Lecture 22
12/04/19

Let x, y be r.v's with finite $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$

let $W = (x - cy)^2$ where $c \in \mathbb{R}$ constant

Note: W is non-negative $\Rightarrow E(W) \geq 0$

$$E[(x-cy)^2] = E[x^2 - 2cxy + c^2y^2]$$

$$= E[x^2] - 2cE[xy] + c^2E[y^2] \geq 0$$

let $c = \frac{E[xy]}{E[y^2]}$

$$E[x^2] - 2\frac{E[xy]}{E[y^2]}E[xy] + \frac{E[xy]^2}{E[y^2]}E[y^2] \geq 0$$

$$\Rightarrow E[x^2] - \frac{E[xy]^2}{E[y^2]} \geq 0$$

$$\Rightarrow E[x^2]E[y^2] - E[xy]^2 \geq 0$$

$$\Rightarrow E[xy]^2 \leq E[x^2]E[y^2]$$

$$\Rightarrow |E[xy]| \leq \sqrt{E[x^2] - E[y^2]}$$

Cauchy-Schwartz inequality

we wanto show:

$\text{corr}[x,y] \in (-1,1)$

let $Z_x = \frac{x - \mu_x}{\sigma_x} \Rightarrow x = \sigma_x Z_x + \mu_x$

②

$$z_y = \frac{y-\mu_y}{\sigma_y} \Rightarrow y = \sigma_y z_y + \mu_y$$

$$E[z_x] = E[z_y] = 0$$

$$SE[z_x] = SE[z_y] = 1$$

$$E[z_x^2] = E[z_y^2] = 1$$

Using Cauchy-Schwartz Inequality

$$|E[z_x z_y]| \leq \sqrt{\underset{\underset{1}{\parallel}}{E[z_x^2]}\ \underset{\underset{1}{\parallel}}{E[z_y^2]}} = 1$$

$$\Rightarrow E[z_x z_y] \in [-1, 1]$$

$$Corr[X,Y] = \frac{E[XY] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$= \frac{E[(\sigma_x z_x + \mu_x)(\sigma_y z_y + \mu_y)] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$\frac{E[\sigma_x \sigma_y z_x z_y + \mu_x \sigma_y z_y + \mu_y \sigma_x z_x + \mu_x \mu_y] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

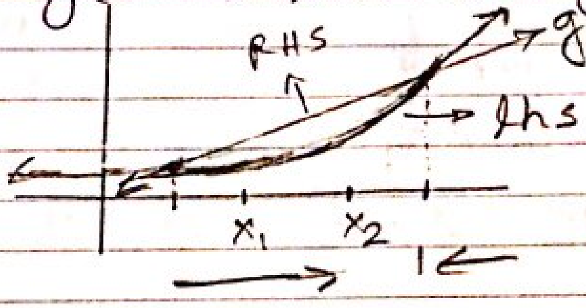
$$= \frac{\sigma_x \sigma_y E[z_x z_y] + 0 + 0 + \cancel{\mu_x \mu_y} - \cancel{\mu_x \mu_y}}{\sigma_x \sigma_y}$$

$$= \frac{\sigma_x \sigma_y E[z_x z_y]}{\sigma_x \sigma_y}$$

A Function g is "convex" on interval $I \subseteq R$ if $\forall \{x_1, x_2 \cdots\} \subseteq I$ and $\forall \{w_1, w_2 \cdots\}$ s.t $\sum w_i = 1$, $w_i > 0$ $\forall i$ then

$$g(w_1 x_1 + w_2 x_2 + \cdots) \leq w_1 g(x_1) + w_2 g(x_2) + \cdots$$

or $g(\sum w_i x_i) \leq \sum w_i g(x_i)$



$w_1 x_1 + w_2 x_2 = x^*$

Let X be a discrete r.v s.t $supp[X] = \{x_1, x_2, \cdots\} = I$

let $p(x_1), p(x_2) \cdots$ be considered the "weights"

For convex g, we know by definition

$$g\Big(\sum_{x \in supp[X]} p(x)\, x\Big) \leq \sum_{s \in supp[X]} p(x)\, g(x)$$

$\Rightarrow g(E[X]) \leq E[g(X)]$ for convex function g

"Jensen's Inequality"

let $g(t) = t^2$

by Jensen's Inequality

$E[X^2] \leq E[X^2] \Rightarrow \mu^2 \leq \sigma^2$

$\boxed{\text{calculus theorem: if } g''(x) \geq 0 \text{ for } x \in I \text{ then } g \text{ is con}}$

④

let $a, b > 0$ and $p, q > 0$ s.t $\frac{1}{p} + \frac{1}{q} = 1$

consider $X \sim \begin{cases} a^p & \text{w.p } \frac{1}{p} \\ b^q & \text{w.p } \frac{1}{q} \end{cases}$

$E[X] = \frac{a^p}{p} - \frac{b^q}{q}$

let $g(t) = -\ln(t)$ which is one to one and convex

$g(X) \sim \begin{cases} -p\ln(a) & \text{wp } \frac{1}{p} \\ -q\ln(b) & \text{w.p } \frac{1}{q} \end{cases}$

$E[g(X)] = -\ln(a) + -\ln(b) = -\ln(ab)$

By Jensen's

$g(E[X]) \leq E[g(X)]$

$= -\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq -\ln(a,b)$

$\Rightarrow \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq ab \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Young's Inequality

Let X, Y be non-negative r.v's

let $a = X$, $b = Y$ and take expectation both sides

$E[XY] \leq \frac{E[X^p]}{p} + \frac{E[Y^q]}{q}$

let $a = \frac{X}{E[X^p]^{\frac{1}{p}}}$, $b = \frac{Y}{E[Y^q]^{\frac{1}{q}}}$

and take expectation of both sides

$$\frac{E[XY]}{E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}} \le \frac{E\left[\left(\frac{X}{E[X^p]^{\frac{1}{p}}}\right)^p\right]}{p} + \frac{E\left[\left(\frac{Y}{E[Y^q]^{\frac{1}{q}}}\right)^q\right]}{q}$$

$$= \frac{\frac{E[X^p]}{E[X^p]}}{p} + \frac{\frac{E[Y^q]}{E[Y^q]}}{q} = \frac{1}{p} + \frac{1}{q} = 1$$

$$= E[XY] \le E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}$$

Holder's Inequality

let $r, s > 0$ and $s > r$

let $p = \frac{s}{r}$, $q = \frac{s}{s-r}$

$$\frac{1}{p} + \frac{1}{q} = \frac{r}{s} + \frac{s-r}{s} = 1$$

let $Y = 1$

$$E[|V|^r] \le E[(|V|^r)^{\frac{s}{r}}]^{\frac{r}{s}} = E[|V|^s]^{\frac{r}{s}}$$

$E[|V|^r] \le E[|V|^s]$. If $E[|V|^s]$ is finite then $E[|V|^r]$ is finite [for $r < s$ and $r, s > 0$]

Convergence in distribution:

$X_n \xrightarrow{d} X$ means CDF of X_n converges to CDF X.

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \ \forall x$$

$X_n \sim \begin{cases} \frac{1}{n+1} & wp\ \frac{1}{3} \\ 1-\frac{1}{n+1} & wp\ \frac{2}{3} \end{cases}$ | $X_n \xrightarrow{d} Bern\left(\frac{2}{3}\right) = \begin{cases} 1\ wp\ \frac{1}{3} \\ 0\ wp\ \frac{2}{3} \end{cases}$

e.g $X_3 \sim \begin{cases} \frac{1}{4} & wp\ \frac{1}{3} \\ \frac{3}{4} & wp\ \frac{2}{3} \end{cases}$, $X_{100} \sim \begin{cases} \frac{1}{100} & w.p\ \frac{1}{3} \\ 99 & wp\ \frac{2}{3} \end{cases}$

Is PMF convergence equivalent to CDF convergence?

Theorem: If $supp[X] \leq \mathbb{Z}$ and $supp[X] \leq \mathbb{Z}$

then PMF convergence $\Leftrightarrow$ CDF convergence

Proof: $\lim P_{X_n}(x)$

Fact: $P_{X_n}(X) = F_{X_n}(X+\frac{1}{2}) - F_{X_n}(X+\frac{1}{2})$

$= P(X_n \in [X \pm \frac{1}{2}])$

$\lim P_X(x) = \lim F_{X_n}(X+\frac{1}{2}) - \lim F_{X_n}(X-\frac{1}{2})$

$= F_X(X+\frac{1}{2}) - F_X(X+\frac{1}{2})$

$= P_X(x)$

Fact: $P_{X_1}(x) = F_{X_1}(X+\frac{1}{2}) - F_{X_n}(X-\frac{1}{2})$

$= P(X_n \in [X \pm \frac{1}{2}])$

prove $\Rightarrow \lim F_{X_n}(x) = \lim P(X_n \leq x)$

$$= \lim \sum_{y=-\infty}^{x} P_{X_n}(y) = \sum_{y=-\infty}^{x} \lim P_{X_n}(y)$$

dominant convergence theorem

$$= \sum_{y=-\infty}^{x} P_X(y) = P(X \leq x) = F_X(x)$$

let $X_n \sim \text{Binm}(n, \frac{\lambda}{n})$, $X \sim \text{Poisson}(\lambda)$

we showed $\lim_{n\to\infty} P_{X_n}(x) = P_X(x) \Rightarrow X_n \xrightarrow{d} X$

IS PDF convergence equivalent to CDF convergence?

No, only PDF convergence $\Rightarrow$ CDF convergence

For a counterexample to the converse

consider $X \sim U(0, \frac{1}{n}) = n \, \mathbb{1}_{x \in [0, \frac{1}{n}]} = f_{X_n}(x)$

$$\lim_{n\to\infty} f_{X_n}(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} = f_{X_1}(x)$$

$$\text{CDF} = \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } x \in [0, \frac{1}{n}] \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases} \xrightarrow{d}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} = \text{CDF of } \deg(0)$$

convergence in probability

To a constant if c is the constant, $c \in \mathbb{R}$

$X_n \xrightarrow{P} c$ if $\forall \varepsilon > 0$, $\lim_{n \to \infty} P(|X_n - c| \geq \varepsilon) = 0$

and $P(|X_n - c| < \varepsilon) = 1$

Let $X_n \sim U(-\frac{1}{n}, \frac{1}{n}) = \frac{n}{2} \mathbb{1}_{x \in [-\frac{1}{n}, \frac{1}{n}]}$

Prove $X_n \xrightarrow{P} 0$