$\boxed{8}$

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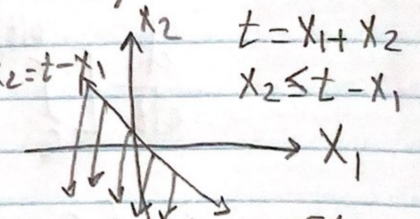
$\vec{X}$ contilum rv $P(\vec{X} \in A) = \int_A \cdots \int f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_k$

let $T = X_1 + X_2 \sim f_T(t) = ?$

First case $f_T(t) = F'(t)$ CDF method

Usually it is difficult to find the CDF of countinuous rv's, so this is not the usual method. The usual method is to use the convolution formula (which we will now derive)



$t = x_1 + x_2$

$x_2 \le t - x_1$

$F_T(t) = P(T \le t) = P(\vec{X} \in A_t)$

$$= \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$A_t = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \le t - x_1 \right\}$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2\, dx_1 = \iint_{R\ -\infty}^{\ \ t} f_{X_1,X_2}(x, v-x)\, dv\, dx$$

Let $x_1 = x$

$x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$\overset{*}{=} \int_{-\infty}^{t} \left(\int_R f_{X_1,X_2}(x, v-x)\, dx \right) dv$$

$$f_T(t) = \frac{d}{dt}\left[\quad \right]$$

Leibnitz's Rule

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x,b(x))\,b'(x) + g(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[g(x,y)\right] dy$$

If the outer derivative is a third variable, then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + g(x,a(t))\,a'(t) + \cancelto{0}{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\left[g(x,y)\right] dy}$$

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\,dy\right] = \underline{g(x,b(t))\,b'(t) + \cancelto{0}{g(x,c)\,\frac{d}{dt}[c]}}$$

$$= \int_{R} f_{X_1,X_2}(x,t-x)\,dx = f_T(t) = f_{X_1}(x) * f_{X_2}(x)$$

general convolution formula

if X_1, X_2 independent

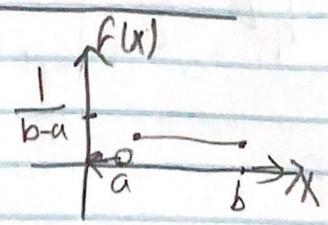
$$= \int_R f_{X_1}(x)\, f_{X_2}(t-x)\,dx = \int_{supp[X_1]} f^{old}_{X_1}(x)\, f^{old}_{X_2}(t-x)\, 1_{t-x \in supp(X_2)}\,dx$$

X_1, X_2 iid

$$= \int_R f(x)\, f(t-x)\,dx = \int_{supp[X]} f^{old}(x)\, f^{old}(t-x)\, 1_{t-x\in supp[X]}\,dx$$

continuous uniform rv

$$X \sim U(a,b) = \underbrace{\underbrace{\frac{1}{b-a}\,1_{x\in[a,b]}}_{f^{old}(x)}}_{f(x)}$$

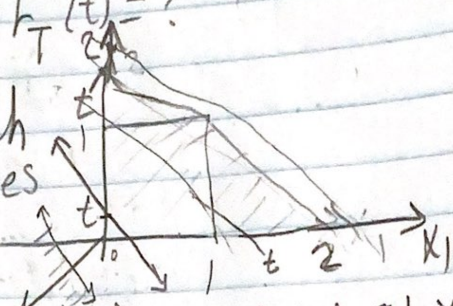


standard uniform rv is when $a=0, b=1$

$X \sim U(0,1) = \mathbb{1}_{X \in [0,1]}$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$

we want to compute CDF which means we want to find volumes in regions under the diagonal line.



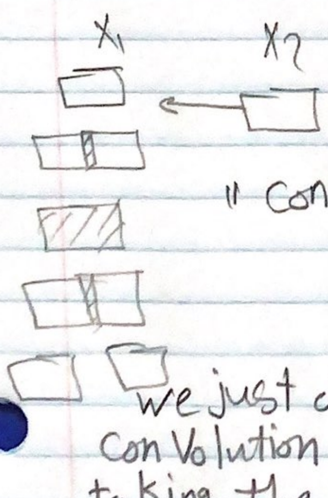
$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1 & \text{if } X_1 \in [0,1] \\ & \& X_2 \in [0,1] \\ 0 & o/t \end{cases}$

$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2/2 & \text{if } t \in [0,1) \\ -\frac{t^2}{2}+2t-1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \geq 2 \end{cases}$$

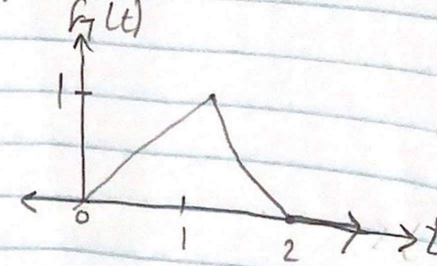
if $t \in (1,2)$

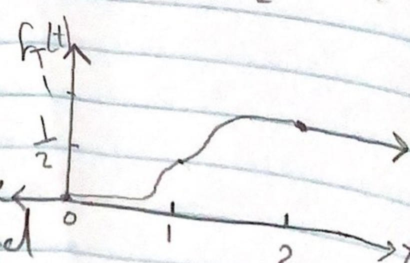
$$F_T(t) = \frac{t^2}{2} - 2\frac{(t-1)^2}{2} = \frac{t^2}{2} - (t^2-2t+1) = -\frac{t^2}{2}+2t-1$$

$$\Rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in (0,1) \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$



"convolving"





we just derived the PDF of the convolution by finding its CDF and taking the derivative. why can't we just use our fancy formula?

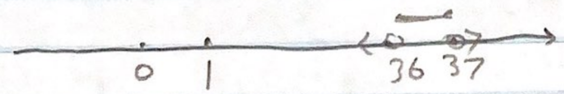
iid version

$$f_T(t) = \int_{supp[X]} f^{old}(x)\, f^{old}(t-x)\, 1_{t-x \in supp[X]}\, dx = \int_0^1 (1)(1)\, 1_{\underset{\substack{x-t\in[-1,0] \\ x\in[t-1,t]}}{t-x\in[0,1]}}\, dx$$

$$= \int_0^1 1_{x\in[t-1,t]}\, dx$$

Let's do some examples. How about $t=-37$? $\int_0^1 1_{x\in[-38,-37]}\, dx$

How about $+37$? $\int_0^1 1_{x\in[36,37]}\, dx = 0$



$$= \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t\in[0,1] \\ 2-t & \text{if } t\in(1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$

e.g. 0.7

$$\int_0^1 1_{x\in[-0.3,0.7]}\, dx$$

e.g. $t=1.63$

$$\int_0^1 1_{x\in(0.63,1.63)}\, dx = \int_{0.63}^1 dx$$

$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) := \lambda e^{-\lambda x} 1_{x\in[0,\infty)}$

$T_2 = X_1 + X_2 \sim f_T(t) = ?$

$$f_T(t) = \int_{supp[X]} f^{old}(x)\, f^{old}(t-x)\, 1_{t-x\in supp[X]}\, dx$$

$$= \int_0^\infty \lambda e^{-\lambda x}\, \lambda e^{\overbrace{-\lambda(t-x)}^{-\lambda t + \lambda x}}\, 1_{\underbrace{t-x\in(0,\infty)}_{\substack{x-t\in(-\infty,0) \\ x\in(-\infty,t)}}}\, dx = \lambda^2 e^{-\lambda t} \int_0^\infty 1_{x\in(-\infty,t)}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = t\lambda^2 e^{-\lambda t} 1_{t\in(0,\infty)} = Erlang(2,\lambda)$$

$\to 2$