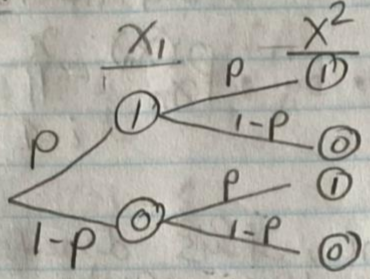
8/31/2020
Lecture 2
Math 621

Recall:



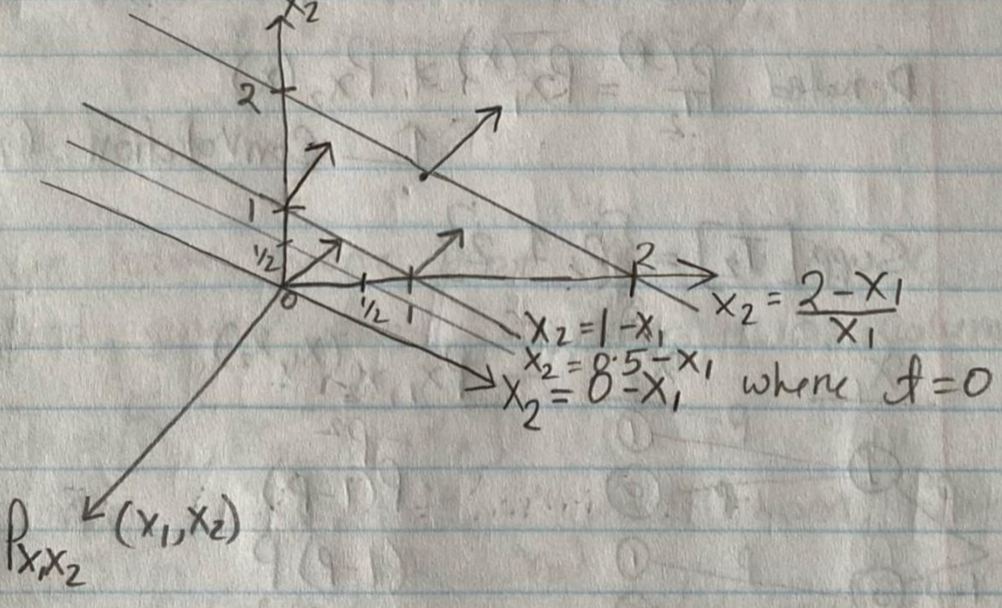
$P_{\vec{X}}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

$$T \sim \begin{cases} 2 & \text{wp } p^3 \\ 1 & \text{wp } 2p(1-p) \\ 0 & \text{wp } (1-p)^2 \end{cases}$$

$$P(T) = P(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2) \mathbb{1}_{x_1+x_2=t}$$

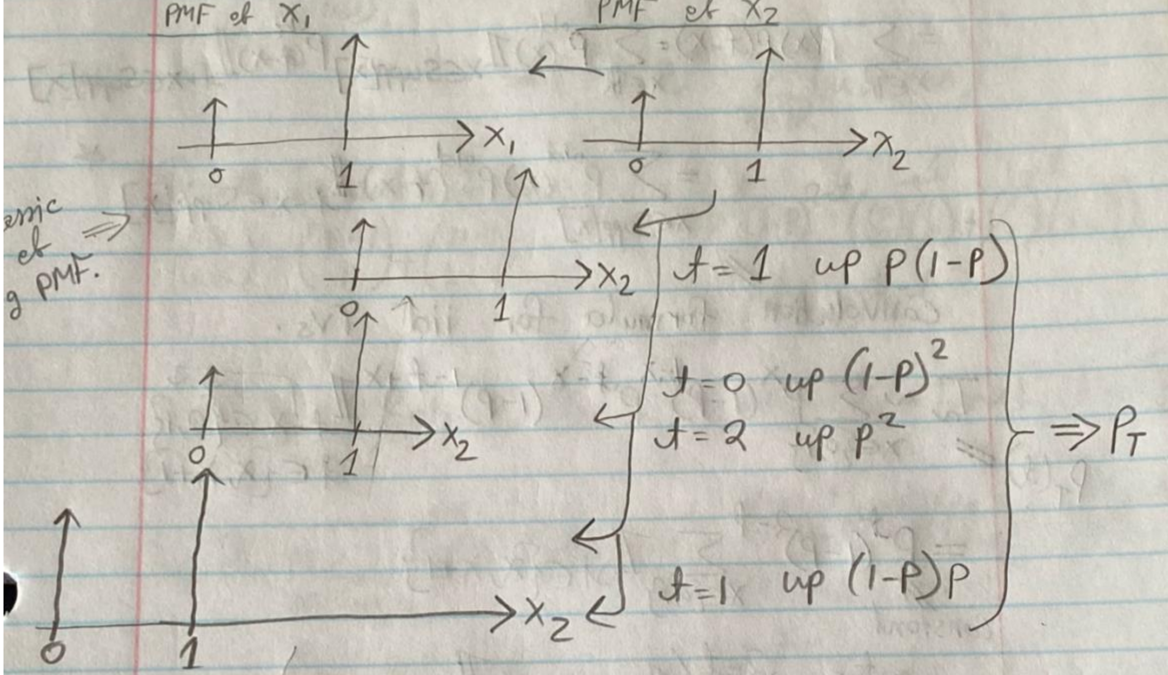
$$\overset{\shortparallel}{P_{X_1}(x) * P_{X_2}(x)} \qquad \overset{\Downarrow}{x_2} = t - x_1$$

Example: $P(T=1) = P_{X_1,X_1}(1,0) + P_{X_1,X_2}(0,1)$



$$\text{If } t=2, \quad P_T(2) = \sum_{x_1 \in \mathbb{R}} P_{X_1X_2}(x_1, 2-x_1)$$

"Convolve" means to "roll or coil together/entwine"



$$p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1, x_2) \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x_1 \in \mathbb{R}} p_{X_1,X_2}(x, t-x) \Leftarrow \text{general convolution formula.}$$

If X_1, X_2 are independent,

$$= \sum_{x_1 \in \mathbb{R}} p_{X_1}(x) p_{X_2}(t-x) \Leftarrow \text{convolution formula for independent rvs.}$$

$$\parallel$$

$$\star \quad \sum_{x \in \mathbb{R}} p^{old}_{X_1}(x) \mathbb{1}_{x \in Supp[X_1]} p^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2}$$

$$\parallel$$

$$\sum_{x \in Supp[X_1]} p^{old}_{X_1}(x) \, p^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$

If x_1, x_2 iid,

$$= \sum_{x\in\mathbb{R}} P(x)P(t-x) = \sum_{x\in\mathbb{R}} P^{old}(x)\mathbb{1}_{x\in Supp[X]} P^{old}(t-x)\mathbb{1}_{t-x\in Supp[X]} \quad \star$$

$$= \sum_{x\in Supp[X]} P^{old}(x)P^{old}(t-x)\mathbb{1}_{t-x\in Supp[X]}$$

$\nearrow$ convolution formula for iid rvs.

$$\underset{p_T(t) =}{T_2 \sim} \sum_{x\in\{0,1\}} p^x(1-p)^{1-x}p^{t-x}(1-p)^{1-t+x}\,\underset{t\in\{x,\,x+1\}}{\mathbb{1}_{t-x\in\{0,1\}}}$$

$$= \underset{\substack{\nearrow \\ \text{constant}}}{p^t(1-p)^{2-t}}\sum_{x\in\{0,1\}}\mathbb{1}_{t\in\{x,x+1\}}$$

$$= p^t(1-p)^{2-t}\Big(\underbrace{\mathbb{1}_{t\in\{0,1\}}+\mathbb{1}_{t\in\{1,2\}}}_{\substack{t=0\ \Rightarrow 1 \\ t=1\ \Rightarrow 2 \\ t=2\ \Rightarrow 1}}\Big) \qquad = \binom{2}{t}$$

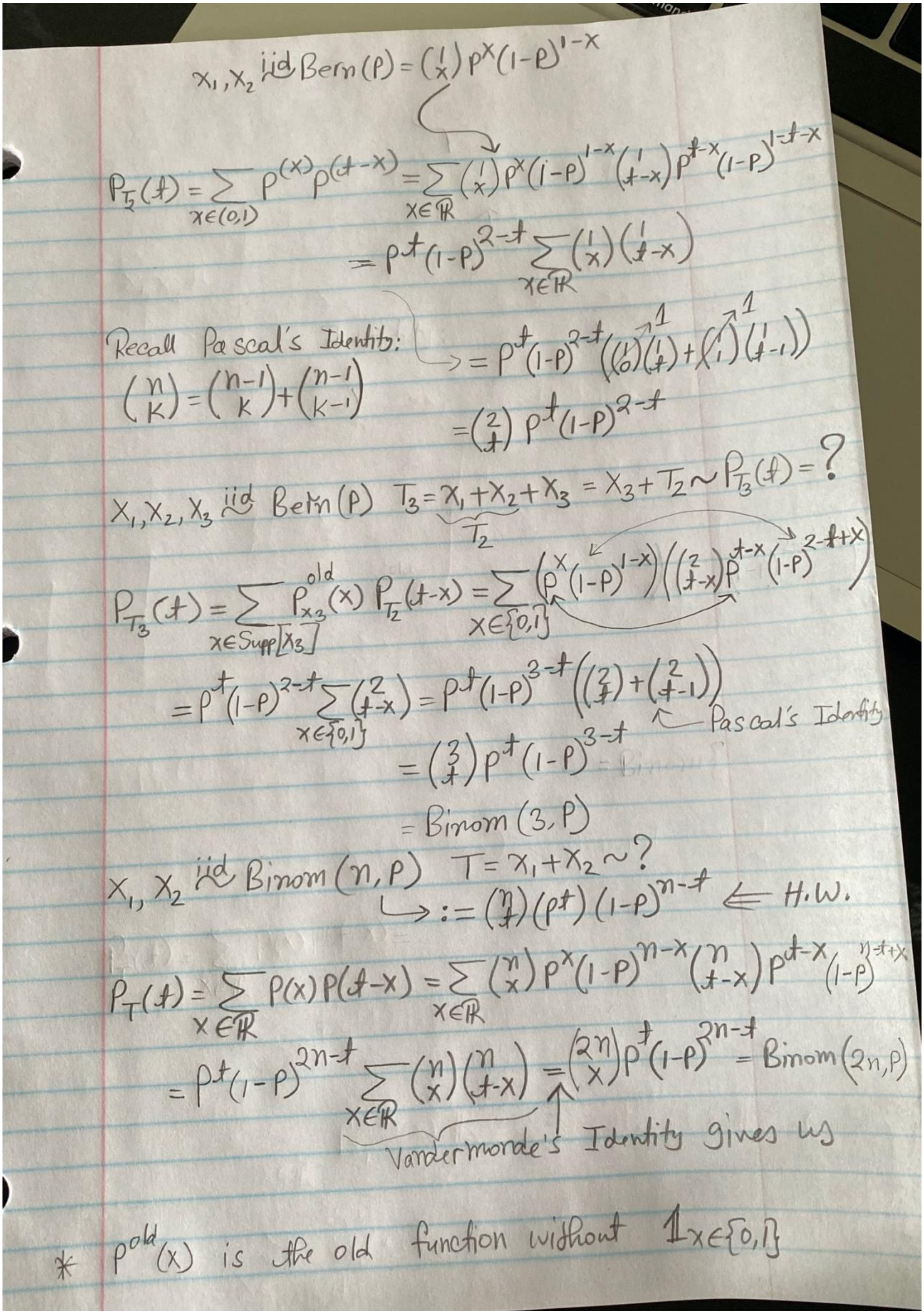
Recall,

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}\mathbb{1}_{n\in\mathbb{N}}\mathbb{1}_{k\in\{0,1,\dots n\}}$$

$$= \binom{2}{t}p^t(1-p)^{2-t} = \text{Binom}(2,p), \quad Supp[T_2] = \{0,1,2\}$$

Generally $Supp[T] = Supp[X_1] + Supp[X_2]$,

$$A+B := \{a+b : a\in A,\ b\in B\}$$



$$X_1, X_2 \overset{iid}{\sim} \mathrm{Bern}(p) = \binom{1}{x} p^x (1-p)^{1-x}$$

$$P_{T_2}(t) = \sum_{x\in(0,1)} p^{(x)} p^{(t-x)} = \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t-x}$$

$$= p^t (1-p)^{2-t} \sum_{x\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$

Recall Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= p^t (1-p)^{2-t}\left(\overset{1}{\binom{1}{0}}\binom{1}{t} + \overset{1}{\binom{1}{1}}\binom{1}{t-1}\right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

$$X_1, X_2, X_3 \overset{iid}{\sim} \mathrm{Bern}(p) \quad T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$$

$$P_{T_3}(t) = \sum_{x\in \mathrm{Supp}[X_3]} P^{old}_{X_3}(x)\, P_{T_2}(t-x) = \sum_{x\in\{0,1\}} \left(p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}\right)$$

$$= p^t (1-p)^{2-t} \sum_{x\in\{0,1\}} \binom{2}{t-x} = p^t (1-p)^{3-t}\left(\binom{2}{t} + \binom{2}{t-1}\right)$$

← Pascal's Identity

$$= \binom{3}{t} p^t (1-p)^{3-t}$$

$$= \mathrm{Binom}(3, p)$$

$$X_1, X_2 \overset{iid}{\sim} \mathrm{Binom}(n, p) \quad T = X_1 + X_2 \sim ?$$

$$\hookrightarrow := \binom{n}{t} (p^t) (1-p)^{n-t} \Leftarrow \text{H.W.}$$

$$P_T(t) = \sum_{x\in\mathbb{R}} P(x) P(t-x) = \sum_{x\in\mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x\in\mathbb{R}} \binom{n}{x}\binom{n}{t-x}}_{\text{Vandermonde's}} \overset{\uparrow}{=} \binom{2n}{x} p^t (1-p)^{2n-t} = \mathrm{Binom}(2n, p)$$

Identity gives us

* $p^{old}(x)$ is the old function without $\mathbb{1}_{x\in\{0,1\}}$