} = \begin{bmatrix} x \\ \vdots \\ x_n \end{bmatrix}$

If $X_1 \overset{d}{=} X_2 \overset{d}{=} \dots \overset{d}{=} X_n \Rightarrow P_{X_1}(x) = P_{X_2}(x) = \dots P_{X_n}(x)$

(identically distributed)

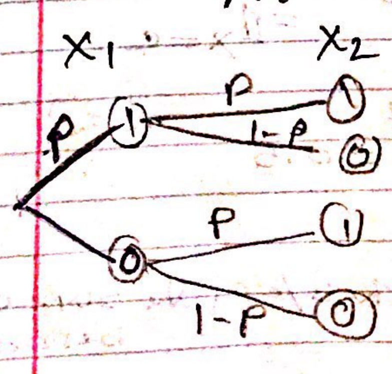
Which means they share the same PMF $\Rightarrow \prod_{i=1}^{n} P(x_i)$

Let $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

Let $T := X_1 + X_2 \sim P_T(t) = ?$

$\text{Supp}(T) = \text{Supp}[X_1] + \text{Supp}[X_2] = \{0, 1, 2\}$

$$A + B = \{a + b : a \in A, b \in B\}$$



(x_1, x_2)	$P_{X_1,X_2}(x_1,x_2)$	T	$P_T(t)$
(1,1)	p^2	2	p^2
(1,0)	$p(1-p)$	1	$2p(1-p)$
(0,1)	$(1-p)$	1	
(0,0)	$(1-p)^2$	0	$(1-p)^2$

Scanned with CamScanner

Thus, $p_T(t) = \begin{cases} 0 & \text{wp } (1-p)^2 \\ 1 & \text{wp } 2p(1-p) \\ 2 & \text{wp } p^2 \end{cases}$

$$\sum_{t \in \text{supp}[T]} p_T(t) = (1-p)^2 + 2p(1-p) + p^2$$
$$= 1 - \cancel{2p} + \cancel{p^2} + \cancel{2p} - \cancel{2p^2} + p^2 = 1$$

$$P(T=t) = p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1 X_2}(x_1, x_2) \mathbb{1}_{t = x_1 + x_2}$$
$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1 X_2}(x_1, t - x_1) \mathbb{1}_{x_2 = t - x_1}$$
$$= \sum_{x_1 \in \mathbb{R}} p_{X_1, X_2}(x_1, t - x_1)$$ general discrete convolution formula

if X_1, X_2 independent $= \sum_{x \in \mathbb{R}} p_{X_1, X_2}(x_1, t - x)$

if $X_1 \overset{d}{=} X_2 = \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum_{x \in \text{supp}[X]} p(x)\, p(t-x) \mathbb{1}_{t-x \in \text{supp}[X]}$

our case:

$$p_T(t) = \sum_{x \in \mathbb{R}} \left(p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}\right)\left(p^{t-x}(1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}\right)$$
$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}$$
$$= \sum p^t (1-p)(1-p)^{1-x} \mathbb{1}_{t-x \in \{0,1\}}$$
$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$
$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t-1 \in \{0,1\}}\right)$$
$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}}\right)$$
$$= \begin{cases} 1(1-p) & \text{if } t = 0 \\ 2p(1-p) & \text{if } t = 1 \\ 1p^2 & \text{if } t = 2 \\ 0 & \text{otherwise} \end{cases} = \binom{2}{t} p^t (1-p)^{2-t} = \text{Binm}(2, p)$$

$$\binom{2}{t} = \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \\ 0 & \text{if otherwise} \end{cases}$$

Scanned with CamScanner