MA368

09/02/2020.

Lecture 03

Let $B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$

possibly an infinite sequence of iid Bernoullis

Let $X :=$ # of zero realization before the first realization of one occurs.

Also, $X := \min\{t : B_t = 1\} - 1$

$p(0) = P(X=0) = P(\{\text{no 0's, just a 1}\}) = p$

$p(1) = P(X=1) = P(\{0, \text{ then a } 1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0, 0, 1\}) = (1-p)^2 p$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0, 0, \ldots, 0}_{x \text{ 0's}}, 1\}) = (1-p)^x p$

$Supp[X] = \{0, 1, 2, \ldots\}$.

possibly infinite.

$X \sim Geom(p) := \underbrace{(1-p)^x p}_{p \text{ old}} \mathbb{1}_{x \in \{0,1,2,\ldots\}}$.

"geometric"

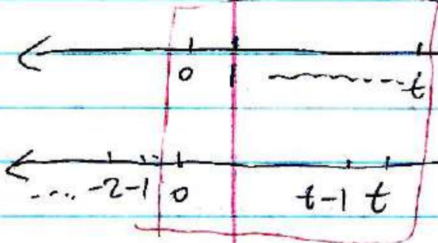
$X_1, X_2 \overset{iid}{\sim} Geom(p)$, $T_2 = X_1 + X_2 \sim p_T(t) = ?$

$$p_T(t) = \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$x - t \in \{0, -1, -2, \ldots\}$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{t, t-1, t-2, \ldots\}}$$



$\{0, 1, \ldots\}$

$\cap$ intersect $= \{0, 1, \ldots, t\}$.

$\{\ldots, t-1, t\}$

Con't $\rightarrow$ $= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{t, t-1, \ldots\}}$

$= (1-p)^t p^2 \sum_{x \in \{0,\ldots,t\}} 1$ indicator is 1

Negative Binomial rv

$= (t+1)(1-p)^t p^2 = \text{NegBinom}(2, p) = \binom{t+1}{1}(1-p)^t p^2$

$\overbrace{0 \; 0 \; \ldots \; 0}^{(1-p)} \; \overset{p}{1} \; \overbrace{0 \; 0 \; \ldots \; 0}^{(1-p)^t} \; \overset{p}{1}$

$t+1$ realization. $t+2$ t plus 2nd realization.

$t+1$ locations to put 1 ones in $\Rightarrow \binom{t+1}{1}$

thus $t+1$ possible locations for the first 1.

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$ ORIG. $\text{Supp}[T_2] = \{0, 1, \ldots\}$.

$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p_T(t) = ?$

$p_{T_3}(t) = \sum_{x \in \text{Supp}[X]} p^{old}_{X_3}(x)\, p^{old}_{T_2}(t-x)\, \mathbb{1}_{t-x \in \text{Supp}[T_2]}$

Plug in $(t-x)$ for t to $(t+1)(1-p)^t p^2$.

$\sum a+b = \sum a + \sum b$

$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$

$\sum (t-x+1)$

$x \in \{t, t-1, t-2, \ldots\}$

$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\ldots,t\}} \cancel{(1)}^{(t+1)} - \sum_{x \in \{0,\ldots,t\}} x \right)$ $x \in \{\ldots, t-1, t\}$.

$= (1-p)^t p^3 \left((t+1)^2 - \frac{t(t+1)}{2} \right) = \boxed{\binom{t+2}{2}(1-p)^t p^3 = \text{NegBin}(3, p)}$

$$\sum_{x\in S} a+bx = \sum_{x\in S} a + \sum_{x\in S} bx = a\sum_{x\in S} 1 + b\sum_{x\in S} x$$

Note $(t+1)^2 - \dfrac{t(t+1)}{2}$

$t^2+2t+1 - \dfrac{t^2+t}{2}$

$\dfrac{t^2+3t+2}{2}$

$\dfrac{(t+2)(t+1)}{2}$

$\dfrac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$

$$\sum_{x\in\{0,1,\dots\}} (t+1-x)\, \mathbb{1}_{x\in\{\dots,t-1,t\}}$$

$$= \sum_{x\in\{0,1,\dots t\}} (t+1) + -x$$

$$= (t+1)\sum_{x\in\{0,1,\dots t\}} 1 - \sum_{x\in\{0,1,\dots t\}} x$$

$$= (t+1)(t+1) - \frac{t(t+1)}{2}$$

$$\underbrace{\overbrace{0\ 0\ \dots\ 0}^{1-p}\ \overset{p}{1}\ \overbrace{0\ 0\ \dots\ 0}^{1-p}\ \overset{p}{1}\ \overbrace{0\ 0\ \dots\ 0}^{1-p}}_{t+2 \text{ realization}}\ \underset{t+3}{\overset{p}{1}}$$

$t+2$ locations to put 2 ones in.

pick 2 position for the first two 1's. $\Rightarrow \binom{t+2}{2}$

$X_1, \dots, X_r \overset{iid}{\sim} Geom(p)$

$T_r = X_1 + X_2 + \dots + X_r \sim NegBin(r,p) := \binom{t+r-1}{r-1}(1-p)^t p^r$

$$\underbrace{\overbrace{0\ 0\ 1\ 0\ \ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \dots}^{t\ 0\text{'s}}}_{(r-1)\ 1\text{'s}}\ \underset{t+r}{1}$$

$t+r-1$ realization

$P_{T_r}(t) = \binom{t+r-1}{r-1}(1-p)^t p^r\, \mathbb{1}_{t\in\{0,1,\dots\}}$

t: # of 0's

r: # of 1's

$t+r-1$ locations to put $r-1$ ones in.

$\Rightarrow \binom{t+r-1}{r-1}$

$X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$.

let $n \to \infty$, $p \to 0$ but $\lambda := np$.

$\Rightarrow p = \frac{\lambda}{n}$ let $n \to \infty$.

Goal: Get the PMF of X under this limit.

$$\lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$$

$p \nearrow$ constant

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x \in \{0,1,\dots n\}}.$$

constant

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}}_{\to 1} \underbrace{\lim_{n\to\infty} \mathbb{1}_{x \in \{0,1,\dots n\}}}_{\mathbb{1}_{x \in \{0,1,\dots\infty\}}}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{(n)(n-1)(n-2)\cdots(n-x+1)}^{x \text{ terms}}}{\underbrace{n \cdot n \cdot n \cdots n}_{x \text{ terms}}} \mathbb{1}_{x \in \{0,1,\dots\}}.$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,\dots\}} = \text{Poisson}(\lambda); \ \lambda \in (0,\infty) \text{ parameter space.}$$

Note $n = 10$ $x = 4$

$$\frac{10!}{(10-4)!} = \frac{10!}{6!}$$

$$= 10 \cdot 9 \cdot 8 \cdot 7$$

$$\lim_{n\to\infty} \frac{n}{n} = 1$$

$$\lim_{n\to\infty} \frac{n-1}{n} = 1$$

$\vdots$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a.$$

$$\underbrace{\overbrace{n(n-1)\cdots(n-x+1)}^{x \text{ terms}}\overbrace{(n-x)\cdots(1)}^{(n-x)!}}_{n!}$$

So,

$$\frac{n!}{(n-x)!\,n^x} = \frac{n(n-1)\cdots(n-x+1)\cancel{(n-x)}\cdots 1}{\cancel{(n-x)}\cancel{(n-x-1)}\cdots n \cdot n \cdot n}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{n \cdot n \cdot n \cdot n}.$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$

$T = X_1 + X_2 \sim p_T(t) = ?$

$$p_T(t) = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \{0,1,\dots\}}.$$

$$= \lambda^t e^{-2\lambda} \frac{t!}{t!} \sum_{x \in \{0,1,\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x \in \{\dots, t-1, t\}}.$$

$$\boxed{= \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x \in (0,\dots,t)} \binom{t}{x}}_{2^t} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,\dots,t\}} \frac{t!}{x!(t-x)!} \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

Note $\binom{t}{x} = \frac{t!}{x!(t-x!)}$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0,\dots,t)} \binom{t}{x}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$