Lecture #5.

$X \sim \text{Multin}_2\ (n, \vec{p})$

$\quad\quad\ \ \hat{}\ k=2$

Deg $(n-x_2) = P_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \dfrac{P(x_1, x_2)}{P(x_2)}$

Last time $P(X_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

$$\to = \frac{\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2}} = \frac{\frac{n!}{x_1!\,x_2!} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} p_2^{x_2}}{\frac{n!}{x_2!\,(n-x_2)!} \mathbb{1}_{x_2 \in J_n}\ p_2^{x_2}\ p_1^{n-x_2}}$$

$$\left[\begin{array}{l} \text{Define } J_n := \{0, 1, \dots, n\} \\ \mathbb{1}_A^u = \dfrac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \text{undefined} & \text{if } A' \end{cases} \end{array}\right.$$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{=1 \text{ if } x_1 = n-x_2} \mathbb{1}_{x_1 = n-x_2} \underbrace{\mathbb{1}^n_{x_2 \in J_n} \mathbb{1}_{x_1 \in J_n}}_{\substack{=1 \\ \text{if } x_1 = n-x_2}} \underbrace{p_1^{x_1+x_2-n}}_{=1 \text{ if } x_1 = n-x_2}$$

$= \text{Deg}(n-x_2)\, \mathbb{1}^n_{x_2 \in J_n}$

$$\boxed{P(A|B) = \frac{P(A, B)}{P(B)} \quad \text{if } P(B)=0 \Downarrow\ P(A|B) \text{ undefined}}$$

★ Let's generalize ~~this~~ ~~conditional~~ conditional probability a little bit:

$\vec{X} \sim \text{Multin}_K(n, \vec{p})$

the vector w/o the jth component

$$P_{\vec{X}_{-j}|X_j}(\vec{x}_{-j}, x_j) = \frac{P_{\vec{X}}(\vec{x})}{P_{X_j}(x_j)}$$

$$= \text{Multin}_{K-1}(n - x_j, ?) = \frac{\text{Multin}_K(n, \vec{p})}{\text{Bin}(n, p_j)}$$

cancel each other

$$= \frac{\binom{n}{x_1, \dots, x_j, \dots x_K} p_1^{x_1} \cdots p_j^{x_j} \cdots p_K^{x_K}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-x_j}}$$

$$= \frac{\frac{n!}{x_1! \cdots x_j! \cdots x_K!} \mathbb{1}_{x_1+\cdots+x_j+\cdots+x_K=n} \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_j \in J_n} \cdots \mathbb{1}_{x_K \in J_n} \left(p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cdots p_{j+1}^{x_{j+1}} \cdots p_K^{x_K}\right)}{\frac{n!}{x_j!\,(n-x_j)!} \mathbb{1}_{x_j \in J_n} (1-p_j)^{n-x_j}}$$

$\Big\{$ $\boxed{\text{Note: } p_1 + \cdots + p_K = 1 \Rightarrow p_1 + \cdots + p_{j-1} + p_{j+1} + \cdots + p_K = 1 - p_j}$

if we divide both sides by $1-p_j$ $= \dfrac{p_1}{1-p_j} + \cdots + \dfrac{p_{j-1}}{1-p_j} + \dfrac{p_{j+1}}{1-p_j} + \cdots + \dfrac{p_K}{1-p_j} = 1$

Note: $n - x_j = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_K$ → prob zero $\Big\}$

Let $n' := n - x_j$

$$= \frac{n'!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_K!} \mathbb{1}_{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_K=n'} \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_{j-1} \in J_n} \mathbb{1}_{x_{j+1} \in J_n}$$

$$\cdot \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}}\ p_{j+1}^{x_{j+1}} \cdots p_K^{x_K}}{(1-p_j)^{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_K}} \mathbb{1}^n_{x_j \in J_n}$$

HW: $Cov[X_1, X_2] = E[(X_1-\mu_1)(X_2-\mu_2)]$

$= \binom{n'}{x_1, \dots, x_{j-1}, x_{j+1}, \dots x_k} \left(p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}} \, p_{j+1}'^{x_{j+1}} \cdots p_k'^{x_k} \right) \mathbb{1}^n_{x_j \in J_n}$

$= Multi_{k-1}(n', \vec{p}') \, \mathbb{1}^n_{x_j \in J_n}$

$\vec{X} \sim Multin_k(n, \vec{p})$, what is $E[\vec{x}]$=? $Var[\vec{x}]$ = ?

* Review from Math 2H. Let $X_1, \dots, X_n$ be rv's and $a, c \in \mathbb{R}$

$E[aX + c] = aE[X] + c$

$E[\sum x_i] = \sum E[X_i] \overset{\text{identically dist.}}{=} n \cdot \mu$

$E[\prod X_i] \overset{\text{if indep.}}{=} \prod E[X_i]$

$\delta^2 := Var[X] := E[(X-\mu)^2]$, $\delta := \sqrt{Var(X)}$

$\quad = E[X^2] - \mu^2$

$Var[X_1 + X_2] = E\left[\left((X_1+X_2) - (\mu_1+\mu_2)\right)^2\right]$

$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2$

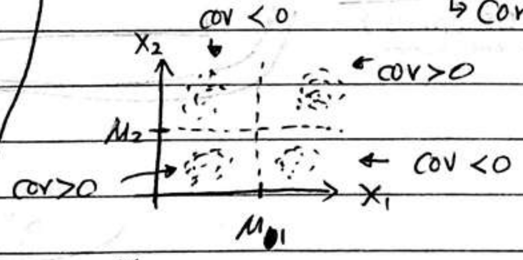
$+ 2\mu_1\mu_2] = E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 + 2E[X_1X_2]$

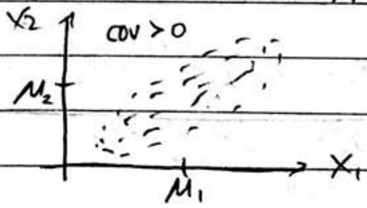
$- 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_1\mu_2 - 2\mu_2^2 + 2\mu_1\mu_2$

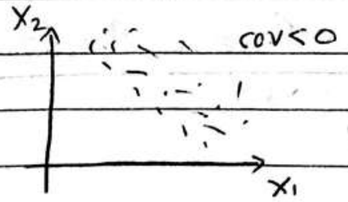
$= \delta_1^2 + \mu_1^2 + \delta_2^2 + \mu_2^2 + \mu_1^2 + \mu_2^2 + 2E[X_1X_2] - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_2^2$

$= \delta_1^2 + \delta_2^2 + 2\underbrace{(E[X_1X_2] - \mu_1\mu_2)}_{} = \delta_1^2 + \delta_2^2 + 2\,cov[X_1, X_2] \overset{\text{if } X_1, X_2 \text{ indep.}}{=} \delta_1^2 + \delta_2^2$

↳ $Cor(X_1 X_2)$: covariance of X_1 with X_2







Covariance Rules:

$Cov[X, X] = \delta^2$

$Cov[X_1, X_2] = Cov[X_2, X_1]$

$Cov[X_1 + X_2, X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$

$Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \delta_{1,2}$

$Var[X_1 + \dots + X_n] = \sum_{i=1}^n \sum_{j=1}^n Cov[X_i, X_j]$

$$E[\vec{x}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}, \quad \text{let } m = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$$

$$E[m] := \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

$$Var[\vec{x}] := E[\underbrace{\vec{x}\vec{x}^T}_{\text{outer product}}] - \underbrace{\vec{\mu}\vec{\mu}^T}_{} = \begin{bmatrix} Var[X_1] & Cov[X_1X_2] & \cdots & Cov[X_1X_k] \\ Cov[X_2X_1] & Var[X_2] & & Cov[X_2X_k] \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & \\ Cov[X_kX_1] & Cov[X_kX_2] & \cdots & Var[X_k] \end{bmatrix}$$

$(k\times1)(1\times k) = k\times k$ matrix

variance - covariance matrix
and it is symmetric

If $X_1, \cdots, X_k$ are independent, what is the varcov matrix?

↳ $\Sigma = \text{diag}\{\delta_1^2 \cdots \delta_k^2\} := \begin{bmatrix} \delta_1^2 & & & & 0 \\ & \delta_2^2 & & & \\ & & \delta_3^3 & & \\ 0 & & & \ddots & \\ & & & & \delta_k^k \end{bmatrix}$

Rules about vector rv expectations

$$E[aX + \vec{c}] = \begin{bmatrix} a\mu_1 + c_1 \\ a\mu_2 + c_2 \\ \vdots \\ a\mu_k + c_k \end{bmatrix} = a\vec{\mu} + \vec{c}$$

$$E[\vec{a}^T X] = E[a_1X_1 + \cdots + a_kX_k] = a_1\mu_1 + \cdots + a_k\mu_k = \vec{a}^T\vec{\mu}$$