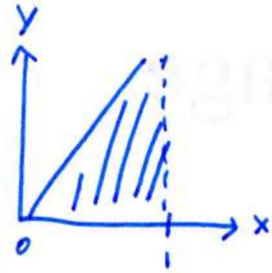
Lecture #14

- P155. Conditional Density:

$X \sim U(0,1)$

$Y|X=x \sim U(0,x)$



Goal: $\underset{\boxed{XY}}{\int} f_Y \cdot f_{X|Y}$

First formula:

- (I) Marguing $f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx$ or $f_Y(y) = \sum_{x \in \mathbb{R}} f_{Y|X}(x,y) P_X(x)$

$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy$ $\qquad P_Y(y) = \int P_{Y|X}(x,y) f_X(x)\,dx$

- (II) Def. of conditional density (Bessel(?))

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)} \text{ for } f_X(x)>0 \Rightarrow f_{XY}(x,y) = f_X(x) f_{Y|X}(x,y)$$

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} \text{ for } f_Y(y)>0 \Rightarrow f_{XY}(x,y) = f_Y(y) f_{X|Y}(x,y)$$

(III) Bager Rule: $f_{Y|X}(x,y) = \dfrac{f_{X|Y}(x,y) f_Y(y)}{f_X(x)}$

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y) f_X(x)}{f_Y(y)}$$

(IV) Bager Theorem:

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{\int_{\mathbb{R}} f_{XY}(x,y)\,dy} = \frac{f_{X|Y}(x,y) f_Y(y)}{\int_{\mathbb{R}} f_{X|Y}(x,y) f_Y(y)\,dy}$$

- $f_X(x) = \mathbb{1}_{x \in [0,1]}$

$f_{Y|X}(x,y) = \frac{1}{x} \mathbb{1}_{y \in [0,x]}$

* $f_{XY}(x,y) = \frac{1}{x} \mathbb{1}_{y \in [0,x]} \mathbb{1}_{x \in [0,1]}$

$= \frac{1}{x} \mathbb{1}_{0 \le y \le x \le 1}$

$= \frac{1}{x} \mathbb{1}_{y \in [0,1]} \mathbb{1}_{x \in [y,1]}$

$f_Y(y) = \int_{\mathbb{R}} \frac{1}{x} \mathbb{1}_{y \in [0,1]} \mathbb{1}_{x \in [y,1]}\,dx$

$= \mathbb{1}_{y \in [0,1]} \int_y^1 \frac{1}{x}\,dx$

$= [\ln(x)]_y^1 \mathbb{1}_{y \in [0,1]}$

$= -\ln(y) \mathbb{1}_{y \in [0,1]}$

Now, $f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{\frac{1}{x} \mathbb{1}_{y \in [0,1]} \mathbb{1}_{x \in [y,1]}}{-\ln(y) \mathbb{1}_{y \in [0,1]}}$

$= -\dfrac{1}{x \ln(y)} \mathbb{1}_{x \in [y,1]}$

e.g. $f_{X|Y}(x, \frac{3}{4}) \approx \frac{3.5}{x} \mathbb{1}_{x \in [\frac{3}{4}, 1]}$

- $f_{X|Y}(1, \frac{3}{4}) \approx 3.5$

  $f_{X|Y}(\frac{3}{4}, \frac{3}{4}) \approx 4.6$

- ~~Multilevel~~ mixture & Compound distribution:

pic: e.g. If there is no traffic download, take $Exp(\frac{1}{5})$ seconds. If there is traffic, downloads take $Exp(\frac{1}{20})$. How long does it take to download?

so, let $X \sim Bern(\frac{2}{3})$ if $X=0 \Rightarrow$ traffic

if $X=1 \Rightarrow$ no traffic

Let $T|X=1 \sim Exp(\frac{1}{5})$ be download time with no traffic

$T|X=0 \sim Exp(\frac{1}{20})$ " with traffic.

Draw the diagram:

	X	$T|X$
$1/3$	0	$Exp(\frac{1}{20})$
$2/3$	1	$Exp(\frac{1}{5})$

I? mixture model, mixture distribution, multilevel ", hiererachical model

Goal
- $f_T(t) = ?$

$$\underbrace{f_T(t) = \sum_{x\in\mathbb{R}} f_{T|X}(x,t)\, P_X(x)}_{\Rightarrow f_{X,T}(x,t)}$$

$$= \sum_{x\in\mathbb{R}} \left(\frac{1}{5}e^{-\frac{1}{5}t}\,\mathbb{1}_{x=1} + \frac{1}{20}e^{-\frac{1}{20}t}\,\mathbb{1}_{x=0}\right)\cdot\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{1-x}\mathbb{1}_{x\in\{0,1\}}$$

$$= \sum_{x\in\{0,1\}} \left(\frac{1}{5}e^{-\frac{1}{5}t}\,\mathbb{1}_{x=1} + \frac{1}{20}e^{-\frac{1}{20}t}\,\mathbb{1}_{x=0}\right)\cdot\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{1-x}$$

verting.

$$= \frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}$$

- Given a download time, what is the traffic probabilities? Bern

$$P_{X|T}(x,t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{\left(\frac{1}{5}e^{-\frac{1}{5}t}\,\mathbb{1}_{x=1} + \frac{1}{20}e^{-\frac{1}{20}t}\,\mathbb{1}_{x=0}\right)\cdot\frac{2}{3}^x\left(\frac{1}{3}\right)^{1-x}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}$$

$$= Bern\left(\frac{\frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}t}}\right)$$

- If download took 25sec, what's the prob. of the traffic?

$$P_{X|T}(0, 25) = \frac{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25}}{\frac{1}{3}\cdot\frac{1}{20}e^{-\frac{1}{20}\cdot 25} + \frac{2}{3}\cdot\frac{1}{5}e^{-\frac{1}{5}\cdot 25}} \approx 84\%.$$

- $X \sim U(0,1)$

$Y \mid X = x \sim U(0, x)$

$\underline{X}$	$\underline{Y \mid X}$	$\underline{Y}$
$U(0,1)$	$U(0,1)$	?

$$f_Y(y) = \int_{\mathbb{R}} \underbrace{f_{Y|X}(x,y)}_{\text{model}} \underbrace{f_X(x)\,dx}_{\text{mixing distribution}}$$

- If the mixing distribution is continuous, then Y is called "compound."

- let $X \sim \text{Gamma}(\alpha, \beta)$

$Y \mid X = x \sim \text{Poisson}(x)$

$\underline{X}$	$\underline{Y \mid X}$	$\underline{Y}$
$\text{Gamma}(\alpha,\beta)$	$\vdots$ Poisson(λ)	? discrete

Poisson

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y)\, f_X(x)\, dx = \int_{\mathbb{R}} \left(\frac{x^y e^{-x}}{x!} \mathbb{1}_{y \in (0,\infty)} \right) \left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)} \right) dx$$

$$\propto \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

$$= \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \mathbb{1}_{y \in \mathbb{N}_0}$$

$$\overset{\text{H.W}}{\propto} \text{ExtNegbin}\left(\alpha, \frac{\beta}{\beta+1}\right)$$

n is known

- $X \sim \text{Beta}(\alpha, \beta)$

$Y \mid X = x, n \sim \text{Bin}(n, x)$

$\underline{X}$	$Y \mid X$	$\underline{Y}$
$\text{Beta}(\alpha,\beta)$	$\text{Bin}(n,x)$	?

$$P_Y(y) = \int_{\mathbb{R}} P_{Y|X}(x,y)\, f_X(x)\, dx = \int_{\mathbb{R}} \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,1,\dots,n\}} \left(\frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]} \right) dx$$

$\Rightarrow$ Beta function

$$= \frac{\binom{n}{y} \mathbb{1}_{y \in \{0,1,\dots n\}}}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1} \, dx}$$

$$= \frac{\binom{n}{y} \cancel{\mathbb{1}_{y\in\{0,1,\dots n\}}}}{B(\alpha,\beta)} B(y+\alpha, n-y+\beta) \, \mathbb{1}_{y \in \{0,1,\dots n\}} = \text{Betabinomial}(\alpha,\beta,n)$$

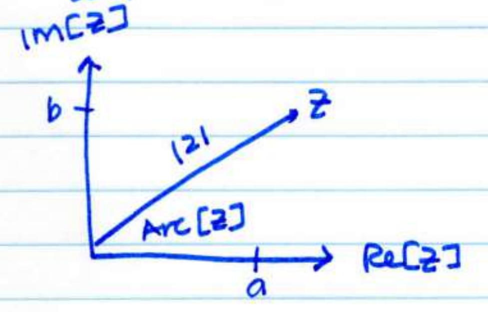
- $X \sim \text{Gamma}(\alpha,\beta)$
  $Y \mid X = x \sim \text{Exp}(X)$

X	$Y \mid X$	Y
$\text{Gamma}(\alpha,\beta)$	$\text{Exp}(X)$	?

$\Rightarrow Y \sim \text{Lomax}(\beta,\alpha)$
wikepedia

- Review of Complex #s:
  $z: a + bi \in \mathbb{C}$ where $i = \sqrt{-1}$
  a: real component, bi: imaginary component

  $|z| = \sqrt{a^2+b^2}$
  $\text{Arc}[z] = \underset{4 \text{ pieces}}{\dots} = \arctan(\frac{b}{a})$



$$\left.\begin{aligned} i^2 &= -1 \\ i^3 &= i \cdot i^2 = -i \\ i^4 &= i^2 \cdot i^2 = 1 \\ i^5 &= i^4 \cdot i = i \\ &\vdots \end{aligned}\right\} \text{cycle of 4 it repeats...}$$