Lec 2 Math 621 9/4/19

convolution operator

$$T = X_1 + X_2 \sim p(t) = p_{X_1}(x) * p_{X_2}(x) = ?$$

collect all the prs

$$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_1 + x_2 = t}$$

$$= \sum \sum p_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$

if indep

$$\overset{\downarrow}{=} \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x)$$

if ident. dist

$$= \sum_{x \in \mathbb{R}} p(x)\, p(t-x)$$

$$= \sum_{x \in \mathbb{R}} p_{old}(x)\, \mathbb{1}_{x \in Supp(X)}\, p_{old}(t-x)\, \mathbb{1}_{t-x \in Supp(X)}$$

$$= \sum_{x \in Supp(X)} p_{old}(x)\, p_{old}(t-x)\, \mathbb{1}_{t-x \in Supp(X)}$$

for $X_1, X_2 \overset{iid}{\sim} Bern(p)$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x}\, p^{t-x} (1-p)^{1+x-t}\, \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t-1 \in \{0,1\}} \right)$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right)$$

$$= p^t (1-p)^{2-t} \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \\ 0 & \text{o/w} \end{cases}$$

$$= \binom{2}{t} p^t (1-p)^{2-t} = Binom(2, p)$$

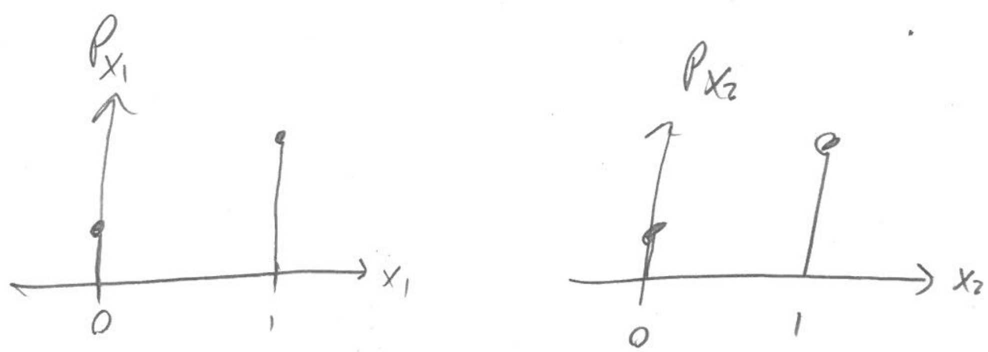
We did a "discrete convolution"

$T = X_1 + X_2 \Rightarrow$

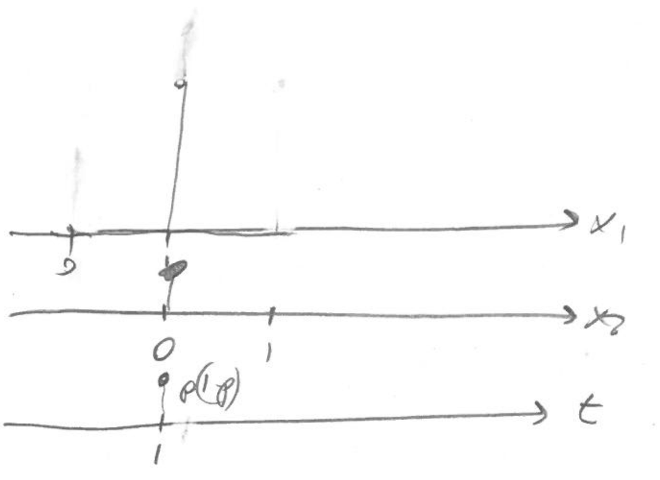
$p(t) = p(x_1) * p(x_2)$

We will develop an explicit easy formula for this ~~soon~~.

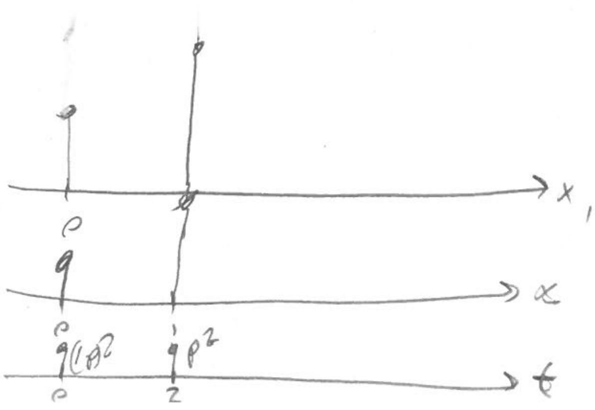
Let's see this pictorially...



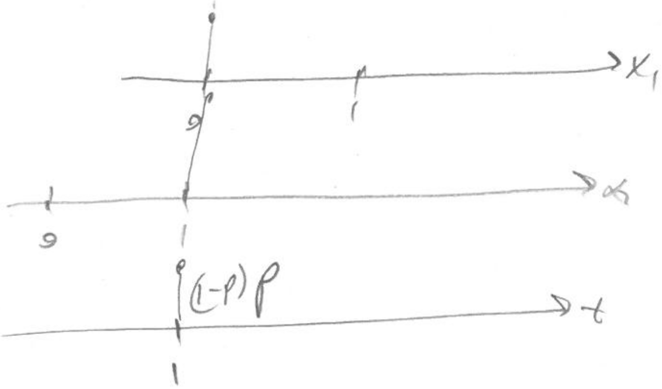
You can imagine a convolution as one pmf passing through the other and adding



First pt. of answer

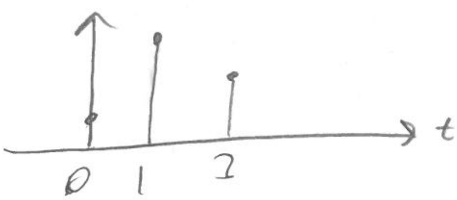


Second pt. of answer



Third pt of iteration

Aggregating these together...



Convolutions are a form of the "sum-product" operator

$T = X_1 + X_2$ and X_1, X_2 independent (not necessarily ident. distr.)

$$p(t) = p_{X_1}(x) * p_{X_2}(x) = \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x)$$

$$= \sum_{x \in \mathrm{supp}(X_1)} p_{X_1}(x)\, p_{X_2}(t-x)$$

written w/out indicator function because it is accounted for in

Recall $X \sim Bin(n,p) = \binom{n}{x} p^{t}(1-p)^{n-x}$

$$\Rightarrow X \sim Bin(1,p) = \binom{1}{x} p^{t}(1-p)^{1-x} = Bern(p)$$

$$\binom{1}{x} = \begin{cases} 1 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ 0 & \text{o/t} \end{cases} \quad \text{e.g} \quad \binom{1}{7}=0 \quad \binom{1}{\pi}=0 \quad \binom{1}{-3}=0 \quad \text{etc...}$$

$$\Rightarrow \binom{1}{t} = \mathbb{1}_{x \in \{0,1\}}$$

$\Rightarrow$ $Bern(p) = \binom{1}{x} p^{x}(1-p)^{1-x}$ as the combinatoric term has the same function as the indicator

$X_1, X_2 \overset{iid}{\sim} Bern(p)$, $T = X_1 + X_2 \sim ?$

$$p(t) = \sum_{x_1 \in Supp(X_1)} p(x_1)\, p(t-x_1) = \sum_{x \in Supp(X_1)} \binom{1}{x} p^{x}(1-p)^{1-x} \binom{1}{t-x} p^{t-x}(1-p)^{1-(t-x)}$$

$$= p^{t}(1-p)^{2-t} \underbrace{\sum_{x \in Supp(X_1)} \binom{1}{x}\binom{1}{t-x}}$$

$$\underbrace{\sum_{x \in \{0,1\}} \binom{1}{t-x}}$$

$$\binom{1}{t} + \binom{1}{t-1}$$

$$\parallel$$

$$\binom{2}{t}$$

Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$

$$T_3 = \underbrace{X_1 + X_2}_{Y \sim \text{Bin}(2,p)} + X_3 = X_3 + Y = \sum_{x_3 \in \text{Supp}[X_3]} p_{X_3}(x_3)\, p_Y(t - x_3)$$

$$= \sum_{x \in \{0,1\}} \binom{1}{x} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}$$

$$= p^t (1-p)^{3-t} \underbrace{\sum_{x \in \{0,1\}} \binom{1}{x}\binom{2}{t-x}}_{\binom{2}{t} + \binom{2}{t-1}}$$

$$\binom{2}{t} + \binom{2}{t-1} = \binom{3}{t} \quad \text{(Pascal's)}$$

In HW you'll get the general Binomial PMF.

$X_1, X_2 \overset{iid}{\sim} \text{Bin}(n,p)$ A cute example

$T = X_1 + X_2$

$$p(t) = \sum_{x \in \text{Supp}[X_1]} \left(\binom{n}{x} p^x (1-p)^{n-x} \right)\left(\binom{n}{t-x} p^{t-x} (1-p)^{n-t+x} \right)$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x=0}^{n} \binom{n}{x}\binom{n}{t-x}}_{\binom{2n}{t}}$$ Vandermonde's Identity

Consider $B_1, B_2, \ldots \overset{\text{iid}}{\sim} \text{Bern}(p)$. Let X be the # of natural 0's before the first natural 1.

infinite

let $X := \min_t \{B_t = 1\} - 1$

$$P(X=0) = P(\{\text{no } 0\text{'s, just a } 1\}) = p$$
$$P(X=1) = P(\{\text{one } 0, \text{ and a } 1\}) = (1-p)p$$
$$P(X=2) = P(\{\text{two } 0\text{'s and a } 1\}) = (1-p)^2 p$$
$$\vdots$$
$$P(X=x) = P(\{x \; 0\text{'s and a } 1\}) = (1-p)^x p$$

$p \in (0,1)$

$\text{Supp}[X] = \{0, 1, 2, \ldots\} = \mathbb{N}_0$

Consider $X_1, X_2, \ldots \overset{\text{iid}}{\sim} \text{Geom}(p)$

Let $T = X_1 + X_2$ # of 0's before the 2^{nd} success.

$$p(t) = \sum_{x \in \text{Supp}[X_1]} p(x)\, p(t-x)$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p (1-p)^{t-x} p \, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= p^2 (1-p)^t \underbrace{\sum_{x=0}^{\infty} \mathbb{1}_{t-x \in \{0,1,\ldots\}}}$$

$$= \sum_{x=0}^{\infty} \mathbb{1}_{x \leq t} = \mathbb{1}_{0 \leq t} + \mathbb{1}_{1 \leq t} + \ldots \mathbb{1}_{t \leq t} + \cancel{\mathbb{1}_{t+1 \leq t} + \ldots}$$
$$= t+1$$

$$\mathbb{1}_{t-x \geq 0} = \mathbb{1}_{t \geq x} = \mathbb{1}_{x \leq t}$$

$$\mathbb{1}_{t-x \in \{0,1,\ldots\}} = \mathbb{1}_{-x \in \{-t, -t+1, -t+2, \ldots\}} = \mathbb{1}_{x \in \{t, t-1, t-2, \ldots, 0, -1, -2, \ldots\}}$$

$$= (t+1)\, p^2 (1-p)^t$$

Why is this true? two 1's $\Rightarrow p^2$

$$\underset{1}{\underline{0}}\ \ \underline{0}\ \ \dots\ \underline{1}\ \dots\ \underset{t+1}{\underline{0}}\ \ \underset{t+2}{\underline{1}}$$

t 0's $\Rightarrow (1-p)^t$

Why $\cdot(t+1)$? Because the first 1 can be in any of the $t+1$ positions.

$$\text{Supp}[T] = \{0, 1, \dots\}$$

Same!

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2$$

all places first two 1's go ; t 0's ; 3 1's

$$p(t) = \sum_{x \in \text{Supp}(X)} p(x)\, p(t-x) = \dots = \binom{t+2}{2}(1-p)^t p^3$$

$$= \sum_{x=0}^{\infty} \left((1-p)^x p\right)\left((t-x+1)\, p^2 (1-p)^{t-x}\, \mathbb{1}_{t-x \in \{0,1,\dots\}}\right)$$

$$= (1-p)^t p^3 \underbrace{\sum_{x=0}^{\infty} (t-x+1)\, \mathbb{1}_{x \le t}}$$

$$(t+1)\underbrace{\sum_{x=0}^{\infty} \mathbb{1}_{x\le t}}_{t+1} - \sum_{x=0}^{\infty} x\, \mathbb{1}_{x \le t}$$

$$= (t+1)^2 - \sum_{x=1}^{t} x = (t+1)^2 - \frac{t(t+1)}{2}$$

$$t^2 + 2t + 1 - \frac{t^2}{2} - \frac{t}{2}$$

$$= \frac{t^2 - 3t + 2}{2}$$

$$= \frac{(t+2)(t+1)}{2}$$

$$= \frac{(t+2)!}{t!\,2!}$$

$$= \binom{t+2}{2}$$