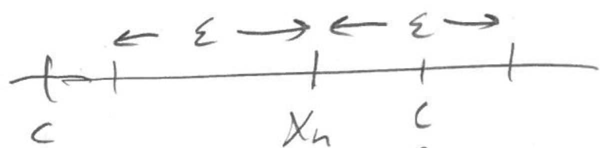
Lec 23 12/9/19 Math 621

Convergence In Probability (to a constant, we will look at conv to a r.v.): $X_n \xrightarrow{P} c$ read

"X_n converges in prob to a constant c" if by definition

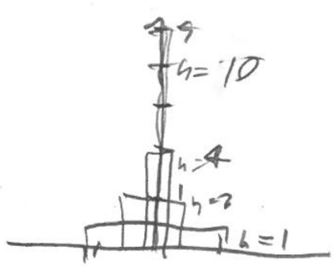
$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - c| \ge \varepsilon) = 0 \iff \lim_{n\to\infty} P(|X_n - c| < \varepsilon) = 1$$

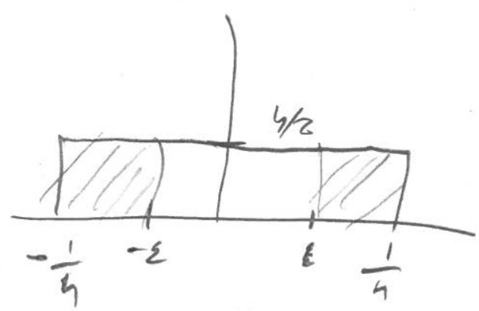


prob. zero away as $n\to\infty$

prob 1 here as $n \to \infty$

e.g. let $X_n \sim U(-\frac{1}{n}, \frac{1}{n}) = \frac{n}{2} \mathbb{1}_{x \in [-\frac{1}{n}, \frac{1}{n}]}$





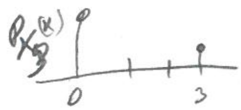
Prove $X_n \xrightarrow{P} 0$

$$\lim_{n\to\infty} P(|X_n - 0| \ge \varepsilon) = \lim P(|X_n| \ge \varepsilon) = \lim \left(P(X_n \le -\varepsilon) + P(X_n \ge \varepsilon) \right) =$$

$$\lim \left(\left(-\varepsilon - -\frac{1}{n}\right) \frac{n}{2} \mathbb{1}_{-\varepsilon > -\frac{1}{n}} + \left(\frac{1}{n} - \varepsilon\right) \frac{n}{2} \mathbb{1}_{\varepsilon < \frac{1}{n}} \right) = \lim 2 \left(\frac{1}{n} - \varepsilon\right) \frac{n}{2} \mathbb{1}_{\varepsilon < \frac{1}{n}}$$

$$= \lim (1 - n\varepsilon) \mathbb{1}_{\varepsilon < \frac{1}{n}} = 0$$

Pick ε, find n s.t. $\varepsilon \ge \frac{1}{n} \Rightarrow 0$



$$X_n \sim \begin{cases} 0 & \text{wp } 1-\frac{1}{n} \\ n & \text{wp } \frac{1}{n} \end{cases}$$ Prove $X_n \xrightarrow{P} 0$

$$\lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = \lim P(X_n \geq \varepsilon) = \lim \frac{1}{n} \mathbb{1}_{\varepsilon < n} = 0 \checkmark$$

$X_n \sim N(0, \frac{1}{n})$ Prove $X_n \xrightarrow{P} 0$ Chebyshev's

$$\lim P(|X_n - 0| \geq \varepsilon) = \lim P(|X_n| \geq \varepsilon) \leq \lim \frac{\sigma^2}{\varepsilon^2} = \lim \frac{1}{n\varepsilon^2} = 0 \checkmark$$

Let $X_1, X_2, \ldots \overset{iid}{\sim}$ with mean μ, variance σ^2

Consider $\bar{X}_1 = X_1$, $\bar{X}_2 = \frac{X_1+X_2}{2}$, ... $\bar{X}_n = \frac{X_1+\dots+X_n}{n}$

Note: $\mu_n = \mu$, $\sigma_n^2 = \frac{\sigma^2}{n}$

Show $\bar{X}_n \xrightarrow{P} \mu$ i.e. the "weak Law of Large Numbers" (LLN).

$$\lim P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \lim \frac{\sigma^2}{n\varepsilon^2} = 0$$

We assumed finite variance. this is actually unnecessary (see HW).

LLN is a <u>major</u> thm. It says that the the mean can never escape from $\bar{X}$ given enough data.

(III) Convergence in "L^r norm" where $r \geq 1$

$X_n \xrightarrow{L^r} c$ means $\lim_{n\to\infty} E[|X_n - c|^r] = 0$

e.g. $X_n \xrightarrow{L^1} c$ means $\lim_{n\to\infty} E(|X_n - c|) = 0$ "convergence in mean"

e.g. $X_n \xrightarrow{L^2} c$ means $\lim_{n\to\infty} E[(X_n - c)^2] = 0$ "mean square convergence"

e.g. $X_n \sim U(0, \frac{1}{n}) = n \mathbb{1}_{x \in (0, \frac{1}{n})}$

WTS $X_n \xrightarrow{L^r} 0$ for all r

non-neg

$\lim E[|X_n - 0|^r] = \lim E[X_n^r] = \lim \int_{\mathbb{R}} x^r n \mathbb{1}_{x \in (0, \frac{1}{n})} dx$

$= \lim n \int_0^{\frac{1}{n}} x^r dx = \lim n \left[\frac{x^{r+1}}{r+1}\right]_0^{\frac{1}{n}} = \lim \frac{n}{r+1} \frac{1}{n^{r+1}} = \frac{1}{r+1} \lim \frac{1}{n^r} = 0$ ✓

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let $1 \leq r < s$

Prove $X_n \xrightarrow{L^s} c \Rightarrow X_n \xrightarrow{L^r} c$

$\lim E[|X_n - c|^r] \underset{\text{Holders}}{\leq} \lim E[|X_n - c|^s]^{\frac{r}{s}} = \left(\lim E[|X_n - c|^s]\right)^{\frac{r}{s}} = 0^{\frac{r}{s}} = 0$

since $\lim E[|X_n - c|^r] \geq 0$ if its ≤ 0 then it must be $= 0$ ✓

Which behavior is stronger?

Prove $X_n \xrightarrow{L^r} c \Rightarrow X_n \xrightarrow{P} c$

Markov's

$$\lim P(|X_n - c| \geq \varepsilon) = \lim P(|X_n - c|^r \geq \varepsilon^r) \leq \lim \frac{E[|X_n - c|^r]}{\varepsilon^r} = 0 \checkmark$$

But $X_n \xrightarrow{P} c \not\Rightarrow X_n \xrightarrow{L^r} c$. Here's a counterexample:

$$X_n \sim \begin{cases} n^2 & \text{wp } \frac{1}{n} \\ 0 & \text{wp } 1 - \frac{1}{n} \end{cases}$$

$X_n \xrightarrow{P} 0$ Proof $\lim P(|X_n| \geq \varepsilon) = \lim \frac{1}{n} = 0$

But $X_n \overset{L^r}{\not\longrightarrow} c$ since $\lim E[|X_n|^r] = \lim E[X_n^r] = \lim \sum_{x \in \{0, n^2\}} x^r p(x)$

$$= \lim \left(0^r\left(1 - \tfrac{1}{n}\right) + (n^2)^r \tfrac{1}{n}\right) = \lim n^{2r-1} = \infty \neq 0 \quad \forall r \geq 1$$

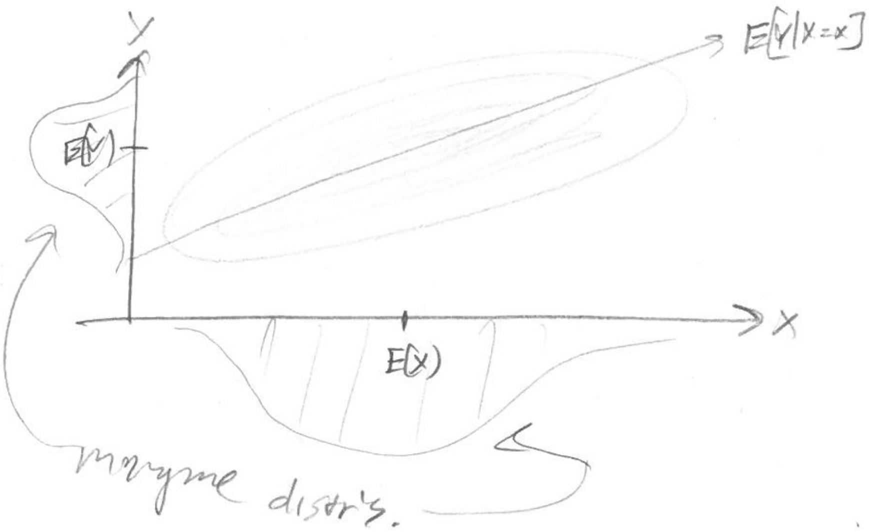
Conv in mean is stronger than conv. in prob.

But expectation is different

$E(X_n) = \frac{n^2}{n} = n$

the expectation goes to ∞

Imagine two r.v.'s creating a joint density $f_{XY}(x,y)$



We can derive a nice identity:

$$E(Y) = \int_{Supp(Y)} y f_Y(y)\, dy = \int_{Supp(Y)} y \int_{Supp(X)} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{Supp(Y)} \int_{Supp(X)} y f_{Y|X}(x,y)\, f_X(x)\, dx\, dy = \int_{Supp(X)} \int_{Supp(Y)} y\, f_{Y|X}(x,y)\, f_X(x)\, dy\, dx$$

$$= \int_{Supp(X)} f(x) \int_{Supp(Y)} y\, f_{Y|X}(x,y)\, dy\, dx = \int_{Supp(X)} f(x)\, \underbrace{E[Y|X=x]}_{g(x)}\, dx = E_X\Big[\overbrace{E_Y[Y|X=x]}^{\text{Shorthand } E(Y|X)}\Big]$$

Law of Iterated Expectation

Using the Law of Iterated Expectations

$$\mathrm{Var}(Y) = E[Y^2] - E(Y)^2$$

$$= E_X\left[E_Y[Y^2|X]\right] - \left[E_X\left(E_Y(Y|X)\right)\right]^2$$

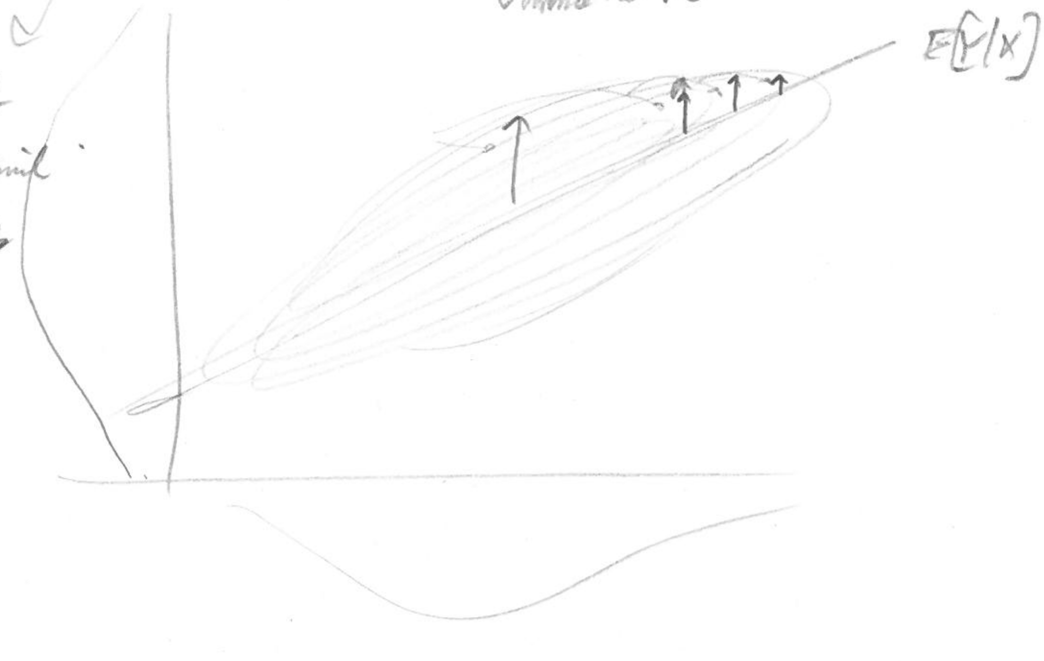
$$= E_X\left[\mathrm{Var}_Y[Y|X] + E_Y(Y|X)^2\right] - \left[E_X\left(E_Y(Y|X)\right)\right]^2$$

$$= E_X\left[\mathrm{Var}_Y(Y|X)\right] + \underbrace{E_X\left[\underbrace{E_Y(Y|X)^2}_{Q^2}\right] - E_X\left[\underbrace{E_Y(Y|X)}_{Q}\right]^2}_{\mathrm{Var}(Q)}$$

$$\Rightarrow \mathrm{Var}(Y) = \overbrace{E_X\left(\mathrm{Var}_Y(Y|X)\right)}^{\mathrm{I}} + \overbrace{\mathrm{Var}_X\left[E_Y(Y|X)\right]}^{\mathrm{II}} \quad \text{Law of Total Variance}$$

variance over the cond. mean

Avg of the conditional variances



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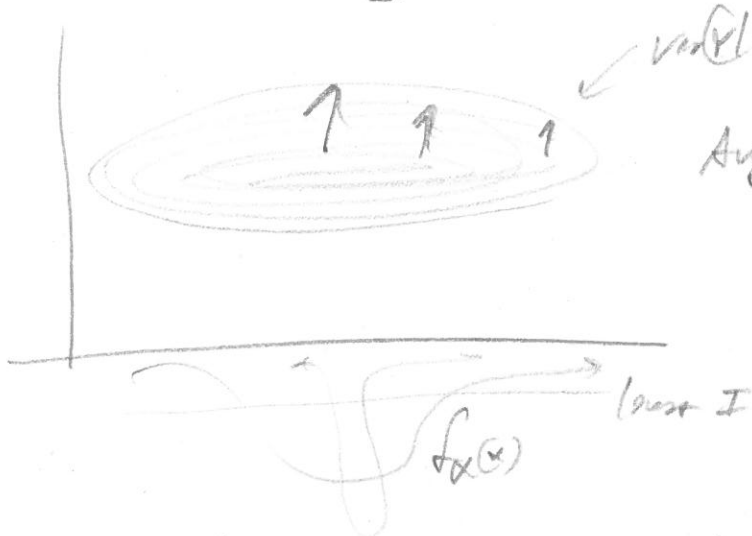
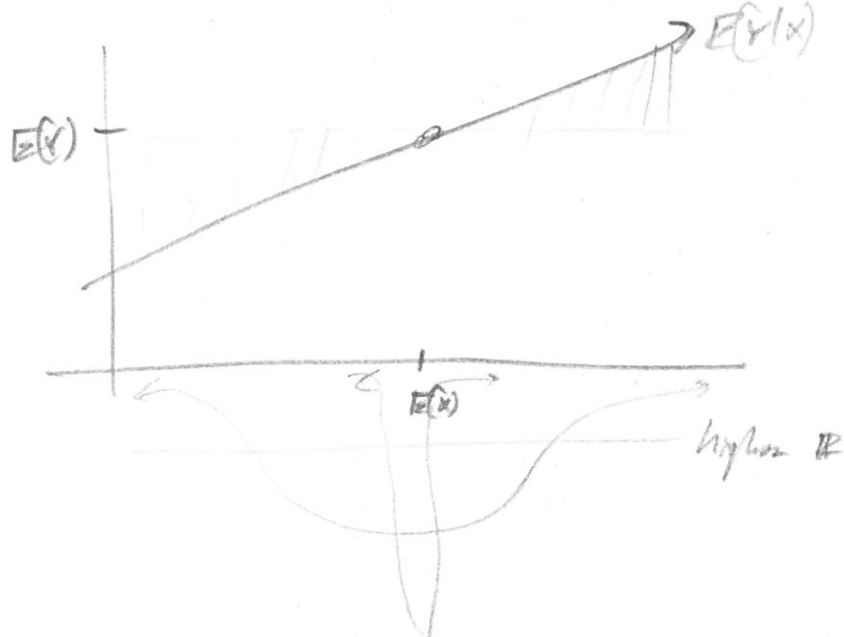


Avg of all these variances weighted by $f_X(x)$

higher I

$$Var_X(X) = E_X[(E_Y(Y|X))^2 - E(Y))^2]$$

II



lower II