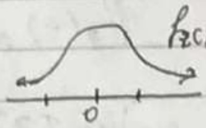
Monday November 16th 2020

Lecture 18



$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$Z \sim N(0,1)$, $Y = Z^2 \sim f_Y(y) = ?$ Not 1:1

$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2\, P(Z \in [0, \sqrt{y}])$

$= 2(F_Z(\sqrt{y}) - \cancel{F_Z(0)}^{\frac{1}{2}}) = 2 \underset{\substack{\uparrow \\ \text{cdf of } Y}}{F_Z} \sqrt{z} - 1$

$f_Y(y) = \frac{d}{dy}[2 F_Z(\sqrt{y}) - 1] = \cancel{2}(\frac{1}{\cancel{2}} y^{-\frac{1}{2}}) f_Z(\sqrt{y}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underset{y \ge 0}{\sqrt{y} \in \mathbb{R}}}$

$\propto y^{\overset{\frac{1}{2}-1}{-\frac{1}{2}}} e^{-\frac{1}{2} y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$Z_1, Z_2, \dots, Z_k \overset{iid}{\sim} N(0,1)$, $Y = \overset{\sim G(\frac12,\frac12)}{Z_1^2} + \overset{\sim(\frac12,\frac12)}{Z_2^2} + \dots + \overset{\sim(\frac12,\frac12)}{Z_k^2} \sim \text{Gamma}(\frac{k}{2}, \frac{1}{2})$

Note the beta is always 1/2 and the alpha is always k/2 so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}(\frac{k}{2}, \frac{1}{2}) = \chi^2_k$ the "Chi Squared distribution with k degrees of freedom" $k \in \mathbb{N}$

$E[Y] = k \underset{\substack{\| \\ 1}}{E[Z^2]} = k$

$\chi^2_k = \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} \overset{k=1,\ \Gamma(\frac12)=\sqrt{\pi}}{=} \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} = \chi^2_1$

$X \sim \chi^2_k$, $Y = \sqrt{X} \Rightarrow X = Y^2 = g^{-1}(y)$. $\left|\frac{d}{dy}[g^{-1}(y)]\right| = |2y| = 2y$

$$f_Y(y) = f_X(y^2)\, 2y = \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{k-2} e^{-y^2/2} (2y)\, \mathbb{1}_{\overset{y \geq 0}{y^2 \geq 0}}$$

$$= \frac{(\frac{1}{2})^{k/2-1}}{\Gamma(\frac{k}{2})} y^{k-1} e^{-y^2/2}\, \mathbb{1}_{y \geq 0} = \chi_k$$ the chi distribution with k degrees of freedom.

$Z \sim N(0,1)$, $|Z| = \sqrt{Z^2} = \sqrt{X_1^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\, \mathbb{1}_{y \geq 0}$

$$= 2\Big(\underbrace{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}_{f_Z}\Big)\, \mathbb{1}_{y \geq 0}$$

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{y}{c}\right)^{\alpha-1}}_{\frac{y^{\alpha-1}}{c^{\alpha-1}}} e^{-\overset{(\beta/c)y}{\beta \frac{y}{c}}}\, \mathbb{1}_{\overset{y>0}{\frac{y}{c}>0}}$$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\beta/c)y}\, \mathbb{1}_{y \geq 0} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$X = \chi^2_k$, $Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{\frac{1}{2}}{\frac{1}{2}}\right) = \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

Let $X_1 \sim \chi^2_{k_1}$ indep of $X_2 \sim \chi^2_{k_2}$

Let $U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\overset{a}{\frac{k_1}{2}}, \overset{a}{\frac{k_1}{2}}\right)$ ind of $V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\overset{b}{\frac{k_2}{2}}, \overset{b}{\frac{k_2}{2}}\right)$

$$R = \frac{U}{V} \sim f_R(r) = \int_{\text{Supp}[V]} f_U(rt)\, \mathbb{1}_{rt \in \text{Supp}[U]}\, f_V(t)\, |t|\, dt$$

$$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\, \mathbb{1}_{\overset{r \in (0,\infty)}{rt \in [0,\infty)}}\, \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\, t\, dt$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)} \left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

Student's T distribution with k degrees of freedom discovered in 1908 by William Gosset while he was working at a beer factory.

If $k \to \infty$ $T_k \to Z$

Student's T distribution has $N(0,1)$ shape but just thicker tails.

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\, du = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} (-|u|)\, du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du \right)$$

$$= \frac{1}{2\pi}\left(\int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} u\, du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u} dt$, $u=0 \Rightarrow t=0$, $u \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{r^2+1}{2}t} \cancel{u} \frac{1}{\cancel{2u}} dt = \frac{1}{\cancel{2}\pi} \frac{1}{\frac{r^2+1}{\cancel{2}}} \int_0^{\infty} \frac{r^2+1}{2} e^{-\frac{r^2+1}{2}t} dt$$

$\int_0^{\infty} \lambda e^{-\lambda x} dx = 1$

PDF exponent r.v

$$= \frac{1}{\pi} \frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

---

let $X = C + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(C, \sigma) = \frac{1}{\sigma\pi} \frac{1}{1+\left(\frac{x-C}{\sigma}\right)^2}$$

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r\geq 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} \, dt$$

$$= a^a b^b r^{a-1} \mathbb{1}_{r\geq 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)} \Gamma(a+b)}_{\frac{1}{B(a,b)}} \cdot \frac{1}{(ar+b)^{a+b}} = \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\frac{b^{-(a+b)}}{\cancel{b^a b^{-b}}}(1+\frac{a}{b}r)^{-(a+b)}} \mathbb{1}_{r\geq 0}$$

$$= \frac{(a/b)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)} \mathbb{1}_{r>0} = \frac{(k_1/k_2)^{k_1/2}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1+\frac{k_1}{k_2}r\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r>0}$$

$= F_{k_1,k_2}$ thus is the "F distribution" or "Fisher-Snedecor Distribution" with k_1 numerator degrees of freedom and k_2 denominator degree of freedom. $k_1, k_2 \in \mathbb{N}$

Let $Z \sim N(0,1)$, $X \sim \chi^2_k$, $W = \frac{Z}{\sqrt{X/k}} \sim f_W(w) = ?$

symmetric around 0

Consider $W^2 = \frac{Z^2/1}{X/k} \sim F_{1,k}$

$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$

Take derivatives...

$\frac{d}{dw}[F_{W^2}(w^2)] = \frac{d}{dw}[F_W(w) - \frac{d}{dw}[F_W(-w)]$

$\cancel{2}w f_{W^2}(w^2) = f_W(w) - (-f_W(-w)) = \cancel{2} f_W(w)$

$$f_W(w) = \cancel{w} \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{\underbrace{B\left(\frac{1}{2}, \frac{k}{2}\right)}_{\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2})}}} \underbrace{(w^2)^{\frac{1}{2}-1}}_{\cancel{w^{-1}}} \left(1+\frac{w^2}{k}\right)^{-(1+k)/2} \mathbb{1}_{\underbrace{w^2 \geq 0}_{w \in \mathbb{R}}}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)} \left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

Student's T distribution with k degrees of freedom discovered in 1908 by William Gosset while he was working at a beer factory.

If $k \to \infty$ $T_k \to Z$

Student's T distribution has $N(0,1)$ shape but just thicker tails.

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\, du = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} (-|u|)\, du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du \right)$$

$$= \frac{1}{2\pi}\left(\int_0^{-\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} u\, du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u} dt$, $u = 0 \Rightarrow t = 0$, $u \to 0 \Rightarrow t \to \infty$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{r^2+1}{2}t} \cancel{u} \frac{1}{\cancel{2u}} dt = \frac{1}{\cancel{2\pi}} \frac{1}{\frac{r^2+1}{\cancel{2}}} \underbrace{\int_0^{\infty} \frac{r^2+1}{2} e^{-\frac{r^2+1}{2}t} dt}_{}$$

$\int_0^1 \lambda e^{-\lambda x} dx = 1$
PDF exponent r.v

$$= \frac{1}{\pi} \frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

---

let $X = C + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(C, \sigma) = \frac{1}{\sigma\pi} \frac{1}{1+\left(\frac{x-c}{\sigma}\right)^2}$$