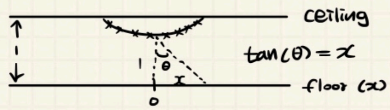
Lecture 19

$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi} \cdot \frac{1}{x^2+1}$ $E[X] = \int_{\mathbb{R}} x \frac{1}{\pi} \frac{1}{x^2+1}\, dx = \infty \to$ doesn't exist

$M_X(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\pi} \frac{1}{x^2+1}\, dx \to$ doesn't exist $\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{1}{\pi} \frac{1}{x^2+1}\, dx = \cdots = e^{-|t|}$

$\phi_X'(t) = -\frac{t}{|t|} e^{-|t|}$, $\phi_X'(0) = \text{DNE}$

Let's derive the Cauchy distribution like the physicists found it :



$\theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$, $X = g(\theta)$, $\theta = g^{-1}(x) = \arctan(x)$, tangent is invertible btw $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

$f_X(x) = f_\theta(g^{-1}(x)) \left| \frac{d}{dx}[g^{-1}(x)] \right|$

$= \frac{1}{\pi} \mathbb{1}_{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{1}{x^2+1}$

$= \text{cauchy}(0,1)$

* Let $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \to \frac{X_i - \mu}{\sigma} = Z_i \sim N(0,1)$

$T_n \sim N(n\mu, n\sigma^2)$, $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$,

$S^2 = \frac{1}{n-1} \Sigma (X_i - \bar{X})^2 \sim f_{S^2}(s^2) = ?$
↑ sample variance

* $Z_1, \cdots, Z_n \overset{iid}{\sim} N(0,1)$

$\vec{Z}^T \vec{Z} = \Sigma_{i=1}^n Z_i^2 \sim \chi_n^2 = \Sigma_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\Sigma (X_i - \mu)^2}{\sigma^2}$

$(X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2$

$= \Sigma (X_i - \bar{X})^2 + \Sigma 2 (X_i - \bar{X})(\bar{X} - \mu) + \Sigma (\bar{X} - \mu)^2$

$= \Sigma (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$

$\to \frac{\Sigma (X_i - \mu)^2}{\sigma^2} = \frac{n-1}{\sigma^2} S^2 + \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi_n^2$

$U_1 \sim \chi_{K_1}^2$ indep. of $U_2 \sim \chi_{K_2}^2$
$\Rightarrow U_1 + K_2 \sim \chi_{K_1 + K_2}^2$

In order for this "maybe" to be true, we need independence of those two terms
i.e. we need S^2 and $\bar{X}$ to be independent.

We need Cochran's Theorem to prove this.

* $\vec{Z}^T \vec{Z} = \underbrace{\vec{Z}^T I \vec{Z}}_{\text{quadratic form}} \sim \chi_n^2$

Consider $\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{B_1 \text{ Matrix}} \vec{Z} = Z_1^2 \sim \chi_1^2$ Consider $\vec{Z}^T \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \ddots \, 0 \end{bmatrix} \vec{Z} = Z_i^2 \sim \chi_1^2$
Rank $[B_i] = 1$
$\Sigma \text{rank}[B_i] = n$

$\vec{Z}^T I \vec{Z} = \vec{Z}^T (B_1 + B_2 + \cdots + B_n) \vec{Z} = \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z} + \cdots + \vec{Z}^T B_n \vec{Z} \sim \chi_n^2$

each of these quadratic forms is independent.

* Cochran's Theorem: If $B_1 + B_2 + \cdots + B_K = I$, $K \le n$ and the sum of their ranks is n, then you have two powerful results: (a) $\vec{Z}^T B_i \vec{Z} \sim \chi^2_{\text{rank}[B_i]}$ and
(b) $\vec{Z}^T B_{i_1} \vec{Z}$ is indep of $\vec{Z}^T B_{i_2} \vec{Z}$, $\forall i_1 \ne i_2$

* Consider $\Sigma (Z_i - \bar{Z})^2 = \Sigma Z_i^2 - 2 \Sigma Z_i \bar{Z} + \Sigma \bar{Z}^2 = \Sigma Z_i^2 - 2n\bar{Z}^2 + n\bar{Z}^2 = \Sigma Z_i^2 - n\bar{Z}^2$

let $\vec{1}_n$ = n-dim column vector of all ones $\bar{Z} = \frac{1}{n} \vec{1}^T \vec{Z} = \frac{1}{n} \vec{Z}^T \vec{1}$

$n\bar{Z}^2 = n\bar{Z}\bar{Z} = \cancel{n} \frac{1}{\cancel{n}} \vec{Z}^T \vec{1} \frac{1}{n} \vec{1}^T \vec{Z} = \frac{1}{n} \vec{Z}^T \vec{1} \vec{1}^T \vec{Z} = \vec{Z}^T (\frac{1}{n} J_n) \vec{Z}$

let $J_n = \vec{1}\vec{1}^T$, which is an $n \times n$ matrix of all ones

$\Sigma (Z_i - \bar{Z})^2 = \vec{Z}^T I \vec{Z} - \vec{Z}^T (\frac{1}{n} J_n) \vec{Z} = \vec{Z}^T (I - \frac{1}{n} J_n) \vec{Z}$

$\vec{Z}^T \vec{Z} = \Sigma (Z_i - \bar{Z})^2 + n(\bar{Z}^2) = \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z}$

I want to use Cochran's Theorem on the above expression. So I need to make sure $B_1 + B_2 = I$ and Rank $[B_1]$ + rank $[B_2] = n$

$B_1 + B_2 = (I - \frac{1}{n} J_n) + \frac{1}{n} J_n = I$

Rank $[B_2]$ = rank $[\frac{1}{n} J_n]$ = rank $[J] = 1$

Rank $[B_1]$ = rank $[I - \frac{1}{n} J] = ?$

$(I - \frac{1}{n} J)^T = I^T - \frac{1}{n} J^T = I - \frac{1}{n} J$

$(1 - \frac{1}{n} J)(1 - \frac{1}{n} J) = II - \frac{1}{n} JI - \frac{1}{n} IJ + \frac{1}{n^2} JJ$
$= 1 - \frac{1}{n} J$

$\text{tr}[I - \frac{1}{n} J] = n - 1 = \text{rank}[B_1]$

$\to \text{rank}[B_1] = n - 1 \to \Sigma \text{rank}[B_i] = 1 + n - 1 = n$

$\vec{Z}^T B_1 \vec{Z} = \Sigma (Z_i - \bar{Z})^2 \sim \chi_{n-1}^2$ indep of
$\vec{Z}^T B_2 \vec{Z} = n\bar{Z}^2 \sim \chi_1^2$

* $\bar{Z}_i = \frac{Z_1 + \cdots + Z_n}{n}$ $\frac{X_1 - \mu}{\sigma} + \cdots + \frac{X_n - \mu}{\sigma}$

$= \frac{\Sigma X_i - n\mu}{\sigma n} = \frac{\bar{X} - \mu}{\sigma}$

$\Sigma (Z_i - \bar{Z})^2 = \Sigma \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma}\right)^2 = \Sigma \left(\frac{X_i - \bar{X}}{\sigma}\right)^2$

$= \frac{1}{\sigma^2} \Sigma [(X_i - \bar{X})^2 = \frac{n-1}{\sigma^2} S^2$

$n\bar{Z}^2 = n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 = \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2 = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$

$\underbrace{\frac{n-1}{\sigma^2} S^2}_{\sim \chi_{n-1}^2} + \underbrace{\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{\chi_1^2} \sim \chi_n^2$
indep

Fisher proved this w/o cochran's Theorem in 1925 and Geary proved in 1936 that this decomposition is exclusive to the iid normal rv model $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, $\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim ?$