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$$= \sum_{j=k}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

Note: $j=n$

$$= \sum_{j=k}^{n} \frac{n!}{j!\,(n-j)!} j f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!\,(n-j)!} (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$j! = j \cdot (j-1)!$, $(n-j)! = (n-j)(n-j-1)!$, $n-j-1 = n-(j+1)$

if $j=n-1$: $L-1=n-1$, $L=n$

reindexing: let $L=j+1 \Rightarrow j=L-1$ only

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{L=k+1}^{n} \frac{n!}{(L-1)!\,(n-L)!} f(x) F(x)^{L-1} (1-F(x))^{n-L}$$

These are equal now

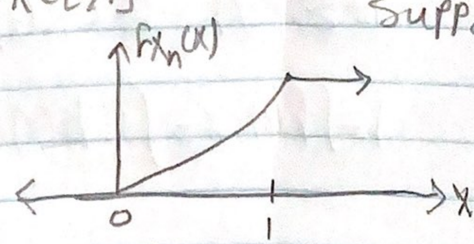
$$= \frac{n!}{(k-1)!\,(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x)$$ (Done) for iid case

Check min, max:

$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1} \checkmark$$

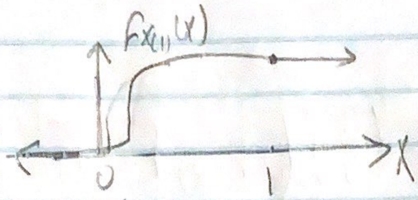
$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1} \checkmark$$

$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1) = \underbrace{1\,\mathbb{1}_{x\in[0,1]}}_{f(x)} \Rightarrow F(x) = x$ in the support



$$F_{X_{(n)}} = F(x)^n = x^n$$

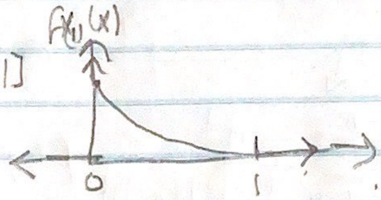
$$F_{X_{(1)}}(x) = 1 - (1-F(x))^n = 1-(1-x)^n$$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in[0,1]} \Rightarrow$$

$$\Rightarrow f_{X_{(n)}}(x) = n x^{n-1}\mathbb{1}_{x\in[0,1]},\ f_{X_{(1)}}(x) = n-(1-x)^{n-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in[0,1]}$$



$$= \text{Beta}(k, n-k+1)$$

we will see the general beta distribution later

$X \sim \text{Gamma}(\alpha_1, \beta)$ indep if $Y \sim \text{Gamma}(\alpha_2, \beta) \Rightarrow$

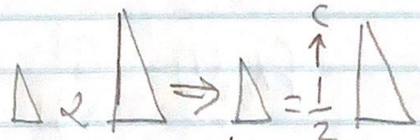
this seems right

$\Rightarrow X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$

$P(X) = c\,k(x)$, $f(x) = c\,k(x)$ normallizing constant

kernel

$\Rightarrow p(x) \propto k(x)$, $f(x) \propto k(x)$



If you know $k(x)$, you can resolve c via the following:

$$1 = \sum p(x) = \sum c\,k(x) \Rightarrow \sum k(x) = \frac{1}{c} \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int f(x)\,dx = \int c\,k(x)\,dx \Rightarrow \int k(x)\,dx = \frac{1}{c} \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

$X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} 1_{x \in \{0,1,\ldots n\}}$

$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x 1_{x \in \{0,\ldots n\}}}_{K(x)}$

$y \sim \text{Weibull}(k,\lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} 1_{y \geq 0} = \overbrace{k\lambda^y}^{c} \overbrace{y^{k-1} e^{-(\lambda y)^k} 1_{y \geq 0}}^{K(x)}$

$X \sim \text{Gamma}(\alpha,\beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \overbrace{x^{\alpha-1} e^{-\beta x}}^{K(x)} 1_{x \geq 0} \quad \propto x^a e^{-bx}$

Let's add the gamms

$f_{X+Y}(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \overbrace{e^{-\beta(t-x)}}^{e^{-\beta t} e^{\beta x}} \underbrace{1_{t-x \in [0,\infty)}}_{\substack{x-t \in (-\infty,0] \\ x \in (-\infty, t)}} dx$

Let's find this density's kernel, $K(t)$

$= \overbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}}^{c} \overbrace{e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx \, 1_{t \geq 0}}^{K(t)} \propto e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx \, 1_{t \geq 0}$

$= e^{-\beta t} t^{\alpha-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1} \left(1-\frac{x}{t}\right)^{\alpha_2-1} dx \, 1_{t \geq 0}$

Let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du \Rightarrow x=0 \Rightarrow u=0,$
$x=t \Rightarrow u=1$

$e^{-\beta t} t^{\alpha_1+\alpha_2-2} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} t \, du \, 1_{t \geq 0} =$ This integral is proven to be impossible

$= e^{-\beta t} t^{\alpha_1+\alpha_2-1} 1_{t \geq 0} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du$

$$\propto t^{\alpha_1+\alpha_2-1} e^{-\beta t} 1_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du \cdot \underbrace{t^{\alpha_1+\alpha_2-1} e^{-\beta t} 1_{t\geq 0}}_{k(t)}$$

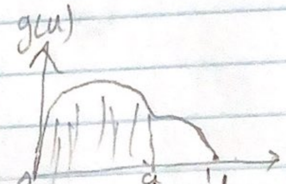
$$B(\alpha_1,\alpha_2) := \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du$$

we can use probability theory to get an integral identity:

$$X \sim \text{Gamma}(\alpha_1+\alpha_2,\beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\beta x} 1_{x\geq 0}$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2)\, t^{\alpha_1+\alpha_2-1} e^{-\beta x} 1_{x\geq 0}$$

$$\Rightarrow B(\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$



$$B(a,\alpha_1,\alpha_2) := \int_0^a \underbrace{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}_{g(u)} du$$ incomplete beta functio

$$I_a(\alpha_1,\alpha_2) = \frac{B(a,\alpha_1,\alpha_2)}{B(\alpha_1,\alpha_2)} \in [0,1]$$ regularized incompleted beta function

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$$X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} 1_{x\in[0,1]}$$ where $\alpha,\beta > 0$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} dy \overset{f(x)}{=} \frac{\int_0^x y^{\alpha-1}(1-y)^{\beta-1} dy}{B(\alpha,\beta)}$$

$$= \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} =: I_x(\alpha,\beta)$$

$$1 = \int_{\text{Supp}[X]} f(x)dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx}_{\text{Beta}} = \frac{B(\alpha,\beta)}{B(\alpha,\beta)} = 1$$