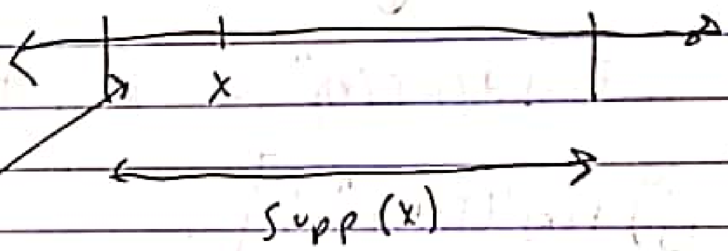
10/28 $X_1, X_2, \dots, X_n \overset{iid}{\sim} f(x)$ with CDF $= F(x)$.

$$\Rightarrow F_{X_{(k)}}(x) = \underbrace{\sum_{j=k}^{n} \underbrace{\binom{n}{j} F(x)^j (1-F(x))^{n-j}}_{\text{pmf Binomial}}}_{\text{comp CDF Binomial}}$$

let $W \sim \text{Binomial}(n, p = F(x))$



As X gets close to min CDF $\to 0$

how many X_i's land $\leq x$

The available space to fit in realizations is shrinking.

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum \binom{n}{j} F(x)^j (1-F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx}\left[F(x)^j (1-F(x))^{n-j}\right]$$

$$F(x)^j(-f(x))(n-j)(1-F(x))^{n-j-1} + (1-F(x))^{n-j} f(x) j F(x)^{j-1}$$

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10/28.

$$= f(x)\left(j\,F(x)^{j-1}(1-F(x))^{n-j} - (n-j)\,F(x)^{j}(1-F(x))^{n-j-1} \right)$$

LHS ⇩

$$f(x)\left(\sum_{j=k}^{n} \frac{n!}{j!(n-j)!}\, j\, F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n} \frac{n!}{j!(n-j)!}(n-j)F(x)^{j}(1-F(x))^{n-j-1} \right)$$

cancel $j/j!$

$n-(j+1)$

when $j=n$ the whole thing goes to zero.

$\sum_{j=k}^{n-1}$ ← only evaluate up to $n-1$.

$$-\sum_{j=k}^{n-1} \frac{n!}{j!(n-(j+1))!} F(x)^{j}(1-F(x))^{n-(j+1)}$$

Let $l=j+1 \Rightarrow j=l-1$

↓

At $j=k \Rightarrow l=k+1$

$$\sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!} F(x)^{l-1}(1-F(x))^{n-l}$$

↑

Looks exactly like LHS. everything cancels ~~exce~~ except when $j=k$.

So we end up with. ⇒

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$$\boxed{= \frac{n!}{(k-1)!\,(n-k)!} f(x)\, F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x)}$$

want formulas for min and max.

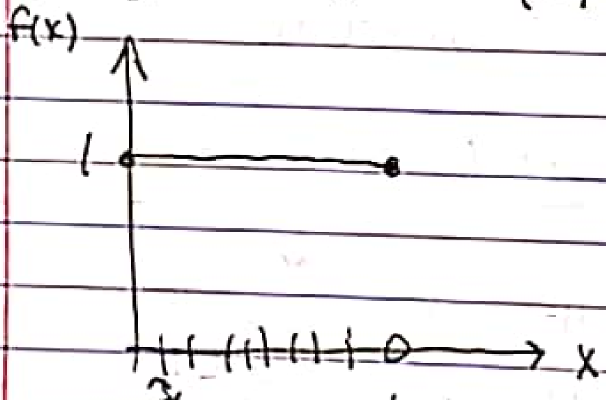
min:

$\longrightarrow f_{X_{(1)}} x = n f(x) (1-F(x))^{n-1}$

max:

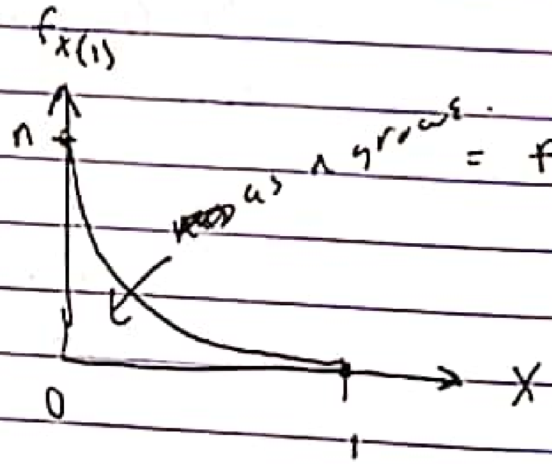
$\longrightarrow f_{X_{(n)}} x = n f(x)\, F(x)^{n-1}$

$X_1, X_2, \ldots X_n \overset{iid}{\sim} U(0,1) = 1, \quad F(x) = x$



realizations

$X_{(1)}$ = minimum.



order statistics have the same support.

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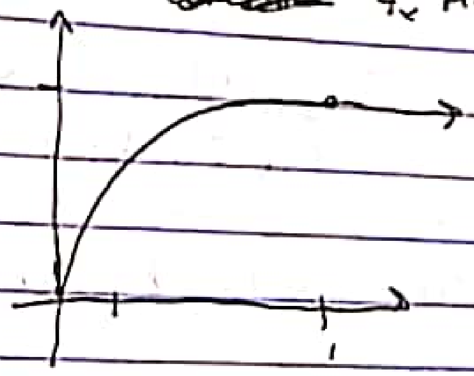
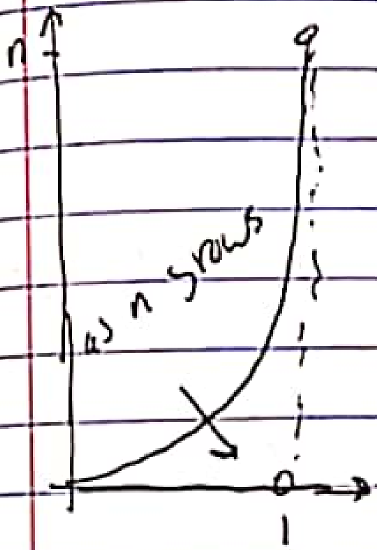
$$f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

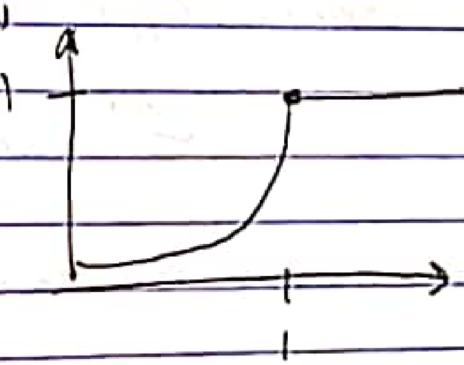
$$F_{X_{(1)}}(x) = 1-(1-x)^{n-1}$$

f_x min

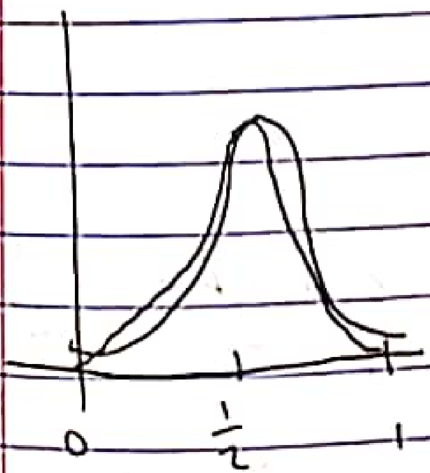


f_x max

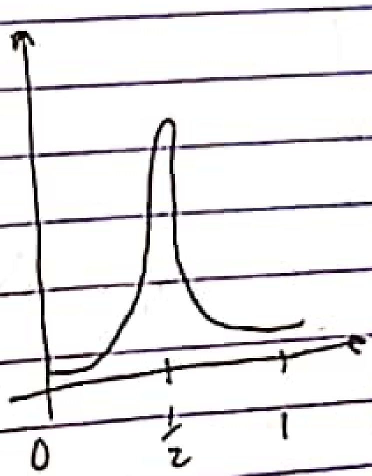




Say $X \approx \frac{1}{2}n$



As n grows larger →



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$$f_{Y_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \overset{\because}{f(x)} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]}$$

$$\boxed{Y_{(k)} \sim = \beta eta(k, n-k+1).}$$

order statistic

let $X \sim$ Gamma(α_1, β) independent of $Y \sim$ gamma(α_2, β)

$$T = X + Y \overset{\text{conjecture}}{\sim} \text{Gamma}(\alpha_1 + \alpha_2, \beta).$$

we will prove this now....

$$t \sim \int_{\text{Supp}(X)} f_X(x) f_Y(t-x) \mathbb{1}_{t-x \in \text{Supp}(Y)} dx.$$

$$= \int_0^\infty \left(\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \cancel{e^{-\beta x}}\right) \frac{\beta^{\alpha_2}}{\Gamma \alpha_2} (t-x)^{\alpha_2 - 1} \underbrace{e^{\beta(t-x)}}_{e^{\beta t} \cancel{e^{-\beta x}}} \underbrace{\mathbb{1}_{t-x \in [0,\infty)}}_{x \le t} dx$$

indicator into the limit

10/28

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} \, dx$$

let $u = \frac{x}{t} \Rightarrow x = ut \Rightarrow \cancel{x} = 0 \Rightarrow \cancel{x}\, u = 0$,

$\cancel{x} = t \Rightarrow u = 1$, $\frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$

$$\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{\beta t} \int_0^1 (ut)^{\alpha_1-1} (t-ut)^{\alpha_2-1} \, t\,du$$

can factor out t now.

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left(\int_0^1 u^{\alpha_1-1} (1-u)^{\alpha_2-1} \, du \right) t^{\alpha_1+\alpha_2-1} e^{\beta t}$$

we can decompose $p(x) = c\,k(x)$ or $\pi(x) = c\,k(x)$ where c is not a function of x (normalized constant) and k is a function of x (kernel)

$\propto k(x)$ $\qquad \propto k(x)$

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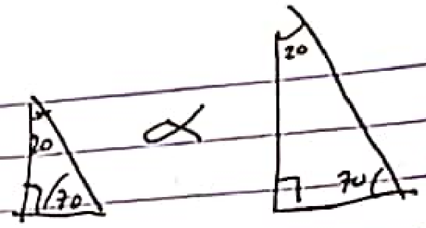

Ex. $\sum p(x) = 1 \Rightarrow \sum c\,k(x) = 1 \Rightarrow c = \frac{1}{\sum k(x)}$

$$\int f(x)\,dx = 1 \Rightarrow \int c\,f(x)\,dx = 1 \Rightarrow c = \frac{1}{\int f(x)\,dx}$$

k(x) specifies p(x) or f(x). The kernel tells you what random variable you have.

Example.

$$X \sim Bin(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x}_{k(x) = \text{kernel}}$$

(water for space field.)

$$X \sim Weibull(k, \lambda) = (k\lambda)(\lambda x)^{k-1} e^{-(\lambda x)^k}$$

$$= \cancel{(k\lambda)}\ \underbrace{k\lambda^k}_{c}\ \underbrace{x^{k-1} e^{-(\lambda x)^k}}_{k(x)}$$

10/28 $X \sim \text{Gamma}(\alpha, \beta) := \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{\beta x}}_{f(x)} \propto x^{\alpha-1} e^{\beta x} \propto$

$\uparrow$ Gamma (α, β)

Back to the original problem.

$$\underbrace{\left(\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du \right)}_{c} \underbrace{t^{(\alpha_1+\alpha_2)-1} e^{\beta t}}_{= f(x) \text{ of gamma}}$$

$\propto$ Gamma $(\alpha_1+\alpha_2, \beta)$

$$\Rightarrow \frac{\cancel{\beta^{\alpha_1+\alpha_2}}^{\,1}}{\Gamma(\alpha_1+\alpha_2)} = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}^{\,1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du \Rightarrow$$

$$\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} =: B(\alpha_1, \alpha_2)$$ Beta function.

$$\int_0^\infty x^{\alpha_2-1} e^{-x}\, dx$$

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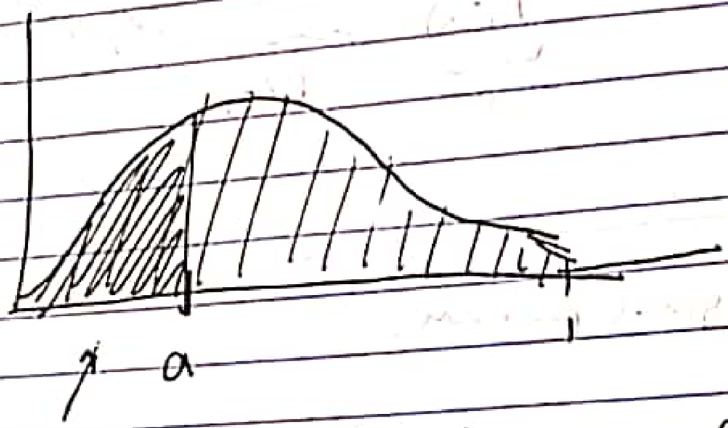

$$B(a, \alpha_1, \alpha_2) := \int_0^a u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\,du$$

↳ incomplete Beta function.

I may be interested in the proportion of the shaded area vs. the whole.

$$I_a(\alpha_1, \alpha_2) := \frac{B(a, \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)}$$

Regularized incomplete Beta function

$$X \sim \text{Beta}(\alpha, \beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x \in [0,1]}$$

$$\frac{1}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

parameter space $\alpha, \beta > 0$

if $\alpha, \beta \in \mathbb{N}$ then $\Rightarrow$

$$= \frac{(\alpha+\beta+1)!}{(\alpha-1)!\,(\beta-1)!}$$

10/28 CDF:

$$F(x) = \int_0^x \frac{1}{\beta(\alpha,\beta)} t^{\alpha-1} (1-t)^{\beta-1} \, dt$$

$$= \frac{\int_0^x t^{\alpha-1} (1-t)^{\beta-1} \, dt}{\beta(\alpha,\beta)} = \frac{\beta(x,\alpha,\beta)}{\beta(\alpha,\beta)}$$

$$= I_x(\alpha,\beta).$$