(lecture 5) September 11^{th}, 2019

$\vec{x} \sim \text{Multin}(n, \vec{p}) \Rightarrow x_j \sim Bin(n, p_j)$

$$E(\vec{x}) = \begin{bmatrix} n p_1 \\ n p_2 \\ \vdots \\ n p_k \end{bmatrix} = n\vec{p}$$

$$\text{let } \vec{\mu} := E[\vec{x}] := \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_k) \end{bmatrix}$$

$$\text{Let } M = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}, \quad E(M) = \begin{bmatrix} E(x_{11}) & \cdots & E(x_{1m}) \\ \vdots & & \\ E(x_{n1}) & \cdots & E(nm) \end{bmatrix}$$

$$\sigma^2 := Var(x) = E(x^2) - m^2$$

$$\sigma_{12} := Cov[x_1, x_2] := E[x_1 x_2] - m_1 m_2 = \cdots = E[(x_1 - m_1)(x_2 - m_2)]$$

If $x_1 x_2 \overset{ind}{\sim} \Rightarrow \sigma_{12} = 0$

$Var[x_1 + x_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$

Rules for Covariance

1) $Cov[x, x] = Var(x) = \sigma^2$
2) $Cov[x_1, x_2] = Cov[x_2, x_1]$
3) $Cov[x_1 + x_2, x_3] = Cov[x_1, x_3] + Cov[x_2, x_3]$
4) $Cov[a_1 x_1, a_2 x_2] = a_1 a_2 \sigma_{12}$

5) $\mathrm{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}[X_i, X_j]$

Example: $\mathrm{Var}(X_1+X_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} \mathrm{Cov}[X_i, X_j] = \sigma_1^2 + \underbrace{\sigma_{12} + \sigma_{21}}_{2\sigma_{12}} + \sigma_1^2$

$$\Sigma := \mathrm{Var}[\vec{x}] = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_k] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}(X_2) & & \\ \vdots & & \ddots & \\ \mathrm{Cov}[X_k,X_1] & \cdots & & \mathrm{Var}[X_k] \end{bmatrix}$$

Σ is symmetric, $k \times k$ matrix
diagonal is non-negative

$= E[\vec{x}\vec{x}^T] - \vec{M}\vec{M}^T$

If $X_1, \ldots, X_k \overset{ind}{\sim}$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}$$

Σ is called a Variance / covariance / Variance - Covariance matrix

$$\text{Var}(\vec{x}) = \begin{bmatrix} np_1(1-p_1) & & & \\ & np_2(1-p_2) & & \sigma_{ij} \\ & & \ddots & \\ - & & & \vdots \\ & & & np_k(1-p_k) \end{bmatrix}$$

We need to compute σ_{ij} where $i \neq j$
we know $\sigma_{ij} < 0 \quad \forall i \neq j$

$$\sigma_{ij} := E[X_i X_j] - \mu_i \mu_j$$

$$= \sum_{x_1 \in Supp[x]} \sum_{x_2 \in Supp[x]} x_1 x_2 \underbrace{P_{X_i X_j}(x_i x_j)}_{\text{E.C. on H.W}} - n^2 p_i p_j$$

Recall

$X_i \sim Bin(n, p_i)$
$X_j \sim Bin(n, p_j)$

$X_i = X_{1i} + X_{2i} + \cdots + X_{ni}$ where $X_{1i}, \ldots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \cdots + X_{nj}$ where $X_{1j}, \ldots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

$\vec{x} = \vec{X}_{1\cdot} + \cdots + \vec{X}_{n\cdot}$ Such that $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n \overset{iid}{\sim} Multi_k(1, \vec{p})$

$$\sigma_{ij} = Cov[X_i, X_j] = Cov[X_{1i} + X_{2i} + \cdots + X_{ni}, X_{1j} + X_{2j} + \cdots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

$$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$$ because

$l \neq m \rightarrow$ Covariance $= 0$ due to independence

$$= \sum_{l=1}^{n} E[X_{li}, X_{lj}] - P_i P_j]$$

$$= \left(\sum_{l=1}^{n} E[X_{li}, X_{lj}]\right) - n P_i P_j$$

$$E[X_{li}, X_{li}] = \sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} x_1 x_2 P_{X_{li}, X_{lj}}(x_1, x_2)$$

$$= P_{X_{li}, X_{lj}}(1,1) = P(X_{li} = 1 \,\&\, X_{lj} = 1) = 0$$

So $\left(\sum_{l=1}^{n} E\left(X_{l_i}, X_{l_j}\right)\right) - n p_i p_j = -n p_i p_j$

So $$\mathrm{Var}(\vec{X}) = \begin{bmatrix} np_1(1-p_1) & & -np_ip_j \\ & np_2(1-p_2) & \\ -np_ip_j & \ddots & \\ & & np_k(1-p_k) \end{bmatrix}$$

What if $\vec{p} = \frac{1}{k}$?

(side note)

Let $X_1, \dots, X_k$ be r.v's for "asset" yearly returns
Let $\mu_1, \dots, \mu_k$ be the expected returns

Let $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$ be a set of weights such that $\vec{n}^T \vec{1} = 1$.

Your portfolio is $f = w_1 X_1 + \dots + w_k X_k = \vec{w}^T \vec{x}$

"Markowitz Optimal Theory"

Goal: $M_f = M_0$ and find $\vec{w}$ such that $Var(f)$ is minimal

$= Var[\vec{w}^T \vec{X}]$

Rules for vector expectation and variance

$E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a}$

where $\vec{a} \in \mathbb{R}^k$ constant

$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + \dots + a_k X_k]$

$= a_1 M_1 + \dots + a_k M_k = \vec{a}^T \vec{\mu}$

$Var[\vec{a}^T \vec{X}] = Var[a_1 X_1 + \dots + a_k X_k]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[a_i X_i, a_j X_k] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}$?

= nice linear algebra expression.

$E[A\vec{x}]$ such that $A \in \mathbb{R}^{l \times k}$ constants

$$= \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \cdots + a_{1k}X_k] \\ E[a_{21}X_1 + a_{22}X_2 + \cdots + a_{2k}X_k] \\ \vdots \\ E[a_{l1}X_1 + a_{l2}X_2 + \cdots + a_{lk}X_k] \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_l \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

Consider $\underbrace{\vec{a}^T}_{1\times k} \overbrace{V}^{k\times k} \underbrace{\vec{a}}_{k\times 1}$ where $V \in \mathbb{R}^{k\times k}$, $\vec{a} \in \mathbb{R}^k$

quadratic form

$$\vec{a}^T V \vec{a} = [a_1 \cdots a_k] \begin{bmatrix} a_1 V_{11} + \cdots + a_k V_{1k} \\ a_1 V_{21} + \cdots + a_k V_{2k} \\ \vdots \\ a_1 V_{k1} + \cdots + a_k V_{kk} \end{bmatrix}$$

$$= a_1 a_1 V_{11} + a_1 a_2 V_{12} + \cdots + a_1 a_k V_{1k} + a_2 a_1 V_{21} + a_2 a_2 V_{22} + \cdots + a_2 a_k V_{2k} + \cdots + a_k a_1 V_{k1} + a_k a_2 V_{k2} + \cdots + a_k a_k V_{kk}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j V_{ij}$$

so $\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij} = \boxed{\vec{a}^T \Sigma \vec{a}}$, Variance Covariance Matrix $\vec{X}$

so $Var[\vec{w}^T \vec{X}] = \vec{w}^T \Sigma \vec{w}$

$$\vec{w}^* = \arg\min_{\vec{w}} \{\vec{w}^T \Sigma \vec{w}\}$$