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Lecture-19
Nov25

Let $\bar{x}$ be a vector r.v of dimension

$$\vec{\mu} := E[X]$$

$$\vec{c} \in \mathbb{R}^n \quad E[\vec{x} + \vec{c}] = \vec{\mu} + \vec{c}$$

$$E[\vec{c}^{\,T} \vec{x}] = \vec{c} + \vec{\mu}$$

$$\Sigma := var[\vec{x}] = E[\vec{x}\vec{x}^T] - \underbrace{E[\vec{x}]E[\vec{x}]^T}_{\vec{\mu}\vec{\mu}^T}$$

$$\begin{bmatrix} var[x_1] & cov[x_1, x_2] & \cdots & \\ cov[x_1, x_2] & var[x_2] & \cdots & \vdots \\ & & \ddots & \\ cov[x_1 x_n] & \cdots & & var[x_n] \end{bmatrix}$$

$$var[\vec{x} + \vec{c}] = var[\vec{x}]$$

$$var[\vec{c}\vec{x}] = \vec{c}\,\Sigma\,\vec{c}$$

Let $A \in \mathbb{R}^n$ constant

$$E[A\vec{x}] = E\begin{bmatrix} \vec{a}_1 \vec{x} \\ \vec{a}_2 \vec{x} \\ \vdots \\ \vec{a}_n \vec{x} \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \vec{x}] \\ E[\vec{a}_2 \vec{x}] \\ \vdots \\ E[\vec{a}_n \vec{x}] \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 \vec{\mu} \\ \vec{a}_2 \vec{\mu} \\ \vdots \\ \vec{a}_n \vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \vec{\mu} = A\vec{\mu}$$

$$var[A\vec{x}]$$

$$= E[(Ax)(Ax)^T] - E[A\vec{x}]E[A\vec{x}]$$

$$= E[A\vec{x}\vec{x}^T A^T] - A\vec{\mu}\vec{\mu}^T A^T$$

$$= AE[\vec{x}\vec{x}^T A^T] - A\vec{\mu}\vec{\mu}^T A^T$$

$= A\,E[\vec{x}\vec{x}^T]A^T - A\vec{\mu}\vec{\mu}A^T$

$= A[E[\vec{x}\vec{x}^T]]A^T - \vec{\mu}\vec{\mu}A^T$

$= A[E[\vec{x}\vec{x}^T] - \vec{\mu}\vec{\mu}^T]A^T = A\Sigma A^T$

Let $U \sim \chi^2_k$,

$E[U] = E[Z_1]^2 + \cdots + E[Z_k]^2 = kE[Z^2] = k.1 = k$

$Z_1, \cdots Z_n \overset{iid}{\sim} N(0,1)$

$\downarrow E(Z) = 0 \quad var(Z) = 1$

$U = Z_1^2 + \cdots + Z_k^2$

$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim N_n(\vec{O}_n, I_n) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{Z}^T\vec{Z}}$

standard multivariate Normal.

$Z_1, \cdots, Z_n \overset{iid}{\sim} N(0,1)$

Let $A \in R^{n\times n}$ constants

$\vec{u} \in R^n$ constants

$\vec{X} = A\vec{Z} + \vec{\mu} \sim f_{\vec{X}}(x) = ?$

$x = \sigma z + \mu \sim N(\mu, \sigma^2)$

$E[\vec{X}] = E[A\vec{Z} + \vec{\mu}] = AE[\vec{Z}] + \vec{\mu} = A\vec{O}_n + \vec{\mu} = \vec{\mu}$

$var[\vec{X}] = var[A\vec{Z} + \vec{\mu}] = var[A\vec{Z}]$

$= A\,var[\vec{Z}]A^T = AIA^T = AA^T = \Sigma$

$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}\, h(\vec{x})\ |J_n|$

$X = A\vec{Z} + \vec{\mu} \Rightarrow \vec{x} - \vec{\mu} = A\vec{Z}$

Assume $A$ is variable

$A^{-1}(A\vec{Z}) \Rightarrow \vec{Z} = A^{-1}(X-\mu)$ [multiply by $A^{-1}$]

let $B = A^{-1}$

$= B(\vec{x}-\vec{\mu}) = B\vec{x} - B\vec{\mu} = h(\vec{x})$

$$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_3}{\partial x_3} & \cdots & \frac{\partial h_n}{\partial x_n} \\ \vdots & & & & \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & & \frac{\partial h_n}{\partial x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{b}_1\vec{x} \cdots \vec{b}_1\vec{\mu} \\ \vec{b}_2\vec{x} \cdots \vec{b}_2\vec{\mu} \\ \vdots \\ b_n\vec{x} \cdots \vec{b}_n\vec{\mu} \end{bmatrix} \begin{matrix} = h_1(\vec{x}) = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_1 - b_{11}\mu_1 - b_{11}\mu_1 - b_{12}\mu_2 - \cdots - b_n\mu_n \\ = h_2(\vec{x}) \\ \vdots \\ = h_n[\vec{x}] \end{matrix}$$

$\vec{X} = A\vec{Z} + \vec{\mu} \sim f_{\vec{X}}(x) = f_{\vec{Z}}(A^{-1}(\vec{x}-\vec{\mu}))\ |\det(A^{-1})|$

$= \frac{1}{\sqrt{(2\pi)^n}}\, e^{\frac{1}{2}(A^{-1}(\vec{x}-\vec{\mu}))^T A^{-1}(\vec{x}-\vec{\mu})}\ |\det(A^{-1})|$

$= \frac{|\det(A^{-1})|}{\sqrt{(2\pi)^n}}\, e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})}$

Fact # 1

$I = AA^{-1} \Rightarrow \det(I) = \det(AA^{-1})$

$1 = \det(A)\det(A^{-1})$

$\det[A^{-1}] = \frac{1}{\det[A]}$

$E[A\vec{X}] = A\vec{\mu}$

$var[A\vec{X}] = A\Sigma A^T$

Fact # 2

$\Sigma = AA^T$

$\det(\Sigma) = \det(A)\det(A^T)$

$\Rightarrow \det(\Sigma) = \det(A)^2$

$\Rightarrow \det(A) = \sqrt{\det(\Sigma)}$

$\Rightarrow \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})}$

$= N_n(\vec{\mu}, \Sigma) \rightarrow$ general multivariate normal

$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Fact # 3

$I = AA^{-1}$

$I = I^T = (AA^{-1})^T$

$= [A^{-1}]^T A^T \Rightarrow I = (A^T)^{-1} A^T \Rightarrow (A^T)^{-1} = (A^{-1})^T$

① Let $\vec{\mu} = \vec{0}, \Sigma = I$

$f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{1}{2}\vec{x}^T\vec{x}}$

② $n = 1$

$\Sigma = [\sigma^2]$ $f_X(x) = \frac{1}{\sqrt{(2\pi)^n \sigma^2}} e^{-\frac{1}{2}(x-\mu)\frac{1}{\sigma^2}(x-\mu)}$

$= N(\mu, \sigma^2)$

Fact # 4

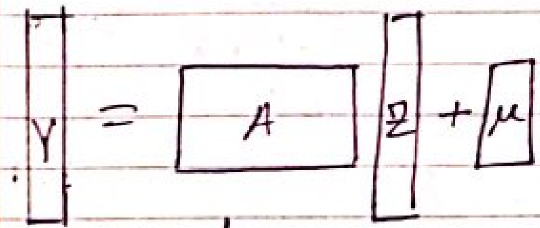
$\Sigma = AA^T \Rightarrow \Sigma^{-1}(AA^T)^{-1}$

$\Rightarrow \Sigma^{-1} = (A^T)^{-1}A^{-1} \Rightarrow \Sigma^{-1} = (A^{-1})^T A^{-1}$

Let $A \in R^{m \times m}$, $\vec{\mu} \in R^m$, $m < n$

let $\vec{X} = A\vec{Z} + \vec{\mu}$

A has rank $\mu$ [Full rank]

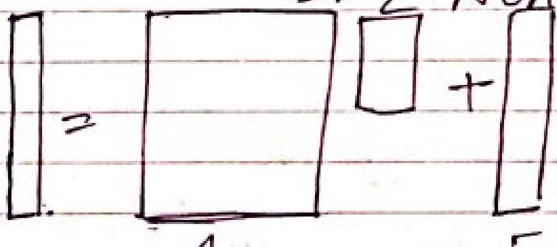


consider $m > n \Rightarrow A$ has at most rank $n$

$\Rightarrow \Sigma = AA^T$ at most rank $n < m$

$\Rightarrow \Sigma$ is not full rank

$\Rightarrow \Sigma$ Non-invertible $\Rightarrow \det(\Sigma) = 0$



$\tilde{A} = \begin{bmatrix} A \\ \vec{v}_1 \\ \vdots \\ v_{n-m} \end{bmatrix}$, $\vec{\mu} = \begin{bmatrix} \vec{\mu} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\tilde{\vec{X}} = \tilde{A}\tilde{Z} + \tilde{\vec{\mu}} \sim N_n(\tilde{\vec{\mu}}, \tilde{A}\tilde{A}^T)$

where $\vec{v}_1 \cdots v_{n-m}$ are chosen so that $\tilde{A}$ is full rank $\Rightarrow A^{-1}$ exists

$$f_{\vec{X}}(\vec{X}) = \int_R \cdots \int_R \frac{1}{\sqrt{(2\pi)^n \det(\tilde{A}\tilde{A}^T)}} e^{-\frac{1}{2}(\vec{X}-\tilde{\vec{\mu}})^T(\tilde{A}\tilde{A}^T)^{-1}(\tilde{X}-\tilde{\vec{\mu}})} \, dx_1 \cdots dx_{n-\mu}$$

$$= \int_R \cdots \int_R \frac{1}{\sqrt{(2\pi)^n \det(\tilde{A}\tilde{A}^T)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T(\tilde{A}\tilde{A}^T)} dx_1 \cdots dx_{n-\mu}$$ ⑥

we will not do this.

def: multivariate characteristic functions

Let $\phi_{\vec{x}}(\vec{t}) = E[e^{i\vec{t}^T\vec{x}}]$

$= E[e^{i(t_1x_1 + \cdots t_nx_n)}] = E[e^{it_1x_1} \cdots e^{it_1x_n}]$

$\overset{x_1, x_2 \text{ iid}}{=} E[e^{it_1x_1}] \cdots E[e^{it_nx_n}] = \prod_{i=1} \phi_{x_i}(t_i)$

(P0) $\phi_{\vec{x}}(\vec{0}) = E[e^{i\vec{0}_n^T\vec{x}}] = E[e^{i0}] = 1$

(P1) $\phi_{\vec{x}}(\vec{t}) = \phi_y(\vec{t}) \Leftrightarrow \vec{x} \overset{d}{=} \vec{y}$

(P2) $Y = A\vec{x} + \vec{b}$ where $A \in R^{n \times n}$, $\vec{b} \in R^n$

$\dim(\vec{x}) = n$

$\phi_y(\vec{t}) = E[e^{\underbrace{i\vec{t}^T(A\vec{x}+\vec{b})}_{i\vec{t}^TA\vec{x} + i\vec{t}^T\vec{b}}}]$

$= E[e^{i\vec{t}^TA\vec{x}} e^{i\vec{t}\vec{b}}] = e^{i\vec{t}^T\vec{b}} E[e^{i(A^T\vec{t})^T\vec{x}}]$

$e^{(A^T\vec{t})^T}$

$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{x}}(A^T\vec{t})$

$\vec{z} \sim N_n(\vec{0}_n, I_n) \Rightarrow \phi_{\vec{z}}(t)$

$= \prod_{i=1}^n \phi_{z_i}(t_i) = \prod_{i=1}^n e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{\Sigma}}$

Let $A \in R^{n \times n}$ invertible $\vec{\mu} \in R^n$

$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma)$, $\phi_{\vec{X}}(\vec{t})$

$$= e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{Z}}(A^T\vec{t})$$

$$= e^{i\vec{t}\vec{\mu}} e^{-\frac{1}{2}(A^T\vec{t})^T(A^T\vec{t})}$$

$$= e^{i\vec{t}\vec{\mu} - \frac{1}{2}\vec{t}^T A A^T \vec{t}}$$

$$\phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}\vec{\mu} - \frac{1}{2}\vec{t}^T \Sigma \vec{t}}$$

let $B \in R^{n \times n}$ $\vec{c} \in R^m$

$\vec{y} = B\vec{x} + \vec{c}$

$$\phi_{\vec{y}}(\vec{t}) = e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(B^T\vec{t})$$

$$= e^{i\vec{t}^T c}\left(e^{i\underbrace{(B^T\vec{t})^T}_{\vec{t}^T B}\vec{\mu} - \frac{1}{2}\underbrace{(B^T\vec{t})^T}_{\vec{t}^T B}\Sigma(B^T\vec{t})}\right)$$

$$= e^{i\vec{t}^T(\vec{c} + B\vec{\mu}) - \frac{1}{2}\vec{t}^T B \Sigma B^T \vec{t}}$$

$\overset{P1}{\Rightarrow}$ $\vec{Y} \sim N_n(\vec{c} + B\vec{\mu}, B\Sigma B^T)$