PMF is always 0 for continuous r.v.

Continuous r.v.'s X's has $|Supp[X]| = |\mathbb{R}|$

$$\Rightarrow p(x) = P(X = x) = 0$$

The derivative of the CDF is very important

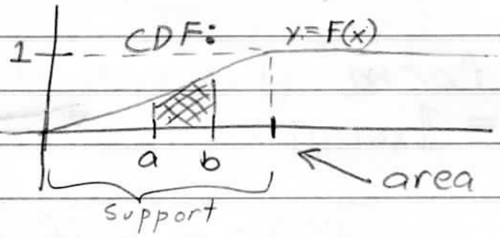
$$f(x) = \frac{d}{dx}\left[F(x)\right] \quad \text{where } F(x) \text{ is the CDF}$$

it is called the probability density function (PDF)

$f(x)$ is PDF

Use: $P(X \in [a,b]) = F(b) - F(a) = \int_a^b f(x)\,dx$

by Fundamental Theorem of Calculus



area is probability

CDF
$F(x) = P(X \leq x)$

Properties of PDF

(I) $\int_{\mathbb{R}} f(x)\,dx = 1$ since $= F(\infty) - F(-\infty)$
$= 0 - 1$
$= 1$

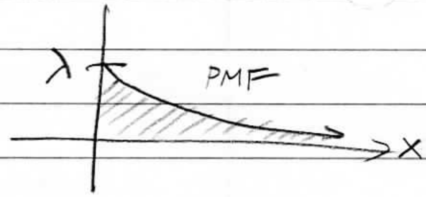
(II) $f(x) \geq 0$ for all x

$$Supp[X] = \{x : f(x) > 0\}$$

Exponential

$$X \sim \mathrm{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$$

$\mathrm{Supp}[X] = [0, \infty)$



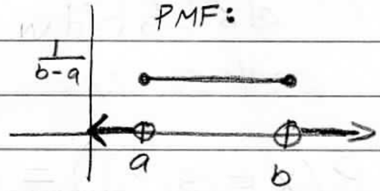
Uniform continuous dist. (a.k.a. Uniform)

$$X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$$

PMF is

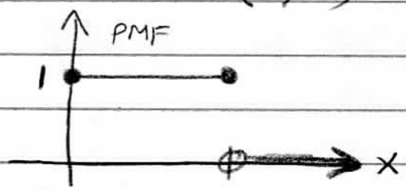
where $a, b \in \mathbb{R}$ but $b > a$

$\mathrm{Supp}[X] = [a, b]$



Standard Uniform (has support $[0,1]$)

$$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$$



$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

If $X_1, X_2, \ldots, X_n$ are indep.

$$f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

And if $X_1, X_2, \ldots, X_n$ are i.i.d.

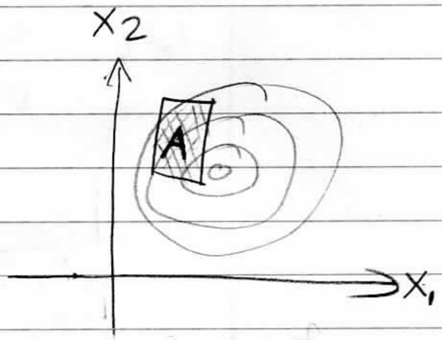
$$f_{\vec{X}}(\vec{x}) = f(x_1) f(x_2) \cdots f(x_n)$$

where f is PMF of $X_1, X_2, \ldots, X_n$

$$\underset{\mathbb{R}^K}{\iint \cdots \int} f_{\vec{X}}(\vec{x})\, dx_1 dx_2 \ldots dx_n = 1$$

ex: if $K = 2$

$$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2)\, dx_1 dx_2$$



p.145 in Book

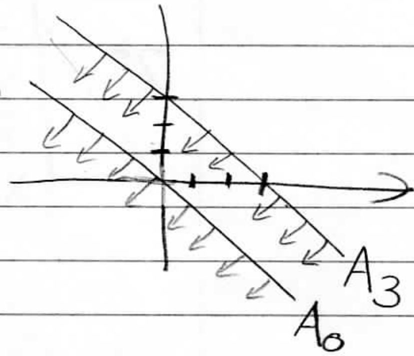
convolution for continuous r.v. :

$$T = X_1 + X_2 \sim f_T(t) = ?$$

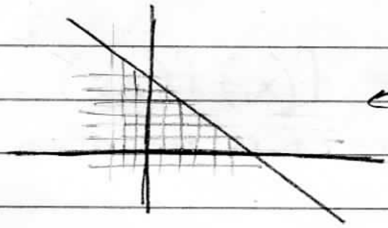
F is CDF

$$F(t) = P(T \leq t) = P(A_t)$$

$$A_t \underset{\text{defined as}}{:=} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underbrace{x_1 + x_2 \leq t}_{x_2 \leq t - x_1} \right\} \subset \mathbb{R}^2$$



strategy:
Use CDF
get convolution,
take derivative
to get PDF



divide A into horizontal or vertical strips

$$P(A_t) = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty, t - x_1)} f_{X_1 X_2}(x_1, x_2)\, dx_1\, dx_2$$

let $x = x_1 \Rightarrow dx = dx_1$

$v = x_2 + x_1 \Rightarrow x_2 = v - x_1$

$\Rightarrow dv = dx_2$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x_1 \Rightarrow v = t$

$$= \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1 X_2}(x, v - x)\, dv\, dx$$

$$F(t) = \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1 X_2}(x, v - x)\, dx\, dv$$

$T = X_1 + X_2 \sim f_T(t) = ?$

$$\underset{\text{PDF}}{f(t)} = \frac{d}{dt}\Big[\underset{\uparrow \\ \text{CDF}}{F(t)}\Big] = \underbrace{\int_{\mathbb{R}} f_{X_1, X_2}(x, t-x)\, dx}_{\text{general convolution formula}}$$

if X_1, X_2 are indep.

$$f(t) = \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\, dx$$

indicators needed

if X_1, X_2 are i.i.d.

$$f(t) = \int_{\mathbb{R}} f(x)\, f(t-x)\, dx$$

Old style notation (with indicator functions)

if X_1, X_2 are indep.

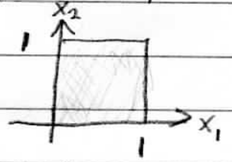
$$f(t) = \int_{Supp[X_1]} f_{X_1}(x)\, f_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}\, dx$$

if X_1, X_2 are i.i.d.

$$f(t) = \int_{Supp[X]} f(x)\, f(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

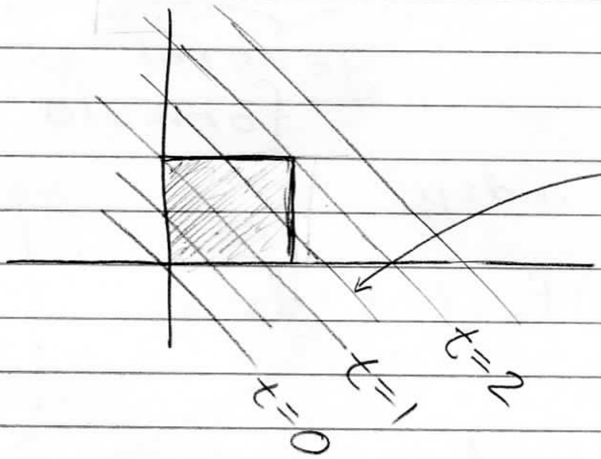
$X_1, X_2 \overset{\text{i.i.d.}}{\sim} \underbrace{U}_{\text{uniform}}(0,1)$

$T = X_1 + X_2 \sim f(t)$ pdf is $f(t)$, cdf is $F(t)$ $\quad$ $\text{Supp}(T) = [0,2]$



joint PMF

$$f(x_1, x_2) = \mathbb{1}_{x_1 x_2}(x_1, x_2) = \mathbb{1}_{x_1 \in [0,1]} \mathbb{1}_{x_2 \in [0,1]}$$

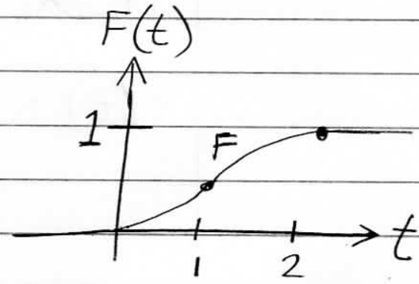


$A_{\square}$ = square − triangle

$$A_{\square} = \frac{t^2}{2} - 2\left(\frac{(t-1)^2}{2}\right)$$
$$= \frac{t^2}{2} - t^2 + 2t - 1$$
$$= -\frac{t^2}{2} + 2t + 1$$

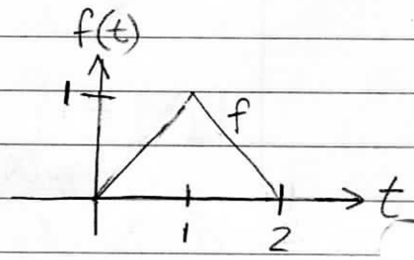
CDF

$$F(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t^2}{2} & \text{if } 0 < t \le 1 \\ -\frac{t^2}{2} + 2t - 1 & \text{if } 1 < t \le 2 \\ 1 & \text{if } t > 2 \end{cases}$$



take deriv. to get PDF $\quad f(t) = F'(t)$

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } 0 < t \le 1 \\ 2-t & \text{if } 1 < t \le 2 \\ 0 & \text{if } t > 2 \end{cases}$$



$$X_1, X_2 \overset{i.i.d.}{\sim} U(0,1) \qquad T = X_1 + X_2 \sim f(t)$$

Using $f(t) = \int_{Supp[X]} f(x) f(x-t) \mathbb{1}_{t-x \in Supp[X]}$

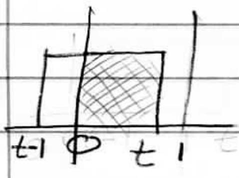
$$= \int_0^1 (1)(1) \mathbb{1}_{t-x \in [0,1]}$$

rewrite: $x - t \in [-1, 0]$

rewrite: $x \in [t-1, t]$

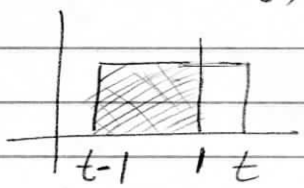
$$= \int_0^1 \mathbb{1}_{x \in [t-1, t]} \, dx$$

do by cases



$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$

video on wikipedia



$$1 - (t-1) = 2 - t$$

Convolution of Exponential r.v.'s

$T_2 = X_1 + X_2 \sim f_T(t) = ?$ $\quad X_1, X_2 \overset{i.i.d.}{\sim} Exp(\lambda)$

Use $f_{T_2}(t) = \int_{Supp[X_1]} f(x) f(t-x) \mathbb{1}_{t-x \in Supp[X]} dx$

f is PMF of X

f_{T_2} is PMF of T_2

$$f_{T_2}(t) = \int_0^\infty (\lambda e^{-\lambda x})(\lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{t-x \in (0,\infty)}) dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \le t} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t} \neq Exp(\lambda)$$

$$Supp[T_2] = (0, \infty)$$

Next:

$$T_3 = X_1 + X_2 + X_3 = T_2 + X_3$$

$$f_{T_3}(t) = \int_0^\infty (x \lambda^2 e^{-\lambda x})(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}) dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x \, dx$$

$$= \frac{1}{2} t^2 \lambda^3 e^{-\lambda t}$$

Next:

$$T_4 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4$$

$$f_{T_4}(t) = \int_0^\infty (\tfrac{1}{2} t^2 \lambda^3 e^{-\lambda x})(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}) dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2 dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \frac{1}{3} t^3$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$$

$$f_4(t) = \frac{1}{2\cdot 3} x^3 \lambda^4 e^{-\lambda x}$$

$$f_{T_5}(t) = \int_0^\infty \left(\frac{1}{2\cdot 3} x^3 \lambda^4 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t}\right) dx$$

$$= \frac{1}{2\cdot 3} \lambda^5 e^{-\lambda t} \int_0^t x^3 dx$$

$$= \frac{1}{2\cdot 3\cdot 4} \lambda^5 e^{-\lambda t} t^4$$

$f_{T_5}(t) =$

$$f_{T_2}(t) = \lambda^2 e^{-\lambda t} t$$

$$f_{T_3}(t) = \frac{1}{2} \lambda^3 e^{-\lambda t} t^2$$

$$f_{T_4}(t) = \frac{1}{2\cdot 3} \lambda^4 e^{-\lambda t} t^3$$

$$f_{T_5}(t) = \frac{1}{2\cdot 3\cdot 4} \lambda^5 e^{-\lambda t} t^4$$

in general,

$f_{T_K}(t) = \frac{1}{(K-1)!} \lambda^K e^{-\lambda t} t^{K-1}$ defines Erlang(K, λ)

this is the Erlang distribution

$$\text{Supp}[T_K] = (0, \infty)$$

parameters: $k \in \mathbb{N}$ $\lambda \in (0, \infty)$