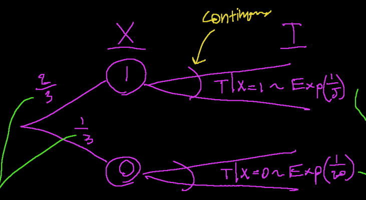
Mixture and compound distributions

e.g. 1/3 of the time, you get bad Internet traffic and your download speeds are T ~ Exp(1/20) i.e. E[T] = 20s and 2/3 of the time you have good Internet traffic and your download speeds are T ~ Exp(1/5) i.e. E[T] = 5st. What is the distribution of T "overall"?

Let X ~ Bern(2/3), a rv modeling traffic. If X = 1, then we have good traffic and if X = 0, we have bad traffic. So now we have T | X = 1 ~ Exp(1/5) and T | X = 0 ~ Exp(1/20). Now we essentially use marginalization to get T "unconditional" (meaning overall). First let's draw a tree:



Marginalization:

$$h(x) = \int_{\mathbb{R}} h(x,y)\,dy$$

— or —

$$h(x) = \sum_y h(x,y)$$

$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\, p_X(x) = \sum_{x \in \{0,1\}} f_{T|X}(t,x)\, p_X(x)$$

$$= f_{T|X}(t,0)\, p_X(0) + f_{T|X}(t,1)\, p_X(1) = \frac{1}{20} e^{-\frac{1}{20}t}\, \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t}\, \frac{2}{3}$$

This was a our first "mixture model" where generally Y|X is the model and X is the mixing distribution.

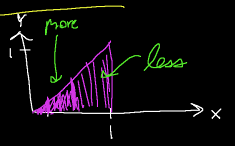
If the download took t = 25s, what is the probability you had bad traffic? Let's find the distribution of traffic conditional on t=25s.

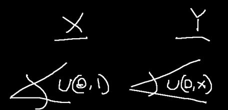
$$p_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, p_X(x)}{f_T(t)} = Bern(\,?\,) \qquad X|T \sim$$

$$p_{X|T}(1,t) = \frac{f_{T|X}(t,1)\, p_X(1)}{f_T(t)} = \underbrace{\frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}}$$

$$p_{X|T}(0,25) = 1 - p_{X|T}(1,25) = 1 - 0.158 = 0.842.$$

$$X \sim U(0,1), \quad Y|X=x \sim U(0,x)$$





The model Y|X is continuous and the mixing distribution is also continous. Thus, Y is a "compound distribution".

$$f_Y(y) = \int_{Supp[X]} f_{Y|X}(y,x)\, f_X(x)\,dx$$

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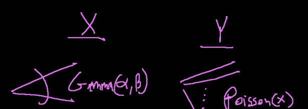
p156-157 let $Y|X=x \sim Poisson(x)$ and $X \sim Gamma(\alpha, \beta)$, $Y \sim ?$



$$p_Y(y) = \int_{Supp[X]} p_{Y,X}(y,x)\,dx = \int_{Supp[X]} p_{Y|X}(y,x)\, f_X(x)\,dx = \int_0^\infty \frac{x^y e^{-x}}{y!} \mathbb{1}_{y \in \{0,1,\ldots\}}$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \overbrace{\int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\,dx}^{\text{lec. 9}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} \mathbb{1}_{y \in \mathbb{N}_0} \overset{HW}{=} \ldots \propto ExtNegBin\left(\alpha, \frac{\beta}{1+\beta}\right)$$

this is a "more flexible" count distribution than the Poisson.

$Y|X=x \sim Bin(n, \overset{p}{x})$, n fixed, $X \sim Beta(\alpha, \beta)$. $Y \sim ?$

this is analogue to the problem above because binomial is also a count distribution with a fixed upper bound, n.

$$p_Y(y) = \int_{Supp[X]} p_{Y,X}(y,x)\,dx = \int_{Supp[X]} p_{Y|X}(y,x)\, f_X(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y}\, \mathbb{1}_{y \in \{0,\ldots,n\}}$$

$$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y \in \{0,\ldots,n\}} \overbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{\beta+n-y-1}\,dx}^{B(y+\alpha,\, \beta+n-y)}$$

$$= \frac{B(y+\alpha, \beta+n-y)}{B(\alpha,\beta)} \binom{n}{y} \mathbb{1}_{y \in \{0,\ldots,n\}} = BetaBinomial(n, \alpha, \beta)$$

HW: $Y|X=x \sim Exp(x)$, $X \sim Gamma(\alpha, \beta) \Rightarrow Y \sim Lomax(\beta, \alpha)$

Midterm II ↑

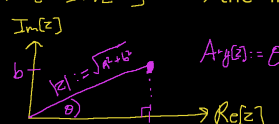
Final ↓

Moment Generating Functions (mgf's) and Characteristic Functions (chf's). We first need to review imaginary #'s and high school trig.

$a, b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, the complex #'s, i := sqrt(-1)

let $Re[z] := a$, the "real" component of the imaginary # z.

let $Im[z] := b$, the "imaginary" component of the imaginary # z



$Arg[z] := \theta \overset{\text{usually}}{=} \arctan\left(\frac{b}{a}\right)$ modulo the signs of a,b.

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = i^2 i^2 = 1, \quad i^c \ldots = i^{c \bmod 4} \text{ clock 4 system}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \ldots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots -$$

$$i\sin(tx) = itx - i\frac{t^3x^3}{3!} + i\frac{t^5x^5}{5!} - \ldots$$

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \Rightarrow e^{i\theta} = i\sin(\theta) + \cos(\theta) \overset{\theta=\pi}{\Rightarrow} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$

Euler's Formula