Lecture 17

$X_1, \dots, X_n \overset{iid}{\sim}$ with μ, σ^2

$\rightarrow \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) := \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$X = \sigma Z + \mu \sim N(\mu, \sigma^2) := \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(x-\mu)^2}$

$X_1 \sim N(\mu_1, \sigma_1^2)$, ind of $X_2 \sim N(\mu_2, \sigma_2^2)$

$\rightarrow X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$Z_i^2 \sim \chi_1^2 := \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$\underbrace{Z_1^2 + \dots + Z_m^2}_{\chi_m^2} + \underbrace{Z_{m+1}^2 + \dots + Z_k^2}_{\chi_{k-m}^2} \sim \chi_k^2 := \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

$X \sim \chi_k^2, \quad \dfrac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$ $\quad \frac{k}{2} = \dfrac{\frac{1}{2}}{\frac{1}{k}}$

$X_1 \sim \chi_{k_1}^2$ ind of $X_2 \sim \chi_{k_2}^2$

$R = \dfrac{X_1 / k_1}{X_2 / k_2} \qquad \text{Supp}[R] = (0, \infty)$

Let $U = \frac{X_1}{K_1}$, $V = \frac{X_2}{K_2}$

$\sim \text{Gamma}\left(\frac{K_1}{2}, \frac{K_1}{2}\right)$ (a) $\quad \text{Gamma}\left(\frac{K_2}{2}, \frac{K_2}{2}\right)$ (b)

$$f_{\frac{U}{V}} = \int_{supp[U]} f_U(rt) f_V(t) \underbrace{\mathbb{I}_{t \in supp[V]}}_{\text{always } 1} |t| \, dt$$

always > 0

$$= \int_0^\infty \left(\frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\right)\left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\right) t \, dt$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \underbrace{\int_0^\infty t^{a+b-1} e^{-(b+ar)t}}_{\frac{\Gamma(a+b)}{(b+ar)^{a+b}}} = \frac{a^a b^b}{B(a,b)} r^{a-1} (b+ar)^{-(a+b)}$$

$$(b(1+\frac{a}{b}r))^{-(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \frac{(1+\frac{a}{b}+r)^{-(a+b)}}{b^a b^b}$$

plug back in

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b} r\right)^{-(a+b)}$$

$$= \frac{\left(\frac{K_1}{K_2}\right)^{K_1/2}}{B\left(\frac{K_1}{2}, \frac{K_2}{2}\right)} r^{\frac{K_1}{2}-1} \left(1+\frac{K_1}{K_2} r\right)^{-\frac{K_1+K_2}{2}} \mathbb{I}_{r>0}$$

$\boxed{\text{F distribution with } K_1, K_2 \text{ deg of freedom}}$ *

Parameter Space

K_1, K_2 are $\mathbb{N}$ because you've summed chi squares

$Z \sim N(0,1)$ ind of $X \sim \chi^2_K$

$$W = \frac{Z}{\sqrt{X/K}} \sim f_W(w) = ?$$

Note: $f_W(w) = f_W(-w)$, sym around zero, even.

$$W^2 = \frac{Z^2}{\frac{X}{K}} = \frac{Z^2/1}{X/K} \sim F_{1,K}$$

$$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w])$$
$$= F_W(w) - F(-w)$$

$\frac{d}{dw}$ both sides: $2w f_{W^2}(w^2) = f_W(w) - - f_W(w) = 2 f_W(w)$

$$\longrightarrow f_W(w) = w f_{W^2}(w^2) = w \frac{\left(\frac{1}{K}\right)^{1/2}}{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{K}{2}\right)}{\Gamma\left(\frac{K+1}{2}\right)}} \underbrace{(w^2)^{\frac{1}{2}-1}}_{\underbrace{(w^2)^{-1/2}}_{w^{-1}}} \left(1 + \frac{1}{K} w^2\right)^{-\frac{K+1}{2}}$$

$\sqrt{\pi}$

$$= \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)} \left(1 + \frac{w^2}{K}\right)^{-\frac{K+1}{2}} := T_K$$

- Student's T distribution with K deg of freedom

- If you square T, get F

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$$R = \frac{Z_1}{Z_2} \sim Cauchy(0,1) := \frac{1}{\pi} \cdot \frac{1}{1+r^2}$$

$$X = c + \sigma R \sim Cauchy(c, \sigma) = \frac{1}{\sigma \pi} \cdot \frac{1}{1 + \left(\frac{r-a}{\sigma}\right)^2}$$

with $\sigma > 0$

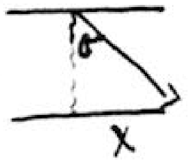
$$\phi_R(t) = E[e^{itR}] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{itr}}{1+r^2} dr = \underset{\text{complex analysis}}{\ldots} = e^{-|t|}$$

$$\phi_R(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ -e^{-t} & \text{if } t < 0 \\ \text{und} & \text{if } t = 0 \end{cases}$$

$$E[R] = \frac{\phi_R'(0)}{i} = \text{undefined, no expected value}$$

Physics example

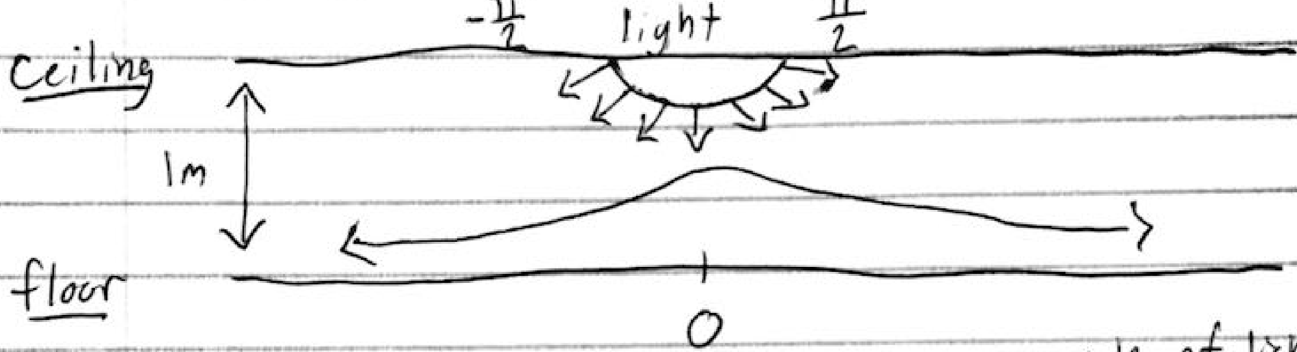
$\downarrow$



$\tan(\theta) = \frac{x}{1} = g(\theta)$

$\to \theta = \arctan(x) = g^{-1}(x)$

$$\frac{d}{dx}[\arctan^{-1}(x)] = \frac{1}{x^2+1}$$



angle of light

Find distribution of light. Assume $\theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$

$$f_X(x) = f_\theta(g^{-1}(\theta)) \left| \frac{d}{d\theta}[\theta^{-1}(x)] \right|$$

$$\frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$$

• indicator goes away

$$\frac{1}{\pi} \mathbb{1}_{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{1}{x^2+1}$$

$$= \frac{1}{\pi} \frac{1}{x^2+1} = \text{Cauchy}(0,1)$$

Applications to Statistics Z, χ^2, F, T

$X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

$T_n \sim N(n\mu, n\sigma^2)$

$\bar{X}$ is the "estimator" for μ

$\bar{x}$ is the "estimate" for μ

estimate for $\sigma^2 \longrightarrow s^2 := \frac{1}{n-1} \sum (x_i - \bar{x})^2$

estimator $\longrightarrow S^2 := \frac{1}{n-1} \sum (X_i - \bar{X}_n)^2$

Want to know:

1. $S^2 \sim ?$
2. Relationship between $\bar{X}_n, S^2$

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$\sum Z_i^2 \sim \chi^2_n$

$\vec{Z}^T Z^T \longrightarrow$ chi-squared where $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$

$Z_i = \dfrac{X_i - \mu}{\sigma} \longleftrightarrow X_i = \mu + \sigma Z_i$

$\sum_{i=1}^{n} \left(\dfrac{X_i - \mu}{\sigma}\right)^2 = \dfrac{1}{\sigma} \sum_{i=1}^{n} (X_i - \mu)^2$

$\sim \chi^2$

Note: $X_i - \mu = X_i - \bar{X} + X - \mu$

$2(n\bar{x}^2 - n\bar{x}^2 - \mu n \bar{X} + n\bar{X}\mu) \rightarrow (X_i - \mu)^2 = ((X_i - \bar{x}) + (\bar{X} - \mu))^2$

$= (X_i - \bar{X})^2 + 2(X_i - X)(\bar{X} - \mu) + (\bar{X} - \mu)^2$

$= \dfrac{1}{\sigma^2}\left(\Sigma(X_i - X)^2 + 2\Sigma(X_i - \bar{X})(\bar{X} - \mu) + \Sigma(\bar{X} - \mu)^2\right)$

$$= \frac{\sum (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_n$$