Hint for HW question:

$$\sum_{\vec{x}_{-j} \in \mathbb{R}^{k-1}} \binom{n}{x_1, x_2, \ldots, x_n} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

$$= \binom{n}{x_j} p_j^{x_j} \sum \binom{n - x_j}{x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k} p_1^{x_1} p_2^{x_2} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k} \cdot \frac{(1-p_j)^{n-x_j}}{(1-p_j)^{n-x_j}}$$

$$= \text{Multinomial}\left(n - x_j, \frac{\vec{p}}{1-p_j}\right)$$

$$X_1, X_2 \overset{\text{i.i.d.}}{\sim} \underset{}{\overset{\text{Uniform}}{U}}(\{0,1,2,3\}) \overset{\text{defined as}}{:=} \begin{cases} 0 & \text{w.p. } \overset{\text{with probability}}{\tfrac{1}{4}} \\ 1 & \text{w.p. } \tfrac{1}{4} \\ 2 & \text{w.p. } \tfrac{1}{4} \\ 3 & \text{w.p. } \tfrac{1}{4} \end{cases}$$

Uniform Discrete Dist.

Generally,

$$X \sim \underset{\text{uniform}}{U}(A) := \frac{1}{|A|} \mathbb{1}_{x \in A}$$

$|A|$ means number of elements of set A

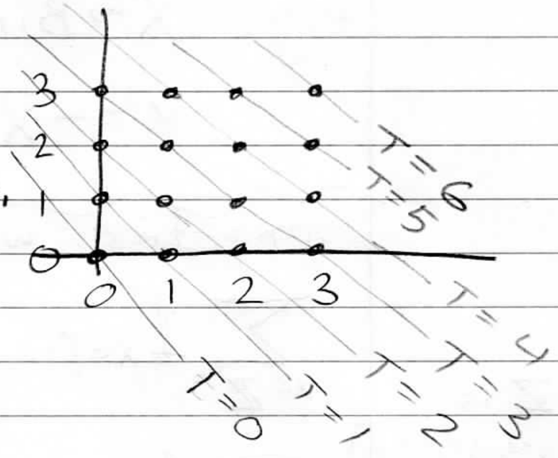
and $Supp[X] = A$

parameter space $A \subset \mathbb{R}$ such that A is finite

$X_1 \sim U(\{0,1,2,3\})$
$X_2 \sim U(\{0,1,2,3\})$

$$p_{X_1, X_2}(x_1, x_2) = \frac{1}{16} \mathbb{1}_{x_1 \in \{0,1,2,3\}} \mathbb{1}_{x_2 \in \{0,1,2,3\}}$$

$$T = X_1 + X_2 \sim p_T(t)$$



$$p(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{x_1, x_2}(x_1, x_2) \mathbb{1}_{x_2 = t - x_1}$$

$$p(1) = \sum \sum p \, \mathbb{1}_{x_2 = 1 - x_1} = 2\left(\frac{1}{16}\right)$$

$$p(0.5) = 0$$

$$p(3) = 4\left(\frac{1}{16}\right)$$

$$p(6) = 1\left(\frac{1}{16}\right) = \frac{1}{16}$$

$$\text{let } Y = \underbrace{-X}_{g(X)} \sim p_Y(y)$$

so $X = -Y$

$$X \sim \overset{\text{uniform}}{U}(\{0,1,2,3\})$$

$$\left.\begin{array}{llll} X=0 & \Rightarrow Y=0 & \text{w.p. } \frac{1}{4} \\ X=1 & \Rightarrow Y=-1 & \text{w.p. } \frac{1}{4} \\ X=2 & \Rightarrow Y=-2 & \text{w.p. } \frac{1}{4} \\ X=3 & \Rightarrow Y=-3 & \text{w.p. } \frac{1}{4} \end{array}\right\} \Rightarrow Y \sim U(\{0,-1,-2,-3\})$$

PMF of Y

$$p_Y(y) := P(Y=y) = P(-Y=-y) = P(X=-y)$$
$$= p_X(-y)$$

$$\Rightarrow \text{Supp}[Y] = -\text{Supp}[X]$$

e.g. $X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$

$$Y = -X \sim \binom{n}{-x} p^{-y} (1-p)^{n+y}$$

practice with indicator functions

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1 \qquad \underbrace{\{-c, \ldots -1, 0, 1, 2, \ldots c\}}_{2c-1 \text{ numbers}}$$

$\mathbb{Z}$ is $\{\ldots,-2,-1,0,1,2,\ldots\}$

$$\sum_{x \in \{-d,-d+1,\ldots,d-1,d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2d+1 & \text{if } c \geq d \\ 2c+1 & \text{if } c < d \end{cases}$$

$$= 2\min(c,d)+1$$

practice with indicator functions:

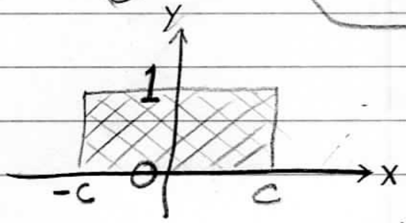
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [-c,c]} = 2c+1 \leftarrow \underbrace{\{-c, \ldots, -2, -1, 0, 1, 2, 3, \ldots, c\}}_{2c-1 \text{ numbers}}$$

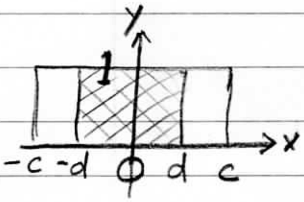
$$\sum_{x \in \{-d, -d+1, \ldots, d-1, d\}} \mathbb{1}_{x \in [-c,c]} = \begin{cases} 2d+1 & \text{if } c \geq d \\ 2c+1 & \text{if } c < d \end{cases}$$

$$= 2\min(c,d)+1$$

$$\int_{\mathbb{R}} \mathbb{1}_{x \in [-c,c]}\, dx = 2c$$



$$\int_{-d}^{d} \mathbb{1}_{x \in [-c,c]}\, dx = \begin{cases} 2d & \text{if } c \geq d \\ 2c & \text{if } c < d \end{cases}$$



$$= 2\min(c,d)$$

$X_1, X_2 \overset{i.i.d.}{\sim} Poisson(\lambda)$

We saw $T = X_1 + X_2 \sim Poisson(2\lambda)$

also written $p_{X_1|T=t}(x)$

$$p_{X_1|T}(x,t) = P(X_1 = x_1 \mid T = t) \neq Poisson(\lambda)$$

because limited by $T=t$

$$= \frac{p_{X_1|T}(x,t)}{p_T(t)}$$

has one-to-one correspondance

$$= \frac{p_{X_1 X_2}(x, t-x)}{p_T(t)}$$

X_1 and X_2 are indep.

$$= \frac{p_{X_1}(x)\, p_{X_2}(t-x)}{p_T(t)}$$

$$= \frac{\left(\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\right)\left(\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{t!}\right)}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} =$$

$$= \binom{t}{x}\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

$$= Bin(t, \tfrac{1}{2})$$

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$$

Let $D = X_1 - X_2 = X_1 + (-X_2) = X_1 + Y$

$\text{Supp}[D] = \mathbb{Z}$

Use $Y = -X_2$

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$$

same as $p_X(-y)$ so

$D \sim p_D(d)$

$$D = X_1 + Y \sim \sum_{x \in \text{Supp}[X]} p_X(x) p_Y(d-x) \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!}\right) \cdot \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{x-d}}{(x-d)!}\right) \mathbb{1}_{d-x \in \{0,-1,-2,\ldots\}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,2,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \geq d}$$

$$= \begin{cases} e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \\ e^{-2\lambda} \sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \geq 0 \end{cases}$$

let $x' = x - d \Rightarrow x = x' + d$

$$= \begin{cases} e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d < 0 \\ e^{-2\lambda} \sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'-d)!x'!} \end{cases}$$

in case "$d \geq 0$" $x' = x - d$ so $x = x' + d$

$$\sum_{x=d}^{\infty} \frac{\lambda^{2x-d}}{x!\,(x-d)!} = \sum_{x'=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{(x'-d)!\,x'!} = \sum_{x'=0}^{\infty} \frac{\lambda^{2x'+d}}{x'!\,(x'-d)!}$$

if $d>0$
$d = |d|$

Use $d' = |d|$ if $d<0$, $d' = -d$

pmf of D is
$$\begin{cases} e^{-2\lambda} \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x+d'}}{x!\,(x+d)!} & \text{if } d<0 \\ e^{-2\lambda} \displaystyle\sum_{x=0}^{\infty} \frac{\lambda^{2x'+d}}{(x'-d)\,x'!} & \text{if } d \geq 0 \end{cases}$$

these look the same so

pmf of D is
$$e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}$$

modified Bessel Function of the 1st kind

$$= e^{-2\lambda}\, I_{|d|}(2\lambda)$$

$$= \text{Skellam}(\lambda, \lambda)$$
(1946)

end of Midterm I stuff

$X_1 \sim Geom(p) := \underbrace{(1-p)^x p \, \mathbb{1}_{x \in \{0,1,2,\ldots\}}}_{p(x)}$

A cdf (cumulative distribution function) is defined

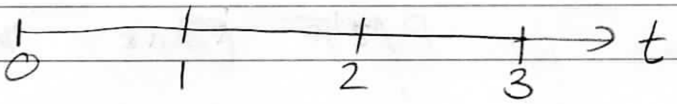
$F(x) = P(X \leq x) = 1 - P(X > x)$

here, for $X_1 \sim Geom(p)$

$F(x) = 1 - P(X > x) = 1 - (1-p)^{x+1}$

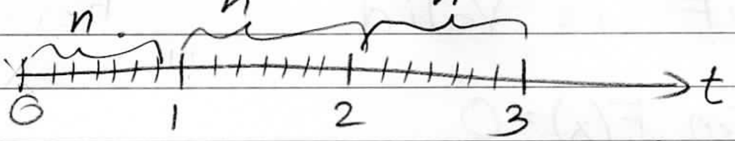
ex:

$F(10) = 1 - P(X \geq 11) = 1 - (1-p)^{12}$



← 1 trial in each period

Now, we run n trials within each period.



X_n is the number of 0's until a successes using the new trial def

$X_n \sim (1-p)^{nx} p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\}}$

$F_{X_n}(x) = \left(1 - (1-p)^{nx+1}\right) \mathbb{1}_{x \geq 0}$

↖ step function?

$$p_{X_\infty}(x) = \lim_{n\to\infty} p_{X_n}(x) =$$

$$= \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda}} \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} \underbrace{\lim_{n\to\infty} \mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\mathbb{1}_{x\in[0,\infty)}}$$

$$= e^{-\lambda}\cdot 0 \cdot 1$$

PMF $p_{X_\infty}(x) = 0$

CDF $F_X(x) = \lim_{n\to\infty} F_{X_n}(x)$

$$= 1 - \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right) \mathbb{1}_{x\geq 0}$$

$$= 1 - e^{-\lambda x}$$

$$\mathrm{Supp}[X_\infty] = [0, \infty)$$

r.v. $|\mathrm{Supp}[X_\infty]| = |\mathbb{R}| \rightarrow X_\infty$ is a continuous r.v.

random variable X_∞ has no PMF (does have a PDF)

but

CDF is Valid

(I) $\lim_{x\to\infty} F(x) = 0$

(II) $\lim_{x\to\infty} F(x) = 1$

(III) $F(x)$ is monotonically increasing

PDF is derivative of CDF

PDF:

$$F'(x) = \lambda e^{-\lambda x} > 0 \checkmark$$

so $X_\infty \sim \exp(\lambda)$ ← X_∞ is an exponential r.v.