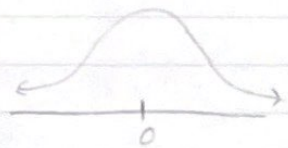
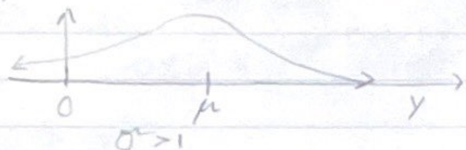
$X \sim (0,1) \approx N(0,1)$



Standard logistic $E[X] = 0$, $SD[X] = \frac{\pi}{\sqrt{3}}$

$Y = \mu + \sigma X \sim \text{Logistic}(\mu, \sigma) := f_Y(y) = \frac{1}{\sigma} \frac{e^{\frac{y-\mu}{\sigma}}}{(e^{\frac{y-\mu}{\sigma}}+1)^2}$

$\mu \in \mathbb{R}$, $\sigma > 0$ all reals

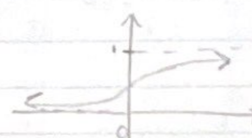


Why is this called the "logistic distribution"?
There's a function called the "logistic function" and it has 3 parameters:
L (max. value), K (steepness), μ (center)

$\ell(x) := \frac{L}{1 + e^{-K(x-\mu)}} = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$

if $L=1, K=1, \mu=0$

the standard logistic function



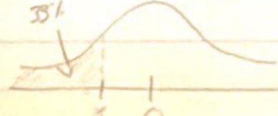
$X \sim \text{Logistic}(0,1) = \frac{e^y}{(1+e^y)^2}$

$F_Y(y) := P(Y \le y) = \int_{-\infty}^{y} \frac{e^t}{(1+e^t)^2} dt = \int_1^{1+e^y} \frac{u-1}{u^2} \frac{1}{u-1} du = \left[-\frac{1}{u}\right]_1^{1+e^y} = \left[1 - \frac{1}{1+e^y}\right] = \frac{e^y}{1+e^y}$

let $u = 1+e^t \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t} du = \frac{1}{u-1} du$, $t = -\infty \Rightarrow u = 1$, $t = y \Rightarrow u = 1+e^y$

$e^t = u-1$

The q^{th} "quartile" or $100q$ "percentile" of a distribution. Definition: Solve for x where $q \ge P(X \le x) = F_X(x)$.



I want 33^{rd} %ile. If $q = 0.5$, that quartile is called the "median", $Md[X]$.

$X \sim U(\{2, 4, \dots, 20\})$

x	$p(x)$	$F(x)$
2	0.1	0.1
4	0.1	0.2
6		0.3
8		0.4
10		0.5
12		0.6
14		0.7
16		0.8
18		0.9
20		1.0

$Q[X, 0.3] = 6$

$Q[X, 0.9] = 18$

$Q[X, 0.85] = 16 \quad \not\forall_x \; F(x) = 0.85$

if X is a continuous rv with "continuous support" i.e. one interval with no gaps e.g. $[0,10]$, the real numbers but not e.g. $[0,1] \cup [2,3]$ (gap from 1 to 2), then $F(x)$ is strictly increasing thus invertible and the minimum x s.t. $q \geq F(x)$ would be $F^{-1}(q) \geq x \Rightarrow x = \underbrace{F^{-1}(q)}_{\text{quantile function}} = Q[X, q]$

$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \Rightarrow F(x) = 1 - e^{-\lambda x}$. Find the quantile function, $F^{-1}(q)$

$q = 1 - e^{-\lambda x} \Rightarrow 1 - q = e^{-\lambda x} \Rightarrow \ln(1-q) = -\lambda x \Rightarrow x = -\frac{1}{\lambda}\ln(1-q)$

$\Rightarrow x = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right) = F^{-1}(q).$

$X \sim Exp(1) \Rightarrow Med[X] = F^{-1}(0.5) = \ln(2)$

$Q[X, 0.8] = \ln(5)$

it's rare to have a quantile function in closed form since it's rare to have a CDF in closed form. e.g.

$X \sim Erlang(K, \lambda)$, $F(x) = P(K, \lambda x)$. $Q[X, q]$ can be found by solving for x in the following equation: $q = P(K, \lambda x)$.

$X \sim Exp(\lambda)$, $Y = Ke^{X}$ $=g(X)$, $K>0$. Find $f_Y(y)$

$$\frac{Y}{K} = e^{X} \Rightarrow X = \ln\left(\frac{Y}{K}\right) = \ln(y) - \ln(K) = g^{-1}(y)$$

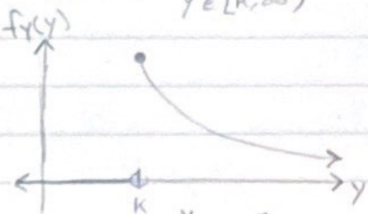
$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|} \underset{\uparrow}{=} \frac{1}{y}, \qquad f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| \Rightarrow$$

Since y always pos.

$$\Rightarrow = \lambda e^{-\lambda \ln\left(\frac{y}{K}\right)}\frac{1}{y}\mathbb{1}_{\ln y - \ln K \in [0,\infty)} = \frac{\lambda}{y} e^{\ln\left(\left(\frac{y}{K}\right)^{-\lambda}\right)}\mathbb{1}_{\ln y \in [\ln(K),\infty)}$$

$$= \frac{\lambda}{y}\left(\frac{y}{K}\right)^{-\lambda}\mathbb{1}_{y\in[K,\infty)} = \text{Paredo I}(K,\lambda)$$

$K \in (0,\infty)$, $\lambda \in (0,\infty)$



$$F_Y(y) = \int_K^y \frac{\lambda K^{\lambda}}{t^{\lambda+1}}\,dt = \lambda K^{\lambda}\left[-\frac{1}{\lambda t^{\lambda}}\right]_K^y = K^{\lambda}\left[\frac{1}{K^{\lambda}} - \frac{1}{y^{\lambda}}\right] = 1-\left(\frac{K}{y}\right)^{\lambda} = q$$

$$F^{-1}_Y(q) = K(1-q)^{-\frac{1}{\lambda}}$$

Remember, exponential is a survival/waiting time r.v. So is the Pareto. The Pareto is used to model population spread, hard-drive time-to-failure. There is also the "Pareto Principle". In 1896, Vilfredo Pareto noticed that 80% of the land in Italy was owned by 20% of the people. That is a property of a specific Pareto distribution, Pareto I (1, 1.161)

$X, Y \overset{iid}{\sim} \underset{\underset{x \in [0,\infty)}{e^{-x}\mathbb{1}}}{\underset{\shortparallel}{Exp(1)}}$, let $D = X - Y = X + \overset{Z}{(-Y)}$

$$Z \sim f_Z(z) = e^{z}\,\mathbb{1}_{z \in (-\infty, 0]}$$

$$f_D(d) = \int_{Supp[X]} f_X^{old}(x)\, f_Z^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Z]}\, dx = \int_0^{\infty} e^{-x} e^{d-x}\, \mathbb{1}_{\substack{d-x \in (-\infty, 0] \\ x \in [d, \infty)}}\, dx =$$

$$= e^{d} \begin{cases} \int_d^{\infty} e^{-2x}\,dx & \text{if } d \geq 0 \\ \int_0^{\infty} e^{-2x}\,dx & \text{if } d < 0 \end{cases} = e^{d} \begin{cases} \left[-\frac{1}{2}e^{-2x}\right]_d^{\infty} & \text{if } d \geq 0 \\ \left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} & \text{if } d < 0 \end{cases}$$

$$= \frac{1}{2}e^{d} \begin{cases} e^{-2d} & \text{if } d \geq 0 \\ 1 & \text{if } d < 0 \end{cases} = \begin{cases} \frac{1}{2}e^{-d} & \text{if } d \geq 0 \\ \frac{1}{2}e^{d} & \text{if } d < 0 \end{cases}$$

$$= \frac{1}{2}e^{-|d|} = \underset{\text{Standard Laplace}}{Laplace(0,1)}$$

$$X = \mu + \sigma D \sim Laplace(\mu, \sigma) := \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

$\mu \in \mathbb{R}$, $\sigma > 0$

this is also a famous rv and it has another name: the "double exponential".
Laplace published this distribution in 1774 calling it the "first law of errors".