11/18

$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$

$Y = cX \sim \text{Gamma}(\alpha, \frac{\beta}{c})$ where $c > 0$

This will be on the final exam

$g^{-1}(y) = \frac{y}{c}$ $\quad \frac{d}{dy}\left[g^{-1}(y)\right] = \frac{1}{c}$

$f_X\left(\frac{y}{c}\right)\frac{1}{c}$

$\frac{\beta^{\alpha}}{\Gamma\alpha}\left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)} \mathbb{1}_{\frac{y}{c} \geq 0} \frac{1}{c}$

$\frac{1}{c^2}\frac{\beta^{\alpha}}{\Gamma\alpha} y^{\alpha-1} e^{-\beta\frac{y}{c}} \mathbb{1}_{\frac{y}{c} \geq 0}$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ with μ, σ^2

$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$X = \sigma Z + \mu \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$\sigma > 0$

Scanned by CamScanner

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$$\underbrace{Z_1^2 + \cdots + Z_m^2}_{\chi^2_m} + \underbrace{Z_{m+1}^2 + \cdots Z_n^2}_{\chi^2_{n-m}} \sim \chi^2_n = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

$Z_i^2 \sim \chi^2_1 := \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$X \sim \chi^2_k$

$\Rightarrow \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

$$\parallel$$

$$\frac{\frac{1}{2}}{1/k}$$

$X_1 \sim \chi^2_{k_1}$ indep of $X_2 \sim \chi^2_{k_2}$

$R = \frac{X_1/k_1}{X_2/k_2} \sim f_R(r) = ??$ $\quad U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\overset{a}{\widetilde{\frac{k_1}{2}}}, \frac{k_1}{2}\right)$

$V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\overset{b}{\widetilde{\frac{k_2}{2}}}, \frac{k_2}{2}\right)$

$R = \frac{U}{V}$ Now use ratio formula to get the pdf.

$$\int_{\text{Supp}\,U} f_U(rt)\, f_V(t)\, \underbrace{\cancel{1}}_{\substack{t \in \text{Supp}\,V \\ \sim (0,\infty) \\ \parallel \\ 1 \text{ always}}} \underbrace{|t|}_{t} \, dt.$$

11/18 $\int_0^\infty \left(\frac{a^a}{\Gamma(a)} r t^{a-1} e^{-art}\right)\left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\right) t\, dt$

$$\frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \int_0^\infty \underbrace{t^{a+b-1} e^{-(b+ar)t} dt}_{= \frac{\Gamma(a+b)}{(b+ar)^{a+b}}}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \underbrace{(b+ar)^{-(a+b)}}_{= \left(b\left(1+\frac{a}{b}r\right)\right)^{-(a+b)}}$$

$$= \frac{\frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)}}{b^a \cancel{b^b}} = \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)}$$

Sub back in values for a, b

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1+\frac{k_1}{k_2} r\right)^{-\frac{k_1+k_2}{2}} \Bigg/\!\!\Bigg/_{r \geq 0} =$$

$= f_{k_1, k_2}$ F-dist with k_1, k_2 degrees of freedom

Fisher-Snedecor

Parameter space: $(k_1, k_2 \in \mathbb{N})$

11/18

$Z \sim N(0,1)$ ind of $X \sim \chi^2_k$

$W = \dfrac{Z}{\sqrt{X/k}} \sim f_W(w) = ??$

symmetric

positive

$f_W(w) = f_W(-w)$

$$W^2 = \frac{Z^2}{X/k} = \frac{Z^2/1}{X/k} \sim F_{(1,k)}$$

Note $f_W(w) = f_W(-w)$

$$F_{W^2}(w^2) = P(W^2 \le w^2) = P\big(W \in [-w, w]\big) = F_W(w) - F_W(-w)$$

take $\frac{d}{dw}$ of both sides.

$$2w\, f_{W^2}(w^2) = f_W(w) - (-f_W(-w)) = 2 f_W(w) \Rightarrow$$

$$w\, f_{W^2}(w^2) = f_W(w)$$

Note that $k_1 = 1$ $k_2 = k$

$$\frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} (w^2)^{\frac{1}{2}-1} \left(1 + \frac{1}{k} w^2\right)^{-\frac{1+k}{2}}$$

$$\boxed{\text{Note: } \frac{1}{B\left(\frac{1}{2}, \frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}\Gamma\left(\frac{k}{2}\right)} \qquad (w^2)^{\frac{1}{2}-1} = \left(w^{-1}\right)}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

(Students T dist with k degree's of freedom.)

11/18 $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$R = \frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) := \frac{1}{\pi}\frac{1}{r^2+1}$

$X = c + \sigma R = \frac{1}{\sigma\pi}\frac{1}{\left(\frac{r-c}{\sigma}\right)^2+1} = \text{Cauchy}(c,\sigma)$

$\sigma > 0$ $\qquad \Gamma(1) = 1$

Note $T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}(1+w^2)^{-1} = \frac{1}{\pi}\frac{1}{1+w^2} = \text{Cauchy}(0,1)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\phi_R(t) = E\left[e^{itR}\right] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{itr}}{r^2+1}\,dr = \underset{\text{analysis}}{\text{complex}} = e^{-|t|}$

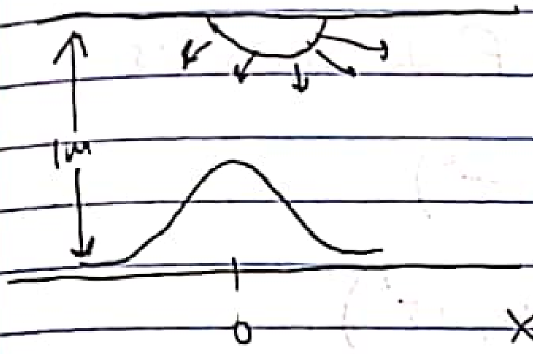
$$\phi_R'(t) = \begin{cases} -e^{-t} & \text{if } t > 0 \\ +e^{-t} & \text{if } t < 0 \\ \text{undef} & \text{if } t = 0 \text{ (no derivative of } |t| \text{ at } 0) \end{cases}$$

$\phi_R'(0)$ undefined $\Rightarrow E[R]$ undefined.

11/18

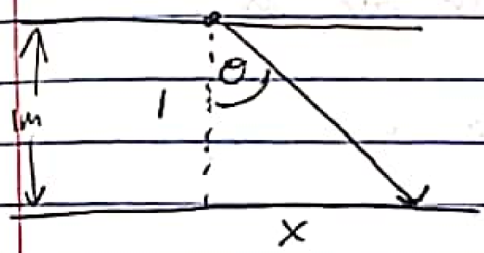
(uniformly shining)

light source



ceiling what is the distribution of light brightness on the floor?

Floor Assume



$\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi}\,\mathbb{1}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

$f_X(x) = f_\theta\, g^{-1}(x)\, \frac{d}{dx}\left[g^{-1}(x)\right]$

$g(\theta) = \tan\theta = \frac{x}{1} \Rightarrow \theta = \text{ARCTAN}(x) = g^{-1}(x)$

This is 1-1 in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that's why it was defined that way.

$f_\theta\, g^{-1}(x)\left[\frac{d}{dx} g^{-1}(x)\right] = \frac{1}{\pi}\,\mathbb{1}_{\arctan(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \cdot \frac{1}{1+x^2}$

$x \in \{\tan(-\frac{\pi}{2}), \tan(\frac{\pi}{2})\}$

$(-\infty, \infty) = \mathbb{R}$

$= \frac{1}{\pi}\,\frac{1}{1+x^2} \sim \text{Cauchy}(0,1)$

11/18 New unit. How does this relate to statistics?

Applications of Z, T, χ^2, F to statistics

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$T = X_1 + \ldots + X_n \sim N(n\mu, n\sigma^2)$

distribution $\rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ = Estimator for μ

number (realization from dist) $\rightarrow \bar{x} = \frac{1}{n}\sum x_i \leftarrow$ ~~estimator~~ estimate for μ.

$S_n^2 := \frac{1}{n-1}\sum(X_i - \bar{X}_n)^2$ estimator

$s^2 := \frac{1}{n-1}\sum(x_i - \bar{x}_n)^2$ estimate

We want to know: ① how is S^2 distributed?
② relationship between $\bar{X}_n$ and S_n^2

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

$Z^T Z = \sum Z_i^2 \sim \chi_n^2$

where $\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ Note $Z_i = \frac{X_i - \mu}{\sigma}$

$\Rightarrow \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$

11/18 $\frac{1}{\sigma^2}\sum (X_i - \mu)^2$ note $X_i - \mu = X_i - \bar{X} + \bar{X} - \mu$

$$(X_i-\mu)^2 = \left((X_i - \bar{X}) + (\bar{X}-\mu)\right)^2 = (X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2$$

$$\frac{1}{\sigma^2}\sum (X_i-\mu)^2 = \frac{1}{\sigma^2}\left(\sum (X_i - \bar{X})^2 + 2\sum (X_i - \bar{X})(\bar{X}-\mu) + \sum (\bar{X}-\mu)^2\right)$$

$$\underbrace{\sum X_i}_{n\bar{X}}\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu$$

$$= n\bar{X}^2 - n\bar{X}^2 - \mu n\bar{X} + n\bar{X}\mu$$

$$= \underbrace{\frac{\sum (X_i - \bar{X})^2}{\sigma^2}}_{} + \underbrace{\frac{n(\bar{X}-\mu)^2}{\sigma^2}}_{} \sim \chi^2_n$$

$$\frac{(n-1)S^2}{\sigma^2} \qquad \underbrace{\left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)^2}_{}$$

Recall $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

χ^2_{n-1} (it is true this is a χ^2_{n-1})

$$= \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 = Z^2 \sim \chi^2_1$$

need 2 things

need a theorem for proof:

Final Dec 16th

2 more HWs