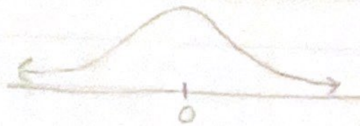
M328 11/16

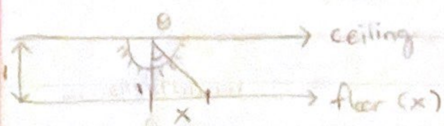
$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi}\frac{1}{x^2+1}$

$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi}\frac{1}{x^2+1}\,dx = \infty$



$M_X(t) = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = \infty$ mgf dne

$\phi_X(t) = \int_{\mathbb{R}} e^{itx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx = \dots\dots = e^{-|t|}$, $\phi_X'(t) = -\frac{t}{|t|}e^{-|t|}$



$\phi_X'(0)$ dne

because 1:1 in support

$\theta \sim U(-\frac{\pi}{2},\frac{\pi}{2}) = \frac{1}{\pi}\mathbb{1}_{\theta\in[\frac{\pi}{2},\frac{\pi}{2}]}$, $x = \tan(\theta) \Rightarrow \theta = \arctan(x)$

$d/dx[\arctan(x)] = \frac{1}{x^2+1}$

$f_X(x) = f_\theta(\arctan(x))\frac{1}{x^2+1} = \frac{1}{\pi}\mathbb{1}_{\substack{\arctan(x)\in[\frac{-\pi}{2},\frac{\pi}{2}] \\ x\in\mathbb{R}}}\frac{1}{x^2+1} = \text{Cauchy}(0,1)$

$X_1,\dots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$, $T_n \sim N(n\mu, n\sigma^2)$, $\bar{X}_n \sim N(\mu,\frac{\sigma^2}{n})$ "sample mean" or "average"

$S_n^2 = \frac{1}{n-1}\sum(X_i-\bar{X})^2 \sim f_{S_n^2}(s^2) = ?$ "sample variance"

$Z_1,\dots,Z_n \overset{iid}{\sim} N(0,1)$, $\sum Z_i^2 \sim \chi^2_n$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $\vec{Z}^T\vec{Z} \sim \chi^2_n$

$Z_1 = \frac{X_1-\mu}{\sigma},\dots,Z_n = \frac{X_n-\mu}{\sigma} \Rightarrow \sum\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$

$(X_i-\mu)^2 = ((X_i-\bar{X})+(\bar{X}-\mu))^2 = (X_i-\bar{X})^2 + 2\underbrace{(X_i-\bar{X})(\bar{X}-\mu)}_{X_i\bar{X}-\bar{X}^2-X_i\mu+\bar{X}\mu} + (\bar{X}-\mu)^2$

$\sum(X_i-\mu)^2 = \underbrace{\sum(X_i-\bar{X})^2}_{(n-1)S^2} + 2(\bar{X}\underbrace{\sum X_i}_{n\bar{X}} - n\bar{X}^2 - \mu\underbrace{\sum X_i}_{n\bar{X}} + n\bar{X}\mu) + n(\bar{X}-\mu)^2$

$\sum\frac{(X_i-\mu)^2}{\sigma^2} = \underbrace{\frac{n-1}{\sigma^2}S^2}_{\substack{\text{Conjecture:} \\ \sim\chi^2_{n-1}}} + \underbrace{\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{\substack{Z \\ Z^2\sim\chi^2_1}} \sim \chi^2_n$

this would be true if $\bar{X}$ is independent of S^2

$$\vec{Z}^T\vec{Z} = \underbrace{\vec{Z}^T I \vec{Z}}_{\text{"quadratic form"}} \sim \chi^2_n$$

Consider: $\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = Z_1^2 \sim \chi^2_1$

$$\overbrace{\vec{Z}^T \begin{array}{cc} & i \\ & \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & 0 & & \\ & & 1 \; 0 & \\ 0 & & \ddots & 0 \end{bmatrix} \end{array} i}^{B_i} \; \vec{Z} = Z_i^2 \sim \chi^2 \qquad \text{rank}[B_i] = 1$$

$$\vec{Z}^T I \vec{Z} = \vec{Z}^T(B_1 + B_2 + \ldots + B_n)\vec{Z} = \vec{Z}^T B_1 \vec{Z} + \ldots + \vec{Z}^T B_n \vec{Z} \sim \chi^2_n$$

Cochran's theorem: let $B_1 + B_2 + \ldots + B_K = I$ s.t. $\sum_{j=1}^{K} \text{rank}[B_j] = n$

then (a) $\vec{Z}^T B_j \vec{Z} \sim \chi^2_{\text{rank}[B_j]}$

(b) $\vec{Z}^T B_{j_1} \vec{Z}$ is indep. of $\vec{Z}^T B_{j_2} \vec{Z}$ $\forall j_1 \neq j_2$

$$\vec{Z}^T\vec{Z} = \sum Z_i^2 = \sum((Z_i - \bar{Z}) + \bar{Z})^2 = \sum(Z_i - \bar{Z})^2 + 2\sum(Z_i - \bar{Z})\bar{Z} + n\bar{Z}^2$$

$+ \sum(Z_i - \bar{Z})\bar{Z} = n\bar{Z}^2 - n\bar{Z}^2 = 0$

$$= \sum(Z_i - \bar{Z})^2 + n\bar{Z}^2$$

Let $\vec{1}_n$ is a column vector of all ones $\Rightarrow \bar{Z} = \frac{1}{n}\vec{Z}^T\vec{1}$

$n\bar{Z}^2 = n(\frac{1}{n}\vec{Z}^T\vec{1})^2 = \cancel{n}\cancel{\tfrac{1}{n}}\vec{Z}^T\vec{1}\,\frac{1}{n}\vec{Z}^T\vec{1} = \vec{Z}^T(\underbrace{\tfrac{1}{n}\vec{1}\vec{1}^T}_{B_2 = \frac{1}{n}J_n})\vec{Z}$ $\qquad$ rank$[B_2] = 1$

let $J_n = \vec{1}_n\vec{1}_n^T$ which is $n\times n$ matrix of all 1s.

$\sum(Z_i - \bar{Z})^2 = \sum Z_i^2 - 2n\bar{Z}^2 + n\bar{Z}^2 = \sum Z_i^2 - n\bar{Z}^2 = \vec{Z}^T I \vec{Z} - \vec{Z}^T(\frac{1}{n}J_n)\vec{Z}$

$= \vec{Z}^T(\underbrace{I - \tfrac{1}{n}J}_{B_1})\vec{Z}$ $\qquad$ $B_1 + B_2 = (I - \frac{1}{n}J) + \frac{1}{n}J_n = I$

Thm from 231: if A is a symmetric matrix and indempotent which means $AA = A$ then rank$[A]$ = tr$[A]$ = sum of the diagonal of A.

rank$[B_1]$ = tr$[B_1]$ $\qquad$ $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix} = 3J$

$B_1^T = (I - \frac{1}{n}J)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J = B_1$ ✓

$B_1B_1 = (I - \frac{1}{n}J)(I - \frac{1}{n}J) = II - \frac{1}{n}JI - \frac{1}{n}IJ + \frac{1}{n^2}JJ = I - \frac{2}{n}J + \frac{1}{n}J = I - \frac{1}{n}J = B_1$ ✓

$B_{1_{(i)}} = 1 - \frac{1}{n} 1 = 1 - \frac{1}{n}$ $\qquad = \sum_{i=1}^{n} 1 - \frac{1}{n} = n-1$

Putting it together using Cochran's theorem:

$\sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ indep of $n\bar{Z}^2 \sim \chi^2_1$

$$\bar{Z} = \frac{Z_1 + \cdots + Z_n}{n} = \frac{\frac{X_1-\mu}{\sigma} + \ldots + \frac{X_n - \mu}{\sigma}}{n} = \frac{\sum X_i - n\mu}{n\sigma} = \frac{\bar{X}-\mu}{\sigma}$$

$$\Rightarrow n\bar{Z}^2 = (\sqrt{n}\,\bar{Z})^2 = \left(\frac{\bar{Z}}{\frac{1}{\sqrt{n}}}\right)^2 = \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_1 \leftarrow \text{indep. of}$$

$$\sum (Z_i - \bar{Z})^2 = \sum\left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum (X_i - \bar{X})^2 = \frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$$

First to prove was Fisher in 1925 then in 1936 Geary proved the iid normal r.v is the "only" distribution that has the indep of $\bar{X}$ and S^2.

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ but what about $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim ?$ Not $N(0,1)$!