Lecture 04

9/9/20

Let $X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X>Y) = ?$

$P(X>Y) = P(Y>X)$

$P(X>Y) + P(Y>X) + P(X=Y) = 1$

$\to 2P(X>Y) + P(X=Y) = 1$

$\to P(X>Y) = \frac{1 - P(X=Y)}{2} < \frac{1}{2}$

(since $P(X=Y) > 0$)

$$P(X>Y) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} \underbrace{p_{X,Y}(x,y)}_{p_X(x)\, p_Y(y)} \mathbb{1}_{x>y}$$

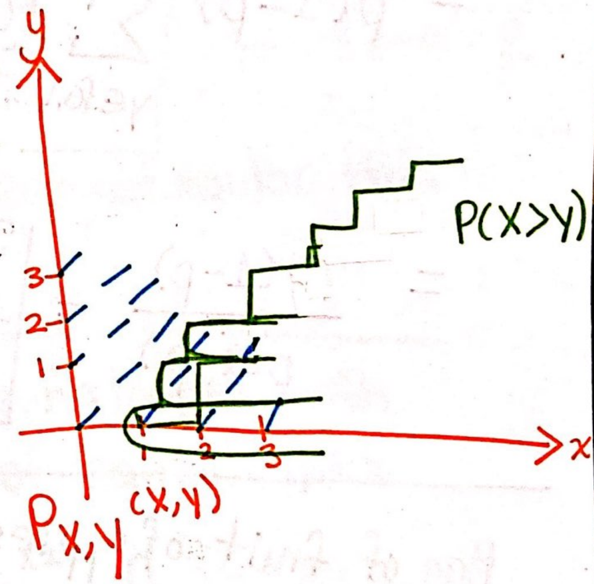
$$= \sum_{x \in \mathbb{R}} p_X(x) \sum_{y \in \mathbb{R}} p_Y(y) \mathbb{1}_{x>y}$$

$$\cancel{= \sum_{x \in \{0,1,\dots,n\}} p_X^{old}(x) \sum_{y \in \{0,1,\dots,n\}} p_Y^{old}(y)} \mathbb{1}_{x>y}$$

$$= \sum_{y \in \mathbb{R}} p_Y(y) \sum_{x \in \mathbb{R}} p_X(x) \mathbb{1}_{x>y}$$

$$= \sum_{y \in \{0,1,\dots,n\}} p(1-p)^y \sum_{x \in \{0,1,\dots\}} p(1-p)^x \mathbb{1}_{x \geq y+1}$$

$$= p^2 \sum_{y \in \{0,1,\dots\}} (1-p)^y \sum_{x \in \{y+1, y+2, \dots\}} (1-p)^x$$



let $x' = x - (y+1)$

$\to x' \in \{0,1,\dots\}$

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$$\hookrightarrow = p^2 \sum_{y \in \{0,1,\dots\}} (1-p)^y \sum_{x' \in \{0,1,\dots\}} (1-p)^{x'} (1-p)^y (1-p)^1$$ "reindexing trick"

$$= p^2 (1-p) \sum_{y \in \{0,1,\dots\}} (1-p)^{2y} \boxed{\sum_{x' \in \{0,1,\dots\}} (1-p)^{x'}}$$

Geometric Series

$$\hookrightarrow \frac{1}{1-(1-p)} = \frac{1}{p}$$

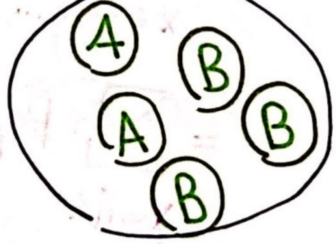
$$= p^{\cancel{2}} (1-p) \sum_{y \in \{0,1,\dots,\}} (1-p)^{2y} \left(\frac{1}{\cancel{p}}\right)$$

$$= p(1-p) \boxed{\sum_{y \in \{0,1,\dots\}} \left((1-p)^2\right)^y} \rightarrow \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)}$$

$$= \frac{1}{2p-p^2} = \frac{1}{p(2-p)}$$

$$= \frac{p(1-p)}{p(2-p)} = \boxed{\frac{1-p}{2-p}} < \frac{1}{2}$$

Bag of fruit of apples and bananas



· Draw with replacement n times

Let X_1 = number of apples

$p_1 = P(\text{apples})$

$\rightarrow X_1 \sim \text{Bin}(n, p_1)$

· Draw with~~out~~ replacement

X_1 = number of apples

X_2 = number of bananas

$X_1 \sim \text{Bin}(n, p_1)$,

$X_2 \sim \text{Bin}(n, p_2)$

Are X_1 and X_2 independent?

Since $X_1 + X_2 = n \rightarrow X_1, X_2$ dependent.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{X} \sim P_{\vec{X}}(\vec{x}) = P_{\vec{X}}(x_1, x_2) =$$

$$\underbrace{\frac{n!}{\underbrace{x_1!}_{\text{apples}}\underbrace{x_2!}_{\text{bananas}}} \underbrace{p_1^{x_1}}_{\text{apples}}}_{\text{vector rv}} \underbrace{p_2^{x_2}}_{\text{bananas}} \cancel{\mathbb{1}_{n \in \mathbb{N}}} \underbrace{\mathbb{1}_{x_1+x_2=n} \underbrace{\mathbb{1}_{x_1 \in \{0,1,\ldots,n\}}}_{\text{apples}} \underbrace{\mathbb{1}_{x_2 \in \{0,1,\ldots,n\}}}_{\text{bananas}}}$$

vector rv

$$\binom{n}{x_1, x_2}$$ multichoose notation

$$\Rightarrow \vec{X} \sim \text{Multi}(n, \vec{p}) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$ Multinominal rv of dim=2

$$\vec{p} = \begin{bmatrix} p \\ p_2 \end{bmatrix}$$

- Since X_1, X_2 are dept. → we cannot factor this JMF.

Bag of fruit has cantaloupes. You draw cantaloupes with probability p_3 and X_3 is the count of cantaloupes

$$\vec{X} \sim \text{Multi}(n, \vec{p}) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_2} \mathbb{1}_{x_1+x_2+x_3=n}$$

$$\mathbb{1}_{x_1 \in \{0,1\ldots,n\}} \mathbb{1}_{x_2 \in \{0,1,\ldots,n\}} \mathbb{1}_{x_3 \in \{0,1,\ldots,n\}}$$

- In general, if there are K types of fruits (# of categories) then the general multinomial rv of dim K is:

$$\vec{X} \sim \text{Multi}(n, \vec{p}) = \binom{n}{x_1, x_2, \ldots, x_K} \prod_{k=1}^{K} p_k^{x_k}$$

$$\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$$

Parameter space: $n \in \mathbb{N}$, $\vec{p} \in \{\vec{r} : \vec{p} \cdot \vec{1} = 1, r_1 \in (0,1), \ldots, r_k \in (0,1)\}$

Support: ~~X~~ $Supp[\vec{X}] = \{\vec{x} : \vec{x} \cdot \vec{1} = n, x_1 \in \{0,1,\ldots,n\} \ldots\ldots x_k \in \{0,1,\ldots,n\}\}$

Let's say $\vec{X} \sim Multi(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$

$k = 2$

dependent?

$P(X_1 = x_1 \mid X_2 = x_2) \overset{?}{=} P(X_1 = x_1) = Bin(n, p_1)$

"

$Deg(n - x_2) \Rightarrow$ dependent. ✓

Conditional PMF.

JMF

$$P_{X_1|X_2}(x_1, x_2) = \frac{P_{X_1, X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

marginal PMF of X_2

$\boxed{\text{want to show: } X_2 \sim Bino(n, p_2)}$

$$P_{X_2}(x_2) = P(X_2 = x_2) = \sum_{x_1 \in \mathbb{R}} P_{X_1, X_2}(x_1, x_2) = \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2} \rightarrow$$

"Margining out x_1"

$$\rightarrow = \sum_{x_1 \in \mathbb{R}} \frac{n!}{x_1! x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\ldots,n\}} \mathbb{1}_{x_2 \in \{0,1,\ldots,n\}}$$

$\cancel{\frac{n!}{x_2!} p \sum_{x_1 \in \{p, n\}}}$

$$\frac{n!}{x_2!} (1-p)^{x_2} \mathbb{1}_{x_2 \in \{0,1,\ldots,n\}} \sum_{x \in \{p \ldots n\}} \frac{p^{x_1}}{x!} \mathbb{1}_{x_1 + x_2 = n} \text{ or } x_1 = n - x_2$$

$$= \frac{n!}{x_2!}(1-p)^{x_2} \prod_{x_2 \in \{0,1,\dots,n\}} \frac{p^{n-x_2}}{(n-x_2)!} = \binom{n}{x_2} p^{n-x_2}$$

combine

$(1-p)^{x_2} = \text{Binom}(n, 1-p)$

$\blacksquare$

- Margining a multinomial to yield one dimension is binomial.