- $X \sim Beta(\alpha, \beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}$

Kernel: $\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \propto x^{\alpha-1}(1-x)^{\beta-1}$

- $Y = g(X)$ & g is 1-1,

$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$

Multi-dimensional r.v's

- Let $g: \mathbb{R}^n \to \mathbb{R}^n$, g is 1-1,
Let $\vec{X}, \vec{Y}$ be r.v vectors w/ dim n.

$\vec{Y} = g(\vec{X})$

$$\left.\begin{aligned} Y_1 &= g_1(X_1, \dots, X_n) \\ Y_2 &= g_2(X_1, \dots, X_n) \\ &\vdots \\ Y_n &= g_n(X_1, \dots, X_n) \end{aligned}\right\} \qquad g_i : \mathbb{R}^n \to \mathbb{R} \quad \forall i$$

Since g is 1-1, it has an inverse 'h' s.t $\vec{X} = h(\vec{Y})$. Then

$$\left.\begin{aligned} X_1 &= h_1(Y_1, \dots, Y_n) \\ X_2 &= h_2(Y_1, \dots, Y_n) \\ &\vdots \\ X_n &= h_n(Y_1, \dots, Y_n) \end{aligned}\right\}$$

Review of Math 202:

$$\boxed{f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\,\left| J_h(\vec{y}) \right|} \quad (*)$$

"Multi. change of variable"

where $J_h := \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{vmatrix}$

"Jacobian Determinant"

$n \times n$ matrix (scalar)

- $T = X_1 + X_2$

Recipe:

1. Find a clever g
2. Find $g^{-1} = h$
3. Compute J_h
4. Plug h, J_h into $(*)$ formula
5. Integrate to get target

ex.

1. Let $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$
$Y_2 = X_2 = g_2(X_1, X_2)$

2. $X_1 = Y_1 - X_2$
$= Y_1 - Y_2$
$= h_1(Y_1, Y_2)$
$X_2 = Y_2 = h_2(Y_1, Y_2)$

3.
$$J_h = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = \boxed{1}$$

no stretch

4. $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)\,|1|$

5. Recall $f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\,dy_2$

$$\Rightarrow f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\,du$$

$$(\text{if } X_1, X_2 \sim \text{ind}) = \int_{\mathbb{R}} f_{X_1}(t-u)\, f_{X_2}(u)\,du$$

$$(\text{if } X_1, X_2 \sim \text{iid}) = \int_{\mathbb{R}} f(t-u)\, f(u)\,du$$

$$\int_{Supp[X_1]} f_{X_1}(t-u) f_{X_2}(u)\, \mathbb{I}_{u \in Supp[X_2]}\,du$$

- $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

ex. 1. Let $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$ Target

$Y_2 = X_2 = g_2(X_1, X_2)$

2. $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$
$X_2 = Y_2 = h_2(Y_1, Y_2)$

3. $J_h = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$

4. $f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2) |y_2|$

5. $f_R(r) = \int_{\mathbb{R}} f_{X_1, X_2}(ru, u) |u| \, du$

if $X_1, X_2 \sim$ ind $= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u) |u| \, du$

$= \int_{Supp[X_1]} f_{X_1}(ru) f_{X_2}(u) \mathbb{1}_{u \in supp[X_2]} |u| \, du$

if $X_1, X_2 \sim$ iid $= \int_{Supp[X_1]} f(ru) f(u) \mathbb{1}_{u \in supp[X_2]} |u| \, du$

//

ex. $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$ (target)

1. Let $Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$

$Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

2. $X_1 = Y_1(X_1 + X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

3. $J_h = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) + y_1 y_2 = y_2$

4. $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|$

5. $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u - ru)\,|u|\,du$

if $X_1, X_2 \sim$ ind $= \int_{supp[X_1]} f_{X_1}(ru)\, f_{X_2}(u - ru)\, \mathbb{1}_{u - ru \in supp[X_2]}\,|u|\,du$

- Let $\left.\begin{array}{l} X_1 \sim \text{Gamma}(\alpha_1, \beta_1) \\ X_2 \sim \text{Gamma}(\alpha_2, \beta_2) \end{array}\right\}$ ind.

$$R = \frac{X_1}{X_1 + X_2} \sim \qquad \cdot\ \text{Supp}[R] = [0, 1]$$

$$\sim \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u - ru)^{\alpha_2} e^{-\beta(u-ru)} \overbrace{\mathbb{1}_{\underbrace{u-ru\in(0,\infty)}_{u \in (0,\infty)}}}^{\text{always } 1}\right) u\,du$$

$$= \alpha \int_0^\infty r^{\alpha_1 - 1} u^{\alpha_1 - 1} u^{\alpha_2 - 1} (1-r)^{\alpha_2 - 1} u\,du$$

$$= r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 + 1} e^{-\beta u}\,du}_{\text{not a function of } r}$$

$$\alpha\, r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \alpha \ \text{Beta}(\alpha_1, \alpha_2)$$

//

Rerivation of Beta Prime Distribution

ex. X_1, X_2 Same as last example

$R = \frac{X_1}{X_2}$ $\quad \text{supp}[R] = (0, \infty)$

$$\sim \int_{\text{supp}[X_1]} f_{X_1}(ru) f_{X_2}(u) |u| \, du$$

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u}\right)\left(\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} \underbrace{\mathbb{1}_{u \in (0,\infty)}}_{\text{always } 1}\right) u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty r^{\alpha_1 - 1} u^{\alpha_1 - 1} e^{-\beta u (r+1)} u^{\alpha_2 - 1} u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u (r+1)} u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \cdot \frac{\Gamma(\alpha_1 + \alpha_2)}{[\beta(r+1)]^{\alpha_1 + \alpha_2}}$$

$$= \boxed{\frac{1}{B(\alpha_1, \alpha_2)} \frac{r^{\alpha_1 - 1}}{(r+1)^{\alpha_1 + \alpha_2}} \mathbb{1}_{r \in (0,\infty)}}$$

$:= \text{Beta Prime}(\alpha_1, \alpha_2)$

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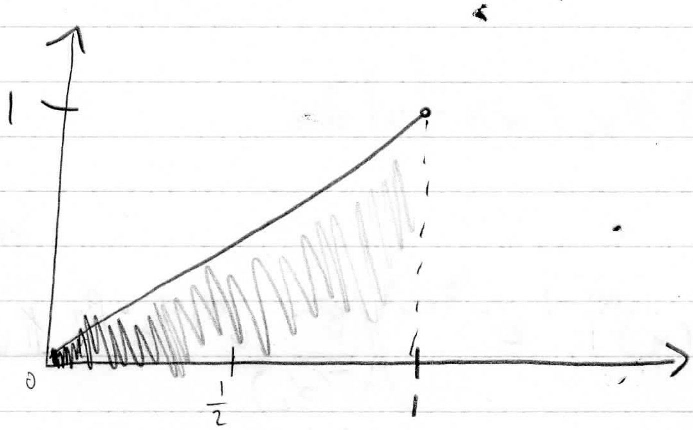
Conditional Densities

Let $X \sim U(0,1)$
$Y \mid X=x \sim U(0,x)$



The shading represents density