Given

$\vec{X} \sim Multinomial(n, \vec{p})$

$$\mu = E[\vec{X}] \overset{\text{defined as}}{:=} \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_K] \end{bmatrix}$$

$X_j \sim Bin(n, p_j)$

for this,

$$E[\vec{X}] = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$$

for a matrix M,

$$M = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1M} \\ X_{21} & X_{22} & \cdots & X_{2M} \\ \vdots & \vdots & & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NM} \end{bmatrix}$$

$$E[M] = \begin{bmatrix} E[X_{11}] & E[X_{12}] & & E[X_{1M}] \\ E[X_{21}] & E[X_{22}] & & E[X_{2M}] \\ & & & \\ E[X_{N1}] & E[X_{N2}] & & E[X_{NM}] \end{bmatrix}$$

Variance

$$\sigma^2 := Var[X] = E[X^2] - \mu^2$$

covariance

$$\sigma_{12} = Cov[X_1 X_2] = E[X_1 X_2] - \mu_1 \mu_2$$
$$= E[(X_1 - \mu_1)(X_2 - \mu_2)]$$

$$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

If X_1, X_2 are indep. then $\sigma_{12} = 0$

Rules for covariance

① $cov[X, X] = E[X^2] - \mu^2 = Var(X) = \sigma^2$ $\quad \sigma_{11} = \sigma_1^2$

② $cov[X_1, X_2] = Cov[X_2, X_1]$

③ $cov[X_1 + X_2, X_3] = Cov[X_1, X_2] + Cov[X_1, X_3]$

④ $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 Cov[X_1, X_2]$
$= a_1 a_2 \sigma_{12}$

where a_1, a_2 are constants, $a_1, a_2 \in \mathbb{R}$

⑤ $Var[X_1 + X_2 + \dots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$

for example:

$$Var[X_1 + X_2] = \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} = \sigma_{11} + \underbrace{\sigma_{12} + \sigma_{21}} + \sigma_{22}$$
$$= \sigma_1^2 + 2\sigma_{12} + \sigma_2^2$$

covariance matrix is called Σ

$$\text{define } \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{mm} \end{bmatrix}$$

← also called variance matrix $Var[\vec{X}]$

Matrix Σ is defined:

also called $Var[\vec{X}]$

$$\Sigma := \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & & Cov[X_1, X_K] \\ Cov[X_2, X_1] & Var[X_2] & & Cov[X_2, X_K] \\ & & \ddots & \\ Cov[X_K, X_1] & Cov[X_K, X_2] & & Var[X_K] \end{bmatrix},$$

$$= \{Cov[X_i, X_j]\}$$

$$:= E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}_T$$

If X_1 and X_2 are indep then $\sigma_{12} = 0$

the diagonal is all the variances

this is called the "variance matrix"

or "covariariance matrix"

or "variance-covariance matrix"

K×K matrix

symmetric

digonals are non-negative

If $X_1, X_2, \ldots, X_K$ are independent then

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & & 0 \\ 0 & \sigma_2^2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \sigma_3^2 \end{bmatrix} = \sigma^2 I_K$$

K×K identity matrix

only diagonals (variances) are positive

all other entries (covariances) are 0

Rules for expectation and variance of vectors of r.v.'s

$$E[\vec{X} + \vec{a}] = \begin{bmatrix} E[X_1 + a_1] \\ E[X_2 + a_2] \\ \vdots \\ E[X_K + a_K] \end{bmatrix} = \vec{\mu} + \vec{a}$$

dot product

$\vec{a} \cdot \vec{X}$
$= \vec{a}^T \vec{X}$

$$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + a_2 X_2 + \dots + a_K X_K]$$
$$= a_1 \mu_1 + a_2 \mu_2 + \dots + a_K \mu_K$$
$$= \vec{a}^T \cdot \vec{\mu}$$

$\vec{a}_{1\cdot}$ means 1st row of A

$$E[\underset{\text{matrix}}{A}\vec{X}] = \begin{bmatrix} E[a_{11}X_1 + \dots + a_{1K}X_K] \\ E[a_{21}X_1 + \dots + a_{2K}X_K] \\ \\ E[a_{L1}X_1 + \dots + a_{LK}] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot}^T \vec{\mu} \\ \vec{a}_{2\cdot}^T \vec{\mu} \\ \\ \vec{a}_{L\cdot}^T \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$A \in \mathbb{R}^{L \times K}$ (A is an $L \times K$ matrix of constants)

$$\Sigma = Var[\vec{X}]$$

$$Var[\vec{a}^T X] = Var[a_1 X_1 + \dots + a_K X_K]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} Cov[a_i X_i, a_j X_j]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \sigma_{ij} = \vec{a}^T \underset{\substack{\uparrow \\ \text{variance-covariance matrix}}}{\Sigma} \vec{a}$$

Variance is sum of all entries in matrix Σ

(Linear Algebra stuff)

$\vec{c} \in \mathbb{R}^K$

$V \in \mathbb{R}^{K \times K}$

Consider $\underset{1\times K}{\vec{c}^T} \underset{K\times K}{V} \underset{1\times K}{\vec{c}}$ ← this is a quadratic form

$$= \begin{bmatrix} c_1 & c_2 & \dots & c_K \end{bmatrix} \begin{bmatrix} c_1 V_{11} + \dots + c_K V_{1K} \\ c_1 V_{21} + \dots + c_K V_{2K} \\ \vdots \\ c_1 V_{K1} + \dots + c_K V_{KK} \end{bmatrix}$$

$$\begin{aligned} = \; & c_1 c_1 V_{11} + \dots + c_1 c_K V_{1K} \\ & + c_2 c_1 V_{21} + \dots + c_2 c_K V_{2K} \\ & + \dots \\ & + c_K c_1 V_{K1} + \dots + c_K c_K V_{KK} \end{aligned}$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} c_i c_j V_{ij}$$

so $\vec{c}^T V \vec{c} = \sum_{i=1}^{K} \sum_{j=1}^{K} c_i c_j V_{ij}$

Therefore we have $Var[\vec{a}^T X] = \vec{a}^T \underset{\substack{\uparrow \\ \text{variance-covariance matrix}}}{\overset{Var[\vec{X}]}{\Sigma}} \vec{a}$

Let $X_1, X_2, \ldots, X_K$ be random variable models for the yearly return of assets $1, \ldots, K$

Let $\vec{w} = [w_1 \; w_2 \ldots w_K]$ be the weigths of the K assets (as a proportion of the total), as a vector, such that $w_1 + w_2 + \ldots + w_K = 1$

$\vec{w}^T \vec{1} = 1$

Let

$F = w_1 X_1 + \ldots + w_K X_K = \vec{w}^T \vec{X}$

be the yearly return on your total portfolio

$E[F] = \mu_F$

target mean return

I want $\mu_F = \mu_0$ with minimal variance

select $\vec{w}$ such that $Var[F]$ is minimal

Σ is Variance-Covariance matrix

$$\vec{w}^* = \underset{\vec{w}^T \cdot \vec{1} = 1}{\operatorname{argmin}} \left\{ \vec{w}^T \Sigma \vec{w} \right\}$$

Markowitz Optimal Portfolio Theory

for

$\vec{X} \sim Multinomial(n, \vec{p})$

$X_j \sim Bin(n, p_j)$

mari — marginal dist of X_j

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & \sigma_{21} & \cdots & \sigma_{2K} \\ \sigma_{12} & np_2(1-p_2) & & \\ \vdots & & \ddots & \\ \sigma_{1K} & & & np_K(1-p_K) \end{bmatrix}$$

except for diagonal, entries
entries are σ_{ij}
Find this...

know
$\sigma_{ij} < 0$

$$\sigma_{ij} = Cov[X_i, X_j] = E[X_i, X_j] - \mu_i \mu_j$$

$$= \left(\sum_{X_1 \in Supp[X_1]} \sum_{X_2 \in Supp[X_2]} x_1 x_2 \, p_{x_1, x_2}(x_1, x_2) \right) - n^2 p_i p_j$$

too hard to do this way

Recall

$X_i \sim Bin(n, p_i)$
$X_j \sim Bin(n, p_j)$

Binomial r.v. — Bernoulli r.v.

$X_i = X_{1i} + X_{2i} + \dots + X_{ni}$
where $X_{1i}, X_{2i}, \dots, X_{ni} \overset{i.i.d.}{\sim} Bern(p_1)$

$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$
where $X_{1j}, X_{2j}, \dots X_{nj} \overset{i.i.d.}{\sim} Bern(p_2)$

X_{1i} and X_{1j} are dependent
→ if $X_{1i} = 1$ then X_{1j} must be 0
(but if $X_{1i} = 0$ then don't know X_{1j}
→ could be 0 or 1)

↓ combine

$$\vec{X} = \vec{X}_{1\cdot} + \vec{X}_{2\cdot} + \dots + \vec{X}_{n\cdot}$$

where $\vec{X}_{1\cdot}, \vec{X}_{2\cdot}, \dots, \vec{X}_{n\cdot} \sim Mult(1, \vec{p})$

Bin

Bernoulli

$$X_i = X_{1i} + X_{2i} + \dots + X_{ni}$$
$$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$$

$$\vec{X}_{f\cdot} = \begin{bmatrix} X_{f1} \\ X_{f2} \\ \vdots \\ X_{fK} \end{bmatrix}$$

$\downarrow$ combine

$$\vec{X} = \vec{X}_{1\cdot} + \vec{X}_{2\cdot} + \dots + \vec{X}_{n\cdot}$$

$\uparrow$ k-dimensional vector

where $\vec{X}_{1\cdot}, \vec{X}_{2\cdot}, \dots, \vec{X}_{n\cdot} \sim \text{Multinomial}(1, \vec{p})$

$$\text{Cov}[X_i, X_j] = \text{Cov}[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$$
$$= \sum_{\ell=1} \sum_{m=1} \text{Cov}[X_{\ell i}, X_{mj}]$$

If $\ell \neq m$ $\text{Cov}[X_\ell, X_m] = 0$
(different draws are independent)

so

$$\text{Cov}[X_i, X_j] = \sum \text{Cov}[X_{\ell i}, X_{\ell j}]$$
$$= \sum_{\ell=1}^{n} (-p_i p_j) = -n p_i p_j$$

$$\text{Cov}[X_{\ell i}, X_{\ell j}] := \left(\sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} x_i x_j \, p_{X_{\ell i} X_{\ell j}}(x_i, x_j) \right) - p_i p_j$$
$$= p_{X_{\ell i}, X_{\ell j}}(1,1) - p_i p_j$$
$$= -p_i p_j$$

As $K \to \infty$
$\text{Cov} \to 0$
for $i \neq j$

So $\text{Cov}[X_i, X_j] = -n p_i p_j$

If $\vec{X} \sim \text{Multinomial}(n, \vec{p})$ and
X_i's are identically distributed, $\vec{p} = \frac{1}{K}\vec{1}$
As $K \to \infty$, $\vec{p} = \frac{1}{K}\vec{1} \to 0$