$\vec{X}$ is a vector r.v of dim. n.

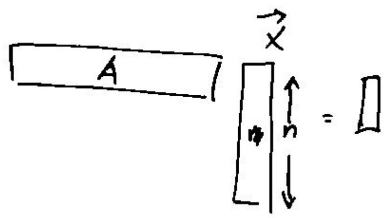
$\vec{\mu} = E[\vec{X}]$ $\qquad \Sigma = Var[\vec{X}] = E[\vec{X}X^T] - \overbrace{E[\vec{X}]\,E[\vec{X}]^T}^{\vec{\mu}\vec{\mu}^T}$

Let $\vec{c} \in \mathbb{R}^n$

$E[\vec{c}^T\vec{X}] = \vec{c}^T\vec{\mu}$

$Var[\vec{c}^T\vec{X}] = \vec{c}^T\Sigma\vec{c}$

$$= \begin{bmatrix} Var[X_1] & cov(X_1X_2) & \cdots & \\ & Var[X_2] & & \\ & & \ddots & \\ & & & Var[X_n] \end{bmatrix}$$



Let $A \in \mathbb{R}^{m\times n}$ matrix of constants

$$\underset{m\times n\ n\times 1}{E[A\vec{X}]} = \begin{bmatrix} E[\vec{a}_{1\cdot}\vec{X}] \\ E[\vec{a}_{2\cdot}\vec{X}] \\ \vdots \\ E[\vec{a}_{m\cdot}\vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot}\vec{\mu} \\ \vec{a}_{2\cdot}\vec{\mu} \\ \\ \vec{a}_{m\cdot}\vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot} \\ \vec{a}_{2\cdot} \\ \\ \vec{a}_{m\cdot} \end{bmatrix}\vec{\mu} = A\vec{\mu}$$

$$\begin{aligned} Var[A\vec{X}] &= E[(AX)(AX)^T] - E[AX]E[AX]^T \\ &= E[A\underbrace{XX^TA^T}] - A\vec{\mu}\vec{\mu}^TA^T \\ &= AE[XX^TA^T] - A\vec{\mu}\vec{\mu}^TA^T \\ &= AE[XX^T]A^T - A\vec{\mu}\vec{\mu}^TA^T \\ &= A(E[XX^T]A^T - \vec{\mu}\vec{\mu}^TA^T) \\ &= A(E[XX^T] - \vec{\mu}\vec{\mu}^T)A^T \\ &= A\Sigma A^T \end{aligned}$$

$U \sim \chi^2_k$

$E[U] = k$

Let $Z_1, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$U = Z_1^2 + \dots + Z_k^2 \sim \chi^2_k$ $\qquad Var[Z] + E[Z]^2$

$E[U] = E[Z_1^2] + \dots + E[Z_k^2] = kE[Z^2] = k$

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim \underbrace{N_n(\vec{0}, I_n)}_{\text{"Multivariate Normal"}} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\vec{Z}^T\vec{Z}}$$

$A \in \mathbb{R}^{n \times n}$ matrix of constants $\quad A = \begin{bmatrix} \vec{a}_{1\cdot} \\ \vdots \\ \vec{a}_{n\cdot} \end{bmatrix} = [\vec{a}_{1\cdot} | \vec{a}_{2\cdot} | \dots | \vec{a}_{n\cdot}]$

$\vec{X} = A\vec{Z} + \vec{\mu} = g(\vec{Z})$

$E[\vec{X}] = AE[\vec{Z}] = A\vec{\mu}$

$Var[A\vec{Z} + \mu] = Var[A\vec{Z}] = Var[\vec{X}] = A\Sigma_x A^T = AI_nA^T = AA^T$

$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_h|$

$\vec{X} - \vec{\mu} = A\vec{Z}$

Assume A is invertible...

$$\vec{Z} = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{\mu}) = h(\vec{X}) = B\vec{X} - B\vec{\mu} = \begin{bmatrix} \vec{b}_{1\cdot}\vec{X} - \vec{b}_{1\cdot}\vec{\mu} \\ \vec{b}_{2\cdot}\vec{X} - \vec{b}_{2\cdot}\vec{\mu} \\ \vdots \\ \vec{b}_{n\cdot}\vec{X} - \vec{b}_{n\cdot}\vec{\mu} \end{bmatrix} \begin{matrix} = h_1(\vec{x}) \\ = h_2(\vec{x}) \\ \vdots \\ = h_n(\vec{X}) \end{matrix}$$

$$J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \det \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{1n} & \cdots & & b_{nn} \end{bmatrix} = \det[A^{-1}]$$

$$\Rightarrow f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(A^{-1}(\vec{X} - \vec{\mu}))\,|\det[A^{-1}]|$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}(\vec{X} - \mu)^T(A^{-1})^T A^{-1}(\vec{x} - \vec{\mu})}\,|\det[A^{-1}]|$$

Fact #1

$AA^{-1} = I$

$\Rightarrow (AA^{-1})^T = I^T = I$

$(A^{-1})^T A^T = I$

$(A^T)^{-1}A^T = I$

$(A^{-1})^T = (A^T)^{-1}$

Fact #2

$\Sigma_x = AA^T$

$(\Sigma_x)^{-1} = (AA^T)^{-1}$

Recall $(AB)^{-1} = B^{-1}A^{-1}$

$(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$

$\Sigma_x^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$

Fact #3

$\det[\Sigma_x] = \det[AA^T]$

$= \det[A]\det[A^T]$

$= \det[A]^2$

$\Rightarrow \det[A] = \sqrt{\det[\Sigma_x]}$

Fact #4

$AA^{-1} = I$

$\det[AA^{-1}] = \det[I] = 1$

$\det[A]\det[A^{-1}] = 1$

$\Rightarrow \det[A^{-1}] = \frac{1}{\det[A]}$

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\underbrace{\sqrt{(2\pi)^n \det[\Sigma]}}_{\Sigma \text{ must be full rank or else divide by } 0.}} e^{-\frac{1}{2}(\vec{x}-\mu)^T \Sigma^{-1} (\vec{x}-\vec{\mu})\ \Sigma^{-1}(\vec{x}-\vec{\mu})} = N_n(\vec{\mu}, \Sigma)$$

"general Multivariate Normal"

$$\vec{X} = A\vec{Z} + \vec{\mu}$$

rank deficient

$\hookrightarrow \det = 0$

$\cancel{X = \sigma Z + \mu}$ → assuming this

$\begin{array}{cc} & \leftarrow n \\ m \updownarrow & \begin{bmatrix} \quad \end{bmatrix} \end{array}$

→ Let $A \in \mathbb{R}^{m \times n}$ where $m < n$

$\Sigma = AA^T$

Let $\tilde{A} = \cancel{[A | \vec{v}_1 | \dots | \vec{v}_{n-m}]} =$

$= \begin{bmatrix} A \\ \vec{v}_1 \\ \vdots \\ \vec{v}_{n-m} \end{bmatrix}$ where the $\vec{v}$'s chosen so the $\tilde{A}$ is full rank.

$\tilde{\vec{\mu}} = \begin{bmatrix} \vec{\mu} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \tilde{\vec{X}} = \tilde{A}\vec{Z} + \tilde{\vec{\mu}} \approx f_{\tilde{X}}(\tilde{\vec{x}}) = \dfrac{1}{\sqrt{2\pi^n \det(\tilde{\Sigma})}} e^{-\frac{1}{2}(x-\tilde{\vec{\mu}})^T \tilde{\Sigma}^{-1}(\tilde{\vec{x}}-\vec{\mu})}$

$$\Rightarrow f_{\vec{X}}(\vec{x}) = \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n-m} f_{\tilde{\vec{X}}}(\tilde{\vec{x}})\, dx_1\, dx_2 \dots dx_{n-m}$$

For r.v $\vec{X}$ define the characteristic function as

$$\phi_{\vec{X}}(\vec{t}) = E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1 + \dots + t_nX_n)}]$$

$\phi : \mathbb{R}^n \to \mathbb{C}$

$$= E[e^{it_1X_1} e^{it_2X_2} \dots e^{it_nX_n}]$$

if $X_1 \dots X_n$ iid

$$= E[e^{it_1X_1}] \dots E[e^{it_nX_n}] = \phi_{X_1}(t_1) \cdots \phi_{X_n}(t_n)$$

(P0) $\phi_{\vec{X}} = E[e^{i\vec{0}_n^T\vec{X}}] = E[e^{i0}] = 1$

(P1) holds...

(P3) $\vec{Y} = A\vec{X} + \vec{b}$ $\quad A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$

$$\phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{i\vec{t}^TA\vec{X}} e^{i\vec{t}^T\vec{b}}] = e^{i\vec{t}^T\vec{b}} E[e^{i(A^T\vec{t})^T\vec{X}}]$$

$$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{X}}(A^T\vec{t}) \quad (3)$$

Find chart. func. of $\vec{Z} \sim N_n(\vec{0}_n, I_n)$

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n} \phi_Z(t_i) = \prod_{i=1}^{n} e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let $\vec{X} = A\vec{Z} + \vec{\mu} \overset{P2}{\Rightarrow} \phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu}} \phi_Z(A^T t) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\vec{t}^T \overbrace{AA^T}^{\Sigma}\vec{t}}$

$$= e^{i\vec{t}^T\mu + \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{Y} = B\vec{X} + \vec{c} \sim ?$ $\quad B \in \mathbb{R}^{l \times m}$, $\vec{c} \in \mathbb{R}^l$

$$\overset{P2}{} \phi_{\vec{Y}}(\vec{t}) = e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}} e^{i\vec{t}^T B\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T \Sigma (B^T\vec{t})}$$

$$= e^{it^T(\vec{c} + B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}}$$

$$\overset{P1}{\Rightarrow} \vec{Y} \sim N_l(B_{\vec{\mu}+c}, B\Sigma B^T)$$

$\vec{Y} = a\vec{X} \sim N_M(a\vec{\mu}, a^2\Sigma)$

$a \in \mathbb{R}$