Lec 10 Math 621 10/16/19

Formula: $f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|$

$Y = -\frac{1}{\lambda}\ln(X) \overset{HW}{\sim} \text{Exp}(\lambda)$

Hard one...

$X \sim \text{Exp}(1)$ $\quad Y = -\ln\left(\frac{e^{-X}}{1-e^{-X}}\right) = \ln\left(\frac{1-e^{-X}}{e^{-X}}\right) = \ln(e^X - 1) = g(X)$

$\Rightarrow e^Y = e^X - 1 \Rightarrow e^X = e^Y + 1 \Rightarrow X = \ln(e^Y+1) = g^{-1}(Y)$

$\Rightarrow \left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\left(\frac{1}{e^y+1}\right)e^y\right| = \frac{e^y}{e^y+1} > 0 \;\forall y \in \mathbb{R}$

$f_Y(y) = \frac{e^y}{e^y+1} \underbrace{e^{-\ln(e^y+1)}}_{e^{\ln\left(\frac{1}{e^y+1}\right)}} \mathbb{1}$

$\underbrace{\ln(e^y+1) \in (0,\infty)}$

$\underbrace{e^y+1 \in (1,\infty)}$

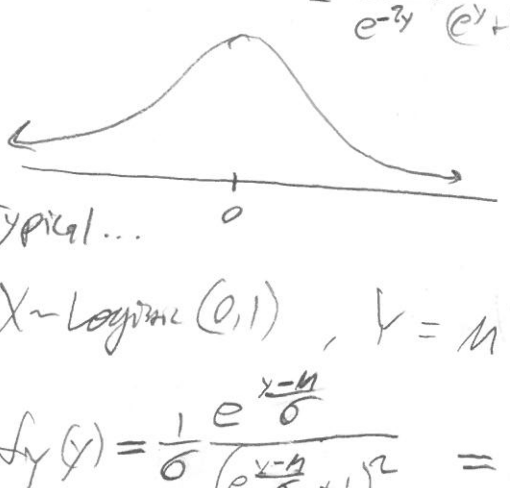
$e^y \in (0,\infty)$

$y \in \mathbb{R}$

no need for indicator anymore! Always $= 1$

$= \frac{e^y}{(e^y+1)^2} = \text{Logistic}(0,1)$

$= \frac{e^{-2y}}{e^{-2y}} \cdot \frac{e^y}{(e^y+1)^2} = \frac{e^{-y}}{(e^{-y}+1)^2}$

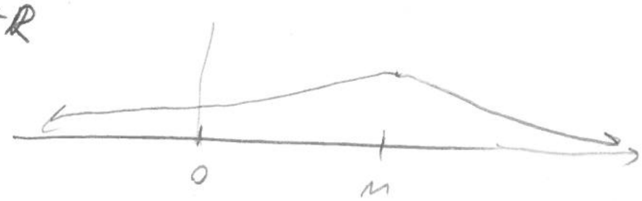


Looks just like normal except with thicker tails. Used in chess rankings (Elo). Used in deep learning. Implicitly used in "logistic regression".

Typical...

$X \sim \text{Logistic}(0,1)$, $Y = \mu + \sigma X$ s.t $\sigma > 0, \mu \in \mathbb{R}$

$f_Y(y) = \frac{1}{\sigma}\frac{e^{\frac{y-\mu}{\sigma}}}{\left(e^{\frac{y-\mu}{\sigma}}+1\right)^2} = \text{Logistic}(\mu,\sigma)$



Shift and scale

$E[Y] = \mu$, $SE[Y] = \sigma\frac{\pi}{\sqrt{3}}$

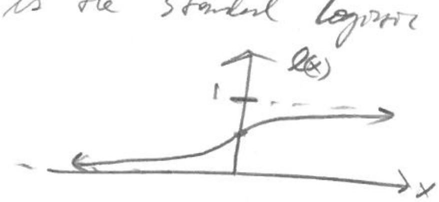
$X \sim U(0,1)$, $Y = -\ln\left(\frac{1}{X}-1\right) \sim \text{Logistic}(0,1)$ HW

Why is it called "logistic"? There's a famous function called the Logistic function

$$\ell(x) := \frac{L}{1+e^{-k(x-\mu)}}$$ where L: max val, k: steepness, μ: center

$$= \frac{1}{1+e^{-x}}$$ is the "standard logistic function"

$$= \frac{e^x}{1+e^x}$$

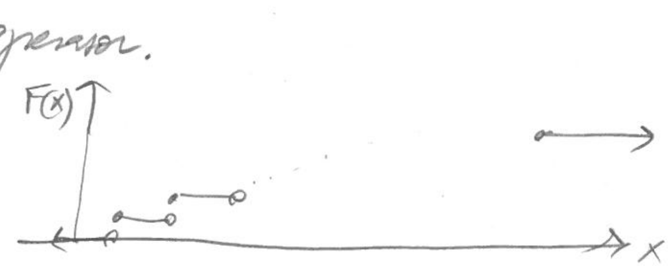


$$F_Y(y) = \int_{-\infty}^{y} \overbrace{\frac{e^t}{(1+e^t)^2}}^{f_Y(t)}\, d(t) = \int_{1}^{1+e^y} \frac{\cancel{u-1}}{u^2}\frac{1}{\cancel{u-1}}\,du = \left[-\frac{1}{u}\right]_1^{1+e^y} = 1-\frac{1}{1+e^y} = \frac{e^y}{1+e^y} = \text{the standard logistic function}$$

let $u = 1+e^t \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du = \frac{1}{u-1}du$ $\quad t=y \Rightarrow u = 1+e^y$, $t=-\infty \Rightarrow u=1$

New Question: Solve for min x s.t. $q \geq P(X \leq x) = F(x)$. This x is called the q^{th} quantile or $100q$ "percentile".
If $q = 0.5$, the x is called the "median". Denoted:
$x = Q[X, q]$, the quantile operator.

$X \sim U(\{2,4,6,8,10,12,14,16,18,20\})$



X	p(x)	F(x)
2	0.1	0.1
4		0.2
6		0.3
8		0.4
10		0.5
12		0.6
14		0.7
16		0.8
18		0.9
20		1

$Q[X, 0.3] = 6$. "6 is the 30%ile of the X distr"
$Q[X, 0.9] = 18$
$Q[X, 0.85] = 18$. $\nexists x$ s.t. $F(x) = 0.85$ but $F(18) = 0.9$, the smallest x s.t. $F(x) \geq 0.85$.

If X is a cont. distr, with a "contiguous support", i.e. one interval with no gaps e.g. $[0,10]$, $(0,\infty)$, $\mathbb{R}$ but not $[0,1] \cup [2,3]$, then $F(x)$ is strictly increasing on the support $\Rightarrow$ is 1:1 mapping, an inverse exists:

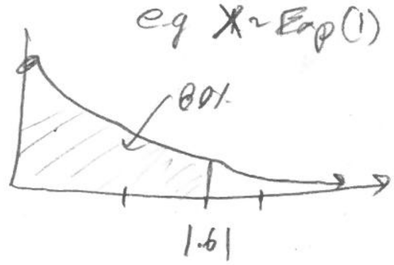
$$\min x \text{ s.t. } q \geq F(x) \Rightarrow F^{-1}(q) \geq x \Rightarrow x = F^{-1}(q).$$

$$\Rightarrow Q[X, q] = F^{-1}(q)$$ the "quantile function"

Let $X \sim \text{Exp}(\lambda)$, $F(x) = 1 - e^{-\lambda x}$ What is $F^{-1}(q)$?

$$q = 1 - e^{-\lambda x} \Rightarrow 1 - q = e^{-\lambda x} \Rightarrow -\ln(1-q) = \lambda x \Rightarrow x = \lambda \ln\left(\frac{1}{1-q}\right)$$

e.g $X \sim \text{Exp}(1)$



$F^{-1}(0.8) = \ln(5) \approx 1.61$

Median? $F^{-1}(0.5) = \lambda \ln(2)$

The quantile function usually isn't in closed form e.g.

$X \sim \text{Erlang}(k, \lambda)$, $F(x) = P(k, \lambda x)$

$Q[X, q]$ can be found if you solve for x: $q = P(k, \lambda x)$ with a computer

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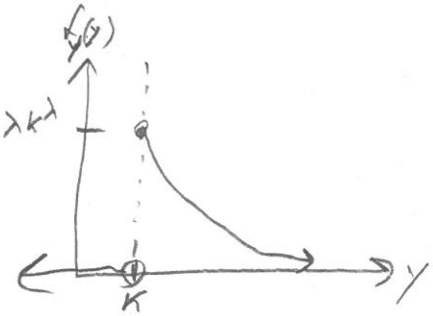
$X \sim \text{Logistic}(0, 1)$ $F(x) = \frac{1}{1+e^{-x}}$. Find quantile function: for HW.

$X \sim \text{Exp}(\lambda)$, $Y = ke^X$ $\quad g^{-1}(Y) = \ln\left(\frac{Y}{k}\right) = \ln(Y) - \ln(k)$

$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{y}$ $\quad f_Y(y) = \lambda e^{-\lambda(\ln(\frac{y}{k}))} \frac{1}{y} \mathbb{1}_{\ln(y) - \ln(k) \in (0,\infty)}$



$$= \frac{\lambda}{y} e^{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)} \mathbb{1}_{\ln(y) \in (\ln(k), \infty)}$$

$$= \frac{\lambda}{y}\left(\frac{y}{k}\right)^{-\lambda} \mathbb{1}_{y \in (k,\infty)} = \frac{\lambda k^{\lambda}}{y^{\lambda+1}} \mathbb{1}_{y \in (k,\infty)} = \text{ParetoI}(k, \lambda)$$

Param space: $k \in (0,\infty)$ b/c density negative
$\lambda \in (0,\infty)$ (from the exp.)

$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}} dt = \lambda k^\lambda \left[\frac{t^{-\lambda}}{-\lambda}\right]_k^y = \left(1 - \left(\frac{k}{y}\right)^\lambda\right) \mathbb{1}_{y \in (k,\infty)}, \quad \bar{F}_Y(y) = \left(\frac{k}{y}\right)^\lambda \mathbb{1}_{y \in (k,\infty)} + \mathbb{1}_{y \in (-\infty, k)}$$

$F_Y^{-1}(q) = Q[Y, q] = k(1-q)^{-\frac{1}{\lambda}}$

Another waiting time / survival distribution used to model
- pop. spread - HD disk failure - sizes of sand particles - And...

the "pareto principle". In 1896, Vilfredo pareto noticed 80% of land owned by 20% of people. Let $k = 1$ $\quad f_Y(y) = \lambda y^{-\lambda - 1}$

$F_Y^{-1}(q) = (1-q)^{-\frac{1}{\lambda}}$

Y models amt of land individual owns

Let $L(a)$ be the % of land owned if land is owned by people that own less than a.

$$L(a) = \frac{\text{Amt owned} \le a}{\text{Total amt owned}} = \frac{\int_1^a y\, f_Y(y)\, dy}{\underbrace{\int_1^\infty y\, f_Y(y)\, dy}_{E(Y)}} = \frac{\cancel{\lambda} \int_1^a y^{-\lambda}\, dy}{\cancel{\lambda} \int_1^\infty y^{-\lambda}\, dy}$$

$$= \frac{\left[\frac{y^{-\lambda+1}}{\cancel{-\lambda+1}}\right]_1^a}{\left[\frac{y^{-\lambda+1}}{\cancel{-\lambda+1}}\right]_1^\infty} = \frac{a^{-\lambda+1} - 1}{0 - 1} = 1 - a^{-\lambda+1}$$

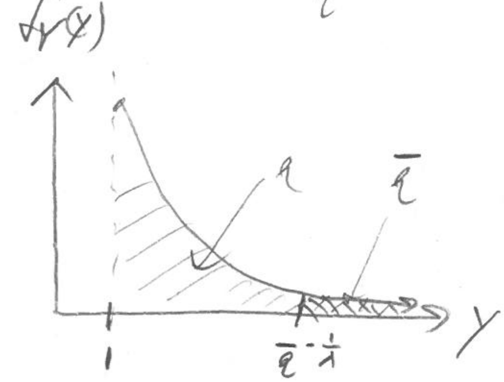
q prop

What is the land amt that ~~...~~ of people have less than?

$Y_q := F_Y^{-1}(q)$ q prop of people own this amt or less.

$= (1-q)^{-\frac{1}{\lambda}}$ let $\bar{q} := 1-q$

$= \bar{q}^{-\frac{1}{\lambda}}$



Calculate $L(Y_q)$ is the % land owned by bottom q of people

$$= 1 - Y_q^{1-\lambda} = 1 - \left(\bar{q}^{-\frac{1}{\lambda}}\right)^{1-\lambda} = 1 - \bar{q}^{\frac{\lambda-1}{\lambda}} = 1 - \bar{q}^{\left(1-\frac{1}{\lambda}\right)}$$

If the bottom q prop of people own $\bar{q}$ % of the land:

$$\bar{q} \overset{\text{set}}{=} 1 - \bar{q}^{\frac{\lambda-1}{\lambda}} \Rightarrow 1-\bar{q} = \bar{q}^{\frac{\lambda-1}{\lambda}} \Rightarrow \ln(1-\bar{q}) = \left(1-\frac{1}{\lambda}\right)\ln(\bar{q})$$

$$\Rightarrow 1 - \frac{1}{\lambda} = \frac{\ln(1-\bar{q})}{\ln(\bar{q})} \Rightarrow \frac{1}{\lambda} = 1 - \frac{\ln(1-\bar{q})}{\ln(\bar{q})} \Rightarrow \lambda = \frac{1}{1 - \frac{\ln(1-\bar{q})}{\ln(\bar{q})}}$$

$$= \frac{\ln(\bar{q})}{\ln(\bar{q}) - \ln(q)} = \frac{\ln(\bar{q})}{\ln\left(\frac{\bar{q}}{q}\right)} = \log_{\bar{q}/q}(\bar{q})$$

If $q = 0.8$

Pareto principle: $\lambda = \log_{\frac{1}{4}}\left(\frac{1}{5}\right) = 1.161$

$Y \sim \text{Pareto}(1, 1.161)$

Let $X, Y \overset{iid}{\sim} Exp(1) := e^{-x} \mathbb{1}_{x \in (0,\infty)}$

$$D = X - Y = X + Z \quad \text{where} \quad Z := -Y \sim e^{x} \mathbb{1}_{x \in (-\infty, 0)}$$

$$f_D(d) = \int_{Supp(X)} f_X(x) f_Z(d-x) \mathbb{1}_{d-x \in Supp(Z)} dx = \int_0^\infty e^{-x} e^{d-x} \mathbb{1}_{\underset{\substack{x-d \in (0,\infty) \\ x \in (d,\infty)}}{\underbrace{d-x \in (-\infty,0)}}} dx$$

$$= e^d \begin{cases} \int_0^\infty e^{-2x} dx & \text{if } d \le 0 \\ \int_d^\infty e^{-2x} dx & \text{if } d > 0 \end{cases} = e^d \begin{cases} \left[-\frac{1}{2} e^{-2x}\right]_0^\infty & \text{if } d \le 0 \\ \left[-\frac{1}{2} e^{-2x}\right]_d^\infty & \text{if } d > 0 \end{cases}$$

$$= \frac{1}{2} e^d \begin{cases} 1 & \text{if } d \le 0 \\ e^{-2d} & \text{if } d > 0 \end{cases} = \frac{1}{2} \begin{cases} e^d & \text{if } d \le 0 \\ e^{-d} & \text{if } d > 0 \end{cases} = \frac{1}{2} e^{-|d|}$$

$$= \text{Laplace}(0,1)$$

AKA double-exponential

$\sigma > 0, \mu \in \mathbb{R}$

$$X = \sigma D + \mu \sim \frac{1}{|\sigma|} f_D\left(\frac{x-\mu}{\mu}\right) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} \sim \text{Laplace}(\mu, \sigma)$$