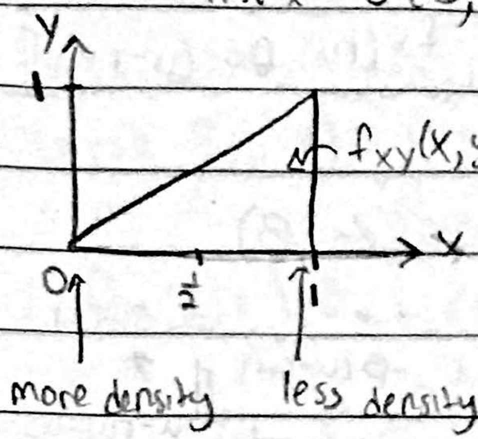
Conditional Densities p.155

consider $X \sim U(0,1)$

$Y|X=x \sim U(0,x)$



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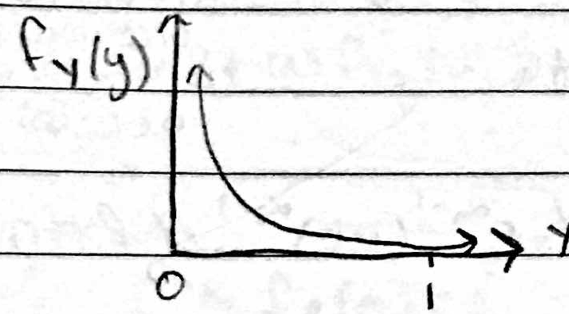
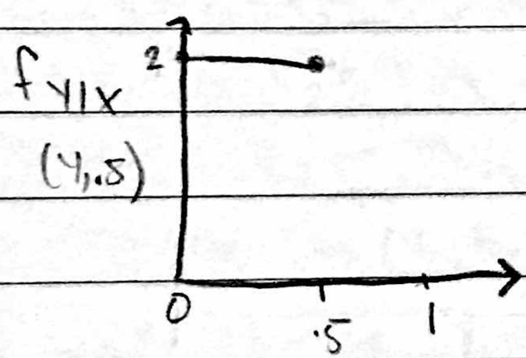
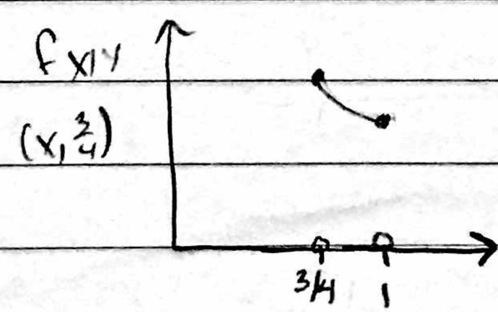






Formulas:

- Margining → $f_Y(y)\int_{\mathbb{R}} f_{XY}(x,y)\,dy$, $f_X(x)=\int_{\mathbb{R}} f_{XY}(x,y)\,dx$

- Def. of conditional probability / density →

$$f_{X|Y}(x,y)=\frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y)>0$$

↳ $f_{XY}=f_{X|Y}\cdot f_Y(y)$

- Baye's Rule → $f_{X|Y}(x,y)=\dfrac{f_{Y|X}(x,y)\,f_X(x)}{f_Y(y)}$

- Baye's Theorem → $f_{X|Y}(x,y)=\dfrac{f_{Y|X}(x,y)\,f_X(x)}{\int_{\mathbb{R}} f_{Y|X}(x,y)\,f_X(x)\,dx}$

back to example - $f_X(x)=\mathbb{1}_{x\in[0,1]}$

$f_{Y|X}(x,y)=\frac{1}{x}\mathbb{1}_{y\in[0,x]}$

$f_{X,Y}(x,y)=f_{Y|X}(x,y)\,f_X(x)=\frac{1}{x}\mathbb{1}_{y\in[0,x]}\mathbb{1}_{x\in[0,1]}=\frac{1}{x}\mathbb{1}_{0\le y\le x\le 1}$

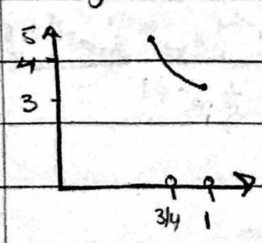
$$f_Y(y)=\int_{\mathbb{R}}\frac{1}{x}\underbrace{\mathbb{1}_{0\le y\le x\le 1}}_{\mathbb{1}_{x\in[y,1]}\mathbb{1}_{y\in[0,1]}}dx = \mathbb{1}_{y\in[0,1]}\int_y^1\frac{1}{x}\,dx=[\ln(x)]_y^1\,\mathbb{1}_{y\in[0,1]}$$

$$=-\ln(y)\,\mathbb{1}_{y\in[0,1]}$$

$$f_{X|Y}(x,y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\frac{\frac{1}{x}\mathbb{1}_{0\le y\le x\le 1}}{\underbrace{-\ln(y)\mathbb{1}_{y\in[0,1]}}_{\text{undefinedness of } f_{XY}}}=\frac{-1}{x\ln(y)}\mathbb{1}_{x\in[y,1]}$$

- let $y=3/4$ $\qquad f_{X|Y}(x,3/4)\approx\frac{3.5}{x}\mathbb{1}_{x\in[3/4,1]}$

$f_{X|Y}(3/4,3/4)=4.6 \quad f_{X|Y}(1,3/4)=3.5$



$\angle$ discrete $\quad$ Δ continuous

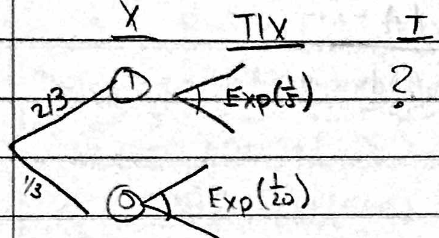
~ Mixture Distributions:

- 2/3 of the time, no internet traffic
and download speeds are $T \sim \text{Exp}(\frac{1}{5})$ ... fast.
But 1/3 of the time, $T \sim \text{Exp}(\frac{1}{20})$ ... slow.
What is the marginal distr. of T? This is called a "mixture model" or "mixture distr." or "multi-level model"

Let $X \sim \text{Ber}(\frac{2}{3})$

where $x = 1 \to$ no traffic

$x = 0 \to$ traffic

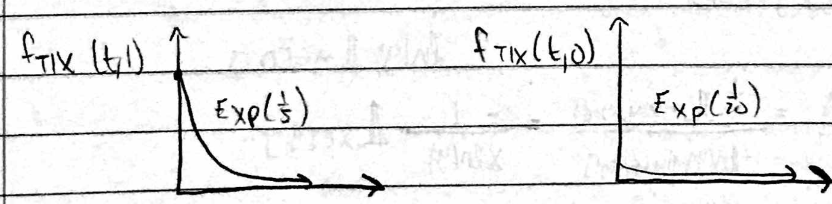


$$f_T(t) = \sum_{x \in \text{Supp}[X]} f_{X,T}(x,t) = \sum_{x \in \text{Supp}[X]} p_X(x) \cdot f_{T|X}(x,t) = \sum_{x \in \{0,1\}} \overbrace{\left(\left(\tfrac{2}{3}\right)^x \left(\tfrac{1}{3}\right)^{1-x}\right)\left(\tfrac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \tfrac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}\right)}^{f_{X,T}(x,t)}$$

$$= \frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}$$

$T \sim \text{Exp}(\lambda) = \lambda e^{-\lambda t}$

$E[T] = \frac{1}{\lambda}$



If $t = 25$ min, what is the probability there was traffic?

$\downarrow$

$$p_{X|T}(x,t) \overset{\text{Bayes Rule}}{=} \frac{f_{T|X}(t,x)\, p_X(x)}{f_T(t)} = \frac{\left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{1-x} \left(\frac{1}{5} e^{-\frac{1}{5}t} \mathbb{1}_{x=1} + \frac{1}{20} e^{-\frac{1}{20}t} \mathbb{1}_{x=0}\right)}{\frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t} + \frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}}$$

$$= \text{Bern}\left(\frac{\frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t}}{\frac{2}{3} \cdot \frac{1}{5} e^{-\frac{1}{5}t} + \frac{1}{3} \cdot \frac{1}{20} e^{-\frac{1}{20}t}}\right)$$

$\text{Supp}[X|T] = \{0,1\}$

$$= \frac{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}+\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}$$

traffic

$$P_{X|T}(0, 25) = 1 - \overbrace{\frac{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}}{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}+\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}} \approx .84$$

- $X \sim U(0,1)$ $\quad Y|X=x \sim U(0,x)$

$\underline{X}$ $\quad \underline{Y|X}$ $\quad \underline{Y}$

$U(0,1)$ $\quad U(0,x)$ $\quad ?$

$$f_Y(y) = \int_{\mathbb{R}} \underbrace{f_{Y|X}(x,y)}_{\text{model}} \underbrace{f_X(x)}_{\text{mixing distribution}}\, dx$$

If mixing distr. is continuous, $f_Y(y)$ is sometimes called a "compound distribution".

- $X \sim Gamma(\alpha,\beta)$, $\quad Y|X=x \sim Poisson(x)$

$\underline{X}$ $\quad \underline{Y|X}$ $\quad \underline{Y}$

$Gamma(\alpha,\beta)$ $\quad Poisson(x)$ $\quad ?$

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y) f_X(x)\, dx = \int_{\mathbb{R}} \left(\frac{e^{-x}x^y}{y!}\mathbb{1}_{y\in\mathbb{N}_0}\right)\left(\underset{\uparrow\text{constant}}{\frac{\beta^\alpha}{\Gamma(\alpha)}} x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x\in(0,\infty)}\right) dx$$

$$\propto \frac{1}{y!}\mathbb{1}_{y\in\mathbb{N}_0}\int_0^\infty x^{y+\alpha-1}e^{-(\beta+1)x}\, dx$$

$$= \frac{1}{y!}\frac{\Gamma(y+2)}{\Gamma(\beta+1)^{y+\alpha}}\mathbb{1}_{y\in\mathbb{N}_0} \propto ExtNegBin\left(\alpha, \frac{\beta}{\beta+1}\right)$$

- $X \sim Beta(\alpha,\beta)$ $\quad Y|X=x,n \sim Bin(n,x)$

$\underline{X}$ $\quad \underline{Y|X,n}$ $\quad \underline{Y}$

$Beta(\alpha,\beta)$ $\quad Bin(n,x)$ $\quad ?$

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y) f_X(x)\, dx = \int_{\mathbb{R}} \left(\binom{n}{y}x^y(1-x)^{n-y}\right)\left(\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in(0,1)}\right) dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)}\underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{y+\beta+1}\, dx}_{\text{Beta function}} = \frac{\binom{n}{y}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) = \underline{BetaBinomial(\alpha,\beta,n)}$$

- $X \sim Gamma(\alpha, B)$ $\quad Y|X=x \sim Exp(x)$

$\underline{X}$	$\underline{Y\|X=x}$	$\underline{Y}$	
$\sim Gamma(\alpha,B)$	$\sim Exp(\lambda)$	?	$\rightarrow Y \sim Lomax(B, x)$

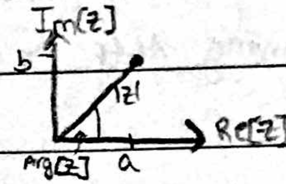
~ $a, b \in \mathbb{R}$

complex #'s

$z := a + bi \in \mathbb{C}$

where $i := \sqrt{-1}$

$Re(z) = a \quad Im[z] = b$



$|z| = \sqrt{a^2+b^2}$, $Arg[z] = \arctan\left(\frac{b}{a}\right)$

$i^2 = -1$

$i^3 = i i^2 = -\sqrt{-1} = -i$

$i^4 = i^2 i^2 = 1$

$i^5 = i i^4 = i$