wed
in KY412

r.v.'s
random variables

$\sim$
has distribution (or has PMF)

X_1, X_2 are discrete r.v.'s

$$T = X_1 + X_2 \sim p(t) = ?$$

usually denoted $p_{X_1}(t) * p_{X_2}(t)$

convolution operator

$$p(t) = \sum_{x \in \mathbb{R}} \sum_{x \in \mathbb{R}} p_{X_1, X_2}(x_1, x_2) \mathbb{1}_{x_1 + x_2 = t}$$

$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1, X_2}(x_1, x_2) \mathbb{1}_{x_2 = x_1 - t}$$

get rid of Σ for x_2 since only one value of x_2 (has a non-zero prob.) and that prob. is 1

$$= \sum_{x \in \mathbb{R}} p_{X_1 X_2}(x, t-x)$$

if X_1, X_2 are indep.

$$= \sum_{x \in \mathbb{R}} p_{X_1 \in \mathbb{R}}(x)\, p_{X_2 \in \mathbb{R}}(t-x)$$

$$= \sum_{x \in \mathbb{R}} p_{X_1\,OLD}(x) \mathbb{1}_{x \in Supp[X_1]}\, p_{X_2\,OLD}(t-x) \mathbb{1}_{x \in Supp[X_2]}$$

$$= \sum_{x_1 \in Supp[X_1]} p_{X_1\,OLD}(x)\, p_{X_2\,OLD}(t-x) \mathbb{1}_{t-x_1 \in Supp[X_2]}$$

use $x = x_1$

$$= \sum_{x \in Supp[X]} p_{OLD}(x)\, p_{OLD}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

Given

$X_1, X_2 \overset{i.i.d.}{\sim} Bern(p)$ so

$$p(t) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right)\left(p^{t-x}(1-p)^{1-(t-x)}\right) \mathbb{1}_{t-x \in \{0,1\}}$$

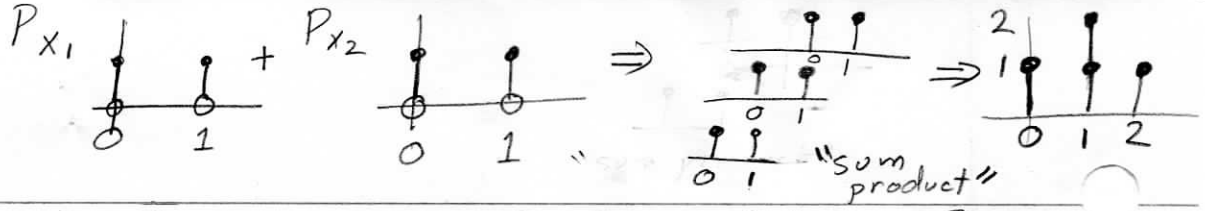
$$= p^t (1-p)^{2-t} \sum \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}}\right)$$

$$\begin{cases} 1 & \text{if } t=0 \\ 2 & \text{if } t=1 \\ 1 & \text{if } t=2 \\ 0 & \text{otherwise} \end{cases}$$

$$= p^t (1-p)^{2-t} \binom{2}{t} = Binom(2, p)$$

Look at Wikipedia Convolution animation



sum of products of p's at the values of the sums

So if X_1, X_2 are i.i.d. Bern(p) then $T = X_1 + X_2$ is Binom$(2, p)$

also,

if $X_1, X_2, X_3 \overset{i.i.d.}{\sim}$ Bern(p)

$$p(t) = \sum \left(\binom{1}{x} p^x (1-p)^{1-x}\right)\left(\binom{t}{x} p^{t-x} (1-p)^{\overbrace{1-(t-x)}^{1+x-t}} \mathbb{1}_{t-x \in \{0,1\}}\right)$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \underbrace{\binom{1}{x}}_{\text{this } = 1} \binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1}\right)$$

$$= p^t (1-p)^{2-t} \binom{2}{t} \text{ by Pascal's}$$

Pascal's Identity

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

or

$$\binom{a}{k} + \binom{a}{k-1} = \binom{a+1}{k}$$

Bern(p) has PMF $\binom{1}{x} p^x (1-p)^{1-x} = \mathbb{1}_{x \in \{0,1\}} p^x (1-p)^{1-x}$

Use $T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = T_2 + X_3$ (mathematical induction type thing)

where $X_1, X_2, X_3 \overset{i.i.d.}{\sim}$ Bern(p)

redundant

$$p(t) = \sum_{x \in \{0,1\}} \left(\binom{1}{x} p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x} (1-p)^{2-(t-x)} \mathbb{1}_{t-x \in \{0,1,2\}}\right)$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \underbrace{\binom{1}{x}}_{=1} \binom{2}{t-x}$$

$$\binom{2}{t} + \binom{2}{t-1} = \binom{3}{t}$$

$$= p^t (1-p)^{3-t} \binom{3}{t}$$

$$p(t) = \binom{3}{t} p^t (1-p)^{3-t}$$

$$= \text{Binom}(3, p)$$

if $X_1, X_2, X_3 \overset{i.i.d.}{\sim}$ Bern(p)

So for $T_3 = X_1 + X_2 + X_3$ is Binom$(3, p)$

So if $T = X_1 + X_2 + \dots + X_n$
where $X_1, X_2, \dots, X_n \overset{i.i.d.}{\sim} Bern(p)$
then $T \sim Binom(n, p)$

Given $X_1, X_2 \overset{i.i.d.}{\sim} Binom(n,p) \overset{\text{PMF is}}{=} \binom{n}{x} p^x (1-p)^{n-x}$
and $T = X_1 + X_2 \sim$

$$\sum_{x \in \{0,1,\dots,n\}} \left(\binom{n}{x} p^x (1-p)^{n-x} \right) \left(\binom{n}{t-x} p^{t-x} (1-p)^{n-(t-x)} \right) \underset{t-x \in \{0,1,\dots,n\}}{\overset{\text{redundant}}{\Uparrow}}$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum \binom{n}{x}\binom{n}{t-x}}_{} = \binom{2n}{t} p$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t}$$

by Vandermonde's Identity

$$= Binom(2n, p) \qquad \left\{ \sum_{\text{all } x} \binom{n}{x}\binom{n}{t-x} = \binom{2n}{t} \right.$$

Given $B_1, B_2, \dots \overset{i.i.d.}{\sim} Bern(p)$

X = # of zeroes realized berore first one is definition of $X \sim Geom(p)$ ← geometric r.v. model

"number of failures before a success"

$Supp[X] = \{0, 1, 2, \dots\}$ ← this set is also known as $\mathbb{N}_0$ "natural numbers" with 0

$p(0) = p$
$p(1) = (1-p)p$
$p(2) = (1-p)^2 p$
$p(3) = (1-p)^3 p$
$\vdots$
$p(x) = (1-p)^t p$

if $X \sim Geom(p)$
PMF is
$p(x) = (1-p)^t p$

$$X_1, X_2 \overset{i.i.d.}{\sim} \text{Geom}(p)$$

$T = X_1 + X_2$
has PMF

$$p(t) = \sum_{x \in \{0,1,\dots\}} \left((1-p)^x p\right)\left((1-p)^{t-x} p \, \mathbb{1}_{t-x \in \{0,1,\dots\}}\right)$$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{t-x}$$

$$= (t+1)(1-p)^t p^2$$

because

$$= \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{t \geq x} = \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \leq t}$$

$$= \mathbb{1}_{0 \leq t} + \mathbb{1}_{1 \leq t} + \mathbb{1}_{2 \leq t} + \mathbb{1}_{3 \leq t} + \mathbb{1}_{4 \leq t} + \mathbb{1}_{5 \leq t} + \dots$$

ex:

$$p(4) = 5(1-p)^4 p^2$$

number of ways of getting $T = 4$

$$\underset{1}{0} \; \underset{2}{0} \; \underset{3}{0} \; \underset{4}{1} \; \underset{5}{0} \; \underset{6}{1}$$

need 4 0's here

5 ways to do this $\binom{5}{1} = 5$

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p)$

$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim$ has PMF

$$p(t) = \sum \left((1-p)^x p\right)\left((t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x\in\{0,1,\ldots\}}\right)$$

$$= (1-p)^t p^3 \underbrace{\sum (t-x+1)}\mathbb{1}_{t-x\in\{0,1,\ldots\}}$$

$$= \sum_{x\in\{0,1,\ldots\}} (t+1)\mathbb{1}_{t-x\in\{0,1,\ldots\}} - \sum_{x\in\{0,1,\ldots\}} x \mathbb{1}_{t-x\in\{0,1,\ldots\}}$$

$$= (t+1)\underbrace{\sum_{x\in\{0,1,\ldots\}} \mathbb{1}_{x\le t}}_{(t+1)} - \underbrace{\sum_{x\in\{0,1,\ldots\}} x\mathbb{1}_{x\le t}}_{\substack{0 + 1\mathbb{1}_{1\le t} + 2\mathbb{1}_{2\le t} + 3\mathbb{1}_{3\le t} \\ + 4\mathbb{1}_{4\le t} + 5\mathbb{1}_{5\le t} + \ldots}}$$

$$= 1 + 2 + \ldots + t$$

$$= \frac{t(t+1)}{2}$$

$$= (t+1)(t+1) - \frac{t(t+1)}{2}$$

$$= (t+1)^2 - \tfrac{1}{2}t(t+1)$$

$$= t^2 + 2t + 1 - \tfrac{1}{2}t^2 - \tfrac{1}{2}t$$

$$= \tfrac{1}{2}t^2 + \tfrac{3}{2}t - \tfrac{1}{2}t$$

$$= \frac{t^2 + 3t - t}{2}$$

$$= \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!2!} = \binom{t+2}{2}$$

so $p(t) = \binom{t+2}{2}(1-p)^t p^3$ for T_3

for T_3
$p(4)$ means
get 3rd 1 (success)
after 4
0's (failures)

for T_3,

$$p(4) = \binom{6}{3} (1-p)^4 p^3$$

get 3 0's
in 1st 6.

$$\begin{array}{ccccccc} \underline{1} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{1} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

need 3 0's
in 1st 6

for $T_3 = X_1 + X_2 + X_3$

PMF

$$p(t) = \binom{t+2}{2} (1-p)^t p^3$$

in general,

if $T = X_1 + X_2 + \dots + X_r$

where $X_1, X_2, \dots, X_r \overset{i.i.d.}{\sim} Geom(p)$

then PMF

$$p(t) = \binom{t+r-1}{r-1} (1-p)^t p^r$$

this is the definition of

$$T \sim NegBin(r, p)$$

Negative Binomial r.v.