10/2 $T_4 = X_4 + T_3$

$f_T(t) = \int_0^\infty \left(\frac{x^2}{2}\lambda^3 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x<t}\right) dx$

$= \frac{1}{2}\lambda^4 e^{-\lambda t}\int_0^\infty x^2 \mathbb{1}_{\underbrace{t-x\in(0,\infty)}_{x<t}} dx = \frac{1}{2}\lambda^4 e^{-\lambda t}\int_0^t x^2 dx = \frac{1}{2\cdot 3}t^3\lambda^4 e^{-\lambda t}$

- $T_5 = T_4 + X_5$

$f_{T_5}(t) = \int_0^\infty \left(\frac{x^3}{2\cdot 3}\lambda^4 e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)}\mathbb{1}_{x<t}\right) dx$

$= \frac{1}{2\cdot 3}\lambda^5 e^{-\lambda t}\int_0^\infty x^3 \mathbb{1}_{\underbrace{t-x\in(0,\infty)}_{x<t}} = \frac{1}{2\cdot 3}\lambda^5 e^{-\lambda t}\int_0^t x^3 dx = \frac{1}{2\cdot 3\cdot 4}t^4\lambda^5 e^{-\lambda t}$

~ $f_{T_k}(t) = \frac{1}{(k-1)!}t^{k-1}\lambda^k e^{-\lambda t} = \text{Erlang}(k,\lambda)$

$\text{Supp}[T_k] = (0,\infty)$, $k\in\mathbb{N}$, $\lambda\in(0,\infty)$ $\quad$ $\text{Erlang}(1,\lambda) = \text{Exp}(\lambda)$

like Geom + NegBin

waiting until ya get k successes

~ Gamma Function: $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$

$a\in(0,\infty)$ (it's an area)

$= \underbrace{\int_0^a t^{x-1}e^{-t}\,dt}_{\substack{\text{lower incomplete Gamma Function} \\ \gamma(x,a)}} + \underbrace{\int_a^\infty t^{x-1}e^{-t}\,dt}_{\substack{\Gamma(x,a) \\ \text{upper incomplete Gamma Function}}}$

$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\gamma(x,a)}{\Gamma(x)} + \frac{\Gamma(x,a)}{\Gamma(x)} = P(x,a) + Q(x,a)$

upper regularized Gamma Function: $Q(x,a)$

lower regularized Gamma Function: $P(x,a)$

- $\Gamma(1) = \int_0^\infty \underbrace{t^{1-1}}_{1} e^{-t}\,dt = -\left(e^{-t}\right)_0^\infty = 1$

$\Gamma(x+1) = x\Gamma(x)$ $\quad$ ex: $\Gamma(2) = 1\Gamma(1)$ $\quad$ $\Gamma(3) = 2\Gamma(2) = 2\cdot 1$ $\quad$ $\Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1$

$\Gamma(n) = (n-1)!$

$\Gamma(n+1) = n!$

- $X \sim \text{Erlang}(k,\lambda) := \overbrace{\frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}}^{f(x)} \mathbb{1}_{x\geq 0}$

$F(x) = \int_0^x f(y)\,dy = \int_0^x \frac{\lambda^k e^{-\lambda y}y^{k-1}}{(k-1)!}\,dy = \frac{\lambda^k}{(k-1)!}\int_0^x e^{-\lambda y}y^{k-1}\,dy$

- $\int_0^\infty t^{x-1} e^{-ct}\, dt \quad c \in \mathbb{R}$

  let $u = ct \quad t = \frac{u}{c} \quad dt = \frac{1}{c}\, du$

  $= \int_0^\infty \frac{u}{c} e^{-u} \left(\frac{1}{c} du\right) = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \left(\frac{1}{c} du\right) = \frac{1}{c^x} \overbrace{\int_0^\infty u^{x-1} e^{-u}\, du}^{\Gamma(x)} = \frac{\Gamma(x)}{c^x}$

- $\int_0^a t^{x-1} e^{-ct}\, dt$ = (same u subsitution as above)

  $\int_0^{ca} \frac{u^{x-1}}{c^{x-1}} e^{-u} \left(\frac{1}{c} du\right) = \frac{1}{c^x} \int_0^{ac} u^{x-1} e^{-u}\, du = \frac{\gamma(x, ac)}{c^x}$

- $\int_a^\infty t^{x-1} e^{-ct}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, ac)}{c^x} = \frac{\Gamma(x, ac)}{c^x}$

- $\lim_{a \to \infty} \gamma(x, a) = \Gamma(x) \qquad \lim_{a \to 0} \gamma(x, a) = 0 \qquad \lim_{a \to \infty} P(x, a) = 1$

- $\lim_{a \to 0} \Gamma(x, a) = \Gamma(x) \qquad \lim_{a \to \infty} \Gamma(x, a) = \infty \qquad \lim_{a \to 0} Q(x, a) = 1$

- back to CDF of Erlang:

  $\frac{\lambda^k}{(k-1)!} \int_0^x e^{-\lambda y} y^{k-1}\, dy = \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}} \frac{\gamma(k, \lambda x)}{\lambda^k} = \frac{\gamma(k, \lambda k)}{\Gamma(k)} = P(k, \lambda x)$

  $= F(x) = P(X \leq x)$

  $P(X > x) = 1 - P(X \leq x) = Q(k, \lambda x)$

- If $n \in \mathbb{N}$, $\Gamma(n, a) = \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}}_{dv} dt \Rightarrow v = -e^{-t} \quad du = (n-1) t^{n-2}\, dt$

  $= [uv]_a^\infty - \int_a^\infty v\, du = \left[t^{n-1} e^{-t}\right]_\infty^a + \int_a^\infty (e^{-t})(n-1) t^{n-2}\, dt = a^{n-1} e^{-a} + (n-1) \int_a^\infty t^{(n-1)-1} e^{-t}\, dt$

  $= a^{n-1} e^{-a} + (n-1)\Gamma(n-1, a)$

  $= a^{n-1} e^{-a} + (n-1)\left(a^{n-2} e^{-a} + (n-2)\Gamma(n-2, a)\right) \qquad \int_a^\infty t^{1-1} e^{-t} dt = e^{-a}$

  $= e^{-a}\left(a^{n-1} + (n-1) a^{n-2} \underset{(n-1)}{+} (n-2) a^{n-3} + \ldots \underset{(n-1)!}{+} (1) \underbrace{\Gamma(1, a)}_{\substack{e^{-a} \\ e^0}}\right)$

  $= e^{-a} (n-1)! \left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \ldots + \frac{a^0}{0!}\right) = e^{-a} (n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$

  $\Gamma(n+1, a) = e^{-a}\, n! \sum_{i=0}^{n} \frac{a^i}{i!}$

  $\frac{\Gamma(n+1, a)}{n!} = e^{-a} \sum_{i=0}^{a} \frac{a^i}{i!}$

- $X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}$

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{x}\frac{\lambda^i}{i!} = \frac{\Gamma(x+1,\lambda)}{x!} = \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

$X \sim \text{Erlang}(k,\lambda) \Rightarrow 1 - F(x) = Q(k, \lambda x)$

- Rate of events is λ. Time is measured in seconds.

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

What is the probability of zero events before 1 second?

$T_1 \sim \text{Erlang}(1,\lambda) = \text{Exp}(\lambda)$

$$P(T>1) = Q(1,\lambda) = \frac{\Gamma(1,\lambda)}{\Gamma(\lambda)} = \frac{\int_1^\infty t^{1-1}e^{-\lambda t}dt}{\int_0^\infty t^{1-1}e^{-t}dt} = \frac{e^{-\lambda}}{1} = e^{-\lambda}$$

Imagine $N \sim \text{Poisson}(\lambda)$

$1 - F_{T_1}(1) = Q(1,\lambda) = F_N(0)$

What is the probability at most one event occurs by 1 second?

$T_2 \sim \text{Erlang}(2,\lambda)$

$P(T_2>1) = 1 - F_{T_2}(1) = 1 - \underset{e^{-\lambda}(1+\lambda)}{Q(2,\lambda)} = F_N(1)$

What is the probability at most K events occur by 1 second?

$T_k \sim \text{Erlang}(k,\lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k,\lambda) = F_N(k)$

↳ Poisson Process: If exponential waiting times, then the number of events that happen per unit time is Poisson-distributed.