Transformations of densities (more dim)

we've done these transformations in 1 Dimension:

X is a continuous random variable

g is a one-to-one function

and $Y = g(X)$

then:

$$f_Y(y) = f_X(g^{-1}(y)) \underbrace{\left| \frac{d}{dy}[g^{-1}(y)] \right|}_{\text{controls stretching or compression}}$$

PDF = density

$g^{-1} = h$

Let $\vec{X}$ be a vector random variabe with dimension n $\vec{X} = (X_1, X_2, \dots, X_n)$

let its density $f_{\vec{X}}(\vec{X})$ be known $f_{\vec{X}}(x_1, x_2, \dots, x_n)$

Let $\vec{Y} = g(\vec{X})$ where $g: \mathbb{R}^n \to \mathbb{R}^n$ $g(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$

with an inverse function h (i.e. $\vec{X} = h(\vec{Y})$)

then,

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y})) \left| J_h(\vec{y}) \right|$$

where J_h, the Jacobian Determinant is defined

$$J_h = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \cdots & \frac{\partial h_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \frac{\partial h_n}{\partial y_2} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix} \right)$$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

note that $g(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$ and $h(y_1, y_2, \dots, y_n) = (x_1, \dots, x_n)$

so $Y_1 = g_1(X_1, X_2, \dots, X_n)$ and $X_1 = h_1(Y_1, Y_2, \dots, Y_n)$

$Y_2 = g_2(X_1, X_2, \dots, X_n)$ etc. $X_2 = h_2(Y_1, Y_2, \dots, Y_n)$ etc.

Steps to take:

for

$T = X_1 + X_2$ want to find pdf of T: $f_T(t) = ?$

1. Find a "clever" g so that we can...
2. Find h (inverse of g)
3. Compute J_h (Jacobian)
4. Substitute into change of variables formula
   $$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\,|J_h(\vec{y})| = f_{\vec{X}}(h_1(y_1), h_2(y_2))\,|J_h(\vec{y})|$$
5. Integrate out the nuisance dimensions.

notice $T = Y_1$

① $Y_1 = X_1 + X_2$ so $g_1(X_1, X_2) = X_1 + X_2$ $\quad y_1 = x_1 + x_2$

$Y_2 = X_2$ $\quad g_2(X_1, X_2) = X_2$ $\quad y_2 =$

can use lowercase for all of these

② $X_1 = Y_1 - X_2 = Y_1 - Y_2$ so $h_1(Y_1, Y_2) = Y_1 - Y_2$

$X_2 = Y_2$ so $h_2(Y_1, Y_2) = Y_2$

③
$$J_h = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) = (1)(1) - (-1)(0)$$

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

$$= (1)(1) - (-1)(0) = 1$$

$$J_h = 1$$

④ $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)\,|1| = f_{X_1,X_2}(y_1 - y_2, y_2)$

joint density

⑤ Want marginal density of Y_1 (since $T = Y_1$ here)

$$f_{Y_1}(y) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy_2 \quad \text{"margining"}$$

$t = x_1 + x_2$

$x_2 = t - x_1$

Let $x_1 = u$

$x_1 = u$

$x_2 = t - u$

if X_1, X_2 indep.

$$\Rightarrow f_T(y) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\,du = \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du$$

If X_1, X_2 are i.i.d $f_T(y) = \int_{supp[X_1]} f_{X_1}(t-u) f_{X_2}(u)\, \mathbb{1}_{t-u \in supp[X_2]}\,du$

ratio → $R = \frac{X_1}{X_2} \sim f_R(r) = ?$ (Find density of R)

density = PDF

1) Find a clever g:

$Y_1 = \frac{X_1}{X_2}$ so $g_1(X_1, X_2) = \frac{X_1}{X_2}$

$Y_2 = X_2$ so $g_2(X_1, X_2) = X_2$

$g(X_1, X_2) = \left(\frac{X_1}{X_2}, X_2\right)$

2) Find h, the inverse of G

$X_1 = Y_1 X_1 = Y_1 Y_2$ so $h_1(Y_1, Y_2) = Y_1 Y_2$

$X_2 = Y_2$ so $h_2(Y_1, Y_2) = Y_2$

3) $$J_h = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}\right) = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}\right)$$

$$= y_2 \cdot 1 - y_1 \cdot 0 = y_2$$

Substitute into change of variables

4) $f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(y_1 y_2, y_2) |y_2|$

5) $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1 Y_2}(y_1, y_2)\, dy_2$ Integrate out extra dimensions

$$= \int_{\mathbb{R}} f_{X_1, X_2}(ru, u)\, |u|\, du$$

these work for any ratio

if are indep.

$$f_{Y_1}(y_1) = \int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u) \mathbb{1}_{u \in Supp[X_2]} |u|\, du$$

$$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ? \quad \text{(find this density)}$$

① $Y_1 = g_1(X_1, X_2) = \dfrac{X_1}{X_1 + X_2}$

$Y_2 = g_2(X_1, X_2) = X_1 + X_2$

② $X_1 = Y_1(X_1 + X_2) = Y_1 Y_2$ so $h_1(Y_1, Y_2) = Y_1 Y_2$

$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2$ so $h_2(Y_1, Y_2) = Y_2 - Y_1 Y_2$

③ $$J_h = \det\left(\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}\right) = \det\left(\begin{bmatrix} y_2 & y_1 \\ y_2 & 1-y_1 \end{bmatrix}\right)$$

$J_h = y_2(1-y_1) - y_1(-y_2) = y_2 - y_2 y_1 + y_1 y_2 = y_2$

$J_h = y_2$

④ $f_{Y_1 Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2 - y_1 y_2) \, |y_2|$

⑤ $f_R(r) = \int_{\mathbb{R}} f_{X_1 X_2}(ru, u - ru) \, |u| \, du$

if X_1, X_2 indep. $= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u - ru) \, |u| \, du$

$$= \int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u-ru) \mathbb{1}_{u - ru \in Supp[X_2]} |u| \, du$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$
$X_2 \sim \text{Gamma}(\alpha_2, \beta)$ $>$ X_1, X_2 are indep.

$\text{Supp}[X_1] = (0, \infty)$
$\text{Supp}[X_2] = (0, \infty)$
$\text{Supp}[X_1 + X_2] = (0, \infty)$

proportion of a Gamma rel. to total of Gammas

$$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$$

$\text{Supp}[R] = (0, 1)$

Using previous formula...

PDF

$$f_R(r) = \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u - ru)^{\alpha_2 - 1} e^{-\beta(u - ru)} \cdot \mathbb{1}_{u - ru \in (0,\infty)} u \, du\right)$$

same as $u(1-r) \in (0, \infty)$
same as $u \in (0, \infty)$
since $(1-r) \in (0,1)$

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} r^{\alpha_1 - 1} u^{\alpha_1 - 1}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} (1-r)^{\alpha_2 - 1} e^{-\beta u}\right) u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_2 - 1} (1-r)^{\alpha_2 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u} \, du}_{\frac{\Gamma(\alpha_1 + \alpha_2)}{\beta^{\alpha_1 + \alpha_2}}} =$$

$$= \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} (1-r)^{\alpha_2 - 1}$$

$$= \frac{1}{B(\alpha_1, \alpha_2)} = \text{Beta}(\alpha_1, \alpha_2)$$

$$X_1 \sim Gamma(\alpha_1, \beta) \quad X_2 \sim Gamma(\alpha_2, \beta) \quad > X_1, X_2 \text{ are indep.}$$

$$R = \frac{X_1}{X_2} \sim \int_{Supp[X_1]} f_{X_1}(ru)\, f_{X_2}(u)\, \mathbb{1}_{u \in Supp[X_2]} |u|\, du$$

$u \in (0,\infty)$ so $|u| = u$

Supp[R] is $(0,\infty)$

Support does not match Beta dist. so this is not Beta

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}(ru)}{\Gamma(\alpha_1)} e^{-\beta u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2-1} e^{-\beta u} \mathbb{1}_{u\in(0,\infty)}\right) u\, du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha-1} \int u^{\alpha_1+\alpha_2-1} e^{-\beta(r+u)}\, du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\underbrace{(\beta(r+1))^{\alpha_1+\alpha_2}}_{\beta^{\alpha_1+\alpha_2}(r+1)^{\alpha_1+\alpha_2}}}$$

$$= \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r\in(0,\infty)}$$

$$= BetaPrime(\alpha_1, \alpha_2)$$

called a BetaPrime distribution

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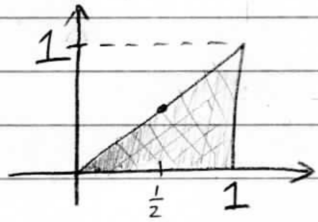
Conditional Densities

Consider $\overset{\text{r.v.}}{X} \sim \overset{\text{uniform}}{U}(0,1)$

$Y \mid X=x \sim U(0,x)$

r.v. X is parameter needed for this conditional density



$f_{Y|X=1}(y) = 1$ also written $f_{Y|X}(y;1) = 1$

$f_{Y|X=0.1}(y) = 10$ also written $f_{Y|X}(y;0.1) = 10$

$f_{X,Y} = ?$ $f_Y = ?$ $f_{X|Y} = ?$

will use Bayes Theorem to do this