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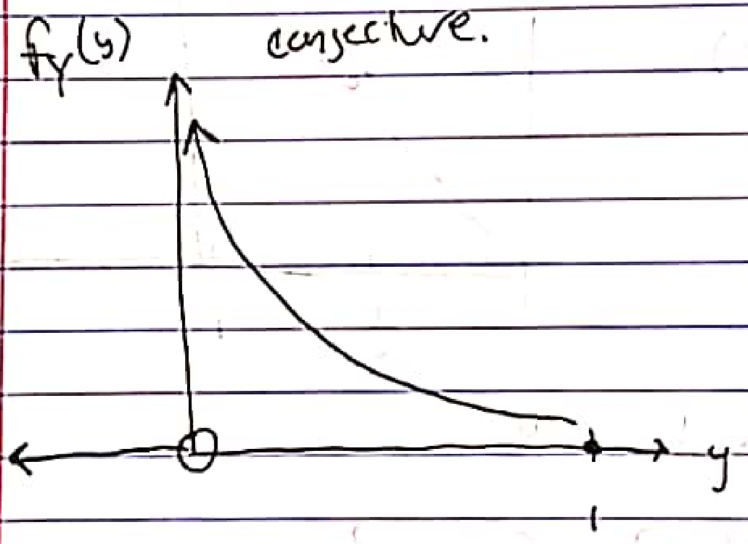
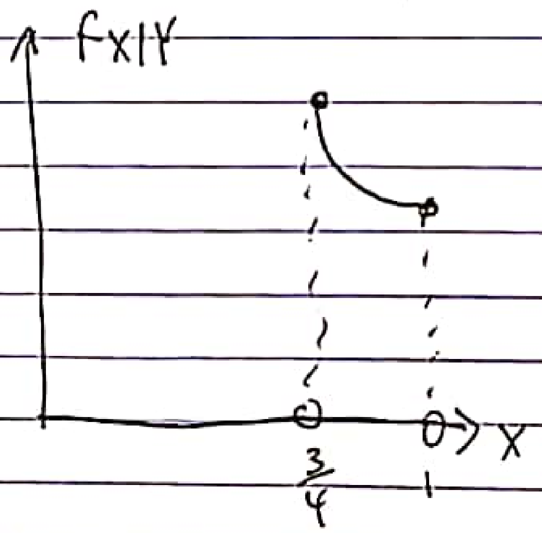
Conditional Densities

$X \sim U(0,1)$

$Y | X = x \sim U(0, x)$

Goal: f_{XY}, f_Y, $f_{X|Y}$





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I

Margining

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\, dx$$

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\, dy$$

II Def of conditional density.

$$f_{Y|X}(x,y) := \frac{f_{XY}(x,y)}{f_X(x)} \qquad f_X(x) > 0$$

$$\Rightarrow f_{XY}(x,y) = f_{Y|X}(x,y)\, f_X(x)$$

$$f_{X|Y}(x,y) := \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_Y(y) > 0$$

$$\Rightarrow f_{XY}(x,y) = f_{X|Y}(x,y)\, f_Y(y)$$

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III Bayes Rule

$$f_{Y|X}(x,y) = \frac{f_{X|Y}(x,y)\, f_Y(y)}{f_X(x)}$$

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y)\, f_X(x)}{f_Y(y)}$$

IV Bayes Theorem

might not be used

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{\int_{\mathbb{R}} f_{XY}(x,y)\,dy} = \frac{f_{X|Y}(x,y)\, f_Y(y)}{\int_{\mathbb{R}} f_{X|Y}(x,y)\, f_Y(y)\,dy}.$$

(X uniform)

$$f_X(x) = \mathbb{1}_{x \in [0,1]}$$

$$f_{Y|X}(x,y) = \frac{1}{x}\mathbb{1}_{y \in [0,x]}$$

$$f_{XY}(x,y) = \frac{1}{x}\mathbb{1}_{y \in [0,x]}\mathbb{1}_{x \in [0,1]}$$

$$= \frac{1}{x}\mathbb{1}_{0 \le y \le x \le 1}$$

$$= \frac{1}{x}\mathbb{1}_{y \in [0,1]}\mathbb{1}_{x \in [y,1]}$$

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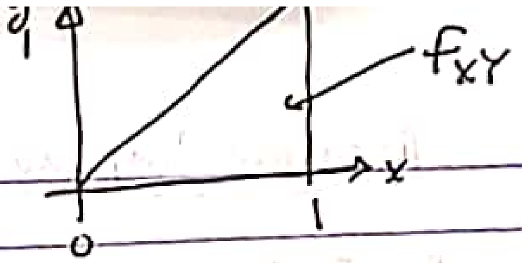

Limits of integration

$$f_Y(y) = \int_{\mathbb{R}} \frac{1}{x} \mathbb{1}_{y\in[0,1)} \mathbb{1}_{x\in[y,1]} \, dx$$

$$\mathbb{1}_{y\in[0,1]} \int_y^1 \frac{1}{x} \, dx = \left[\ln(x)\right]_y^1 \mathbb{1}_{y\in[0,1]}$$

$$= -\ln(y) \mathbb{1}_{y\in[0,1]}$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(x)} = \frac{\frac{1}{x} \mathbb{1}_{y\in[0,1)} \mathbb{1}_{x\in[y,1]}}{-\ln(y) \mathbb{1}_{y\in[0,1]}}$$

$$= -\frac{1}{x\ln(y)} \mathbb{1}_{x\in[y,1]}$$ eg $f_{X|Y}(x, \frac{3}{4}) \approx \frac{3.5}{x} \mathbb{1}_{x\in[\frac{3}{4},1]}$

$f_{X|Y}(1, \frac{3}{4}) \approx 3.5$

$f_{X|Y}(\frac{3}{4}, \frac{3}{4}) \approx 4.6$

Example of whats called a mixture distribution

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11/4 <u>Mixture and Compound distributions</u>

eg If there is no internet traffic, downloads take $Exp(\frac{1}{5})$ seconds. If there is traffic downloads take $Exp(\frac{1}{20})$ seconds. How long do downloads take in general?
└ traffic is $\frac{1}{3}$ of the time.

let $X \sim Bernoulli(\frac{2}{3})$
if $x=0 \Rightarrow$ traffic
$x=1 \Rightarrow$ no traffic

let $T|X=1 \sim Exp(\frac{1}{5})$ download w/ no traffic

$T|X=0 \sim Exp(\frac{1}{20})$ with traffic.

$U \sim Exp(\lambda) = \lambda e^{-\lambda u}$

$E[u] = \frac{1}{\lambda}$

mixture model

	X	$T\|X$	T
$1/3$	0	$Exp(\frac{1}{20})$	
$2/3$	1	$Exp(\frac{1}{5})$	

$f_{X,T}(x,T)$

$$f_T(t) = \sum_{x \in \mathbb{R}} f_{T|X}(x,t)\, P_X(x)$$

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$$= \sum_{x \in \mathbb{R}} \left(\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0} \right) \frac{2}{3}^{x} \left(\frac{1}{3}\right)^{1-x} \mathbb{1}_{x \in \{0,1\}}$$

becomes

$$\sum_{x \in \{0,1\}} \text{plug in } 0 \Rightarrow \frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \underset{1}{\text{plug in}}\ \frac{2}{3} \frac{1}{5} e^{-\frac{1}{5}t}$$

Given a download time, what is the traffic probabilities?

$$P_{X|T}(x,t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{\left(\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0} \right)}{\frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}}$$

$$\left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{1-x} \mathbb{1}_{x \in \mathbb{B}}$$

this equals

$$\text{Bern}\left(\frac{\frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}}{\frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}} \right)$$

If download took 25 seconds what is the probability of traffic?

$$P_{X|T}(0,25) \approx 84\%$$

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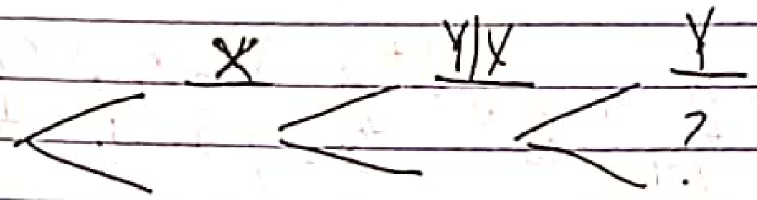
$X \sim U(0,1)$, $Y | X=x \sim U(0,x)$

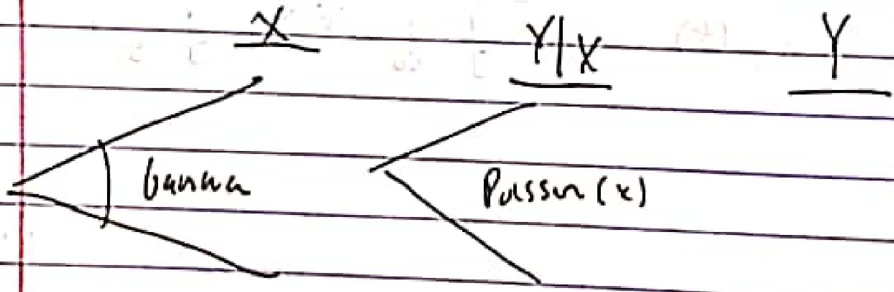


$$f_Y(y) = \int_{\mathbb{R}} \overbrace{f_{Y|X}(x,y)}^{\text{model}} \cdot \overbrace{f_X(x)}^{\text{mixing}} \, dx$$

If the mixing distribution is continuous then Y is called "compound."

$X \sim \text{Gamma}(\alpha, \beta)$

$Y|X = x \sim \text{Poisson}(x)$



$$p_Y(y) = \int_{\mathbb{R}} p_{Y|X} \, f_X(x) \, dx = \int_{\mathbb{R}} \left(\frac{x^y e^{-x}}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \right) x \left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)} \right) d$$

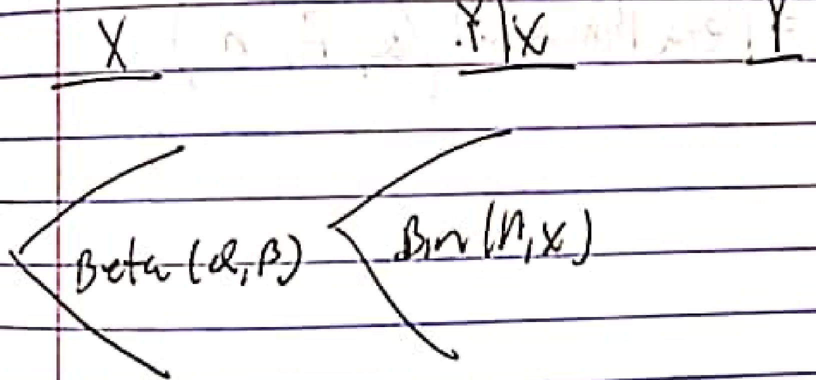
11/4 $\propto \frac{1}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} \, dx = \frac{3}{4}$

$$= \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \mathbb{1}_{y\in\mathbb{N}_0} \overset{HW}{\propto} \text{ExtNegBin}\left(\alpha, \frac{\beta}{\beta+1}\right)$$

n is known

$X \sim \text{Beta}(\alpha, \beta)$

$Y \mid X = x, n \sim \text{Bin}(\overset{n}{\cancel{z}}, \overset{x}{\cancel{\alpha}})$



$$p_Y(y) = \int_{\mathbb{R}} p_{Y|X}(x,y) f_X(x)\, dx = \int_{\mathbb{R}} \left(\binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y\in\{0,\dots,n\}} \right) \times$$

$$\left(\frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x\in[0,1]} \right) dx =$$

pull out all constants.

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$$\frac{\binom{n}{y}\mathbb{1}_{y\in\{0,1,\dots,n\}}}{B(\alpha,\beta)}\underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\,dx}_{\text{Beta Function}}$$

$$=\frac{\binom{n}{y}}{B(\alpha,\beta)}\,B(y+\alpha,\,n-y+\beta)\,\mathbb{1}_{y\in\{0,1,\dots,n\}}$$

$$=\text{BetaBinomial}(\alpha,\beta,n)$$

$X\sim\text{Gamma}(\alpha,\beta)$

$Y|X=x\sim\text{Exp}(x)$

HW $\Rightarrow Y\sim\text{Lomax}(\beta,\alpha)$

New Unit. Moment Generating / characteristic functions.

$a,b\in\mathbb{R}$

$z:=a+bi\in\mathbb{C}$ where $i:=\sqrt{-1}$

$Re[z]=a$, $Im[z]=b$

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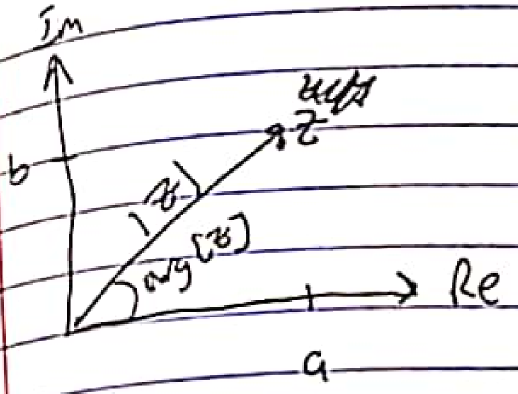

$|z| = \sqrt{a^2+b^2}$

$Arg[z] = \arctan\left(\frac{b}{a}\right)$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

Vote $\boxed{\text{90 min this Fri - 10 AM}}$

90 min next Tues -