$\sqrt[3]{59372}$

$X_1, X_2 \overset{iid}{\sim} U(\{0,1,2,3\}) = \begin{cases} 0 & \text{w.p. } \frac{1}{4} \\ 1 & \vdots \\ 2 & \vdots \\ 3 & \frac{1}{4} \end{cases} = \frac{1}{4} \mathbb{1}_{x \in \{0,1,2,3\}}$

$\uparrow$ uniform discrete

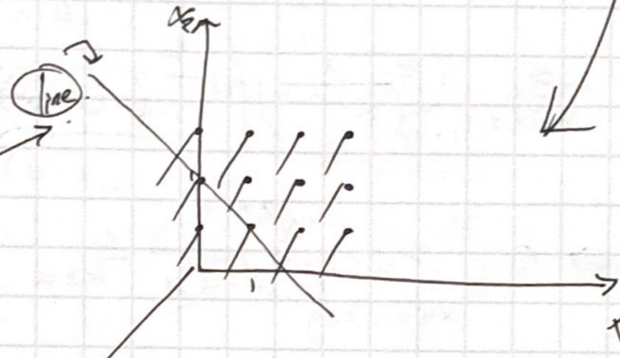
Legal random variable

In general $X \sim U(A) = \frac{1}{|A|} \mathbb{1}_{x \in A}$

Parameter Space: A is finite set $\subset \mathbb{R}$

$\text{Supp}[X] = A$

line



$P_{X_1,X_2}(x_1,x_2) = \frac{1}{16} \mathbb{1}_{x_1 \in \{0,1,2,3\}} \mathbb{1}_{x_2 \in \{0,1,2,3\}}$

$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2) \mathbb{1}_{x_2 = t - x_1}$

$p(1) = \sum \sum p \mathbb{1}_{x_2 = 1 - x_1}$ ← two hairs touching line.

$p(0.5)$ = none hair is touching line.

Let $Y = \underbrace{-X}_{g(x)} \Rightarrow X = -Y$

$X \sim U(\{0,1,2,3\}) \Rightarrow Y \sim U(\{0,-1,-2,-3\})$

$X = 0 \Rightarrow Y = 0$

$X = 1 \Rightarrow Y = -1$

$X = 2 \Rightarrow Y = -2$

$X = 3 \Rightarrow Y = -3$

all places where there is + prob.

$\text{Supp}[Y] = -\text{Supp}[X]$

We know $\text{Supp}[X]$, $P_X[X]$

We know $\text{Supp}[Y]$, $P_Y[Y]$

$P_Y(y) = P(Y = y) = P(-Y = -y) = P(X = -y) = P_X(-y)$

let $z' = -z \Rightarrow -z' = z$

$\text{Supp}[Y] = \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z' : p_X(z') > 0\}$

$X \sim \text{Bin}(n,p)$, $Y = -X \sim \binom{n}{-y} p^{-y} (1-p)^{n+y}$

$= \binom{n}{x} p^x (1-p)^{n-x}$

$$\sum_{\substack{x \in \cancel{[-d,d]} \\ \mathbb{Z}}} \mathbb{1}_{x \in [-c,c]} = 2c+1$$

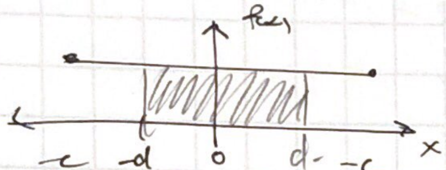
$\mathbb{Z} \leftarrow \{\ldots -2,-1,0,1,2\ldots\}$

$$\sum_{x \in \{-d,-d+1,\ldots,0,\ldots,d-1,d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2d+1 & \text{if } d \le c \\ 2c+1 & \text{if } d > c. \end{cases}$$

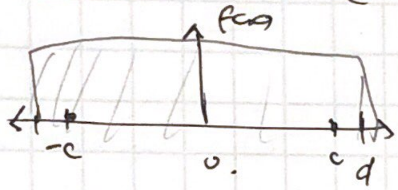
and $c \in \mathbb{N}_0$
$d \in \mathbb{N}_0$

i.e $\sum \mathbb{1}_{x \in [-3,3]} = 7$ (c, c)
if $x \in \{-4,-3,-2,-1,0,1,2,3,4\}$.

$\sum \mathbb{1}_{x \in [-3,3]} = 5$
if $x \in \{-2,-1,0,1,2\}$.



$$\int_{\mathbb{R}} \mathbb{1}_{x \in [-c,c]}\, dx = 2c$$



$$\int \mathbb{1}_{x \in [-c,c]}\, dx = \begin{cases} 2d & \text{if } d<c \\ 2c & \text{if } d>c \end{cases}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$ probability of geting 5,10 or 10,10.

$$P_{X_1|T}(x,t) = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x, t-x)}{P_T(t)}$$

$\cancel{\sim}$ Poisson

$$\Rightarrow \frac{P(x)\,p(t-x)}{P_T(x)} = \frac{\dfrac{\cancel{e^{-\lambda}}\lambda^x}{x!}\dfrac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\dfrac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}}$$

λ^t

$$\text{Bin}\left(t,\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{t-x} = \binom{t}{x}\cancel{\binom{t}{x}}\left(\tfrac{1}{2}\right)^t \Leftarrow$$

$X_1, X_2 \overset{iid}{\sim}$ Poisson (λ)

Supp$[X] = +\mathbb{Z}_0$

$Y = -\mathbb{Z}_0$

$$D = X_1 - X_2 = \overbrace{X_1}^{X} + \overbrace{(-X_2)}^{Y}$$

$$P_x(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P_y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$$

Supp$[D] = \mathbb{Z}$

$$\sim \sum_{x\in\{0,1,\dots\}} \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{\underbrace{(-(d-x))!}_{(x-d)!}} \mathbb{1}_{d-x\in\{0,-1,-2,\dots\}}\right)$$

$$\Rightarrow e^{-2\lambda}\sum_{x\in\{0,1,2,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x\ge d}$$

i.e. $\mathbb{1}_{x-d\in\{0,1,2,\dots\}}$

$\mathbb{1}_{x\ge d}$

$$= e^{-2\lambda}\begin{cases} \displaystyle\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d\ge 0\\[2ex] \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d<0\end{cases}$$

Let $x' = x - d$

$\Rightarrow x = x' + d$

$$\Rightarrow \sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!(x')!}$$

Let $d' = -d$

$\Rightarrow d' = |d|$

$$\Rightarrow \sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!}$$

$$\Rightarrow e^{-2\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}}_{I_{|d|}(2\lambda)} = e^{-2\lambda} I_{|d|}(2\lambda)$$

$=$ Skellam (λ, λ)

Modified Bessel Function of the 1 kind

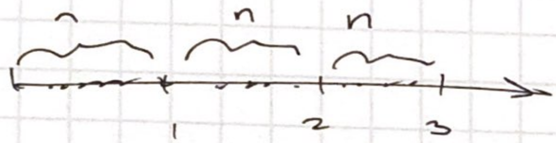
Geom ⇒ Bernoulli → until get 1 success

Continuous Random Variables.

Let $X_1 \sim \text{Geom}(p) = (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\dots\}}$

$$F_{X_1}(x) = P(X_1 \leq x) = 1 - P(X_1 > x)$$
$$= 1 - (1-p)^x$$

Let n Bernoulli experiments occur between each time period of x scale.



Let X_n be the waiting time.

$\text{Supp}[X_n]$.

$$P_{X_n}(x) = (1-p)^{nx} p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}} = \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx} = 1 - \left(1 - \frac{\lambda}{n}\right)^{nx}$$

Let $n \to \infty$, $p \to 0$ s.t. $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

Similar to the Poisson construction.

$$P_{X_\infty} = \lim_{n\to\infty} P_{X_n}(x) = \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda}} \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0} = 0 \;\; \forall x.$$

$$F_{X_\infty} = \lim_{n\to\infty} F_{X_n}(x) = 1 - \left(\lim\left(1 - \frac{\lambda}{n}\right)^n\right)^x$$
CDF
$$= \left(1 - e^{-\lambda x}\right) \mathbb{1}_{x \geq 0}$$

* PMF stated in 0.

$\text{Supp}[X_\infty] = [0, \infty)$ ← continuous.

$\sum_{x \in \text{Supp}[X_\infty]} P_{X_\infty}(x) = 0 \neq 1$ ⇒ X_∞ has no PMF

$|\text{Supp}[X_\infty]| = |\mathbb{R}|$ ⇒ X_∞ a cont r.v.

- Is $\lim_{x\to-\infty} F(x) = 0$
- Is $\lim_{x\to\infty} F(x) = 1$
- Is $F(x)$ monotonic increasing?

  $\frac{d}{dx}[F(x)] = \lambda e^{-\lambda x} > 0 \;\; \forall x \geq 0$ ✓
- $X \sim \text{Exp}(\lambda)$

  exponential r.v.

→ It is a CDF.