$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ From previous class $X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$D = X_1 - X_2 \sim ?$
(difference)
$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$

$p_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!}$ $\text{Supp}[X] = \{0, 1, 2, \ldots\}$, $\text{Supp}[Y] = \{\ldots, -2, -1, 0\}$

$\text{Supp}[X+Y] = \text{Supp}[X] + \text{Supp}[Y] = \overset{\text{all integers}}{\mathbb{Z}}$

convolution formula for independent discrete rv's

$$p_T(t) = \sum_{x \in \text{Supp}[X]} p_X^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{\overbrace{-(d-x)}^{x-d}} e^{-\lambda}}{(\underbrace{-(d-x)}_{x-d})!} \mathbb{1}_{d-x \in \{\ldots,-1,0\}}$$

$x - d \in \{0, 1, \ldots\}$, $x \in \{d, d+1, \ldots\}$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \in \{d, d+1, \ldots\}}$$

let $d' = -d = |d'|$

$$= e^{-2\lambda} \begin{cases} \text{if } d \le 0 & \displaystyle\sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} \\ \text{if } d > 0 & \displaystyle\sum_{x \in \{d,d+1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x' \in \{0,1,\ldots\}} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!\,x'!} \end{cases} = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$$

let $x' = x - d \Leftrightarrow x = x' + d$

$d = |d|$

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d \in \mathbb{Z}}$$

$$= \text{Skellam}(\lambda, \lambda)$$ discovered (1946)

$$I_{|d|}(2\lambda) := \sum_{x \in \{0,1,\ldots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$

Modified Bessel Function of
the First Kind (it's a solution to a
famous differential equation)

this is used to model
point spreads in sport games, photon noise, ...

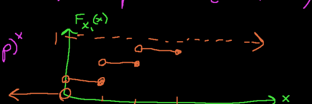
$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

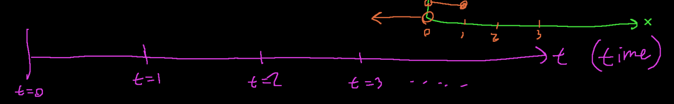
$$p_{X_1|T}(x,t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} = \frac{p_{X_1}(x)\, p_{X_2}(t-x)}{p_T(t)}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\cancel{\lambda^x}}{x!} \frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}\left(t, \frac{1}{2}\right)$$

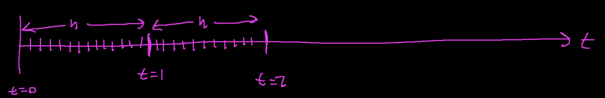
$$X_1 \sim \text{Geom}(p) := \overbrace{(1-p)^x p}^{p_{X_1}(x)}\, \mathbb{1}_{x \in \{0,1,\ldots\}}, \quad \text{Supp}[X_1] = \{0, 1, \ldots\}$$

$$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = 1 - (1-p)^x$$





In every "second", let's do n iid Bernoulli(p) experiments.



Let's call the resulting geometric rv X_n and its unit of realization is t

$$p_{X_n}(x) = (1-p)^{nx} p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots, 2, \ldots\}}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx}$$ where $\lambda \in (0, \infty)$

let $n \to \infty$, $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ same as Poisson

$$p_{X_\infty}(x) := \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$$

Not a valid PMF!

$$= \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda x}} \cdot \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} \cdot \underbrace{\lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\ldots\}}}_{\mathbb{1}_{x\in[0,\infty)}} = 0 \quad \forall x!$$

$$\sum_{x \in \text{Supp}[X_\infty]} p_{X_\infty}(x) = 0$$

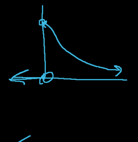
$\text{Supp}[X_\infty] = [0, \infty)$

$$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nx} = \left(1 - e^{-\lambda x}\right)\mathbb{1}_{x \in [0,\infty)}$$

The PMF wasn't valid. Is the CDF valid? If so, I need to check three properties. (1) It's 0 as I go to negative infinity, (2) it's 1 as I go to positive infinity and (3) it's an increasing function.

(1) $\lim_{x\to-\infty} (1 - e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 0$ ✓

(2) $\lim_{x\to\infty} (1 - e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 1 - \lim_{x\to\infty} \frac{1}{e^{\lambda x}} = 1$ ✓



(3) $\frac{d}{dx}\left[(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)} \ge 0$ ✓

$\Rightarrow$ Valid CDF!

We now have a continuous rv X. Continuous rv's have the following properties:

$|\text{Supp}[X]| = |\mathbb{R}|$ uncountable infinity (the size of the continuum)

They do not have PMF's (because the probability of the rv being at any specific number is zero) but they do have CDF's.

The derivative of the CDF is a very useful function, it is called the probability density function (PDF) denoted f(x).
(Note: discrete rv's do not have PDF's).

$$f(x) := F'(x), \quad P(X \in [a,b]) = \underbrace{P(X \le b)}_{F(b)} - \underbrace{P(X \le a)}_{F(a)} = \int_a^b f(x)\,dx$$

by fundamental thm. calculus

$$\int_{\mathbb{R}} f(x) = 1 = \text{"}\overset{1}{F(\infty)} - \overset{0}{F(-\infty)}\text{"}$$

$f(x) \ge 0$ since CDF's are increasing functions

$\Rightarrow \text{Supp}[X] = \{x : f(x) > 0\}$

$X_1, \ldots, X_n$ indep. $\Rightarrow f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)$ joint density function

---

$$X \sim \text{Exp}(\lambda) := \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{old}(x)}\mathbb{1}_{x\in[0,\infty)}}_{f(x)}, \quad F(x) = (1 - e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}$$

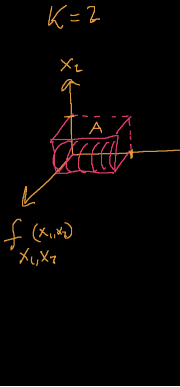
Exponential rv

$\lambda \in (0, \infty)$

its parameter space

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x}). \qquad \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{K} f_{\vec{X}}(\vec{x})\,dx_1 \ldots dx_K = 1$$

$K = 2$



$$P(A) = \iint_A f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2$$