Lec 17 Math 621 10/30/19

Arbitrary Transformations!

$X \sim \text{Beta}(\alpha,\beta) = \underbrace{\frac{1}{B(\alpha,\beta)}}_{c} \underbrace{x^{\alpha-1}(1-x)^{\beta-1}}_{k(x)} \mathbb{1}_{x\in(0,1)}$

$\propto x^{\alpha-1}(1-x)^{\beta-1}$

Let $g: \mathbb{R}^n \to \mathbb{R}^n$ and 1:1

Let $\vec{X}, \vec{Y}$ be r.v. vectors both with dimension n and $\vec{Y} = g(\vec{X})$.

Given $f_{\vec{X}}(\vec{x})$, find $f_{\vec{Y}}(\vec{y})$. This is the generalization of what we did before.

Recall what a multi-dimensional function does:

$$Y_1 = g_1(X_1, \ldots, X_n)$$
$$Y_2 = g_2(X_1, \ldots, X_n)$$
$$\vdots$$
$$Y_n = g_n(X_1, \ldots, X_n)$$

Since its 1:1 $\exists\, h = g^{-1}$ that "undoes" the function: $\vec{X} = h(\vec{Y})$

$$X_1 = h_1(Y_1, \ldots, Y_n)$$
$$X_2 = h_2(Y_1, \ldots, Y_n)$$
$$\vdots$$
$$X_n = h_n(Y_1, \ldots, Y_n)$$

The multivariate change-of-variables formula is:

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\,|J_h(\vec{y})|$$

where $J_h := \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$ the "Jacobian determinant"

Recipe

① Find a clever g

② Find h

③ Compute J_h

④ Plug and Chug

⑤ Integrate

Let's first verify the convolution formula

$$T = X_1 + X_2 \quad \text{a function of } X_1 \ \& \ X_2$$

① let $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, let $Y_2 = X_2 = g_2(X_1, X_2)$ we will see why soon...

② $\Rightarrow X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$

$\Rightarrow X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $$J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$$

④ $$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 - y_2, y_2)\,|1|$$

⑤ we are looking for f_T which is only f_{Y_1} so.

$$f_{Y_1}(y) = \int_{\mathbb{R}} f_{Y_1,Y_2}(\vec{y})\, dy_2$$

exactly the conv. form!

if $X_1, X_2 \overset{ind}{\sim}$

$X_1, X_2 \overset{iid}{\sim}$

$$\Rightarrow f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\, du = \int_{\mathbb{R}} f_{X_2}(u)\, f_{X_1}(t-u)\, du = \int_{\mathbb{R}} f(u)\, f(t-u)\, du$$

OLD

$$= \int_{Supp(X_2)} f_{X_2}(u)\, f_{X_1}(t-u)\, \mathbb{1}_{X_1 \in Supp(X_1)}\, du$$

Let's do the ratio!

$$R = \frac{X_1}{X_2} \sim f_R(r) = ?$$

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\, |y_2|$

⑤ $f_R(r) = \int_{\mathbb{R}} f_{X_1, X_2}(ur, u)\, |u|\, du \overset{\text{if } X_1, X_2 \text{ ind}}{=} \int_{\mathbb{R}} f_{X_1}(ur)\, f_{X_2}(u)\, |u|\, du \overset{\text{if } X_1, X_2 \text{ iid}}{=} \int_{\mathbb{R}} f(ur) f(u) |u|\, du$

$= \int_{\text{Supp}(X_1)} \underbrace{f_{X_1}(ur)\, f_{X_2}(u)}_{\text{OLD}}\, \mathbb{1}_{u \in \text{Supp}[X_2]}\, |u|\, du$

How about $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$, $Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 (X_1 + X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{bmatrix} = y_2(1 - y_1) - {-y_1 y_2} = y_2 - y_1 y_2 + y_1 y_2 = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\, |y_2|$

if X_1, X_2 iid ↓ ... if X_1, X_2 iid ↓ ... [illegible]

(5) $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ur, u-ur)\,|u|\,du = \int_{\mathbb{R}} f_{X_1}(ur) f_{X_2}(u-ur)\,|u|\,du = \int_{\mathbb{R}} f(ur) f(u-ur)\,|u|\,du$

$= \int_{Supp(X_1)} f_{X_1}(ur) f_{X_2}(u-ur)\, \mathbb{1}_{u-ur \in Supp(X_2)}\,|u|\,du$

$X_1 \sim Gamma(\alpha_1, \beta)$ indep. of $X_2 \sim Gamma(\alpha_2, \beta)$

$R \sim \frac{X_1}{X_1+X_2} \sim \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ur)^{\alpha_1 - 1} \cancel{e^{-\beta ur}}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u-ur)^{\alpha_2-1} e^{-\beta(u-ur)} \mathbb{1}_{u-ur \in (0,\infty)}\right) u\,du$ $\quad e^{-\beta u}\cancel{e^{+\beta ur}}$

$\Rightarrow Supp(R) = (0,1)$

$\propto \int_0^\infty (ur)^{\alpha_1-1}(u-ur)^{\alpha_2-1} e^{-\beta u}\, \mathbb{1}_{u-ur\in(0,\infty)}\, u\,du$

$= r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta u}\, \mathbb{1}_{u(1-r)\in(0,\infty)}\,du}_{\text{not a function of } r} \propto r^{\alpha_1-1}(1-r)^{\alpha_2-1} \propto Beta(\alpha_1,\alpha_2)$

$u(1-r)\in(0,\infty)$: $u \in (0,\infty)$ since $r \in (0,1)$.

% of time to wait for one event out of time for both events is Beta-distr.

p152 $X_1 \sim Gamma(\alpha_1,\beta)$ indep. of $X_2 \sim Gamma(\alpha_2,\beta)$

$Supp(R) = (0,\infty)$

$R = \frac{X_1}{X_2} \sim \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ur)^{\alpha_1-1} e^{-\beta ur}\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2-1} e^{-\beta u}\, \mathbb{1}_{u\in(0,\infty)}\right) u\,du$

$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (ur)^{\alpha_1-1} u^{\alpha_2} e^{-\beta u(r+1)}\,du$

$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u}\,du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{(\beta(r+1))^{\alpha_1+\alpha_2}}} = \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}}\, \mathbb{1}_{r\in(0,\infty)} = BetaPrime(\alpha_1,\alpha_2)$