(Lecture 18) November 20th, 2014

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\rightarrow Z_1 = \frac{X_1 - \mu}{\sigma}, Z_2 = \frac{X_2 - \mu}{\sigma}, \ldots, Z_n = \frac{X_n - \mu}{\sigma} \overset{iid}{\sim} N(0,1)$

$\vec{Z}^T Z = \sum Z_i^2 = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \text{algebra} = \frac{(n-1) S_n^2}{\sigma^2} + n\frac{(\bar{x} - \mu)}{\sigma^2} \sim \chi_n^2$

Fact: $U_1 \sim \chi^2_{k_1}$ indep. of $U_2 \sim \chi^2_{k_2} \rightarrow U + V \sim \chi^2_{k_1 + k_2}$

Recall from MATH 241

$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \rightarrow \frac{\vec{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

$\rightarrow \frac{(\bar{x} - \mu)^2}{\frac{\sigma^2}{n}} \sim \chi_1^2$

$\rightarrow n\frac{(\bar{x} - \mu)^2}{\sigma^2} \sim \chi_1^2$

Conjecture:

1) $\frac{(n-1) S_n^2}{\sigma} \sim \chi^2_{n-1}$

2) $\frac{(n-1) S_n^2}{\sigma^2}$ indep of $n\frac{(\bar{x} - \mu)^2}{\sigma^2}$

$\longleftrightarrow S_n^2$ indep of $\bar{X}_n$

since n, μ, σ are constants

$\vec{Z}^T \vec{Z} \sim \chi^2_n$

$= \underbrace{\vec{Z}^T I_n \vec{Z}}_{\text{quadratic}}$

$$\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = Z_1^2 \sim \chi^2_1$$

$$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & \ddots & \\ & & & 0 \end{bmatrix}}_{B_2} \vec{Z} = Z_2^2 \sim \chi^2_1$$

$\vdots$

$$\vec{Z}^T \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & & 0 \\ 0 & & 1 \end{bmatrix} \vec{Z} = Z_n^2 \sim \chi^2_1$$

All of these are indep.

$\vec{Z}^T B_1 \vec{Z} + \cdots + \vec{Z} B_n \vec{Z} \sim \chi^2_n$

$\vec{Z}^T \left(B_1 \vec{Z} + B_2 \vec{Z} + \cdots + B_n \vec{Z} \right)$

$\vec{Z}^T \underbrace{\left(B_1 + B_2 + \cdots + B_n \right)}_{I_n} \vec{Z}_n$

$\text{rank}[B_1] = \text{rank}[B_2] = \cdots = \text{rank}[B_n] = 1$

$\sum \text{rank}[B_i] = n$

Cochran's Theorem

If (a) $B_1 + \cdots + B_k = I_n$ and

(b) $\sum_{j=1}^{k} \text{rank}[B_j] = n$

then

(a) $\vec{Z}^T B_j \vec{Z} \sim \chi^2_{\text{rank}[B_j]}$

(b) $\vec{Z}^T B_{j_1} \vec{Z}$ indep of $\vec{Z}^T B_{j_2} \vec{Z}$ for all $j_1 \neq j_2$

$$\vec{Z}^T \vec{Z} = \sum Z_i^2 = \sum \left((Z_i - \bar{Z}) + \bar{Z} \right)^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2(Z_i - \bar{Z})\bar{Z} + \bar{Z}^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2\left(\sum Z_i \bar{Z} - \sum \bar{Z}^2\right) + \sum \bar{Z}^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2(n\bar{Z}^2 - n\bar{Z}^2) + n\bar{Z}^2$$

Let $\vec{1} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ dim $= n$

$$\bar{Z} = \frac{1}{n} \sum Z_i = \frac{1}{n} \vec{1}^T \vec{Z} = \frac{1}{n} \vec{Z}^T \vec{1}_n$$

$$n\bar{Z}^2 = n\bar{Z}\bar{Z} = n\left(\frac{1}{n} \vec{Z}^T \vec{1}_n\right)\left(\frac{1}{n} \vec{1}_n \vec{Z}\right)$$

$$= \vec{Z} \frac{1}{n} \vec{1} \vec{1}^T \vec{Z}$$

Let $J_n := \underset{n\times 1}{\vec{1}_n}\ \underset{1\times n}{\vec{1}_n^{\intercal}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$

$$\vec{z}^{\intercal} \frac{1}{n} \vec{1}\vec{1}^{\intercal} \vec{z} = \vec{z}^{\intercal} \underbrace{\left(\frac{1}{n} J_n\right)}_{B_2} \vec{z}$$

$$B_2 = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \quad \text{rank}[B_2] = 1$$

$$\sum (z_i - \bar{z})^2 = \sum z_i^2 - 2\underbrace{\sum z_i \bar{z}}_{n\bar{z}} + \underbrace{\sum \bar{z}^2}_{n\bar{z}^2} = \sum z_i^2 - n\bar{z}^2$$

$$= \vec{z}^{\intercal} I_n \vec{z} - \vec{z}^{\intercal}\left(\frac{1}{n} J_n\right)\vec{z}$$

$$= \vec{z}^{\intercal} \underbrace{\left(I_n - \frac{1}{n} J_n\right)}_{B_1} \vec{z}$$

$$B_1 = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \ddots & & \vdots \\ \vdots & & & \\ -\frac{1}{n} & & -\frac{1}{n} & 1-\frac{1}{n} \end{bmatrix}$$

Theorem If square matrix A is symmetric and idempotent i.e. $AA = A$ then rank$[A]$ = tr$[A]$

$$\left(I_n - \frac{1}{n} J_n\right)^{\intercal} = I_n^{\intercal} - \frac{1}{n} J_n^{\intercal} =$$

$(I_n - \frac{1}{n} J_n)(I_n - \frac{1}{n} J_n)$

$= I_n I_n - \frac{1}{n} J_n I_n - \frac{1}{n} I_n J_n + \frac{1}{n^2} J_n J_n$

$= I_n - \frac{2}{n} J_n + \frac{1}{n} \cdot n \cdot J_n$

$= I_n - \frac{1}{n} J_n$

So $\text{rank}[B_i] = \sum_{i=1}^{n} 1 - \frac{1}{n} = n-1$

By cochran's Thm,

$\sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ indep of $n\bar{Z}^2 \sim \chi^2_1$

$$\bar{Z} = \frac{Z_1 + \cdots + Z_n}{n} = \frac{\frac{X_1 - \mu}{\sigma} + \cdots + \frac{X_n - \mu}{\sigma}}{n}$$

$$= \frac{X_1 + \cdots + X_n - n\mu}{n\sigma}$$

$$= \frac{\bar{X} - \mu}{\sigma}$$

$$\rightarrow n\bar{Z}^2 \rightarrow \frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2_1$$

$$\sum (Z_i - \bar{Z})^2 = \sum \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma} \right)^2$$

$$= \sum \frac{(X_i - \bar{X})^2}{\sigma^2}$$

$= \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ which indep of χ^2_1

$\rightarrow S^2$ and $\bar{X}$ are independent

Fisher proved this in 1925

Geary (1936) showed that only the normal distribution has this property.

If σ is known and you're testing

$H_0 : \mu = \text{value}$

$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ "One-Sample Z-Test"

If we wish to test

$H_0 : \sigma^2 = \text{value}$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ "One-Sample χ^2 Test"

If we have two independent samples and wish to test

$H_0 : \sigma_1^2 = \sigma_2^2$ $\quad \frac{\frac{(n_1-1)S_1^2}{\sigma_1^2} \Big/ (n_1 - 1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2} \Big/ (n_2 - 1)}$

$$= \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \frac{s_1^2}{s_2^2} \sim f_{n_1,n_2}$$ "Two Sample F-Test for variance equality"

$$\frac{\bar{x}-\mu}{S} = \frac{\bar{x}-\mu}{\sqrt{\frac{S^2}{n}}} = \frac{\bar{x}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2}} = \frac{\bar{x}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2}}$$

σ unknown

If testing $H_0: \mu = \text{Value}$

$$\left.\frac{\frac{\bar{x}-\mu}{\sigma}}{\frac{\sigma}{\sqrt{n}}}\right\} \sim N(0,1)$$

"One-Sample T-test"

$$\left.\sqrt{\frac{(n-1)S^2}{\sigma^2}\Big/ n-1}\right\} \sqrt{\frac{\chi^2_{n-1}}{n-1}}$$

And the numerator and denominator are independent. (By Cochran's Theorem)

Multivariate Normal Distribution

$$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$$

Let $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}_n$

$$\text{Var}[\vec{Z}] = I_n$$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{Z}) = ?$$

$$f_{Z_1 \dots Z_n}(Z_1, \dots, Z_n) \overset{iid}{=} \prod_{i=1}^{n} f(Z_i)$$

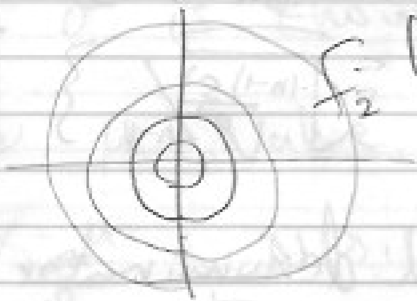
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z_i^2}{2}}$$

$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2}$

$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}, I_n)$, Standard Multivariate Normal Distribution

n: dimension; $\vec{0}$: mean; I_n: variance

$\vec{z} \sim N_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$



$f_2(\vec{z})$

Consider $\vec{\mu} \in \mathbb{R}^n$

$$\vec{x} = \vec{z} + \vec{\mu} = \begin{bmatrix} z_1+\mu_1 \\ z_2+\mu_2 \\ \vdots \\ z_n+\mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1, 1) \\ \sim N(\mu_2, 1) \\ \vdots \\ \sim N(\mu_n, 1) \end{matrix}$$

$\vec{x} \sim N_n(\vec{\mu}, I_n)$

Let $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$

$$\vec{x} = A\vec{z} = \begin{bmatrix} z_1 \\ z_1+z_2 \\ z_1+z_2+z_3 \\ \vdots \\ \\ z_1+\cdots+z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \\ \\ \sim N(0,n) \end{matrix} \Bigg\} \text{ not independent}$$

$$\begin{aligned} \mathrm{Cov}[z_1+z_2, z_1] &= \mathrm{Cov}[z_1, z_1] + \mathrm{cov}[z_1, z_2] \\ &= \mathrm{Var}[z_1] + 0 \\ &= 1 \neq 0 \Rightarrow \text{dependent} \end{aligned}$$

$E[\vec{x}] = E[A\vec{z}]$, $\mathrm{Var}[\vec{x}] = \mathrm{Var}[A\vec{z}]$.