# Derivation of Standard Logistic Distribution 10/16

$$f_Y(y) = f_X\underbrace{(g^{-1}(y))}_{\text{Step 1}} \left| \underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{Step 2}} \right|$$

$$X \sim Exp(1) = e^{-x}\,\mathbb{1}_{x\in(0,\infty)}$$

$$Y = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{-x}-1) = g(x)$$

$$\Rightarrow e^{Y} = e^{x}-1 \Rightarrow e^{X} = e^{Y}+1 \Rightarrow X = \ln(e^{Y}+1) = g^{-1}(Y)$$

$$\frac{d}{dy}\left[\ln(e^{y}+1)\right] = \frac{e^{y}}{e^{y}+1} > 0 \quad \forall y$$

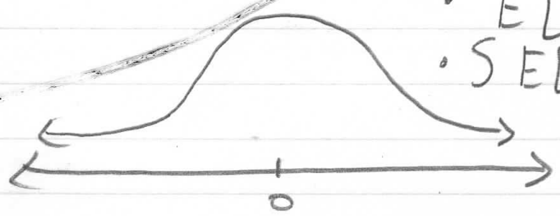
$$f_Y(y) = \underbrace{e^{-\ln(e^{y}+1)}}_{e^{\ln\left(\frac{1}{e^{y}+1}\right)}}\,\mathbb{1}_{\ln(e^{y}+1)\in(0,\infty)} \cdot \frac{e^{y}}{e^{y}+1}$$

$$= \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{e^{-2y}}{(e^{-y})^2} = \boxed{\frac{e^{-y}}{(1+e^{-y})^2}} = Logistic(0,1)$$

$$L = \sigma Y + \mu \sim \boxed{\frac{1}{\sigma} \cdot \frac{e^{-\left(\frac{\ell-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{\ell-\mu}{\sigma}\right)}\right)^2}} \sim Logistic(\mu,\sigma)$$

where $\sigma > 0$

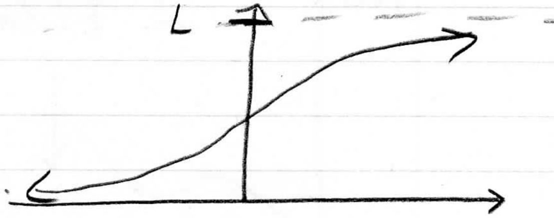
$\mu \in \mathbb{R}$



- $E[L] = \mu$
- $SE[L] = \frac{\sigma\pi}{\sqrt{3}}$

- **Logistic Function**

- $\ell(x) := \dfrac{L}{1+e^{-k(x-c)}}$




where: L is max value
k is steepness
c is center

- Substitution

let $u = e^t + 1$
$\Rightarrow e^t = u - 1$
$\Rightarrow t = \ln(u-1)$

$\dfrac{du}{dt} = e^t$

$\Rightarrow dt = \dfrac{1}{e^t} du$

$\Rightarrow dt = \dfrac{1}{u-1} du$

$t = -\infty \Rightarrow u = 1$, $t = y$
$\Rightarrow u = e^y + 1$

$$F_Y(y) = \int_{-\infty}^{y} \frac{e^t}{(e^t+1)^2}\, dt$$

$$= \int_1^{e^y+1} \frac{u-1}{u^2} \frac{1}{u-1}\, du$$

$$= \left[-u^{-1}\right]\Big|_1^{e^y+1}$$

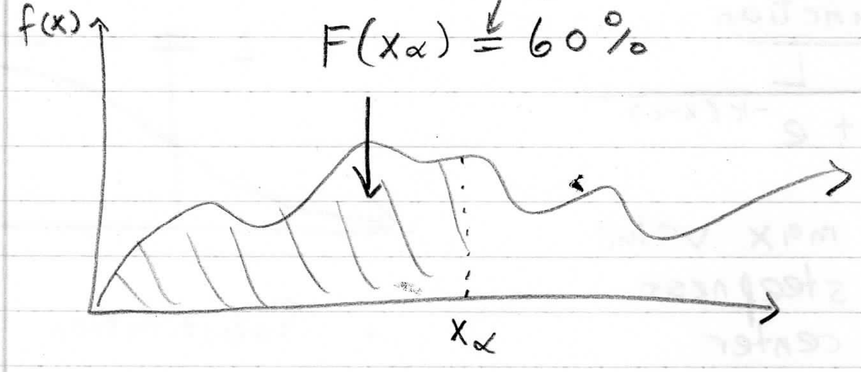
$$= 1 - \frac{1}{e^y+1}$$

$$= \frac{e^y}{e^y+1}$$

$$= \frac{1}{1+e^{-y}}$$

e.g

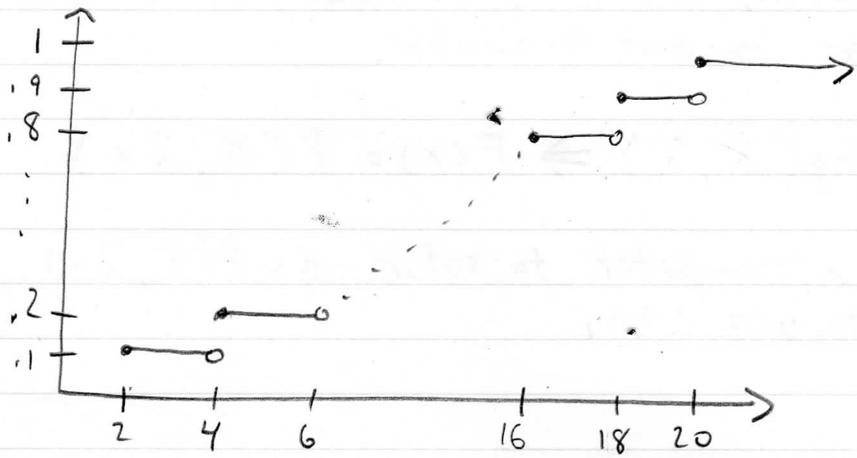
$F(x_\alpha) \stackrel{\downarrow}{=} 60\%$




Find minimum $x^*$ s.t $P(X \le x) \ge q \in (0,1)$.
This is called the <u>quantile operator</u> $Q[X, q]$
where $q$ is the "quantile" + $100 \cdot q$ is the "percentile"

- Quantile Operator (Discrete)

ex. $X \sim U(\{2, 4, 6, \ldots, 20\})$




$Q[X, 0.1] = 2$
$Q[X, 1] = 20$
$Q[X, 0.9] = 18$
$Q[X, 0.85] = 18$

- Define <u>median</u> of r.v $X := Med[X] = Q[X, \frac{1}{2}]$

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- If $X$ is continuous w/ monotonic increasing CDF, then $Q[X, q] = F_X^{-1}(q)$ "the percentile function."

ex. $X \sim Exp(\lambda) = \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x} = q$

$\Rightarrow e^{-\lambda x} = 1 - q \Rightarrow \lambda x = \ln\left(\frac{1}{1-q}\right)$

$\Rightarrow F_X^{-1}(q) = \frac{1}{\lambda} \ln\left(\frac{1}{1-q}\right)$

$Med[X] = \frac{1}{\lambda} \ln(2)$

Often times, the CDF $F(x)$ is not available in closed form. If it is available, often times its inverse is not available.

ex. $X \sim Erlang(K, \lambda) \Rightarrow F(x) = P(K, \lambda x)$

We need a computer to solve $q = P(K, \lambda x)$ for $x$ as best as we can.

# Derivation of Pareto

Let $X \sim Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)}$

$Y = k e^X$

$$\Rightarrow \frac{y}{k} = e^x \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k)$$
$$= g^{-1}(y)$$

$$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{1}{y}$$

$$f_Y(y) = \lambda e^{-\lambda \ln\left(\frac{y}{k}\right)} \mathbb{1}_{\underbrace{\ln(y) - \ln(k) \in (0,\infty)}} \cdot \frac{1}{y}$$

$\ln y \in (\ln(k), \infty)$

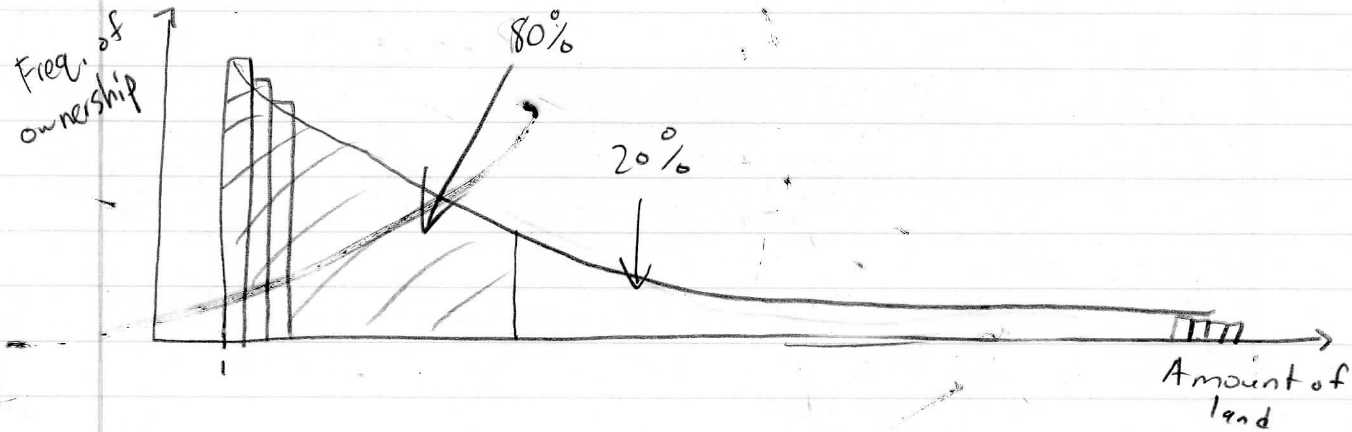
$\Rightarrow y \in (k, \infty)$

$$\left( \text{Note: } -\lambda \ln\left(\frac{y}{k}\right) = \ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right) = \ln\left(\frac{k^\lambda}{y^\lambda}\right) \right)$$

$$= \frac{\lambda}{y}\frac{k^\lambda}{y^\lambda} = \boxed{\frac{\lambda k^\lambda}{y^{\lambda+1}} \mathbb{1}_{y \in (k,\infty)}} = \text{Pareto } I(k, \lambda)$$

//

Let $k = 1$



CDF

$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}}\,dt = \lambda k^\lambda \left[\frac{-t^{-\lambda}}{\lambda}\right]\Big|_k^y$$

$$= k^\lambda\left(k^{-\lambda} - y^{-\lambda}\right) = \left(1 - \left(\frac{k}{y}\right)^\lambda\right) \mathbb{1}_{y\in(k,\infty)}$$

Let $q = 1 - \left(\frac{k}{y}\right)^\lambda$

$$\Rightarrow 1-q = \frac{k^\lambda}{y^\lambda}$$

$$\Rightarrow y^\lambda = \frac{k^\lambda}{1-q}$$

$$\Rightarrow y = \frac{k}{(1-q)^{\frac{1}{\lambda}}}$$

$$= k(1-q)^{-\frac{1}{\lambda}}$$

$$= F_Y^{-1}(q)$$

Note: If $k=1 \Rightarrow f_Y(y) = \frac{\lambda}{y^{\lambda+1}} \mathbb{1}_{y\in(1,\infty)}$

$$F_Y^{-1}(q) = (1-q)^{-\frac{1}{\lambda}}$$

- Let $L(q)$ be the proportion of land owned by all the people who themselves own $\leq q$.

$$L(q) = \frac{\int_1^q y f_Y(y)\,dy}{\int_1^\infty y f_Y(y)\,dy}$$

$$= \frac{\lambda \left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]\Big|_1^q}{\lambda \left[\frac{y^{-\lambda+1}}{-\lambda+1}\right]\Big|_1^\infty}$$

$$= \frac{q^{1-\lambda} - 1}{0 - 1}$$

$$= \boxed{1 - q^{1-\lambda}}$$

- Set $q = F^{-1}(q)$

Set $L(q) = 1 - q = \bar{q}$

$$\Rightarrow \bar{q} = 1 - \left(\bar{q}^{-\frac{1}{\lambda}}\right)^{1-\lambda}$$

$$\Rightarrow q = \bar{q}^{\frac{\lambda-1}{\lambda}} = \bar{q}^{1-\frac{1}{\lambda}}$$

$$\Rightarrow \ln(q) = \left(1 - \frac{1}{\lambda}\right)\ln(\bar{q})$$

$$= \ln(\bar{q}) - \frac{1}{\lambda}\ln(\bar{q})$$

$$\Rightarrow \ln(\bar{q}) - \ln(q) = \frac{1}{\lambda}\ln(\bar{q})$$

$$\Rightarrow \frac{1}{\lambda} = \frac{\ln(\bar{q}) - \ln(q)}{\ln(\bar{q})}$$

$$\Rightarrow \lambda = \frac{\ln(\bar{q})}{\ln(\bar{q}) - \ln(q)} = \frac{\ln(\bar{q})}{\ln\left(\frac{\bar{q}}{q}\right)} = \log_{\frac{\bar{q}}{q}}(\bar{q})$$

## Pareto's 80-20 Principle

Let $q = 0.8$

$\bar{q} = 0.2$

$\log_{0.25}(0.2) \approx 1.161$

$Y \sim$ Parero I $(1, 1.161)$ – you get Pareto Principle.

# Derivation of Laplace

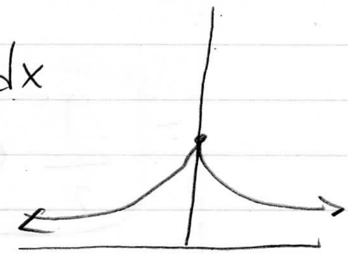

Let $X_1, X_2 \overset{iid}{\sim} Exp(1) = e^{-x}\mathbb{1}_{x\in(0,\infty)}$

$$D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} = X + Y$$


$$Y \sim e^{-(-y)}\mathbb{1}_{-y\in(0,\infty)} = e^{y}\mathbb{1}_{y\in(-\infty,0)}.$$

$$f_D(d) \overset{(ind)}{=} \int_{Supp[X]} f_X(x)\, f_Y(d-x)\,\mathbb{1}_{d-x\in Supp[Y]}\,dx$$

$$= \int_0^\infty (e^{-x})\left(e^{d-x}\,\mathbb{1}_{\underbrace{\underbrace{d-x\in(-\infty,0)}_{x-d\in(0,\infty)}}_{x\in(d,\infty)}}\right)dx$$



Double Exponential

$$= e^{d}\begin{cases}\int_d^\infty e^{-2x}\,dx & \text{if } d \geq 0\\ \int_0^\infty e^{-2x}\,dx & \text{if } d < 0\end{cases}$$

$$= e^{d}\begin{cases}-\frac{1}{2}\left[e^{-2x}\right]\Big|_d^\infty & \text{if } d \geq 0\\ -\frac{1}{2}\left[e^{-2x}\right]\Big|_0^\infty & \text{if } d < 0\end{cases}$$

$$= \frac{1}{2}e^{d}\begin{cases}e^{-2d} & \text{if } d \geq 0\\ 1 & \text{if } d < 0.\end{cases}$$

$$= \frac{1}{2} \begin{cases} e^{-d} & \text{if } d \geq 0 \\ e^{d} & \text{if } d < 0 \end{cases}$$

$$\left( \begin{array}{l} \text{if } d \geq 0 \Rightarrow d = |d| \\ \text{if } d < 0 \Rightarrow -d = |d| \Rightarrow d = -|d| \end{array} \right)$$

$$= \boxed{\frac{1}{2} e^{-|d|}} = \text{Laplace}(0, 1)$$

Set $L = \mu + \sigma D \sim \boxed{\frac{1}{2\sigma} e^{-\frac{|l-\mu|}{\sigma}}} = \text{Laplace}(\mu, \sigma)$

where $\mu \in \mathbb{R}$
$\sigma > 0$