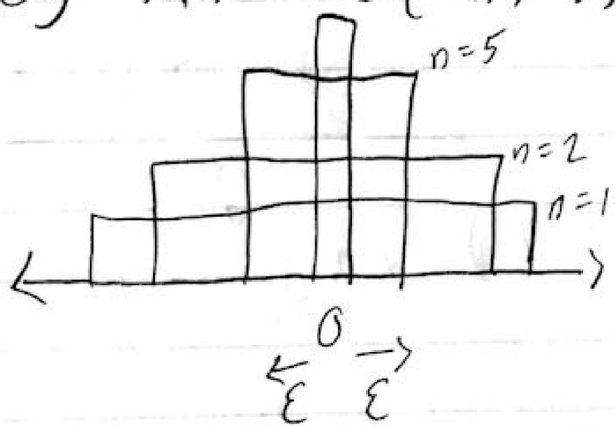
12/9/19

Convergence in P to a constant

$X_n \xrightarrow{P} c$ this means for $\forall \varepsilon > 0$

$$\lim_{n \to \infty} P(|X_n - c| \ge \varepsilon) = 0$$

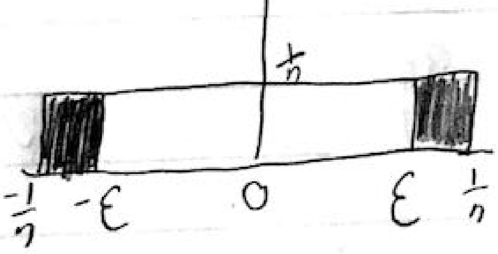
eg $X_n \sim U(-\frac{1}{n}, \frac{1}{n})$



WTS $X_n \xrightarrow{P} 0$

· $\lim P(|X_n| \ge \varepsilon)$

· $\lim P((X_n \le \varepsilon) + P(X \ge \varepsilon))$



$$= \lim \left((-\varepsilon - -\tfrac{1}{n})\tfrac{n}{2}\,\mathbb{I}_{-\varepsilon \ge -\frac{1}{n}} + (\tfrac{1}{n} - \varepsilon)\tfrac{n}{2}\,\mathbb{I}_{\varepsilon \le \frac{1}{n}} \right)$$

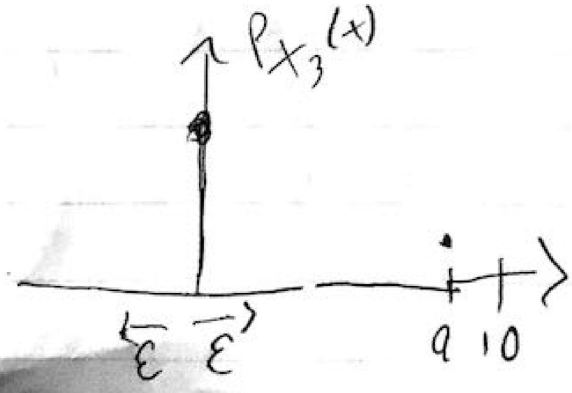
$$= \lim 2(\tfrac{1}{n} - \varepsilon)\tfrac{n}{2}\,\mathbb{I}_{\varepsilon \le \frac{1}{n}}$$

$$= \lim (1 - n\varepsilon)\,\mathbb{I}_{\varepsilon < \frac{1}{n}}$$

$= 0$ ✓

$$X_n \sim \begin{cases} n^2 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1 - \frac{1}{n} \end{cases}$$

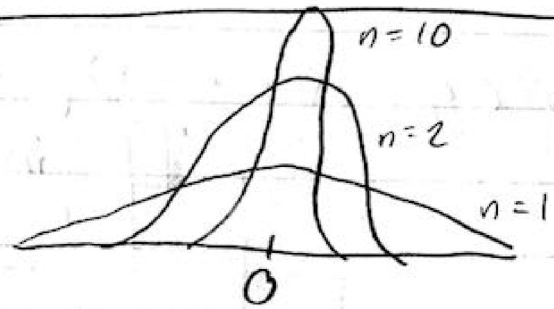
$$X_3 \sim \begin{cases} 9 & \text{wp } \frac{1}{3} \\ 0 & \text{wp } \frac{2}{3} \end{cases}$$



12/9/19

$$\lim P(|X_n| \ge \varepsilon)$$
$$= \lim \frac{1}{n} \underbrace{\mathbb{1}_{\varepsilon < n}}_{\text{always 1 with small } \varepsilon\text{'s}}$$

$$X_n \sim N(0, \tfrac{1}{n})$$



skinnier bell curve

$$\lim P(|X_n| \ge \varepsilon) = \lim \int_{\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi \frac{1}{n}}} e$$ fill in later

<u>Chebyshev Ineq</u>

$$P(|X - m| \ge a) \le \frac{\sigma^2}{a}$$

$$\lim P(|X_n| \ge \varepsilon) \le \lim \frac{1}{n\varepsilon^2} = 0 \checkmark$$

12/9/19

Let $X_1, X_2, \ldots,$ iid with μ, σ^2

Consider $\bar{X}_1 = \frac{X_1}{1}, \bar{X}_2 = \frac{X_1 + X_2}{2}, \ldots \bar{X}_n = \frac{X_1 + \ldots X_n}{n}$

$$\lim P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} \lim \operatorname{Var}[\bar{X}_n] = \frac{1}{\varepsilon^2} \lim \frac{\sigma^2}{n}$$
$$= 0$$

for very large values $\longrightarrow \bar{X}_n \xrightarrow{P} \mu$

- Weak law of large numbers (WLLN)

<u>Convergence in L^r norm to a c</u>

$r \geq 1$ $\quad X_n \xrightarrow{L^r} c$ also means $\lim E[|X_n - c|^r] = 0$

e.g. $X_n \xrightarrow{L^1} c$ this means $\lim E[|X_n - c|] = 0$

"convergence in mean"

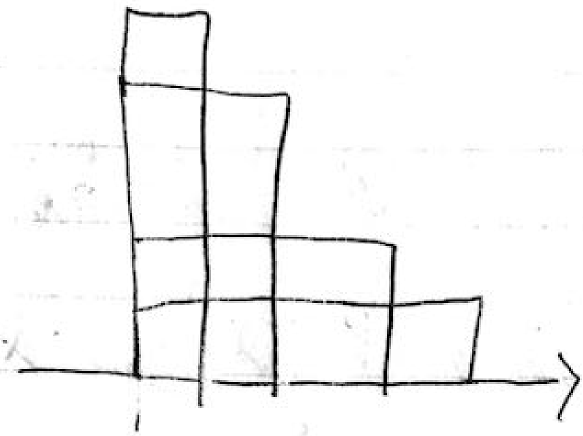
$X_n \xrightarrow{L^2} c$ this means $\lim E[(X_n - c)^2] = 0$

"mean square convergence"

$X_n \sim U(0, \frac{1}{n}) = n \, \mathbb{1}_{x \in [0, \frac{1}{n}]}$

WTS $X_n \xrightarrow{L^r} 0 \quad \forall r \geq 1$

12/9/19



$$\lim E[|X_n - c|^r] = \lim E[X_n^r]$$
$$= \lim \int_{\mathbb{R}}$$
$$= \lim n \int_0^{\frac{1}{n}} x^n dx$$
$$= \lim n \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{1}{n}} = \lim n \frac{1}{n+1} ($$

$$= \frac{1}{n+1} \lim \frac{1}{n^r} = 0$$

Let $r, s > 0$, $s > r$

WTS $X_n \xrightarrow{L^s} c \implies X_n \xrightarrow{L^r} c$

$$\lim \underbrace{E[|X_n - c|^r]}_{\geq 0} \overset{\text{Holders}}{\leq} \lim \left(E[|X_n - c|^s]\right)^{r/s}$$
$\forall r \geq 1$
$$= \left(\lim E[|X_n - c|^s]\right)^{r/s} = 0^{r/s} = 0$$

12/9/19

$X_n \xrightarrow{L^r} c \implies X_n \xrightarrow{p} c$ but not the converse

$$\lim P(|X_n - c| \geq \varepsilon) = \lim P(|X_n - c|^r \geq \varepsilon^r)$$

$$\leq \lim \frac{E[|X_n - c|^r]}{\varepsilon^r} = \frac{1}{\varepsilon^r} \lim E[|X_n - c|^r]$$

$$= 0 \checkmark$$

$$X_n \sim \begin{cases} n^2 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1 - \frac{1}{n} \end{cases} \qquad E[X_1] = 0(1 - \tfrac{1}{n}) + n^2\left(\tfrac{1}{n}\right)$$

$$= 0 + n = n$$

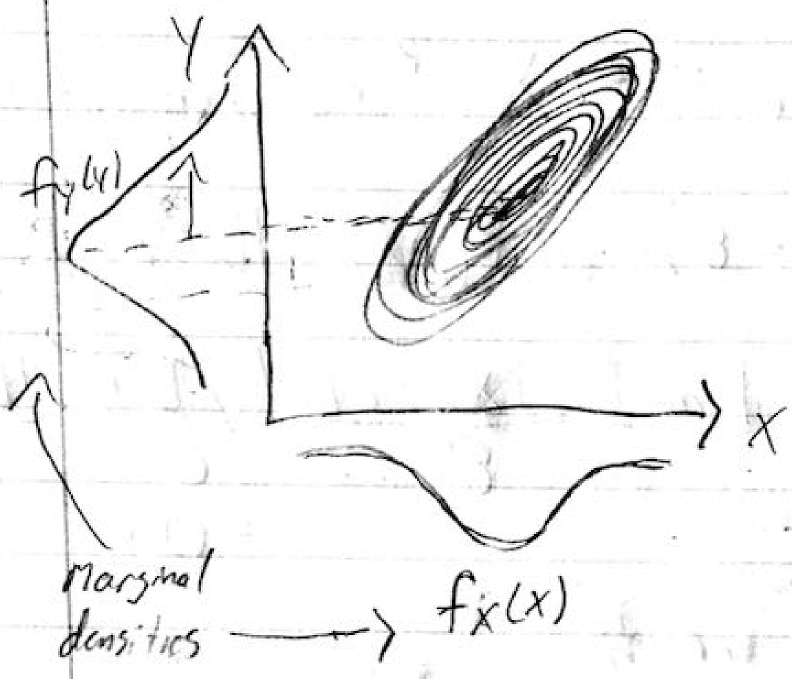
$$E[X_n^r] = 0^r(1 - \tfrac{1}{n}) + (n^2)^r\left(\tfrac{1}{n}\right) = 0 + n^{2r}$$

$$= n^{2r}$$

$$\lim E[|X_n - 0|^r] = \lim E[X_n^r] = \lim n^{2r} = \infty \neq 0$$

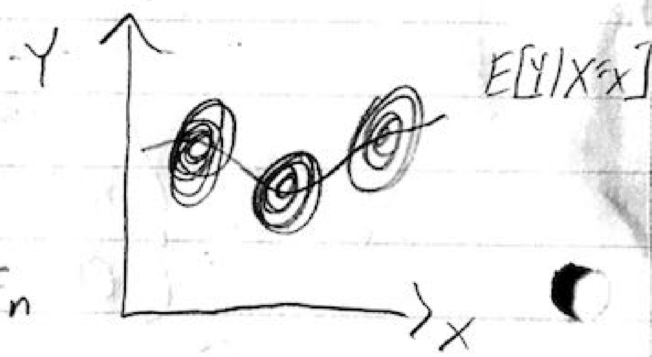
$$\rightarrow X_n \overset{L^r}{\not\rightarrow} 0 \quad \forall r \geq 1$$

12/9/19



$f_{XY}(x,y)$

$E[Y|X=x] = g(x)$



CEF Cond Expec Fn

Assume finite

$$E[Y] = \int_{\mathbb{R}} Y f_X(y)\, dy$$

$$= \int_{\text{Supp}\,Y} Y \int_{\text{Supp}\,X} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{\text{Supp}[Y]} Y \int_{\text{Supp}[X]} f_{Y|X}(x,y) f_X(x)\, dx\, dy$$

$$= \int_{\text{Supp}\,X} f_X(x) \underbrace{\int_{\text{Supp}\,Y} Y f_{Y|X}(x,y)\, dy\, dx}_{E[Y|X=x]\,\}\,g(x)}$$

12/9/19

$$= \int_{supp[X]} E[Y|X=x] f_X(x)\, dx$$

$$= E[E[Y|X=x]]$$

$$\longrightarrow E[Y] = E_X[E_Y[Y|X]]$$

Law of Iterated Expectation

$$Var[Y] = E_Y[Y^2] - E_Y[Y]^2$$

Law of iterative expec

$$\overset{?}{=} E_X[E_Y[Y^2|X]] - E_X[E_Y[Y|X]]^2$$

$$Var[Y|X] = E[Y^2|X] - E[Y|X]^2$$

$$\longrightarrow E[Y^2|X] = Var[Y|X] + E[Y|X]^2$$

$$= E_X[Var_Y[Y|X] + E_Y[Y|X]^2] - E_X[E_Y[Y|X]]^2$$

$$= E_X[Var[Y|X]] + E_X[\underbrace{E_Y[Y|X]}_{Q}{}^2] - E_X[\underbrace{E_Y[Y|X]}_{Q}]^2$$

$$E_X[Q^2] - E_X[Q]^2$$

Law of total variance

$$Var[Y] = \underbrace{E_X[Var_Y[Y|X]]}_{I} + \underbrace{Var_X[E_Y[Y|X]]}_{II}$$

12/9/19