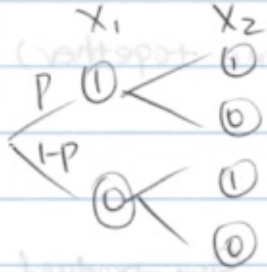
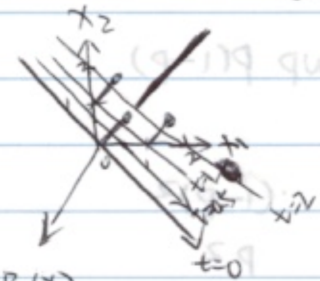
Lecture 2.



X_1	X_2	$P_{X_1,X_2}(x_1,x_2)$	T	
1	1	p^2	2	mutually exclusive
1	0	$p(1-p)$	1	
0	1	$(1-p)p$	1	
0	0	$(1-p)^2$	0	

$$P_T(t) = P = (T=t) = \begin{cases} 2 & \text{wp } p^2 \\ 1 & \text{wp } 2p(1-p) \\ 0 & \text{wp } (1-p)^2 \end{cases}$$

Slope = -1



$t = X_1 + X_2 \Rightarrow x_2 = t - x_1$

$P(T=0)$

$x_2 = 0 - x_1 = -x_1$

PMF: $P_X(x)$

$$P_T(t) = \underbrace{\sum_{x \in \mathbb{R}} \sum_{x \in \mathbb{R}}}_{\text{Search thru } \mathbb{R}^2} \underbrace{P_{X_1,X_2}^{(x_1,x_2)}}_{\text{Add up all probability}} \underbrace{\mathbb{1}_{x_1+x_2=t}}_{\text{select event}} = \left\{ \begin{array}{l} x_2 = t - x_1 \\ \sum_{x_2 \in \mathbb{R}} \end{array} \right. = \sum_{x \in \mathbb{R}} P_{X_1,X_2}^{(x,t-x)}$$

General convolution Formula

① IF X_1 X_2 Independent Formula

$$= \sum_{x \in \mathbb{R}} P_{X_1}^{(x)} P_{X_2}^{(t-x)} = \underline{\sum_{x \in \mathbb{R}}} P_{X_1}^{old}(x) \underline{\mathbb{1}_{x \in Supp[X_1]}} P_{X_2}^{old}(t-x) \underline{\mathbb{1}_{t-x \in Supp[X_2]}}$$

$$= \sum_{x \in Supp[X_1]} P^{old}_{X_1}(x) P^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$

convolution Formula For Independent rvs

② IF X_1, X_2 iid Formula

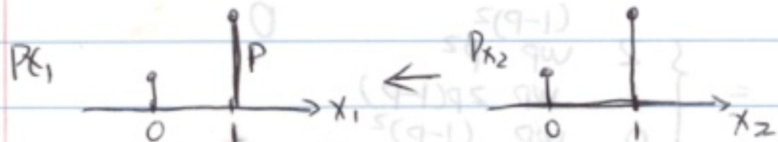
$$= \sum_{x \in \mathbb{R}} P^{(x)} P^{(t-x)} = \sum_{x \in \mathbb{R}} P^{old}(x) \mathbb{1}_{x \in Supp[X]} P^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in Supp[X]} P^{(x)} P^{(t-x)} \mathbb{1}_{(t-x \in Supp[X])}$$

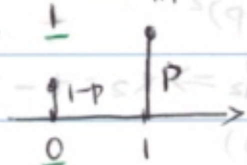
convolution Formula for iid rvs

("Convolve" means to "roll, coil or entwine together")

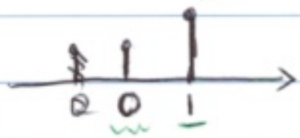
$$p_T^{(t)} = p_{X_1}^{(x)} \overset{\text{Convolution}}{*} p_{X_2}$$



Sum-product

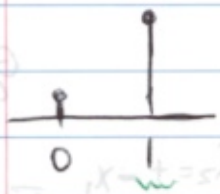


$\Rightarrow t=1$ wp $p(1-p)$



$\Rightarrow t=0$ wp $(1-p)^2$

$t=2$ wp p^2



$\Rightarrow t=1$ wp $(1-p)p$

$$p_{T_2}^{(t)} = \sum_{x\in\{0,1\}} p^x(1-p)^{1-x} \; p^{t-x}(1-p)^{1-t+x} \, \mathbb{1}_{\underbrace{t-x\in\{0,1\}}_{t\in\{x,\,x+1\}}}$$

$$= p^t(1-p)^{2-t} \sum_{x\in\{0,1\}} \mathbb{1}_{t\in\{x,x+1\}}$$

$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}}\right) \quad \binom{2}{t}$$

Let $t = T_2$

$$T_2 \overset{\text{equal}}{\sim} \begin{cases} 0 & \text{wp } (1-p)^2 \\ 1 & \text{wp } 2p(1-p) \\ 2 & \text{wp } p^2 \end{cases}$$

Review 241 $\left| \; \binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n\in\mathbb{N}} \mathbb{1}_{k\in\{0,1,\dots,n\}} \right.$

$$\Downarrow$$

$$= \binom{2}{t} p^t (1-p)^{2-t} = \text{Binom}(2, p)$$

$\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2]$

$A+B := \{a+b;\ a\in A,\ b\in B\}$

$\binom{1}{x} = \mathbb{1}_{x\in\{0,1\}}$

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) = p^x(1-p)^{1-x}\mathbb{1}_{x\in\{0,1\}}$

$$= \binom{1}{x} p^x (1-p)^{1-x}$$

$$P_{T_2}(t) = \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x\in\mathbb{R}} \underbrace{\binom{1}{x}}_{\mathbb{1}_{x\in\{0,1\}}} \binom{1}{t-x}$$

$$= p^t(1-p)^{2-t}\sum \binom{1}{t-x} = p^t(1-p)^{2-t}\left(\binom{1}{t} + \binom{1}{t-1}\right)$$

Pascals Identity

$$\star \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{use this}$$

$$\Downarrow$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p) \qquad T_3 = X_1 + X_2 + X_3$

$= X_3 + T_2 \sim P_{T_3}(t) = ?$

① Indepedent Formula

$$P_{T_3}(t) = \sum_{x\in\{0,1\}} \overbrace{p^x(1-p)^{1-x}}^{\text{PMF}} \binom{2}{t-x} p^{t-x}(1-p)^{2-t+x}$$

$$= p^t(1-p)^{3-t} \sum_{x\in\{0,1\}} \binom{2}{t-x} = p^t(1-p)^{3-t}\left(\binom{2}{t} + \binom{2}{t-1}\right)$$

use ★ $= \binom{3}{t} p^t (1-p)^{3-t} = \text{Binom}(3, p)$

HW: Find PMF of Binom (n, p) via induction

$$X_1, X_2 \overset{iid}{\sim} \text{Binomial}(n, p) \quad T = X_1 + X_2 \sim ?$$

iid Formula $P_T(t) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$

$$= p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t-x}$$

Vandermonde's Identity

$$= \binom{2n}{t} p^t (1-p)^{2n-t} = \text{Binom}(2n, p)$$