11/18/19
Lecture-17

For Final exam

$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$

$Y = cX \sim \text{Gamma}(\alpha, \frac{\beta}{c}) = \frac{(\frac{\beta}{c})^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\frac{\beta}{c} y)} \mathbb{1}_{y \geq 0}$

where $c > 0$

$g(x) = cx$

$g^{-1}(y) = \frac{y}{c}$

$\frac{d[g^{-1}(y)]}{dy} = \frac{1}{c}$

$f_Y(y) = f_X(\frac{y}{c}) \frac{1}{c}$

$= \frac{\beta^\alpha}{\Gamma(\alpha)} (\frac{y}{c})^{\alpha-1} e^{-\beta(\frac{y}{c})}$

$= \frac{1}{c} + \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta y}{c}} \mathbb{1}_{y \geq 0}$

$\frac{(\frac{\beta}{c})^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta y}{c}} \mathbb{1}_{y \geq 0}$

CLT → Central Limit Theorem

$X_1, \ldots, X_n \overset{iid}{\sim}$ with μ, σ^2

$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$X \sim \sigma Z + \mu \sim N(\mu, \sigma^2)$

$\sigma > 0$ $\quad$ PDF $= f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Z_1, Z_2 \cdots\cdots Z_n \overset{iid}{\sim} N(0,1)$ ②

$Z_i^2 \sim \chi^2 := \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

chi squared

$\underbrace{Z_1^2 + \cdots Z_m^2}_{\chi^2_m} + \underbrace{Z_{m+1}^2 + Z_n^2}_{} \sim \underbrace{\chi^2_n}_{\chi^2_n} = \text{Gamma}(\frac{n}{2}, \frac{1}{2})$

$\chi^2_m \quad \chi^2_m + \chi^2_{n-m} = \chi^2_n$

$x \sim \chi^2_k \Rightarrow \frac{x}{k} \sim \text{Gamma}(\frac{k}{2}, \frac{k}{2})$

$\frac{k}{2} = \frac{\frac{1}{2}}{\frac{1}{k}}$

$x_1 \sim \chi^2_{k_1}$ ind of $x_2 \sim \chi^2_{k_2}$

$R = \frac{\frac{x_1}{k_1}}{\frac{x_2}{k_2}} = \frac{U}{V}$

let $u = \frac{x_1}{k_1} \sim \text{Gamma}(\frac{k_1}{2}, \frac{k_1}{2})$

$V = \frac{x_2}{k_2} \sim \text{Gamma}(\frac{k_2}{2}, \frac{k_2}{2})$

$\int f_u(rt) f_v(t) \mathbb{1}_{t \in \text{supp}[V]} |t| dt$

$\text{supp}(u)$

$= \int_0^\infty \left(\frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\right)\left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\right) t\, dt$

$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \int_0^\infty \underbrace{t^{a+b-1} e^{-(b+ar)t} dt}_{\frac{\Gamma(a+b)}{(b+ar)^{a+b}}}$

③

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \underbrace{(b+ar)^{-(a+b)}}_{\left(b\left(1+\frac{a}{b}r\right)\right)^{-(a+b)}}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \left(b\left(1+\frac{a}{b}r\right)\right)^{-(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \frac{\left(1+\frac{a}{b}r\right)^{-(a+b)}}{b^a b^b}$$

$$= \frac{\left(\frac{a}{b}\right)^{a}}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2},\frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1+\frac{k_1}{k_2}r\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r \geq 0}$$

$= F_{k_1,k_2} =$ F Distribution with k_1 and k_2 degress of freedom. "Fisher-snecedor dist"

$Z \sim N(0,1)$ ind of $X \sim \chi^2_k$

$$W = \frac{Z}{\sqrt{\frac{X}{k}}} \sim f_W(w) = ?$$

$$W^2 = \frac{Z^2}{\frac{X}{k}} = \frac{\frac{Z^2}{1}}{\frac{X}{k}} \sim \frac{\chi^2_1}{\chi^2_k} \sim F_{1,k}$$

F dist

note: $f_W(w) = f_W(-w) \rightarrow$ Fundamental theorem of calculus.

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$F_{W^2}(w^2) = P(W \leq w^2) = P(W \in [-w, w)$

$= P(-w \leq W \leq w)$

$CDF = F_{W^2}(w^2) = F_W(w) - F_W(-w)$

Take $\frac{d}{dw}$ both sides

$= 2w f_{W^2}(w^2) = 2 f_W(w) = w f_{w^2}(w^2)$

Note := $k_1 = 1$, $k_2 = k$

$$= \frac{w \left(\frac{1}{k}\right)^{\frac{1}{2}}}{\beta\left(\frac{1}{2}, \frac{k}{2}\right)} (w^2)^{\frac{1}{2}-1} \left(1 + \frac{1}{k} w^2\right)^{-\frac{1+k}{2}}$$

$$\boxed{= \frac{1}{\beta\left(\frac{1}{2}, \frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} \Gamma\left(\frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{k}{2}\right)} \quad (w^2)^{\frac{1}{2}-1} = w^{2 - \frac{1}{2}} = w^{-1}}$$

$$= w w^{-1} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\ \Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

This is called student's "T" Distribution with k degree of snecedor

Cauchy :-

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$$R = \frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) := \frac{1}{\pi}\frac{1}{r^2+1}$$

$$X = c + \sigma R = \frac{1}{\sigma\pi}\frac{1}{\left(\frac{r-c}{\sigma}\right)^2+1} = \text{Cauchy}(c,\sigma)$$

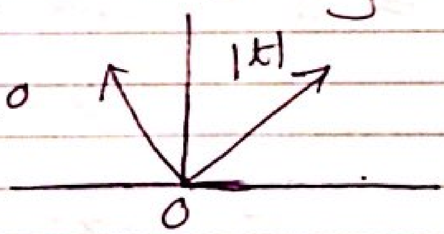
where $\sigma > 0$

Note: $$T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}(1+w^2)^{-1} = \frac{1}{\pi}\cdot\frac{1}{1+w^2} = \text{Cauchy}(0,1)$$

$$\phi_R(t) = E\left[e^{itR}\right] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{itr}}{r^2+1}\,dr = \cdots =$$

complex analysis

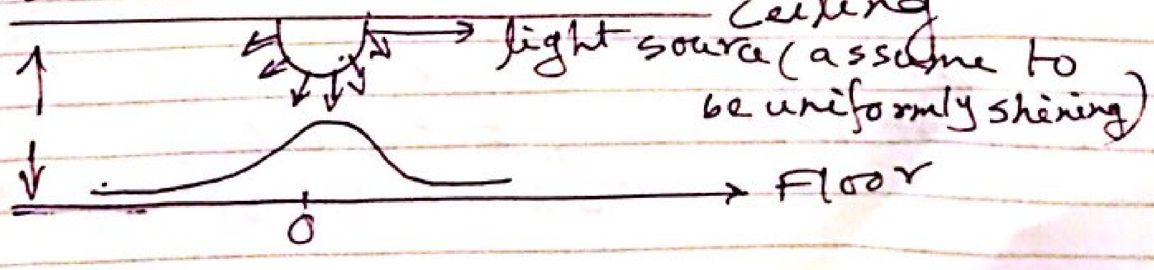
$$\phi_R(t) = e^{-|t|} = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ e^{t} & \text{if } t \leq 0 \end{cases}$$



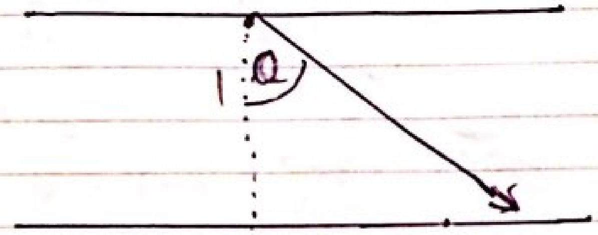
$$\phi_R'(t) = \begin{cases} -e^{-t} & \text{if } t > 0 \\ e^{t} & \text{if } t < 0 \\ \text{undefined} & \text{if } t = 0 \end{cases}$$

$\phi_R'(0)$ undefined $\Rightarrow E(R)$ is undefined

Cauchy dist has no mean.



What is the distribution of light brightness on the floor?



$\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$

$f_X(x) = f_\theta\, g^{-1}(x)\, \frac{d}{dx}[g^{-1}(x)]$

$g(\theta) = \tan(\theta) = \frac{X}{1} = X \Rightarrow \theta = \arctan(X)$
$= g^{-1}(x)$

$= \frac{1}{\pi} \mathbb{1}_{\underbrace{\arctan(X) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}_{X \in [\tan(-\frac{\pi}{2}), \tan(\frac{\pi}{2})]}} \cdot \frac{1}{1+X^2}$

$= \frac{1}{\pi} \cdot \frac{1}{1+x^2} = \text{cauchy}(0,1)$

Next unit

Application of Z, T, χ^2, F to statistics

Let $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$T = X_1 + X_2 + \cdots + X_n \sim N(n\mu, n\sigma^2)$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

estimator for μ

$\bar{x} = \frac{1}{n} \sum x_i$

estimate for specific realization

$$S_n^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x}_n)^2$$

estimate for σ^2

$$S^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2$$

estimate for σ

want to know

① $S_n^2 \sim ?$

② Relationship between $\bar{X}_n$ & S_n^2

$Z_1, \cdots, Z_n \overset{iid}{\sim} N(0,1)$ let $\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$

$\vec{Z}^T \vec{Z} = \Sigma Z_i^2 \sim \chi_n^2$

note: $Z_i = \frac{X_i - \mu}{\sigma}$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$$

$$= \frac{1}{\sigma^2}\Sigma(X_1 - \mu)^2$$

note: $X_i - \mu = X_i - \bar{X} + \bar{X} - \mu$

$$(X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2$$

$$= (X_i - \bar{X})^2 + 2(X_i - \bar{X})\cdot(\bar{X} - \mu) + (\bar{X} - \mu)^2$$

$$= \frac{1}{\sigma^2}\left(\Sigma(X_i - \bar{X})^2 + 2\Sigma(X_i - \bar{X})(\bar{X} - \mu) + \Sigma(\bar{X} - \mu)^2\right.$$

$$= \Sigma(X_i - \bar{X})(\bar{X} - \mu)$$

$$= \Sigma \underbrace{X_i}_{n\bar{X}} \bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu$$

$$= n\bar{x}^2 - n\bar{x}^2 - \mu n\bar{x} - n\bar{x}n$$

$$= 0$$

$$\therefore \underbrace{\frac{\Sigma(x_i - \bar{x})^2}{\sigma^2}} + \frac{n(\bar{x}-\mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\frac{(n-1)S^2}{\sigma^2} + \frac{\sqrt{n}\,(\bar{x}-\mu)^2}{\sigma}$$

$$\frac{(n-1)S^2}{\sigma^2} + \underbrace{\left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{z^2 \sim \chi^2_1}$$

$$= \frac{(n-1)S^2}{\sigma^2} + z^2$$

conjecture: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Recall: $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ b/c $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$= \frac{(x-\bar{x})^2}{\sigma^2} + \left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$$ let $S_n^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x}_n)^2$

and $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = z \sim N(0,1)$