Consider B_1, B_2, ... ~iid Bern(p)

possibly infinite sequence of iid Bernoullis

Let X := # of zeroes before the first one occurs $= \min\{t : b_t = 1\} - 1$

$p(0) = P(X=0) = P(\{1\}) = p$

$p(1) = P(X=1) = P(\{0,1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$ $\qquad Supp[X] = \{0,1,2,\dots\}$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0,0,\dots,0}_{x},1\}) = (1-p)^x p$

$X \sim Geom(p) := (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\dots\}}$

geometric r.v.

$X_1, X_2 \overset{iid}{\sim} Geom(p) \qquad T_2 = X_1 + X_2 \sim p_{T_2}(t) = ?$

$x \in \{\dots, t-1, t\}$

$x - t \in \{\dots, -1, 0\}$

$p_{T_2}(t) = \sum_{x \in Supp[X]} p^{old}(x) p^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]} = \sum_{x \in \{0,1,\dots\}} (1-p)^x p (1-p)^{t-x} p \, \mathbb{1}_{t-x \in \{0,1,\dots\}}$

$= (1-p)^t p^2 \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \in \{\dots, t-1, t\}}$

$\{0,1,\dots\} \cap \{\dots, t-1, t\}$

$(t+1)$

$\{0,1,\dots\}$

$\{\dots, t-1, t\}$

the negative binomial rv

$= (1-p)^t p^2 \sum_{x \in \{0,1,\dots,t\}} 1 = (t+1)(1-p)^t p^2 = NegBin(2,p)$

$Supp[T_2] = \{0,1,\dots\}$

$t \qquad p \; p$

$0 \; 0 \dots 1 \; 0 \; 0 \; 0 \dots 0 \; 1$

$t+2$

$t+1$ realizations

thus $t+1$ possible locations for the first 1

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p), \quad T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 \sim p_{T_3}(t) = ?$

$= X_3 + T_2$

$p_{T_3}(t) = \sum_{x \in Supp[X_3]} p^{old}_{X_3}(x) \, p^{old}_{T_2}(t-x) \mathbb{1}_{t-x \in Supp[T_2]} = \sum_{x \in \{0,1,\dots\}} (1-p)^x p \, (t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \{0,1,\dots\}}$

$= (1-p)^t p^3 \sum_{x \in \{0,1,\dots\}} (t+1-x) \mathbb{1}_{x \in \{\dots, t-1, t\}}$

$\sum_{x \in S} a + bx = \sum_{x \in S} a + \sum_{x \in S} bx = a \sum_{x \in S} 1 + b \sum_{x \in S} x$

$= (1-p)^t p^3 \sum_{x \in \{0,1,\dots,t\}} (t+1) - x = (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\dots,t\}} 1 - \sum_{x \in \{0,1,\dots,t\}} x \right)$

$= (1-p)^t p^3 \left((t+1)(t+1) - \frac{t(t+1)}{2} \right) = \binom{t+2}{2} (1-p)^t p^3 = NegBin(3,p)$

$t^2 + 2t + 1 - \frac{t^2}{2} - \frac{t}{2}$

$\frac{t^2 + 3t + 2}{2} = \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$

$0 \; 0 \dots 0 \; 1 \; 0 \; 0 \; 0 \dots 0 \; 1 \; 0 \; 0 \dots 0 \; 1$ last success

$t+2$ realizations $\qquad t+3$

t+2 locations to put 2 ones in => t+2 choose 2

$X_1, X_2, \dots, X_r \overset{iid}{\sim} Geom(p), \quad T_r := X_1 + X_2 + \dots + X_r \sim NegBin(r,p) = p_{T_r}(t)$

$p_{T_r}(t) = \binom{t+r-1}{r-1} (1-p)^t p^r \, \mathbb{1}_{t \in \{0,1,\dots\}}$

$\frac{10!}{(10-4)!} = \frac{10!}{6!} = \underbrace{10 \cdot 9 \cdot 8 \cdot 7}_{4}$

$n = 10, x = 4$

$X \sim Bin(n,p)$

Let n be really large and p be really small,
n -> +infinity, p -> 0, but lambda = np.
Our goal is to get the PMF of X under this limit.

$\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$\lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}} = \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}} = \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$

$\mathbb{1}_{x \in \{0,1,\dots\}}$

$= \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!\,n^x} \lim \left(1 - \frac{\lambda}{n}\right)^n \lim \left(1 - \frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x \in \{0,1,\dots\}} = \frac{\lambda^x}{x!} \lim \left(\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-x+1}{n}\right) e^{-\lambda} (1) \mathbb{1}_{x \in \{0,1,\dots\}}$

$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,\dots\}} \lim \frac{n}{n} \lim \frac{n-1}{n} \cdots \lim \frac{n-x+1}{n}$

$\lim \left(1 + \frac{1}{n}\right)^n = e$

$\lim \left(1 + \frac{a}{n}\right)^n = e^a$

$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,\dots\}} = Poisson(\lambda). \quad \lambda \in (0, \infty)$ parameter space

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$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda), \quad T = X_1 + X_2 \sim p_T(t) = ?$

$\frac{0/1}{1} \; \frac{0/1}{2} \dots \frac{0/1}{t}$

$p_T(t) = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \{0,1,\dots\}}$

$\binom{t}{0} + \binom{t}{1} + \dots + \binom{t}{t} = 2^t$

$x \in \{\dots, t-1, t\}$

$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,\dots,t\}} \frac{1}{x!(t-x)!} = \lambda^t e^{-2\lambda} \sum_{x \in \{0,\dots,t\}} \frac{t!}{t!} \frac{1}{x!(t-x)!} = \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,\dots,t\}} \binom{t}{x}$

$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = Poisson(2\lambda).$