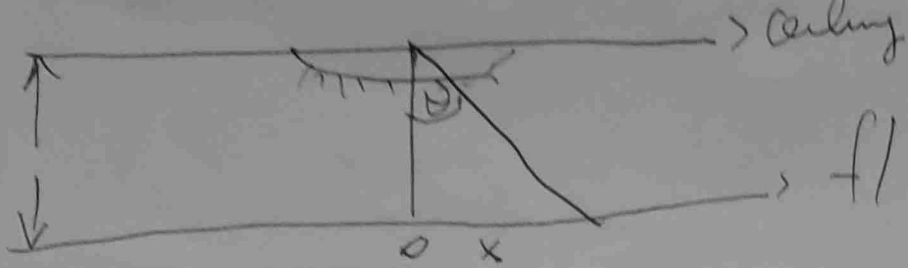
Wed November 18th 2020

$$M_X(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\pi} \frac{1}{1+x^2} dx = \infty \text{ mgf dnem } t \in$$

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{1}{\pi} \frac{1}{x^2+1} dx = \dots = \phi_X'(t) = -\frac{t}{|t|} e$$



$$\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}, \quad x = \tan(\theta) \to \theta = \arctan(x)$$

$$\int \frac{d}{dx}[\arctan(x)] = \frac{1}{x^2+1}$$

$$f_X(x) = f_\theta(\arctan(x)) \frac{1}{x^2+1} = \frac{1}{\pi} \overbrace{\mathbb{1}_{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}}^{x \in \mathbb{R}} \frac{1}{x^2+1} = \text{Cauchy}(0,1)$$

Let $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$T_n \sim N(n\mu, n\sigma^2)$

$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ "sample mean" or "average"

$$S_n^2 = \frac{1}{n-1} \Sigma (x_i - \bar{x})^2 \sim f_{S_n^2}(s) = ? \quad \text{"Sample Variance".}$$

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$ $\Sigma Z_i^2 \sim \chi_n^2$ $Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $Z^T Z \sim \chi_n^2$

$Z_1 = \frac{X_1-\mu}{\sigma}, \dots Z_n = \frac{X_n-\mu}{\sigma} \Rightarrow \Sigma \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$ ②

$$(X_i-\mu)^2 = ((X_i-\bar{X})+(\bar{X}-\mu))^2 = (X_i-\bar{X})^2 + 2\underbrace{(X_i-\bar{X})(\bar{X}-\mu)}_{X_i\bar{X}-\bar{X}^2-X_i\mu+\bar{X}\mu} + (\bar{X}-\mu)^2$$

$$\Sigma(X_i-\mu)^2 = \Sigma(X_i-\bar{X})^2 + 2(\bar{X}\underbrace{\Sigma X_i}_{n\bar{X}} - n\bar{X}^2 - \mu\Sigma X_i + n\bar{X}\mu) + n(\bar{X}-\mu)^2$$

$$\Sigma\left(\frac{X_i-\mu}{\sigma^2}\right)^2 = \frac{n-1}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_n$$

this would be true if $\bar{X}$ is in depend ant of S^2

"quadratic form"

$\vec{Z}^T\vec{Z} = \vec{Z}^T I \vec{Z} \sim \chi^2_n$

Consider: $\vec{Z}^T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \vec{Z} = Z_1^2 \sim \chi^2_1$ rank$[B_i] = 1$

$\vec{Z}^T \begin{bmatrix} 0 & \cdots & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix} \vec{Z} = Z_i^2 \sim \chi^2_1$

$\vec{Z}^T I \vec{Z} = \vec{Z}^T(B_1+B_2+\dots+B_n)\vec{Z} = \vec{Z}^T B_1 \vec{Z} + \dots + \vec{Z}^T B_n \vec{Z} \sim \chi^2_n$

Cochran's thm: If $B_1+B_2+\dots+B_k = I$ s.t $\sum_{j=1}^{k} \text{rank}[B_j] = n$

then (a) $\vec{Z}^T B_j \vec{Z} \sim \chi^2_{\text{rank}[B_j]}$ and (b) $\vec{Z}^T B_{j_1} \vec{Z}$ is independent

of $\vec{Z}^T B_{j_2} \vec{Z}$ $\forall j_1 \neq j_2$

$\vec{Z}^T\vec{Z} = \Sigma Z_i^2 = \Sigma((Z_i-\bar{Z})+\bar{Z})^2 = \Sigma(Z_i-\bar{Z})^2 + 2\Sigma(Z_i-\bar{Z})\bar{Z} + n\bar{Z}^2 =$

$\Sigma(Z_i-\bar{Z})^2 + n\bar{Z}^2$

let $\vec{1}_n$ is a column vector of all ones $\Rightarrow \bar{z} = \frac{1}{n}\vec{z}^T\vec{1}$ (3)

$n\bar{z}^2 = n\left(\frac{1}{n}\vec{z}^T\vec{1}\right)^2 = n\frac{1}{n}\vec{z}^T\vec{1}\,\frac{1}{n}\vec{z}^T\vec{1} = \vec{z}^T\left(\frac{1}{n}\vec{1}\vec{1}^T\right)\vec{z}$ rank$[B_2]=1$

let $J_n = \vec{1}_n\vec{1}_n^T$ which is an $n\times n$ matrix of all entries $=1$

$\sum(z_i-\bar{z})^2 = \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2 = \sum z_i^2 - n\bar{z}^2$

$= \vec{z}^T I\vec{z} - \vec{z}^T\left(\frac{1}{n}J_n\right)\vec{z} = \vec{z}^T\left(I - \frac{1}{n}J\right)\vec{z}$

$B_1 + B_2 = \left(I - \frac{1}{n}J\right) + \frac{1}{n}J = I$

Thm from Math 231: if A is symmetric matrix and idempotent which mean $AA = A$ then rank$[A] = tr[A] =$ sum of the diagonal of A.

$B_1^T = \left(I - \frac{1}{n}J\right)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J = B_1$ ✓

$B_1B_1 = \left(I - \frac{1}{n}J\right)\left(I - \frac{1}{n}J\right) = II - \frac{1}{n}JI - \frac{1}{n}IJ + \frac{1}{n^2}JJ$.

$= I - \frac{2}{n}J + \frac{1}{n} = I - \frac{1}{n}J = B_1 \Rightarrow$ rank$[B_1] =$

$= \sum_{i=1}^{n} 1 - \frac{1}{n} = n-1$

$B_{1\,i,i} = 1 - \frac{1}{n}1 = 1 - \frac{1}{n}$

Puting it all together, we can use Cochran's theorem

$\sum(z_i-\bar{z})^2 \sim \chi^2_{n-1}$ indep of $n\bar{z}^2 \sim \chi^2_1$

$\bar{z} = \frac{z_1 + \dots + z_n}{n} = \frac{\frac{x_1-\mu}{\sigma} + \dots + \frac{x_n-\mu}{\sigma}}{n} = \frac{\sum x_i - n\mu}{n\sigma} = \frac{\bar{x}-\mu}{\sigma}$

$\Rightarrow n\bar{z}^2 = \left(\sqrt{n}\,\bar{z}\right)^2 = \left(\frac{\bar{z}}{\frac{1}{\sqrt{n}}}\right)^2 = \left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_1$

$$\Sigma (z_i - \bar{z})^2 = \Sigma \left(\frac{x_i - \mu}{\sigma} - \frac{\bar{x} - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \Sigma (x_i - \bar{x})^2 =$$

(4)

$$= \frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$$

First to prove this was fisher in 1925 and then in 1936, Geary proved the iid normal rv is the "only" distribution that has the independent of $\bar{X}$ and S^2

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$ but what about $$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim ?$$

Not $N(0,1)$