Lecture 9:

Poisson Process:

$T_k \sim \text{Erlang}(k, \lambda)$

$N \sim \text{Poisson}(\lambda)$

$P(T_k > 1) = P(N \leq k-1)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$

r.v's for counting & waiting:

	Fixed time, count #	Fixed #, measure time
Discrete	Bern/Bin	Geom/NegBin
Continuous	Poisson	Exp/Erlang

Experiments occur in discrete time.

what's the prob. of zero successes by time $t=50$ if the prob. of success is 0.1?

$N \sim \text{Bin}(50, 0.1)$ So, $P(N=0) = P(T>49)$

$T \sim \text{Negbin}(1, 0.1)$ failed 50 times

What's the prob. of $\leq k$ successes by the time t if the prob. success is p?

$N \sim \text{bin}(t, p)$ So, $P(N \leq k) = P(T > t-k-1) = P(T \geq t-k)$

$T \sim \text{Negbin}(k+1, p)$

$\Rightarrow F_N(k) = 1 - F_T(t-k-1)$

$$\sum_{i=0}^{k} \binom{t}{i} p^i (1-p)^{t-i} = 1 - \sum_{i=0}^{t-k-1} \binom{k+i}{k} (1-p)^i p^k$$

- $T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0} = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$ (Gamma)

↳ Gamma $X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$T \sim \text{Negbin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

$\binom{k+t-1}{k-1} = \frac{(k+t-1)!}{(k-1)!\, t!}$

$\Rightarrow \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

↳ extended negbin

Think about Gamma as waiting time.

· Consider $k \in (0,\infty)$

Transformation of Discrete r.v's:

$X \sim Bern(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

$Y = X+3 \sim \begin{cases} 3 & \text{w.p } 1-p \\ 4 & \text{w.p } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y\in\{3,4\}}$

· $X = g^{-1}(Y) = Y-3$

If g is 1 to 1,

Def: $P_Y(y) = P(Y=y) = P(g(X)=y) = P(X = g^{-1}(y)) = P_X(g^{-1}(y))$

· $X \sim U(\{1,2,\cdots,10\}) = 0.1 \, \mathbb{1}_{x \in \{1,2,\cdots 10\}}$

· $Y = g(X) = \min\{X,3\}$

$P(X)$	X	Y	$P(Y)$
0.1	1	1	0.1
0.1	2	2	0.1
0.1	3	3	0.8
0.1	4	3	
⋮	⋮	⋮	
0.1	10	3	

General formula:

If g is invertible

$\Rightarrow P_Y(y) = \sum_{\{x: y = g(x)\}} P_X(x) = P_X(g^{-1}(y))$

· $X \sim bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x\in\{0,1,\ldots,n\}}$

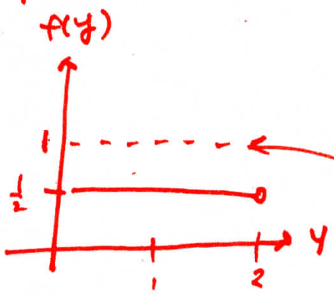
$Y = X^3 \sim \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y}\in\{0,\cdots,n\}}$

$g^{-1}(y) = \sqrt[3]{y}$

• Transformation of Continuous r.v's:

· $X \sim U(0,1)$

$Y = 2X = g(X) \Rightarrow X = \frac{Y}{2} = g^{-1}(Y)$

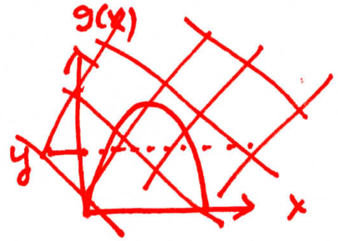


which is

Is $f_Y(y) \overset{?}{=} f_X(g^{-1}(y)) = \mathbb{1}_{\frac{y}{2}\in[0,1]} = \mathbb{1}_{y\in[0,2]}$ No

So, $f_Y(y) = \frac{1}{2} \cdot \mathbb{1}_{\frac{y}{2}\in[0,1]} = \frac{1}{2} \cdot \mathbb{1}_{y\in[0,2]}$

We know that if g is 1-1 $\Leftrightarrow$ g is strictly increasing or strictly decreasing



Case:

Ⓐ strictly increasing:

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

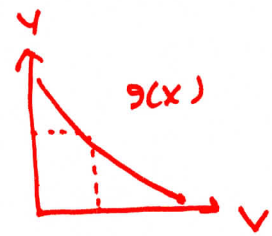
To get PDF, just differentiate.

$$f_Y(y) = \frac{d}{dy}[F_X(g^{-1}(y))]$$

positive

$$= F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$
$$= f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|$$

Ⓑ strictly decreasing:



$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[1 - F_X(g^{-1}(y))]$$

negative

$$= -F_X'(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$
$$= f_X'(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|$$

Summary: PDF is the same for both, but CDF is different for strictly increasing or decreasing.

• $Y = g(X) = aX + c$ where $a, c \in \mathbb{R}$ constants
(the linear transformation/shifts and/or scales)

→ inverse f-n

$$\boxed{\begin{aligned} &g^{-1}(y) = \\ &y = aX + c \\ &X = \frac{y-c}{a} \end{aligned}}$$

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} \frac{y-c}{a} = \frac{1}{a}$$

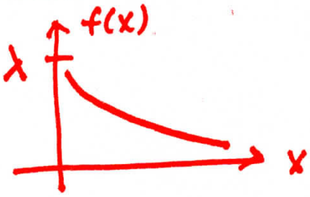
$$f_Y(y) = f_X\left(\frac{y-c}{a}\right) \cdot \frac{1}{a}$$

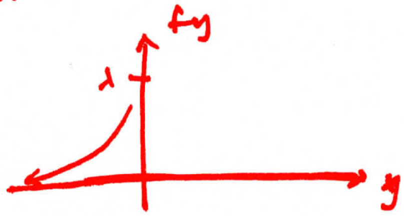
• $Y = X + c \sim f_X(y-c)$

• $Y = -X \sim f_X(-y)$

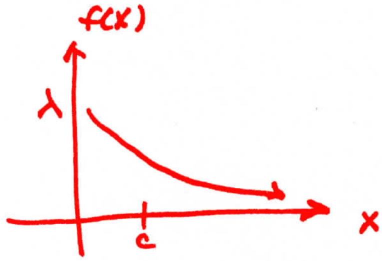
• $Y = aX \sim f_X\left(\frac{y}{a}\right) \cdot \frac{1}{|a|}$

• $\boxed{X \sim Exp(\lambda)}$ & $Y = -X \sim f_X(-y) = \lambda e^{-\lambda(-y)} \mathbb{1}_{-y \in (0,\infty)}$
$= \lambda e^{\lambda y} \mathbb{1}_{y \in (-\infty, 0)}$





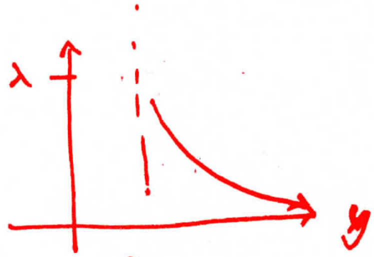
• $Y = X + c$, $c > 0$



$$Y = X + c \sim f_X(y-c) = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y-c \in (0,\infty)}$$

$$= \left(e^{\lambda c}\right) \lambda e^{-\lambda y} \mathbb{1}_{y \in (c,\infty)}$$

↑ scale



$Exp\left(\frac{\lambda}{a}\right)$

$\parallel$

• $Y = aX \overset{a>0}{\sim} f_X\left(\frac{y}{a}\right) \frac{1}{a} \mathbb{1}_{\frac{y}{a} \in (0,\infty)} = \frac{\lambda}{a} e^{-\frac{\lambda}{a} y} \mathbb{1}_{y \in (0,\infty)}$