$$f_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j}\left(j f(x) F(x)^{j-1}(1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

$n-1$ if $j=n$

$$= \sum_{j=k}^{n} \frac{n!}{\cancel{j!}_{(j-1)!}(n-j)!} \cancel{j} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{\cancel{n}\, n-1} \frac{n!}{j!\,\cancel{(n-j)!}_{(n-j-1)!}} \underbrace{\cancel{(n-j)}}_{0} f(x) F(x)^{j}(1-F(x))^{n-j-1}$$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!\,\underbrace{(n-j-1)!}_{(n-(j+1))!}} f(x) F(x)^{j}(1-F(x))^{\overbrace{n-j-1}^{n-(j+1)}}$$

reindexing trick for. Let $l = j+1 \Rightarrow j = l-1 \Rightarrow j=k \Rightarrow l = k+1$, $\Rightarrow j = n-1 \Rightarrow l = n$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{l=k+1}^{n} \frac{n!}{(l-1)!\,(n-l)!} f(x) F(x)^{l-1}(1-F(x))^{n-l}$$

note that both sum expressions are exactly the same, so when we subtract we're left with just the expression when j=k

$$= \frac{n!}{(k-1)!\,(n-k)!} f(x) F(x)^{k-1}(1-F(x))^{n-k} = f_{X_{(k)}}(x) \checkmark$$

Let's make sure we can uncover the min/max formulas:

$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!\,(n-1)!} f(x) F(x)^{1-1}(1-F(x))^{n-1} = n f(x)(1-F(x))^{n-1} \checkmark$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x) F(x)^{n-1}(1-F(x))^{n-n} = n f(x) F(x)^{n-1} \checkmark$$

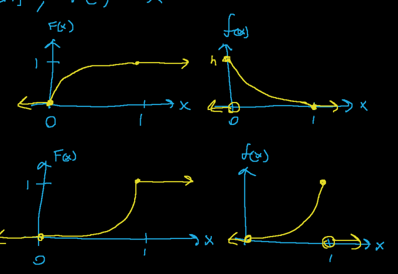
$$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1) = \overbrace{1\, \mathbb{1}_{x\in[0,1]}}^{f(x)}, \quad F(x) = x$$

$$F_{X_{(1)}}(x) = 1-(1-F(x))^n = 1-(1-x)^n$$

$$f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

$$F_{X_{(n)}}(x) = F(x)^n = x^n$$

$$f_{X_{(n)}}(x) = n x^{n-1}$$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in[0,1]} = \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k+1-1} \mathbb{1}_{x\in[0,1]}$$

$$= \text{Beta}(k, n-k+1)$$

$X \sim \underbrace{\text{Gamma}(\alpha_1, \beta)}_{\text{"Erlang}(\alpha_1,\beta)\text{"}}$ independent of $Y \sim \underbrace{\text{Gamma}(\alpha_2,\beta)}_{\text{Erlang}(\alpha_2,\beta)}$, $T = X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

reasonable...

To prove this, we develop a new tool that makes it easier for us. That's "kernels", k(x). For any PMF or PDF, we can decompose it into a normalization constant c and a kernel k(x)

$$p(x) = c\,k(x) \quad \text{and} \quad f(x) = c\,k(x) \quad \Rightarrow \quad p(x) \propto k(x), \quad f(x) \propto k(x)$$

not a function of x

$$1 = \sum_{Supp} p(x) = \sum c\,k(x) \Rightarrow \frac{1}{c} = \sum k(x) \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int_{Supp} f(x)\,dx = \int c\,k(x)\,dx \Rightarrow \frac{1}{c} = \int k(x)\,dx \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

this means that k(x) is 1:1 with the PMF or PDF. If you know k(x). you know the distribution of the rv. Let's see some examples:

$$X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x\in\{0,1,\ldots,n\}} = \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!\,(n-x)!}\left(\frac{p}{1-p}\right)^x \mathbb{1}_{\ldots}}_{k(x)}$$

$$X \sim \text{Weibull}(k,\lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y\geq 0} = \underbrace{k\lambda^k}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y\geq 0}}_{k(x)}$$

$$X \sim \text{Gamma}(\alpha,\beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\geq 0}}_{k(x)} \propto \underbrace{x^a e^{-bx}}\, \mathbb{1}_{x\geq 0}$$

$X \sim \text{Gamma}(\alpha_1,\beta)$ ind. of $Y \sim \text{Gamma}(\alpha_2,\beta)$, $T = X+Y \sim f_T(t) = ?$

$$f_T(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t}\cancel{e^{\beta x}}} \mathbb{1}_{t-x\in[0,\infty)}\,dx$$

$$x-t \in (-\infty, 0]$$
$$x \in (-\infty, t]$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}\,dx\, \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}\,dx\, \mathbb{1}_{t\geq 0} = e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1}\left(1-\frac{x}{t}\right)^{\alpha_2-1} dx\, \mathbb{1}_{t\geq 0}$$

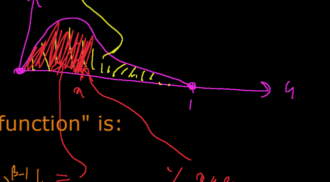
let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$, $x=0 \Rightarrow u=0$, $x=t \Rightarrow u=1$

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, t\,du\, \mathbb{1}_{t\geq 0} = e^{-\beta t} t^{\alpha_1+\alpha_2-1} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du}_{B(\alpha_1,\alpha_2)} \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t} t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta) \checkmark$$

Let's talk about the "beta function", a famous ubiquitous function.

$$B(\alpha,\beta) := \int_0^1 \overbrace{u^{\alpha-1}(1-u)^{\beta-1}}^{g(u)}\,du =$$ not available in closed form



The "incomplete beta function" is:

$$B(a,\alpha,\beta) = \int_0^a u^{\alpha-1}(1-u)^{\beta-1}\,du =$$

%age

The regularized incomplete beta function is:

$$I_a(\alpha,\beta) := \frac{B(a,\alpha,\beta)}{B(\alpha,\beta)} \in [0,1]$$

not a Bessel function!

Let's derive a beta function - gamma function identity

$$T \sim \text{Gamma}(\alpha_1+\alpha_2,\beta) = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)} \cancel{t^{\alpha_1+\alpha_2-1}} \cancel{e^{-\beta t}} \cancel{\mathbb{1}_{t\geq 0}} = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} B(\alpha_1,\alpha_2) \cancel{t^{\alpha_1+\alpha_2-1}} \cancel{e^{-\beta t}} \cancel{\mathbb{1}_{t\geq 0}}$$

$$\Rightarrow B(\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$ Cool identity!

$$X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in[0,1]} \qquad \alpha,\beta > 0$$

$$1 = \int_{Supp(X)} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{\cancel{B(\alpha,\beta)}} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}_{\cancel{B(\alpha,\beta)}} = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)}$$

$$= I_x(\alpha,\beta)$$