Lec 9 Math 621 10/7/19

Models for Running independent experiments

$T_K \sim \text{Erlang}(K, \lambda)$, $N \sim \text{Poisson}(\lambda)$

$$P(T_K > 1) = P(N \le K-1)$$

$$1 - F_{T_K}(1) = F_N(K-1) = Q(K, \lambda)$$

Poisson Process

	Fixed time, count # successes	Fixed # successes, measure time
Discrete	Bern / Binomial	Geom / Neg Bin. / Ext Neg Bin
Continuous	Poisson	Exp / Erlang / Gamma

There's a relationship between Poisson & Erlang (Exp). Is there an analogous relationship between Binomial & Neg. Binomial?

YES!

What is the prob. there are 0 events by time $= 50$ if prob. success is 0.1?

$N \sim \text{Binomial}(50, 0.1)$, $T \sim \text{NegBin}(1, 0.1)$

$$P(T > 49) = 1 - F_T(49) = F_N(0)$$

What is the prob. of $\le K$ events by time $= t$ if prob success is p?

$N \sim \text{Binom}(t, p) = \binom{t}{x} p^x (1-p)^{t-x}$, $T \sim \text{NegBin}(K+1, p) := \binom{K+x}{K}(1-p)^x p^{K+1}$

$$P(T \ge t-K) = 1 - F_T(t-K-1) = F_N(K)$$

$$\Rightarrow 1 - \sum_{i=0}^{t-K-1} \binom{K+i}{K} (1-p)^i p^K = \sum_{i=0}^{K} \binom{t}{i} p^i (1-p)^{t-i}$$

A new combinatorial identity!

$$T \sim \text{Erlang}(k,\lambda) := \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t\geq 0} = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t\geq 0}$$

$k \in \mathbb{N}, \lambda \in (0,\infty)$

$$T \sim \text{NegBin}(k,p) := \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$

$k \in \mathbb{N}, p \in (0,1)$

$$= \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$

What if $k \in (0,\infty)$? What would a fractional # of successes

Erlang = "Gamma"

NegBin = "ExtNegBin"

"extended"

$$X \sim \text{Gamma}(k,\lambda) = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t\geq 0}$$

Usually parameterized and written as:

$$X \sim \text{Gamma}(\alpha,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\geq 0}$$

$\underbrace{\alpha > 0}_{\text{\# of successes to wait for must be +}},\ \underbrace{\beta > 0}_{\text{prob of success per unit time positive}}$

Transformations of Discrete Variables

$y = x+3 = g(x)$
$x = y-3 = g^{-1}(y)$

$X \sim \text{Bern}(p) = p^x(1-p)^{1-x}\mathbb{1}_{x\in\{0,1\}} = p_X(x)$

$$Y = X+3 \sim \begin{cases} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)}\mathbb{1}_{y\in\{3,4\}} = p_Y(y)$$

If $Y = g(X)$, $\text{Supp}(Y) = g(\text{Supp}(X))$ and maybe $p_Y(y) = p_X(g^{-1}(y))$? Proof...

$$p_Y(y) = P(Y=y) = P(g(X)=y) = P(X = g^{-1}(y)) = p_X(g^{-1}(y))$$

This assumes... g has an inverse and if it doesn't:

Consider $X \sim U(\{1,2,3,4,5,6,7,8,9,10\}) = \frac{1}{10}$, $Y = g(X) = \min\{X, 3\}$

y	$p_Y(y)$
1	$\frac{1}{10}$
2	$\frac{1}{10}$
3	$\frac{8}{10}$

If no inverse, If inverse

$$p_Y(y) = \sum_{\{x:\, g(x)=y\}} p_X(x) = \sum_{x\in\{g^{-1}(y)\}} p_X(x) = p_X(g^{-1}(y))$$

e.g. $$p_Y(3) = \sum_{\{x:\, g(x)=3\}} p_X(x) = \sum_{x\in\{3,4,\dots,10\}} p_X(x) = p_X(3)+\dots+p_X(10) = \frac{8}{10}$$

e.g. $X \sim Bin(n,p)$, $Y = X^3$

$p_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}}$ looks weird... true!

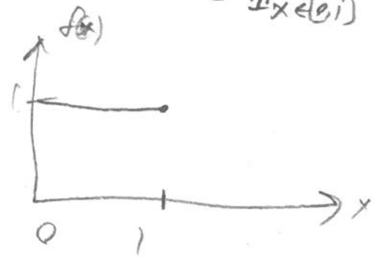
Transformations of Cont. r.v.'s

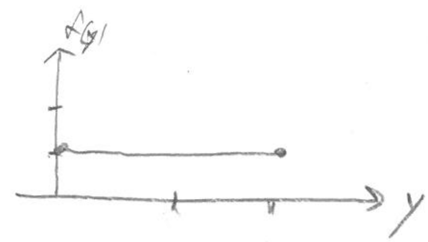
But does...

$$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$$

let $X \sim U(0,1)$, $Y = 2X = g(X)$, $\Rightarrow X = \frac{Y}{2} = g^{-1}(Y)$

$= \mathbb{1}_{x \in [0,1]}$





Likely $f_Y(y) = \frac{1}{2} \mathbb{1}_{y \in [0,2]}$

$f_Y(y) = f_X\left(\frac{y}{2}\right) = 1$ Something is wrong! This doesn't work for densities since densities are not probabilities. But CDF's represent prob's.

First consider g is a 1:1 function, strictly increasing ... why??

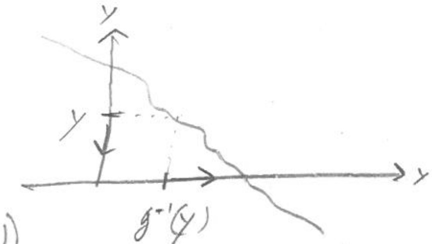
$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}\left[F_X(g^{-1}(y))\right] \overset{\text{Chain Rule}}{=} F_X'(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$

$$= f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$ Note $\frac{d}{dy}[g^{-1}(y)] > 0$ by assumption

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

If g is 1:1 strictly decreasing



$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = -F_X'(g^{-1}(y))\, \frac{d}{dy}[g^{-1}(y)] \qquad \text{Note: } \frac{d}{dy}[g^{-1}(y)] < 0 \text{ by assumption}$$

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

Derive some rules! "Shifting and/or scaling"

let $Y = g(X) = aX + c$ where $a, c \in \mathbb{R}$ constants but $a \neq 0$ o/w $Y \sim Deg(c)$

$$\Rightarrow X = g^{-1}(Y) = \frac{Y-c}{a} \qquad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|a|}$$

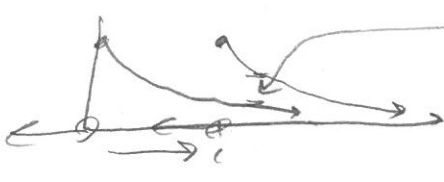
$$\Rightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-c}{a}\right)$$

If $Y = -X \Rightarrow a = -1, c = 0 \Rightarrow f_Y(y) = \frac{1}{|-1|} f_X(-y) = f_X(-y)$

If $Y = X + c \Rightarrow a = 1$, $f_Y(y) = f_X(y - c)$ shifted distr.

e.g. $X \sim Exp(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)} \Rightarrow f_Y(y) = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y-c \in (0,\infty)}$

$$= \underline{e^{\lambda c}}\, \lambda e^{-\lambda y} \mathbb{1}_{y \in (c,\infty)}$$



constant multiple

$X \sim U(0,1)$, $Y = aX + c \sim ?$

if $c < c+a \Rightarrow a > 0$

$$f_Y = \frac{1}{|a|} f_X\left(\frac{y-c}{a}\right) = \frac{1}{|a|} \mathbb{1}_{\frac{y-c}{a} \in [0,1]} = \frac{1}{|a|} \mathbb{1}_{y \in [c, c+a]}$$

HW

if $c+a < c \Rightarrow a < 0$ " $\frac{1}{|a|} \mathbb{1}_{y \in [c+a, c]}$