Uniform Discrete

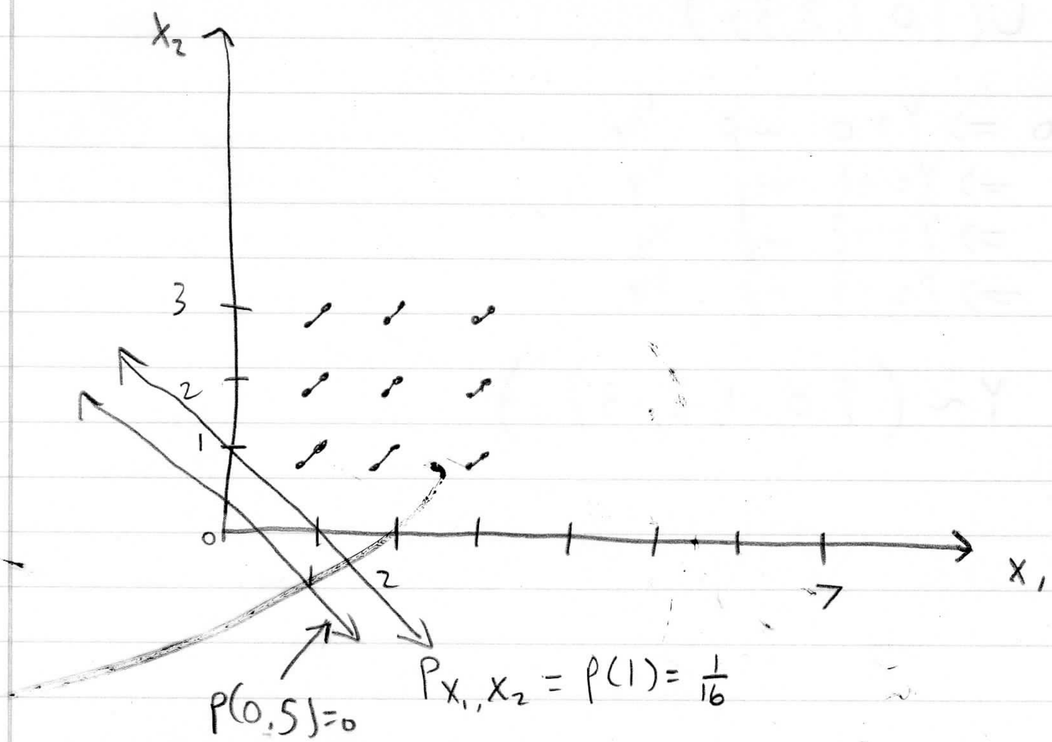
$$X, Y \overset{iid}{\sim} U(\{0,1,2,3\}) := \begin{cases} 0 & wp\ 1/4 \\ 1 & wp\ 1/4 \\ 2 & wp\ 1/4 \\ 3 & wp\ 1/4 \end{cases}$$

$$= \frac{1}{4} \mathbb{1}_{x \in \{0,1,2,3\}}$$

- Generally: $U(A) := \frac{1}{|A|} \mathbb{1}_{x \in A}$

  - $Supp[x] = A$
  - Parameter space $(A) \subset \mathbb{R}$ s.t $|A| < \mathbb{N}$

$$P_{X_1, X_2}(x_1, x_2) = \frac{1}{16} \mathbb{1}_{x_1 \in \{0,1,2,3\}} \mathbb{1}_{x_2 \in \{0,1,2,3\}}$$



- $T = X_1 + X_2 \sim p_T(t)$

$$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1, X_2}(x_1, x_2)\, \mathbb{1}_{x_2 = t - x_1}$$

$$p(1) = \sum \sum p\, \mathbb{1}_{x_2 = 1 - x_1} = 2\left(\frac{1}{16}\right) = \frac{1}{8}$$

$p(0.5) = 0$
$p(3) = 4\left(\frac{1}{16}\right) = \frac{1}{4}$
$p(6) = \frac{1}{16}$

$p(7) = 0$

- Let $Y = \underbrace{-X}_{g(X)} \sim p_Y(y)$

$X \sim U(\{0, 1, 2, 3\})$

$X = 0 \Rightarrow Y = 0$ wp $1/4$
$X = 1 \Rightarrow Y = -1$ wp $1/4$
$X = 2 \Rightarrow Y = -2$ wp $1/4$
$X = 3 \Rightarrow Y = -3$ wp $1/4$

Thus $Y \sim (\{0, -1, -2, -3\})$

- PMF of Y

- $P_Y(y) := P(Y=y) = P(-Y=-y) = P(X=-y)$

$= P_X(-y)$

$\Longrightarrow \text{Supp}[Y] = -\text{Supp}[X]$

ex. $X \sim \text{Binom}(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

$Y = -X \sim \binom{n}{-y} p^{-y} (1-p)^{n+y}$

- Revisiting Indicator Functions

- $$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1$$

$c \in \mathbb{N}_0$

- $$\sum_{x = \{-d,-d+1,\ldots,d-1,d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2d+1 & \text{if } c \geq d \\ 2c+1 & \text{if } c < d \end{cases} = 2\min\{c,d\}+1$$

- $$\int_{\mathbb{R}} \mathbb{1}_{x \in [-c,c]}\, dx = 2c$$

- $$\int_{-d}^{d} \mathbb{1}_{x \in [-c,c]}\, dx = \begin{cases} 2d & \text{if } c \geq d \\ 2c & \text{if } c < d \end{cases} = 2\min\{c,d\}$$

Poisson Revisited

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$P_{X_1|T}(x,t) = P(X_1 = x_1 \mid T = t) \neq \text{Poisson}(\lambda)$

$$P_{X_1|T} = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x,t-x)}{P_T(t)}$$

i.i.d

$$\overset{iid}{=} \frac{P_{X_1}(x)\,P_{X_2}(t-x)}{P_T(t)} = \frac{\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}\right)}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}}$$

$$= \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}\left(t, \frac{1}{2}\right)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$D = X_1 - X_2 = \overbrace{X_1}^{X} + \overbrace{(-X_2)}^{Y}$

$\text{Supp}[D] = \mathbb{Z}$

$P_X(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$P_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!} = P_X(-y)$

$X + (-X_2) = X + Y = \sum_{x \in \text{Supp}[X]} P_X(x)\, P_Y(d-x)\, \mathbb{1}_{d-x \in \text{Supp}[Y]}$

$= \sum_{x \in \mathbb{N}_0} \left(\dfrac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\dfrac{e^{-\lambda}\lambda^{-(d-x)}}{[-(d-x)]!}\right) \mathbb{1}_{d-x \in -\mathbb{N}_0}$

$= e^{-2\lambda} \sum_{x \in \mathbb{N}_0} \dfrac{\lambda^{2x-d}}{x!(d-x)!} \mathbb{1}_{x \ge d}$

$$= e^{-2\lambda} \begin{cases} \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \\ \displaystyle\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \ge 0 \end{cases}$$

1. Let $x' = x - d \Rightarrow x = x' + d$, 2. Let $d' = -d = |d|$

$$= e^{-2\lambda} \begin{cases} \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!} & \\ \displaystyle\sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,x'!} & \text{if } \underset{|d|}{\overset{\shortparallel}{d}} \ge 0 \end{cases}$$

$$= e^{-2\lambda} \begin{cases} \sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!} & \text{if } d<0 \\ \sum_{x'=0}^{\infty} \frac{\lambda^{2x'+d}}{(x'+d)!\,x!} & \text{if } d \geq 0 \end{cases}$$

$$= e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$ Modified Bessel function of the first kind

$$= e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda)$$

End of Midterm 1 material

Midterm 2 Material - Continuous Random Variables

Derivation of Exponential R.V

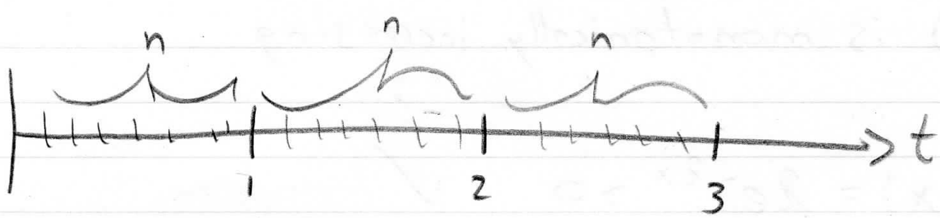
$X_1 \sim \text{Geom}(p) := \underbrace{(1-p)^x p \, \mathbb{1}_{x \in \mathbb{N}_0}}_{p(x)}$

- CDF

$F(x) := P(X \le x) = 1 - P(X > x) = 1 - (1-p)^{x+1}$

$F(10) = 1 - P(X \ge 11) = 1 - (1-p)$

We run n trials w/ each "original" period
X_n is the # of 0's until a success using the new trial def.



$X_n \sim (1-p)^{nx} p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}}$

Let $n \to \infty$, $p \to 0$, but $\lambda = np \Rightarrow p = \lambda/n$
(similar to derivation of Poisson from Binomial).

$F_{X_n}(x) = 1 - (1-p)^{nx+1}$

$p_{X_\infty}(x) := \lim_{n\to\infty} p_{X_n}(x) = \left[\underbrace{\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\right]^x \cdot \cancel{\lim_{n\to\infty} \frac{\lambda}{n}}^{0} \cdot \lim \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}} = 0$

$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = 1 - \left[\lim\left(1 - \frac{\lambda}{n}\right)^n\right]^x \lim 1 - \frac{\lambda}{n} \, \mathbb{1}_{x \ge 0}$

$= (1 - e^{-\lambda x}) \, \mathbb{1}_{x \ge 0}$

$\text{Supp}[X_\infty] = [0, \infty)$

$|\ \ | = \mathbb{R} \Rightarrow X_\infty$ is a continuous r.v

Has no PMF

CDF is valid

1. $\lim_{x \to -\infty} F(x) = 0$

2. $\lim_{x \to \infty} F(x) = 1$

3. $F(x)$ is monotonically increasing

$F'(x) = \lambda e^{-\lambda x} > 0$ ✓

$\lambda = np \in (0, \infty)$