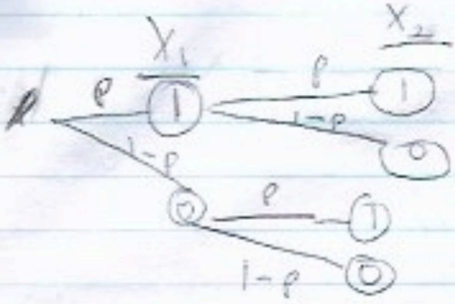
8-31-20

Lec 2

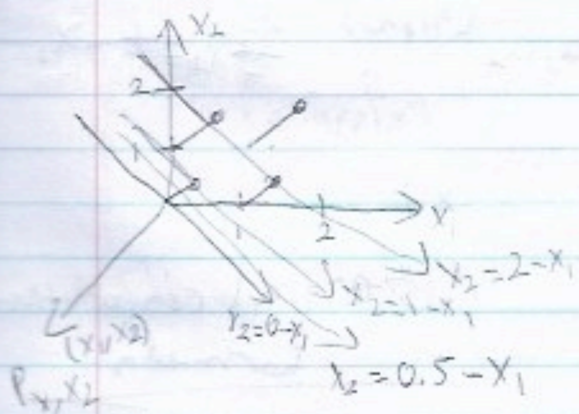


$p_{\vec{x}}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

$$T \sim \begin{cases} 2 & \text{w.p } p^2 \\ 1 & \text{w.p } 2p(1-p) \\ 0 & \text{w.p } (1-p)^2 \end{cases}$$

$$p(t) = P(T=t) = \sum_{x_1\in\mathbb{R}}\sum_{x_2\in\mathbb{R}} p_{x_1,x_2}(x_1,x_2)\,\mathbb{1}_{x_1+x_2=t}$$

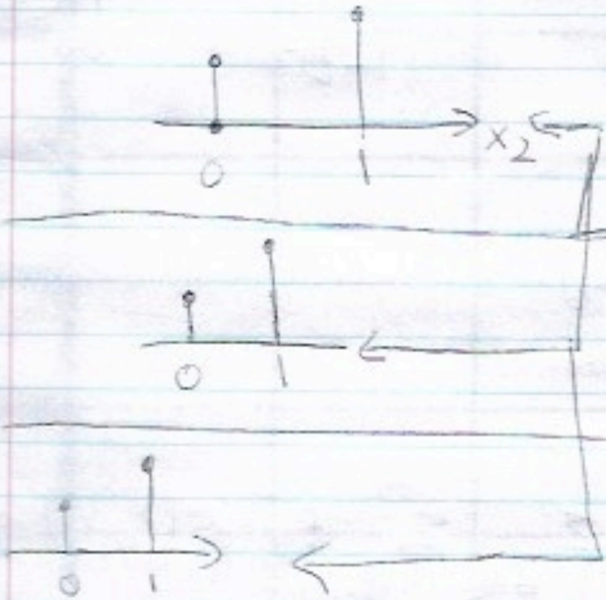
$$= p_{X_1}^{(x)} * p_{X_2}^{(x)}$$



- If $t=2$

$$p_T(2) = \sum_{x_1\in\mathbb{R}} p_{X_1,X_2}(x_1, 2-x_1)$$

"Convolve" means to "roll or coil together/entwine"



$t=2$ wp p^2

$t=1$ wp $(1-p)p$

$$p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_2=t-x_1}$$

$$= \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$

General convolution formula

If X_1, X_2 indep.

$$= \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x)$$

convolution formula for indep. rvs

$$\parallel$$

$$\sum_{x \in \mathbb{R}} p^{old}_{X_1}(x)\, \mathbb{1}_{x \in \mathrm{supp}[X_1]}\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in \mathrm{supp}[X_2]}$$

$$\parallel$$

$$\sum_{x \in \mathrm{supp}[X_1]} p^{old}_{X_1}(x)\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in \mathrm{supp}[X_2]}$$

$X_1, X_2 \overset{iid}{}$

Convolution formula for iid rvs

$$= \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in \mathrm{supp}[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in \mathrm{supp}[X]}$$

$$= \sum_{x \in \mathrm{supp}[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in \mathrm{supp}[X]}$$

* These are all the convolution formulas we will use. Highlight them later.

Note: $t - x \in \{0,1\} = t \in [x, x+1]$

This was just playing in stuff → $T_2 \sim \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \cdot p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}$

now we change → $= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$

$\rightarrow = p^t (1-p)^{2-t} \underbrace{\left(\mathbb{1}_{t \in (0,1)} + \mathbb{1}_{t \in \{1,2\}} \right)}$

$t = 0 \Rightarrow 1$
$t = 1 \Rightarrow 2$ $= \binom{2}{t}$
$t = 2 \Rightarrow 1$

$= \binom{2}{t} p^t (1-p)^{2-t} = \text{Binom}(2, p)$, $\text{Supp}[T_2] = \{0,1,2\}$

$\boxed{\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0, \ldots, n\}}}$

Generally $\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2]$, $A + B := \{a + b : a \in A, b \in B\}$

First new style PMF formula

$p_{T_2}(t) = \sum_{x \in \mathbb{R}} p(x) p(t-x) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$

equal to indicator 1 in first notes

X_1, X_2 iid $\text{Bern}(p) = \binom{1}{x} p^x (1-p)^{1-x}$

$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x} \binom{1}{t-x}$

Use Pascal identity:

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$= p^t (1-p)^{2-t} \left(\cancel{\binom{1}{0}}^{1} \binom{1}{t} + \cancel{\binom{1}{1}}^{1} \binom{1}{t-1} \right)$ →

$$= \binom{2}{t} p^{t} (1-p)^{2-t} = \text{Binom}(0,1)$$

X_1, X_2, X_3 iid Bern(p) $\quad T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3$

$$= X_3 + T_2 \sim p_{T_3}(t) = ?$$

Need a formula now
(use Convolution formula)

Good ex of plug in

$$p_{T_3}(t) = \sum_{x \in \text{supp}[X_3]} p_{X_3}(x)\, p_{T_2}(t-x) \cancel{= \sum_{x\in\{0,1\}}}$$

$$= \sum_{x\in\{0,1\}} \left(p^{x}(1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x}(1-p)^{2-t+x}\right)$$

$$= p^{t}(1-p)^{2-t} \sum_{x\in\{0,1\}} \binom{2}{t-x} = p^{t}(1-p)^{3-t}\left(\binom{2}{t}+\binom{2}{t-1}\right)$$

$$= \binom{3}{t} p^{t}(1-p)^{3-t} = \text{Binom}(3,p)$$

if you multiply 2 Binomial then

= Binom(2n, p) ? maybe

$X_1, X_2 \stackrel{iid}{\sim} \text{Binom}(n,p) \quad T = X_1 + X_2 \sim ?$

$\text{H.W}: = \binom{n}{x} p^x (1-p)^{n-x}$

$$p_T(t) = \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t-x} = \binom{2n}{x} p^t (1-p)^{2n-t}$$

$$= \text{Binom}(2n, p)$$

What is Vandermonde's identity?