$\boxed{5}$

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$\vec{X} \sim \text{multin}_k(n, \vec{P})$

dimention $k=2$ $\quad \vec{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

$$P_{X_1|X_2}(x_1, x_2) = P(X_1 = x_1 \mid X_2 = x_2) = \frac{P(X_1, X_2)}{P(X_2)}$$

$P(X_2) = Bin(n, P_2) = Bin(n, 1-P_1)$

$$= \frac{\binom{n}{x_1, x_2} P_1^{x_1} P_2^{x_2}}{\binom{n}{x_2} P_2^{x_2} (1-P_2)^{n-x_2}}$$

$J_n = \{0, \ldots, n\}$

Define a ratio of indicators:

$$= \frac{\frac{\cancel{n!}}{x_1!\, \cancel{x_2!}} \mathbb{1}_{\substack{x_1 = n-x_2 \\ x_1 + x_2 = n}} \mathbb{1}_{x_1 \in J_n} \mathbb{1}^{u}_{x_2 \in J_n} P_1^{x_1} \cancel{P_2^{x_2}}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \cancel{\mathbb{1}_{x_2 \in J_n}} \cancel{P_2^{x_2}} P_1^{n-x_2}}$$

$$\mathbb{1}^{u}_{A} = \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \text{undefind} & \text{if } A^c \end{cases}$$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{\substack{=1 \text{ when} \\ x_1 = n-x_2}} \mathbb{1}_{x_1 = n-x_2} \underbrace{\mathbb{1}_{x_1 \in J_n} \mathbb{1}^{u}_{x_2 \in J_n}}_{\substack{=1 \text{ when} \\ x_1 = n-x_2 \\ \& x_2 \in J_n}} = Deg(n-x_2) \mathbb{1}^{u}_{x_2 \in J_n}$$

Hint: $P(A|B) = \frac{P(A,B)}{P(B)}$ which is undefind if $P(B) = 0$

Let's generalize this result alittle bit.

$$\vec{X} \sim \text{Multin}_k(n, \vec{p}) \qquad \vec{p} = \begin{bmatrix} p_1 \\ p_k \end{bmatrix}$$

$$P_{\vec{X}_{-j}|X_j}(\vec{X}_{-j}, X_j) = \frac{P_{\vec{X}}(\vec{x})}{P_{X_j}(X_j)} \sim \text{Multin}_{k-1}(n - X_j, ?)$$

All elements of vector rv X except the jth component

$$= \frac{\text{Multin}_k(n, \vec{p})}{\text{Bin}(n, p_j)} = \frac{\binom{n}{x_1 \cdots x_k} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cancel{p_j^{x_j}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-p_j}}$$

$$= \frac{\frac{\cancel{n!}}{x_1! \cdots \cancel{x_j!} \cdots x_k!} 1_{\overbrace{x_1 + \cdots + x_j + \cdots x_k = n}^{x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k = n - x_j}} 1_{x_1 \in J_n} \cdots \cancel{1_{x_j \in J_n}} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)!} \cancel{1_{x_j \in J_n}} (1-p_j)^{n-x_j}}$$

let $n' = n - x_j$. Note: $x_1 + \cdots x_k = n \Rightarrow n - x_j = x_1 + \cdots x_{j-1} + x_{j+1} + \cdots + x_k$

Note: $p_1 + \cdots p_k = 1$

$$\Rightarrow p_1 + \cdots p_j + p_{j+1} + \cdots p_k = 1 - p_j$$

divide both sides by $1 - p_j$ $\Rightarrow \underbrace{\frac{p_1}{1-p_j}}_{p_1'} + \cdots + \underbrace{\frac{p_{j-1}}{1-p_j}}_{p_{j-1}'} + \underbrace{\frac{p_{j+1}}{1-p_j}}_{p_{j+1}'} + \cdots \underbrace{\frac{p_k}{1-p_j}}_{p_k'} = 1$

$$p_1' + \cdots p_{j-1}' + p_{j+1}' + \cdots p_k' = 1$$

$$\binom{n'}{x_1, \cdots x_{j-1}, x_{j+1} \cdots x_k}$$

$$\frac{n'!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_k!} 1_{x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k = n'} 1_{x_1 \in J_n} \cdots 1_{x_{j-1} \in J_n} \cdots 1_{x_k \in J_n}$$

$$1^{u}_{x_j \in J_n} \underbrace{\frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k}}}_{p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}} p_{j-1}'^{x_{j+1}} \cdots p_k'^{x_k}}$$

$= \text{Multin}_{k-1}(n', \vec{p}')\, \mathbb{1}^{u}_{x_j \in J_u}$

N

$\begin{bmatrix} p' \\ \vdots \\ p_k \end{bmatrix}$

$X \sim \text{Multin}_k(n, \vec{p})$ wher is $E[\vec{X}]$? $Var[\vec{X}]$?

we need definitions for expectation and variance for vector rv's.

X is a sclar $p, c \in \mathbb{R}$

$* \; E[aX + c] = aE[X] + c$

$* \; E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\mu$ if identically distribute

$* \; E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$ if they're independent

$\sigma^2 := Var[X] = E[(X-\mu)^2] \overset{\text{distr}}{=} \sum_{X \in R} (X-\mu)^2 p(x)$

$\overset{\text{continues}}{=} \int_R (x-\mu)^2 f(x)\,dx$

standard deviation

$\sigma = \sqrt{Var[X]} = SD[X]$

$Var[X] = [X^2] - \mu^2$

$Var[X_1 + X_2] = E[(X_1 + X_2) - (\mu_1 + \mu_2))^2]$

$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2\mu_1 X_1 - 2\mu_1 X_2 - 2\mu_2 X_2 + 2X_1X_2 + 2\mu_1\mu_2]$

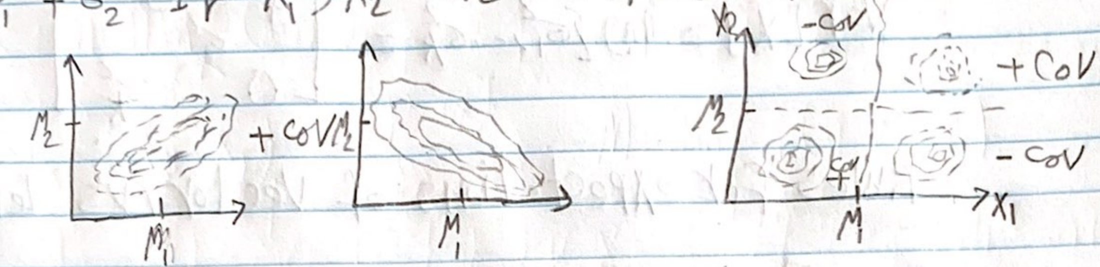
$= E[X_1^2] + E[X_2^2] + \cancel{\mu_1^2} + \cancel{\mu_2^2} - \cancel{2\mu_1^2} - \cancel{2\mu_1\mu_2} - 2\mu_2\mu_1 - \cancel{2\mu_2^2} + 2E[X_1X_2] + \cancel{2\mu_1\mu_2}$

$= \sigma_1^2 + \sigma_2^2 + 6(E[X_1X_2] - \mu_1\mu_2)$

$\boxed{\sigma_{12} = Cov[X_1 X_2] = E[X_1 X_2] - \mu_1\mu_2 \overset{\text{HW}}{=} E(X_1-\mu_1)(X_2-\mu_2)}$

$= \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$ If X_1, X_2 independent $\Rightarrow cov[X_1, X_2] = \mu_1\mu_2 - \mu_1\mu_2 = 0$

$= \sigma_1^2 + \sigma_2^2$ If X_1, X_2 are independent



Rules for covariances $a_1, a_2 \in \mathbb{R}$

① $cov[X, X] = \sigma^2$

② $cov[X_1, X_2] = cov[X_2, X_1]$

③ $cov[X_1 + X_2, X_3] = \sigma_{13} + \sigma_{23}$

④ $cov[a_1X_1, a_2X_2] = a_1a_2\sigma_{12}$

⑤ $Var[X_1 + \cdots X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} cov[X_i, X_j]$

$\vec{\mu} := E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}$, Let $m = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ X_{21} & \cdots & X_{2m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$, $E(m) = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1m}] \\ \vdots & & \vdots \\ E[X_{n1}] & \cdots & E[X_{nm}] \end{bmatrix}$

capital sigma

$\Sigma := Var[\vec{X}] = \underbrace{E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T}_{\text{other products}} = \begin{bmatrix} Var[X_1] & cov[X_1, X_2] & \cdots & cov[X_1, X_k] \\ cov[X_2, X_1] & Var[X_2] & & cov[X_2, X_k] \\ \vdots & & \ddots & \vdots \\ cov[X_k, X_1] & cov[X_k, X_2] & \cdots & Var[X_k] \end{bmatrix}$

$(k \times 1)(1 \times k)$ $k \times k$ HW

The "variance - covariance matrix" It's squar $k \times k$ and symmetric.

If $X_1, \ldots X_K$ are indefendent then the Varcon matrix is:

$$Var[\vec{X}] = diag\{\sigma_1^2, \ldots \sigma_K^2\} = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_K^2 \end{bmatrix}$$

Rules for expectations of vector rv's let $\vec{a} \in R^K$

① $E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a}$

② $E[\vec{a} + \vec{X}] = E[a_1X_1 + a_2X_2 + \cdots a_KX_K] = a_1\mu_1 + \cdots a_K\mu_K = \vec{a}^T\vec{\mu}$