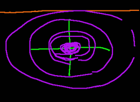
$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z\sim N(0,1)}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{U\sim\chi^2_{n-1}}}{n-1}}} \sim T_{n-1}$$

due to Cochran's thm, if X_i's are iid N(mu, sigs) =>
Xbar and S^2 are independent and thus numerator and denom-
inator here are independent

The Multivariate Normal rv (MVN)



$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \text{ s.t. } Z_1,\dots,Z_n \overset{iid}{\sim} N(0,1), \quad E[\vec{Z}] = \vec{0}_n, \quad Var[\vec{Z}] = I_n$$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}, I)$$

let $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} = \vec{Z} + \vec{\mu} = \begin{bmatrix} Z_1+\mu_1 \\ \vdots \\ Z_n+\mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1,1) \\ \\ \sim N(\mu_n,1) \end{matrix} \sim N_n(\vec{\mu}, I)$

let $A = \underset{n\times n}{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}$, $\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1+Z_2 \\ Z_1+Z_2+Z_3 \\ \vdots \\ Z_1+Z_2+\dots+Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \vdots \\ \sim N(0,n) \end{matrix} \sim N_n(\vec{0}, ?)$

$$\sigma_{1,2} = Cov[X_1,X_2] = Cov[Z_1, Z_1+Z_2] = Cov[Z_1,Z_1] + Cov[Z_1,Z_2] = 1+0 = 1$$

$$Var[A\vec{Z}] = A\,Var[\vec{Z}]\,A^T = AA^T$$

General rule to figure out variance-covariance matrix of matrix A times rv vector X:

$$Var[A\vec{X}] := E[(A\vec{X})(A\vec{X})^T] - E[A\vec{X}]E[A\vec{X}]^T$$
$$= E[A\vec{X}\vec{X}^TA^T] - E[A\vec{X}]E[A\vec{X}]^T$$
$$= A\,E[XX^T]A^T - A\,E[\vec{X}](A\,E[\vec{X}])^T$$
$$= A\,E[XX^T]A^T - A\,E[X]E[X]^TA^T$$
$$= A\big(\underbrace{E[XX^T] - E[\vec{X}]E[\vec{X}]^T}_{\Sigma}\big)A^T$$
$$= A\Sigma A^T$$

$$\vec{X} = \underbrace{A\vec{Z}+\vec{\mu}}_{g(\vec{Z})} \sim N_n(\vec{\mu}, AA^T) = f_{\vec{X}}(\vec{x}) = ?$$

$$\Downarrow$$

$$\vec{Z} = \underbrace{A^{-1}}_{B}(\vec{X}-\vec{\mu}) = h(\vec{X})$$

In order for g to be 1:1, the matrix A must be invertible.

$$= B\vec{X} - B\vec{\mu} \Rightarrow \begin{matrix} h_1(\vec{x}) = \vec{b}_{1\cdot}\vec{X} - \vec{b}_{1\cdot}\vec{\mu} \\ \vdots \\ h_n(\vec{x}) = \vec{b}_{n\cdot}\vec{X} - \vec{b}_{n\cdot}\vec{\mu} \end{matrix}$$

$$J_h = \det\begin{bmatrix} \partial h_1/\partial x_1 & \cdots & \partial h_1/\partial x_n \\ \vdots & & \vdots \\ \partial h_n/\partial x_1 & \cdots & \partial h_n/\partial x_n \end{bmatrix} = \det\overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B} = \det[A^{-1}] = \frac{1}{\det[A]}$$

Note: $AA^{-1} = I \Rightarrow \det[AA^{-1}] = 1 \Rightarrow \det[A]\det[A^{-1}] = 1$

$\Rightarrow (AA^{-1})^T = I^T = I \Rightarrow (A^{-1})^TA^T = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))|J_h| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-\vec{\mu}))^T(A^{-1}(\vec{x}-\vec{\mu}))} \frac{1}{|\det[A]|}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T(A^{-1})^TA^{-1}(\vec{x}-\vec{\mu})} = \frac{1}{\sqrt{(2\pi)^n\det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$$\Sigma = Var[\vec{X}] = AA^T \Rightarrow \Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$$

$$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T] = \det[A]^2$$

$$= \frac{1}{\sqrt{(2\pi)^n\det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})} = f_{\vec{X}}(\vec{x}) = N_n(\vec{\mu}, \Sigma)$$

Does this work is A is m x n? The answer is no... but we will solve that another way.

Multivariate chf's.

$$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1+\dots+t_nX_n)}] = E[e^{it_1X_1}e^{it_2X_2}\cdot\ldots\cdot e^{it_nX_n}]$$

if $X_1,\dots,X_n$ indep.

$$= E[e^{it_1X_1}]E[e^{it_2X_2}]\cdot\ldots\cdot E[e^{it_nX_n}] = \phi_{X_1}(t_1)\phi_{X_2}(t_2)\cdot\ldots\cdot\phi_{X_n}(t_n)$$

P0 $\phi_{\vec{X}}(\vec{0}) = 1$

P1 Yes!

P2 $\vec{Y} = A\vec{X}+\vec{b} \Rightarrow \phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{i\underbrace{\vec{t}^TA}_{(A^T\vec{t})^T}\vec{X}}e^{i\vec{t}^T\vec{b}}]$

$$= e^{i\vec{t}^T\vec{b}}\phi_{\vec{X}}(\vec{t}') = e^{i\vec{t}^T\vec{b}}\phi_{\vec{X}}(A^T\vec{t})$$

Let's find the chf of the standard MVN, $\vec{Z} \sim N_n(\vec{0}, I)$

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n}\phi_Z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

$$\vec{X} = A\vec{Z}+\vec{\mu} \sim N(\vec{\mu}, \underset{AA^T}{\Sigma}), \quad \phi_{\vec{X}}(\vec{t}) \overset{P2}{=} e^{i\vec{t}^T\vec{\mu}}\phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}}e^{-\frac{1}{2}\underbrace{(A^T\vec{t})^T}_{\vec{t}^TA}A^T\vec{t}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$$\vec{X} \sim N(\vec{\mu}, \Sigma), \quad \vec{Y} = B\vec{X}+\vec{c}, \quad B \in \mathbb{R}^{m\times n}, \quad \vec{c} \in \mathbb{R}^m$$

$$\phi_{\vec{Y}}(\vec{t}) \overset{P2}{=} e^{i\vec{t}^T\vec{c}}\, e^{i\overbrace{(B^T\vec{t})^T}^{\vec{t}^TB}\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(B\vec{\mu}+\vec{c}) - \frac{1}{2}\vec{t}^TB\Sigma B^T\vec{t}} \overset{P1}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{\mu}+\vec{c}, B\Sigma B^T)$$

$$\vec{X} \sim N_n(\vec{\mu}, \underset{\text{Assume: } AA^T}{\Sigma}), \quad \overbrace{(\vec{X}-\vec{\mu})^T\Sigma^{-1}(\vec{X}-\vec{\mu})}^{\text{scalar rv}} \sim ?$$

If $n=1$, $(x-\mu)\frac{1}{\sigma^2}(x-\mu) = \left(\frac{x-\mu}{\sigma}\right)^2$

the squared "z-score"

$(AA^T)^{-1} = (A^{-1})^TA^{-1}$

$$= (\vec{X}-\vec{\mu})^T(A^{-1})^TA^{-1}(\vec{X}-\vec{\mu})$$
$$= (A^{-1}(\vec{X}-\vec{\mu}))^T(A^{-1}(\vec{X}-\vec{\mu}))$$

$\vec{Z} \sim N_n(\vec{0}, I)$

$$= \vec{Z}^T\vec{Z} \sim \chi^2_n$$

Mahalanobis Distance (1936)