10/30 $X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \propto x^{\alpha-1}(1-x)^{\beta-1}$

Lecture 13

- $Y = g(X)$ and g is 1:1 $\rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$

$g: \mathbb{R}^n \to \mathbb{R}^n$ and 1:1 let $\vec{X}, \vec{Y}$ be r.v. vectors dim n

$\vec{Y} = g(\vec{X})$

$Y_1 = g_1(X_1, \dots, X_n)$

$Y_2 = g_2(X_1, \dots, X_n)$

$\vdots$

$Y_n = g_n(X_1, \dots, X_n)$

$g_i : \mathbb{R}^n \to \mathbb{R}$

Because g is 1:1, it has an inverse, h

$X_1 = h_1(Y_1, \dots, Y_n)$

$X_2 = h_2(Y_1, \dots, Y_n)$

$\vdots$

$X_n = h_n(Y_1, \dots, Y_n)$

Then ..... (from Math 202)

$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\, |J_n(\vec{y})|$

Mult. change of variables

"the Jacobian determinant"

$$J_n := \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

$(\det[5] = 5)$

~ Recipe:

1. Find a clever g.
2. Find its inverse, h.
3. Compute J_h
4. Plug h, J_h into masta formula
5. Integrate to get target

- $T = X_1 + X_2$: target

① let $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$

let $Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - (0 \cdot -1) = 1$

④ $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)|1|$

⑤ Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$

$\hookrightarrow f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\, du$

$=$ if X_1, X_2 ind $\int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\, du \overset{X_1, X_2 \text{ iid}}{=} \int_{\mathbb{R}} f(t-u) f(u)\, du$

$= \int_{Supp[X_1]} f_{X_1}(t-u) f_{X_2}(u) \mathbb{1}_{u \in Supp[X_2]}\, du$

- $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

① let $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$

let $Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(X_1, X_2)$

$X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} Y_2 & Y_1 \\ 0 & 1 \end{bmatrix} = (Y_2)(1) - (Y_1)(0) = Y_2$

④ $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2, y_2)\,|y_2|$

⑤ $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u)\,|u|\,du$

if X_1, X_2 ind $= \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\,|u|\,du \overset{\text{Old style}}{=} \int_{supp[X_1]} f_{X_1}(ru) f_{X_2}(u) \mathbb{1}_{u \in supp[X_2]} |u|\,du$

if X_1, X_2 iid $= \ldots\ldots$

- $R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ?$

① let $Y_1 = \frac{X_1}{X_1+X_2}$ let $Y_2 = X_1 + X_2$

② $X_1 = Y_1(X_1+X_2) = Y_1Y_2 = h_1(Y_1,Y_2)$

$X_2 = Y_2 - X_1 = Y_2 - Y_1Y_2 = h_2(Y_1,Y_2)$

③ $J_h = \det\begin{bmatrix} Y_2 & Y_1 \\ -Y_2 & 1-Y_1 \end{bmatrix} = Y_2(1-Y_1) - Y_1(-Y_2) = Y_2 - Y_1Y_2 + Y_1Y_2 = Y_2$

④ $f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2, y_2 - y_1y_2)\,|y_2|$

⑤ $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u-ru)\,|u|\,du$

if X_1, X_2 ind $\int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u-ru)|u|du \overset{\text{old style}}{=} \int_{supp[X_1]} f_{X_1}(ru)\, f_{X_2}(u-ru) \mathbb{1}_{u-ru \in supp[X_2]} |u|\,du$

if X_1, X_2 iid $\ldots\ldots$

- let $X_1 \sim Gamma(\alpha_1, \beta)$ ind. of $X_2 \sim Gamma(\alpha_2, \beta)$

$R = \frac{X_1}{X_1+X_2}$ $supp[R] = [0,1]$

$R \sim \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1-1} e^{-\beta ru}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u-ru)^{\alpha_2-1} \underbrace{e^{-\beta(u-ru)}}_{e^{-\beta u} e^{\beta ru}} \overset{\text{always}=1}{\underbrace{\mathbb{1}_{u-ru \in (0,\infty)}}_{u(1-r) \in (0,\infty)}}\right) \overset{\text{bec 0 to }\infty}{u}\, du$

since $r \in (0,1)$
$\hookrightarrow 1-r \in (0,1)$
$u \in (0,\infty)$

$\propto \int_0^\infty r^{\alpha_1-1} u^{\alpha_1-1} u^{\alpha_2-1} (1-r)^{\alpha_2-1} e^{-\beta u}\, u\, du$

$= r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2+1} e^{-\beta u}\, du}_{\text{not a function of } r} \propto r^{\alpha_1-1}(1-r)^{\alpha_2-1} \propto Beta(\alpha_1, \alpha_2)$

- X_1, X_2 same as previous example

$R = \frac{X_1}{X_2} \sim \int_{supp[X_1]} f_{X_1}(ru) f_{X_2}(u) |u| \, du$

supp[R] $= (0,\infty)$

$$= \int_0^\infty \left(\frac{\beta^{\alpha-1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1-1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2-1} e^{-\beta u} \underbrace{\mathbb{1}_{u\in(0,\infty)}}_{=1\forall u}\right) u \, du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty r^{\alpha_1-1} u^{\alpha_1-1} e^{-\beta u(r+1)} u^{\alpha_2-1} u \, du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u} \, du}_{\text{gamma} \times \text{constant}}$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{(\beta(r+1))^{\alpha_1+\alpha_2}} = \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r\in(0,\infty)} = \text{BetaPrime}(\alpha_1,\alpha_2)$$

~ **Conditional Densities** p.155

consider $X \sim U(0,1)$

$Y|X=x \sim U(0,x)$