9/18 continuous r.v.'s $|\mathrm{supp}[X]| = |\mathbb{R}| \Rightarrow p(x)=0$

$P(X=x)=0$

$f_X(x) := F'_X(x)$

↳ Probability Density Function

$P(X \in [a,b]) = F(b) - F(a) \overset{\text{by FTC}}{=\!=} \int_a^b f(x)\,dx$

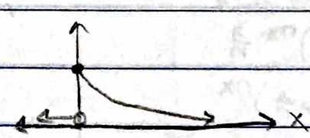
Properties of PDF:

- $\int_{\mathbb{R}} f(x)\,dx = 1 \rightarrow F(\infty) - F(-\infty) = 1 - 0 = 1$
- $f(x) \geq 0$ because F is monotonically increasing

$\mathrm{Supp}[X] = \{x : f(x) > 0\} \rightarrow$ all places that the density is positive

- $X \sim \mathrm{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$

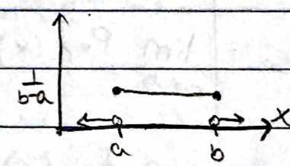
exponential r.v.



$\lambda \in (0,\infty)$ $\mathrm{Supp}[X] = [0,\infty)$

- Another continuous r.v.: $X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$

uniform r.v.



$\mathrm{supp}[X] = [a,b]$

perameter space: $a,b \in \mathbb{R}$ but $b > a$

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$ "standard uniform"

- $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix}$ it has a joint density function (JDF)

$f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_k}(x_k) = f(x_1) \cdot \ldots \cdot f(x_k)$

↑ if $X_1, \ldots X_k$ iid ↑ if iid

$\int \cdots_{\mathbb{R}^k} \int f_{X_1 \ldots X_k}(x_1, \ldots, x_k)\, dx_1 \ldots dx_k = 1$

$P(\vec{X} \in A) = \iint_A f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$ If $k=2$

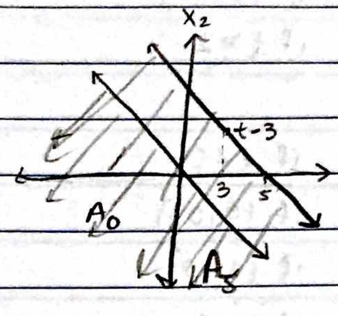
~ $T = X_1 + X_2 \sim f_T(t) = ?$

$= f_{X_1} * f_{X_2}$ → continuous convolution

(p. 145)

$F_T(t) = P(T \le t) = P(A_t)$

$A_t := \{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underbrace{x_1 + x_2 \le t}_{x_2 \le t - x_1} \}$



$$= \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty, t-x_1)} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

let $x_1 = x$ let $x_2 = v - x \Rightarrow v = x_2 + x$

$dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$= \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, t-x)\,dv\,dx = \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dx\,dv$$

$$f_T(t) = \frac{d}{dt}[\quad] = \underbrace{\int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dx}_{\text{general convolution formula}}$$

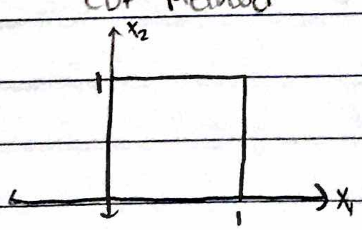
if $X_1, X_2 \overset{ind}{\sim}$ $= f_{X_1}(x) f_{X_2}(t-x)\,dx$ $\left[\text{Old: } \int_{Supp[X_1]} f_{X_1}(x) f_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}\,dx\right]$

if $X_1, X_2 \overset{iid}{\sim}$ $= \int_{\mathbb{R}} f(x) f(t-x)\,dx$ $\left[\text{Old: } \int_{Supp[X]} f(x) f(t-x) \mathbb{1}_{t-x \in Supp[X]}\,dx\right]$

- $X_1, X_2 \overset{iid}{\sim} U(0,1)$

$T = X_1 + X_2 \sim f_T(t) = ?$

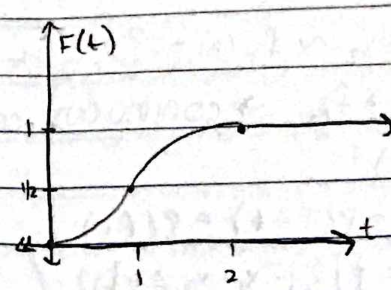
CDF Method



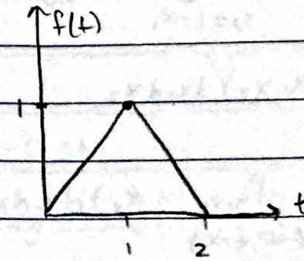
$f_{X_1,X_2}(x_1,x_2) = f(x_1) f(x_2) = \mathbb{1}_{x_1 \in [0,1]} \mathbb{1}_{x_2 \in [0,1]}$

$Supp[T] = [0,2]$

$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{1}{2}t^2 & \text{if } t \in [0,1] \\ -\frac{1}{2}t^2 + 2t - 1 & \text{if } t \in (1,2] \\ 1 & \text{if } t > 2 \end{cases}$$



$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \in [0,1) \\ 2-t & \text{if } t \in [1,2] \\ 0 & \text{if } t > 2 \end{cases}$$



$$f_T(t) = \int_0^1 \underbrace{f(x)}_{1}\, \underbrace{f(t-x)}_{1}\, \underbrace{\mathbb{1}_{t-x \in [0,1]}}_{x-t \in [-1,0] \Rightarrow x \in [t-1,t]}\, dx$$

$$= \int_0^1 \mathbb{1}_{x \in [t-1,t]}\, dx = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$

- $X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

  $T = X_1 + X_2 \sim f_T(t) = ?$

  $Supp[T] = [0, \infty)$

  $$f_T(t) = \int_0^\infty (\lambda \cancel{e^{-\lambda x}})(\lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{t-x \in [0,\infty)})\, dx$$

  $$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \le t}\, dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t\lambda^2 e^{-\lambda t}$$

- $X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda)$

  $T = X_1 + X_2 + X_3 \qquad f_T(t) = ?$

  $T_3 = X_3 + T_2$

  $$f_T(t) = \int_0^\infty (\lambda \cancel{e^{-\lambda x}})((t-x)\lambda^2 \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{x \le t})\, dx = \lambda^3 e^{-\lambda t} \int_0^t (t-x)\, dx$$

  $$= \frac{t^2}{2} \lambda^3 e^{-\lambda t}$$