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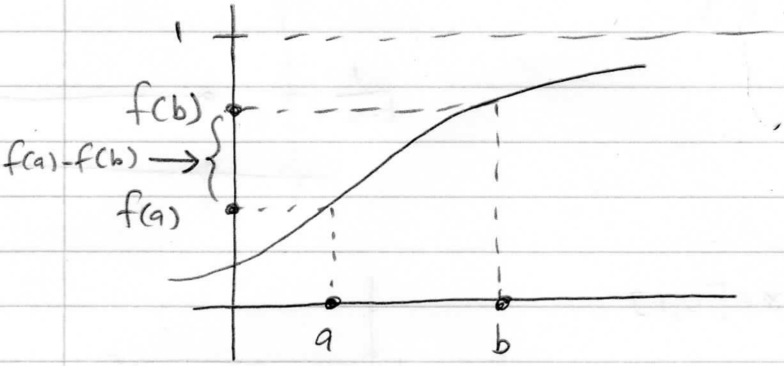
- For continuous r.v's X, $|supp[X]| = |\mathbb{R}|$, $supp[x] = \{x \mid f(x) > 0\}$
$\Rightarrow p(x) = P(X = x) = 0$.
The derivative of the CDF is very important.

- $f(x) := \frac{d}{dx}\left[F(x)\right]$ - Probability Density Function (CPD

$$P(X \in [a,b]) = F(b) - F(a) = \int_a^b f(x)\,dx$$

Fundamental Thm. of Calc.

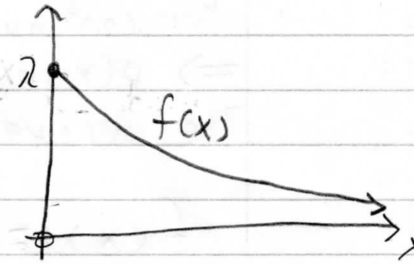


- Properties of $f(x)$ (PDF)

1. $\int_{\mathbb{R}} f(x)\,dx = 1 = \underbrace{F(\infty)}_{1} - \underbrace{F(-\infty)}_{0}$

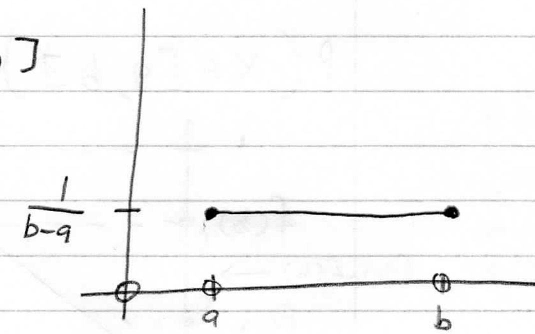
2. $f(x) \geq 0$ ($F(x) = P\ X \leq x$) is monotonically increasing, derivative ≥ 0.

- $X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0}$



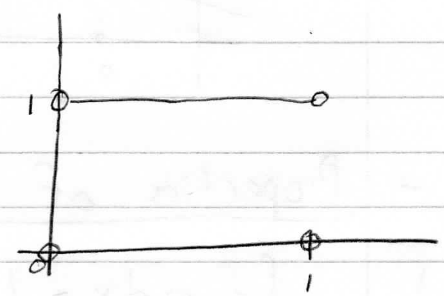
Supp$[X] = [0, \infty)$
Note: Exp(1) ~ *Standard Exponential

- $X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$
*Uniform
(Completely random r.v)



Supp$[X] = [a,b]$
$a, b \in \mathbb{R}$

- $X = U(0,1) = \mathbb{1}_{x \in [0,1]}$
* Standard Uniform



- Let $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix}$

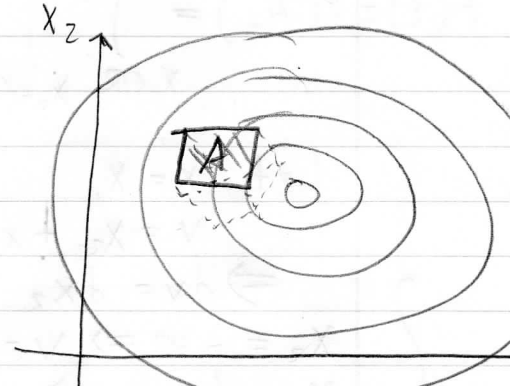
<u>Joint Density Function (JDF)</u>

$$f_{\vec{X}}(\vec{x}) \underset{\text{if ind.}}{=} f_{X_1}(x_1) \cdots f_{X_K}(x_K) \underset{\text{if i.i.d}}{=} f(x_1) \cdots f(x_K)$$

- $$\underset{\mathbb{R}^K}{\int \cdots \int} f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_K = 1$$

ex. Let $K = 2$

$$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

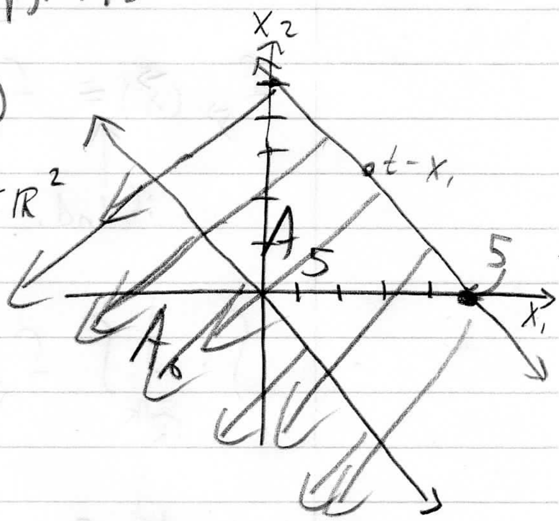


- Convolution (by computing CDF then taking derivative).

Let $T = X_1 + X_2 \sim f_T(t) = ?$ pg. 145

CDF: $F(t) = P(T \leq t) = P(A_t)$

$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \underbrace{x_1 + x_2 \leq t}_{x_2 \leq t - x_1} \right\} \subset \mathbb{R}^2$



$$F(t) = P(A_t) = \int_{x_1 \in \mathbb{R}} \int_{x_2 \in (-\infty, t-x_1)} f_{X_1,X_2}(x_1, x_2)\, dx_2\, dx_1$$

Let $x = x_1$

$v = x_2 + x \Rightarrow x_2 = v - x$

$\Rightarrow dv = dx_2$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x_1 \Rightarrow v = t$

FTC: $\frac{d}{dx} \int_{-\infty}^{a} f(x)\,dx = f(a)$

$$\Rightarrow P(A_t) = \int_{x \in \mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\, dv\, dx$$

nalysis

$$= \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\, dx\, dv$$

$$f(t) = \frac{d}{dt}\left[\quad\right] = \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x) =$$

General Convolution Formula

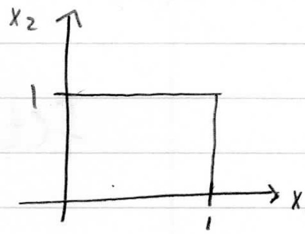
$$\underset{\text{if ind}}{=} \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx \overset{\text{if ind}}{=} \int_{Supp[X_1]} f_{X_1}(x) f_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_1]}\, dx$$

$$\underset{\text{if iid}}{=} \int_{\mathbb{R}} f(x) f(t-x)\,dx = \int_{Supp[X]} f(x) f(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

ex, $X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f(t)$

$Supp[T] = [0,2]$

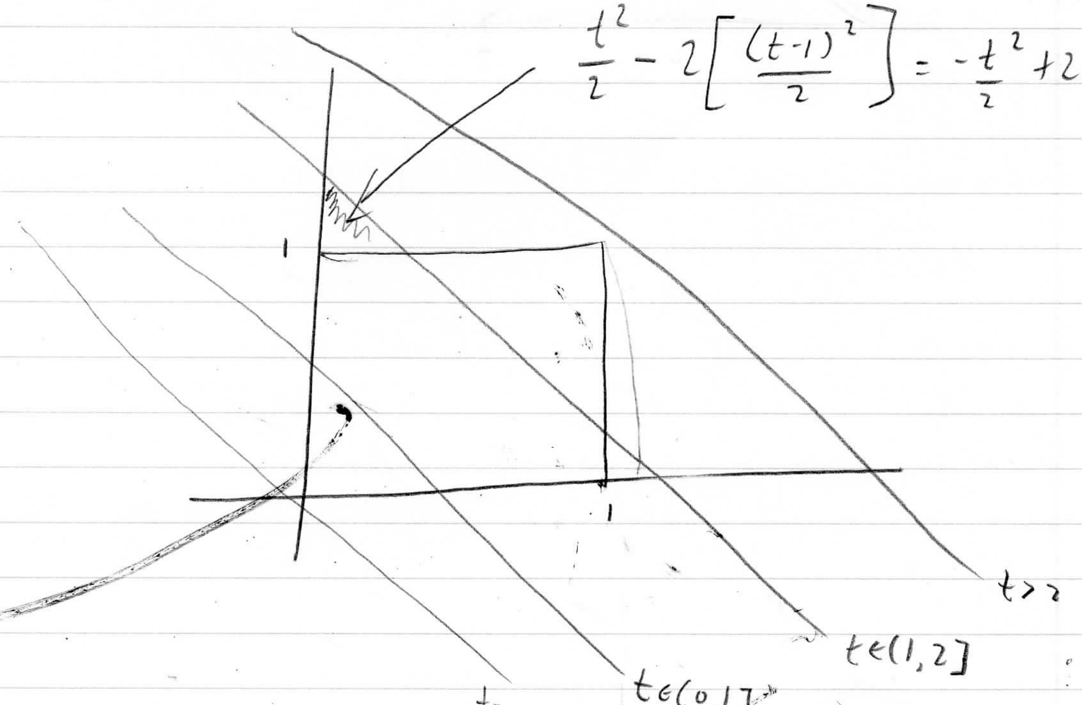
$f_{X_1,X_2}(x_1,x_2) = \mathbb{1}_{x_1 \in [0,1]} \mathbb{1}_{x_2 \in [0,1]}$

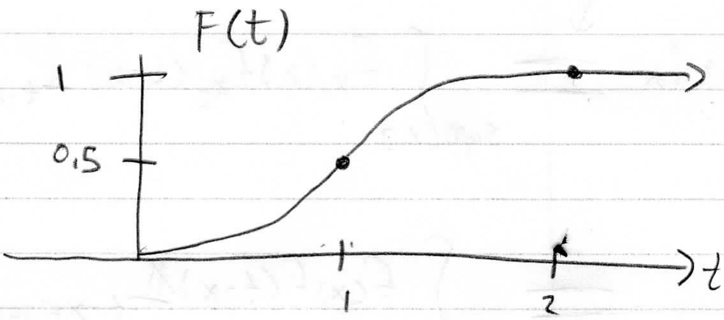


JDF is unit cube

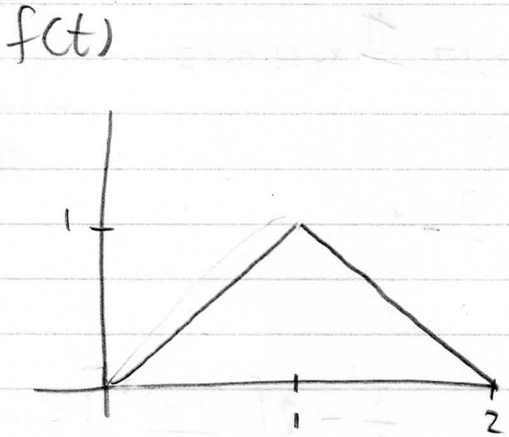
$$F(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{2}t^2 & \text{if } t \in (0,1] \\ -\frac{t^2}{2} + 2t - 1 & \text{if } t \in (1,2] \\ 1 & \text{if } t > 2 \end{cases}$$

$$\frac{t^2}{2} - 2\left[\frac{(t-1)^2}{2}\right] = -\frac{t^2}{2} + 2t - 1$$





$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$



Using i.i.d formula

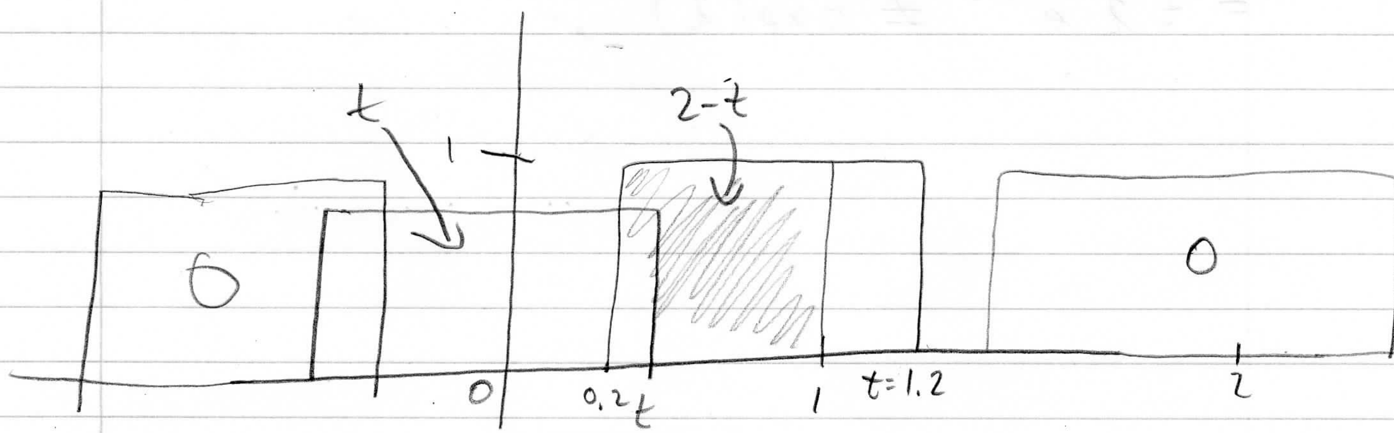
$$f(t) = \int_0^1 \underset{f_{X_1}(x)}{(1)}\ \underset{f_{X_2}(t-x)}{(1)}\ \mathbb{1}_{\underbrace{t-x \in [0,1]}} \, dx$$

$$= \mathbb{1}_{x-t \in [-1,0]}$$

$$= \mathbb{1}_{x \in [t-1, t]}$$

$$= \int_0^1 \mathbb{1}_{x \in [t-1,t]} \, dx = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t > 2 \end{cases}$$

from $1-(t-1)$



- <u>Sum of two exponential r.v's</u>

$X_1, X_2 \sim Exp(\lambda)$

$T = X_1 + X_2 \sim f_T(t) \stackrel{?}{\sim}$

$$f_{T_2}(t) = \int_0^\infty (\lambda e^{-\lambda x})(\lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in (0,\infty)}) dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \le t} \, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx$$

$$= t \lambda^2 e^{-\lambda t} \neq Exp(\lambda)$$

- Let $T_3 = X_1 + X_2 + X_3 = T_2 + X_3$, $Supp[T] = [0,$

$$f_{T_3}(t) = \int_0^\infty (x\lambda^2 e^{-\lambda x})(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t})\,dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x\,dx$$

$$= \frac{1}{2} t^2 \lambda^3 e^{-\lambda t} \quad //$$

- Let $T_4 = X_1 + \dots + X_4 = T_3 + X_4$

$$f_{T_4}(t) = \int_0^\infty \left(\frac{1}{2} x^2 \lambda^3 e^{-\lambda x}\right)(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t})\,dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\,dx$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}$$

- Let $T_5 = X_1 + \dots + X_5 = T_4 + X_5$

$$f_{T_5}(t) = \int_0^\infty \left(\frac{1}{2 \cdot 3} x^3 \lambda^4 e^{-\lambda x}\right)(\lambda e^{-\lambda(t-x)} \mathbb{1}_{x \le t})\,dx$$

$$= \frac{1}{2 \cdot 3} \lambda^5 e^{-\lambda t} \int_0^t x^3\,dx$$

$$= \frac{1}{2 \cdot 3 \cdot 4} \lambda^5 e^{-\lambda t} t^4$$

$\longrightarrow$

- General Form

$$f_{T_k}(t) = \frac{1}{(k-1)!} \lambda^k e^{-\lambda t} t^{k-1} := \underline{\text{Erlang}(k, \lambda)}$$

- $\text{Supp}[T_k] = (0, \infty)$

$k \in \mathbb{N}$
$\lambda \in (0, \infty)$