(Lecture 7) September 18th, 2009

Continuous r.v.'s $|\text{supp}(X)| = |\mathbb{R}| \Rightarrow p(x) = 0$

$P(X=x) = 0$ $\quad \text{Supp}(X) = \{x : f(x) > 0\}$

$f_X(x) := F_X'(x)$

$$P(X \in [a,b]) = F(b) - F(a) \overset{\text{By FTC}}{=} \int_a^b f(x)\,dx$$

Properties of PDF

1) $\int_{\mathbb{R}} f(x)\,dx = 1$

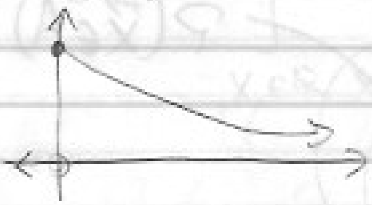
$= F(\infty) - F(-\infty)$

$= 1 - 0 = 1$

2) $f(x) \geq 0$ because F is monotonically increasing

$$X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0} \qquad \text{Supp}[X] = [0, \infty)$$

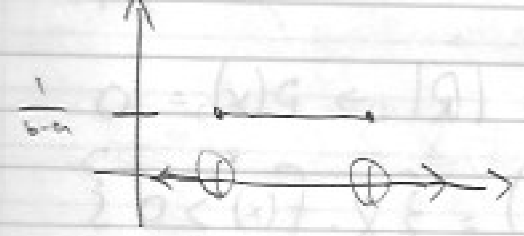
exponential r.v. $\qquad \lambda \in (0, \infty)$



Another continuous random variable

$$X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}, \quad \text{Supp}[X] = [a,b]$$

uniform r.v.



Parameter space $a,b \in \mathbb{R}$
but $b > a$

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$ ← "standard uniform"

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix}$ it has a joint density function (JDF)

$$f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_k}(x_k) = f(x_1) \cdot \ldots \cdot f(x_k)$$

↑ if $X_1, \ldots, X_k$ ind ↑ if iid

$$\int_{\mathbb{R}^k} \cdots \int f_{X_1 \cdots X_k}(x_1 \cdots x_k)\, dx_1 \cdots dx_k = 1$$

If $\vec{X}$ $k=2$

$$P(\vec{X} \in A) = \iint_A f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

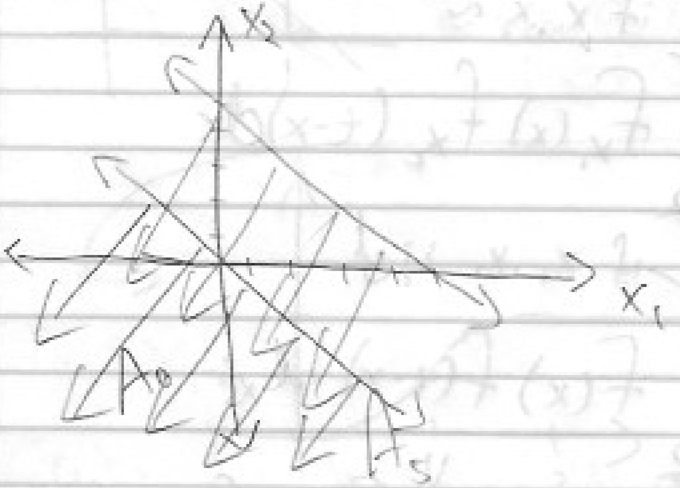
$T = X_1 + X_2 \sim f_T(t) = ?$

$$= \underbrace{f_{X_1}(x) * f_{X_2}(x)}_{\text{Continuous convolution (page 145 on textbook)}}$$

$F_T(t) = P(T \le t) = P(A_t)$

$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 \le t \right\}$

$x_2 \le t - x_1$



$$= \iint_{A_t} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1 = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty, t - x_1]} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$$

$\boxed{\text{Let } X_1 = x}$

$x_2 = v - x \rightarrow v = x_2 + x$

$dx_2 = dv$

if $x_2 = -\infty \rightarrow v = -\infty$, if $x_2 = t - x \rightarrow v = t$

$$F_T(t) = \int_{x\in R} \int_{-\infty}^{t} f_{X_1,X_2}(x, t-x)\, dv\, dx$$

$$= \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\, dx\, dv$$

$$f_T(t) = \frac{d}{dt}\left(\quad\right) = \int f_{X_1,X_2}(x, t-x)\, dx$$ ← general convolution formula

if $X_1, X_2 \overset{ind}{\sim}$

$$= \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\, dx$$

$$\int_{Supp[X_1]} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in Supp(X_2)}\, dx$$

if X_1, X_2 iid

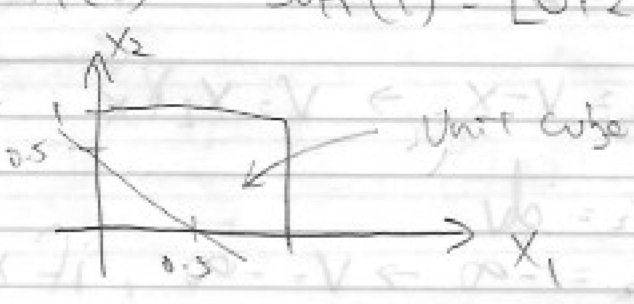
$$= \int_{\mathbb{R}} f(x)\, f(t-x)\, dx$$

$$\int_{Supp(x)} f(x)\, f(t-x)\, \mathbb{1}_{t-x \in Supp(x)}\, dx$$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$

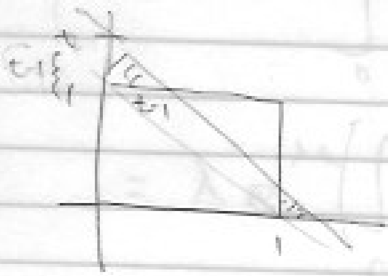
$T = X_1 + X_2 \sim f_T(t)$ $\quad Supp(T) = [0, 2]$

CDF Method

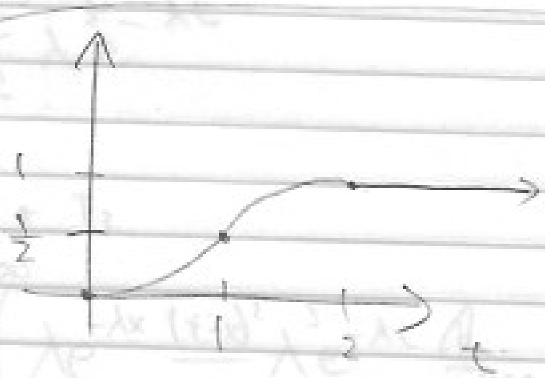


$$f_{X_1,X_2}(x_1,x_2) = f(x_1)\,f(x_2)$$
$$= \mathbb{1}_{x_1\in[a,b]}\,\mathbb{1}_{x_2\in[a,b]}$$

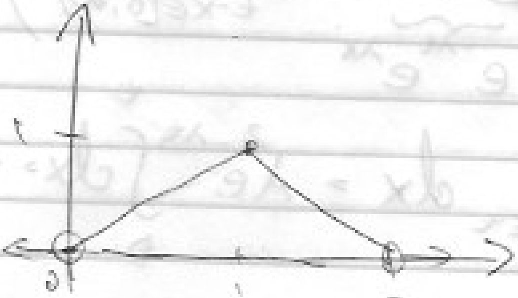
$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1}{2}t^2 & \text{if } t\in(0,1] \\ -\frac{1}{2}t^2+2t-1 & \text{if } t\in(1,2] \\ 1 & \text{if } t>2 \end{cases}$$



$$\frac{1}{2}t^2 - 2\left(\frac{1}{2}(t-1)^2\right) = \frac{1}{2}t^2 - (t^2-2t+1)$$
$$= -\frac{1}{2}t^2 + 2t - 1$$



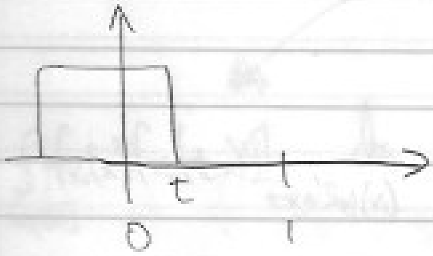
$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ t & \text{if } t\in[0,1] \\ 2-t & \text{if } t\in(1,2] \\ 0 & \text{if } t>2 \end{cases}$$



$$f_T(t) = \int_0^1 \underbrace{f(x)}_{1}\,\underbrace{f(t-x)}_{1}\,\underbrace{\mathbb{1}_{t-x\in[0,1]}}_{\substack{x-t\in[-1,0] \\ \Downarrow \\ x\in[t-1,t]}}\,dx$$

$$= \int_0^1 \mathbb{1}_{x\in[t-1,t]}\,dx$$

$$= \begin{cases} 0 & \text{if } t\le 0 \\ t & \text{if } t\in(0,1] \\ 2-t & \text{if } t\in(1,2] \\ 0 & \text{if } t\ge 2 \end{cases}$$



$X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

$T = X_1 + X_2 \sim f_T(t) = ?$

$Supp(T) = [0,\infty)$

$$f_T(t) = \int_0^\infty \left(\lambda e^{-\lambda x}\right)\left(\lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t}e^{\lambda x}}\,\mathbb{1}_{t-x\in[0,\infty)}\right) dx$$

$$= \lambda^2 e^{-\lambda t}\int_0^\infty \mathbb{1}_{x\le t}\,dx = \lambda^2 e^{-\lambda t}\int_0^t dx = t\lambda^2 e^{-\lambda t}$$

$X_1, X_2, \dots \overset{iid}{\sim} Exp(\lambda)$

$T_3 = X_3 + T_2$

$$f_T(t) = \int_0^\infty (\lambda e^{-\lambda x})\left((t-x)\lambda^2 \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{x \le t}\right) dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t (t-x)\,dx$$

$$= \lambda^3 e^{-\lambda t} \left(\int_0^t t - \int_0^t x\right) dx$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t}$$

$T_4 = X_4 + T_3$

$$f_T(t) = \int_0^\infty \lambda e^{-\lambda x} \frac{(t-x)^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{x \le t}\, dx$$

$$= \frac{\lambda^4 e^{-\lambda t}}{2} \int_0^t \dots$$

(Continue in next class)