Wednesday November 11 2020

Lecture 17

Consider $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ R.V.'s of unknown PMF/PDF but we know it has expectation μ and variance σ (both finite).

Let $T_n = X_1 + X_2 + \ldots + X_n$, $E[T_n] = n\mu$, $Var[T_n] = n\sigma^2$

Let $\bar{X}_n := \frac{X_1 + \ldots + X_n}{n} = \frac{T_n}{n}$. $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \frac{\sigma^2}{n}$

Let $Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu$, $E[Z_n] = 0$, $Var[Z_n] = 1$

"$\bar{X}_n$ Standardize"

ident dist + P_1

$$\phi_{T_n}(t) \overset{P_3}{=} \phi_{X_1}(t) \cdot \ldots \cdot \phi_{X_n}(t) = \phi_X(t)^n$$

$$\phi_{\bar{X}_n}(t) \overset{P_3}{=} \phi_{T_n}\left(\frac{t}{n}\right) = \phi_X\left(\frac{t}{n}\right)^n \qquad C = e^{\ln(C)}$$

$$\phi_{Z_n}(t) \overset{P_2}{=} e^{-\frac{it\mu\sqrt{n}}{\sigma}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right) = e^{-\frac{it\mu\sqrt{n}}{\sigma}\left(\frac{\sqrt{n}}{\sqrt{n}}\right)} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n = e^{-\frac{it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{-\frac{it\mu n}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\frac{\frac{1}{\cancel{n}}}{\frac{1}{n}}} = e^{\frac{\frac{-it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{n\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}\right)} = \phi_{Z_n}(t)$$

We want to examine $\lim_{n\to\infty} \phi_Z(t)$ and if we find its limiting ch.f, we can use P8 to show that $Z_n \xrightarrow{d} Z \Rightarrow Z_n \overset{d}{\sim} Z$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}} = e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

Let $u = \frac{t}{\sqrt{n}\sigma}$ If $n\to\infty \Rightarrow u \to 0$

$$\overset{\text{L'Hopital Rule}}{=} e^{\frac{t^2}{\sigma^2}} \lim_{u\to 0} \frac{\frac{\phi_X'(u)}{\phi_X''(u)} - iu}{u} \overset{\text{L'Hopital Rule}}{=} e^{\frac{t^2}{2\sigma^2}} \lim_{u\to 0} \frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}$$

$$= e^{\frac{t^2}{2\sigma^2}} \frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2} \overset{P_0}{=} e^{\frac{t^2}{2\sigma^2}} \left(i^2 E[X^2] - (iE[X])^2\right)$$

$$= e^{-\frac{t^2}{\cancel{2\sigma^2}}\underbrace{(E[X^2] - E[X]^2)}_{\cancel{\sigma^2}}} = e^{-\frac{t^2}{2}} = \phi_Z(t)$$

Is $e^{-t^2/2} \overset{?}{\in} L_1$. What is $\int_{\mathbb{R}} e^{-t^2/2}\, dt = \sqrt{\pi} < \infty$ yes! ← Gaussian Integral

$$f_Z(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \phi_Z(t)\, dt = \frac{1}{2\pi} \int e^{-(itz + \frac{t^2}{2})}\, dz$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 - \left(\frac{\sqrt{2}iz}{2}\right)^2$$

$$\frac{2i^2z^2}{4} = -\frac{z^2}{2}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} e^{-\frac{z^2}{2}}\, dt = \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} dt$$

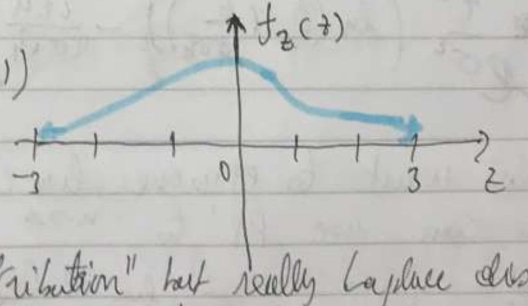
Let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\, dy$, $t \to -\infty \Rightarrow y \to -\infty$, $t \to \infty, y \to \infty$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy \overset{\text{gaussian integral}}{=} \frac{1}{2\pi} e^{-\frac{z^2}{2}} \sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \overset{Z\sim}{=} N(0,1)$$

$$\Rightarrow X_1, \dots, X_n \overset{iid}{\sim} \text{mean } \mu, \text{ Variance } \sigma^2 < \infty \Rightarrow \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

This fact is called the "Central Limit Theorem" and ~~the~~ it is the Crown jewel of an intermediate probability class. The importance of the Theorem Can't be ~~...~~ overstated. All around us we have devices that use it.

$Z \sim f_Z(z) = N(0,1)$



It's called the Gaussian distribution" but really Laplace discovered it and called it his "second law of errors". It's actually the most common error distribution in the world. A lot the field of statistics is derived by assuming Gaussian/normal iid errors.

$$E[Z] \overset{P_4}{=} i\phi'_Z(0) = 0 \checkmark$$

$$\phi'_Z(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -t e^{-t^2/2}, \quad \phi''_Z(t) = \frac{-d}{dt}\left[t e^{-t^2/2}\right] = -\left(-t^2 e^{-t^2/2} + e^{-t^2/2}\right)$$

$$Var[Z] = E[Z^2] - \cancel{E[Z]^2} \overset{P_4}{=} i^2 \phi''_Z(0) = -(-1) = 1 = SD[Z]$$

for $\sigma > 0$

$$X = \mu + \sigma Z \sim f_X(x) = ?$$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

"Normal distribution"

$$E[X] = \mu + \sigma E[Z] = \mu, \quad Var[X] = Var[\mu + \sigma Z] = \sigma^2$$

$$\phi_X(t) \overset{P_2}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu - \sigma^2 t^2/2}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep. of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim ?$

$$\phi_T(t) \overset{P_3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 - \sigma_1^2 t^2/2} e^{it\mu_2 - \sigma_2^2 t^2/2}$$

$$= e^{it(\mu_1+\mu_2) - (\sigma_1^2 + \sigma_2^2)t^2/2} \overset{P_1}{\Longrightarrow} X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$X \sim N(\mu, \sigma^2), \quad Y = e^X \sim f_Y(y) = ? \qquad g^{-1}(y) = \ln(y) \quad \frac{d}{dy} \quad |[J]| = \frac{1}{|y|}$$

$$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}\frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} = \text{Log } N(\mu, \sigma^2)$$

Log-normal model

"Not in final"

e.g.: You bring I_0 ~~amount~~ amount of money and each year it goes up/down by a random percentage X_i

$$I_y = I_0 e^{X_1} e^{X_2} e^{X_3} \cdots e^{X_y} = I_0 e^{\overbrace{X_1 + \cdots + X_y}^{\approx N(n\mu, n\sigma^2)}}$$

ex: $X_i \sim N(7\%, 11\%^2)$ $\quad P(I_y < I_0) = \cdots\cdots$