Lecture-5

Given

$\vec{X} \sim \text{Multinomial}(n, \vec{P})$

$X_j \sim \text{Bin}(n, P_j)$

$$E[\vec{X}] = \begin{bmatrix} nP_1 \\ nP_2 \\ \vdots \\ nP_K \end{bmatrix} = n\vec{P}$$

$$\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ E[X_1] \\ \vdots \\ E[X_K] \end{bmatrix}$$

$$M = \begin{bmatrix} X_{11}, X_{12} \cdots X_{1m} \\ \vdots \\ X_{N_1}, X_{N_2} \cdots X_{NM} \end{bmatrix} \quad E[M] = \begin{bmatrix} E[X_{11}] \cdots E[X_{m}] \\ \vdots \\ E[X_{N_1}] \cdots E[X_{NM}] \end{bmatrix}$$

Variance: $\sigma^2 = var[X] = E[X^2] - \mu^2$

Covariance: $\sigma_{12} = cov[X_1 X_2] = E[X_1 X_2] - \mu_1 \mu_2$

$= E[(X_1 - \mu_1)(X_2 - \mu_2)]$

$var(X_1 + X_2) = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$

$X_1\, X_2 \overset{iid}{\sim} \Rightarrow \sigma_{12} = 0$

Rules for covariance

① $cov(X, X] = [E[X^2] - \mu^2] = var(X) = \sigma^2$

② $cov[X_1, X_2] = cov[X_2, X_1]$

③ $cov[X_1 + X_2, X_3] = cov[X_1, X_3] + cov[X_2 + X_3]$

④ $cov(a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12} = a_1 a_2 cov[X_1, X_2]$
where $a_1, a_2 \in \mathbb{R}$ constant.

⑤ $var[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} = \sigma_{11} + \underbrace{\sigma_{12} + \sigma_{21}}_{} + \sigma_{22}$$

$$\sigma_1^2 + 2\sigma_{12} + \sigma_2^2$$

covariance matrix is called Σ

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define $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & & \sigma_{m \times m} \end{bmatrix}$ = also called variance matrix $var[\vec{X}]$

matrix $\Sigma := Var[\vec{X}] = \begin{bmatrix} var[X_1] & cov[X_1,X_2] & \cdots & cov[X_1,X_K] \\ cov[X_2,X_1] & var[X_2] & \cdots & cov[X_2,X_K] \\ & & \ddots & \\ cov[X_K,X_1] & cov[X_K,X_2] & & var[X_K] \end{bmatrix}$

$\Downarrow$

K×K symmetric diagonal non-neg matrix

$:= E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T = cov[X_i, X_j]$

This matrix is called "variance matrix" or "covariance matrix" or "variance-covariance matrix".

$\vec{X}$ multinomial $(n, \vec{P})$, If $X_1, X_2 \cdots X_K$ are independent ($X_j \sim Bin(n, P_j)$)

$\Sigma \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \sigma_K^2 \end{bmatrix} = \sigma^2 I_K$

$\Downarrow$ only diagonals are positive all other entries are 0.

$\Downarrow$ K×K identity matrix

Rules for expectation and variance of r.v's

$E[\vec{X} + \vec{a}] = \begin{bmatrix} E[X_1 + a_1] \\ \vdots \\ E[X_K + a_K] \end{bmatrix}$

vector variables $\vec{a} \in \mathbb{R}^K$ constant

$= \begin{bmatrix} \mu_1 + a_1 \\ \vdots \\ \mu_K + a_K \end{bmatrix} = \vec{\mu} + \vec{a}$

$\vec{a}\cdot\vec{x} : E[\vec{a}^T\vec{x}] = E[a_1x_1 + \cdots + a_kx_k] = a_1\mu_1 + a_2\mu_2 + \cdots + a_k\mu_k$

$= \vec{a}^T\vec{x}$

$$E[\underset{\text{matrix}}{A}\vec{x}] = \begin{bmatrix} E[a_{11}x_1 + \cdots + a_{1k}x_k] \\ E[a_{21}x_1 + \cdots + a_{2k}x_k] \\ \vdots \\ E[a_{L1}x_1 + \cdots + a_{Lk}x_k] \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T\vec{\mu} \\ \vdots \\ \vec{a}_L^T\vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$A \in R^{L\times k}$ (A is an $L\times k$ matrix of constants)

$$\mathrm{var}[\vec{a}^T\vec{x}] = \mathrm{variance}[a_1x_1 + \cdots + a_kx_k]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}[a_ix_i, a_jx_j] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_j\sigma_{ij}$$

$$= \vec{a}^T\underset{\mathrm{var}[\vec{x}]}{\Sigma}\vec{a}$$

$\vec{c} \in R^{k\times k}$

$V \in R^{k\times k}$

consider $\underset{1\times k}{\vec{c}^T}\underset{k\times k}{V}\underset{1\times k}{\vec{c}} = [c_1\, c_2 \cdots c_k]\begin{bmatrix} c_1v_{11} + \cdots + c_kv_{1k} \\ c_1v_2 + \cdots + c_kv_{2k} \\ \vdots \\ c_1v_{k1} + \cdots + c_kv_{kk} \end{bmatrix}$

Quadratic form

$$\text{dot product} = \begin{bmatrix} c_1c_1v_{11} + \cdots + c_1c_kv_{1k} \\ c_2c_1v_{21} + \cdots + c_2c_kv_{2k} \\ \vdots \\ c_kc_1v_{k1} + \cdots + c_kc_kv_{kk} \end{bmatrix} = \sum_{i=1}^{k}\sum_{j=1}^{k} c_ic_jv_{ij}$$

Therefore $\vec{c}^TV\vec{c} = \sum_{i=1}^{k}\sum_{j=1}^{k} c_ic_jv_{ij}$

Then $\mathrm{var}[\vec{a}^T\vec{x}] = \vec{a}^T\Sigma\vec{a}$

Let $\vec{x} = (x_1, x_2 \cdots x_k)$ ba r.v. model for the yearly return of assets $1, \dots, K$

let $\vec{W} = [W_1, W_2 \cdots W_K]$ be the weigths of the K assets (as a proportion of the total) as a vector such that $\vec{W}^T \vec{1} = 1$

let $F = W_1 X_1 + \cdots + W_K X_K = \vec{W}^T \vec{X}$ be the yearly return of your profile.

$E[F] = \mu F$ → target mean return

I want $\mu F = \mu_0$ with minimal variance select $\vec{W}$ such that $Var[F]$ is minimal

$$\vec{W}^* = \underset{\vec{W}^T \cdot \vec{1} = 1}{\arg\min} \left\{ \vec{W}^T \Sigma \vec{W} \right\}$$

↓

Markowitz optimal portfolio Theory.

Variance

$\vec{X} \sim \text{Multinomial}(n, \vec{P})$

$(X_j \sim Bin(n, P_j))$

$$Var[\vec{X}] = \begin{bmatrix} nP_1(1-P_1) & \sigma_{21} & \cdots & \sigma_{2K} \\ \sigma_{12} & nP_2(1-P_2) & \sigma_{ij} & \\ & \sigma_{ij} & \ddots & \\ & & & nP_K(1-P_K) \end{bmatrix}$$

$\sigma_{ij} < 0$

$$\sigma_{ij} = cov[X_i X_j] = E[X_i X_j] - \mu_i \mu_j$$

$$= \Big[\underbrace{\sum_{x_1 \in supp[X_1]} \sum_{x_2 \in supp[X_2]} x_1 x_2 P_{X_1} P_{X_2}(x_1, x_2)}_{\text{impossible}} - n^2 P_i P_j$$

$supp[X_1] = \{0, 1, \dots n\}$, $supp[X_2] = \{0, 1, \cdots n\}$

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Recall $X_i \sim Bin(n, p_i)$ $X_i = X_{1i} + X_{2i} + \cdots + X_{ni}$

where $X_{1i}, X_{2i}, \cdots X_{ni}$ iid $Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \cdots + X_{nj}$

where $X_{1j}, X_{2j}, \cdots, X_{nj} \sim Bern(p_j)$

When X_{1i} and X_{1j} are dependent

if $X_{1i} = 1$ then X_{1j} must be 0, X_{1j} could be 0 or 1)

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \cdots + \vec{X}_n$ where $\vec{X}_1, \cdots \vec{X}_n \sim mult(1, \vec{P})$

$$cov[X_i X_j] = cov[X_{1i} + \cdots + X_{ni}, X_{1j} + \cdots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} cov[X_{li}, X_{mj}]$$

If $l \neq m$ $cov[X_{li}, X_{mj}] = 0$

$$\sum_{l=1}^{n} cov[X_{li}, X_{lj}]$$

$$cov[X_{li}, X_{lj}] := \left[\sum_{x \in \{0,1\}} \sum_{x \in \{0,1\}} x_i x_j P_{X_{li}, X_{lj}}(x_1, x_2) - p_i p_j\right]$$

$$cov[X_{li}, X_{lj}] = P_{X_{li}} X_{lj}(1,1) - p_i p_j = -p_i p_j$$

impossible to get an apple and a banana at the same time.

If $\vec{X} \sim multinomial(n, \vec{P})$ and X_i's are identically distributed, $\vec{P} = \frac{1}{k} \vec{1}$

$k \to \infty$