$B \in \mathbb{R}^{m \times n}$

Now let $X \sim N_n(\vec{\mu}, \Sigma)$ and $\vec{Y} = B\vec{X} + \vec{c} \sim ?$

Use ch-f.'s!

$$\phi_{\vec{Y}}(\vec{t}) = e^{i\vec{t}^T\vec{c}}\,\phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}}\left(e^{i(B^T\vec{t})^T\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T\Sigma(B^T\vec{t})}\right)$$

we need to make sure $B\Sigma B^T$ still legal cov-matrix

$$= e^{i\vec{t}^T(B\vec{\mu}+\vec{c}) - \frac{1}{2}\vec{t}^T B\Sigma B^T \vec{t}} \overset{\text{P1}}{\Rightarrow} \vec{Y} \sim N_m(B\vec{\mu}+\vec{c}, B\Sigma B^T)$$

$\vec{Y} = a\vec{X} \sim N(a\vec{\mu}, a^2\Sigma)$

$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_m(\vec{\mu}, AA^T)$

$A \in \mathbb{R}^{m \times n}$, $\vec{\mu} \in \mathbb{R}^m$

MORE!

Another cool fact about multivariate ch.f's

$$\phi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{X}}\right]$$

let $\vec{t} = \begin{bmatrix} t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{i[t\,0\ldots0]\vec{X}}\right] = E\left[e^{itX_1}\right] = \phi_{X_1}(t) \overset{\text{P1}}{\Rightarrow} X_1 \sim \ldots$

this means you can find marginal distr's! No need for $\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f_{\vec{X}}(\vec{x})\,dx_1\,dx_j\ldots$

e.g. $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$ what is $X_1 \sim ?$

$$\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = e^{i[t\,0\ldots0]\vec{\mu} - \frac{1}{2}[t\,0\ldots0]\Sigma\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = e^{it\mu_1 - \frac{1}{2}\Sigma_{11}} \Rightarrow X_1 \sim N(\mu_1, \underset{\overset{\shortparallel}{\sigma_1^2}}{\Sigma_{11}})$$

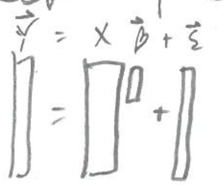
HW: multiple dimension

We will now justify the T test and F tests from linear regression ECON 382, 387 using Math 390 material

NOT COVERED ON FINAL

this was a whole class in grad school called linear model theory

Let $\vec{Y} = X\vec{\beta} + \vec{\mathcal{E}}$ where $\vec{X} \in \mathbb{R}^{n \times p}$, $\vec{\beta} \in \mathbb{R}^p$ constants



$\text{rank}(\vec{X}) = p$ and $\vec{\mathcal{E}}_n \sim N_n(\vec{0}, \sigma I_n)$ Normal errors. Homoskedastic error assumption

B

We want to estimate $\vec{\beta}$, the linear coefficients. Also unknown is σ^2.

$$\vec{Y} \sim N_n(X\beta, \sigma^2 I_n)$$

$$\frac{1}{\sigma}\vec{\varepsilon} \sim N_n(\vec{0}_n, I_n) \Rightarrow \left(\frac{1}{\sigma}\vec{\varepsilon}\right)^T\left(\frac{1}{\sigma}\vec{\varepsilon}\right) = \frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon} \sim \chi^2_n$$

$$\vec{Y} - X\vec{\beta} = \vec{\varepsilon} \Rightarrow \frac{\vec{Y} - X\vec{\beta}}{\sigma} = \frac{1}{\sigma}\vec{\varepsilon} \sim N_n(\vec{0}, I_n)$$

In ECON 382 and Math 390, we prove the least squares estimate

$$\hat{\vec{\beta}} = (X^TX)^{-1}X^T\vec{Y} = (X^TX)^{-1}X^T(X\beta + \varepsilon) = (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\varepsilon$$

$$= \vec{\beta} + (X^TX)^{-1}X^T\varepsilon \sim N_p\left(\vec{\beta}, \underbrace{(X^TX)^{-1}X^T(\sigma^2 I)\left((X^TX)^{-1}X^T\right)^T}_{\sigma^2 \cancel{(X^TX)^{-1}}\cancel{(X^TX)}\left((X^TX)^{-1}\right)^T}\right) = N_p\left(\vec{\beta}, \sigma^2(X^TX)^{-1}\right)$$

Note: $\hat{\vec{\beta}}$ has nice properties e.g.

$E[\hat{\vec{\beta}}] = \vec{\beta}$

and many, many more!

For any element of $\hat{\beta}_k$, we get a <u>univariate</u> normal (marginal distn)

$$\hat{\beta}_k \sim N\left(\beta_k, \sigma^2 (X^TX)^{-1}_{kk}\right) \Longrightarrow \frac{\hat{\beta}_k - \beta_k}{\sigma\sqrt{(X^TX)^{-1}_{kk}}} \sim N(0,1)$$

Student's Problem!

But we don't know σ! So lets estimate it...

$$\text{Let } \vec{E} := \vec{Y} - X\hat{\vec{\beta}} \overset{\text{since } \hat{\vec{\beta}} \approx \vec{\beta}}{\approx} \vec{\varepsilon}. \text{ We know that:}$$

$$\underbrace{\frac{1}{\sigma^2}\vec{\varepsilon}^T\vec{\varepsilon}}_{\chi^2_n} = \frac{1}{\sigma^2}\vec{\varepsilon}^T P\vec{\varepsilon} + \frac{1}{\sigma^2}\vec{\varepsilon}^T(I - P)\vec{\varepsilon}.$$

obviously $(I-P) + P = I$ regardless of the def. of P!

let $P := X(X^TX)^{-1}X^T$.

From 231... we can show $\text{rank}(P) = p$ since this is an orthog. projection matrix we can show rank[$I-P$] = $n-p$ since I - (proj. matrix) has rank n - rank[projection matrix] from thm. in lin. alg.

$$\vec{\varepsilon}^T P \vec{\varepsilon} = \varepsilon^T P P \vec{\varepsilon} = \varepsilon^T P^T P \vec{\varepsilon} = (P\vec{\varepsilon})^T (P\vec{\varepsilon})$$

$$P\varepsilon = P(Y - X\beta) = PY - PX\beta = X\underbrace{(X^TX)^{-1}X^TY}_{\hat{\beta}} - X\cancel{(X^TX)^{-1}X^TX}\beta = X(\hat{\beta}-\beta)$$

$$\Rightarrow \frac{\vec{\varepsilon}^T P \vec{\varepsilon}}{\sigma^2} = \frac{(X(\hat{\beta}-\beta))^T X(\hat{\beta}-\beta))}{\sigma^2} = \frac{(\hat{\beta}-\beta)^T X^T X (\hat{\beta}-\beta)}{\sigma^2} \sim \chi^2_p$$

we can also show $\text{rank}[I-P] = n-p$ since $I-$ orthogonal projection matrix must have n-rank

Hence, we can use Cochran's Thm and we have:

Also... $\frac{1}{\sigma^2}\vec{\varepsilon}^T(I-P)\vec{\varepsilon} \sim \chi^2_{n-p}$

Note: $(I-P)(I-P) = II - PI - IP - PP = I - 2P + PP = I - 2P + P = I - P$

$PP = \left(X(X^TX)^{-1}X^T\right)\left(X(X^TX)^{-1}X^T\right) = X(X^TX)^{-1}X^T = P$ other page

And $P^T = \left(X(X^TX)^{-1}X^T\right)^T = X\left((X^TX)^{-1}\right)^TX^T = X(X^TX)^{-1}X^T = P$ since $(A^{-1})^T = (A^T)^{-1}$

$$\begin{aligned}
&= \vec{\varepsilon}^T(I-P)(I-P)\vec{\varepsilon}\\
&= \vec{\varepsilon}^T(I-P)^T(I-P)\vec{\varepsilon}\\
&= \left((I-P)\varepsilon\right)^T\left((I-P)\vec{\varepsilon}\right)\\
&= \vec{E}^TE \quad \text{"}=SSE\text{"}
\end{aligned}$$

$$\begin{aligned}
(I-P)\vec{\varepsilon} &= (I-P)(Y - X\beta)\\
&= IY - PY - IX\beta + PX\beta\\
&= (I-P)Y - X\beta + \cancel{X(X^TX)^{-1}X^TX}\beta\\
&= (I-P)Y\\
&= Y - X(X^TX)^{-1}X^TY\\
&= Y - X\hat{\beta}\\
&= E
\end{aligned}$$

$\Rightarrow \frac{1}{\sigma^2}\vec{E}^TE \sim \chi^2_{n-p} \Rightarrow E$ and $\hat{\beta}$ are independent

---

$E\left[\frac{\vec{E}^TE}{\sigma^2}\right] = n-p \Rightarrow$ let $MSE := \frac{SSE}{n-p}$

$\Rightarrow E\left[\frac{MSE}{\sigma^2}\right] = 1 \Rightarrow E[MSE] = \sigma^2 \Rightarrow \overset{\text{"}RMSE\text{"}}{\sqrt{MSE}} \approx \sigma$

Some manipulations to show validity of T-~~test~~: If you wish to test $H_0: \beta_K =$ some value

$$\Rightarrow \frac{\hat{\beta}_K - \beta_K}{RMSE\sqrt{(X^TX)^{-1}_{KK}}} = \frac{\hat{\beta}_K - \beta_K}{\sqrt{\frac{SSE}{n-p}}\sqrt{(X^TX)^{-1}_{KK}}} = \frac{\hat{\beta}_K - \beta_K}{\sqrt{\sigma^2\frac{SSE}{\sigma^2}/n-p}\sqrt{(X^TX)^{-1}_{KK}}}$$

$$= \frac{\left.\frac{\hat{\beta}_K - \beta_K}{\sigma\sqrt{(X^TX)^{-1}_{KK}}}\right\} Z}{\left.\sqrt{\frac{SSE}{\sigma^2}/n-p}\right\}\sqrt{\frac{\chi^2_{n-p}}{n-p}}} \sim T_{n-p}$$

and $\hat{\beta}$ and SSE indep. due to Cochran's Thm

squaring this? $F_{1,n-p}$

The omnibus F-test is also homework. If you wish to test $H_0: \vec{\beta}$ = some vector of values

$$\frac{\dfrac{(\hat{\vec{\beta}}-\vec{\beta})^\top X^\top X(\hat{\vec{\beta}}-\vec{\beta})}{\cancel{\sigma^2}}\Big/p}{\dfrac{SSE}{\cancel{\sigma^2}}\Big/n-p} = \frac{(\hat{\vec{\beta}}-\vec{\beta}) X^\top X(\hat{\vec{\beta}}-\vec{\beta})\Big/p}{SSE\Big/n-p} \sim F_{p,n-p}$$

"Partial F-test" Let $X = \begin{array}{cc} \multicolumn{2}{c}{\longleftarrow p \longrightarrow} \\ \leftarrow q \rightarrow & \leftarrow p-q \rightarrow \\ \begin{bmatrix} \\ X_A \\ \\ \end{bmatrix} & \begin{bmatrix} \\ X_{-A} \\ \\ \end{bmatrix} \end{array}$

$$\underset{\sim \chi^2_n}{\frac{\vec{\varepsilon}^\top\vec{\varepsilon}}{\sigma^2}} = \underset{\sim \chi^2_q}{\frac{\vec{\varepsilon}^\top P_A\varepsilon}{\sigma^2}} + \underset{\sim \chi^2_{p-q}}{\frac{\vec{\varepsilon}^\top P_{-A}\vec{\varepsilon}}{\sigma^2}} + \underset{\sim \chi^2_{n-p}}{\frac{\vec{\varepsilon}^\top(I-P)\vec{\varepsilon}}{\sigma^2}}$$

Let $P_A = X_A(X_A^\top X_A)^{-1}X_A^\top$

Let $P_{-A} = P - P_A$

$\text{rank}[X_A] = q$

$\text{rank}[X_{-A}] = p-q$

$$\frac{\vec{\varepsilon}^\top P_A\vec{\varepsilon}\big/q}{\vec{\varepsilon}^\top(I-P)\vec{\varepsilon}\big/n-p} \sim F_{q,n-p}$$

$$P_A\vec{\varepsilon} = X_A(X_A^\top X_A)^{-1}X_A^\top(\vec{Y}-X\vec{\beta}) = X_A\underbrace{(X_A^\top X_A)^{-1}X_A^\top\vec{Y}}_{\hat{\beta}_A} - X_A(X_A^\top X_A)^{-1}X_A^\top[X_A X_A]\vec{\beta}$$

$$= X_A\hat{\vec{\beta}}_A - [X_A \; 0_{n\times(p-q)}]\begin{bmatrix}\vec{\beta}_A \\ \vec{\beta}_{-A}\end{bmatrix} = X_A\hat{\vec{\beta}}_A - X_A\vec{\beta}_A - 0\beta_{-A} = X_A(\hat{\vec{\beta}}_A - \vec{\beta}_A)$$

$H_0: \beta_A$ = some values

$$\rightarrow = \frac{(\hat{\vec{\beta}}_A-\vec{\beta}_A)^\top X_A^\top X_A(\hat{\vec{\beta}}_A-\vec{\beta}_A)\Big/q}{SSE\Big/n-p} \sim F_{q,n-p}$$