Lecture 1 $\forall$ = for All

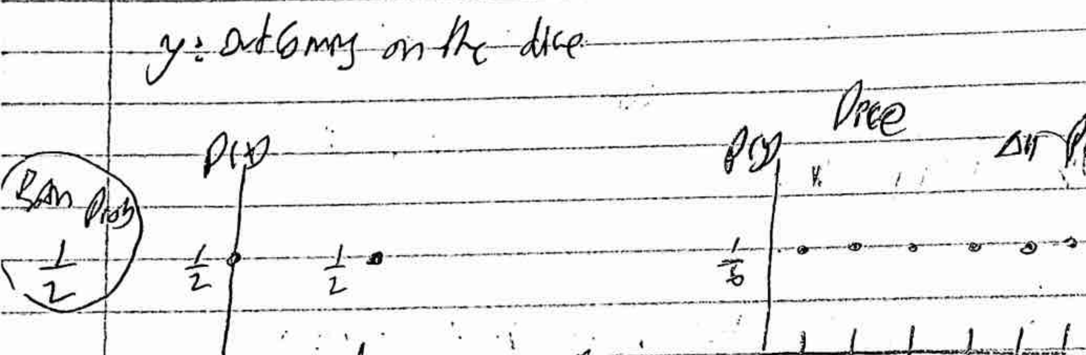
iid random Variables = Independent Identical Distributed Random variables

Random Variable — Independent / Identically Distributed = IID

Ex: Suppose you flip a fair coin (H or T) and also a fair 6-sided die $i = 1, 2, 3, 4, 5, 6$

X: fair coin result $i = 0$ ... heads, 1 ... tails

y: outcomes on the dice



with this in mind what is the probability on getting (X, y)

$\Rightarrow P(\text{Head}, 4)$? (X, y) they are iid unless stated otherwise
(coin) (dice)

So the probability becomes $P(x=0 \cap y=4) = P_x(0) \cdot P_y(4)$

$= \frac{1}{2} \cdot \frac{1}{6} = \boxed{\frac{1}{12}}$

Def: Random Variable X, y with pmf P_x & P_y are Independent if:

$P(X \cap y) = P_X(x) \cdot P_y(y)$ for all pairings (X, y)

or

$P_x = P_y$ then X & y are said to be Identically Distributed

Notation $\therefore$ = Therefore   iid
$\exists$ = Exist

A, B, C,

HW => Email

Review bef final

Notation Indicator

$\mathbb{1}$ indicator are use to explain functions. ex:

$\mathbb{1}$ = Indicator means $\mathbb{1}\left[\begin{matrix}\text{a true or false}\\ \text{expression}\end{matrix}\right] = \begin{cases} 1 & \text{if expression is true} \\ 0 & \text{if false} \end{cases}$

Recall Econ 352 econometrics
Like a dummy variable ex.

Ex: $\mathbb{1}_A = \begin{cases} 1 = A \text{ (true)} \\ 0 = A^c \text{ (false)} \end{cases}$ ex: $f(x)\ \mathbb{1}(x \le 17)$

So, $(x = 18) = 0$

$(x = -37) = 1$

Another example.

$g(x) = \mathbb{1}(x \le 3)$ means

because $\mathbb{1} = \begin{cases} 1 = \text{True} \\ 0 = \text{False} \end{cases}$

Another example => for Densities.

$f(s) = \begin{cases} e^{-s} & \text{if } s \ge 0 \\ 0 & \text{Otherwise} \end{cases}$ using Indicator => $e^{-s}\ \mathbb{1}(s \ge 0) \Rightarrow \left[\begin{matrix}\text{If True} * 1 \\ \text{If false} * 0\end{matrix}\right]$

$e^{-s} * 1 = e^{-s}$

$e^{-s} * 0 = 0$

Another example dummy variable
$U \sim \text{Unif}([a, b])$

$g_U(x) = \frac{1}{b-a}\ \mathbb{1}(x \in [a, b])$

Ω = Sample Space of Outcomes

Supp[X] = Suppor of r.v

$\exists$ =

$\mathbb{1}$ = Indicator

$\in$ = Belongs

$\mathbb{R}$ = All Real Values

Using Indicator function Describing Ineudlities

$$\int_{-\infty}^{\infty} x\,\mathbb{1}(x \geq 0)\, e^{-x}\, dx$$

note any value < 0 will be = 0 so we can elliminate that

$$\Rightarrow \int_{0}^{\infty} x e^{-x}\, dx$$

other exe Backwards

$$\int_{0}^{1} y\, dy = \int_{-\infty}^{\infty} y\,\mathbb{1}(y \in [0,1])\, dy$$

Alternative function Notation

$$\mathbb{1}_{A}(x) = \mathbb{1}(x \in A)$$

This mean that Indicator X belongs to A

note Using indicator you can write

$$\left.\begin{array}{ll} e^{-s} & \text{if } s \geq 0 \\ 0 & \text{otherwise} \end{array}\right\} = e^{-s}\,\mathbb{1}_{[0,\infty)}(s)$$

A Discrete Random Variable X has a Probability mass function (PMF) $p(x) = P(X=x)$

Also notated $X \sim p(x)$ & Cumulative Distribution (CDF)

$$F(x) = P(X \leq x)$$

The r.v (Random Variable) has "support" $Supp(X) = \{x : p(x) > 0, x \in \mathbb{R}\}$

& $|Supp(X)| \leq |\mathbb{N}|$ i.e. finite at most countable infinite (Discrete)

The Support & PMF are relate in $\sum_{x \in Supp(X)} p(x) = 1$

The most fundamental Random Variable is the Bernoulli

$$X \sim Bern(p) = \begin{cases} 1 & wp \; p \\ 0 & wp \; 1-p \end{cases} = p^x (1-p)^{1-x}$$

& Support, $Supp(X) = \{0,1\}$ & Parameter Space $p \in (0,1)$

If $p=0 \Rightarrow Bern(0) = \{0, wp \; 1 = Deg(0)$ degenerate

If $p = 1 \Rightarrow Bern(1) = \{1, wp \; 1 = Deg(1)$

$Deg(c) := \{c \; w.p.$

note all values have to be >0 to work on is zero $Deg(c) := \{c \; wp\} = \mathbb{1}_{x=c}$

Supp (x)

~~Pmf~~

* Indicator function (Notation)

$$\mathbb{1}_A = \begin{cases} 1 & \text{is } A \text{ (True)} \\ 0 & \text{is } A^c \text{ (False)} \end{cases}$$

Indicators functions for every A. are notation to state If condition is true or false

$f(x) = \mathbb{1}_{x < 17}$

$= f(19) = 0$

$= f(-37) = 1$

* Let's us upgrade our Pmf to be valid

$$p(x) = p^x (1-p)^{1-x} \; \mathbb{1}_{x \in \{0,1\}}$$

Supp (x)

$$p_0(x) \; \mathbb{1}_{x \in \text{Supp}(X)}$$

Consider r.v.s $X_1, X_2, \ldots, X_n$ the join mass function (Jmf) is

$$p_{X_1 X_2 \ldots X_n}(x_1 x_2 \ldots x_n) := P[X_1 = x_1, \ldots X_n = x_n]$$

If $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ (Independent) then

$$p_{X_1, \ldots X_n}(x_1, \ldots x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

$\forall \vec{X} \in \mathbb{R}^n$

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

If $X_1 \overset{iid}{=} X_2 \overset{iid}{=} \dots \overset{iid}{=} X_n$ (Identically distributed)$^{+n}$

$$p_{X_1}(x) = p_{X_2}(x) = \dots = p_{X_n}(x) \quad \forall x \in \mathbb{R}$$

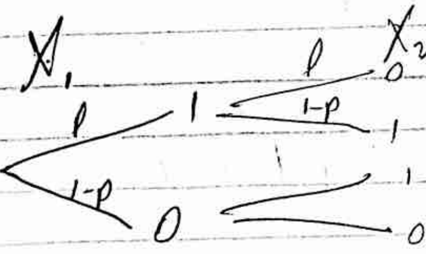
$\to X_1, X_2, \dots X_n \overset{ind}{\sim}$ independent & Identl Dist.

Let $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$.

$T := X_1 + X_2 \sim ?$

$$\text{Supp}(T) = \text{Supp}(X_1) + \text{Supp}(X_2) = \{0, 1, 2\}$$

$$A + B = \{a + b : a \in A, b \in B\}$$



X_1	X_2	(X_1, X_2)		T	$p_T(t)$
1	1	$(1,1)$	p^2	2	p^2
1	0	$(1,0)$	$p(1-p)$	1	$2p(1-p)$
0	1	$(0,1)$	$(1-p)p$	1	
0	0	$(0,0)$	$(1-p)^2$	0	$(1-p)^2$

$$p_T(t) = \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 2 & \text{w.p. } p^2 \end{cases}$$

Binomial & Bernoully Page 134 Sheldon Ross Book.

Add them up

$1 \overset{?}{=} p^2 + 2p(1-p) + (1-p)^2 = \cancel{p^2} + \cancel{2p} - \cancel{2p^2} + 1 - \cancel{2p} + \cancel{p^2}$

We need a formula for Infinite #

Discrete formula

$$p(t) = \sum_{x_1 \in R} \sum_{x_2 \in R} p_{X_1 X_2}(x_1, x_2) \, \mathbb{1}_{x_1 + x_2 = t} = \sum_{x \in R} \sum_{x \in R} p_{X_1, X_2}(x_1, \underbrace{t - x_1}_{x_2})$$

$$\mathbb{1}_{x_2 = t - x_1} = \sum_{x \in R} p_{X_1 X_2}(x, t - x) =$$

$$\Rightarrow \sum_{x \in R} p_{X_1}(x) \, p_{X_2}(t-x) = \sum_{x \in R} p(x) p(t-x)$$

Check 1

$\sum_{x \in R} \mathbb{1}_{x^2 > 1} = 3^2$

$= p^2$

$1 - (t - x)$

$\shortparallel$

$1 - t + x$

$\mathbb{1}_{t-x \in \{0,1\}}$

Sum of Two Bernoullis

$$p(t) = \sum_{x} \left(p^{x} (1-p)^{1-x} \, \mathbb{1}_{x \in (0,1)} \right) \left(p^{t-x} (1-p)^{1-t+x} \right)$$

$$\Rightarrow \sum_{x \in (0,1)} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \, \mathbb{1}_{t - x \in (0,1)} = p^t (1-p)^{2-t} \sum \mathbb{1}_{t-x \in (0,1)}$$

$$\Rightarrow p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in (0,1)} + \mathbb{1}_{t-1 \in (0,1)} \right)$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in (0,1)} + \mathbb{1}_{t \in \{1,2\}} \right) = \begin{Bmatrix} 1 \text{ if } t = 0 \\ 2 \text{ if } t = 1 \\ 1 \text{ if } t = 2 \end{Bmatrix} = \binom{2}{t}$$

$$\sum_{x \in R} p(x)\, p(t-x) = \sum_{x \in R} P_{OLD}^{(x)} \prod_{x \in supp(x)} P_{OLD}^{(t-x)} \prod_{t-x \in supp(x)}$$

$$= \sum_{x \in supp(x)} P_{OLD}(x)\, P_{old}^{(t-x)} \prod_{t-x \in supp(x)}$$

$$\Rightarrow \binom{2}{t} p^{t} (1-p)^{2-t} = \text{Binomial}(2, p)$$

CHECK 1

$$\Rightarrow \begin{cases} 0 & n.. \quad (1-p)^2 \\ 1 & 2p(1-p) \\ 2 & p^2 \end{cases}$$