Let $X, Y \overset{iid}{\sim} \mathrm{Geom}(p)$

$P(X>Y) = ?$

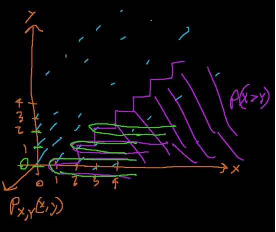
$P(X>Y) = P(Y>X)$

$P(X>Y) + P(Y>X) + P(X=Y) = 1$

$\Rightarrow 2P(X>Y) + P(X=Y) = 1$

$\Rightarrow P(X>Y) = \frac{1-P(X=Y)}{2} < \frac{1}{2}$ since $P(X=Y) > 0$

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \underbrace{p_{X,Y}(x,y)}_{= p_X(x)p_Y(y)} \mathbb{1}_{x>y}$$



$$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p_X(x)\, p_Y(y)\, \mathbb{1}_{x>y}$$

$$= \sum_{y\in\mathbb{R}} \sum_{x\in\mathbb{R}} p_X(x)\, p_Y(y)\, \mathbb{1}_{x>y}$$

$$= \sum_{y\in\mathbb{R}} p_Y(y) \sum_{x\in\mathbb{R}} p_X(x)\, \mathbb{1}_{x>y}$$

$$= \sum_{y\in\{0,1,\dots\}} p(1-p)^y \sum_{x\in\{0,1,\dots\}} p(1-p)^x\, \mathbb{1}_{\underbrace{x>y}_{x \geq y+1}}$$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x\in\{y+1,y+2,\dots\}} (1-p)^x$$

let $x' = x-(y+1) \Rightarrow x' \in \{0,1,\dots\}$

$\Rightarrow x = x' + y + 1$

reindexing trick

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x'\in\{0,1,\dots\}} (1-p)^{x'}(1-p)^y(1-p)^1$$

$$= p^2(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y} \boxed{\sum_{x'\in\{0,1,\dots\}} (1-p)^{x'}}$$

Geometric Series, for $q \in (-1,1)\setminus\{0\}$

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$= \cancel{p^2}(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y} \left(\frac{1}{\cancel{p}}\right)$$

$$= p(1-p) \boxed{\sum_{y\in\{0,1,\dots\}} \left((1-p)^2\right)^y} = \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2} = \frac{1}{p(2-p)}$$

$$= \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \boxed{\frac{1-p}{2-p}} < \frac{1}{2}$$

Bag of fruit of Apples and Bananas

Draw with replacement n times

Let $X_1 = \#$ apples, $p_1 = P(\text{apple})$

$\Rightarrow X_1 \sim \mathrm{Bin}(n, p_1)$

Draw n with replacement

$X_1 = \#$ apples, $X_2 = \#$ bananas

$X_1 \sim \mathrm{Bin}(n,p_1)$, $X_2 \sim \mathrm{Bin}(n,p_2)$

Are X_1 and X_2 independent?

Since $X_1 + X_2 = n \Rightarrow X_1, X_2$ dependent

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{X} \sim p_{\vec{X}}(\vec{x}) = p_{\vec{X}}(x_1,x_2) = \frac{n!}{x_1!\,x_2!}\, p_1^{x_1} p_2^{x_2}\, \mathbb{1}_{x_1+x_2=n}\, \mathbb{1}_{x_1\in\{0,1,\dots,n\}}\, \mathbb{1}_{x_2\in\{0,1,\dots,n\}}$$

vector rv

$\binom{n}{x_1, x_2}$ multichoose notation

$$\Rightarrow \vec{X} \sim \mathrm{Multi}(n, \vec{p}) = \binom{n}{x_1,x_2} p_1^{x_1} p_2^{x_2}, \quad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Multinomial rv of dim=2

Since X_1, X_2 are dependent, we cannot factor this JMF

Bag of fruit now has cantaloupes. You draw cantaloupes with probability p_3 and X_3 is the count of cantaloupes.

$$\vec{X} \sim \mathrm{Multi}(n,\vec{p}) = \binom{n}{x_1,x_2,x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \frac{n!}{x_1!\,x_2!\,x_3!}\, p_1^{x_1} p_2^{x_2} p_3^{x_3}\, \mathbb{1}_{x_1+x_2+x_3=n}\, \mathbb{1}_{x_1\in\{0,\dots,n\}}\, \mathbb{1}_{x_2\in\{0,\dots,n\}}\, \mathbb{1}_{x_3\in\{0,\dots,n\}}$$

In general, if there are K types of fruit (# categories) then the general multinomial rv of dim K is:

$$\vec{X} \sim \mathrm{Multi}(n,\vec{p}) = \binom{n}{x_1,x_2,\dots,x_K} \prod_{k=1}^{K} p_k^{x_k}, \quad \vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$$

Parameter Space: $n \in \mathbb{N}$, $\vec{p} \in \{\vec{v} : \vec{v}\cdot\vec{1} = 1,\ v_1 \in (0,1), \dots, v_K \in (0,1)\}$

Support: $\mathrm{Supp}[\vec{X}] = \{\vec{x} : \vec{x}\cdot\vec{1} = n,\ x_1 \in \{0,1,\dots,n\}, \dots, x_K \in \{0,1,\dots,n\}\}$

$$\vec{X} \sim \mathrm{Multi}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) = \binom{n}{x_1,x_2} p^{x_1}(1-p)^{x_2}$$

$K=2$

$P(X_1 = x_1 \mid X_2 = x_2) \overset{?}{=} P(X_1 = x_1) = \mathrm{Bin}(n,p_1)$ Dependent?

$= \mathrm{Deg}(n-x_2) \Rightarrow$ Dependent!

Conditional PMF

$$p_{X_1|X_2}(x_1,x_2) := \frac{p_{X_1,X_2}(x_1,x_2)}{p_{X_2}(x_2)}$$

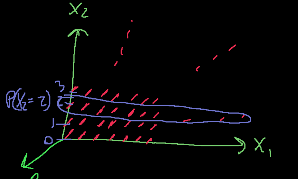
JMF

Marginal PMF of X_2

Want to show: $X_2 \sim \mathrm{Bin}(n, p_2)$

$$p_{X_2}(x_2) = P(X_2 = x_2) = \sum_{x_1\in\mathbb{R}} p_{X_1,X_2}(x_1,x_2)$$

"Margining out X_1"



$$= \sum_{x_1\in\mathbb{R}} \binom{n}{x_1,x_2} p^{x_1}(1-p)^{x_2}$$

$$= \sum_{x_1\in\mathbb{R}} \frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2}\, \mathbb{1}_{x_1+x_2=n}\, \mathbb{1}_{x_1\in\{0,\dots,n\}}\, \mathbb{1}_{x_2\in\{0,1,\dots,n\}}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2}\, \mathbb{1}_{x_2\in\{0,1,\dots,n\}} \sum_{x_1\in\{0,\dots,n\}} \frac{p^{x_1}}{x_1!}\, \mathbb{1}_{x_1=n-x_2}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2}\, \mathbb{1}_{x_2\in\{0,1,\dots,n\}} \frac{p^{n-x_2}}{(n-x_2)!} = \binom{n}{x_2} p^{n-x_2}(1-p)^{x_2} = \mathrm{Bin}(n, 1-p)$$

Margining a multinomial to yield one dimension is binomial.