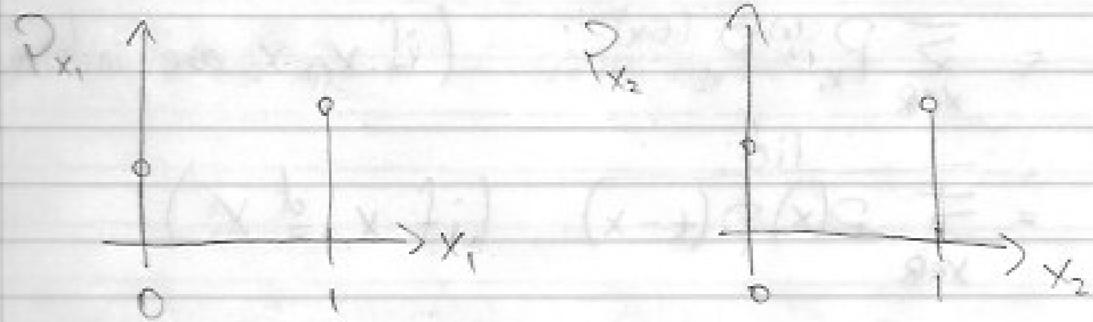
$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t-1\in\{0,1\}}\right)$$

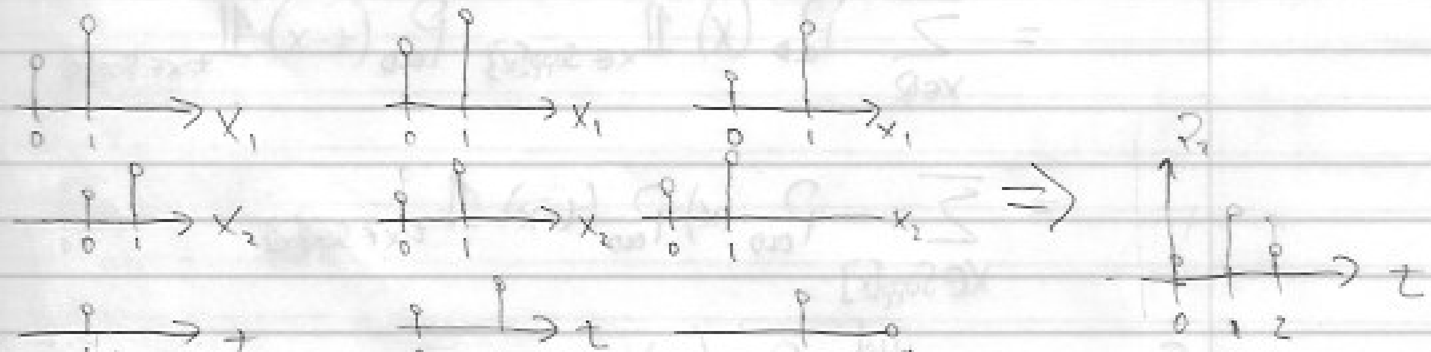
$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}}\right)$$

$$= p^t(1-p)^{2-t}\underbrace{\begin{cases} 1 & \text{if } t=0 \\ 2 & \text{if } t=1 \\ 1 & \text{if } t=2 \\ 0 & \text{otherwise} \end{cases}}_{\binom{2}{t}}$$

$$= \binom{2}{t} p^t(1-p)^{2-t}$$

$$= \text{Binom}(2, p)$$





$T = x_1 + x_2$, $x_1, x_2 \overset{iid}{\sim} \text{Bern}(p) := \binom{1}{x} p^x (1-p)^{1-x}$

$$P(t) = \sum_{x\in[0,1]} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-(t-x)}$$

$$= p^t (1-p)^{2-t} \sum_{x\in\{0,1\}} \binom{1}{x}\binom{1}{t-x}$$

$$\sum_{x\in\{0,1\}} \binom{1}{t-x} = \binom{1}{t} + \binom{1}{t-1} = \binom{2}{t}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$X_1 X_2 X_3 \overset{iid}{\sim} Bern(p)$$

$$T = X_1 + X_2 + X_3 \sim ?$$

$$T_3 = X_3 + T_2 \sim \sum_{x \in Supp[X_1]} P_{X_3}(x) P_{T_2}(t-x) \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum_{x \in [0,1]} \left(p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x} (1-p)^{\overset{2+x-t}{\overset{\shortparallel}{2-(t-x)}}} \underbrace{\mathbb{1}_{t-x \in \{0,1,2\}}}_{\text{not needed}}\right)$$

$$= p^t (1-p)^{3-t} \underbrace{\sum_{x \in [0,1]} \binom{2}{t-x}}_{\binom{2}{t} + \binom{2}{t-1} = \binom{3}{t}}$$

$$= \binom{3}{t} p^t (1-p)^{3-t}$$

$$= Binom(3, p)$$

$$X_1 X_2 \overset{iid}{\sim} Bin(n, p)$$

$$p(t) = \sum_{x \in [0,1,\cdots,n]} \left(\binom{n}{x} p^x (1-p)^{n-x}\right)\left(\binom{n}{t-x} p^{t-x} (1-p)^{\overset{n+x-t}{\overset{\shortparallel}{n-(t-x)}}} \underbrace{\mathbb{1}_{x \in [0,1,\cdots,n]}}_{\text{not needed}}\right)$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x \in [0,1,\cdots,n]} \binom{n}{x}\binom{n}{t-x}}_{= \binom{2n}{t} \text{ by Vandermonde's Identity}}$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t}$$

$$= \text{Binom}(2n, p)$$

Note: $p(t) = \sum_{x \in \text{Supp}(x)} p_{OLD}(x)\, p_{OLD}(t-x)\, \mathbb{1}_{t-x \in \text{Supp}[x]}$

OLD: PMF from MATH 241
NEW: OLD · Indicator function.

$B_1, B_2, \cdots \overset{iid}{\sim} \text{Bern}(p)$

Let $X = \arg\min_t \{B_t = 1\} - 1$

# of 0's before you get a 1.

$\text{Supp}(x) = \{0, 1, 2, \cdots\} = \mathbb{N}_0$

$p(0) = p$
$p(1) = (1-p)p$
$p(2) = (1-p)^2 p$
$\vdots$

$p(x) = (1-p)^x p$

$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p)$

$T = X_1 + X_2 \sim p(t) = \sum_{x \in [0,1,\cdots]} \left((1-p)^x p\right)\left((1-p)^{t-x} p\, \mathbb{1}_{t-x \in [0,1,\cdots]}\right)$

$\text{Supp}(T) = \mathbb{N}_0$ (how many 0's you get before you get two 1's)

$$= \sum_{x \in [0,1,\cdots]} \left((1-p)^x p \right) \left((1-p)^{t-x} p \, \mathbb{1}_{t-x \in [0,1,\cdots]} \right)$$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\cdots\}} \mathbb{1}_{t-x \in [0,1,\cdots]}$$

Note: $t-x \in [0,1 \cdots]$

$\Downarrow$

$t-x \geq 0$

$\Downarrow$

$t \geq x$

$\Downarrow$

$x \leq t$

$$= (1-p)^t p^2 \underbrace{\sum_{x \in \{0,1,\cdots\}} \mathbb{1}_{x \leq t}}_{\mathbb{1}_{0\leq t} + \mathbb{1}_{1 \leq t} + \mathbb{1}_{2 \leq t} + \mathbb{1}_{3 \leq t} + \mathbb{1}_{4 \leq t} + \mathbb{1}_{5 \leq t} + \cdots = t+1}$$

$$= (t+1)(1-p)^t p^2 = p(t)$$

$$p(4) = (5)(1-p)^4 p^2$$

	0	1	0	0	0	1
Index	1	2	3	4	5	6

4 0's are associated with $(1-p)^4$
2 1's are " " p^2

$$X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$$

$$T = X_1 + X_2 + X_3$$

$$= X_3 + T_2$$

$$= p(t) = \sum_{x \in [0,1,\cdots]} (1-p)^x p \left((t-x+1)(1-p)^{t-x} p^2 \, \mathbb{1}_{t-x \in [0,1\cdots]} \right)$$

$$= p^3(1-p)^t \sum_{x\in[0,1,\dots]} (t-x+1) \underbrace{\mathbb{1}_{t-x\in[0,1,\dots]}}_{\mathbb{1}_{x\le t}}$$

$$= \sum_{x\in[0,1,\dots]} \mathbb{1}_{x\le t} - \sum_{x\in[0,1,\dots]} x\mathbb{1}_{x\le t} \quad 0+\mathbb{1}_{1\le t}+2\mathbb{1}_{2\le t}+\cdots$$

$$= (t+1)\underbrace{\sum_{x\in[0,1,\dots]}\mathbb{1}_{x\le t}}_{(t+1)} - (1+2+\cdots+t)$$

$$= (t+1)^2 - \left(\frac{t(t+1)}{2}\right)$$

$$= t^2+2t+1 - \frac{t^2-t}{2}$$

$$= \frac{t^2+3t+2}{2}$$

$$= \frac{(t+2)(t+1)}{2}$$

$$= \frac{(t+2)!}{t!\,2!}$$

$$= \binom{t+2}{2}$$

$$= \binom{t+2}{2}(1-p)^t p^3$$

$$= \text{NegBin}(3,p)$$

prob. of 4 0's

$$P(5) = \binom{6}{2}(1-p)^4 p^3 = 15(1-p)^4 p^3$$

prob. of 3 1's

1	0	1	0	0	0	1
1	2	3	4	5	6	7

$$X_1 + \cdots + X_r \sim \text{NegBin}(r, p)$$

$$= \binom{t+r-1}{r-1} (1-p)^t p^r$$

(Lecture 2)

September 4th, 2019

(From lecture 1)

$$T = X_1 + X_2 \sim p(t) = p_{X_1}(x) * p_{X_2}(x)$$ ← Convolution operator

$$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{x_1,x_2}(x_1, x_2)\, \mathbb{1}_{x_1+x_2=t}$$

$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{x_1,x_2}(x_1, x_2)\, \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x \in \mathbb{R}} p_{x_1,x_2}(x, t-x)$$ ← general formula for discrete convolution.

$$= \sum_{x \in \mathbb{R}} p_{x_1}(x)\, p_{x_2}(t-x) \quad (\text{if } x_1, x_2 \text{ are independent})$$

$$\overset{iid}{=} \sum_{x \in \mathbb{R}} p(x)\, p(t-x) \quad (\text{if } x_1 \overset{d}{=} x_2)$$

$$= \sum_{x \in \mathbb{R}} p_{OLD}(x)\, \mathbb{1}_{x \in Supp[x]}\, p_{OLD}(t-x)\, \mathbb{1}_{t-x \in Supp[x]}$$

$$= \sum_{x \in Supp[x]} p_{OLD}(x)\, p_{OLD}(t-x)\, \mathbb{1}_{t-x \in Supp[x]}$$

For $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$

$T = X_1 + X_2$

$$p(t) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x} \right) \left(p^{t-x} (1-p)^{\overset{1+x+t}{\underset{\shortparallel}{1-(t-x)}}}\, \mathbb{1}_{t-x \in \{0,1\}} \right)$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$