Not covere on final

Justify T-test, Omnibus F-test, partial F-test in Linear Regression.

$\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$ Typical assumption $\vec{\varepsilon} \sim N_n(\vec{0}, \sigma^2 I_n) \Rightarrow \vec{z} = \frac{1}{\sigma}\vec{\varepsilon} \sim N_n(\vec{0}, I_n)$

$$\underset{n\times 1}{[Y]} = \underset{n\times p}{\Big[\ X\ \Big]}\underset{p\times 1}{[\beta]} + \underset{n\times 1}{\begin{bmatrix}\vdots\\ \varepsilon\\ \vdots\end{bmatrix}}$$ } Homoskedasticity (same variance)

$\Rightarrow \vec{Y} - X\vec{\beta} = \vec{\varepsilon}$

$\Rightarrow \dfrac{\vec{Y} - X\vec{\beta}}{\sigma} \sim N_n(\vec{0}_n, I_n)$

Using vector densities, the least square estimate of unknown β is:

$$\hat{\vec{\beta}} = (X^TX)^{-1}X^T\vec{Y} = (X^TX)^{-1}X^T(X\beta + \varepsilon) = (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\varepsilon$$

$$= \vec{\beta} + \underbrace{(X^TX)^{-1}X^T\sigma\frac{1}{\sigma}\vec{\varepsilon}}_{\vec{z}}$$

$$\Rightarrow \hat{\vec{\beta}} \sim N\Big(\vec{\beta}, \underbrace{(\sigma(X^TX)^{-1}X^T)^T(\sigma(X^TX)^{-1}X^T)}_{\sigma^2 \cancel{(X^TX)^{-1}(X^TX)}(X^TX)^{-1}}\Big)$$

$$= N(\vec{\beta}, \sigma^2(X^TX)^{-1})$$

$$\Rightarrow \beta_K \sim N(\beta_K, \sigma^2(X^TX)^{-1}_{KK})$$

$\dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ $\Rightarrow \dfrac{\hat{\beta}_K - \beta_K}{\sigma\sqrt{(X^TX)^{-1}_{KK}}} \sim N(0,1)$

$\dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$

But we have student's problem, σ is unknown.

$$\underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon}}_{\sim \chi^2_n} = \underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T P\vec{\varepsilon}}_{\sim \chi^2_p} + \underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T(I-P)\vec{\varepsilon}}_{\sim \chi^2_{n-p}}$$

$\underset{n\times p}{X}$, rank$[X] = p < n$

$P = X(X^TX)^{-1}X^T$ orthogonal projection matrix

rank$[P] = p$

$I - P$ orthogonal projection onto set of $\mathbb{R}^n$

rank$[I-P] = n-p$.

$(X(X^TX^{-1})X^T)(X(X^TX)^{-1}X^T)$

$PP = P$

$(I-P)(I-P) = P$

$$\vec{\varepsilon}^T P\vec{\varepsilon} = \vec{\varepsilon}^T PP\vec{\varepsilon} = \vec{\varepsilon}^T P^T P\varepsilon = (P\varepsilon)^T P\varepsilon \xrightarrow{\times\left(\frac{1}{\sigma^2}\right)}$$

$$P\varepsilon = P(Y - X\beta) = PY - PX\beta$$

$$= X(X^TX)^{-1}X^TY - X(X^TX)^{-1}X^TX\beta$$

$$= X\hat{\beta} - X\beta = X(\vec{\beta} - \beta)$$

$$\frac{1}{\sigma^2} \Rightarrow \frac{(X(\hat{\vec\beta}-\beta))^T(X(\hat{\vec\beta}-\vec\beta))}{\sigma^2} = \frac{(\hat{\vec\beta}-\beta)X^TX(\hat{\vec\beta}-\vec\beta)}{\sigma^2} \sim \chi^2_p$$

$$\vec{\mathcal{E}}^T(I-p)(I-p)\vec{\mathcal{E}} = ((I-p)\vec{\mathcal{E}})^T((I-p)\vec{\mathcal{E}})$$

$$(I-p)\vec{\mathcal{E}} = (I-p)(\vec{Y}-X\vec\beta) = IY - pY - IX\vec\beta + pX\vec\beta$$
$$= Y - pY - X\vec\beta + X(X^TX)^{-1}X^TX\vec\beta$$
$$= Y - X\underbrace{(X^TX)^{-1}X^TY}_{\hat{\vec\beta}} = Y - X\hat{\vec\beta} = E$$

"residual"

$$\frac{1}{\sigma^2}\vec{\mathcal{E}}^T\vec{\mathcal{E}} = \frac{1}{\sigma^2}\vec{\mathcal{E}}^T p\vec{\mathcal{E}} + \frac{1}{\sigma^2}\vec{\mathcal{E}}^T(1-p)\vec{\mathcal{E}}$$
$$= \frac{1}{\sigma^2}(\hat{\vec\beta}-\vec\beta)X^TX(\hat{\vec\beta}-\vec\beta) + \frac{1}{\sigma^2}SSE$$

$\Rightarrow \vec{E}, \hat{\vec\beta}$ independent

$$E\Big[\overbrace{\frac{E^TE}{\sigma^2}}^{SSE}\Big] = \frac{n-p}{\sigma^2}$$

Let $MSE = \frac{SSE}{n-p}$

$$E\Big[\frac{MSE}{\sigma^2}\Big] = E\Big[\frac{SSE}{(n-p)\sigma^2}\Big] = \frac{1}{\sigma^2}$$

$\Rightarrow E[MSE] = \sigma^2$

$\sqrt{MSE} \approx \sigma$

"

RMSE "root mean sd error"

Back $\Rightarrow$
$$\frac{\hat\beta_k - \beta_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}} \sim N(0,1)$$

"T test in Linear regression"

$$\Rightarrow \frac{\hat\beta_k - \beta_k}{RMSE\sqrt{(X^TX)^{-1}_{kk}}} = \frac{\hat\beta_k-\beta_k}{\sqrt{\frac{SSE}{n-p}}\sqrt{(X^TX)^{-1}_{kk}}} = \frac{\left.\frac{\hat\beta_k-\beta_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}}\right\}}{\sqrt{\underbrace{\frac{\frac{SSE}{\sigma^2}}{n-p}}_{\chi^2_{n-p}}}} \sim T_{n-p}$$

$H_0 : \vec\beta =$ some vector q values.

$$= \frac{\frac{\vec{\mathcal{E}}^T p\vec{\mathcal{E}}}{\sigma^2 p}}{\frac{\vec{\mathcal{E}}^T(I-p)\vec{\mathcal{E}}}{\sigma^2(n-p)}} \sim F_{p,n-p} \quad \frac{\chi^2}{\chi^2}$$

"Omnibus F-test In Linear Regression"

$$= \frac{\frac{(\hat{\vec\beta}-\beta)^TX^TX(\hat{\vec\beta}-\beta)}{p}}{\frac{SSE}{n-p}}$$