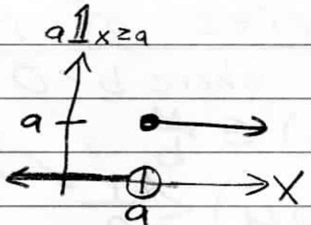
Markov's Inequality

Let X be a non-negative random variable
(i.e. $Supp(X) = (0, \infty)$)
with finite expectation μ.
Let $a > 0$ (a is a constant)
and consider the following inequality

$$a \mathbb{1}_{X \geq a} \leq a$$



Verify the inequality:
if $x \geq a$ then $a(1) \leq X \Rightarrow x \geq a$ ✓
if $x < a$ then $a(0) \leq X \Rightarrow X \geq 0$ (this because X is non-neg.)

Let's take $E[\cdot]$ on both sides

$$E[a\mathbb{1}_{X \geq a}] \leq E[X]$$

$$\Rightarrow a\, E[a\mathbb{1}_{X \geq a}] \leq \mu$$

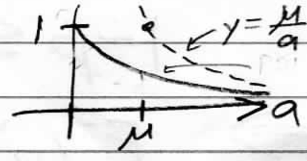
$$E[\mathbb{1}_{X \geq a}] = 1P(X \geq a) + 0P(X < a) = P(X \geq a)$$

$$\Rightarrow E[\mathbb{1}_{X \geq a}] \leq \frac{\mu}{a}$$

$$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$$ ← Markov's Inequality

$$\Rightarrow 1 - \overset{\text{CDF}}{F}(a) \leq \frac{\mu}{a}$$

"Markov's Tail bound"
"crude bound is $\frac{\mu}{a}$"



only useful if $a > \mu$

$X \sim Exp(1) \Rightarrow \mu = 1, \ \sigma^2 = 1$

$\Rightarrow P(X \geq a) = e^{-a}$

a	$P(X \geq a)$	Markov $\frac{\mu}{a}$
2	0.1353	0.5
5	0.0667	0.2
10	0.00004	0.1

← shows how crude this bound is

Corrolaries of Markov's Inequality

① $b = a\mu$ where $b > 0$

$P(X \geq b) \leq \frac{\mu}{b}$ $\qquad \frac{\mu}{\mu a} = \frac{1}{a}$

$\Rightarrow P(X \geq a\mu) \leq \frac{1}{a}$

② Let $h(x)$ be one-to-one

$$P(h(X) \geq h(a)) \leq \frac{E[h(x)]}{h(a)}$$

$\Rightarrow P(X \geq a) \leq \frac{E[h(x)]}{h(a)}$ for one-to-one function h

③ Let $a = Quantile[X, p]$ (this means $P(X \leq a) = p$) $F(a) = p$

if X is continuous with connected support

$a = Quantile[X, p] = F_X^{-1}(p)$ $\qquad a = F_X^{-1}(p)$

$$P(X \geq a) \leq \frac{\mu}{a}$$

$$1 - F_X(a) \leq \frac{\mu}{a}$$

$$1 - F_X(F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$$

$$1 - p \leq \frac{\mu}{F_X^{-1}(p)}$$

$$\Rightarrow F^{-1}(p) \leq \frac{\mu}{1-p}$$

notice

if $p = \frac{1}{2}$

$F^{-1}(\frac{1}{2}) \leq \frac{\mu}{\frac{1}{2}}$

$Med[X] \leq 2\mu$

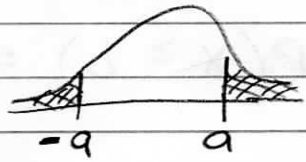
more Corrolaries of Markov's Inequality

(4) Let X be any r.v., and $E[X]$ is finite

Note $|X|$ is non-negative

$$P(|X| \geq a) \leq \frac{E[X]}{a} \quad \text{Note: } E[|X|] < \infty$$

right and left tail



(5) Let X be a random variable with mean μ, variance σ^2

Let $Y = (X-\mu)^2$. Note Y is non-negative

$$P(Y \geq a^2) \leq \frac{E(Y)}{a^2}$$

$$P((X-\mu)^2 \geq a^2) \leq \frac{E[(X-\mu)^2]}{a^2}$$

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ Chebyshev's Inequality

$$P(-a \leq X-\mu < a) = \frac{\sigma^2}{a^2}$$

or $P(|X-\mu|<a) = 1-\frac{\sigma^2}{a^2}$

Assume X is non-negative

$P(X-\mu \geq a \text{ OR } X-\mu \leq a)$

$= P(X-\mu \geq a) + P(X-\mu \leq a)$

$= P(X \geq \mu+a) + P(X \leq \mu-a)$

$= P(X \geq \mu+a) + P(X<0)$

$= P(X \geq \mu + a) + P(X < 0)$
$= P(X \geq \mu + a) + 0$ ← since X is non-neg.
$= P(X \geq \mu + a)$

let $b = \mu + a \Rightarrow b > 2\mu$
$a = b - \mu$

$= P(X \geq b) \leq \dfrac{\sigma^2}{(b-\mu)^2}$ ← so chebychev better than Markov

If $X \sim Exp(1) \Rightarrow \mu = 1, \sigma^2 = 1$
$\Rightarrow P(X \geq a) = e^{-a}$

$= P(X \geq b) \leq \dfrac{1}{(b-1)^2}$

a	$P(X \geq x)$	Markov	Chebychev
2	0.1353	0.5	1
5	0.0067	0.2	0.0635
10	0.00004	0.1	0.0125

Markov's Inequality

$P(X \geq a) \leq \dfrac{\mu}{a}$

X is non-neg. r.v.
a is constant, $a > 0$
$\mu = E(X)$

Chebychev's Inequality

$P(|X - \mu| \geq a) \leq \dfrac{\sigma^2}{a^2}$

X is any r.v.
a is a constant, $a > 0$
$\mu = E(X)$, $\sigma^2 = Var[X]$

Chebychev's Inequality:

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

(can also be written $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$)

k = # of standard deviatio

Corrolaries of Markov's Inequality

⑥ Let X be any r.v.

$Y = e^{tX}$ where $t \neq 0$

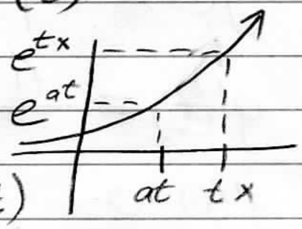
note Y is nonnegative for all t

$\Rightarrow P(Y \geq c) \leq \frac{E[Y]}{c}$ let $c = e^{at} > 0$

$\Rightarrow P(e^{tX} \geq e^{at}) \leq \frac{E[e^{tX}]}{e^{at}} = e^{-at} M_X(t)$

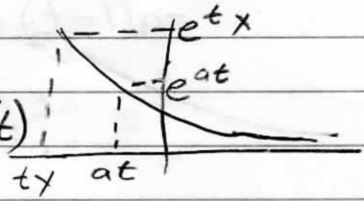
moment generating function

$P(e^{tX} \geq e^{at}) \leq M_X(t)$



if $t > 0$

$P(X \geq a) \leq e^{-at} M_X(t)$



if $t < 0$

$P(X \leq a) \leq e^{-at} M_X(t)$

Negative for both left and right tails

$\Rightarrow$ if $t > 0$: $P(X \geq a) \leq \min_{t>0} \{e^{-at} M_X(t)\}$

if $t < 0$: $P(X \leq a) \leq \min_{t<0} \{e^{-at} M_X(t)\}$

Chernoff's Inequality

Chernoff's Inequality

Chernoff's Bounds

$$P(X \geq a) \leq \min_{\text{for } t>0} \left(e^{-at} M_X(t) \right)$$

$$P(X \leq a) \geq \max_{\text{for } t<0} \left(e^{-at} M_X(t) \right)$$

$X \sim Exp(\lambda)$

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \frac{\lambda}{t-\lambda} \left[e^{-(t-\lambda)x} \right]_0^\infty = \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda - t} =$$

if $t-\lambda<0$
same as $t<\lambda$

if $\lambda = 1$ (i.e. $X \sim Exp(1)$)

get $M_X(t) = \frac{1}{1-t}$

let $g(t) = \frac{e^{-at}}{1-t}$

$$g'(t) = \frac{(1-t)(-a)e^{-at} - e^{-at}(-1)}{(1-t)^2}$$

set $g'(t) = 0$

$$-a(1-t)e^{-at} + e^{-at} = 0$$

$$e^{-at}(a(t-1)+1) = 0$$

$$at - a + 1 = 0$$

$$at = a - 1$$

$$t = \frac{a-1}{a}$$

$$t = 1 - \tfrac{1}{a} \in (0,1) \text{ if } a>1$$

$$P(X \leq a) = e^{-a(1-\frac{1}{a})} \frac{1}{1-(1-\frac{1}{a})}$$

Chernoff's inequality

$$P(X \leq a) \leq e^{-a\left(1-\frac{1}{a}\right)} \frac{1}{1-\left(1-\frac{1}{a}\right)}$$

$$P(X \leq a) \leq e^{-a+1} \frac{1}{\frac{1}{a}}$$

$$P(X \leq a) \leq a e^{-a+1} = \frac{ea}{e^{a}}$$

Chernoff's Inequality application to Exp(1)

$$X \sim Exp(1) \Rightarrow \mu = 1, \sigma^2 = 1 \Rightarrow P(X \geq a) = e^{-a}$$

$z = \frac{x-1}{1}$

a	$P(X \geq a)$	Markov	Chebyshev	Chernoff
2	0.1353	0.5	1	0.73536
5	0.0067	0.2	0.0635	0.09158
10	0.00004	0.1	0.0123	0.00123

X, Y are r.v.'s with means μ_X, μ_Y
variances σ_X^2, σ_Y^2

Cov[X,Y] denoted → σ_{XY}

$$Cov[X,Y] = E[XY] - \mu_X \mu_Y$$

this is a measure of linear dependence in terms of X and Y

Corr[X,Y] denoted → ρ_{XY}

$$Corr[X,Y] = \frac{Cov[X,Y]}{SE[X]\,SE[Y]}$$

also written $\rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

"Correlation"
which is unitless
and bounded $Corr[X,Y] \in [-1,1]$

let $Y = cX$

$$Corr[X,Y] = \frac{Cov[X,cX]}{SE[X]\,SE[cX]} = \frac{c\,Cov[X,X]}{}$$

$$= \frac{c\,Var[X]}{SE[X]\,|c|\,SE[X]} = \frac{c}{|c|}\,\frac{Var[X]}{Var[X]} = \frac{c}{|c|}$$

$(SE[X])^2 = Var[X]$

$$Corr[X,Y] = \frac{c}{|c|} = \begin{cases} 1 & \text{if } c > 0 \\ -1 & \text{if } c < 0 \end{cases}$$

So if Y is a multiple of X,
Corr[X,Y] is 1 or -1