· ~~show~~ Let $X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \ge 0}$ ★ PDF of Gamma

show $Y = cX \sim \text{Gamma}(\alpha, \frac{\beta}{c})$.

$$= \frac{(\frac{\beta}{c})^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\frac{\beta}{c} y)} \mathbb{1}_{y \ge 0}$$

$g(x) = cX$

$g^{-1}(y) = \frac{y}{c}$

$\frac{d}{dy} = \frac{1}{c}$

$f_X(g^{-1}(y)) \frac{1}{c} =$

$$f_Y(y) = f_X\left(\frac{y}{c}\right) \cdot \frac{1}{c}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta(\frac{y}{c})} \mathbb{1}_{\frac{y}{c} > 0} \cdot \frac{1}{c}$$

$$= \frac{1}{c^{\alpha-1}} \cdot \frac{1}{c} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta}{c} y} \mathbb{1}_{y \ge 0}$$

$$= \frac{(\frac{\beta}{c})^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta}{c} y} \mathbb{1}_{y \ge 0}$$

· **CLT** $X_1 \cdots X_n \overset{iid}{\sim}$ with μ, σ^2

$$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$X = \sigma Z + \mu \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$\sigma > 0$

$Z_i^2 \sim \chi_1^2 := \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$Z_1 \cdots Z_n \overset{iid}{\sim} N(0,1)$

$$\underbrace{Z_1^2 + \cdots + Z_m^2}_{\chi_m^2} + \underbrace{Z_{m+1}^2 + \cdots + Z_n^2}_{\chi_{n-m}^2} \sim \chi_n^2 = \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

· $X \sim \chi_k^2$

$$\Rightarrow \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$$

$$\frac{k}{2} = \frac{\frac{1}{2}}{\frac{1}{k}}$$

$X_1 \sim \chi_{k_1}$, indep. $X_2 \sim \chi_{k_2}$

$R = \frac{X_1/k_1}{X_2/k_2} \sim f_R(r) = ?$ $\Rightarrow$ So it becomes $\frac{U}{V} \sim \int_{supp[U]} f_U(rt) f_V(t) \mathbb{1}_{t \in supp[V]} |t| dt$

Let $U = \frac{X_1}{k_1} \sim \text{Gamma}(\underset{\text{call } a}{\frac{k_1}{2}}, \underset{\text{call } b}{\frac{k_1}{2}})$

Let $V = \frac{X_2}{k_2} \sim \text{Gamma}(\frac{k_2}{2}, \frac{k_2}{2})$

$$= \int_0^\infty \left(\frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\right)\left(\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\right)$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \underbrace{\int_0^\infty t^{(a+b)-1} e^{-(b+ar)t} dt}_{\frac{\Gamma(a+b)}{(b+ar)^{a+b}}}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} \underbrace{(b+ar)^{-(a+b)}}_{(b(1+\frac{a}{b}r))^{-(a+b)}}$$

$$= \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \frac{(1+\frac{a}{b}r)^{-(a+b)}}{b^a \cancel{b^b}}$$

$$= \frac{(\frac{a}{b})^a}{B(a,b)} r^{a-1} (1+\frac{a}{b}r)^{-(a+b)}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B(\frac{k_1}{2}, \frac{k_2}{2})} r^{\frac{k_1}{2}-1} (1+\frac{k_1}{k_2}r)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r \geq 0}$$

Name "Fisher-Snedecor..." $= F_{k_1,k_2}$ F-distribution with k_1 & k_2 degree of freedom

Note: k_1 & $k_2 \in \mathbb{N}$

- $Z \sim N(0,1)$ ind. of $X \sim \chi_k^2$

$W = \frac{Z}{\sqrt{X/k}} \sim f_W(w) = ?$ → symmetric

$W^2 = \frac{Z^2}{X/k} = \frac{Z^2/1}{X/k} \sim F_{1,k}$ check

Note: $f_W(w) = f_W(-w)$

$F_{W^2}(w^2) = P(W^2 = w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$

take d/dw both sides

$\Rightarrow 2w f_{W^2}(w^2) = f_W(w) - f_W(-w)$ chain rule

$= 2 f_W(w)$

$\Rightarrow f_W(w) = w f_{W^2}(w^2)$

Note that $k_1 = 1$ & $k_2 = k$.

Now, plug this in the formula that we just derived:

$$= w \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} (w^2)^{\frac{1}{2}-1} \left(1 + \frac{1}{k} w\right)^{-\left(\frac{1+k}{2}\right)}$$

Note that $\frac{1}{B\left(\frac{1}{2}, \frac{k}{2}\right)} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} \Gamma\left(\frac{k}{2}\right)}$, $(w^2)^{\frac{1}{2}} = w^{2-\frac{1}{2}} = w^{-1}$

So, $\cancel{w \cdot w^{-1}} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\, \Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$ student's T distribution with k degree of freedom

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- Z_1 & $Z_2 \overset{iid}{\sim} N(0,1)$

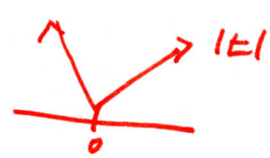
$R = \frac{Z_1}{Z_2} \sim \text{Cauchy}(0,1) = \frac{1}{\pi} \cdot \frac{1}{r^2+1}$

$Y = c + \sigma R$ where $\sigma > 0$

$= \frac{1}{\sigma\pi} \cdot \frac{1}{\left(\frac{r-c}{\sigma}\right)^2 + 1} = \text{Cauchy}(\sigma, \sigma)$

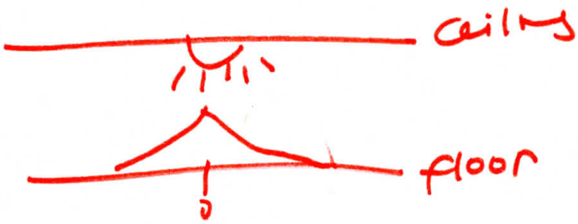
- Note: $T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{2}\right)} \left(1 + \frac{w^2}{1}\right)^{-\frac{1+1}{2}} = \frac{1}{\pi}(1+w^2)^{-1} = \frac{1}{\pi} \cdot \frac{1}{1+w^2} = \text{Cauchy}(0,1)$

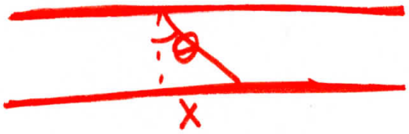
$\phi_R(t) = E[e^{itR}] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{itr}}{r^2+1}\, dr = \underbrace{\ldots}_{\text{complex analysis}} = e^{-|t|}$



$\phi_R'(t) = \begin{cases} -e^{-t} & \text{if } t > 0 \\ e^{-t} & \text{if } t < 0 \\ \text{undefined} & t = 0 \end{cases}$

What's the dist. of light brightness of the floor?





$\theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$

$$f_X(x) = f_\theta(g^{-1}(x)) \frac{d}{dx}[g^{-1}(x)]$$

$$g(\theta) = \tan(\theta) = \frac{x}{1} = x \Rightarrow \theta = \tan^{-1}(x) = g^{-1}(x)$$

$$\text{so, } f_X(x) = \frac{1}{\pi} \mathbb{1}_{\underbrace{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}_{x \in [\tan(-\frac{\pi}{2}), \tan(\frac{\pi}{2})]}} \cdot \frac{1}{1+x^2} = \frac{1}{\pi} \cdot \frac{1}{1+x^2} = \text{Cauchy}(0,1)$$

- Application: of Z, T, χ^2, F to statistics:

Let $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$T = X_1 + \dots + X_n \sim N(n\mu, n\sigma^2)$

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

the estimator for μ

$$S_n^2 = \frac{1}{n-1} \Sigma (X_i - \bar{X}_n)^2$$

$\bar{X} = \frac{1}{n} \Sigma X_i$

estimate for μ

$$S^2 = \frac{1}{n-1} \Sigma (X_i - \bar{X})^2$$

estimate for σ

estimator for σ^2

- Want to know ① $S_n^2 \sim ?$

② Relationship b/w $\cancel{\bar{X}_n}$ & S_n^2

• $c_1 \ldots c_n \sim N(0,1)$

$\vec{z}^T \vec{z} = \sum z_i^2 \sim \chi_n^2$ where $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

Note: $z_i = \frac{x_i - \mu}{\sigma}$

$$\Rightarrow \sum \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$$

$$= \frac{1}{\sigma^2} \sum (x_i - \mu)^2$$

Note: $x_i - \mu = x_i - \bar{x} + \bar{x} - \mu$

$$(x_i - \mu)^2 = ((x_i - \bar{x}) + (\bar{x} - \mu))^2$$

$$= (x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - \mu) + (\bar{x} - \mu)^2$$

$$= \frac{1}{\sigma^2}\left(\sum (x_i - \bar{x})^2 + \underbrace{2(x_i - \bar{x})(\bar{x} - \mu)}_{\sum \underbrace{x_i}_{n\bar{x}} \bar{x} - \bar{x}^2 - x_i \mu + \bar{x}\mu} + (\bar{x} - \mu)^2\right)$$

$$= n\bar{x}^2 - n\bar{x}^2 - \mu n \bar{x} + n \bar{x} \mu$$

$$= 0$$

$$= \underbrace{\frac{\sum (x_i - \bar{x})^2}{\sigma^2}}_{\frac{(n-1)s^2}{\sigma^2}} + \underbrace{\frac{n(\bar{x} - \mu)^2}{\sigma^2}}_{\underbrace{\left(\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma}\right)^2}_{\underbrace{\left(\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{z^2 \sim \chi_1^2}}} \sim \chi_n^2$$

Recall $\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

b/c $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

• Conjecture: $\frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$