Lecture #4 :

- Bag of fruit :

p_1 : prob. of apple
p_2 : prob. of banana
p_3 : prob. of cantaloupe

$p_1 + p_2 + p_3 = 1$

Draw n with replacement

let X_1 : # of apple

X_2 : # of banana

X_3 : # of cantaloupe

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim P_{\vec{X}}(\vec{x}) = \frac{n!}{x_1!\,x_2!\,x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n}$$

where $\mathbb{1}_{x_1 \in \{0,1 \ldots n\}} \mathbb{1}_{x_2 \in \{0,1 \ldots n\}} \mathbb{1}_{x_3 \in \{0,1 \ldots n\}}$

~~= multi~~

$$= \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \text{multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}\right)$$

Generally with k types of objects,

$$\vec{X} \sim \text{multinomial}(n, \vec{p}) = \binom{n}{x_1, x_2 \ldots x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

jmf

so $\dim[\vec{X}] = k$

$\text{supp}[\vec{X}] = \{\vec{x} : \vec{x} \in \mathbb{N}^k, \vec{x} \cdot \vec{1} = n\}$

$= \{\vec{x} : \vec{x} \in \{0, 1, \ldots n\}^k, \vec{x} \cdot \vec{1} = n\}$

$\vec{p} \in \{\vec{p} : (0,1)^k, \vec{p} = \{\vec{p} \cdot \vec{1} = \vec{1}\}$

$n \in \mathbb{N}$

- Two dimension case:

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \text{multi}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right)$$

$p_1 = p$

$p_2 = 1-p$

How is X_1 distributed? $X_1 \sim \text{bin}(n,p)$

$X_2 \sim \text{bin}(n, 1-p)$

Is $X_1 \overset{d}{=} X_2$? No

Is $X_1, X_2 \overset{ind}{\sim}$? No b/c if there are 10 fruit & 3 apples, then 7 bananas

independent

Multinomial.

* They are dependent on each other in multinomial.

check

- For two independent r.v's

$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1), \quad \forall x_1 \in \text{supp}[X_1] \ \& \ \forall x_2 \in \text{supp}[X_2]$$

rainfall is an example.

Ex $P(X_1 = 1 \mid X_2 = n) \overset{?}{=} P(X_1 = 1) = np(1-p)^{n-1}$

Not true b/c $P(X_1 = 1 \mid X_2 = n) = 0$

dependent

therefore, not independent.

- Conditional PMF/JMF

*** $p_{X_1|X_2}(x_1, x_2) = P(X_1 = x_1 \mid X_2 = x_2) = \dfrac{p_{X_1,X_2}(x_1,x_2)}{p_{X_2}(x_2)}$

Def. of conditional probability

☆ Marginalization

$p_{X_1,X_2}(x_1,x_2)$

X_2 \ X_1	0	1	2	3	4	5
0						
1						
2						

row

$$p_{X_2}(x_2) = \sum_{x_1 \in \text{supp}[X_1]} p_{X_1,X_2}(x_1,x_2)$$

$$= \sum_{x_1 \in \{0..n\}} \frac{n!}{x_1!\,x_2!}\, p^{x_1} (1-p)^{x_2}\, \mathbb{1}_{x_1+x_2=n}\, \mathbb{1}_{x_2 \in \{0..n\}}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2}\, \mathbb{1}_{x_2 \in \{0,1..n\}} \underbrace{\sum_{x_1 \in \{0 \dots n\}} \frac{1}{x_1!} p^{x_1}\, \mathbb{1}_{x_1 = n-x_2}}_{\frac{1}{(n-x_2)!} p^{n-x_2}}$$

$$= \binom{n}{x_2}(1-p)^{x_2} p^{n-x_2}$$

$$= Bin(n, 1-p)$$

So, going back to $\overset{*}{**1}$ $= \dfrac{\frac{n!}{x_1!x_2!} p^{x_1}(1-p)^{x_2}\, \mathbb{1}_{x_1+x_2=n}}{\frac{n!}{x_2!(n-x_2)!} p^{n-x_2}(1-p)^{x_2}}$

$$= \frac{(n-x_2)!}{x_1!} p^{x_1+x_2-n}\, \mathbb{1}_{x_1+x_2=n}$$

$$= \begin{cases} \frac{x!}{x!} p^0 = 1 & \text{if } x_1+x_2=n \\ 0 & \text{if } x_1+x_2 \neq n \end{cases}$$

$$= Deg(n-x_2) = \begin{cases} n-x_2 & \text{w.p } 1 \end{cases}$$

check

Now, we're proving $P_{\vec{X}_{-j}|X_j}(\vec{x}_{-j}, x_j)$

• Find expectation of multinomial: Variance as well.

$E[\vec{x}]$? $Var[\vec{x}]$?

$M_i = E[X]$

discrete: $\sum_{x \in \mathbb{R}} x\,p(x)$

cont.: $\int_{\mathbb{R}} x\,f(x)\,dx$

• $E[aX+c] = a\mu + c$ where a & c are constants.

• $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E(X_i)$ always true

• $E[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} E[X_i]$ if X_1 & X_2 are $\overset{iid}{\sim}$ (independent)

σ^2: $Var[X] = E[(X-\mu)^2] = E[X^2] - \mu^2$.

σ: $\{E[X] = \sqrt{\sigma^2}$

standard error = standard deviation

$$Var[X_1 + X_2] = E[(X_1+X_2) - (\mu_1+\mu))^2]$$
$$= [X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2X_1X_2 + 2\mu_1\mu_2]$$
$$= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 - 2\mu_1^2 - 2\mu_2\mu_1 - 2\mu_1\mu_2 - 2\mu_2^2 + 2E[X_1X_2] + 2\mu_1\mu_2$$
$$= \sigma_1^2 + \sigma_2^2 + 2(E[X_1, X_2] - \mu_1\mu_2)$$

If $X_1, X_2 \overset{iid}{\sim}$ $= \sigma_1^2 + \sigma_2^2$

$$\sigma_{12} = Cov[X_1, X_2] = E[X_1, X_2] - \mu_1\mu_2 = E(X_1 - \mu_1)(X_2 - \mu_2)$$

Covariance:

of multinomial

↳ negative covariance.