$$X_1, X_2, \ldots, X_r \overset{i.i.d.}{\sim} Geom(p)$$

defined by PMF

$$T = \sum_{i=1}^{r} X_i \sim NegBin(r,p) := \binom{t+r-1}{r-1}(1-p)^t p^r$$

$$\underbrace{\overset{0}{\underset{1}{\sqcup}}\ \overset{1}{\underset{2}{\sqcup}}\ \overset{0}{\underset{3}{\sqcup}}\ \overset{1}{\underset{4}{\sqcup}}\ \cdots\ \overset{1}{\underset{t+r+1}{\sqcup}}}_{\text{in here have } t\ 0\text{'s},\ r-1\ 1\text{'s}}\ \overset{1}{\underset{t+r}{\sqcup}}$$

$\rightarrow$ # of ways to do this in $t+r+1$ slots $\binom{t+r-1}{r-1}$

$$X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

let n get large and p get small but peg/fix

$$\lambda = np \Rightarrow p = \frac{\lambda}{n}$$

$n \in \mathbb{N}$

$p \in (0,1)$ $\searrow$ get $\lambda \in (0,\infty)$

$Supp[X] = \{0,1,\ldots\} = \mathbb{N}_0$

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{n(n-1)(n-2)\cdots(n-x+1)}^{\text{have } x \text{ factors}}}{n(n)(n)\cdots(n)}\, e^{-\lambda}(1)$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

because $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right) = 1-0 = 1$

$$\xrightarrow{} \text{PMF} \quad p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$X \sim Poisson(\lambda) \qquad Supp[X] = \{0, 1, \ldots\} = \mathbb{N}_0$$

$$X_1, X_2 \overset{i.i.d.}{\sim} Poisson(\lambda)$$

$$T = X_1 + X_2 \sim \sum_{x \in \{0,1,\ldots\}} \left(\frac{\lambda^{x} e^{-\lambda}}{x!}\right)\left(\frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \{0,1,\ldots\}}\right)$$

$$= \lambda^{t} e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x \leq t}$$

$$= \lambda^{t} e^{-2\lambda} \sum_{x \in \{0,1,\ldots,t\}} \frac{1}{x!(t-x)!}$$

$$= \lambda^{t} e^{-2\lambda} \sum_{x \in \{0,\ldots,t\}} \frac{t!}{x!(t-x)!} \cdot \frac{1}{t!}$$

Note that

$$\boxed{\sum_{x=0}^{t} \binom{t}{x} = 2^{t}}$$

sum of # of all possible subsets of a set of t distinct elements

$$= \lambda^{t} e^{-2\lambda} \left(\sum_{x \in \{0,1,2,\ldots t\}} \binom{t}{x}\right) \frac{1}{t!}$$

$$= \frac{\lambda^{t} e^{-2\lambda}}{t!} \left(2^{t}\right)$$

$$= \frac{(2\lambda)^{t} e^{-2\lambda}}{t!}$$

This is Poisson (2λ)

$A = \{a_1, a_2, \ldots, a_n\}$ $\quad |A| = n$

set A has n elements

$$2^A = \{B : B \subseteq A\} = \{B : B \subseteq A, |B| = 0\} \cup \quad \leftarrow \text{all empty subsets}$$

power set of A $\quad$ set of all subsets of A

$$\{B : B \subseteq A, |B| = 1\} \cup \quad \leftarrow \text{all subsets of 1 element}$$

$$\{B : B \subseteq A, |B| = 2\} \cup \quad \leftarrow \text{all subsets of 2 elements}$$

$$\vdots$$

know $|2^A| = 2^n$

$$\{B : B \subseteq A, |B| = n\} \quad \leftarrow \text{all subsets of } n \text{ elements}$$

So

$$2^n = |2^A| = |\{B : B \subseteq A, |B| = 0\}| + |\{B : B \subseteq A, |B| = 1\}| + |\{B : B \subseteq A, |B| = 2\}| + \cdots + |\{B : B \subseteq A, |B| = n\}|$$

$\binom{n}{k} = {}_nC_k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

note that $|\{B : B \subseteq A, |B| = k\}| = \binom{n}{k}$

The number of subsets with k elements is $\binom{n}{k}$

So

$$2^n = |2^A| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$= 1 + n + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

Therefore,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$\binom{n}{n} = 1$

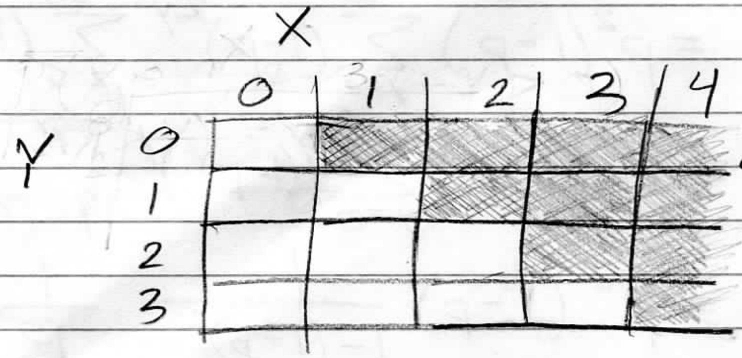
$\binom{n}{n-k} = \binom{n}{k}$

$X, Y \overset{i.i.d}{\sim} Geom(p)$ Find $P(X>Y)$

know

$P(X>Y) = P(Y>X) < \frac{1}{2}$

$1 = P(X>Y) + P(Y>X) + \underbrace{P(X=Y)}_{>0}$



$p_{X,Y}(x,y) = p(x)p(y)$

$$P(X>Y) = \sum_{y\in\mathbb{R}} \sum_{x\in\mathbb{R}} p_{X,Y}(x,y)\, \mathbb{1}_{x>y}$$

$$P(X>Y) = \sum_{y\in\mathbb{R}} \sum_{x\in(y,\infty)} p(x)p(y)$$

here, $p(x) = p(1-p)^x \mathbb{1}_{x\in\{0,1,..\}}$ $p(y) = p(1-p)^y \mathbb{1}_{y\in\{0,1,\dots\}}$

$$P(X>Y) = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} p(1-p)^x \mathbb{1}_{x\in\{0,1,..\}}\, p(1-p)^y \mathbb{1}_{y\in\{0,1,\dots\}}$$

$$= p^2 \sum_{y\in\{0,1,..\}} (1-p)^y \sum_{x\in\{y+1,y+2,\dots\}} (1-p)^x$$

Note: $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

$$= p^2 \sum_{y=0}^{\infty} (1-p)^y \sum_{x=y+1}^{\infty} (1-p)^x$$

Let $z = x-(y+1)$

so $x = z + y + 1$

$$\boxed{\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}}$$

$$= p^2 \sum_{y=0}^{\infty} (1-p)^y \sum_{z=0}^{\infty} (1-p)^{z+y+1}$$

re-indexing

$$= p^2 \sum_{y=0}^{\infty} (1-p)^y \sum_{z=0}^{\infty} (1-p)^z (1-p)^y (1-p)$$

$$= p^2(1-p) \sum_{y=0}^{\infty} (1-p)^{2y} \underbrace{\sum_{z=0}^{\infty} (1-p)^z}$$

$$= p^2(1-p) \sum_{y=0}^{\infty} ((1-p)^2)^y \frac{1}{1-(1-p)}$$

$$= p^2 (1-p) \frac{1}{1-(1-p)^2} \frac{1}{p}$$

$$= p^2(1-p) \frac{1}{1-(1-2p+p^2)} \frac{1}{p}$$

$$= p^2(1-p) \frac{1}{2p-p^2} \frac{1}{p}$$

$$= p^2(1-p) \frac{1}{p(2-p)} \frac{1}{p}$$

$$= \frac{\cancel{p^2}(1-p)}{\cancel{p^2}(2-p)}$$

$$P(X>Y) = \frac{1-p}{2-p} \quad \text{where } X, Y \overset{iid}{\sim} \text{Geom}(p)$$

expected value is mean; an average weighted by prob.

if X is a discrete r.v.

$$E(X) = \sum_{x \in \mathbb{R}} x\, p(x)$$

expected value of X

$$E(g(x)) = \sum_{x \in \mathbb{R}} g(x)\, p(x)$$

expected value of $g(x)$

if X, Y are discrete r.v.'s

$$E[XY] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} xy\, p(x,y)$$

$$E[g(X,Y)] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p(x,y)\, p(x,y)$$

$p(x,y)$ means $P(X=x, Y=y)$

useful case:

$$E[\mathbb{1}_{x \in A}] = \sum_{x \in \mathbb{R}} \mathbb{1}_{x \in A}\, p(x) = \sum_{x \in A} p(x) = P(X \in A)$$

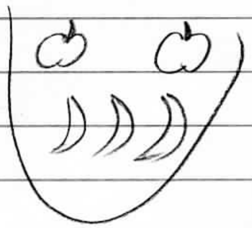
x is Bern(p)

our case

$$P(X > Y) = E[\mathbb{1}_{X>Y}] = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p(x,y)\, \mathbb{1}_{X>Y}$$

Multinomial Distributions

Bag of apples and bananas



p_1 = prob. of drawing an apple
p_2 = prob. of drawing a banana

$$p_1 + p_2 = 1$$

Draw n with replacement
X_1 = # of apples
X_2 = # of bananas

$$X_1 + X_2 = n$$

$$X_1 \sim Bin(n, p_1) \quad p_{X_1}(x) = \binom{n}{x} p_1^x (1-p_1)^{n-x}$$

$$X_2 \sim Bin(n, p_2) = Bin(n, 1-p_1)$$

vector form

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{X_1,X_2}(x_1, x_2) = \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{\substack{x_1 + x_2 = n \\ x_1 \geq 0, x_2 \geq 0}}$$

$$\binom{n}{x} = \frac{n!}{x_1!(n-x_1)!} = \frac{n!}{x_1! x_2!}$$

$$p_{X_1 X_2}(x_1, x_2) = \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\ldots\}} \mathbb{1}_{x_2 \in \{0,1,\ldots\}}$$

Multichoose or multinomial coefficient

$:=$ is defined as

$$\boxed{\binom{n}{x_1, x_2} := \frac{n!}{x_1! x_2!} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\ldots\}} \mathbb{1}_{x_2 \in \{0,1,\ldots\}}}$$

$$\vec{X} \sim \text{Multinomial}\left(n, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) \sim \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$

defines $\vec{X}$ is a Multinomial r.v. (if $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$)