Poisson Process

$T_K \sim \text{Erlang}(K, \lambda)$

$N \sim \text{Poisson}(\lambda)$

$P(T_K > 1) = P(N \leq K-1)$

$\Rightarrow 1 - F_{T_K}(1) = F_N(K-1) = Q(K, \lambda)$

"Counting + waiting r.v's"

	Fixed time, Count #	Fixed #, Measure time
Discrete	Bern / Binom	Geom / Neg. Bin.
Continuous	Poisson	Exp / Erlang

Denoted "(1 event) / (n>1 events)"

Q: What is the prob. there are 0 successes by 50 trials each w.p 0.1 of success?

$N \sim \text{Bin}(50, 0.1)$

OR

$T \sim \text{NegBin}(1, 0.1)$

$P(N=0) = P(T>49)$

$F_N(0) = 1 - F_T(49)$

Q: What is the prob. of $\leq k$ events by experiment # t if prob. success is p?

$N \sim Binom(t, p)$

OR

$T \sim NegBin(k+1, p)$

$P(N \leq k) = P(T > t-k-1)$

$\Rightarrow F_N(k) = 1 - F_T(t-k-1)$

$$\Rightarrow \sum_{i=0}^{k} \binom{t}{i} p^i (1-p)^{t-i} = 1 - \sum_{i=0}^{t-k-1} \binom{k+i}{k} (1-p)^i p$$

- Let $T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0}$

$$\| $$

* Gamma Distribution: $= \frac{\lambda^k}{\Gamma(k)} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0}$

Parameter space
$k \in \mathbb{N}$
$\lambda \in (0, \infty)$

$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

* Extended Neg. Bin: $= \frac{\Gamma(k+t)}{\Gamma(k) t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

$k \in \mathbb{N}$
$p \in (0,1)$

- $X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$

$\alpha, \beta > 0$

Transformations of Discrete R.V's

- $X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = P_X(x)$

$Y := g(X) = X + 3 \sim \begin{cases} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y \in \{3,4\}}$

$\Rightarrow X = g^{-1}(Y) = Y - 3$

- Assume $\exists g^{-1}$, then

$$
\begin{aligned}
P_Y(y) &= P(Y = y) \\
&= P(g(X) = y) \\
&= P(X = g^{-1}(y)) \\
&= P_X(g^{-1}(y))
\end{aligned}
$$

Let $X \sim U(\{1, \dots, 10\}) = \frac{1}{10} \mathbb{1}_{x \in \{1, \dots, 10\}}$

$Y = g(X) = \min\{X, 3\}$

y	$P_Y(y)$
1	$1/10$
2	$1/10$
3	$8/10$

General:

$$P_Y(y) = \sum_{\{x \mid g(x) = y\}} P_X(x) \underset{\text{if } \exists g^{-1}}{=} p_X(g^{-1}(y))$$

- Let $X \sim Binom(n, p)$

$Y = X^3$

$\Rightarrow X = \sqrt[3]{Y}$

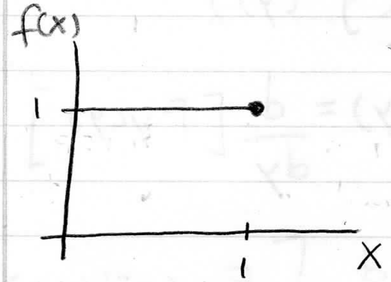
$$p_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n - \sqrt[3]{y}} \mathbb{1}_{y \in \{0, 1, 8, \dots, n^3\}}$$

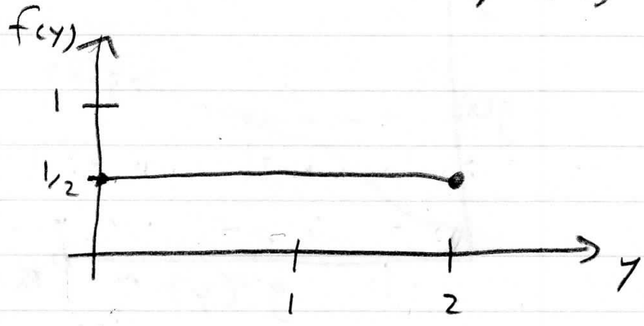
$$\mathbb{1}_{x \in \{0, \dots, n\}} = \mathbb{1}_{\sqrt[3]{y} \in \{0, \dots, n\}} = \text{''}$$

Transformations of Continuous R.V's

$$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$$

$X \sim U(0,1)$, $Y = 2X \sim \frac{1}{2}\mathbb{1}_{y\in[0,2]}$





$Y = 2X \Rightarrow g^{-1}(y) = \frac{Y}{2} = X$

$$f_Y(y) \overset{?}{=} \underbrace{f_X\left(\frac{Y}{2}\right)}_{\text{Stating } \frac{1}{2} \text{ between } 0,2} = \underbrace{\mathbb{1}_{y\in[0,2]}}_{\text{Stating } 1 \text{ between } 0,2}$$

Incorrect

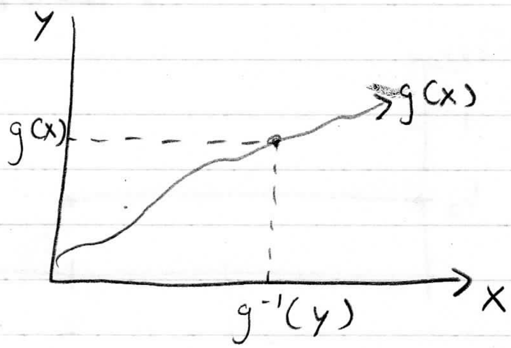
- From upcoming notes,

$$Y = 2X \sim f_Y\left(\frac{y}{2}\right)\frac{1}{2} = \frac{1}{2}\mathbb{1}_{y\in[0,2]}$$

CDF of Y: First consider g monotone increasing.

Assume $\exists g^{-1}$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(Y))$$



$$= F_X(g^{-1}(y))$$

$$\Rightarrow f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right]$$

$$= \frac{d}{dy}\left[F_X(g^{-1}(y))\right]$$

(Note (Chain Rule): $\frac{d}{dt}\left[h(j(t))\right] = h'(j(t))\, j'(t)$)

$$= F'_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

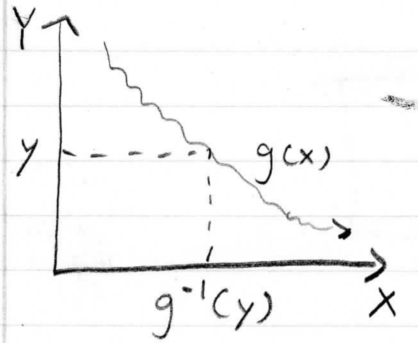
$$= f_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$= f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| \quad \text{Since } g^{-1}(y) > 0$$

Assume g is strictly decreasing

Complement CDF

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y))$$



$$= 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -F_X'(g^{-1}(y)) \frac{d}{dy}\left[g^{-1}(y)\right]$$

<0 w/o neg, >0 w/ neg

$$= F_X'(g^{-1}(y))\left(-\frac{d}{dy}\left[g^{-1}(y)\right]\right)$$

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|$$

Thus * $\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right|}$ *

We have $f_Y(y)$ whether our function is monotone increasing or decreasing.

Note: $f_Y(y)$ is density of Y

- Let $Y = g(X) = aX + c$, linear transformation (shift by scale)

$$\Rightarrow X = g^{-1}(Y) = \frac{Y - c}{a}$$

$$\Rightarrow \boxed{f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}}$$

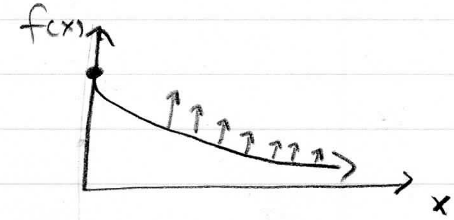
$$\boxed{\begin{array}{l} Y = -X \Rightarrow a = -1,\ c = 0 \\ \Rightarrow f_Y(y) = f_X(-y) \end{array}}$$

$$\boxed{Y = X + c \Rightarrow f_Y(y) = f_X(y - c)}$$

$$= f_X\left(\frac{y-c}{3}\right)\frac{1}{|a|}\mathbb{1}_{\frac{y-c}{a} \in supp[X]}$$

$$= f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}\mathbb{1}_{y \in a\,supp[X] + c}$$ //

- Let $X \sim \text{Exp}(\lambda)$

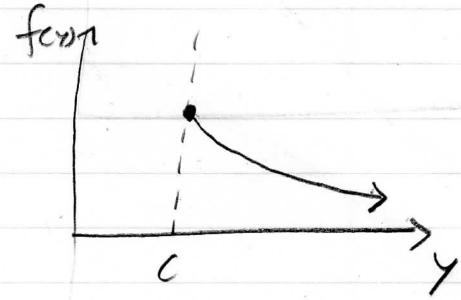


$Y \sim X + c$

$\sim \lambda e^{-\lambda(y-c)} \mathbb{1}_{y-c \in (0,\infty)}$

$= \lambda e^{-\lambda y + \lambda c} \mathbb{1}_{y \in (c,\infty)}$

$= \underbrace{e^{\lambda c}}_{\text{Scaling}} \lambda e^{-\lambda y} \mathbb{1}_{y \in (c,\infty)}$



//

- Let $X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$

$Y \sim -\ln(X) = g(X)$

$\Rightarrow X = e^{-Y} = g^{-1}(y)$

Note $\left| \frac{d}{dy}[e^{-y}] \right| = e^{-y}$

Density: $f_Y(y) = f_X(e^{-y}) e^{-y}$

$= \mathbb{1}_{e^{-y} \in [0,1]} e^{-y}$

$= e^{-y} \mathbb{1}_{y \in (0,\infty)}$

$= \text{Exp}(1)$

//