Lec 7 Math 621 9/18/19

Continuous r.v.'s X have $|Supp[X]| = |\mathbb{R}|$

they have no valid PMF. the prob $P(X=x) = 0 \ \forall x$

They have valid CDF's. And, the derivative of the CDF is useful:

$$f_X(x) := \frac{d}{dx}\left[F_X(x)\right]$$ By FTC

prob. density function (PDF)

$$P(X \in [a,b]) = F(b) - F(a) = \int_a^b f_X(x)\,dx$$

Most often, the PDF is available in closed form but the CDF is not

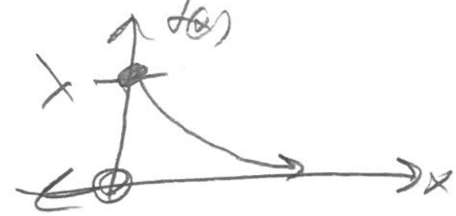
Properties:

$\int_{\mathbb{R}} f(x)\,dx = 1$ since $\underbrace{\lim_{x \to \infty} F(x)}_{1} - \underbrace{\lim_{x \to -\infty} F(x)}_{0} = 1$ by def of CDF

$f(x) \geq 0$ since $F(x)$ is monotonically increasing.

$$Supp(X) := \{x : f(x) > 0\}$$

$$X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}}_{f(x),\ \text{the PDF}}$$

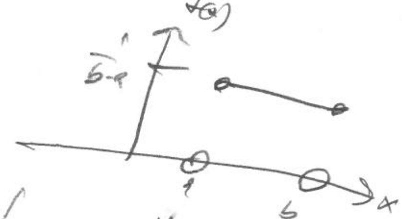


Cont r.v. notation is defined by PDF and discrete r.v. notation, PMF.

continuous uniform ("uniform")

Another cont. r.v. is $X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in (a,b)}$

param space: $a \in \mathbb{R}$, $b > a$, $\text{Supp}(X) = [a,b]$



If $a=0, b=1$ $\quad X \sim U(0,1) = \mathbb{1}_{x \in (0,1)}$, the standard uniform

Uniform can be built from the limit of discrete uniforms.

jdf

if $X_1, \ldots, X_k$ ind

iid

For $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix}$ $\quad f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdots f_{X_k}(x_k) = f(x_1) \cdots f(x_k)$

$$\int \cdots \int_{\mathbb{R}^k} f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k = 1$$

To get prob's from a jdf, we use integrations over sets.

For ex. in 2-d ...

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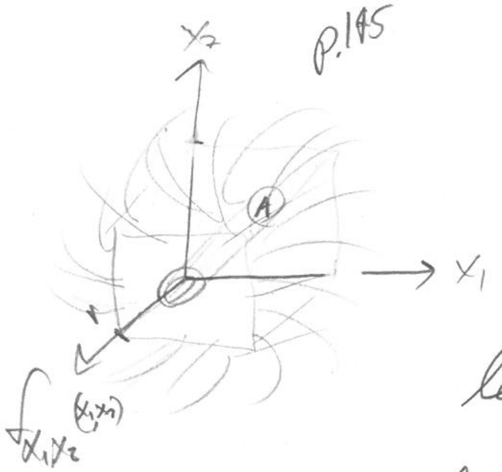
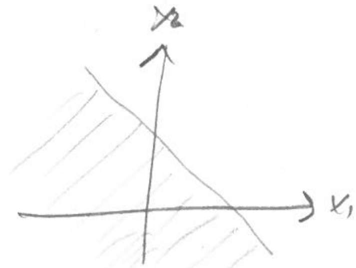


$$P(A) = \iint_A f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

let $T = X_1 + X_2 \sim f_T(t) = ?$



let $A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underset{\substack{\| \\ x_2 \le t - x_1}}{x_1 + x_2 \le t} \right\}$

$$F_T(t) = P(T \le t) = P(A_t) = \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1 = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty,\, t-x_1]} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

reindex... let $x_1 = x$, $x_2 = v - x \Rightarrow v = x_2 + x$, $dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$= \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\,dv\,dx$$

$$F_T(t) = \int_{-\infty}^{t} \left(\int_{x \in \mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx \right) dv$$

$$f_T(t) = \frac{d}{dt}\left(F_T(t)\right) = \frac{d}{dt}\left[\int_{-\infty}^{t} \left(\int_{x \in \mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx \right) dv \right] = \int_{x \in \mathbb{R}} f_{X_1,X_2}(x, t-x)\,dx$$

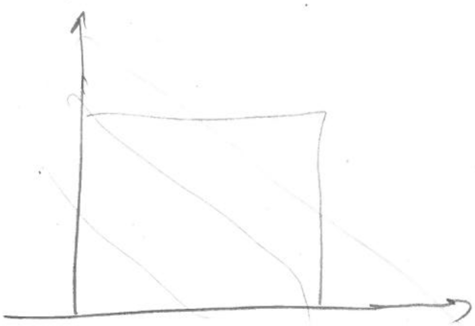
general conv. formula

$$= \int_{x \in \mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\,dx \quad \text{for } X_1, X_2 \overset{ind}{\sim} \quad \overset{!!}{=} f_{X_1}(x) * f_{X_2}(x)$$

$$= \int_{Supp(X_1)} f_{X_1}(x)\, f_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp(X_2)}\,dx \quad \text{for dd-style PDF notation}$$

$$= \int_{Supp(X)} f(x)\, f(t-x)\, \mathbb{1}_{t-x \in Supp(X)}\,dx \quad \text{if } X_1, X_2 \overset{iid}{\sim}, \text{ dd-style}$$

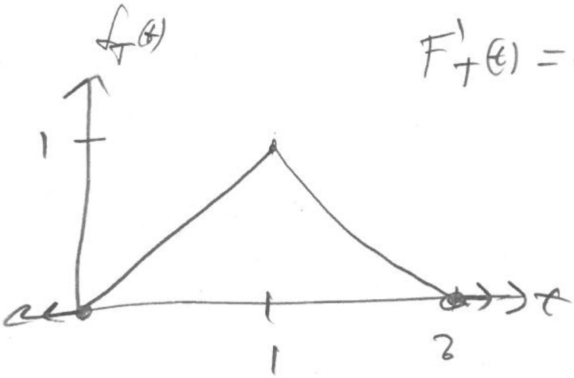
$X_1, X_2 \overset{iid}{\sim} U(0,1)$ $\quad T = X_1 + X_2 \sim f_T(t) = ?$

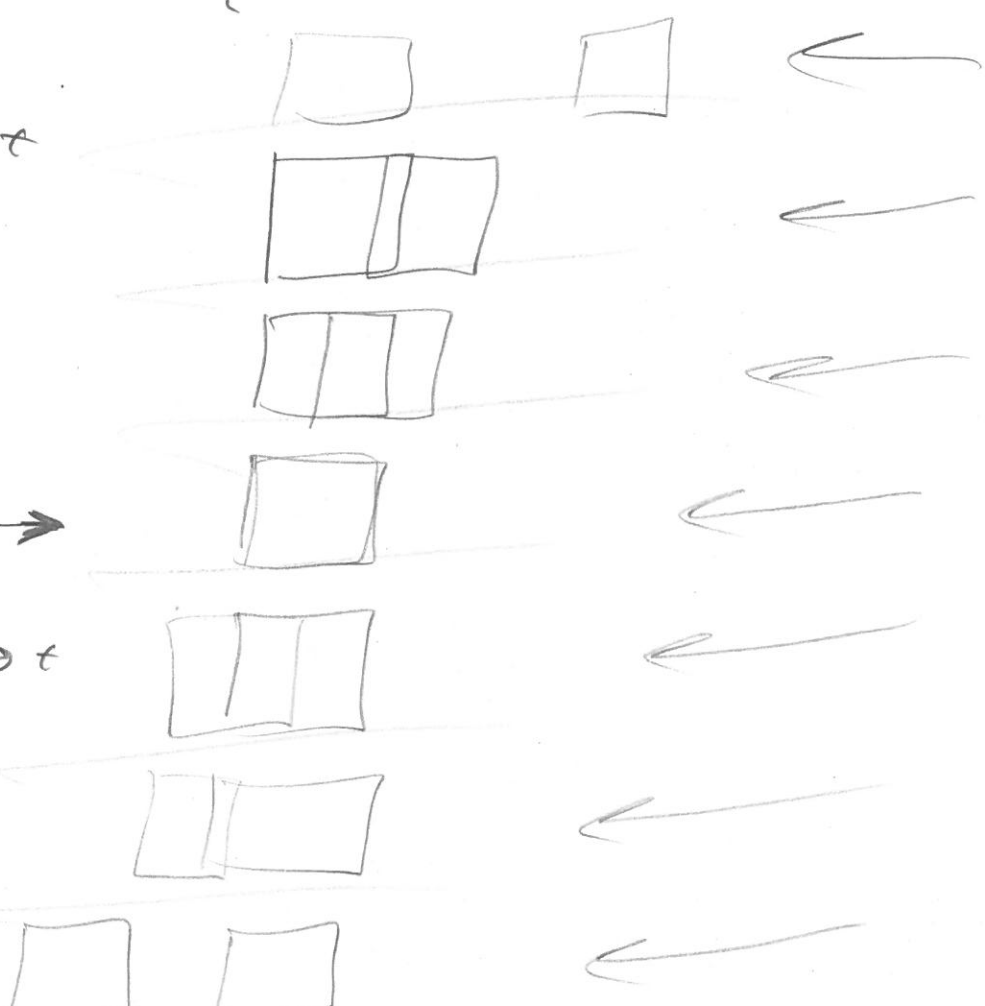


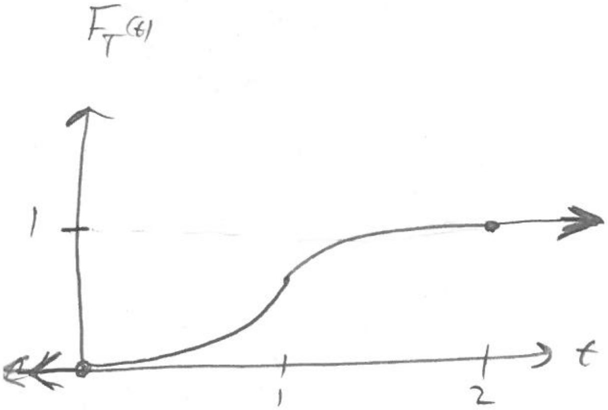
CDF method

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2/2 & \text{if } t \in [0,1] \\ t^2/2 - 2\frac{(t-1)^2}{2} & \text{if } t \in (1,2] \\ 1 & \text{if } t > 2 \end{cases} \qquad -\frac{t^2}{2} + 2t + 1$$

$$F_T'(t) = f_T(t) = \begin{cases} t & \text{if } t \in (0,1) \\ 2-t & \text{if } t \in [1,2] \\ 0 & \text{o/t} \end{cases}$$



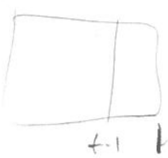




CDF method dirty... Straight from def? Not much easier...

$$f_T(t) = \int_{\underbrace{Supp(X)}_{(0,1)}} \underbrace{f(x)}_{1}\ \underbrace{f(t-x)}_{1}\ \underbrace{1\!\!1_{t-x \in \underbrace{Supp(X)}_{(0,1)}}}_{1\!\!1_{x-t \in [-1,0)}}\ dx$$

$$= \int_0^1 1\!\!1_{x \in [t-1,t]}\, dx = \begin{cases} 0 \text{ if } t<0 \\ t \text{ if } t \in [0,1) \\ 1-(t-1) = 2-t \text{ if } t \in [1,2) \\ 0 \text{ if } t \geq 2 \end{cases}$$



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$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) \qquad T_2 = X_1 + X_2 \sim f_T(t) = ?$ $\qquad Supp(T) = (0,\infty)$

$$f_{T_2}(t) = \int_0^\infty (\lambda e^{-\lambda x})\, \lambda e^{-\lambda(t-x)}\, 1\!\!1_{t-x \in (0,\infty)}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty 1\!\!1_{x \leq t}\, dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t}$$

$T = X_1 + X_2 + X_3 \sim ? = T_2 + X_3$

$$f_{T_3}(t) = \int_0^\infty (x \lambda^2 e^{-\lambda t})(\lambda e^{-\lambda(t-x)}\, 1\!\!1_{x \leq t})\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x\,dx = \frac{1}{2} t^2 \lambda^3 e^{-\lambda t}$$

$$T = X_1 + X_2 + X_3 + X_4 = T_3 + X_4$$

$$f_{T_4}(t) = \int_0^\infty \left(\frac{1}{2} x^2 \lambda^3 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\,dx$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$$

$$f_{T_5}(t) = \int_0^\infty \left(\frac{1}{2 \cdot 3} x^3 \lambda^4 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2 \cdot 3} \lambda^5 e^{-\lambda t} \int_0^t x^3\,dx$$

$$= \frac{1}{2 \cdot 3 \cdot 4} t^4 \lambda^5 e^{-\lambda t}$$

$$T = X_1 + \dots + X_k \sim \text{Erlang}(k, \lambda) := \frac{1}{(k-1)!} t^{k-1} \lambda^k e^{-\lambda t}$$

Param Space $k \in \mathbb{N}$, $\underbrace{\lambda \in (0, \infty)}_{\text{same as before}}$

$$\text{Erlang}(1, \lambda) = \text{Exp}(\lambda)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\text{NegBin}(1, p) = \text{Geom}(p)$$