NOT ON

11/27 EXAM FROM THIS PART: ~~[illegible]~~ 2ND PDF. start

11/27

Justify T-test in linear regression.

$$\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$$

$$n\begin{bmatrix} Y \end{bmatrix}_1 = n\begin{bmatrix} X \end{bmatrix}_p \begin{bmatrix} \beta \end{bmatrix}^p_1 + n\begin{bmatrix} \varepsilon \end{bmatrix}_1$$

p ← full rank — columns are not duplicate, span = p dimensions.

Standard assumption: $\vec{\varepsilon} \sim N_n(\vec{0}, \sigma^2 I)$

Independent, normal, homoskedastic.

$\vec{\beta}, \sigma^2$ are unknown parameters.

$$\frac{1}{\sigma}\vec{\varepsilon} \sim N_n(\vec{0}, I) \qquad \frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon} \sim \chi^2_n$$

least squares minimal estimate for $\vec{\beta}$ as:

Estimator for $\vec{\beta}$

$$\hat{\vec{\beta}} = (X^TX)^{-1}X^T\vec{Y} = (X^TX)^{-1}X^T(X\vec{\beta} + \vec{\varepsilon})$$

$$= (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\vec{\varepsilon}$$

$$= \vec{\beta} + (X^TX)^{-1}X^T\vec{\varepsilon}$$

11/27 $\hat{\beta} \sim N_p\left(\vec{\beta}, \underbrace{\left((X^TX)^{-1}X^T\right)(\sigma^2 I)\left((X^TX)^{-1}X^T\right)^T}_{\sigma^2 (X^TX)^{-1} X^T X \left((X^TX)^T\right)^{-1}}\right)$

$$= N_p\left(\vec{\beta}, \sigma^2 (X^TX)^{-1}\right) \Rightarrow E\left[\hat{\vec{\beta}}\right] = \vec{\beta}$$

unbiased estimator.

Based on marginy $\hat{\beta}_K \sim N\left(\beta_K, \sigma^2 (X^TX)^{-1}_{KK}\right)$

$$\Rightarrow \frac{\hat{\beta}_K - \beta_K}{\sigma\sqrt{(X^TX)^{-1}_{KK}}} \sim N(0,1)$$

"Students" problem. σ is unknown. Use estimate instead.

$$\frac{1}{\sigma^2} \vec{\varepsilon}^T \vec{\varepsilon} = \frac{1}{\sigma^2} \vec{\varepsilon}^T (P) \vec{\varepsilon} + \frac{1}{\sigma^2} \vec{\varepsilon}^T (I-P) \vec{\varepsilon}$$

χ^2_n

$P := X(X^TX)^{-1}X^T$ orthogonal projection matrix.

$n \times n$ $\quad$ $\text{rank}[P] = p$

$I-P :=$ Another orthogonal projection onto $n-p$ "missing dimensions"

$$PP = X(X^TX)^{-1}X^T X(X^TX)^{-1}X^T = P$$

$$(I-P)(I-P) = II - PI - IP + PP = I - P - P + P = I - P$$

both idempotent

11/3/7

Cochran's Theorem applys.

$$\frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon} = \chi_p^2 + \chi_{n-p}^2$$

independent

$$\vec{\varepsilon}^T P \vec{\varepsilon} = \vec{\varepsilon}^T P P \vec{\varepsilon} = (P\vec{\varepsilon})^T P\vec{\varepsilon} = (\hat{\vec{\beta}}-\vec{\beta})^T X^T X(\hat{\vec{\beta}}-\vec{\beta})$$

$$\vec{\varepsilon}(I-P)\vec{\varepsilon} = \vec{\varepsilon}^T(I-P)(I-P)\vec{\varepsilon} = ((I-P)\vec{\varepsilon})^T(I-P)\vec{\varepsilon} = E^TE = SSE$$

$$P\vec{\varepsilon} = P(\vec{Y}-X\vec{\beta}) = P\vec{Y} - PX\vec{\beta} = \cancel{X(X^TX)^{-1}X^T}$$

$$= \underbrace{X(X^TX)^{-1}X^T\vec{Y}}_{\hat{\vec{\beta}}} - X(X^TX)^{-1}X^TX\beta = X(\hat{\vec{\beta}}-\vec{\beta})$$

$$(I-P)\vec{\varepsilon} = (I-P)(\vec{Y}-X\vec{\beta}) = I\vec{Y} - P\vec{Y} - IX\vec{\beta} + PX\vec{\beta}$$

$$= \vec{Y} - P\vec{Y} - X\vec{\beta} + X\vec{\beta}$$

$$= \vec{Y} - X\hat{\vec{\beta}} = \vec{E}$$ residuals

$$E\left[\frac{1}{\sigma^2}SSE\right] = n-p \qquad E\left[\frac{SSE}{n-p}\right] = \sigma^2$$

$$E\left[\frac{1}{\sigma^2}\frac{SSE}{n-p}\right] = 1 \qquad \Rightarrow MSE \approx \sigma^2$$

$$RMSE = \sqrt{MSE} \approx \sigma$$

11/27

SSE is independent of $(\hat{\vec{B}}-\vec{B})^T X^T X (\hat{\vec{B}}-\vec{B}) \Rightarrow$ RMSE $\hat{\vec{B}}$ indep

$$\frac{\hat{B}_k - B_k}{RMSE\sqrt{(X^TX)^{-1}_{kk}}} = \frac{\hat{\beta}_k - B_k}{\sqrt{MSE(X^TX)^{-1}_{kk}\frac{\sigma^2}{\sigma^2}}} = \frac{\frac{\hat{\beta}_k - \beta_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}}}{\sqrt{\frac{MSE}{\sigma^2}\cdot\frac{n-p}{n-p}}} = \frac{\overbrace{\frac{\hat{\beta}_k - B_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}}}^{Z}}{\underbrace{\sqrt{\frac{SSE}{\sigma^2}\Big/n-p}}_{\sqrt{\chi^2_{n-p}/n-p}}}$$

$\sim T_{n-p}$ justification for student T-test in linear regression.

I

$$\frac{\frac{1}{\sigma^2}\overbrace{\vec{\varepsilon}^T P \vec{\varepsilon}}^{\chi^2_p}/p}{\frac{1}{\sigma^2}\underbrace{\vec{\varepsilon}(I-P)\vec{\varepsilon}}_{\chi^2_{n-p}}/n-p} \sim F_{p,n-p}$$

$$\parallel$$

$H_0: \vec{\beta} =$ values.

$$\frac{(\hat{\vec{B}}-\vec{B})^T X^T X(\hat{\vec{B}}-\vec{B})/p}{SSE/n-p}$$

justification for omnibus F-test.