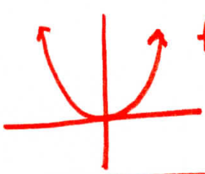
Lecture:

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5!} + \cdots$
- $\sin(x) = x - \frac{x^3}{3} + \frac{x^5}{5!} - \cdots$
- $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \cdots$
- $e^{itx} = 1 + itx \quad \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$
- $i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} + \cdots$
- $\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} + \cdots$

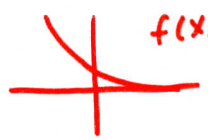
$\Rightarrow e^{itx} = \cos(tx) + i\sin(tx)$

Let $\theta = tx \Rightarrow e^{i\theta} = \cos(\theta) + i\sin(\theta)$ if $\theta = \pi \Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$ Euler identity

- Define $L^1 := \{f : \int_{\mathbb{R}} |f(t)|\, dt < \infty\}$ "L" integrable functions or "all-one" which is the set of all absolutely integrable functions.



$f = x^2 \notin L^1$



$f(x) = e^{-x} \notin L^1$ $\quad f(x) = e^{-x}\mathbb{1}_{x>0} \in L^1$

$\boxed{\text{All PDF's} \in L^1}$

$\int_{\mathbb{R}} |f(x)|\, dx = \int_{\mathbb{R}} f(x)\, dx = 1 < \infty$

- If $f(t) \in L^1$ then $\exists \hat{f}(\omega)$ which can be found via the "Fourier transformation

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-2\pi i w t} f(t)\, dt$$

Further, if $\hat{f}(w) \in L^1$ then we can use the inverse Fourier transform to recover $f(t)$.

$$f(t) = \int_{\mathbb{R}} e^{2\pi i w t} \hat{f}(w)\, dw \Rightarrow f, \hat{f} \in L^1 \Rightarrow f \,\&\, \hat{f} \text{ are } 1{:}1$$

· $f(t)$ is known as the "time domain." → instruments are diff.

$\hat{f}(w)$ is known as the "frequency domain."

• Let X be a r.v. Define: $\phi_X(t) := E[e^{itX}]$ which is called the characteristic function of X (ch.f.)

$$\int_{\mathbb{R}} e^{itx} f(x)\,dx \qquad \sum_{x \in \mathbb{R}} e^{itx} p(x)$$

We care about this b/c it gives us more tools to solve problems & we can prove new theorems.

Property 0: $\phi_X(0) = E[e^{i(0)X}] = E[1] = 1, \ \forall X$

P1: If $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$ → distribution

P2: If $Y = aX + b \Rightarrow \phi_Y(t) = E[e^{it(aX+b)}] = E[e^{itaX} e^{itb}]$

$$= e^{itb} E[e^{(i(at)X}] = e^{itb} \phi_X(at)$$

P3: If $X_1 \,\&\, X_2 \overset{ind}{\sim}$ & $T = X_1 + X_2$

$\Rightarrow \phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}] = E[e^{itX_1}]E[e^{itX_2}]$

$$= \phi_{X_1}(t)\,\phi_{X_2}(t) = \phi_X(t)^2$$

$X_1 \,\&\, X_2 \overset{iid}{\sim}$

↳ Sometimes, it makes convolution easier.

P4: "Moment Generation"

real analysis

$\phi_X'(t) = \frac{d}{dt}[E[e^{itX}]] = E[\frac{d}{dt}[e^{itX}]] = E[iXe^{itX}] \Rightarrow \phi_X'(0) = E[iX] \Rightarrow \cancel{iE[X]}$ (constant i)

$$\Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$

$\phi_X''(t) = \frac{d}{dt}[\phi_X'(t)] = E[\frac{d}{dt}[iXe^{itX}]]$

$= E[i^2X^2e^{itX}] \Rightarrow \phi_X''(0) = E[i^2X^2] \Rightarrow E[X^2] = \frac{\phi_X''(0)}{i^2}$

⋮

$$\Rightarrow E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}$$

P5: Existence:

$|\phi_X(t)| = |E(e^{itX})| \overset{c}{=} |\int_{\mathbb{R}} e^{itx} f(x)\,dx| \le \int_{\mathbb{R}} |e^{itx}|\, f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = 1$

$\overset{d}{=} |\sum_{x \in \mathbb{R}} e^{itx} p(x)| \le \sum_{x \in \mathbb{R}} |e^{itx}|\, p(x)\,dx = \sum_{x \in \mathbb{R}} p(x) = 1$

$|a+b| \le |a| + |b|$

$|\int_{\mathbb{R}} h(y)\,dy| \le \int_{\mathbb{R}} |h(y)|\,dy$

$|e^{itx}| = |\cos(tx) + i\sin(tx)| = \sqrt{\cos^2(tx) + \sin^2(tx)} = 1$

$\Rightarrow \phi_X(t) \in [-1, 1], \ \forall x, t$

(P6) : Inversion: If $\phi_X(t) \in L^1$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t) dt$$

$t = -2\pi w$

(P7) : Levy's CDF formula:

$$\forall \phi_X(t)\text{'s}, \; P(X \in [a,b]) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t) dt$$

- Consider a sequence of r.v's $X_1, X_2, \dots, X_n$

if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x$, we say "$X_n$ converges in distribution to X" & the shorthand is $X_n \xrightarrow{d} X$.

(P8) Levy's Continuity theorem:

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X$$

- Moment Generating Function (mgf) for r.v. X is

$M_X(t) = E[e^{tX}]$ (Looks like ch.f, but except $\underline{\underline{i}}$ here) similar properties to ch.f.

(P0) $M_X(0) = 1$

(P1) $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

(P2) $Y = aX + b \Rightarrow M_Y(t) = e^{tb} M_X(at)$

(P3) If X_1 & $X_2 \overset{ind}{\sim}$ $T = X_1 + X_2$

$\Rightarrow M_T(t) = M_{X_1}(t) M_{X_2}(t) = M_X(t)^2$

↑ if X_1 & $X_2 \sim iid$

(P4) $E[X^n] = M_X^{(n)}(0)$

$\boxed{\text{No P5}}$ $\Rightarrow$ which means they might not exist, although ch.f exists always. mgf is however more useful.

(Ex1) $X \sim Gamma(\alpha, \beta)$

time domain

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\in(0,\infty)} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\int_0^\infty x^{\alpha-1} e^{-(\beta-it)x} dx}_{} = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(\beta-it)^\alpha}$$

$$= \left(\frac{\beta}{\beta - it}\right)^\alpha$$

↑

Frequency domain

$T = X_1 + X_2 \sim ?$ The answer is $\text{Gamma}(\alpha_1 + \alpha_2, \beta)$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$$

$$= \left(\frac{\beta}{\beta - it}\right)^{\alpha_1}\left(\frac{\beta}{\beta - it}\right)^{\alpha_2} = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1+\alpha_2}$$

P1

$\Rightarrow X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$

EX3 $X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = \sum_{x\in\mathbb{R}} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\mathbb{N}_0} = e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!}}_{\text{power series}} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$(e^{it})^x$

EX4 $X_1 \sim \text{Poisson}(\lambda_1)$ ind. of $X_2 \sim \text{Poisson}(\lambda_2)$

$T = X_1 + X_2 ?$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t)$$

$$= e^{\lambda_1(e^{it}-1)} e^{\lambda_2(e^{it}-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$\Rightarrow X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$

midterm #2

- Let $X_1 \dots X_n \overset{iid}{\sim}$ r.v's with finite expectation μ & finite variance σ.

Let $T_n = X_1 + \dots + X_n$, the sum r.v.

Let $\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + \dots + X_n}{n}$, the average r.v.

From math 241, $E[\bar{X}_n] = \mu$ & $Var[\bar{X}_n] = \frac{\sigma^2}{n} \Rightarrow SE(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$

$\phi_{T_n}(t) = (\phi_X(t))^n$

P3

$Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu$

$\Rightarrow aX+b$ situation

$\phi_{\bar{X}_n}(t) = (\phi_X(\frac{t}{n}))^n$

P2

"standardization"

$E[Z_n] = 0$, $SE[Z_n] = 1$

$\phi_{Z_n}(t) = ?$

P3