Lecture - 03 09/02/2020

Let $B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$

Possibly an infinite sequence of iid rv's (Bernoulli)

Let $X :=$ # of zero realization before the first realization of one.

also; $X := \min\{t : B_t = 1\} - 1$

derive PMF

$P(0) = P(X=0) = P(\{\text{No } 0\text{'s, just a } 1\}) = p$

$P(1) = P(X=1) = P(\{0, \text{then a } 1\}) = (1-p)\,p$

$P(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2\,p$

$P(x) = P(X=x) = P(\{\underbrace{0,0,\ldots,0}_{x},1\}) = (1-p)^x\,p$

$Supp[X] = \{0,1,2,\ldots\}$

$X \sim Geom(p) := (1-p)^x\,p\,\mathbb{1}_{x \in \{0,1,\ldots\}}$

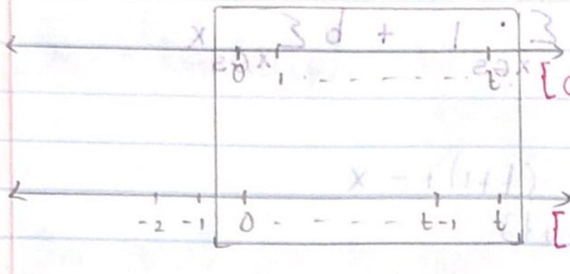
"geometric r.v"

$X_1, X_2 \overset{iid}{\sim} Geom(p)$; $T_2 = X_1 + X_2 \sim P_{T_2}(t) = ?$

$$P_{T_2}(t) = \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(t-x)\,\mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x\,p\,(1-p)^{t-x}\,p\,\underbrace{\underbrace{\mathbb{1}_{t-x \in \{0,1,\ldots\}}}_{x-t \in \{\ldots,-1,0\}}}_{x \in \{\ldots,t-1,t\}}$$

$$= (1-p)^t \cdot p^2 \cdot \sum_{x \in [0,1,\dots]} \mathbb{1}_{x \in [\dots,t-1,t]}$$



$[0,1,\dots]$

$[\dots,t-1,t]$

$$[0,1,\dots] \cap [\dots,t-1,t] = [0,1,\dots,t]$$

$$= (1-p)^t p^2 \sum_{x \in [0,1,\dots,t]} 1$$

The negative binomial rv

$$= \underbrace{(t+1)}_{\binom{t+1}{1}} (1-p)^t p^2 = NegBin(2,p) = P_T(t)$$

$supp[T_2] = [0,1,\dots]$

$$\underbrace{0\ 0\ \dots\ 0\ 1\ 0\ 0\ \dots\ 0}_{t+1 \text{ realizations}}\ \underset{t+2}{1}$$

thus t+1 possible locations for the first 1

$$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p); \quad T_3 = X_1 + X_2 + X_3 \sim P_{T_3}(t) = ?$$

$$= X_3 + T_2$$

$$P_{T_3}(t) = \sum_{x \in Supp[X_3]} P^{old}_{X_3}(x) \cdot P^{old}_{T_2}(t-x) \, \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum_{x \in [0,1,\dots]} (1-p)^x p (t-x+1)(1-p)^{t-x} p^2 \, \mathbb{1}_{t-x \in [0,1,\dots]}$$

$$= (1-p)^t p^3 \sum_{x \in [0,1,\dots]} (t+1-x) \, \mathbb{1}_{x \in [\dots,t-1,t]}$$

$$\sum_{x\in S} a+bx = \sum_{x\in S} a + \sum_{x\in S} bx$$

$$= a\sum_{x\in S} 1 + b\sum_{x\in S} x$$

Precalculus

$$= (1-p)^t p^3 \sum_{x\in\{0,1,\dots,t\}} (t+1)-x$$

$$= (1-p)^t p^3 \left[(t+1)\sum_{x\in\{0,1,\dots,t\}} 1 - \sum_{x\in\{0,1,\dots,t\}} x \right]$$

$$= (1-p)^t t^3 \left(\underbrace{(t+1)(t+1) - \frac{t(t+1)}{2}}_{t^2+2t+1-\frac{t^2}{2}-\frac{t}{2}}\right)$$

$$\frac{t^2+3t+2}{2} = \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$= \binom{t+2}{2}(1-p)^t p^3 = \text{NegBin}(3,p)$$

0 0 . . . . 0 1 0 0 . . . . 0 1 0 0 . . . . 0 1

t+2 realizations

t+ 3

t+2 locations to put 2 ones in $\Rightarrow$ t+2 choose 2

$X_1, X_2, \dots, X_r \overset{iid}{\sim} \text{Geom}(p)$

$T_r := X_1 + X_2 + \dots + X_r \sim \text{NegBin}(r,p) = P_{T_r}(t)$

$$\boxed{\begin{aligned}&\lim \left(1+\tfrac{1}{n}\right)^n = e\\ &\lim \left(1+\tfrac{a}{n}\right)^n = e^a\end{aligned}}$$

$$P_{T_r}(t) = \binom{t+r-1}{r-1}(1-p)^t p^r \, \mathbb{I}_{t \in \{0,1,\ldots\}}$$

$X \sim Bin(n,p)$ Let n be really large and p be really small;
$n \longrightarrow +\infty$, $p \longrightarrow 0$, but lambda $= np$

Our goal is to get PMF of X under this limit.

$$\lambda = np \implies p = \lambda/n$$

$$\lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} \, \mathbb{I}_{x\in\{0,\ldots,n\}}$$

Example $\frac{10!}{(10-4)!} = \frac{10!}{6!} = \underbrace{10.9.8.7}^{4}$

$n=10$, $x=4$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\lambda/n\right)^{n-x} \mathbb{I}_{x\in\{0,1,\ldots,n\}}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{I}_{x\in\{0,1,\ldots n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!\,n^x} \lim\left(1-\frac{\lambda}{n}\right)^n \lim \left(1-\frac{\lambda}{n}\right)^{-x} \lim \mathbb{I}_{x\in\{0,1,\ldots n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x}}{\underbrace{n.n.\cdots\cdots n}_{x}} e^{-\lambda}(1)\,\mathbb{I}_{x\in\{0,1,\ldots\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{I}_{x\in\{0,1,\ldots\}} \quad \cancel{\lim \frac{n}{n}}^{\,1} \; \cancel{\lim \frac{n-1}{n}}^{\,1}, \ldots, \cancel{\lim \frac{n-x+1}{n}}^{\,1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{I}_{x\in\{0,1,\ldots\}} = \text{Poisson}(\lambda).$$

$\lambda \in (0,\infty)$ parameter space

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda), \quad T = X_1 + X_2 \sim P_T(t) = ?$

$$P_T(t) = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \underbrace{\mathbb{I}_{t-x \in \{0,1,\ldots\}}}_{x \in \{\ldots, t-1, t\}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\ldots t\}} \frac{1}{x!(t-x)!}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\ldots,t\}} \frac{t!}{t!} \cdot \frac{1}{x!(t-x)!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,\ldots,t\}} \binom{t}{x}$$

$$\binom{t}{0} + \binom{t}{1} + \ldots + \binom{t}{t} = 2^t$$

$$\frac{0/1}{1} \quad \frac{0/1}{2} \quad \ldots \quad \frac{0/1}{t}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t$$

$$= \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$