Monday 26 October 2020

Lecture 14

We'll be doing "arbitrary multivariable transformations" of variables.

$g: \mathbb{R}^n \to \mathbb{R}^n$ and invertible. Let $\vec{X}, \vec{Y}$ be vector rv's both with dimension n and $\vec{Y} = \vec{g}(\vec{X})$

Given the jdf of the vector X rv, find the jdf of the vector Y rv. This generalizes that we did previously with univariate change of variable. Let's recall what this multivariate function looks like.

$$Y_1 = g_1(X_1, \dots, X_n)$$
$$Y_2 = g_2(X_1, \dots, X_n)$$
$$\vdots$$
$$Y_n = g_n(X_1, \dots, X_n)$$

$$\vec{g} = [g_1, \dots, g_n]^T$$
$$\vec{g}^{-1} = \vec{h} = [h_1, \dots, h_n]^T$$

$$X_1 = h_1(Y_1, \dots, Y_n)$$
$$X_2 = h_2(Y_1, \dots, Y_n)$$
$$\vdots$$
$$X_n = h_n(Y_1, \dots, Y_n)$$

From multivariable calculus, you can show that the multivariate change of variables formula is:

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h(\vec{y})|$$

where $J_h := \det \begin{bmatrix} \partial h_1/\partial y_1 & \dots & \partial h_1/\partial y_n \\ \vdots & & \\ \partial h_n/\partial y_1 & \dots & \partial h_n/\partial y_n \end{bmatrix}$ this is called the "Jacobian determinant"

Let's use this formula to prove the convolution formula for

$$T = X_1 + X_2 \sim f_T(t) = ?$$

There's a recipe for these types of problems

1) Find a g (set the first dimension Y_1 = your target) so that...
2) you can find the h.
3) Compute the Jacobian Determinant J_h
4) Substitute 2-3 into the multivariate change of variables formula
5) Integrate the "nuisance dimension(s)".

"nuisance"

(1) $T = Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1 - \cancel{X_2} = Y_1 - Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

(3) $$J_h = \det \begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1\cdot 1 - (1\cdot 0) = 1$$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 - y_2, y_2)\,|1|$

(5) $$f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\,dy_2$$
$$= \int_{\mathbb{R}} f_{\vec{X}}(t-u, u)\,du$$

$$\overset{X_1, X_2 \text{ ind}}{=} \int_{\mathbb{R}} f_{X_1}(t-u)\, f_{X_2}(u)\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(t-u)\, f(u)\,du = \int_{Supp[X]} f^{old}(t-u)\, \mathbb{1}_{t-u \in Supp[X]}\, f^{old}(u)\,du$$

$$= \int f^{old}_{X_1}(t-u)\, \mathbb{1}_{t-u \in Supp[X_1]}\, f^{old}_{X_2}(u)\,du$$

$R = \frac{X_1}{X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2), \quad Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2), \quad X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_n = \det \begin{bmatrix} Y_2 & Y_1 \\ 0 & 1 \end{bmatrix} = Y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2) |y_2|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2) dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2) |y_2| dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u)|u| \, du$ general formula

X_1, X_2 ind

if iid

$= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)|u| \, du = \int_{\mathbb{R}} f(ru) f(u)|u| \, du = \int_{Supp[X]} f^{old}(ru) \mathbb{1}_{ru \in Supp[X]} f^{old}(u)|u| du$

$\int f_{X_1}^{old}(ru) \mathbb{1}_{ru \in Supp[X_1]} f_{X_2}^{old}(u)|u| \, du$

- $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

① $R = Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2), \quad Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1(X_1 + X_2) = Y_1(Y_2) = h_1(Y_1, Y_2)$
$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

③ $J_n = \det \begin{bmatrix} Y_2 & Y_1 \\ -Y_2 & 1 - Y_1 \end{bmatrix} = Y_2(1 - Y_1) - (Y_1 \cdot (-Y_2))$
$= Y_2 - Y_1 Y_2 + Y_1 Y_2$
$= Y_2$

$\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{Y}}(y_1 y_2, y_2 - y_1 y_2)|y_2|$

⑤ $f_R(r) = f_{Y_1}(y) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|\,dy_2$

$$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)|u|\,du$$

if x_1, x_2 indep

$$\overset{}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u - ru)|u|\,du \overset{iid}{=} \int_{\mathbb{R}} f(ru) f(u-ru)|u|\,du = \int_{\mathbb{R}} f^{old}(ru) \mathbb{1}_{ru \in Supp[X]} f^{old}(u-ru) \mathbb{1}_{u-ru \in Supp[X]} |u|\,du$$

$$\| $$

$$\int f^{old}_{X_1}(ru) \mathbb{1}_{ru \in Supp[X_1]} f^{old}_{X_2}(u-ru) \mathbb{1}_{u-ru \in Supp[X_2]} |u|\,du$$

$X_1 \sim Gamma(\alpha, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$, $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

R is the proportion of the waiting time for the first gamma and thus $Supp[R] = [0, 1]$

$$f_R(r) = \int_{\mathbb{R}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1-1} u^{\alpha_1-1}} \cancel{e^{-\beta r u}}\, \mathbb{1}_{\underbrace{ru \in [0,\infty)}_{u \in [0,\infty)}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} \underbrace{(u - ru)^{\alpha_2 - 1}}_{u(1-r)} \underbrace{e^{-\beta(u - ru)}}_{e^{-\beta u} e^{\beta r u}} \mathbb{1}_{\underbrace{u - ru \in [0,\infty)}_{u(1-r) \in [0,\infty)}} |u|\,du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u}\, du \; \mathbb{1}_{r \in [0,1]}$$

$u^{\alpha_2 - 1}(1-r)^{\alpha_2 - 1}$

$$\frac{1}{B(\alpha_1, \alpha_2)} \longrightarrow \frac{\Gamma(\alpha_1 + \alpha_2)}{\beta^{\alpha_1 + \alpha_2}}$$

$$= \frac{1}{B(\alpha_1, \alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \mathbb{1}_{r \in [0,1]} = Beta(\alpha_1, \alpha_2)$$

Page 152: $X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$, $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

$$f_R(r) = \int_{Supp[X_2]} f^{old}_{X_1}(ru) \mathbb{1}_{ru \in Supp[X_1]} f^{old}_{X_2}(u) |u|\,du$$

$$= \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u} \mathbb{1}_{\underbrace{ru \in [0,\infty)}_{u \in [0,\infty)}} \cancel{\mathbb{1}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} u\,du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \mathbb{1}_{r>0} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u}\, du}$$

$$\longrightarrow \qquad \longrightarrow \frac{\Gamma(\alpha_1+\alpha_2)}{\underbrace{(\beta(r+1))^{\alpha_1+\alpha_2}}_{\beta^{\alpha_1+\alpha_2}(r+1)^{\alpha_1+\alpha_2}}} \quad \text{(See lecture 9)}$$

$$= \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0} = \text{Beta Prime } (\alpha_1, \alpha_2)$$