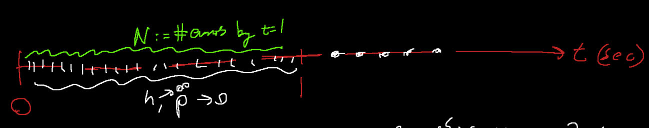
$T_K \sim \text{Erlang}(k,\lambda)$, $N \sim \text{Poisson}(\lambda)$ $\quad X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_K > 1)$
$= 1 - F_{T_K}(1) = Q(k,\lambda)$ $\quad F_N(x) = Q(x+1,\lambda)$

$\Rightarrow 1 - F_{T_K}(1) = F_N(k-1)$ "Poisson Process"



$K=5$ $\quad \{T_5 > 1\} = \{X_1+X_2+X_3+X_4 < 1\} \cup \{X_1+X_2+X_3<1\} \cup$
$\{X_1+X_2<1\} \cup \{X_1<1\} \cup \{X_1>1\}$

$1-F_{T_5}(1) \quad F_N(4)$
$= \{N=4\} \cup \{N=3\} \cup \{N=2\} \cup \{N=1\} \cup \{N=0\}$

$P(T_5>1) = P(N \le 4)$

$$T \sim \text{Erlang}(k,\lambda) = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t\ge 0} = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t \ge 0}$$ "Gamma"

$k \in \mathbb{N}, \lambda \in (0,\infty)$

$$T \sim \text{NegBin}(k,p) = \binom{k+t-1}{k-1}(1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0} = \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$

$k \in \mathbb{N}, p \in (0,1)$ $\quad$ "Extended Negative Binomial"

What if $k \in (0,\infty)$? Is the top PDF legal and the bottom PMF legal? YES

$$\int_0^\infty \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k = 1.$$

which means... these are rv's.

$$X \sim \text{Gamma}(\alpha,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \mathbb{1}_{t\ge 0}$$

$X \sim \text{ExtNegBin}(k,p)$ . . . .

Transformations for Discrete rv's

$$X \sim \text{Bern}(p),\quad Y = X+3 \sim \begin{cases} 3 & \text{wp } 1-p \\ 4 & \text{wp } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y-3\in\{0,1\}}$$

$= p^x(1-p)^{1-x}\mathbb{1}_{x\in\{0,1\}}$

how do I express the transformed PMF using the original PMF?

If $Y = g(X) \sim p_Y(y) = p_X(g^{-1}(y))$

$\Updownarrow$

$g^{-1}(y) = x$

Is this formula general? No. This is only the formula for g invertible. If g non-invertible...

$$X \sim U(\{1,2,\ldots,10\}) = \frac{1}{10}\mathbb{1}_{x\in\{1,2,\ldots,10\}}$$

$$Y = \min\{X,3\} \sim \begin{cases} 1 & \text{wp } \frac{1}{10} \\ 2 & \text{wp } \frac{1}{10} \\ 3 & \text{wp } P(X=3)+P(X=4)+\ldots+P(X=10) = \frac{8}{10} \end{cases}$$

$$p_Y(y) = \sum_{\{x:\, g(x)=y\}} p_X(x) \overset{\text{if } g \text{ invertible for Supp}[X]}{=} \sum_{\{x:\, x=g^{-1}(y)\}} p_X(x) = p_X(g^{-1}(y))$$

$$X \sim \text{Bin}(n,p),\quad Y = X^2 \sim p_Y(y) = p_X(g^{-1}(y)) = p_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}}(1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y}\in\{0,1,\ldots,n\}}$$

$\Downarrow$

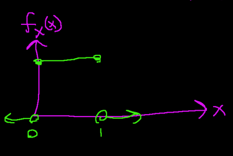
$x = \sqrt{y} = g^{-1}(y)$

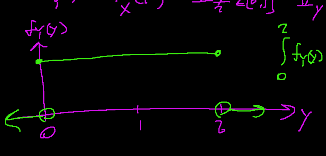
Transformations for Continuous rv's

$Y = g(X)$, X is continuous

for invertible g, $f_Y(y) \overset{?}{=} f_X(g^{-1}y)$ incorrect!

$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]},\quad Y = 2X \sim f_Y(y) = f_X\left(\tfrac{y}{2}\right) = \mathbb{1}_{\frac{y}{2}\in[0,1]} = \mathbb{1}_{y\in[0,2]}$$





$\int_0^2 f_Y(y) = 2-0 = 2 \neq 1$ ILLEGAL

where'd we go wrong? The above formula was derived using probabilities. Densities are not probabilities! But... CDF's are probabilities! Strategy: let's derive the CDF of Y using the CDF of X. And then, like when we did convolutions, take the derivative to get the density for Y.

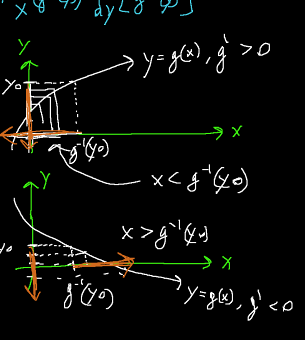
$$F_Y(y) = P(Y\le y) = P(g(X)\le y) \overset{\text{if } g'>0}{=} P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

$$= f_X(g^{-1}(y))\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{Stretching}}$$

$$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$g' > 0 \Rightarrow g^{-1\prime} > 0$



if $g' < 0$

$\downarrow$

$= P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

$$f_Y(y) = \frac{d}{dy}[\quad] = -\frac{d}{dy}[F_X(g^{-1}(y))] = f_X(g^{-1}(y))\left(-\frac{d}{dy}[g^{-1}(y)]\right)$$

$$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$\frac{d}{dy}[g^{-1}(y)] < 0$

$$\Longrightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| \quad \text{for all } g \text{ invertible}$$

Let's derive some more rules! The most common invertible function is.... the straight line! Y = aX + c! $\Rightarrow x = g^{-1}(y) = \frac{y-c}{a}$, $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|a|}$

s.t. $a, c \in \mathbb{R}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$ "shift and scale"

if $c = 0$ just a scale.... $Y = aX$

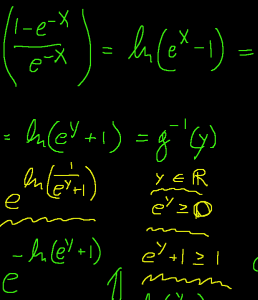
$f_Y(y) = f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$

if $a = 1$ just a shift $Y = X + c$

$f_Y(y) = f_X(y-c)$

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x\ge 0}$

$Y = X + c = \lambda e^{-\lambda(y-c)}\mathbb{1}_{y\ge c}$



$X \sim \text{Exp}(1) = e^{-x}\mathbb{1}_{x\ge 0}$

$$Y = g(X) = -\ln\left(\frac{e^{-X}}{1-e^{-X}}\right) = \ln\left(\frac{1-e^{-X}}{e^{-X}}\right) = \ln(e^X - 1) = y$$

$$\Rightarrow e^y = e^x - 1 \Rightarrow e^y + 1 = e^x \Rightarrow x = \ln(e^y+1) = g^{-1}(y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{e^y}{e^y+1}\right| = \frac{e^y}{e^y+1} \qquad \underbrace{e^{\ln\left(\frac{1}{e^y+1}\right)}} \qquad \underbrace{y\in\mathbb{R}} \quad \underbrace{e^y \ge 0}$$

$$f_Y(y) = f_X(\ln(e^y+1))\frac{e^y}{e^y+1} = e^{-\ln(e^y+1)}\,\mathbb{1}_{\underbrace{\ln(e^y+1)\ge 0}_{\overbrace{e^y+1\ge 1}}}\frac{e^y}{e^y+1}$$

$$= \frac{1}{e^y+1}\frac{e^y}{e^y+1} = \frac{e^y}{(e^y+1)^2}\cdot\frac{e^{-2y}}{e^{-2y}} = \text{Logistic}(0,1)$$

$$\shortparallel$$

$$\frac{e^{-y}}{(e^{-y}+1)^2}$$