(Lecture 13) October 30th, 2019

$$X \sim Beta(\alpha, \beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \propto x^{\alpha-1}(1-x)^{\beta-1}$$

$Y = g(X)$ and g is one to one

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

$g: \mathbb{R}^n \to \mathbb{R}^n$ and one to one. Let $\vec{X}, \vec{Y}$ be r.v. vectors, dim n.

$$\vec{Y} = g(\vec{X})$$

$$Y_1 = g_1(X_1, \cdots, X_n)$$

$$Y_2 = g_2(X_1, \cdots, X_n)$$

$$\vdots$$

$$Y_n = g_n(X_1, \cdots, X_n)$$

$$g_i : \mathbb{R}^n \to \mathbb{R}$$

Because g is one to one, it has an inverse, h such that $\vec{X} = h(\vec{Y})$

$$X_1 = h_1(Y_1, \cdots, Y_n)$$

$$X_2 = h_2(Y_1, \cdots, Y_n)$$

$$\vdots$$

$$X_n = h_n(Y_1, \cdots, Y_n)$$

Then from MATH 202

$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{Y})) \left| J_h(\vec{y}) \right|$, multi change of variables

J_h is the Jacobian determinant

$$J_h := \det \underbrace{\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}}_{n \times n \text{ matrix}}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{scalar.}}$$

$T = X_1 + X_2$

<u>Recipe</u>

1) Find a clever g
2) Find its inverse, h
3) Compute J_h
4) Plug in and check in master formula
5) Integrate to get target.

1) Let $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$
   Let $Y_2 = X_2 = g_2(X_1, X_2)$

2) $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$
   $X_2 = Y_2 = h_2(Y_1, Y_2)$

3) $J_h = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0(-1) = 1$

4) $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)\,|1|$

5) Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$

$$f_{T}(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\,du$$

if $X_1, X_2 \overset{ind}{\sim}$

$$= \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du \quad \left(\int_{Supp[X_2]} f_{X_1}(t-u) f_{X_2}(u) \mathbb{1}_{u \in Supp[X_2]}\,du\right)$$

if $X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(t-u) f(u)\,du$$

$$R = \frac{X_1}{X_2} \sim f_R(r) = ?$$

1) Let $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$

Let $Y_2 = X_2 = g_2(X_1, X_2)$

2) $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

3) $J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} Y_2 & Y_1 \\ 0 & 1 \end{bmatrix} = Y_2 - 0(Y_1)$
$= Y_2$

4) $f_{Y_1,Y_2}(Y_1,Y_2) = f_{X_1,X_2}(Y_1Y_2, Y_2)\,|Y_2|$

5) $f_R(r) = \int_{\mathbb{R}} f_{X_1X_2}(ru, u)\,|u|\,du$

if X_1, X_2 ind

$= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)|u|\,du$

or

$\int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u)\,\mathbb{1}_{u \in Supp[X_2]}\,|u|\,du$

$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ?$

1) Let $Y_1 = \frac{X_1}{X_1+X_2} = g_1(X_1, X_2)$

Let $Y_2 = X_1+X_2 = g_2(X_1, X_2)$

2) $X_1 = Y_1(X_1+X_2) = Y_1Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 - X = Y_2 - Y_1Y_2 = h_2(Y_1, Y_2)$

3) $J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} Y_2 & Y_1 \\ -Y_2 & 1-Y_1 \end{bmatrix}$

$= Y_2(1-Y_1) - Y_1(-Y_1) = Y_2 - Y_1 Y_2 + Y_1 Y_2 = Y_2$

4) $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1 X_2}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|$

5) $f_R(r) = \int_{\mathbb{R}} f_{X_1 X_2}(ru, u-ru)\,|u|$

$X_1, X_2 \overset{ind}{\sim}$

$= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u-ru)\,|u|\,du$

or

$\int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u-ru)\,\mathbb{1}_{u-ru \in Supp[X_2]}\,|u|\,du$

Let $X_1 \sim Gamma(\alpha_1, \beta)$ indep. of $X_2 \sim Gamma(\alpha_2, \beta)$

$R = \frac{X_1}{X_1+X_2} \sim \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1 - 1} e^{-\beta ru}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(u-ru)^{\alpha_2 - 1} e^{-\beta(u-ru)}\,\mathbb{1}_{u-ru \in (0,\infty)}\right.$

$Supp[R] = [0,1]$

$u(1-r) \qquad e^{-\beta u} e^{-\beta ru} \qquad u\,du$

$\propto \int_0^\infty r^{\alpha_1 - 1} u^{\alpha_1 - 1} u^{\alpha_2 - 1}(1-r)^{\alpha_2 - 1} \cdot u$

$u - ru \in (0,\infty)$
Since $r \in (0,1)$
$1-r \in (0,1)$
$u \in (0,\infty)$

$= r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \underbrace{\int_0^\infty u^{\alpha_1 \alpha_2 + 1} e^{-\beta u}\,du}_{\text{not a function of } r}$

$\propto r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \propto Beta(\alpha_1, \alpha_2)$

X_1, X_2 Same as the previous example

$$R = \frac{X_1}{X_2} \sim \int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u) |u| \, du$$

$Supp[R] = (0, \infty)$

$$R = \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u} \right) \left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} \underbrace{\mathbb{1}_{u \in (0,\infty)}}_{=1 \, \forall u} \right) u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty r^{\alpha_1 - 1} u^{\alpha_1 - 1} e^{-\beta u (r+1)} u^{\alpha_2 - 1} u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta(r+1)u} \, du}_{\text{gamma with constant}}$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \cdot \frac{\Gamma(\alpha_1 + \alpha_2)}{(\beta(r+1))^{\alpha_1 + \alpha_2}}$$

$$= \frac{1}{B(\alpha_1 + \alpha_2)} \frac{r^{\alpha_1 - 1}}{(r+1)^{\alpha_1 + \alpha_2}} \mathbb{1}_{r \in (0, \infty)}$$

$= \text{Beta Prime}(\alpha_1, \alpha_2)$

Page 155 Conditional Densities

Consider $X \sim U(0,1)$

$Y|X=x \sim U(0,X)$