Math 621

Lecture 13

10-21-2020

$$f_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} j f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{\cancel{n}\ n-1 \text{ if } j=n} \frac{n!}{j!\underbrace{\cancel{(n-j)!}}_{(n-j-1)!}} \cancel{(n-j)} f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!\underbrace{(n-j-1)!}_{(n-(j+1))!}} f(x) F(x)^{j} (1-F(x))^{\overbrace{n-j-1}^{n-(j+1)}}$$

reindexing trick ... let $l = j+1$ $\Rightarrow j = l-1$ $\Rightarrow j = k \Rightarrow l = k+1$; $j = n-1 \Rightarrow l = n$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!} f(x) F(x)^{l-1} (1-F(x))^{n-l}$$

[note that both sum expressions are exactly the same, so when we subtract we're left with just the expression when $j=k$]

$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x)$$

(2) let's make sure we can uncover the min / max formulas:

$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!\,(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1}$$

$$= n f(x) (1-F(x))^{n-1} \checkmark$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n}$$

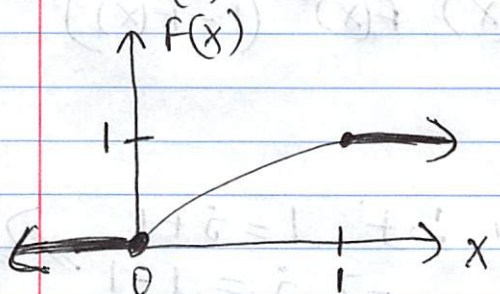
$$= n f(x) F(x)^{n-1} \checkmark$$

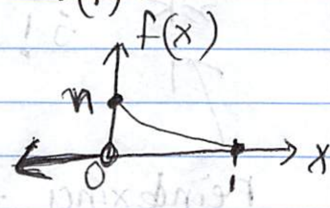
$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1) = \underbrace{1\, \mathbb{1}_{x \in [0,1]}}_{f(x)}$, $F(x) = x$

CDF of minimum:

$$F_{X_{(1)}}(x) = 1 - (1-F(x))^n = 1-(1-x)^n$$

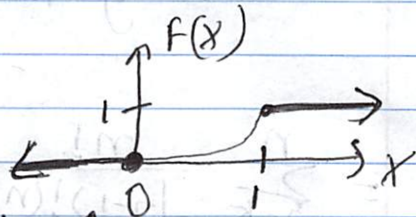
PDF: $f_{X_{(1)}}(x) = n(1-x)$



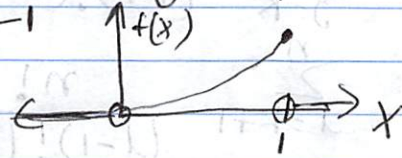


CDF of maximum:

$$F_{X_{(n)}}(x) = F(x)^n = x^n$$



PDF: $f_{X_{(n)}}(x) = n x^{n-1}$



General formula:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x\in[0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k+1-1} \mathbb{1}_{x\in[0,1]}$$

$$= \text{Beta}(k, n-k+1)$$

$X \sim \text{Gamma}(\alpha_1, \beta)$ independent of $Y \sim \text{Gamma}(\alpha_2, \beta)$,

$\underbrace{\text{Gamma}(\alpha_1,\beta)}_{\text{"Erlang}(\alpha_1,\beta)\text{"}}$, $\underbrace{\text{Gamma}(\alpha_2,\beta)}_{\text{"Erlang}(\alpha_2,\beta)\text{"}}$, $T = X + Y \sim \text{Gama}(\alpha_1+\alpha_2, \beta)$

Leomonde

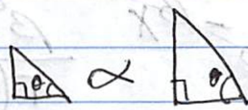
To prove this, we use "kernels", $k(x)$.
For any PMF or PDF, we can decompose it into a normalization constant c & a kernel $k(x)$.

$$P(x) = c\,k(x) \quad \text{and} \quad f(x) = c\,k(x)$$

not a function of x

(Proportional)

$$\Rightarrow P(x) \propto k(x), \quad f(x) \propto k(x)$$



We know, $1 = \sum P(x) = \sum c\,k(x)$

$$\Rightarrow \frac{1}{c} = \sum k(x) \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int_{\text{SUPP}[\,]}^{\text{SUPP}[\,]} f(x)\,dx = \int c\,k(x)\,dx$$

$$\Rightarrow \frac{1}{c} = \int k(x)\,dx \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

this means that $k(x)$ is 1-1 with the PMF or PDF. If you know $k(x)$, you know the distribution of the rv.
Let's see some examples:

$$X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\ldots,n\}}$$

$$= \underbrace{n!\,(1-p)^n}_{c} \underbrace{\frac{1}{x!\,(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1}_{x \in \{0,\ldots,n\}}}_{k(x)}$$

④

$$X \sim \text{Weibull}(k, \lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y=0}$$

$$= \underbrace{k\lambda^k}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y<0}}_{k(x)}$$

$$X \sim \text{Gamma}(\alpha, \beta) = \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}}_{k(x)}$$

Proportional to $\longrightarrow \propto x^{a} e^{-bx} \mathbb{1}_{x \geq 0}$

$X \sim \text{Gamma}(\alpha_1, \beta)$ independent of

$Y \sim \text{Gamma}(\alpha_2, \beta)$, $T = X + Y \sim f_T(t) = ?$

using independent convolution

$$f_T(t) = \int_0^{\infty} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t} \cancel{e^{\beta x}}} \underbrace{\mathbb{1}_{t-x \in [0,\infty)}}_{x-t \in (-\infty, 0)} \, dx$$

$x \in (-\infty, t]$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} \, dx \, \mathbb{1}_{t \geq 0}$$

$$\propto e^{-\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} \, dx \, \mathbb{1}_{t \geq 0}$$

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1} \left(1-\frac{x}{t}\right)^{\alpha_2-1} dx \, \mathbb{1}_{t \geq 0}$$

u-subtitution:

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du,$

$x = 0 \Rightarrow u = 0$

$x = t \Rightarrow u = 1$

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^1 u^{\alpha_1-1} (1-u)^{\alpha_2-1} t \, du \, \mathbb{1}_{t \geq 0}$$

$$= e^{-\beta t} t^{\alpha_1+\alpha_2-1} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du}_{C(\alpha_1,\alpha_2)} \mathbb{1}_{t\geq 0}$$

$C(\alpha_1,\alpha_2)$ not a function of t

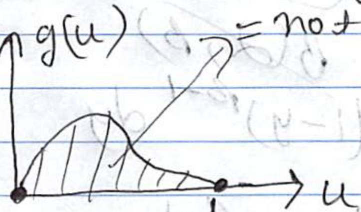
$$\propto e^{-\beta t} t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t \geq 0}$$

$$\propto \text{Gamma}(\alpha_1+\alpha_2, \beta) \checkmark$$

Let's talk about the "beta function". A famous ubiquitous function

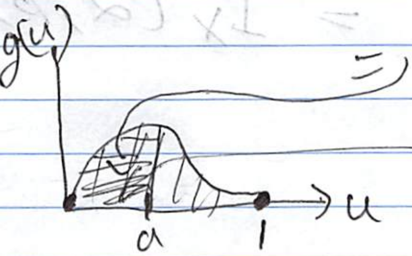
$$B(\alpha,\beta) = \int_0^1 \overbrace{u^{\alpha-1}(1+u)^{\beta-1}}^{g(u)}\, du$$

= not available in closed form.



The "incomplete beta function" is:

$$B(a,\alpha,\beta) = \int_0^a u^{\alpha-1}(1-u)^{\beta-1}\, du$$



The regularized incomplete beta function is:

$$I_a(\alpha,\beta) = \frac{B(a,\alpha,\beta)}{B(\alpha,\beta)} \overset{\in[0,1]}{=} \%$$

percentage

Let's derive a beta function-gamma function identity

$$T \sim \text{Gamma}(\alpha_1+\alpha_2,\beta)$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\geq 0}$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2)\, t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t\geq 0}$$

⑥

$$\Rightarrow B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$ ✓ Identity!!

$$X \sim \text{Beta}(\alpha, \beta) \doteq \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]} \quad (\alpha,\beta>0)$$

$$1 = \int_{\text{supp}(X)} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\,dx$$

$$= \frac{1}{\cancel{B(\alpha,\beta)}} \underbrace{\int_0^1 x^{\alpha-1} (1-x)^{\beta-1}\,dx}_{\cancel{B(\alpha,\beta)}} = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1} (1-y)^{\beta-1}\,dy$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1} (1-y)^{\beta-1}\,dy$$

$$= \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta)$$