Lecture 10 10/07/2020

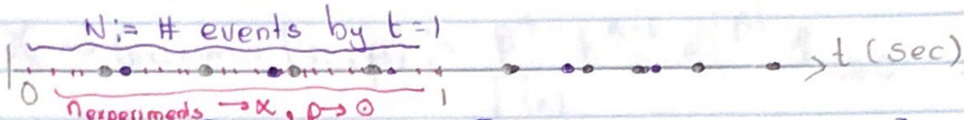
$T_k \sim \text{Erlang}(k, \lambda)$, $N \sim \text{Poisson}(\lambda)$

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_k > 1)$

$= 1 - F_{T_k}(1) = Q(k, \lambda) \qquad F_N(x) = Q(x+1, \lambda)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1)$ "Poisson Process"



$k = 5 \quad [T_5 > 1] = [X_1 + X_2 + X_3 + X_4 < 1] \cup [X_1 + X_2 + X_3 < 1]$
$\cup [X_1 + X_2 < 1] \cup [X_1 < 1] \cup [X_1 > 1]$

$= [N = 4] \cup [N = 3] \cup [N = 2] \cup [N = 1] \cup [N = 0]$

$$1 - F_{T_5}(1) \qquad F_N(4)$$
$$\| \qquad\qquad \|$$
$$P(T_5 > 1) = P(N \leq 4)$$

$$T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t \geq 0}$$

"Gamma"

$$= \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t \geq 0}$$

$k \in \mathbb{N}, \ \lambda \in (0, \infty)$

$$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

$$= \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

"Extented Negative Binomial"

$k \in \mathbb{N}, \ p \in (0, 1)$

What if $k \in (0, \infty)$? Is the top PDF legal and the bottom PMF legal? YES

$$\int_0^\infty \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} dt = 1 \text{ and } \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1$$

which means ... these are rv's.

$$X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \mathbb{1}_{t \geq 0}$$

$X \sim \text{ExtNegBin}(k, p)$

Transformations for Discrete rv's

$$X \sim \text{Bern}(p), \quad Y = X + 3 \sim \begin{cases} 3 & \text{wp } 1-p \\ 4 & \text{wp } p \end{cases}$$

$$\| \qquad\qquad\qquad\qquad = p^{y-3} (1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}}$$

$$p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$$

How do I express the transformed PMF using the original PMF?

If $Y = g(X) \sim P_Y(y) = P_X(g^{-1}(y))$

$\Updownarrow$

$g^{-1}(y) = x$

Is this formula general? No... This is only the formula for g invertible. If g non-invertible .....

$$X \sim U(\{1, 2, ..., 10\}) = \frac{1}{10} \mathbb{1}_{x \in \{1, 2, ..., 10\}}$$

$Y = \text{Min}[X, 3] \sim \begin{cases} 1 \text{ wp } 1/10 \\ 2 \text{ wp } 1/10 \\ 3 \text{ wp } P(X=3) + P(X=4) + \dots + P(X=10) = 8/10 \end{cases}$

$$P_Y(y) = \sum_{[x : g(x) = y]} P_X(x) \overset{\text{if } g \text{ invertible for Supp}[X]}{=} \sum_{[x : x = g^{-1}(y)]} P_X(x)$$

$$= P_X(g^{-1}(y))$$

$$X \sim \text{Bin}(n, p), \quad Y = X^2 \sim P_Y(y) = P_X(g^{-1}(y)) = P_X(\sqrt{y})$$

$$\Downarrow$$

$$x = \sqrt{y} = g^{-1}(y)$$

$$= \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \cdot \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$$

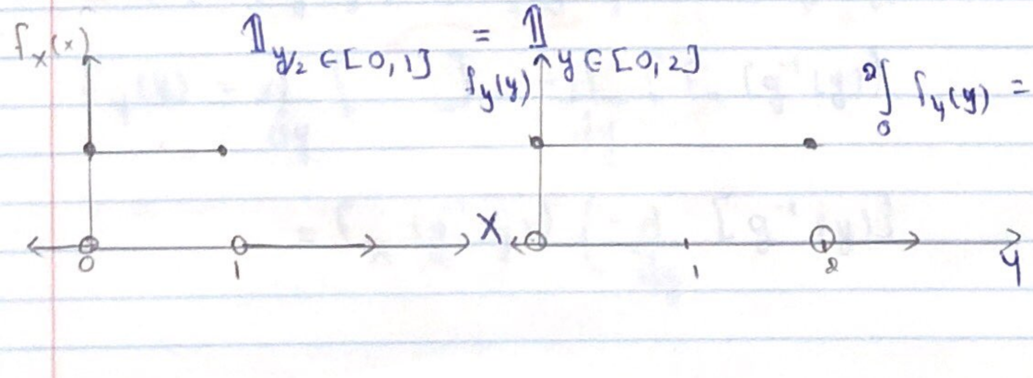
Transformations for continuous rv's

$Y = g(X)$, X is continuous for invertible g,

$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ incorrect.

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$, $Y = 2X \sim f_Y(y) = f_X(y/2) =$

$\mathbb{1}_{y/2 \in [0,1]} = \mathbb{1}_{y \in [0,2]}$



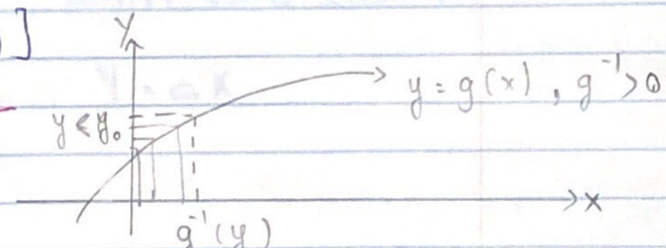
$\int_0^2 f_Y(y) = 2 - 0 = 2 \neq 1$ ILLEGAL.

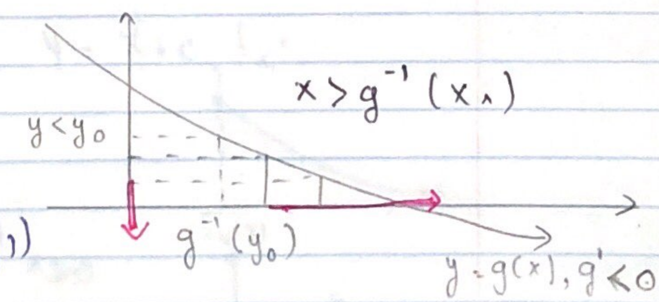
Where'd we go wrong? The above formula was derived using probabilities. Densities are not probabilities! But... CDF's are probabilities! Strategy: let's derive the CDF of Y using the CDF of X. And then, like when we did convolutions, take the derivative to get the density for Y.

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) \overset{\text{if } g'>0}{=} P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y)) = F_X'(g^{-1}(y)) \cdot \frac{d}{dy}[g^{-1}(y)]$$

$$= f_X(g^{-1}(y)) \underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{stretching}}$$





if $g' < 0$.

$$\leq P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[\quad] = -\frac{d}{dy}[F_X(g^{-1}(y))$$

$$= f_X(g^{-1}(y)) \left(-\frac{d}{dy}[g^{-1}(y)]\right.$$

$$= f_X(g^{-1}(y)) \left| \tfrac{d}{dy}[g^{-1}(y)] \right|$$

$$\frac{d}{dy}[g^{-1}(y)] < 0.$$

$$\Longrightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \tfrac{d}{dy}[g^{-1}(y)] \right| \text{ for all } g \text{ invertible}$$

Let's derive some more rules! The most common invertible function is .... the straight line! $Y = aX + c$!

$$\Longrightarrow x = g^{-1}(y) = \frac{y-c}{a}, \quad \left| \frac{d}{dy}[g^{-1}(y)] \right| = \frac{1}{|a|}$$

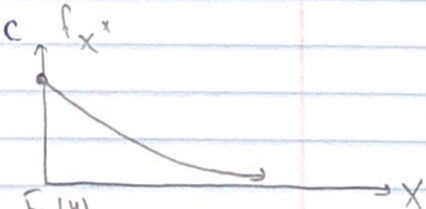
s.t. $a, c \in \mathbb{R}$.

$$f_Y(y) = f_X(y - c/a) \frac{1}{|a|} \quad \text{" shift and scale "}$$

if $c = 0$ just a scale ... $Y = aX$

$$f_Y(y) = f_X(y/a) \frac{1}{|a|}$$

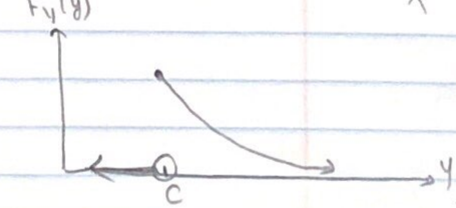
if $a = 1$ just a shift $Y = X + c$



$$f_Y(y) = f_X(y - c)$$

$$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

$$Y \sim X + c = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y \geq c}$$



$$X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x > 0}$$

$$Y = g(X) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = y$$

$$\Rightarrow e^{y} = e^{x} - 1 \Rightarrow e^{y} + 1 = e^{x}$$

$$\Rightarrow x = \ln(e^{y}+1) = g^{-1}(y)$$

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$$

$$\underbrace{e^{\ln(1/(e^{y}+1))}}_{e^{-\ln(e^{y}+1)}} \qquad \underbrace{\underbrace{\underbrace{y \in \mathbb{R}}_{e^{y} \geq 0}}_{e^{y}+1 \geq 1}}_{\ln(e^{y}+1) \geq 0}$$

$$f_Y(y) = f_X(\ln(e^{y}+1))\,\frac{e^{y}}{e^{y}+1} = e^{-\ln(e^{y}+1)}\,\mathbb{I}_{\ln(e^{y}+1) \geq 0}\,\frac{e^{y}}{e^{y}+1}$$

$$= \frac{1}{e^{y}+1}\,\frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^{2}} \cdot \frac{e^{-2y}}{e^{-2y}} = \text{Logistic}(0,1)$$

$$= \frac{e^{-y}}{(e^{-y}+1)^{2}}$$