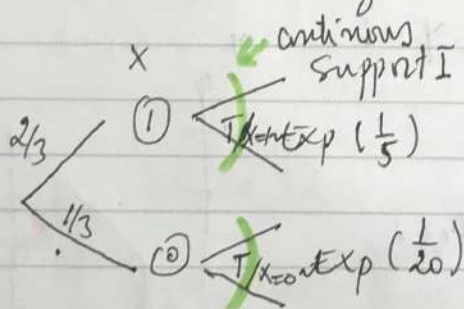


Wednesday October 28, 2020

Lecture 15

Mixture and compound distributions:

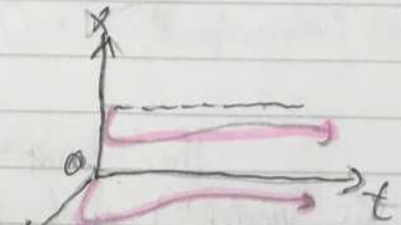
Consider a situation where $2/3$ of the time there is just internet speed so your downloads take $T \sim \text{Exp}(1/5) \Rightarrow E[T] = 5s$ and the other $1/3$ of the time, there is internet traffic, so your downloads take $T \sim (1/20) \Rightarrow (1/20) \Rightarrow E[T] = 20s$. What is the distribution of the $X \sim \text{Bern}(2/3)$ and $X=1$ corresponds to fast internet and $X=0$ corresponds to slow internet. Let's draw a tree diagram:



$$h(x) = \int_{\mathbb{R}} h(x, y) dy$$

$$\parallel \sum_{y \in \mathbb{R}} h(x, y)$$

$$f_T(t) = \sum_{x \in \text{supp}[X]} f_{T|X}(t, x) = \sum_{x \in \text{supp}[X]} f_{T|X}(t, x) P_X(x)$$



$$= \sum_{x \in \{0,1\}} f_{T|X}(t, x) P_X(x) = f_{T|X}(t, 0) P_X(0) + f_{T|X}(t, 1) P_X(1)$$

$$= \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}$$

If download speed was $t = 25s$, what is the probability it is a slow internet day, i.e. $X=0$? $X/T \sim \text{Bern}(?)$

$$P_{X|T}(x|t) = \frac{f_{T|X}(t, x) P_X(x)}{f_T(t)} \quad \text{"Bayes Rule"}$$

Bernoulli param = $P_{X|T}(1, t)$

Bernoulli param = $P_{X|T}(1, t) = \frac{f_{T|X}(t, 1) P_X(t)}{f_T(t)}$

$X|T \sim \text{Bern}(?)$

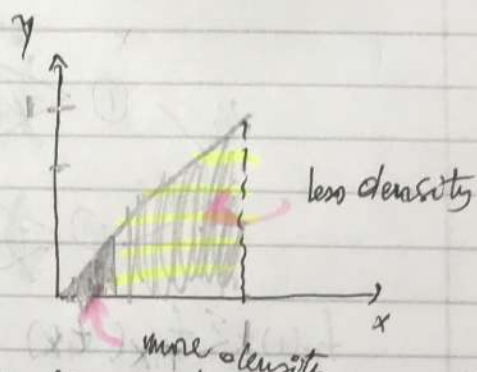
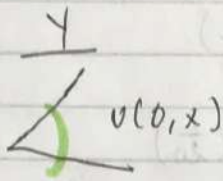
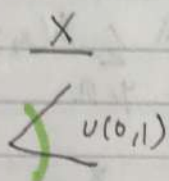
$W \sim \text{Bern}(P)$

$P = P(W=1)$

$$= \frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}$$

$$P_{X|T}(0, 0.25) = 1 - P_{X|T}(1, 0.25) = 1 - \frac{\frac{1}{5} e^{-\frac{1}{5} \cdot 0.25} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20} \cdot 0.25} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5} \cdot 0.25} \cdot \frac{2}{3}} = 0.842$$

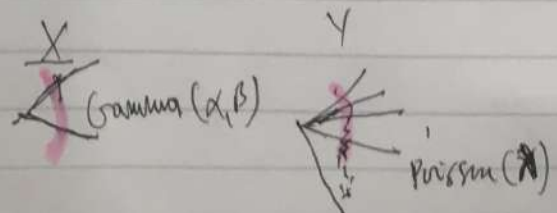
$X \sim U(0, 1), Y|X=x \sim U(0, x) \Rightarrow Y \sim ?$



The first example featured T which was continuous (we call that the "model") and X which is discrete (we call that "mixing" distribution). In the second example Y , the model is continuous and X is discrete. Thus the unconditional distribution T is called a mixture "distribution".

In the second Y , model is continuous and X , the mixing distribution is also continuous and we call the unconditional distribution Y a "compound distribution".

P156 = 157 let $Y|X=x \sim \text{Poisson}(x), X \sim \text{Gamma}(\alpha, \beta), Y \sim ?$



$$\begin{aligned}
 P_Y(y) &= \int_{\text{Supp}[X]} P_{Y|X}(y, x) f_X(x) dx = \int_0^\infty \frac{e^{-x} x^y}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}}
 \end{aligned}$$

let's integral

= ^{HW} Ext Neg Bin $(\alpha, \frac{\beta}{\beta+1})$ This is a more flexible count model than the poisson.

$Y|X=x \sim \text{Bin}(n, x)$ where n is known, $X \sim \text{Beta}(\alpha, \beta)$

X
Beta (α, β)

Y
Bin (n, x)

$$\begin{aligned}
 P_Y(y) &= \int_{\text{Supp}[X]} P_{Y|X}(y, x) f_X(x) dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0, \dots, n\}} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} dx \\
 &= \frac{\binom{n}{y} \mathbb{1}_{y \in \{0, 1, \dots, n\}}}{B(\alpha, \beta)} \int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1} dx = \frac{B(y+\alpha, n-y+\beta)}{B(\alpha, \beta)} \binom{n}{y} \mathbb{1}_{y \in \{0, \dots, n\}}
 \end{aligned}$$

$Y|X=x \sim \text{Exp}(\lambda)$, $X \sim \text{Gamma}(\alpha, \beta) \xRightarrow{\text{HW}} Y \sim \text{Lomax}(\beta, \alpha)$

which is a more flexible waiting time than the exponential

that of Markov III material

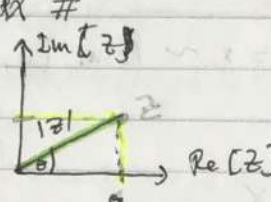
Start "Formal"

Moment generating functions (mgf's) and Characteristic functions (chf's).
To derive these, we need to review Complex / Imaginary numbers. First define $i = \sqrt{-1}$ "imaginary"

Let $a, b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, Complex number

$\text{Re}[z] := a$, $\text{Im}[z] := b$, real component and imaginary component of a

$|z| = \sqrt{a^2 + b^2}$, $\text{Arc}[z] := \theta \stackrel{\text{Complex \#}}{=} \arctan\left(\frac{b}{a}\right)$



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