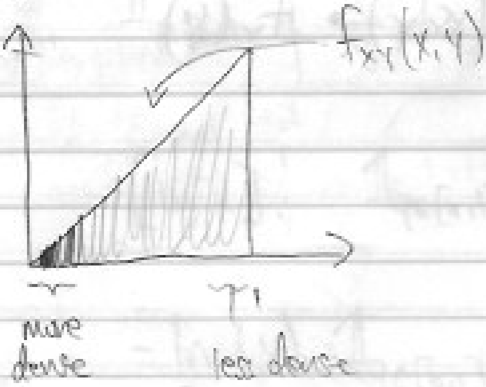
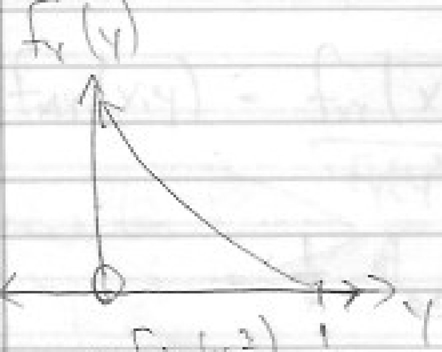
(Lecture 14) November $4^{th}$, 2019

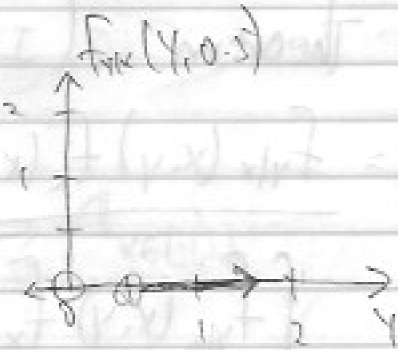
## Conditional Densities (pg 155)

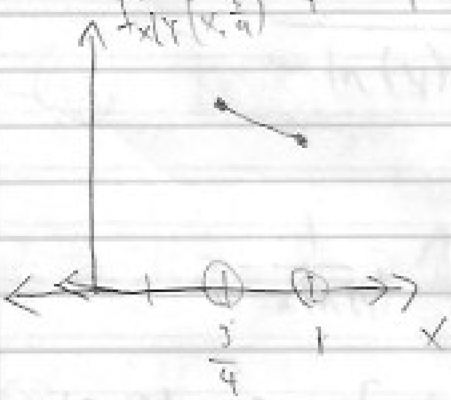
$X \sim U(0,1)$

$Y|X=x \sim U(0,1)$




formulas

I) Margining $f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy$, $f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx$

II) Definition of conditional probability density

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} \text{ if } f_Y(y) > 0$$

$$\Downarrow$$

$$f_{XY}(x,y) = f_{X|Y}(x,y) \cdot f_Y(y)$$

III) Bayes Rule

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y)\, f_X(x)}{f_Y(y)}$$

IV) Bayes theorem (I - III)

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y)\, f_X(x)}{\int_{\mathbb{R}} f_{Y|X}(x,y)\, f_X(x)\, dx}$$

$$f_X(x) = \mathbb{1}_{x \in [0,1]}$$

$$f_{Y|X}(x,y) = \frac{1}{x} \mathbb{1}_{y \in [0,x]}$$

$$f_{XY}(x,y) = f_{Y|X}(x,y)\, f_X(x) = \frac{1}{x} \mathbb{1}_{y \in [0,x]} \mathbb{1}_{x \in [0,1]}$$

$$= \frac{1}{x} \mathbb{1}_{0 \le y \le x \le 1}$$

$$f_Y(y) = \int_{\mathbb{R}} \frac{1}{x} \mathbb{1}_{0\le y\le x\le 1}\, dx$$

$$\mathbb{1}_{0\le y\le x\le 1} = \mathbb{1}_{x\in[y,1]}\, \mathbb{1}_{y\in[0,1]}$$

$$= \mathbb{1}_{y\in[0,1]} \int_y^1 \frac{1}{x}\, dx$$

$$= \left[\ln x\right]_y^1 \mathbb{1}_{y\in[0,1]}$$

$$= -\ln(y)\, \mathbb{1}_{y\in[0,1]}$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

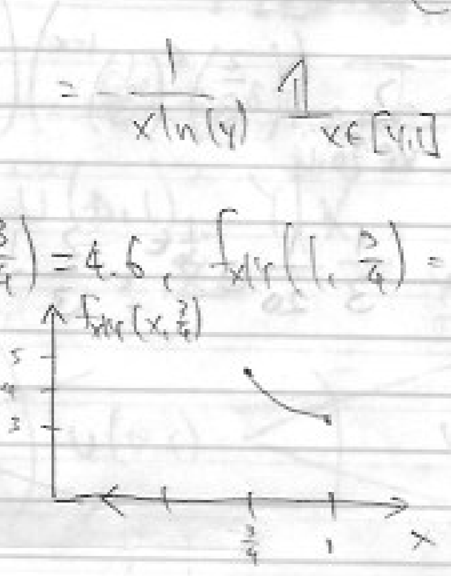
$$= \frac{\frac{1}{x}\, \mathbb{1}_{0\le y\le x\le 1}}{-\ln(y)\, \mathbb{1}_{y\in[0,1]}} \qquad \mathbb{1}_{0\le y\le x\le 1} \to \mathbb{1}_{x\in[y,1]}$$

↑ undefinedness of $f_{X|Y}$

$$= -\frac{1}{x\ln(y)}\, \mathbb{1}_{x\in[y,1]} \overset{\text{If } y=\frac{3}{4}}{\approx} \frac{3.5}{x}\, \mathbb{1}_{x\in[\frac{3}{4},1]}$$

$f_{X|Y}(\frac{3}{4},\frac{3}{4}) = 4.6,\ f_{X|Y}(1,\frac{3}{4}) = 3.5$

let $y=\frac{3}{4}$



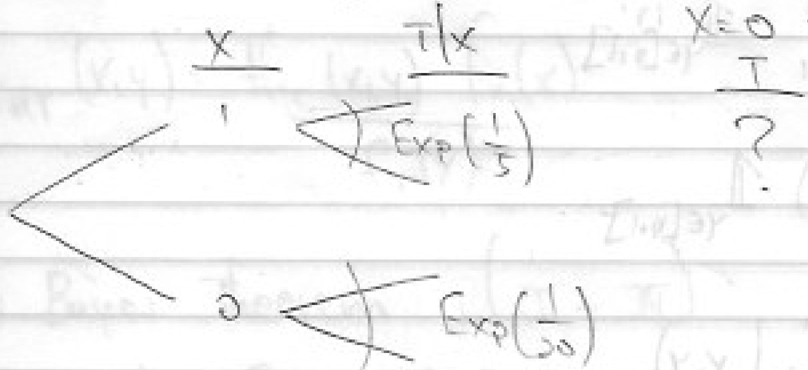
## Mixture Distributions

Example: $\frac{2}{3}$ of the time, no Internet traffic and download speeds are $T \sim \text{Exp}\left(\frac{1}{5}\right)$, ... fast. But $\frac{1}{3}$, there is traffic, $T \sim \text{Exp}\left(\frac{1}{20}\right)$, ... slow

What is the marginal distribution of $T$? this is called a "mixture model" or "mixture distribution" or "multilevel model"

Let $X \sim \text{Bern}\left(\frac{2}{3}\right)$ where $X = 1 \to$ no traffic

$X = 0 \to$ traffic



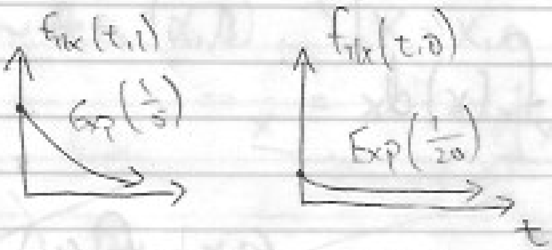

Note: $T \sim \text{Exp}(\lambda) = \lambda e^{-\lambda t}$

Note: } Discrete, $\to$ continuous

$$f_T(t) = \sum_{\text{Supp}[X]} f_{X,T}(x,t) = \sum_{x \in \text{Supp}[X]} p_X(x) \cdot f_{T|X}(x,t)$$

$$= \sum_{x \in \{0,1\}} \left(\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}\right)\left(\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}$$




If $t=25$ min, what is the probability there is a traffic.

$$p_{X|T}(x,t) \underset{\uparrow}{=} \frac{f_{T|X}(t,x)\,p_X(x)}{f_T(t)} = \frac{\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{1-x}\left(\frac{1}{5}e^{-\frac{1}{5}t}\mathbb{1}_{x=1}+\frac{1}{20}e^{-\frac{1}{20}t}\mathbb{1}_{x=0}\right)}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t}+\frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}}$$

Bayes Rule

$$= \mathrm{Bern}\left(\frac{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}+\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{20}t}}\right)$$

$Supp[X|T] = \{0,1\}$

$X|T \sim \mathrm{Bern}(\quad)$

$$p_{X|T}(0,25) = 1 - \frac{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}}{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}+\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}$$

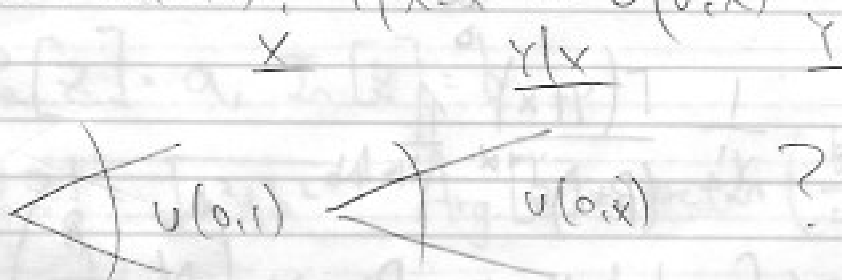
traffic

$$= \frac{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}+\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}$$

$\approx 0.84$

$X \sim U(0,1)$, $Y|X=x \sim U(0,x)$

$\underline{X}$ $\quad$ $\underline{Y|X}$ $\quad$ $\underline{Y}$

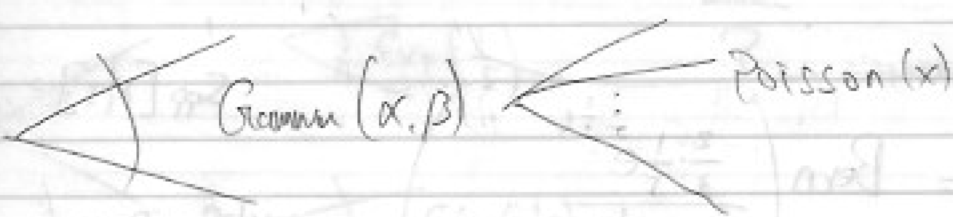


$$f_Y(y) = \int_{\mathbb{R}} f_{Y|X}(x,y)\, f_X(x)\, dx = \cdots$$

If mixing distribution is continuous, $f_Y(y)$ is sometimes called a "compound distribution"

$X \sim \text{Gamma}(\alpha, \beta)$, $Y|X=x \sim \text{Poisson}(x)$

$\underline{X}$ $\qquad$ $Y|X$ $\qquad$ $Y$?




$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y)\, f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \left( \frac{e^{-x} x^y}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \right) \left( \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)} \right) dx$$

$$\propto \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

$$= \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \mathbb{1}_{y \in \mathbb{N}_0}$$

$\propto$ Extended Negative Binomial $\left(\alpha, \frac{\beta}{\beta+1}\right)$

$X \sim Beta(\alpha,\beta)$, $Y|X=x,n \sim Bin(n,x)$

$\underline{X}$ $\quad$ $\underline{Y|x,n}$ $\quad$ $\underline{Y}$

$Beta(\alpha,\beta)$ $\quad$ $Bin(n,x)$ $\quad$ ?

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y)\, f_X(x)\, dx = \left(\binom{n}{y} x^y (1-x)^{n-y}\right)\left(\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in(0,1)}\right) dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\, dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha,\, n-y+\beta)$$

$$= BetaBinomial(\alpha,\beta,n)$$

$$\sum_{y=0}^{n} \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha,\, n-y+\beta) = 1$$

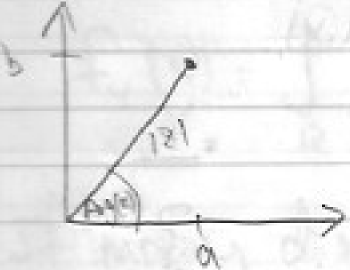
$a,b \in \mathbb{R}$

$z := a+bi \in \mathbb{C}$ ← Complex #'s

where $i := \sqrt{-1}$

$Re[z] = a$, $Im[z] = b$

$|z| = \sqrt{a^2+b^2}$ $\quad$ $Arg[z] = \arctan\left(\frac{b}{a}\right)$




$i^2 = -1$

$i^3 = i i^2 = -\sqrt{-1} = -i$

$i^4 = i^2 i^2 = 1$

$i^5 = i i^4 = i$