Lec 14 Math 621 11/4/14 (1)

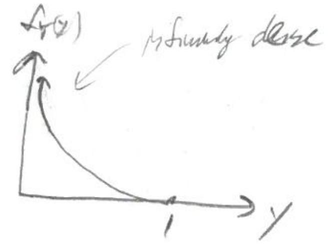
p155 Conditional Densities

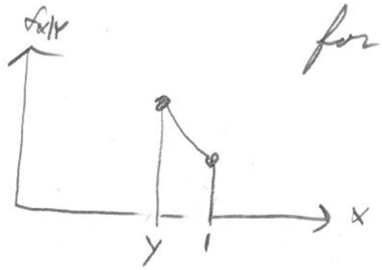
Recall $P(A|B) = \frac{P(AB)}{P(B)}$ is only defined if $P(B) \neq 0$

Let $X \sim U(0,1)$, $Y | X=x \sim U(0,x)$


more density, less density $\Rightarrow$ Not uniform triangle $f_{X,Y}$

Find $f_Y(y)$ and $f_{X|Y}(x,y)$




for y big... x is large

The formulas you need to solve these problems:

(I) marginal

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, dx$$

(II) Def. of cond prob. (Book is wrong!)

if $f_X(x) \geq 0$ Ignore this!

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_{Y|X}(y,x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\Rightarrow f_{X,Y}(x,y) = f_{X|Y}(x,y)\, f_Y(y) = f_{Y|X}(y,x)\, f_X(x)$$

(III) Bayes Rule

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(y,x)\, f_X(x)}{f_Y(y)}$$

(IV) Bayes Thm

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(y,x)\, f_X(x)}{\int_{\mathbb{R}} f_{X,Y}(x,y)\, dx} = \frac{f_{Y|X}(y,x)\, f_X(x)}{\int_{\mathbb{R}} f_{Y|X}(y,x)\, f_X(x)\, dx}$$

HW You can do the same with PMF's. Change integrals $\Rightarrow$ sums

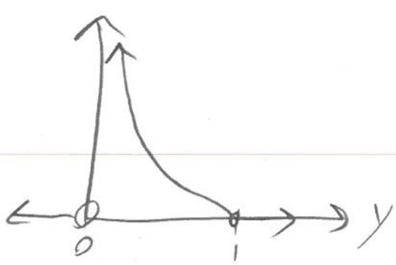
Back to the problem...

$$f_X(x) = \mathbb{1}_{x \in (0,1)}, \quad f_{Y|X}(y,x) = \frac{1}{x} \mathbb{1}_{y \in [0,x]} \quad \text{if } f_X(x) > 0$$

using I, II,
$$f_Y(y) = \int_{\mathbb{R}} f_{Y|X}(y,x) f_X(x)\,dx = \int_{\mathbb{R}} \underbrace{\frac{1}{x} \mathbb{1}_{y \in [0,x]} \mathbb{1}_{x \in (0,1)}}_{f_{X,Y}(x,y)} dx$$

$$\mathbb{1}_{y \in [0,x]} \mathbb{1}_{x \in (0,1)} = \mathbb{1}_{0 \le y \le x \le 1} = \mathbb{1}_{x \in [y,1)} \mathbb{1}_{y \in [0,1)}$$

$$= \int_0^1 \frac{1}{x} \mathbb{1}_{x \in [y,1)} \mathbb{1}_{y \in [0,1)} dx = \int_y^1 \frac{1}{x} dx\, \mathbb{1}_{y \in [0,1)} = \Big[\ln(x)\Big]_y^1 \mathbb{1}_{y \in (0,1)} = -\ln(y)\, \mathbb{1}_{y \in (0,1)}$$




using III...

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(y,x)\, f_X(x)}{f_Y(y)} = \frac{\overbrace{\frac{1}{x} \mathbb{1}_{y \in (0,x)} \mathbb{1}_{x \in (0,1)}}^{\mathbb{1}_{x \in [y,1]}}}{-\ln(y)\, \mathbb{1}_{y \in (0,1)}} = -\frac{1}{\ln(y)} \frac{1}{x} \mathbb{1}_{x \in [y,1]}$$

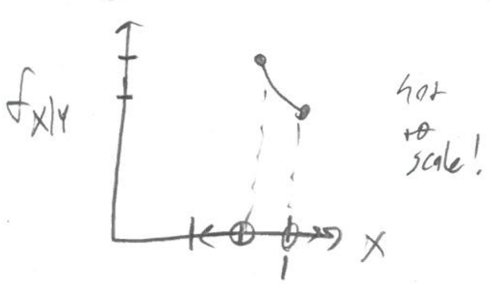
if $f_Y(y) > 0$

technical point: $f_{X|Y}$ undefined if $y \notin (0,1)$. why?

e.g. $y = \frac{3}{4} \Rightarrow \frac{1}{\ln(y)} \approx -3.5$

$$f_{X|Y}\left(x, \tfrac{3}{4}\right) = 3.5 \frac{1}{x} \mathbb{1}_{x \in [\frac{3}{4}, 1]}$$

$f_{X|Y}\left(\frac{3}{4}, \frac{3}{4}\right) \approx 4.6$, $\quad f_{X|Y}\left(1, \frac{3}{4}\right) \approx 3.5$



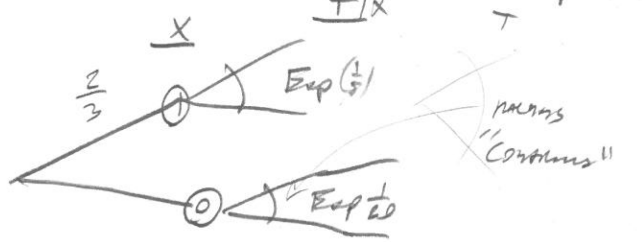
no traffic

**Mixture Distr.'s** $\frac{2}{3}$ of the time, download speeds are $T \sim \text{Exp}\left(\frac{1}{5}\right)$ (i.e. fast)

bad traffic

$\frac{1}{3}$ of the time, download speeds are $T \sim \text{Exp}\left(\frac{1}{20}\right)$ (i.e. slow). What is the distr. of $T$?

Let $X \sim \text{Bern}\left(\frac{2}{3}\right)$




$T \mid X=1 \sim \text{Exp}\left(\frac{1}{5}\right)$, $T \mid X=0 \sim \text{Exp}\left(\frac{1}{20}\right)$

$$f_T(t) = \sum_{Supp(X)} f_{T|X}(t,x)\, p_X(x) = \sum_{x \in \{0,1\}} \left(\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}\right)\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x}$$

$$\frac{2}{3}\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{3}\frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}$$

$$= \frac{1}{3}\left(\frac{1}{20} e^{-\frac{1}{20}t}\right) + \frac{2}{3}\left(\frac{1}{5} e^{-\frac{1}{5}t}\right)$$ $Y$ is a "mixture distr" or "mixture model"

$$= \frac{1}{3}\text{Exp}\left(\frac{1}{20}\right) + \frac{2}{3}\text{Exp}\left(\frac{1}{5}\right)$$

$Y|X$ is the model, $X$ is the mixing distr.

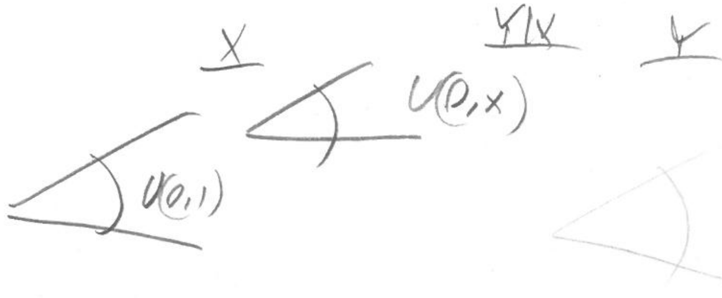
If the download took 25 seconds, what is the traffic distr?

$$p_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, p_X(x)}{f_T(t)} = \frac{\frac{2}{3}\cdot\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{3}\cdot\frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}}{\frac{1}{3}\frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3}\frac{1}{5} e^{-\frac{1}{5}t}}$$

$$p_{X|T}(x,25) = \text{Bern}(0.158) = \text{Bern}\left(\frac{\frac{2}{3}\frac{1}{5} e^{-\frac{1}{5}t}}{\frac{1}{3}\frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3}\frac{1}{5} e^{-\frac{1}{5}t}}\right)$$

Prob of no traffic only 16% ... makes sense!

If $X \sim U(0,1)$, $Y|X=x \sim U(0,x)$




If the mixing distr is continuous, Y is also called a compound distr.

$$\overbrace{f_Y(y)}^{\text{mixture model}} = \int_{supp(X)} \overbrace{f_{Y|X}(y,x)}^{\text{model}} \overbrace{f_X(x)}^{\text{mixing distr}} \, dx \quad \text{or}$$

p156/7 Let $Y|X=x \sim \text{Poisson}(x)$, $X \sim \text{Gamma}(\alpha,\beta)$. Find $f_Y(y)$.

$$p_Y(y) = \int_0^\infty \left(\frac{e^{-x} x^y}{y!} \mathbb{1}_{y\in\mathbb{N}_0}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\right) dx$$

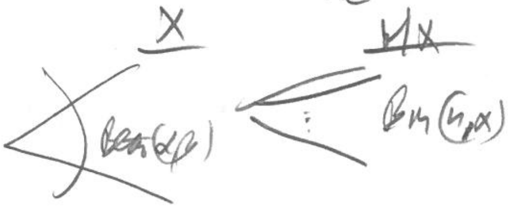

$$\propto \frac{1}{y!}\int_0^\infty x^{y+\alpha-1} e^{-(1+\beta)x} \, dx \, \mathbb{1}_{y\in\mathbb{N}_0} = \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(1+\beta)^{y+\alpha}} \mathbb{1}_{y\in\mathbb{N}_0} \overset{hw}{\propto} \text{ExtNegBin}\left(\alpha, \frac{\beta}{1+\beta}\right)$$

Let $Y|X=x \sim \text{Bin}(n,x)$, $X \sim \text{Beta}(\alpha,\beta)$

$$p_Y(y) = \int_0^1 \left(\binom{n}{y} x^y (1-x)^{n-y}\right)\left(\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\right) dx = \frac{\binom{n}{y}}{B(\alpha,\beta)}\left(\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1} \, dx\right)$$

$\mathbb{1}_{y\in\{0,\dots,n\}}$ inside here

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \text{BetaBin}(n,\alpha,\beta)$$

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$Y|X=x \sim \text{Exp}(x)$, $X \sim \text{Gamma}(\alpha,\beta) \overset{hw}{\Rightarrow} Y \sim \text{Lomax}(\beta,\alpha)$ another generalization of exp. Another mixing time/survival distr.