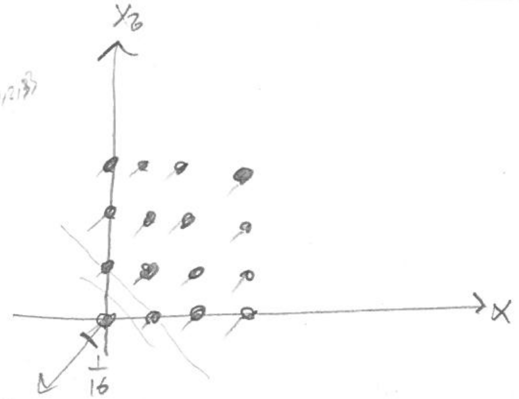
Lec 6 Math 621 9/16/14

$$X_1, X_2 \stackrel{iid}{\sim} U(\{0,1,2,3\}) := \begin{cases} 0 & \text{wp } \frac{1}{4} \\ 1 & \text{wp } \frac{1}{4} \\ 2 & \text{wp } \frac{1}{4} \\ 3 & \text{wp } \frac{1}{4} \end{cases} = \frac{1}{4}\mathbb{1}_{x\in\{0,1,2,3\}}$$

In general $X \sim U(A) = \frac{1}{|A|}\mathbb{1}_{x\in A}$

Param space? A is finite $\quad Supp(X) = A$

$p_{X_1,X_2}(x_1,x_2) = \frac{1}{16}\mathbb{1}_{x_1\in\{0,1,2,3\}}\mathbb{1}_{x_2\in\{0,1,2,3\}}$



$T = X_1 + X_2 \quad Supp(T) = \{0,1,\ldots,6\}$

$$p(t) = \sum_{x_1\in\mathbb{R}}\sum_{x_2\in\mathbb{R}} p_{X_1,X_2}(x_1,x_2)\mathbb{1}_{x_2 = t - x_1}$$

$p(1) = \sum\sum p_{X_1,X_2}\mathbb{1}_{x_2=1-x_1} = \frac{1}{16}+\frac{1}{16}$ two dots

$p(0.5) = \sum\sum p_{X_1,X_2}\mathbb{1}_{x_2=0.5-x_1} = 0$

$p(2) = 3\left(\frac{1}{16}\right)$

$p(3) = 4\left(\frac{1}{16}\right)$

$p(4) = 3\left(\frac{1}{16}\right)$

$\vdots$

Let's ask another question $\rightarrow$ a single transformation of the r.v. $g(X)$

let $Y = -X \sim$? Well... when $X=0 \Rightarrow Y=0$, $X=1 \Rightarrow Y=-1$, $X=2 \Rightarrow Y=-2$, $X=3 \Rightarrow Y=-3$

$\Rightarrow Y \sim U(\{0,-1,-2,-3\})$. $Supp(Y) = -Supp(X)$. What is rule in general?

$$p_Y(y) = P(Y=y) = P(-Y=-y) = P(X=-y) = p_X(-y)$$

True for all discrete r.v.'s!

let $z' = -z$

$$Supp(Y) = \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z' : p_X(z') > 0\} = -\{z' : p_X(z') > 0\} = -Supp(X)$$

e.g. $X \sim Bin(n,p) := \binom{n}{x}p^x(1-p)^{n-x}$, $Y = -X \sim \binom{n}{-y}p^{-y}(1-p)^{n+y}$

Lec 6 9/10/19 Review indicator functions

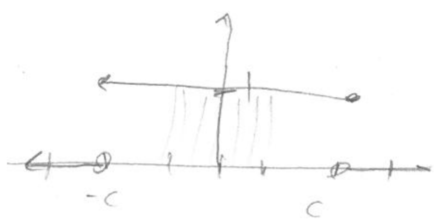
Consider $\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-1,1]} = 3$

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1 \quad , \quad c \in \mathbb{N}_0$$

$$\sum_{x \in \{-d,-d-1,\dots,0,\dots,d-1,d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2c+1 & \text{if } d \geq c \\ 2d+1 & \text{if } d < c \end{cases} = 2\min\{c,d\}+1$$

$d \in \mathbb{Z}$

$$\int_{\mathbb{R}} \mathbb{1}_{x \in [-c,c]}\, dx = 2c$$



$$\int_{-d}^{d} \mathbb{1}_{x \in [-c,c]}\, dx = 2\min\{c,d\}$$

Review Poisson and conditional distr.

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$$p_{X_1|T}(x,t) := \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} = \frac{p_{X_1}(x)\, p_{X_2}(t-x)}{p_T(t)} = \frac{\frac{e^{-\lambda}\lambda^x}{x!}\frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

$$= \binom{t}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{t-x} = \text{Bin}\left(t, \tfrac{1}{2}\right)$$

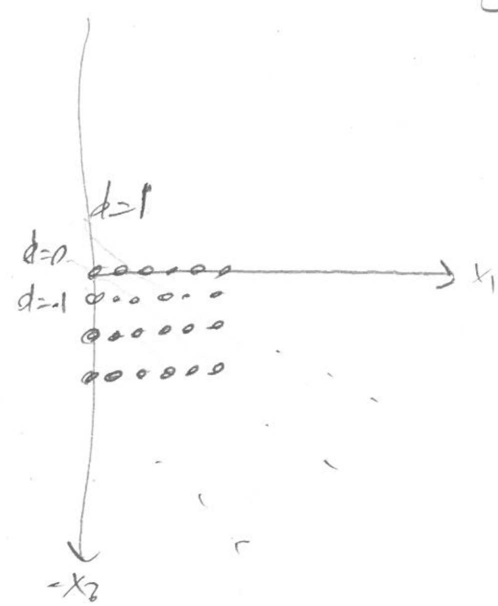
Why does this make sense? If we know $T=10$, this means 10 successes shared between X_1, X_2. They are equally likely to come from both since $\lambda_1 = \lambda_2$. Not so mysterious.

$X \sim \text{Poisson}(\lambda)$, $Y = -X \sim \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$

let $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$$D = X_1 - X_2 = X_1 + (-X_2)$$

$Supp(D) = Supp(X_1) + Supp(-X_2) = \mathbb{Z}$ nice!



$$p_D(d) = \sum_{x \in Supp(X_1)} p_{X_1}(x)\, p_{X_2}(x-d)\, \mathbb{1}_{d-x \in Supp(-X_2)}$$

$$= \sum_{x \in \{0,1,\dots\}} \left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{x-d}}{(x-d)!}\, \underbrace{\mathbb{1}_{d-x \in \{0,-1,-2,\dots\}}}\right)$$

$$\mathbb{1}_{x-d \in \{0,1,\dots\}}$$

$$\mathbb{1}_{x \ge d}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x \ge d} = e^{-2\lambda} \begin{cases} \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \le 0 \\ \sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d > 0 \end{cases}$$

let $d' = -d$ $\quad \sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!} \overset{d'=|d|}{=} \sum_{x=0}^{\infty} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$

let $x' = x-d$ $\Rightarrow x = x'+d$ $\quad \sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!(x'+d-d)!} = \sum_{x'=0}^{\infty} \frac{\lambda^{2x'+d}}{x'!(x'+d)!} \overset{d=|d|}{=} \sum_{x'=0}^{\infty} \frac{\lambda^{2x'+|d|}}{x'!(x'+|d|)!}$

$$= e^{-2\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}} = e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda)$$

(1946)

"Modified Bessel := $I_{|d|}(2\lambda)$
Function of the First Kind"
Sol'n to differential eq.

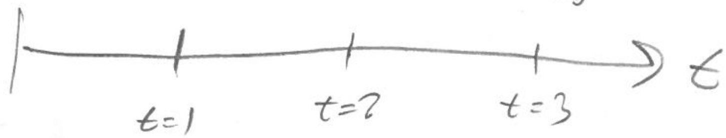
Models pt. spreads in sports, differences in photon noise + more...

Mid I ↑

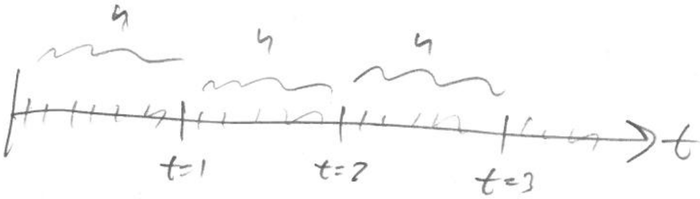
Mid II ↓

Recall $X_1 \sim \text{Geom}(p) := (1-p)^x p \mathbb{1}_{x\in}$, $F_{X_1}(x) = P(X_1 \le x) = 1 - P(X_1 > x)$ (all failures up to x)

$\text{Supp}(X_1) = \{0, 1, \dots\}$ $\quad \overset{\|}{p_{X_1}(x)}$ $\quad = 1 - (1-p)^x$



let n experiments be performed between each original time period, let this r.v. be denoted X_n



$\text{Supp}(X_n) = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots\}$

$p_{X_n}(x) = (1-p)^{nx} p \mathbb{1}_{x\in\dots}$ If $n \to \infty$ and $p \to 0$ s.t. $\lambda = pn \Rightarrow p = \frac{\lambda}{n}$

$F_n(x) = 1 - (1-p)^{nx} \mathbb{1}_{x\in\dots}$

$\Rightarrow p_{X_n}(x) = \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x\in\dots}$ $\quad \text{Supp}[X_\infty] = [0, \infty)$ a Continuous Smear

$F_{X_n}(x) = 1 - \left(1 - \frac{\lambda}{n}\right)^{nx} \mathbb{1}_{x\in\dots}$

No holes!

Not discrete

$p_{X_\infty}(x) = \lim_{n\to\infty} p_{X_n}(x) = \lim \underbrace{\left(\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda x}} \underbrace{\lim \frac{\lambda}{n}}_{0} \mathbb{1}_{x\in\dots} = 0 \quad \forall x$

$|\text{Supp}(X_\infty)| = |\mathbb{R}|$

$\sum_{x\in[0,\infty)} p(x) = 0 \ne 1$. Thus this is not a valid PMF!

$F_{X_\infty}(x) = \lim_{n\to\infty} 1 - \left(1-\frac{\lambda}{n}\right)^{nx} \mathbb{1}_{x\in\dots} = (1 - e^{-\lambda x}) \mathbb{1}_{x\in\dots}$ Valid?

I $\quad F_{X_\infty}(\underset{\overset{\|}{0}}{\min\{\text{Supp}(X_\infty)\}}) = 0$ ✓

II $\quad \lim_{x\to\infty} F_{X_\infty}(x) = 1 - \lim e^{-\lambda x} = 1 - 0 = 1$ ✓