(Lecture 6)

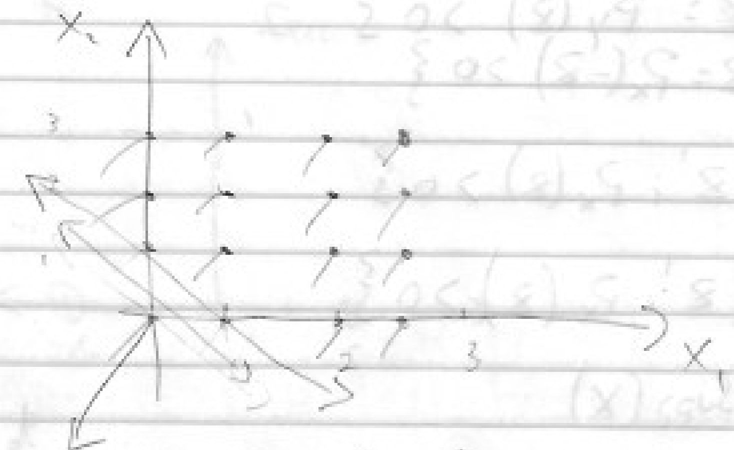
September 16^{th}, 2019

$$X_1, X_2 \overset{iid}{\sim} U(\{0,1,2,3\}) := \begin{cases} 0 & w.p.\ 1/4 \\ 1 & w.p.\ 1/4 \\ 2 & w.p.\ 1/4 \\ 3 & w.p.\ 1/4 \end{cases} = \frac{1}{4}\mathbb{1}_{x \in \{0,1,2,3\}}$$

In general $X \sim U(A) := \frac{1}{|A|}\mathbb{1}_{x \in A}$

Parameter space : A is a finite set $\subset \mathbb{R}$

$Supp(X) = A$



$$P_{X_1,X_2}(x_1,x_2) = \frac{1}{16}\mathbb{1}_{x_1 \in \{0,1,2,3\}}\mathbb{1}_{x_2 \in \{0,1,2,3\}}$$

$$P(t) = \sum_{x_1 \in \mathbb{R}}\sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}\mathbb{1}_{x_2 = t - x_1}$$

$$P(1) = \sum\sum P\,\mathbb{1}_{x_2 = 1 - x_1}$$

Let $Y := \underbrace{-X}_{g(x)}$

$X \sim U(\{0,1,2,3\})$

If $x = 0 \to Y = 0$
If $x = 1 \to Y = -1$ so $X \sim U(\{0,1,2,3\}) = Y \sim U(\{0,1,2,3\})$
If $x = 2 \to Y = -2$
If $x = 3 \to Y = -3$

$\text{Supp}(Y) = -\text{Supp}(X)$

we know $\text{Supp}(X)$, $P_X(x)$, we want $\text{Supp}(Y)$, $P_Y(y)$

$P_Y(y) := P(Y = y) = P(-Y = -y) = P(X = -y) = P_X(-y)$

$$\text{Supp}(Y) := \{z : P_Y(z) > 0\}$$
$$= \{z : P_X(-z) > 0\}$$

let $z' = -z$ $= \{-z' ; P_X(z) > 0\}$

$z = -z'$

$$= -\{z' : P_X(z) > 0\}$$
$$= -\text{Supp}(X)$$

$X \sim \text{Bin}(n, p)$, $Y := -X \sim \binom{n}{-y} p^{-y}(1-p)^{n+y}$
$= \binom{n}{x} p^x (1-p)^{n-x}$

(Review on indicator function problems)

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [c,c]} = 2c+1$$

and
$c \in \mathbb{N}_0$

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-1,1]} = 3$$

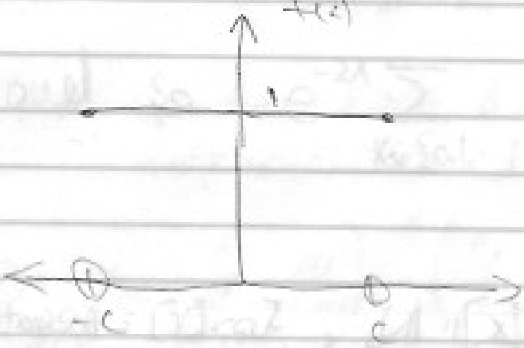
$$\sum_{x\in\{-d,-d+1,\dots,0,\dots,d-1,d\}} \mathbb{1}_{x\in[-c,c]} = \begin{cases} 2d+1 & \text{if } d\le c \\ 2c+1 & \text{if } d>c \end{cases}$$

↑ all integers

and $c\in\mathbb{N}_0,\ d\in\mathbb{R}$

$$\sum_{x\in\{-4,-3,-2,-1,0,1,2,3,4\}} \mathbb{1}_{x\in\{-3,3\}} = 7 \qquad \sum_{x\in\{-2,-1,0,1,2\}} \mathbb{1}_{x\in[-3,3]} = 5$$

$$\int_{\mathbb{R}} \mathbb{1}_{x\in[-c,c]}\,dx = 2c$$



$$\int_{-d}^{d} \mathbb{1}_{x\in[-c,c]}\,dx = \begin{cases} 2d & \text{if } d<c \\ 2c & \text{if } d\ge c \end{cases}$$

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P_{X_1|T}(x,t) \neq \text{Poisson}(\lambda)$$

$$= \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x,t-x)}{P_T(t)} = \frac{P(x)P(t-x)}{P_T(x)}$$

$$= \frac{\left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{(t-x)}}{(t-x)!}\right)}{\frac{e^{-2\lambda}(2\lambda)^{t}}{t!}} = \binom{t}{x}\left(\frac{1}{2}\right)^{t}$$

$$= \mathrm{Bin}\left(t, \tfrac{1}{2}\right)$$

$X_1, X_2 \overset{iid}{\sim} \mathrm{Poisson}(\lambda)$

$D := X_1 - X_2 = X_1 + \underbrace{(-X_2)}_{\sim\, Y}$

$$p_X(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$$

$\mathrm{Supp}(D) = \mathbb{Z}$ $\left(\mathrm{Supp}[X] = \mathbb{N}_0,\ \mathrm{Supp}[Y] = \text{negative integers with } 0.\right)$

$$X_1 + (-X_2) \sim \sum_{x \in \{0,1,\dots\}} \left(\frac{e^{-\lambda}\lambda^{x}}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!}\mathbb{1}_{d-x \in \{0,1,2,3\dots\}}\right)$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots,\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\mathbb{1}_{x \geq d}$$

Note: $\underbrace{\mathbb{1}_{x-d \in \{0,1,2,\dots\}}}_{\mathbb{1}_{x \geq d}}$

$$= e^{-2\lambda} \begin{cases} \sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \geq 0 \\ \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \end{cases}$$

Let $d' = -d \Rightarrow d' = |d|$

$$\text{so } \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!(x+d')!}$$

Let $x' = x-d$, $x = x'+d$

$$\text{So } \sum_{d=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'=0}^{\infty} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!(x')!}$$

So both are equal to $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$

$$\text{and so } e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \geq d} = e^{-2\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}}_{I_{|d|}(2\lambda)}$$

this is modified Bessel function of the 1st kind to a famous differential equation

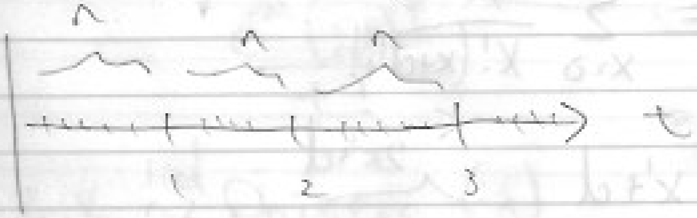
$$= e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda) \qquad (1946)$$

Remark: Anything upto this point is the midterm material.

Let $X_1 \sim \text{Geom}(p) = (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\dots\}}$

$$F_{X_1}(x) := P(X_1 \leq x) = 1 - P(X_1 > x)$$
$$= 1 - (1-p)^x$$

Let n Bernoulli experiments occur between each time period of x scale



Let X_n be the waiting time   Supp[X_n] $\swarrow$

$$P_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\dots\}} = \left(1-\frac{\lambda}{n}\right)^{nx}\frac{\lambda}{n}$$

$$F_{X_n}(x) = 1-(1-p)^{nx} = 1-\left(1-\frac{\lambda}{n}\right)^{nx}$$

Let $n\to\infty$, $p\to 0$ such that $\lambda = np \to p = \frac{\lambda}{n}$

Similar to the Poisson construction

$$P_{X_\infty} = \lim_{n\to\infty} P_{X_n}(x) = \Bigg(\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\Bigg)^x \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} = 0 \quad \forall x$$

$$F_{X_\infty} = \lim_{n\to\infty} F_{X_n}(X) = 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x = 1-e^{-\lambda x}$$

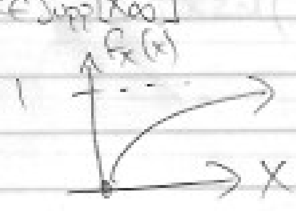
CDF?

Supp[X_∞] $= [0,\infty)$ ← Continuous interval

$|\text{Supp}[X_\infty]| = |\mathbb{R}| \to X_\infty$ is a continuous random variable

$$\sum_{x\in \text{Supp}[X_\infty]} P_{X_\infty}(x) = 0 \neq 1$$

$\Rightarrow X_\infty$ has no PMF



Is $\lim_{x\to-\infty} F(x) = 0$ ✓ Yes

Is $\lim_{x\to\infty} F(x) = 1$

Is $f(x)$ monotonically increasing?

$\frac{d}{dx}[F(x)] = \lambda e^{-\lambda x} > 0 \quad \forall x \geq 0$ Yes

So $F_{X(x)}$ is CDF

$F_{X(x)} \rightarrow X \sim Exp(\lambda)$, exponential r.v.