Lec 19 Math 621 11/25/19

$U \sim \chi^2_k \quad E[U] = E[Z_1^2 + \dots + Z_k^2] = k\,E[Z_1^2]$
$= k\left(Var(Z) + E(Z)^2\right) = k\,Var(Z) = k1 = k$

let $\vec{X}$ be a r.v. vector with dim n. $E[\vec{X}] = \vec{\mu}$

$$\Sigma := Var[\vec{X}] = \begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \cdots & Cov[X_1,X_n] \\ Cov[X_1,X_2] & Var[X_2] & & \\ \vdots & & \ddots & \\ Cov[X_1,X_n] & \cdots & & Var[X_n] \end{bmatrix}$$

the definition

$$:= E[\underbrace{\vec{X}\vec{X}^T}_{n\times n}] - \underbrace{E[\vec{X}]E[\vec{X}]^T}_{n \times n} = Var(\vec{X})$$

$E[\vec{X}]E[\vec{X}]^T = \vec{\mu}\vec{\mu}^T$

let $A \in \mathbb{R}^{m\times n}$ matrix of constants

$$\underbrace{E[A\vec{X}]}_{\substack{m\times n\ n\times 1 \\ m\times 1}} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{X}] \\ E[\vec{a}_2 \cdot \vec{X}] \\ \vdots \\ E[\vec{a}_m \cdot \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_m \cdot \vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \vec{\mu} = A\vec{\mu}$$

$$Var[A\vec{X}] = E[AX(AX)^T] - E[AX]\,E[AX]^T$$

$$\underbrace{\ }_{\substack{m\times n\ n\times 1 \\ m\times n}} = E[AXX^TA^T] - (A\vec{\mu})(A\vec{\mu})^T$$

$m \times m$

$$= A\,E[XX^TA^T] - A\vec{\mu}\vec{\mu}^T A^T$$

$$= A\,E[XX^T]A^T - A\vec{\mu}\vec{\mu}^T A^T$$

$$= A\left(E[XX^T]A^T - \vec{\mu}\vec{\mu}^T A^T\right)$$

$$= A\left(E[XX^T] - \vec{\mu}\vec{\mu}^T\right)A^T$$

$$= A\Sigma A^T$$

this generalizes the rule we learned before

let $A = \vec{a}^T$ where $\vec{a} \in \mathbb{R}^n$

$$Var[\vec{a}^T\vec{X}] = \vec{a}^T \Sigma \vec{a} = A\Sigma A^T$$

From last lecture

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \sim N_n(\vec{0}_n, I_n) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$$\vec{X} = A\vec{Z}$$

$$E(\vec{X}) = E(A\vec{Z}) = A\,E(\vec{Z}) = A\vec{0}_n = \vec{0}_n$$

$$Var(\vec{X}) = Var[A\vec{Z}] = A\,Var[\vec{Z}]\,A^T = A I_n A^T = AA^T$$

Let $A \in \mathbb{R}^{n \times n}$, $\vec{\mu} \in \mathbb{R}^n$

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim f_{\vec{X}}(\vec{x}) = ?$$

Let's use change of variables formula. Assuming g is 1:1 is the same as assuming A is invertible in this case.

$$\Rightarrow A\vec{Z} = \vec{X} - \vec{\mu} \Rightarrow \vec{Z} = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{\mu}) = h(\vec{X}) = B\vec{X} - B\vec{\mu}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_h|$$

$$\Rightarrow h_1(\vec{x}) = \vec{b}_{1\cdot}\vec{X} - \vec{b}_{1\cdot}\vec{\mu}$$

$$\vdots$$

$$h_n(\vec{x}) = \vec{b}_{n\cdot}\vec{X} - \vec{b}_{n\cdot}\vec{\mu}$$

$$J_h = \det\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n}\end{bmatrix} = \det\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix} = \det[A^{-1}]$$

Note:
$AA^{-1} = I$
$\det[AA^{-1}] = 1$
$\det(A)\det(A^{-1}) = 1$
$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

$$\Rightarrow f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(A^{-1}(\vec{x} - \vec{\mu}))\,|\det[A^{-1}]| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-\vec{\mu}))^T(A^{-1}(\vec{x}-\vec{\mu}))} \frac{1}{|\det[A]|}$$

$$= \frac{1}{(2\pi)^{n/2}|\det[A]|} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T(A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})}$$

Recall $\mathrm{Var}[\vec{X}] = AA^T = \Sigma$. So it would be nice to use Σ as a parameter, not A.

Facts: $(AB)^{-1} = B^{-1}A^{-1}$

$$\Sigma = AA^T \Rightarrow \Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$$

$$= (A^{-1})^T A^{-1}$$

and $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I$

$\Rightarrow (A^{-1})^T A^T = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$

by def of inverse

Also... $\det[\Sigma] = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2 \Rightarrow \det(A) = \sqrt{\det[\Sigma]}$

$$\Rightarrow = \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})} = N_n(\vec{\mu}, \Sigma)$$

where $\vec{\mu} \in \mathbb{R}^n$
Σ is a valid cov matrix

general multivariate normal

$\vec{\mu} = \vec{0}$ and $\Sigma = I \Rightarrow$ standard MVN

What about for $A \in \mathbb{R}^{m \times n}$ and $\vec{\mu} \in \mathbb{R}^m$? $\underset{m\times 1}{\vec{X}} = \underbrace{\underset{m\times n}{A}\,\underset{n\times 1}{\vec{Z}}}_{m\times 1} + \underset{m\times 1}{\vec{\mu}}$

We can imagine setting $A_{aug} = \begin{bmatrix} A \\ \vec{v}_{m+1} \\ \vdots \\ \vec{v}_{n-m} \end{bmatrix}$, $\vec{\mu}_{aug} = \begin{bmatrix} \vec{\mu} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\underset{n\times 1}{\vec{X}_{aug}} = \underset{n\times n}{A_{aug}}\,\underset{n\times 1}{\vec{Z}} + \underset{n\times 1}{\vec{\mu}_{aug}}$

The $\vec{v}$'s can't be $\vec{0}$ since we need A_{aug} to be full rank $\Rightarrow$ invertible

where $m+1, \ldots n$ dimensions are nuisance so marginalize them!

$$f_{\vec{X}}(\vec{x}) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_{\vec{X}_{aug}}(\vec{x}_{aug})\, dx_{m+1} \cdots dx_n$$

That integral is HARD! We will solve this problem soon...

Introducing multivariate ch.f.'s

For r.v. $\vec{X}$ with dimension n, the ch.f. is defined as

$$\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right] = E\left[e^{i(t_1X_1 + \ldots + t_nX_n)}\right] = E\left[e^{it_1X_1}e^{it_2X_2}\cdots e^{it_nX_n}\right]$$

$$\underset{\uparrow}{=} E\left[e^{it_1X_1}\right]E\left[e^{it_2X_2}\right]\cdots E\left[e^{it_nX_n}\right] = \phi_{X_1}(t_1)\cdots\phi_{X_n}(t_n)$$

if $X_1, \ldots X_n$ indep

Note: not the same as $X_1 + \ldots + X_n$ since $\vec{t}$ has dimension n!

(P0) $\phi_{\vec{X}}(\vec{0}_n) = 1$ (P1) holds

(P2) $\vec{Y} = A\vec{X} + \vec{b}$ where $A \in \mathbb{R}^{m\times n}$, $\vec{b} \in \mathbb{R}^m$, $\dim(\vec{Y}) = m$

$$\Rightarrow \phi_{\vec{Y}}(\underset{\dim m}{\vec{t}}) = E\left[e^{i\vec{t}^T(A\vec{X}+\vec{b})}\right] = E\left[e^{i\underbrace{\vec{t}^TA}_{(A^T\vec{t})^T}\vec{X}}e^{i\vec{t}^T\vec{b}}\right] = e^{i\vec{t}^T\vec{b}}\,\phi_{\underset{\dim n}{\vec{X}}}(\underset{n\times m\, m\times 1}{A^T\vec{t}})$$

Let's get ch.f. of $\vec{Z}$:

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^n \phi_{Z_i}(t_i) = \prod_{i=1}^n e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let's get ch.f. of $\vec{X}$ by (P2) $\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma)$ let $A \in \mathbb{R}^{n\times n}$ invertible

$$\phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu}}\,\phi_{\vec{Z}}(\vec{t}^TA) = e^{i\vec{t}^T\vec{\mu}}\,e^{-\frac{1}{2}(A^T\vec{t})^TA^T\vec{t}} = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^TAA^T\vec{t}} = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$