9/9 Bag of fruit

p_1: Prob of Apple p_2: prob of Banana p_3: prob of cantaloupe

$p_1 + p_2 + p_3 = 1$ X_k: # of fruit k

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim \overbrace{p_{\vec{X}}}^{\text{JMF}}(\vec{x}) = \frac{n!}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n} \mathbb{1}_{x_1 \in \{0,1,\dots n\}} \mathbb{1}_{x_2 \in \{0,1,\dots n\}} \mathbb{1}_{x_3 \in \{0,1,\dots n\}}$$

$$= \binom{n}{x_1 x_2 x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \text{Multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right)$$

Generally with K categories, $\vec{X} \sim \text{Multinomial}(n, \vec{p}) = \binom{n}{x_1, x_2, \dots x_K} p_1^{x_1} p_2^{x_2} \dots p_K^{x_K}$

$\text{supp}[\vec{X}] = \{\vec{x} : \vec{x} \in \{0,1,\dots n\}^K, \underbrace{\vec{x} \cdot \vec{1} = n}_{\text{They all add up to } n}\}$

$n \in \mathbb{N}$

$\vec{p} \in \{\vec{v} : \vec{v} \in (0,1)^K, \vec{v} \cdot \vec{1} = 1\}$

- $p_1 + p_2 = 1$ $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \text{Multinomial}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$

$p_1 = p$ $p_2 = 1-p$

$X_1 \sim \text{Bin}(n, p)$ $X_2 \sim \text{Bin}(n, 1-p)$ $X_1 \overset{d}{\neq} X_2$ except if they're distributed evenly

Are $X_1, X_2 \overset{ind}{\sim}$? (No - They're for sure dependent)

If so, then

$P(X_1 = x_1 | X_2 = x_2) = P(X_1 = x_1)$ for all $\forall \vec{x} \in \text{supp}[\vec{X}]$

$0 = P(X_1 = 1 | X_2 = n) \neq P(X_1 = 1)$

$\hookrightarrow \binom{n}{1} p^1 (1-p)^{n-1} = np(1-p)^{n-1}$

$\Longrightarrow$ they're dependent

$$p_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 | X_2 = x_2) = \frac{p_{X_1,X_2}(x_1, x_2)}{p_{X_2}(x_2)}$$

Def of conditional probability

we should get $= Deg(n-x_2) = \mathbb{1}_{x_1 = n-x_2}$

Marginal PMF

$$P_{X_2}(x_2) = \sum_{x_1 \in supp[x_1]} P_{X_1,X_2}(x_1,x_2) = \sum_{x_1 \in \{0,1,\ldots,n\}} \frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1+x_2=n}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2} \sum_{x_1 \in \{0,1,\ldots,n\}} \frac{1}{x_1!} p^{x_1} \mathbb{1}_{x_1 = n-x_2} = \frac{n!}{x_2!}(1-p)^{x_2}\frac{1}{(n-x_2)!} p^{n-x_2}$$

$$= \binom{n}{x_2}(1-p)^{x_2} p^{n-x_2} = Bin(n, 1-p)$$

$$\frac{P_{x_1,x_2}(x_1,x_2)}{P_2(x_2)} = \frac{\frac{\cancel{n!}}{x_1!\,\cancel{x_2!}} p^{x_1} \cancel{(1-p)^{x_2}} \mathbb{1}_{x_1+x_2=n}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \cancel{(1-p)^{x_2}} \cdot p^{n-x_2}} = \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n} \mathbb{1}_{x_1+x_2=n}$$

$$Deg(n-x_2) = \begin{cases} 1 & \text{if } x_1+x_2 = n \quad x_1 = n-x_2 \\ 0 & \text{o.w.} \end{cases}$$

$\{n-x_2$ w.p. 1

- $P_{\vec{X}_{-j}|x_j}(\vec{x}_{-j}, x_j)$ let $\vec{X}_{-j} := \begin{bmatrix} x_1 \\ \vdots \\ x_{j-1} \\ \vdots \\ x_{j+1} \\ \vdots \\ x_k \end{bmatrix}$

$$= \frac{Multinom(n, \vec{p})}{Binom(n, p_j)}$$

$$= \frac{\frac{\cancel{n!}}{x_1!\ldots x_{j-1}!\,\cancel{x_j!}\,x_{j+1}!\ldots x_k!} p_1^{x_1}\ldots p_{j-1}^{x_{j-1}}\,\cancel{p_j^{x_j}}\,p_{j+1}^{x_{j+1}}\ldots p_k^{x_k}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)}\,\cancel{p_j^{x_j}}\,(1-p_j)^{n-x_j}}$$

let $n' := n-x_j$

$$= \frac{n'!}{x_1!\ldots x_{j-1}!\,x_{j+1}!\ldots x_k!} \; \frac{p_1^{x_1}\ldots p_{j-1}^{x_{j-1}}\,p_{j+1}^{x_{j+1}}\ldots p_k^{x_k}}{(1-p_j)^{n'}}$$

Note: $n' = x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots x_k$

$\hookrightarrow (1-p_j)^{x_1}\ldots(1-p_j)^{x_{j-1}}(1-p_{j+1})^{x_{j+1}}\ldots(1-p_k)^{x_k}$

$= Multinom(n', \vec{p}')$ where $\vec{p}' := \begin{bmatrix} \frac{p_1}{1-p_j} \\ \vdots \\ \frac{p_{j-1}}{1-p_j} \\ \frac{p_{j+1}}{1-p_j} \\ \vdots \\ \frac{p_k}{1-p_j} \end{bmatrix}$ s.t. $\dim[\vec{p}'] = k-1$

$$\vec{X} := \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \qquad \begin{array}{l} E[\vec{X}] = ? \\ var[\vec{X}] = ? \end{array}$$

- Let $X_1, \dots, X_n$ be r.v.'s

$E[aX + c] = a\mu + c$ where $\mu := E[X]$, $a, c \in \mathbb{R}$ constants

$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$ true always

discrete $= \sum_{x \in \mathbb{R}} (x-\mu)^2 p(x)$

continuous $= \int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx$

$$\sigma^2 = var[X] := E[(X-\mu)^2] = \dots = E[X^2] - \mu^2$$

$$\sigma = SE[X] := \sqrt{\sigma^2}$$

- $var[X_1 + X_2] = E\left[\left((X_1 + X_2) - (\mu_1 + \mu_2)\right)^2\right]$

$$= E\left[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2\mu_1 X_1 - 2\mu_1 X_2 - 2\mu_2 X_2 - 2\mu_2 X_1 + 2X_1X_2 + 2\mu_1\mu_2\right]$$
$$= E[X_1^2] + E[X_2^2] + \cancel{\mu_1^2} + \cancel{\mu_2^2} - \cancel{2\mu_1^2} - 2\mu_1\mu_2 - \cancel{2\mu_2\mu_1} - \cancel{3\mu_2^2} + 2E[X_1X_2] + \cancel{2\mu_1\mu_2}$$
$$= \sigma_1^2 + \sigma_2^2 + 2\left(E[X_1X_2] - \mu_1\mu_2\right) = \sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}$$

covariance $= \sigma_{1,2} := cov[X_1, X_2] := E[X_1X_2] - \mu_1\mu_2$

$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E[X_i]$ if $X_1, \dots, X_2$ iid

$$E[(X_1-\mu_1)(X_2-\mu_2)] =$$