Lecture 06
09/16/2020
Math 621

Let $A \in \mathbb{R}^{L\times K}$ matrix of constant

$\vec{a}_i \Rightarrow$ row i of matrix A

$$\underset{L\times 1}{E[A\vec{X}]} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \dots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \dots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \dots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1, \vec{X}] \\ E[\vec{a}_2, \vec{X}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}] \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1, \vec{\mu} \\ \vec{a}_2, \vec{\mu} \\ \vdots \\ \vec{a}_L \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$$Var[\vec{a}^T\vec{X}] = Var[\overbrace{a_1X_1}^{Y_1} + \dots + \overbrace{a_KX_K}^{Y_K}] = \sum_{i=1}^{K}\sum_{j=1}^{K} cov[Y_i, Y_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} cov[a_iX_i, a_jX_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_j\sigma_{ij} = \underset{(1\times K)}{\vec{a}^T}\ \underset{(K\times K)}{\overset{\to Var[\vec{X}]}{\Sigma}}\ \underset{(K\times 1)}{\vec{a}} = \text{scaler}$$

This is called "quadratic form"

Let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^{K+1}$

$$\vec{a}^T V\vec{a} = \underset{[a_1 \dots a_K]}{\vec{a}^T} \cdot \begin{bmatrix} a_1V_{11} + \dots + a_KV_{1K} \\ a_2V_{21} + \dots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \dots + a_KV_{KK} \end{bmatrix} = \begin{matrix} \overbrace{a_1a_1V_{11} + \dots + a_1a_KV_{1K}}^{i=1} + a_2a_1V_{21} + \\ \dots + a_2a_KV_{2K} + \dots + \\ \underbrace{a_Ka_1V_{K1} + \overset{i=2}{\dots} + a_Ka_KV_{KK}}_{i=K} \end{matrix}$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_jV_{ij}$$

This is an application in finance. Imagine $X_1, \dots X_K$ are financial assets (e.g. different Stocks). Each has mean return μ_i, and each pair have covariance Σ_{ij}. Let W-vector be a vector of "weights" where each components is the percentage you put into each of these assets. Thus the entires of W sum of 1. Your protolio F is

$\vec{w}^T$ and $\vec{x}$:

$$F = \vec{w}^T \vec{x}, \quad \vec{w}^T 1 = 1, \quad E[\vec{x}] = \mu, \quad Var[\vec{x}] = \Sigma$$

$$E[F] = E[\vec{w}^T \vec{x}] = \vec{w}^T \vec{\mu} = \mu_F, \quad Var[F] = Var[\vec{w}^T \vec{x}] = \vec{w}^T \Sigma \vec{w}$$

Goal is to pick μ_F and minimize it's variance by computing $\vec{w}$ optimally.

$$\min \vec{w}^T \Sigma \vec{w} \text{ subject to } \vec{w}^T \vec{\mu} = \mu_F, \ \vec{w}^T \vec{1} = 1,$$

Markowitz optimal Portfolio design.

$$\vec{x} \sim Mult_k(n, \vec{p}), \quad E[\vec{x}] = \begin{bmatrix} E[x_1] \\ \vdots \\ E[x_k] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$x_j \sim Bin(n, p_j)$

$$\Sigma = Var[\vec{x}] = \begin{bmatrix} Var[x_1] & & Cov[x_i, x_j] \\ & \ddots & \\ cov[x_i, x_j] & & Var[x_k] \end{bmatrix} \qquad Var[\vec{x_j}] = np_j(1-p_j)$$

JMF $x_i x_j$ ← difficult to obtain

$i \neq j$

$$Cov[x_i, x_j] = E[x_i, x_j] - \mu_i \mu_j = \sum_{x_i \in \{0 \dots n\}} \sum_{x_j \in \{0 \dots n\}} x_i x_j \, p_{x_i x_j}(x_i\, x_j) - n^2 p_i p_j$$

Since JMF of $x_i x_j$ is difficult, another way will be:

$$x_i \sim Bin(n, p_i); \ x_i = x_{1i} + \dots + x_{ni} \text{ where } x_{1i}, \dots x_{ni} \overset{iid}{\sim} Bern(p_i)$$

$$\vdots$$

$$x_j \sim Bin(n, p_j); \ x_j = x_{1j} + \dots + x_{nj} \text{ where } x_{1i}, \dots x_{ni} \overset{iid}{\sim} Bern(p_j)$$

$$\vdots$$

$$x \sim Multin_k(n, \vec{p}); \ \vec{x} = \vec{x}_1 + \dots + \vec{x}_n \text{ where } \vec{x}_1, \dots, \vec{x}_n \overset{iid}{\sim} Mult_k(1, \vec{p})$$

$$Cov[X_i, X_j] = Cov[X_{1i} + \cdots + X_{ni}, X_{1j} + \cdots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}] \text{ all pairs } = \boxed{\sum_{l=1}^{m} Cov[X_{li}, X_{lj}]} \ *$$

$$= \sum_{l=1}^{n} E[X_{li} - X_{mj}] - \underbrace{E[X_{li}]}_{p_i} \underbrace{E[X_{mj}]}_{p_j}$$

if $l \neq m$ then is X_{li} independent of X_{mj}? Yes

$$E[X_{li}, X_{lj}] = \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \cdot P_{X_{li} X_{lj}}(x_{li}, x_{lj}) = P_{X_{li}, X_{lj}}(1,1) = 0$$

only non-zero if $X_{li} = X_{lj} = 1$

$X_{li} = 1$ means you get an apple, X_{lj} means you get a banana and both being 1 means you get both an apple and banana at the same time (on one draw). Impossible. Probability 0.

$$Cov[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & np_1p_2 \\ -np_1p_2 & np_2(1-p_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ -np_1p_K & \cdots & & np_K(1-p_K) \end{bmatrix}$$

$$X_1, X_2 \overset{iid}{\sim} U(\{0,1,2,3\}) = \begin{cases} 0 & wp \ 1/4 \\ 1 & wp \ 1/4 \\ 2 & wp \ 1/4 \\ 3 & wp \ 1/4 \end{cases} = \frac{1}{4} \mathbb{1}_{x \in \{0,1,2,3\}}$$

uniform discrete

Generally, $X \sim U(A) = \frac{1}{|A|} \mathbb{1}_{x \in A}$

Parameter space $A \subset \mathbb{R}$ and $|A| < \infty$.

$T = X_1 + X_2 \sim P_T(t) = $ H.W. example.

Midterm I material ends Here.!

Midterm II

$\underbrace{Y = -X = g(X)}$ ← Y is a function of the rv X (a very simple function).

$Y \sim \overset{PMF}{P_Y(Y)}$

$X = 0 \Rightarrow Y = 0$
$X = 1 \Rightarrow Y = -1$ $\quad$ $Supp[Y] = -Supp[X]$
$X = 2 \Rightarrow Y = -2$
$X = 3 \Rightarrow Y = -3$

$P_Y(Y) := P(Y = y) \overset{\downarrow}{=} P(-X = y) = P(X = -y) = P_X(-y)$

This is for all discrete rv's. $\quad$ Let $z' = -z$

$Supp[Y] := \{z : P_Y(z) > 0\} = \{z : P_X(-z) > 0\} = \{-z' : P_X(z') > 0\}$

$= -\{z' : P_X(z') > 0\} = -Supp[X]$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ From previous class $X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$X_1 - X_2 \sim ?$

difference → $D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$ $\quad P_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!}$, $Supp[X] = \{0, 1, 2, \ldots\}$

$Supp[Y] = \{\ldots -2, -1, 0\}$

$Supp[X+Y] = Supp[X] + Supp[Y] = \mathbb{Z}$ all integers.

$P_T(t) = \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$