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$a_{i\cdot} :=$ i^{th} row vector of A

$A \in \mathbb{R}^{L\times K} \to$ constants

$$E[A\vec{X}] = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \cdots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \cdots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \cdots a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1, \vec{X}] \\ E[\vec{a}_2, \vec{X}] \\ \vdots \\ E[\vec{a}_L, \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1\vec{\mu} \\ a_2\vec{\mu} \\ \vdots \\ a_L\vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$(L\times K)(K\times 1) = L\times 1$

$\vec{a} \in \mathbb{R}^K$

$$Var[\vec{a}^T\vec{X}] = Var[\underbrace{a_1X_1}_{Y_1} + \underbrace{\cdots}_{Y_2} + a_KX_K]$$
$$= Var[Y_1 + \cdots + Y_K] = \sum_{i=1}^{K}\sum_{j=1}^{K} cov[Y_i, Y_j]$$
$$= \sum\sum cov[a_iX_i, a_jX_j] = \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_j\sigma_{ij}$$
$$= \vec{a}^T\Sigma\vec{a} = Var[\vec{X}]$$

$\hookrightarrow (1\times K)(K\times K)(K\times 1) =$ Scalar

Let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^K$

$$\vec{a}^TV\vec{a} = \vec{a}\cdot(V\vec{a}) = \vec{a}\cdot \begin{bmatrix} a_1V_{11} + \cdots + a_KV_{1K} \\ a_1V_{21} + \cdots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \cdots + a_KV_{KK} \end{bmatrix} = \begin{matrix} a_1a_1V_{11} + \cdots + a_1a_KV_{1K} \\ a_2a_1V_{21} + \cdots + a_2a_KV_{2K} \\ \vdots \\ a_Ka_1V_{K1} + \cdots + a_Ka_KV_{KK} \end{matrix}$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{K} a_ia_jV_{ij}$$

Quadratic Forms w/ V being the "determining Matrix"

* Application in finance. Let $X_1, X_2, \ldots, X_K$ be financial assets (e.g. stocks). So let $W_1, W_2, \ldots, W_K$ be the proportion allocated to each of these assets. Let $F = \vec{W}^T\vec{X}$ a rv modeling your portfolio and let $\vec{\mu} = E[\vec{X}]$, $\Sigma = Var[\vec{X}]$

$\hookrightarrow E[F] = \vec{W}^T\vec{\mu} = \mu_F$, $Var[F] = \vec{w}^T\Sigma\vec{w}$ It's possible to pick W-vector to optimize the portfolio by minimizing the variance of return, $Var[F]$, conditional on μ_F

This is called "Markowitz optimal portfolio theory."

$\min_{\vec{w}} Var[F]$ subject to μ_F being constant and $\vec{w}^T\vec{1} = 1$

$\vec{X} \sim \text{Multin}_k\,(n, \vec{p})$. $\quad E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$

$X_j \sim \text{Bin}\,(n, p_j)$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & & \\ & n(p_2)(1-p_2) & \sigma_{ij} & \\ & & \ddots & \\ \sigma_{ij} & & & n(p_k)(1-p_k) \end{bmatrix}$$

$Cov[X_i, X_j] = E[X_i X_j] - E[X_i]E[X_j]$
$i \neq j$

$= \sum_{x_i \in \mathbb{R}} \sum_{x_j \in \mathbb{R}} x_i x_j \, p_{X_i, X_j}(x_i, x_j) - n^2 p_i p_j$

$\vec{X}$ ↳ complicated → fail...

i = apple
j = banana

Remember $\begin{bmatrix} \vdots \\ X_i \sim \text{Bin}\,(n, p_i) \\ X_j \sim \text{Bin}\,(n, p_j) \\ \vdots \end{bmatrix}$ $\quad X_i = X_{1i} + X_{2i} + \dots + X_{ni}$ where $X_{1i} \dots X_{ni} \overset{iid}{\sim} \text{Bern}(p_i)$

$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} \text{Bern}(p_j)$

↳ We've expressed the multinomial r.v. w/ $n \times k$ Bernoulli.

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \overset{iid}{\sim} \text{Multin}_k\,(1, \vec{p})$

$COV[X_i, X_j] = COV[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$

$= \sum_{l=1}^{n} \sum_{m=1}^{n} COV[X_{li}, X_{mj}]$

A lot of these covariances are zero due to independence.

Which one? If l is different than m, the covariance is zero

$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$

$= \sum_{l=1}^{n} (E[X_{li} X_{lj}] - \underbrace{E[X_{li}]}_{p_i} \underbrace{E[X_{lj}]}_{p_j})$

$\sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \, p_{X_{li}, X_{lj}}(x_{li}, x_{lj})$

$= p_{X_{li}, X_{lj}}(1,1) = 0$ → You can't get an apple AND a banana on one grab

the only term that's nonzero is...

$= -np_i p_j$

PMF of X

uniform discrete

$X \sim U(\{0,1,2,3\}) = \begin{cases} 0 & \text{wp } 1/4 \\ 1 & '' \\ 2 & '' \\ 3 & '' \end{cases}$

Generally $X \sim U(A)$

$\text{Supp}[\cancel{X}] = A$, $A \subset \mathbb{R}$ s.t. $|A| < \infty$ and $|A| \geq 1$ or $A \neq \phi$

$\text{Supp}[X] = \{0,1,2,3\}$

Create a new rv $Y = -X = g(X)$, a very simple function.

$\text{Supp}[Y] = \{-3,-2,-1,0\}$

PMF $P(Y) = \begin{cases} -3 & \text{with prob } 1/4 \\ -2 & '' \\ -1 & '' \\ 0 & '' \end{cases}$

$$P_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) =: p_X(-y)$$

Generally, for discrete rv X, is there a pattern?

$$\text{Supp}[Y] = \{z : P_Y(z) > 0\} = \{z : p_X(-z) > 0\}$$
$$= \{-z : p_X(z) > 0\} = -\{z : p_X(z) > 0\} =: -\text{Supp}[X]$$

$X_1 X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1,\dots\}}$

In class we showed: $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

Let $D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} = X + Y$, $Y \sim p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\dots,-1,0\}}$

$P_D(d) = \sum_{x \in \text{Supp}[X]} P_X^{old}(x) P_Y^{old}(d-x) \mathbb{1}_{d-x \in \text{Supp}[Y]}$

$\text{Supp}[D] = \text{Supp}[X] + \text{Supp}[Y] = \mathbb{Z}$ all integer