$$= \sum_{j=K}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

$$= \sum_{j=K}^{n} \frac{n!}{\underset{(j-1)!}{j!}\,(n-j)!}\, j\, f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=K}^{\overset{n-1}{n}} \frac{n!}{j!\,\underset{(n-j-1)!}{(n-j)!}} \underbrace{(n-j)}_{0} f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

note $j=n \Rightarrow$

reindexing$^{\text{RHS}}$: let $\ell = j+1 \Rightarrow j = \ell - 1$

$$= \sum_{j=K}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{\ell=K+1}^{n} \frac{n!}{(\ell-1)!\,(n-\ell)!} f(x) F(x)^{\ell-1} (1-F(x))^{n-\ell}$$

both are equal now

$$= \frac{n!}{(K-1)!\,(n-K)!} f(x) F(x)^{K-1} (1-F(x))^{n-K} = f_{X_{(K)}}(x) \quad \text{Done for iid}$$

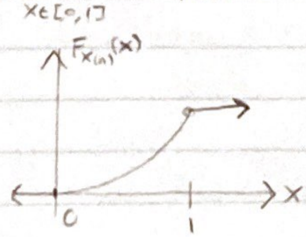
Check min, max:

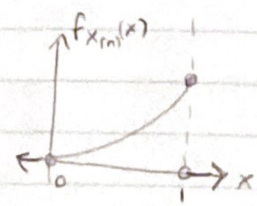
$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!\,(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1} \checkmark$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1} \checkmark$$

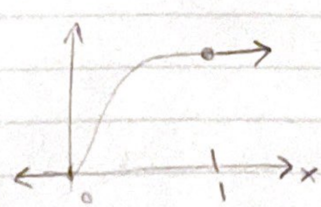
$$X_1, \dots, X_n \overset{iid}{\sim} U(0,1) = \overbrace{1 \, \mathbb{1}_{x \in [0,1]}}^{f(x)} \Rightarrow F(x) = x \text{ in the support}$$

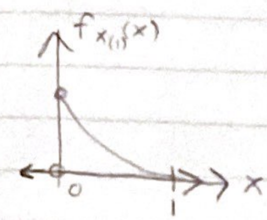
$$F_{X_{(n)}}(x) = F(x)^n = x^n$$





$$F_{X_{(1)}}(x) = 1 - (1-F(x))^n = 1 - (1-x)^n$$





$$f_{X_{(K)}}(x) = \frac{n!}{(K-1)!\,(n-K)!} x^{K-1} (1-x)^{n-K} \mathbb{1}_{x \in [0,1]} \Rightarrow f_{X_{(n)}}(x) = n x^{n-1} \mathbb{1}_{x \in [0,1]}, \; f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(K)\Gamma(n-K+1)} x^{K-1} (1-x)^{n-K} \mathbb{1}_{x \in [0,1]} = Beta(K, n-K+1)$$

$X \sim \text{Gamma}(\alpha_1, \beta)$ indep of $Y \sim \text{Gamma}(\alpha_2, \beta) \Rightarrow X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

Seems right

The easiest proof of this is to employ "Kernels". What's a Kernel?

$p(x) = c\,K(x)$, $f(x) = c\,K(x)$ normalizing constant

Kernel

$\Rightarrow p(x) \propto K(x)$, $f(x) \propto K(x)$

Similar triangles

if you know $K(x)$, you can resolve c via the following:

$1 = \sum p(x) = \sum c\,K(x) \Rightarrow \sum K(x) = \frac{1}{c} \Rightarrow c = \left(\sum K(x)\right)^{-1}$

$1 = \int f(x)\,dx = \int c\,K(x)\,dx \Rightarrow \int K(x)\,dx = \frac{1}{c} \Rightarrow c = \left(\int K(x)\,dx\right)^{-1}$

this means that $K(x)$ identifies a PMF or PDF. So if you "see" a $K(x)$ in the wild, you immediately know the rv because you know that you can solve for c if you had a computer.

$$X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x}\, \mathbb{1}_{x \in \{0,1,\ldots,n\}}$$

$$= \underbrace{n!\,(1-p)^n}_{c}\ \underbrace{\frac{1}{x!\,(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1}_{x \in \{0,1,\ldots,n\}}}_{K(x)}$$

$$Y \sim \text{Weibull}(k,\lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k}\, \mathbb{1}_{y \geq 0} = \underbrace{k\lambda^k}_{c}\ \underbrace{y^{k-1} e^{-(\lambda y)^k}\, \mathbb{1}_{y \geq 0}}_{K(x)}$$

$$X \sim \text{Gamma}(\alpha, \beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c}\ \underbrace{x^{\alpha-1} e^{-\beta x}\, \mathbb{1}_{x \geq 0}}_{K(x)} \qquad \propto x^a e^{-bx}$$

Let's add the gammas:

$$f_{X+Y}(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \overset{= -\beta t + \beta x}{e^{-\beta(t-x)}} \mathbb{1}_{t-x \in [0,\infty)} \, dx$$

$x - t \in (-\infty, 0]$

$x \in (-\infty, t]$

Let's find this density's kernel, $k(t)$

$$= \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}}_{c} \underbrace{e^{-\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} dx \, \mathbb{1}_{t \geq 0}}_{k(t)} \underset{\text{proportional}}{\propto} e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx \, \mathbb{1}_{t\geq 0}$$

"for now"

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1} \left(1-\frac{x}{t}\right)^{\alpha_2-1} dx \, \mathbb{1}_{t\geq 0}$$

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du \Rightarrow x=0 \Rightarrow u=0, \; x=t \Rightarrow u=1$

$$= e^{-\beta t} t^{\alpha_1+\alpha_2-2} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} t\,du \, \mathbb{1}_{t\geq 0} = e^{-\beta t} t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t\geq 0} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du}_{\text{impossible to do}}$$

even if the integral is impossible, what will the result be a function of? α_1 and α_2.

$$\propto t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta) = \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du}_{c} \cdot \underbrace{t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\geq 0}}_{k(t)}$$

that integral is famous, called the "beta function"

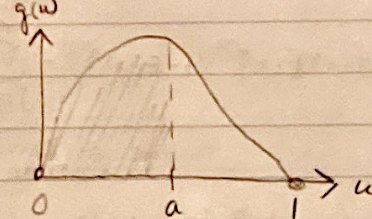
$$B(\alpha_1, \alpha_2) = \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du$$

We can use probability theory to get an integral identity:

$$X \sim \text{Gamma}(\alpha_1+\alpha_2, \beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \cancel{t^{\alpha_1+\alpha_2-1} e^{-\beta x} \mathbb{1}_{x\geq 0}} = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1, \alpha_2) \cancel{t^{\alpha_1+\alpha_2-1} e^{-\beta x} \mathbb{1}_{x\geq 0}}$$

$$\Rightarrow B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

$$B(a, \alpha_1, \alpha_2) := \int_0^a u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}\,du$$ incomplete beta function



$$\underset{a}{I}(\alpha_1, \alpha_2) = \frac{B(a, \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \in [0,1]$$ regularized incomplete beta function

$$X \sim Beta(\alpha, \beta) := \underbrace{\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in[0,1]}}_{f(x)}$$ where $\alpha, \beta > 0$

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\int_0^x x^{\alpha-1}(1-y)^{\beta-1}\,dy}{B(\alpha,\beta)} = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$

$$1 = \int_{supp[X]} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}_{beta} = \frac{B(\alpha,\beta)}{B(\alpha,\beta)} = 1$$