Lecture 9

$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 \sim f_{T_3}(t) = ?$

Independent Formula $= \int_{Supp[T_2]} f^{old}_{T_2}(x) f^{old}_{X_3}(t-x) \mathbb{1}_{t-x \in Supp[X_3]} dx$

$= \int_0^\infty x\lambda^2 \cancel{e^{-\lambda x}} \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}} \overbrace{\mathbb{1}_{t-x \in [0,\infty)}}^{x \le t} dx$

$= \lambda^3 e^{-\lambda t} \int_0^{\infty(t)} x \mathbb{1}_{x \le t} dx = \lambda^3 e^{-\lambda t} \int_0^t x\,dx \, \mathbb{1}_{t \in [0,\infty)}$

$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$

$T_4 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4 \sim f_{T_4}(t) = ?$

$f_{T_4}(t) = \int_{\underbrace{Supp[T_3]}_{[0,\infty)}} f^{old}_{T_3}(x) f^{old}_{X_4}(t-x) \mathbb{1}_{t-x \in Supp[X_4]} dx$

$= \int_0^\infty \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}} \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}} \overbrace{\mathbb{1}_{t-x \in [0,\infty)}}^{x \le t} dx$

$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^\infty x^2 \mathbb{1}_{x \le t} dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2 dx \, \mathbb{1}_{t \in [0,\infty)}$

$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$

$T_k = \sum_{i=1}^{k} X_i \sim \text{Erlang}(k, \lambda) := \overset{PDF}{=} \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in [0,\infty)}$ Does not change

Continue $Supp[T_k] = [0, \infty) \quad \lambda \in (0, \infty), \; k \in \mathbb{N}$

$$\begin{array}{ccc} \text{Exp}(\lambda) & \xrightarrow{\text{add up}} & \text{Erlang}(k, \lambda) \\ \Updownarrow & & \Updownarrow \text{connection / conceptually analogous} \\ \text{Geom}(p) & \xrightarrow{\text{add up}} & \text{NegBin}(k, p) \end{array}$$

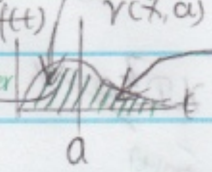
Waiting sth happen — Waiting for many / different thing happen, such as 9

Let's do some pure math, I want to define the gamma family of Function, Beginning with gamma function for x non-neg. $x \geq 0$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt \overset{a>0}{=} \underbrace{\int_0^{a} t^{x-1} e^{-t}\,dt}_{\gamma(x,a)} + \underbrace{\int_a^{\infty} t^{x-1} e^{-t}\,dt}_{\Gamma(x,a)}$$

t does matter can exchang

eg: $\Gamma(3) = \int_0^{\infty} \underbrace{t^2 e^{-t}}_{\text{call } f(t)}\,dt = 2$ Area is Answer



Complete $\int_0^{\infty}$

Lower incomplete gamma Function $\gamma(x,a)$ $\int_0^a$

upper incomplete gamma Function $\Gamma(x,a)$ $\int_a^{\infty}$

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\overset{\text{lower}}{\gamma(x,a)} + \overset{\text{upper}}{\Gamma(x,a)}}{\Gamma(x)} = \underbrace{\frac{\gamma(x,a)}{\Gamma(x)}}_{\text{call } P(x,a)} + \underbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}_{\text{call } Q(x,a)} = P(x,a) + Q(x,a) = 1$$

Percenge Below a — Percenge up a

Lower regularized gamme Function : $P(x,a)$

upper regularized gamma Function : $Q(x,a)$

$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t}\,dt = 1$ PDF consider $X \sim Exp(1)$, $e^{-t}\mathbb{1}_{t\in[0,\infty)}$

$\Gamma(x+1) = x\Gamma(x)$

Let $n \in \mathbb{N}$ natural number

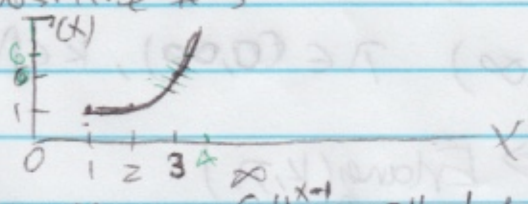
$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)\cdots(3)(2)(1) = (n-1)!$$

Let $x \in (0,\infty)$

$$\Gamma(x) = (x-1)\Gamma(x-1) = \cdots = (x-1)(x-2)\cdots\Gamma(c) \quad \text{where } c \in (0,1)$$

the gamma Function "extends" the factorial Function to all positive #'s

$\Gamma(1) = 0! = 1$
$\Gamma(2) = 1! = 1$
$\Gamma(3) = 2! = 2$
$\Gamma(4) = 3! = 6$



$c > 0$

$$\int_0^{\infty} t^{x-1} e^{-ct}\,dt \underset{\#}{=} \int_0^{\infty} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du = \frac{1}{c^x}\int_0^{\infty} u^{x-1} e^{-u}\,du = \frac{\Gamma(x)}{c^x}$$ IF call is $\Gamma(x)$

Substitution: let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow \frac{du}{dt} = c \Rightarrow dt = \frac{1}{c}du$ $t = 0 \Rightarrow u = 0$ $t \to \infty \Rightarrow u \to \infty$

$u=ct.\ t=\frac{u}{c}\ \frac{du}{dt}=c\ dt=\frac{1}{c}du.\ t=0\Rightarrow u=0\ t\to\infty\Rightarrow u\to\infty.\ t=a\Rightarrow u=ac$

lower $\int_0^a t^{x-1}e^{-ct}dt \overset{\downarrow}{=} \int_{0}^{ac}\frac{u^{x-1}}{c^{x-1}}e^{-u}\frac{1}{c}du = \frac{1}{c^x}\int_0^{ac}u^{x-1}e^{-u}du = \frac{\gamma(x,ac)}{c^x}$

complete ~~complete~~ lower upper

upper $\int_a^\infty t^{x-1}e^{-ct}dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$

IF $n\in\mathbb{N}$

Definition

upper gamma Function $\Gamma(n,a) := \int_a^\infty \underbrace{t^{n-1}}_{u}\underbrace{e^{-t}dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du =$

HW: $v=\int dv=\int e^{-t}dt = -e^{-t}$ $\quad \frac{du}{dt}=(n-1)t^{n-2}\Rightarrow du=(n-1)t^{n-2}dt$

$=\left[t^{n-1}(-e^{-t})\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}dt$

$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}dt = a^{n-1}e^{-a}+(n-1)\Gamma(n-1,a)$ (upper)

Repeat $\Gamma(n-1,a)$

$= a^{n-1}e^{-a} + (n-1)(a^{n-1}e^{-a}+(n-2)\Gamma(n-2,a))$

Repeat again

$a^{n-1}e^{-a}+(n-1)(a^{n-2}e^{-a}+(n-2)(a^{n-3}e^{-a}+(n-3)\Gamma(n-3,a)$

~~$=e^{-a}(1+(n-1)(1+(n-2)).(1+(n-3)\Gamma(n-3,a))$~~

$=e^{-a}(a^{n-1}+(n-1)(a^{n-2}+(n-2)(a^{n-3}+(n-3)\Gamma(n-3,a))))$

$=e^{-a}(a^{n-1}+(n-1)a^{n-2}+(n-1)(n-2)a^{n-3}+(n-1)(n-2)(n-3)\Gamma(n-3,a))$

$=e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!}+\frac{a^{n-2}}{(n-2)!}+\frac{a^{n-3}}{(n-3)!}+\frac{1}{(n-4)!}\Gamma(n-3,a)\right)$

$=e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!}+\frac{a^{n-2}}{(n-2)!}+\cdots\frac{a^1}{1!}+\frac{a^0}{0!}\right) \quad e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$

$n\to\infty \quad \overset{\parallel}{e^{a}}$

$\Gamma(1,a)=\int_a^\infty e^{-t}dt = \left[-e^{-t}\right]_a^\infty = e^{-a}$

Back to probability land...

$X\sim \text{Erlang}(k,\lambda) := \frac{\lambda^k e^{-\lambda x}x^{k-1}}{(k-1)!}\mathbb{1}_{x\ge0}$ $\quad \frac{\gamma(k,\lambda x)}{\lambda^k}$

CDF $F_X(x) := P(X\le x) = \int_0^x \frac{\lambda^k e^{-\lambda t}t^{k-1}}{(k-1)!}dt\,\mathbb{1}_{x\ge0} = \frac{\lambda^k}{(k-1)!}\int_0^x t^{k-1}e^{-\lambda t}dt$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)} \frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = P(k,\lambda x)$$

Survival Function

$$1 - F_X(x) = 1 - P(k,\lambda x) = Q(k,\lambda x)$$

$$X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\mathbb{N}_0} \qquad \sum_{i=0}^{\infty} \frac{a^i}{i!}$$

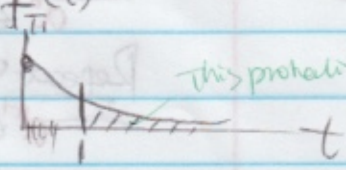
CDF

$$F_X(x) := P(X\le x) = \sum_{t=0}^{x} \frac{\lambda^t e^{-\lambda}}{t!} = e^{-\lambda}\boxed{\sum_{t=0}^{x} \frac{\lambda^t}{t!}} = \cancel{\frac{1}{x!}}$$

$$= \underbrace{\frac{1}{x!}}_{\Gamma(x+1)} \overbrace{e^{-\lambda}\, x! \sum_{t=0}^{x} \frac{\lambda^t}{t!}}^{\Gamma(x+1,\lambda)} = \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

relationship between Erlang and Poisson is known as "Poisson process"

$$T_1 \sim Exp(\lambda) = Erlang(1,\lambda)$$



$\#$ of events $\sim$ Poisson(λ) in one unit time

$$P(T_1 > 1) = Q(\overset{x+1}{1},\lambda)$$

equal

$$N \sim Poisson(\lambda)$$

$$\cancel{P} F_N(0) = P(N\le 0) = P(N=0) = Q(1,\lambda)$$