Lec 6 Marin Azhar

let $A \in \mathbb{R}^{L \times K}$ matrix of constants

What is $E[A\vec{X}]$ ($L \times 1$ = dim)

$$= \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \cdots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \cdots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \cdots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{X}] \\ E[\vec{a}_2 \cdot \vec{X}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}] \end{bmatrix} =$$

first row x a col

$$= \begin{bmatrix} \vec{a}_1 \cdot \mu \\ \vec{a}_2 \cdot \mu \\ \vdots \\ \vec{a}_L \mu \end{bmatrix} = A\vec{\mu}$$

what is $[\vec{a}^T \vec{X}] = Var[\underbrace{a_1X_1}_{y_1} + \cdots + \underbrace{a_KX_K}_{y_K}] = \sum_{j=1}^{K}\sum_{j=1}^{K} cov[Y_i, Y_j]$

$$= \sum\sum cov[a_iX_i, a_jX_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j \sigma_{ij}$$

This is called a "quadratic form" $= \vec{a}^T \overset{var[\vec{X}]}{\Sigma} \vec{a}$

$(1 \times K)(K \times K)(K \times 1) = \text{scalar}$

let $V \in \mathbb{R}^{K \times K}$, $\vec{a} \in \mathbb{R}^{K \times 1}$

$$\vec{a}^T V \vec{a} = \vec{a}^T \begin{bmatrix} a_1V_{11} + \cdots + a_KV_{1K} \\ a_1V_{21} + \cdots + a_KV_{2K} \\ \vdots \\ a_KV_{K1} + \cdots + a_KV_{KK} \end{bmatrix} = \underbrace{a_1a_1V_{11} + \cdots + a_1a_KV_{1K}}_{i=1} + \underbrace{a_2a_1V_{21} + \cdots + a_2a_KV_{2K}}_{i=2} \cdots$$

$\vec{a}^T = [a_1 \ldots a_K]$

$$\underbrace{a_Ka_1V_{K1} + \cdots + a_Ka_KV_{KK}}_{i=K}$$

$$= \sum_{i=1}^{K}\sum_{i=j}^{K} a_ia_jV_{ij}$$

This is an application in finance Imagine $X_1, X_2, \ldots, X_K$ are financial assets (e.g different stocks) Each has mean return $\mu_i$. And each pair have covariance Sigma $_{ij}$ let $\vec{w}$ be a vector of "weights" where each component is the percentage you put into each of these assets. Thus the entries of $\vec{w}$ to 1. your portfolio F is $\vec{w}^T \vec{X}$

$$F = \vec{w}^T \vec{X} \quad \vec{w}^T \vec{1} = 1 \qquad E[\vec{X}] = \mu, \; Var[\vec{X}] = \Sigma$$

$$E[F] = E[\vec{w}^T \vec{X}] = \vec{w}^T \vec{\mu} = \mu_F, \; Var[F] = Var[\vec{w}^T \vec{X}] = \vec{w}^T \Sigma \vec{w}$$

Goal is to pick $\mu_F$ and minimize it's variance by computing the $\vec{w}$ optimally

$$\min \vec{w}^T \Sigma \vec{w} \text{ subject to } \vec{w}^T \vec{\mu} = \mu_F, \; \vec{w}^T \vec{1} = 1.$$

Markowitz optimal portfolio design

$$\vec{X} \sim \text{multin}_K(n, \vec{p}), \; E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$$

$x_j \sim Bin(n, p_j)$

$$\Sigma = Var[X] = \begin{bmatrix} Var[X_1] & & Cov[X_i X_j] \\ & \ddots & \\ Cov[X_i, X_j] & & Var[X_K] \end{bmatrix}$$

$$Var[X_j] = np_j(1-p_j)$$

"difficult to obtain"

for $i \neq j$

Jmt $X_i$ and $X_j$ $(X_i, X_j)$

$$Cov[X_i, X_j] = E[X_i, X_j] - \mu_i \mu_j = \sum_{x_i = \{0,\ldots,n\}} \sum_{x_j = \{0,\ldots,n\}} x_i x_j P_{x_i x_j} - n^2 p_i p_j$$

(2) $X_i \sim Bin(n, p_i)$ $\quad X_i = \underset{\text{yes, no apple}}{X_{1i} + \dots + X_{ni}}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j \sim Bin(n, p_j)$ $\quad X_j = \underset{\text{yes, no bannanas}}{X_{1j} + \dots + X_{nj}}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

$\vec{X} \sim multin_k(n, \vec{p})$ $\quad \vec{X} = \vec{X}_1 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots \vec{X}_n \overset{iid}{\sim} multi_k(1, \vec{p})$

k dimentional but 1 pick
the 1 fruit is the recoreded
fruit from one of the
dimention
multi dementional bern

$$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

all pairs

statement should be here

$$= \sum_{l=1}^{n} \cancel{\sum_{m=1}^{n}} E[X_{li} - X_{mj}] - \underbrace{E[X_{li}]}_{p_i} \underbrace{E[X_{mj}]}_{p_j = p_i}$$

if $l \neq m$ then is $X_{li}$ independent from $X_{mj}$? yes!

$$E[X_{li}, X_{lj}] = \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} P_{x_{li}, x_{lj}}(x_{li}, x_{lj}) = $$ only non zero if $x_{li} = x_{lj} = 1$

$= P_{x_{li} x_{lj}}(1,1) = 0$

$X_{li} = 1$ means you get an apple, $X_{lj}$ means you get a banana and both being 1 means you get both an apple and a banana at the same time (on one draw) impossible, probability 0

more apples means less bannanas

$$\Sigma = Var[\vec{X}] = \begin{bmatrix} np(1-p_1) & -np_1p_2 & & -np_1p_k \\ -np_1p_2 & np_2(1-p_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ -np_1p_k & \cdots & \cdots & np_k(1-p_k) \end{bmatrix}$$

let $X_1, X_2 \overset{iid}{\sim} U(\{0,1,2,3\})$ $= \begin{cases} 0 & \text{wp } 1/4 \\ 1 & \text{wp } 1/4 \\ 2 & \text{wp } 1/4 \\ 3 & \text{wp } 1/4 \end{cases} = \frac{1}{4} \mathbb{1}_{x \in \{0,1,2,3\}}$

uniform discrete

generally $X \sim U(A) = \frac{1}{|A|} \mathbb{1}_{x \in A}$

Prameters space $A \subset \mathbb{R}$ and $|A| < \infty$

$T = X_1 + X_2 \sim P_T(t) =$ " for H.W "

midterm II ↓

y is a function of the rv X (a very simple function)

$Y = -X = y(X)$

If $X = 0$ then $y = 0$

If $X = 1$ then $y = -1$ $\quad Supp[y] = -Supp[X]$

If $X = 3$ then $y = -3$

$P_y(y) := P(Y = y) = P(-X = y) = P(X = -y) = P_x(-y)$

This is for all discrete rv's

let $z' = -z$

$Supp[Y] = \{z : P_Y(z) > 0\} = \{z : P_X(-z) > 0\} = \{-z' : P_x(z') > 0\}$

$= -\{z' : P_x(z') > 0\} = -Supp[X]$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$ from prev class $X_1 + X_2 \sim Poisson(2\lambda)$

Pmf of negative Poisson?

$X_1 - X_2 \sim ?$ $\quad P_y(y) = \frac{e^{-\lambda} \lambda^{-y}}{(-y)!}$ $\quad Supp[Y] = \{\ldots, -2, -1, 0\}$

$\to \{0, 1, 2, 3, 4 \ldots\}$

$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{y} \overset{"}{\sim}$

$Supp[X + Y] \overset{?}{=} Supp[X] + Supp[Y] = \mathbb{Z}$

every element in one with every element in the other

$P_D(t) = \sum_{x \in Supp[X]} P^{old}(x) P^{old}(d - x) \mathbb{1}_{d - x \in Supp[y]}$