Lecture #3

- $X_1, \cdots X_{\cancel{r}} \overset{iid}{\sim} Geom(p)$

$$T = \sum_{\cancel{i=1}}^{n} X_i \sim NegBin(r,p) = \binom{x+r-1}{r-1} p^r (1-p)^t$$

- $X \sim bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

let n get large & p get small, but p

$$\lambda = np \Rightarrow p = \frac{\lambda}{n}, \quad n \in \mathbb{N} \text{ \& } p \in (0,1)$$
$$\lambda \in (0,\infty)$$

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{\cancel{n}!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{\cancel{x}!} \lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \overset{1}{\cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$ is called poisson.

$supp[X] = \{0, 1 \ldots\} = \mathbb{N}_0$ (natural #s with 0)

- $X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$

$T = X_1 + X_2$?

$$p(t) = \sum_{x \in supp(X)} p_{X_1}(x) \, p_{X_2}(t-x) \, \mathbb{1}_{t-x \in supp(x)}$$

$$= \sum_{x\in\{0,1,2\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \cancel{supp\{0,1}\{0,1,2\ldots\}}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x\in\{0,1,2\ldots t\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x\le t}$$

$\cancel{t \lambda^t e^{-}}$

$$\cancel{= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x\in\{0,1,2\ldots\}} \mathbb{1}_{t-x\in\{0,1,2\ldots\}}}$$ same as $\mathbb{1}_{x\le t}$

=

$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\dots,t\}} \frac{1}{x!(t-x)!} \cdot \frac{t!}{t!} = \binom{t}{x}$

$= \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x \in \{0,1,\dots,t\}} \binom{t}{x}}$ → prove that this is equal to 2^t which will not be on the exam though

$= \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$

- $X, Y \overset{iid}{\sim} \text{Geom}(p)$

What is $P(X>Y)$?

$= P(Y>X) < \frac{1}{2}$ because X & Y are arbitrary

$P(X>Y) + P(Y>X) + \underbrace{P(X=Y)}_{>0} = 1$

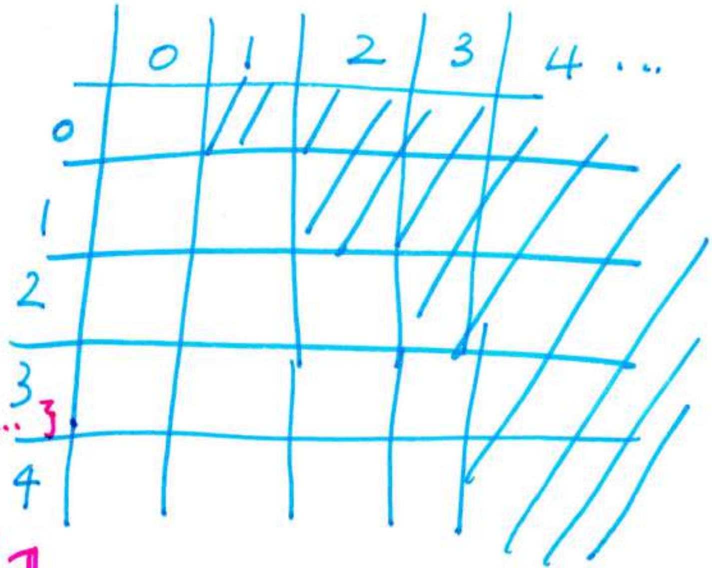
SOLVE

What is $P(X>Y)$?

$\cancel{= P(X+Y>a)}$

$P(X>Y) = \sum_{y \in \mathbb{R}} \sum_{x \in \mathbb{R}} P_{X,Y}(x,y) \mathbb{1}_{x>y}$

$\cancel{= \sum_{y \in \mathbb{R}} \sum_{x \in (y,\infty)} p(1-p)^x p(1-p)^y \mathbb{1}_{x \in \{0,1,\dots\}}}$



$= \sum_{y \in \mathbb{R}} \sum_{x \in (y,\infty)} p(1-p)^x \mathbb{1}_{x \in \{0,1,\dots\}} p(1-p)^y \mathbb{1}_{y \in \{0,1,\dots\}}$

$= p^2 \sum_{y \in \mathbb{R}} \sum_{x \in (y,\infty)} (1-p)^x \mathbb{1}_{x \in \{0,\dots\}} (1-p)^y \mathbb{1}_{y \in \{0,\dots\}}$

$= p^2 \sum_{y \in \{0,1,\dots\}} (1-p)^y \sum_{x \in \{y+1, y+2, \dots\}} (1-p)^x$

* Geometric series: $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

From here, let $x' = x - (y+1)$

$\Rightarrow x = x' + y + 1$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x'\in\{0,1,\dots\}} (1-p)^{x'+y+1}$$

$$= p^2(1-p) \sum_{y\in\{0,\dots\}} (1-p)^{2y} \underbrace{\sum_{x'\in\{0,1,\dots\}} (1-p)^{x}}_{\frac{1}{1-(1-p)} = \frac{1}{p}}$$

$$= p(1-p) \sum_{y\in\{0,\dots\}} (1-p)^{2y}$$

$$= p(1-p) \sum_{y\in\{0,1,\dots\}} \left((1-p)^2\right)^y$$

$$= p(1-p)\frac{1}{1-(1-p)^2} = \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \frac{1-p}{2-p}$$

If p is really small, then it becomes $\frac{1}{2}$.

- X is a discrete r.v.

$$E(X) = \sum_{x\in\mathbb{R}} x\,p(x)$$

Expected value
: Expectation.
= Average

$$E[g(X)] = \sum_{x\in\mathbb{R}} g(x)\,p(x)$$

imp

$$E[g(X,Y)] = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} g(x,y)\,p(x,y)$$

- $$E[\mathbb{1}_{X\in A}] = \sum_{x\in\mathbb{R}} \mathbb{1}_{x\in A}\, p(x) = \sum_{x\in A} p(x) = p(X\in A)$$ important for later

$$p(X>Y) = E[\mathbb{1}_{X>Y}] = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p(x,y)\,\mathbb{1}_{x>y}$$

- multinomial:

Story: p_1 = prob. of drawing an apple
p_2 = " banana

$p_1 + p_2 = 1$

Draw n with replacement

X_1 = # of apples drawn

X_2 = # of banan "

$X_1 \sim \text{bin}(n, p_1)$

$X_2 \sim \text{bin}(n, p_2)$

$$= \frac{n!}{x_1!(n-x_1)!} = \frac{n!}{x_1!\,x_2!}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{X_1,X_2}(x_1, x_2) = \binom{n}{x_1} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$$

If adding 3, then it becomes 3.

- multinormial coefficient

$$\binom{n}{x_1, x_2} := \frac{n!}{x_1!\,x_2!} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,\dots,n\}} \mathbb{1}_{x_2 \in \{0,\dots,n\}}$$

- $X \sim$ multinomial

$$\left(n, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) \sim \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$