Poisson Process:

10/7 $T_k \sim Erlang(k, \lambda)$

$N \sim poisson\ (\lambda)$

$P(T_k > 1) = P(N \leq k-1)$

$1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$

~ Counting + waiting r.v.'s

	Fixed Time, count #	Fixed #, measure time
Discrete	Bern / Binomial	Geometric / Negative Binomial
continuous	Poisson	Exponential / Erlang

What is the probability there are 0 successes by 50 trials, each with probability .1 of success?

$N \sim Bin(50, .1)$ $\quad\quad T \sim NegBin(1, .01)$

$P(N=0) = P(T > 49)$

$F_N(0) = 1 - F_T(49)$

What is the probability of $\leq k$ events by experiment # t is probability of success is p?

$N \sim Binomial(t, p)$ $\quad\quad T \sim NegBin(k+1, p)$

$P(N \leq k) = P(T > t-k-1)$

$F_N(k) = 1 - F_T(t-k-1)$

$\sum_{i=0}^{k} \binom{t}{i} p^i (1-p)^{t-i} = 1 - \sum_{i=0}^{t-k-1} \binom{k+i}{k} (1-p)^i p^k$

- $T \sim Erlang(k, \lambda) = \frac{\lambda^k}{(k-1)!} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0}$

$= \frac{\lambda^k}{\Gamma(k)} e^{-\lambda t} t^{k-1} \mathbb{1}_{t \geq 0}$

Parameter space: $k \in \mathbb{N}$ $\quad \lambda \in (0, \infty)$

$T \sim NegBin(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

$= \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

$k \in \mathbb{N}$ $\quad p \in (0,1)$

can $k \in (0,\infty)$?

Yes

→ Extended Negative Binomial (from NegBin)

→ Gamma Distribution (from Erlang)

↳ special case of Gamma

$$X \sim Gamma(\alpha,\beta) := \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0} \qquad \alpha,\beta > 0$$

~ Transformations

$\boxed{\text{Discrete r.v.'s}}$

$X \sim Bern(p) = p^x (1-p)^{1-x} \mathbb{1}_{x\in\{0,1\}} = p_X(x)$

$$Y = X+3 \sim \begin{cases} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y\in\{3,4\}}$$

$Y = g(X) \qquad X = g^{-1}(Y) = Y-3$

$p_Y(y) = P(Y=y) = P(g(X)=y) = P(X = g^{-1}(y)) = p_X(g^{-1}(y))$ → Assuming $\exists\, g^{-1}$

- $X \sim U(\{1,2,\dots,10\}) = \frac{1}{10} \mathbb{1}_{x\in\{1,2,\dots,10\}}$

$Y = g(X) = \min\{X,3\}$

y	$p_Y(y)$
1	1/10
2	1/10
3	8/10

$$p_Y(y) = \sum_{\{x: g(x)=y\}} p_X(x) = p_X(g^{-1}(y))$$

↑ if $\exists\, g^{-1}$

- $X \sim Bin(n,p)$

$Y = X^3 \Rightarrow X = \sqrt[3]{Y}$

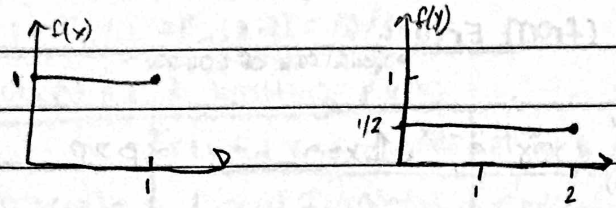
$$p_Y(y) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{y\in\{0,1,8,27,\dots,n^3\}}$$

$\boxed{\text{Continuous r.v.'s}}$

$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$

$g^{-1}(y) = \frac{y}{2} = x$

$X \sim U(0,1)$ $\quad$ $Y = 2X \sim \frac{1}{2}\mathbb{1}_{y \in [0,2]}$



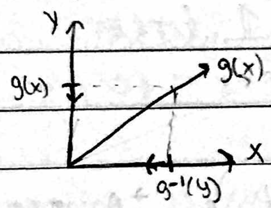
$f_Y(y) \overset{?}{=} f_X(\frac{y}{2}) = \mathbb{1}_{y \in [0,2]}$ $\qquad$ $f_Y(\frac{y}{2})\frac{1}{2} = \frac{1}{2}\mathbb{1}_{y \in [0,2]}$

Incorrect

Assume $\exists g^{-1}$

First consider g strictly increasing

$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$



$\hookrightarrow f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))]$

$= F'_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$

$= f_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$

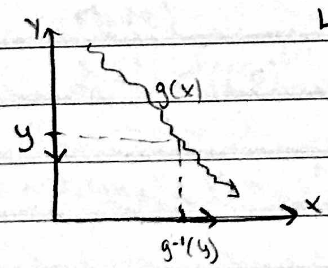
$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$

Assume g is strictly decreasing

$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

$\hookrightarrow f_Y(y) = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = -F_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$ <0

$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$



If g 1:1, $f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$

~ $Y = g(X) = aX + c$, linear transformation (shift + scale)

$\Rightarrow X = g^{-1}(Y) = \frac{Y-c}{a}$

$f_Y(y) = f_X(\frac{y-c}{a})\frac{1}{|a|}$

$= f_X(\frac{y-c}{a})\frac{1}{|a|}\mathbb{1}_{\frac{y-c}{a} \in Supp[X]} = f_X(\frac{y-c}{a})\frac{1}{|a|}\mathbb{1}_{y \in a\,Supp[X]+c}$

- $Y = -X \rightarrow a = -1 \quad c = 0$

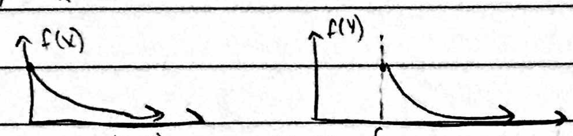
  $f_Y(y) = f_X(-y)$

- $Y = X + c$

  $f_Y(y) = f_X(y - c)$

- $X \sim Exp(\lambda)$

  $Y = X + c$



  $Y \sim \lambda e^{-\lambda(y-c)} \mathbb{1}_{y-c \in (0,\infty)} = \lambda e^{-\lambda y + \lambda c} = \underbrace{e^{\lambda c}}_{\text{scaling}} \lambda e^{-\lambda y} \mathbb{1}_{y \in (c,\infty)}$

- $X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$

  $Y = -\ln(X) = g(X)$

  $X = e^{-Y} = g^{-1}(Y)$

  $\left|\frac{d}{dy}[e^{-y}]\right| = e^{-y}$

  $e^{-y} \in (0,1) \Rightarrow -y \in (-\infty, 0) \Rightarrow y \in (0, \infty)$

  $f_Y(y) = f_X(e^{-y})\, e^{-y} = \mathbb{1}_{e^{-y} \in [0,1]}\, e^{-y} = e^{-y} \mathbb{1}_{y \in (0,\infty)} = Exp(1)$