$X \sim \text{Gamma}(\alpha, \beta)$

Show $Y = cX \sim \text{Gamma}(\alpha, \frac{\beta}{c})$

where $c > 0$

$g(x) = cx$

$g^{-1}(y) = \frac{y}{c}$

$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{c}$

$$X \sim f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{1}{c} \right|$$

$$f_Y(y) = f_X\left(\frac{y}{c}\right) \frac{1}{c}$$

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta\left(\frac{y}{c}\right)} \cdot \frac{1}{c} \mathbb{1}_{\frac{y}{c} \geq 0}$$

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{y^{\alpha-1}}{c^{\alpha-1}} e^{-\frac{\beta y}{c}} \cdot \frac{1}{c} \mathbb{1}_{y \geq 0}$$

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{y^{\alpha-1} \cdot \cancel{c}}{c^\alpha} e^{-\frac{\beta}{c}(y)} \cdot \frac{1}{\cancel{c}} \mathbb{1}_{y \geq 0}$$

$$= \frac{\left(\frac{\beta}{c}\right)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta}{c} \cdot y} \mathbb{1}_{y \geq 0}$$

$$\sim \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

CLT (central Limit Theorem)

$X_1, X_2, \ldots, X_2 \overset{i.i.d.}{\sim}$ with $\overset{\text{known}}{\mu, \sigma^2}$

$$\Rightarrow \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$X \sim \sigma Z + \mu \sim N(\mu, \sigma^2)$$

$\sigma > 0$

with pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Z_1, Z_2, \ldots, Z_n \overset{i.i.d.}{\sim} N(0,1)$

$Z_i^2 \sim \chi_1^2 := \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

"chi squared - 1 d.f."

d.f. degrees of freedom

$Z_1^2 + \cdots + Z_n^2 \sim \chi_n^2 = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

"chi squared - n d.f."

so

$$\chi_m^2 + \chi_{n-m}^2 = \chi_n^2$$

$X \sim \chi^2_K$ Since: if $X \sim \text{Gamma}(\alpha, \beta)$ and $Y = cX$ then $Y \sim \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$

so

$$Y = \frac{X}{K} \sim \text{Gamma}\left(\frac{K}{2}, \frac{K}{2}\right) \qquad \frac{\frac{1}{2}}{\frac{1}{K}}$$

$X_1 \sim \chi^2_{K_1}$ is indep of $X_2 \sim \chi^2_{K_2}$

$$R = \frac{X_1/K_1}{X_2/K_2} = \frac{U}{V}$$

let $U = \frac{X_1}{K_1} \sim \text{Gamma}\left(\overbrace{\frac{K_1}{2}}^{a}, \frac{K_1}{2}\right) = \text{Gamma}(a,a)$ Let $a = \frac{K_1}{2}$

$V = \frac{X_2}{K_2} \sim \text{Gamma}\left(\frac{K_2}{2}, \frac{K_2}{2}\right) = \text{Gamma}(b,b)$ Let $b = \frac{K_2}{2}$

$$R = \frac{U}{V} \sim \int_{\text{Supp}[U]} f_U(rt) f_V(t) \mathbb{1}_{t \in \text{Supp}[V]} |t| dt$$

$$= \int_0^\infty \left(\frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\right)\left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\right) t\, dt$$

randomized experiments invented by Fisher

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \underbrace{\int_0^\infty t^{a+b-1} e^{-(b+ar)t} dt}_{\frac{\Gamma(a+b)}{(b+ar)^{a+b}}}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} (b+ar)^{-(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \left(b\left(1+\frac{a}{b}r\right)\right)^{-(a+b)}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \frac{\left(1+\frac{a}{b}r\right)^{-(a+b)}}{b^a b^b}$$

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)}$$

$$= \frac{\left(\frac{a}{b}\right)^{a}}{B(a,b)} r^{a-1}\left(1+\frac{a}{b}r\right)^{-(a+b)}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2},\frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1}\left(1+\frac{k_1}{k_2}r\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r\geq 0}$$

$$= F_{k_1,k_2}$$

"F distribution with $\underset{\text{numerator}}{k_1}$ and $\underset{\text{denominator}}{k_2}$ degrees of freedom"

"Fisher-Snecedor dist."

$Z \sim N(0,1)$ indep. of $X \sim \chi^2_k$

$$W = \frac{Z}{\sqrt{X/k}} \sim f_W(w) = ?$$

$$W^2 = \frac{Z^2}{X/k} = \frac{Z^2/1}{X/k} \sim \frac{\chi^2_1}{\chi^2_k} \sim \underset{\text{F dist}}{F_{1,k}}$$

Note: $f_W(w) = f_W(-w)$ ← F dist. is symmetric (pdf is even funct.)

$$\underset{\text{cdf of } W^2}{F_{W^2}(w^2)} = P(W \leq w^2) = P(W \in [-w,w])$$
$$= P(-w \leq W \leq w)$$

cdf of w^2 cdf of w

$$F_{w^2}(w^2) = P\left(W \in [-w, w]\right) = F_w(w) - F_w(-w)$$

$$F_{w^2}(w^2) = F_w(w) - F_w(-w)$$

Take $\frac{d}{dw}$ of both sides

$$2w\, f_{w^2}(w^2) = f_w(w) - - f_w(-w)$$

$$2w\, f_{w^2}(w^2) = 2 f_w(w)$$

$$w\, f_{w^2}(w^2) = f_w(w)$$

$$f_w(w) = w\, f_{w^2}(w^2)$$

William Gosset came up with Student's T dist.

$$f_w(w) = w \frac{\left(\frac{1}{K}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{K}{2}\right)} (w^2)^{\frac{1}{2}-1} \left(1 + \frac{1}{K} w^2\right)^{-\frac{1+K}{2}}$$

Note: $K_1 = 1$, $K_2 = K$

$$f_w(w) = \cancel{w}\, w^{-1} \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\, \Gamma\left(\frac{K}{2}\right)} \left(1 + \frac{w^2}{K}\right)^{-\frac{K+1}{2}}$$

is called the Student T distribution with k degrees of freedom

got this from

$$\frac{1}{B\left(\frac{1}{2}, \frac{K}{2}\right)} = \frac{\Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{K}{2}\right)} = \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{K}{2}\right)}$$

and $(w^2)^{\frac{1}{2}-1} = w^{2\cdot\left(-\frac{1}{2}\right)} = w^{-1}$

Note: $\overset{\text{T dist, 1 d.f.}}{T_1} = \dfrac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}\left(1+w^2\right)^{-1} = \dfrac{\Gamma(1)}{\sqrt{\pi}\sqrt{\pi}}\left(1+w^2\right)^{-1} = \dfrac{1}{\pi}\dfrac{1}{1+w^2}$

$= \text{Cauchy}(0,1)$

$Z_1, Z_2 \overset{\text{i.i.d.}}{\sim} N(0,1)$

$$R = \frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) := \overset{\text{PDF}}{\frac{1}{\pi}\frac{1}{r^2+1}}$$

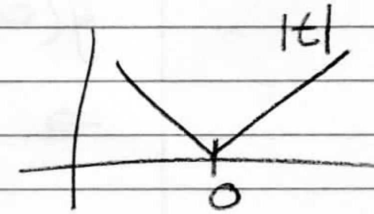
$$X = c + \sigma R = \frac{1}{\sigma\pi}\frac{1}{\left(\frac{r-c}{\sigma}\right)^2+1} = \text{Cauchy}(c,\sigma)$$

where $\sigma > 0$

$$\phi_R(t) = E\left[e^{itR}\right] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{itr}}{r^2+1}\,dr = \ldots$$

, by complex analysis, get

$$\phi_R(t) = e^{-|t|} = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ e^{t} & \text{if } t \leq 0 \end{cases}$$

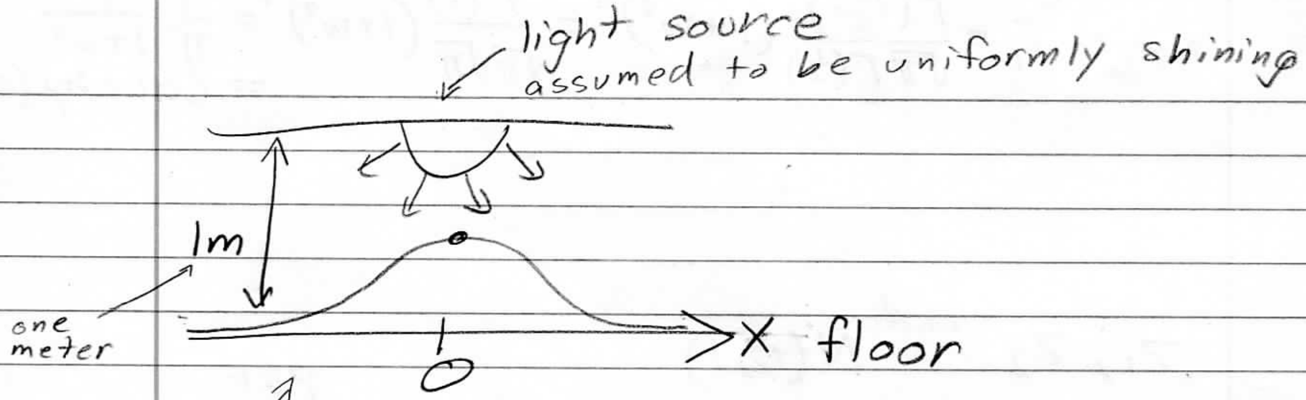


$$\phi_R'(t) = \begin{cases} -e^{-t} & \text{if } t > 0 \\ e^{t} & \text{if } t < 0 \\ \text{undefined} & \text{if } t = 0 \end{cases}$$

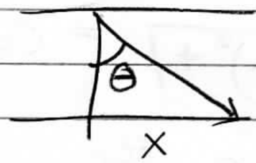
so

$\phi_R'(0)$ undefined $\Rightarrow E[R]$ is undefined

"Cauchy dist. has no mean"



what is the dist. of light brightness on the floor?



$\leftarrow X = \tan\theta$ where

$$\theta \sim \underset{\text{uniform}}{U}\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$$

$$g(\theta) = \tan\theta = \frac{X}{1} = X$$

$$\Rightarrow \theta = \arctan(X) = g^{-1}(X)$$

Cauchy dist. also known as Lorenz. dist. late 1800s

$$g^{-1}(x) = \tan^{-1}(x)$$

$$\frac{d}{dy}\left[g^{-1}(x)\right] = \frac{1}{1+x^2}$$

$$f_X(x) = f_\theta(g^{-1}(x)) \left| \frac{d}{dx}\left[g^{-1}(x)\right] \right|$$

$$= \frac{1}{\pi} \mathbb{1}_{\underbrace{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}_{x \in [\tan(-\frac{\pi}{2}), \tan(\frac{\pi}{2})]}} \cdot \frac{1}{1+x^2}$$

same as $x \in (-\infty, \infty)$

$$= \frac{1}{\pi} \frac{1}{1+x^2} = \text{Cauchy}(0,1)$$

Applications to Statistics

Applications of Z, T, χ^2, F to Statistics

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$

$$T = X_1 + X_2 + \ldots + X_n \sim N(n\mu, n\sigma^2)$$

estimator for μ $\quad \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ $\quad$ ← estimator is r.v.

$\bar{x}$ is estimate for μ

estimate for μ $\quad \bar{x} = \frac{1}{n}\sum x_i$ $\quad$ ← estimate is a specific realization

estimator for σ^2

$$S_n^2 = \frac{1}{n-1}\sum (X_i - \bar{X}_n)^2$$

$$s_n^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2$$

estimate for σ

Want to know

1) $S_n^2 \sim ?$ Find dist.

2) Relationship between $\bar{X}_n$ and S_n^2

$$Z_1, \dots, Z_n \overset{i.i.d.}{\sim} N(0,1) \qquad \text{Let } \vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$$

$$\underbrace{\vec{Z}^T \vec{Z}}_{\substack{\text{dot product} \\ \vec{Z} \circ \vec{Z}}} = \sum Z_i^2 \sim \chi_n^2$$

Note: $Z_i = \dfrac{X_i - \mu}{\sigma}$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$$

$$= \frac{1}{\sigma^2} \sum (X_i - \mu)^2$$

because $X_i - \mu = X_i - \bar{X} + \bar{X} - \mu$

so

$$(X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2 = (X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2$$

$$= \frac{1}{\sigma^2}\left(\sum (X_i - \bar{X})^2 + 2\sum (X_i - \bar{X})(\bar{X} - \mu) + \sum (\bar{X} - \mu)^2\right)$$

notice $\sum (X_i - \bar{X})(\bar{X} - \mu)$

$= \sum X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu$; using $\sum X_i = n\bar{X}$

$= n\bar{X}^2 - n\bar{X}^2 - \mu n\bar{X} - n\bar{X}n$

$= 0$

$$= \frac{\sum (X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi_n^2$$

$$= \frac{(X_i - \bar{X})^2}{\sigma^2} + \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2$$

$$= \frac{(X_i - \bar{X})^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

Recall $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ because $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$$= \frac{(n-1)S^2}{\sigma^2} + Z^2$$

Z^2 is same as χ_1^2

Conjecture

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

$\therefore$ therefore:

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

X_i are r.v.'s

μ, σ, are constants

$$= \frac{(X_i-\overline{X})^2}{\sigma^2} + \left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

using $S_n^2 = \frac{1}{n-1}\Sigma(X_i-\overline{X}_n)^2$

$$\Rightarrow \Sigma(X_i-\overline{X}_n)^2 = (n-1)S_n^2$$

and $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} = Z \sim N(0,1)$

because $\overline{X} \sim N(\mu, \sigma^2)$

get that

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

$$= \frac{(n-1)S_n^2}{\sigma^2} + Z^2$$

S_n^2, Z are r.v.'s

$$= \frac{(n-1)S_n^2}{\sigma^2} + \chi_1^2$$

since $Z^2 = \chi_1^2$

chi-sqared d.f. =1

Conjecture:

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$$

Using conjecture, get:

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

$$= \chi_{n-1}^2 + \chi_1^2 = \chi_n^2$$