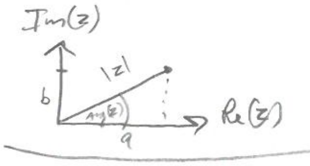
Lec 15 Math 621 11/6/19

Moment generating functions (mgf's) and Characteristic functions (ch.f's).

First... a review of imaginary #'s and trigonometry

$a, b \in \mathbb{R}$ $\quad z := a + bi \in \mathbb{C}$ ← the complex #'s



$Re(z) := a$, $Im(z) := b$ $\quad |z| := \sqrt{a^2+b^2}$, $Arg(z) = \arctan\left(\frac{b}{a}\right)$ usually...

where $i := \sqrt{-1} \Rightarrow i^2 = -1$, $i^3 = ii^2 = -\sqrt{-1} = -i$, $i^4 = (i^2)^2 = 1$, $i^5 = ii^4 = i$, ...

Recall $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - \dots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \dots$$

$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$ $\quad$ if $tx = \theta \Rightarrow e^{i\theta} = i\sin(\theta) + \cos(\theta)$

if $\theta = \pi \Rightarrow e^{i\pi} = \cos(\pi) = -1 \Rightarrow e^{i\pi} + 1 = 0$ (Euler's identity)

Define $L^1 := \left\{ f : \int_{\mathbb{R}} |f(t)|\, dt < \infty \right\}$ "L^1 integrable" or "absolutely integrable"

Are all PDF's $\in L^1$? Yes! (HW)

If $f \in L^1 \Rightarrow \exists \hat{f}$, the "Fourier transform" of f:

$$\hat{f}(\omega) := \int_{\mathbb{R}} e^{-2\pi i \omega t} f(t)\, dx$$

Fourier transformation operation

Further, if $\hat{f} \in L^1$ (which is not guaranteed), then we can do a "reverse Fourier transformation operation" to get the original f back!

$$\int_{\mathbb{R}} e^{2\pi i \omega t} \hat{f}(\omega)\, dt = f(t)$$

Fourier Inversion Thm: if $f, \hat{f} \in L^1$ and f is continuous, then f and $\hat{f}$ are 1:1.

$f(t)$ is known as "time domain" and $\hat{f}(\omega)$ is known as the "frequency domain".

Because $f(t)$ can be decomposed into a sum of sines and cosines with different freq's & and $|\hat{f}(\omega)|$ provides the amplitudes and $\text{Arg}[\hat{f}(\omega)]$ provides phase shifts.

DEMO

Back to prob. Let X ~~be~~ a r.v. [illegible] cont. Therefore,

$$\phi_X(t) := E[e^{itX}] = \int_{\mathbb{R}} e^{itx} f(x)\, dx$$ which is the Fourier transform in a different units $t = 2\pi\omega$ if cont.

$= \sum_{x \in \mathbb{R}} e^{itx}$ $p(x)$ if discrete

Properties of $\phi_X(t)$ (P0) $\phi_X(0) = E[e^{it0}] = E[1] = 1$

(P1) If $\phi_X(t) = \phi_Y(t) \Leftrightarrow X \overset{d}{=} Y$. Uniqueness

(P2) If $Y = aX + b$ then $\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{itaX} e^{itb}] = e^{itb} E[e^{i(at)X}]$
$= e^{itb} \phi_X(at)$

(P3) If X_1, X_2 ind and $T = X_1 + X_2$ then $\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}]$
$= E[e^{itX_1}] E[e^{itX_2}] = \phi_{X_1}(t)\, \phi_{X_2}(t) \underset{\text{if } X_1, X_2 \text{ iid}}{=} (\phi_X(t))^2$

(P4) Moment Generation — conditions smooth/careful

$\phi_X'(t) = \frac{d}{dt} E[e^{itX}] = E[\frac{d}{dt}[e^{itX}]] = E[iXe^{itX}]$, $\phi_X'(0) = E[iX]$; $\Rightarrow E(X) = \frac{1}{i}\phi_X'(0)$

$\phi_X''(t) = \frac{d}{dt}[\phi_X'(t)] = E[i^2 X^2 e^{itX}]$, $\phi_X''(0) = E[i^2X^2]$ $\Rightarrow E[X^2] = \frac{1}{i^2}\phi_X''(0)$

$\vdots$ $\Rightarrow E[X^n] = \frac{1}{i^n}\phi_X^{(n)}(0)$ (if the moment exists)

Existence and Boundedness

(P5) $\phi_X(t) \in [-1, 1]$ and thus always exists.

$|\phi_X(t)| = |E[e^{itX}]| = \left|\int e^{itx} f(x)\,dx\right| \le \int |e^{itx} f(x)|\,dx = \int |e^{itx}| f(x)\,dx = \int f(x)\,dx = 1$

$= \left|\sum e^{itx} p(x)\right| \le \sum |e^{itx} p(x)| = \sum |e^{itx}| p(x) = \sum p(x) = 1$

$|e^{itx}| = |i\sin(tx) + \cos(tx)| = \sqrt{\sin^2(tx) + \cos^2(tx)} = \sqrt{1} = 1$

Inversion

(P6) If $\phi_X(t) \in L^1$ then ... $f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} \phi_X(t)\,dt$ Inversion thm

Levy's CDF thm.

(P7) $P(X \in (a,b)) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\,dt$. No need for $\phi_X(t) \in L$!

(P8) Levy's Continuity Thm

(Not covered on midterm)

Consider a seq of r.v.'s $X_1, X_2, \ldots, X_n$

We define $X_n \xrightarrow{d} X$ i.e. "X_n converges in distribution to X"

if $\lim_{n\to\infty} F_{X_n}(x) = F(x)\ \forall x$ (pointwise function convergence).

If $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Longrightarrow X_n \xrightarrow{d} X$. Proof is on PhD qual. exams!

Define $M_X(t) := E[e^{tX}]$, the moment generating function (MGF)

Properties (P0) $M_X(0)=1$

(P1) $M_X(t)=M_Y(t) \Leftrightarrow X \stackrel{d}{=} Y$

(P2) $Y=aX+b \Rightarrow M_Y(t) = e^{tb} M_X(at)$

(P3) X_1, X_2 ind $\Rightarrow M_T(t) = M_X(t)\, M_Y(t)$

(P4) $E[X^n] = M_X^{(n)}(0)$ this is where it gets its name

The advantage of ch.f.'s is they always exist. MGF's do not have property (P5) $\Rightarrow$ they may not exist. If they do exist, $M_X(t) = \phi_X(-it)$.
$\Rightarrow$ may not exist.

$\Rightarrow$ We will be using ch.f.'s because they always exist and they have more properties $\Rightarrow$ more powerful!

P1, P3 together make convolutions the hard way...

e.g.

$X \sim \text{Bern}(p)$ HW

$$\phi_X(t) = E[e^{itX}] = \sum_{x\in\{0,1\}} e^{itx} p^x (1-p)^{1-x} = e^{it(0)} p^{(0)} (1-p)^{1-(0)} + e^{it(1)} p^{(1)} (1-p)^{1-(1)}$$

$$= 1-p+pe^{it}$$

$X \sim \text{Bin}(n,p)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x (1-p)^{n-x} = (1-p+pe^{it})^n$$

e.g. $X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - it)^\alpha}$$

$$= \left(\frac{\beta}{\beta - it}\right)^\alpha$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ ind. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

$$\phi_{X_1+X_2}(t) \underset{(P3)}{=} \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right)^{\alpha_2} = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1+\alpha_2} \underset{(P1)}{\Rightarrow} X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

Compare this proof to what we did with the convolution!

$X \sim \text{Poisson}(\lambda)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x=0}^\infty e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^\infty \frac{(e^{it}\lambda)^x e^{-\lambda}}{x!} \cdot \frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}} = \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \underbrace{\sum_{x=0}^\infty \frac{(e^{it}\lambda)^x e^{-\lambda e^{it}}}{x!}}_{=1}$$

since its PMF of $\text{Poisson}(e^{it}\lambda)$

$$= e^{-\lambda + \lambda e^{it}} = e^{\lambda(e^{it} - 1)}$$

$X_1 \sim \text{Poisson}(\lambda_1)$ indep of $X_2 \sim \text{Poisson}(\lambda_2)$

$$\phi_{X_1+X_2}(t) \underset{(P3)}{=} e^{\lambda_1(e^{it}-1)} e^{\lambda_2(e^{it}-1)} = e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$$\underset{(P1)}{\Rightarrow} X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

Consider

$X_1, \dots, X_n \overset{iid}{\sim}$ r.v.'s with expectation μ and variance σ^2.

Let $T_n = X_1 + \dots + X_n$

let $\bar{X}_n := \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$ be the r.v. of the sample avg. of n r.v.'s.