Probability Theory — MATH 621 — CLASS. 09/11/2019

$\vec{X} \sim \text{Multinomial}(n, \vec{p}) \Rightarrow X_j \sim \text{Binomial}(n, p_j) \ \forall j$

$$\vec{M} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix} \qquad M \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$$

matrix of r.v's

$$E[M] = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1m}] \\ \vdots & \ddots & \vdots \\ E[X_{n1}] & \cdots & E[X_{nm}] \end{bmatrix}$$

$$\text{If } \vec{X} \sim \text{Multinomial}(n, \vec{p}) \Rightarrow \vec{M} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$E[\vec{X}]$

$$\sigma^2 := Var[X] = E[X^2] - \mu^2$$

$$\sigma_{12} := Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j = E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

☆ If $X_1, X_2 \overset{iid}{\sim} \Rightarrow \sigma_{12} = 0$ ☆

Rules for Covariances : relationship between elements in vector.

1) $Cov[X, X] = Var[X]$ $\quad (\sigma_{11}^2 = \sigma_1^2)$

2) $Cov[X_i, X_j] = Cov[X_j, X_i]$

3) $Cov[X_1 + X_2, X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$

4) $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$ $\quad a_1, a_2 \in \mathbb{R}$

5) $Var[X_1 + \dots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j]$

$$\left\{ \underline{\text{Note}}: \quad Var[X_1 + X_2] = \sum_{i=1}^{2} \sum_{j=1}^{2} Cov[X_i, X_j] = \sigma_1^2 + 2\sigma_{12} + \sigma_2^2 \right\}$$

Variance / Covariance Matrix

$$\Sigma := Var[\vec{X}] = \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_k] \\ Cov[X_2, X_1] & Var[X_2] & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov[X_k, X_1] & \cdots & \cdots & Var[X_k] \end{bmatrix}$$

$$:= E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T$$

Properties of Σ:

1) Symmetric
2) Diagonal non-negative

If $X_1, \dots, X_n \overset{ind}{\sim}$

$$\Rightarrow \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k^2 \end{bmatrix} = I \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_k^2 \end{bmatrix}$$

If $X_1, \dots, X_n \overset{iid}{\sim}$

$$\Rightarrow \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2 \end{bmatrix} = \sigma^2 I$$

(Rules for Expectation and Variance of r.v. vectors)

$$E[\vec{X} + \vec{a}] = \begin{bmatrix} E[X_1 + a_1] \\ E[X_2 + a_2] \\ \vdots \\ E[X_k + a_k] \end{bmatrix} = \vec{\mu} + \vec{a} \qquad \vec{a} \in \mathbb{R}^k$$

$$E[\vec{a}^T \vec{X}] = E[a_1 X_1 + \dots + a_k X_k] = a_1 \mu_1 + \dots + a_k \mu_k = \vec{a}^T \vec{\mu}$$

let $A \in \mathbb{R}^{L \times K}$ — matrix of constants

$$E[A\vec{X}] = \begin{bmatrix} E[a_{11} X_1 + \dots + a_{1k} X_k] \\ E[a_{21} X_2 + \dots + a_{2k} X_k] \\ \vdots \\ E[a_{L2} X_2 + \dots + a_{Lk} X_k] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot}^T \vec{\mu} \\ \vec{a}_{2\cdot}^T \vec{\mu} \\ \vdots \\ \vec{a}_{L\cdot} \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$$Var[\vec{a}^T \vec{X}] = Var[a_1 X_1 + \dots + a_k X_k]$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} Cov[a_i X_i, a_j X_j]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j \sigma_{ij}$$

let $\vec{c} \in \mathbb{R}^k$ and $V \in \mathbb{R}^{k \times k}$

Consider the quantity : $\vec{c}^T V \vec{c}$

quadratic form in c with determining matrix V.

$$\vec{c}^T V \vec{c} = [c_1 \cdots c_k] \begin{bmatrix} c_1 V_{11} + \cdots + c_k V_{1k} \\ c_1 V_{21} + \cdots + c_k V_{2k} \\ \vdots \\ c_1 V_{k1} + \cdots + c_k V_{kk} \end{bmatrix}$$

$$= \left. \begin{matrix} c_1 c_1 V_{11} + c_1 c_2 V_{12} + \cdots + c_1 c_k V_{1k} + \\ c_2 c_1 V_{21} + c_2 c_2 V_{22} + \cdots + c_2 c_k V_{2k} + \\ \vdots \\ c_k c_1 V_{k1} + c_k c_2 V_{k2} + \cdots + c_k c_k V_{kk} \end{matrix} \right\}$$

Summing many terms
NOT a matrix.

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j V_{ij} \quad \star$$

So,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j \sigma_{ij} = \vec{a}^T \underline{\Sigma} \vec{a}$$

$\Sigma := Var[\vec{X}]$

Sidetrack to Finance:

Let $[X_1, \ldots, X_k] = \vec{X}$ be the yearly returns of assets $1, \ldots, k$.

Let $[w_1, \ldots, w_k] = \vec{w}$ be a vector of weights s.t. $\vec{w}^T \cdot \vec{1} = 1$

Let $F = \vec{w}^T \vec{X}$ be your total portfolio yearly return

We want: $\mu_F = \mu_0$ with minimal variance so then select $\vec{w}$ s.t. $Var(\vec{F})$ is minimal.

$$\vec{w}^* = \underset{\vec{w}^T \cdot \vec{1} = 1}{\operatorname{argmin}} \left\{ \vec{w}^T \Sigma \vec{w} \right\}$$

Markowitz Optimal Portfolio Theory.

$$\vec{X} \sim \text{Multinomial}(n, \vec{p}) \Rightarrow X_j \sim \text{Binomial}(n, p_j) \ \forall j$$

$$\Sigma := Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & \sigma_{ij} & \cdots & \sigma_{ij} \\ \sigma_{ij} & \ddots & & \vdots \\ \vdots & & \ddots & \sigma_{ij} \\ \sigma_{ij} & \cdots & \sigma_{ij} & np_k(1-p_k) \end{bmatrix}$$

We know that $\sigma_{ij} < 0$

$$\sigma_{ij} = Cov[X_i, X_j] = E[X_i, X_j] - \mu_i \mu_j$$

$$= \sum_{x_1 \in \{0,\ldots,n\}} \sum_{x_2 \in \{0,1,\ldots,n\}} \left(x_1 x_2 \, p_{X_i X_j}(x_1, x_2) \right) - n^2 p_i p_j$$

We find this value next page.

Recall $X_i \sim \text{Binomial}(n, p_i)$; $X_j \sim \text{Binomial}(n, p_j)$

$$\left\{ \begin{array}{l} X_i = X_{1i} + X_{2i} + \dots + X_{ni} \quad \text{where } X_{1i}, X_{2i}, \dots, X_{ni} \overset{iid}{\sim} \text{Bern}(p_i) \\ X_j = X_{1j} + X_{2j} + \dots + X_{nj} \quad \text{where } X_{1j}, X_{2j}, \dots, X_{nj} \overset{iid}{\sim} \text{Bern}(p_j) \end{array} \right\}$$

$\Rightarrow X_{li}, X_{lj}$ dependent $\forall l$; $X_{li}, X_{mj} \overset{ind}{\sim}$ s.t $l \neq m$ ☆

it follows that $\vec{X}_{1\cdot}, \vec{X}_{2\cdot}, \dots, \vec{X}_{n\cdot} \overset{iid}{\sim} \text{Multinomial}(1, \vec{p})$ ☆

$$\begin{aligned}
\text{Cov}[X_i, X_j] &= \text{Cov}[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{kj}] \\
&= \sum_{l=1}^{n} \sum_{m=1}^{n} \text{Cov}[X_{li}, X_{mj}] \\
&= \sum_{l=1}^{n} \text{Cov}[X_{li}, X_{lj}] \leftarrow \text{b/c } \text{Cov}[X_{li}, X_{mj}] = 0 \text{ if } l \neq m \text{ which signifies difference in draw.} \\
&= \sum_{l=1}^{n} \left(E[X_{li}, X_{lj}] - p_i p_j \right) \\
&= \underline{\left(\sum_{l=1}^{n} E[X_{li}, X_{lj}] \right)} - n p_i p_j \\
&= -n p_i p_j \ \text{☆☆☆}
\end{aligned}$$

$$\begin{aligned}
&\text{Cov}[X_{li}, X_{lj}] \\
&= E[X_{li}, X_{lj}] \\
&= \sum_{x_1 \in \{0,1\}} \sum_{x_2 \in \{0,1\}} \left(x_1 x_2 P_{X_{li} X_{lj}}(x_1, x_2) \right) - p_i p_j \\
&= P_{X_{li} X_{lj}}(1,1) \\
&= 0
\end{aligned}$$

☆ only non zero is when $x_1 = 1$ AND $x_2 = 1$ ☆

So, UPDATED Σ :

$$\begin{bmatrix} n p_1(1-p_1) & (-n p_i p_j) & \cdots & (-n p_i p_j) \\ (-n p_i p_j) & \ddots & & \vdots \\ \vdots & & \ddots & (-n p_i p_j) \\ (-n p_i p_j) & \cdots & (-n p_i p_j) & n p_k(1-p_k) \end{bmatrix}$$

$\sigma_{ij} = -n p_i p_j$

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