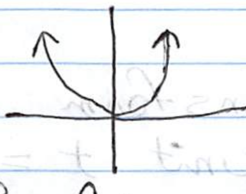
Math 621
Lecture 16
11-09-2020

Define L^1: $\{f : \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$ all functions in this set are called "L1 integrable" or "absolutely integrable."

Are PDF's in the set L1? Yes.

If $f \in L^1$, then $\exists \hat{f}$, the "Fourier transform" of f:



$f(x) = x^2 \notin L^1$

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x)\,dx = \mathcal{F}[f]$$

omega symbol

This is called the "forward fourier transform" or "fourier analysis". X is called the "time domain" and omega is called the "frequency domain". One of Fourier's ideas is that functions in L1 can be decomposed into a sum of sines & cosines with different frequencies, omega, and amplitudes, |f(omega)|, and phase shifts, Arg [f(omega)].

Further, if $\hat{f} \in L^1$, then we can do a "reverse/inverse fourier transform" to restore our original function f:

$$f(x) = \int_{\mathbb{R}} e^{i2\pi x\omega} \hat{f}(\omega)\,d\omega = \mathcal{F}^{-1}[\hat{f}]$$

This is called the "inverse fourier transform" or "fourier synthesis".

(2)

Fourier Inversion theorem:
if f and fhat are in L1, then f and fhat are 1:1.

We defind the Characteristic function (chf) for RV X as:

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} f_X(x)\,dx$$
$$= \sum_{x \in \mathbb{R}} e^{itx} p_X(x)$$

this is the fourier transform with a different frequency unit $t = -2\pi\omega$.

The reason why we bother to take this crazy-looking transformation is that there are really powerful properties of the chf that will enable us to solve problems. Here are the main properties:

P_0 $\phi_X(0) = E[e^{i(0)X}] = E[1] = 1 \quad \forall X, \forall t.$

P_1 $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y,$
"Uniqueness"

P_2 If $Y = aX + b$ where $a, b \in \mathbb{R}$,
then $\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{\overset{t'}{\widetilde{iat}}X} \underbrace{e^{itb}}_{\text{Constant}}]$

So, $e^{itb} E[e^{it'X}]$

$= e^{itb}\phi_X(t') = e^{itb}\phi_X(at)$

P_3 X_1, X_2 are independent, $T = X_1 + X_2$

$\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}]$

$$\overset{\text{independent}}{=} E[e^{itX_1}]\,E[e^{itX_2}] = \phi_{X_1}(t)\,\phi_{X_2}(t)$$

(P4) Moment Generator

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right]$$

we are able to interchange differentiation & integration here

$$= E\left[\frac{d}{dt}[e^{itX}]\right] = E[iXe^{itX}]$$

$$\phi_X'(0) = E[iX] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$

$$\phi_X''(t) = \frac{d}{dt}\left[E[iXe^{itX}]\right] = E\left[iX\frac{d}{dt}[e^{itX}]\right]$$

$$= E[i^2X^2 e^{itX}]$$

second moment

$$\phi_X''(0) = E[i^2X^2] \Rightarrow E[X^2] = \frac{\phi_X''(0)}{i^2}$$

$$\boxed{E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}}$$ Formula

(P5) $\phi_X(t) \in [-1,1]$ i.e. the chf exists $\forall x, \forall t$.

$|\phi_X(t)| \leq 1$, Proof: $|E[e^{itX}]| = \left|\int_{\mathbb{R}} e^{itx} f_X(x)\,dx\right|$

$$\leq \int_{\mathbb{R}} |e^{itx} f_X(x)|\,dx$$

$$\leq \int_{\mathbb{R}} |e^{itx}|\,|f_X(x)|\,dx = \int_{\mathbb{R}} |\overbrace{\cos(tx)}^{a} + i\overbrace{\sin(tx)}^{b}|\, f_X(x)\,dx$$

$$= \int_{\mathbb{R}} \sqrt{\cos^2(tx) + \sin^2(tx)}\; f_X(x)\,dx$$

$$= \int_{\mathbb{R}} f_X(x)\,dx = 1.$$

(P6) Inversion: If $\phi_X(t) \in L'$, then

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-jtx}\phi_X(t)dt$$

(P7) Levy's CDF formula

$$P(X \in [a,b]) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{itb}}{it}\phi_X(t)dt$$

(P8) Levy's Continuity Theorem:

Consider a sequence of rv's $X_1, X_2, \ldots, X_n$. We define "X_n converges in distribution to X" (denoted $X_n \xrightarrow{d} X$) as:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x$$

"Pointwise convergence"

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X.$$

If n large, $\phi_{X_n}(t) \approx \phi_X(t) \Rightarrow X_n \overset{d}{\approx} X$.

Define $M_X(t) = E[e^{tX}]$, the moment generating function (mgf)

Properties:

(P0) $M_X(0) = 1 \quad \forall X$

(P1) $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

(P2) $Y = aX+b$, $M_Y(t) = e^{tb}M_X(at)$

(P3) X_1, X_2 independent, $T = X_1 + X_2$,

$$M_T(t) = M_{X_1}(t)M_{X_2}(t).$$

(P4) $E[X^n] = M_X^{(n)}(0)$

It does not have (P5).

Thus, sometimes mgf's don't exist at all and sometimes it doesn't exist for certain values of t.

Chf's can do everything mgf's can do and more!!

Thus, You don't need mgf's!

$X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_x$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x} \, dx$$

$$= \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it} \right)^\alpha$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, $T = X_1 + X_2$

$$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\, \phi_{X_2}(t)$$

$$= \left(\frac{\beta}{\beta - it} \right)^{\alpha_1} \left(\frac{\beta}{\beta - it} \right)^{\alpha_2}$$

$$= \left(\frac{\beta}{\beta - it} \right)^{\alpha_1 + \alpha_2}$$

$$\overset{P1}{\Longrightarrow} T \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta).$$

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$$\overset{(P1)}{\Longrightarrow} T \sim \text{Gamma}(\alpha_1+\alpha_2, \beta).$$