Lecture 6

9/16/20

Let $A \in \mathbb{R}^{L \times K}$ matrix of constants

$$\underset{L\times 1}{E[A\vec{X}]} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \ldots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \ldots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \ldots + a_{LK}X_K] \end{bmatrix} =$$

$\vec{a_i}$ row i of matrix A

$$= \begin{bmatrix} E[\vec{a_1} \cdot \vec{X}] \\ E[\vec{a_2} \cdot \vec{X}] \\ \vdots \\ E[\vec{a_L} \cdot \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a_1} \cdot \vec{\mu} \\ \vec{a_2} \cdot \vec{\mu} \\ \vdots \\ \vec{a_l} \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$$Var[\vec{a}^T \vec{X}] = Var[\underbrace{a_1X_1}_{Y_1} + \ldots + \underbrace{a_KX_K}_{Y_K}] = \sum_{i=1}^{K} \sum_{j=1}^{K} cov[Y_i, Y_j] =$$

$$\sum \sum cov[a_iX_i, a_jX_j] = \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \sigma_{ij} = \underset{(1\times K)}{\vec{a}^T} \underset{(K\times K)}{\underbrace{\Sigma}_{Var[\vec{X}]}} \underset{(K\times 1)}{\vec{a}}$$

this is called a quadratic form

PROOF:

Let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^{K\times 1}$

$$\vec{a}^T V \vec{\mu} = \vec{a}^T \begin{bmatrix} a_1V_{11} + \ldots + a_KV_{1K} \\ a_1V_{21} + \ldots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \ldots + a_KV_{KK} \end{bmatrix} =$$

$$\hookrightarrow \; = \overbrace{a_1a_1V_{11} + \dots + a_1a_kV_{1k}}^{i=1} + \overbrace{a_2a_1V_{21} + \dots + a_{2k} + }^{j=2} \dots +$$

$$\underbrace{a_ka_1V_{k1} + \dots + a_ka_kV_{kk}}_{i=k} = \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_jV_{ij}$$

This is an application in finance. Imagine $X_1, \dots, X_k$ are financial assets. Each has mean return μ_i, and each pair have covariance Σ_{ij}. Let w vector be a vector of "weights" where each component is the percentage you put into each of these assets. Thus the entries of w sum to 1.

$F = \vec{w}^T\vec{X}$, $\vec{w}^T\vec{1} = 1$, $E[\vec{X}] = \mu$, $Var[\vec{X}] = \Sigma$

$E[F] = E[\vec{w}^T\vec{X}] = \vec{w}^T\vec{\mu} = \mu_F$. The goal is to pick μ_F & minimum variance by computing the w-vector optimally.

$Var[F] = Var[\vec{w}^T\vec{X}] = \vec{w}^T\Sigma\vec{w}$

$\min \vec{w}^T\Sigma\vec{w}$ subject to $\vec{w}^T\vec{\mu} = \mu_F$, $\vec{w}^T\vec{1} = 1$

$\vec{X} \sim Mult._k(n, \vec{p})$, $E[\vec{x}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = X_j \sim Bin(n, p_j) =$

$= \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$

$$\hat{\Sigma} = Var[X] = \begin{bmatrix} Var[X_1] & & Cov[X_i, X_j] \\ & \ddots & \\ Cov[X_i, X_j] & & Var[X_k] \end{bmatrix} \quad Var[X_j] = np_j(1-p_j)$$

for $i \neq j$, $Cov[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j = \sum_{x_i \in \{0,\dots,n\}} \sum_{x_j \in \{0,\dots,n\}} x_i x_j \overbrace{P_{X_i, X_j}(x_i, x_j)}^{\text{JMF } X_1, X_2} \rightarrow$

$-n^2 p_i p_j$

$X_i \sim Bin(n, p_i)$, $X_j \sim Bin(n, p_j)$

$X_i = X_{1i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

$X_i \sim Bin(n, p_i)$

$X_j \sim Bin(n, p_i)$

$\vdots$

$\vec{X} \sim Mult_k(n, \vec{p})$ $\quad \vec{X} = \vec{X_1} + \dots + \vec{X_n}$ where $\vec{X_1}, \dots, \vec{X_n} \overset{iid}{\sim} Mult_k(1, \vec{p})$

$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$

$= \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$ all pairs $= \sum_{l=1}^{m} Cov[X_{li}, X_{lj}]$

$= \sum_{l=1}^{n} \cancel{\sum_{m=1}^{n}} E[X_{li} \cdot X_{mj}] - \underbrace{E[X_{li}]}_{p_i} \underbrace{E[m_j]}_{p_j} = \boxed{-n p_i p_j}$

if $l \neq m$ then is X_{li} independent of X_{mj}? Yes, independent.

$E[X_{li}, X_{lj}] = \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} P_{X_{li} X_{lj}}(x_{li}, x_{lj}) =$ only nonzero if $x_{li} = x_{lj} = 1$

$P_{X_{li}, X_{lj}}(1,1) =$ $X_{li} = 1$ means you get an apple, X_{lj} means you get a banana & both being 1 means you get both an apple & banana at the same time (on one draw). Impossible. Probability 0.

midterm 2 material

$Y = -X = g(x)$ ↳ Y is a function of the rv X (a very simple function).

PMF

$Y \sim P_Y(y)$

$X=0 \rightarrow Y=0$

$X=1 \rightarrow Y=-1$

$X=2 \rightarrow Y=-2$

$X=3 \rightarrow Y=-3$

$\text{Supp}[Y] = -\text{Supp}[X]$

$P_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) = P_X(-y)$

This is for all discrete rv's.

let $z' = -z$

$\text{Supp}[Y] = \{z : P_Y(z) > 0\} = \{z : P_X(-z) > 0\} = \{-z' : P_X(z') > 0\} =$

$= -\{z' : P_X(z') > 0\} = -\text{Supp}[X].$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ From previous class $X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$X_1 - X_2 \sim ?$

difference

$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \overset{ii}{\sim} ?$

$$P_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$$

$\text{Supp}[Y] = \{\ldots, -2, -1, 0\}$

$\text{Supp}[X+Y] = \text{Supp}[X] + \text{Supp}[Y] = \mathbb{Z}$ (all integers)

$$P_T(t) = \sum_{x \in \text{Supp}[X]} P^{old}(x)\, P^{old}(d-x)\, \mathbb{1}_{d-x \in \text{Supp}[Y]}$$