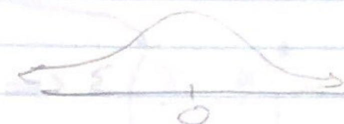
Lecture 19 11/18/2020

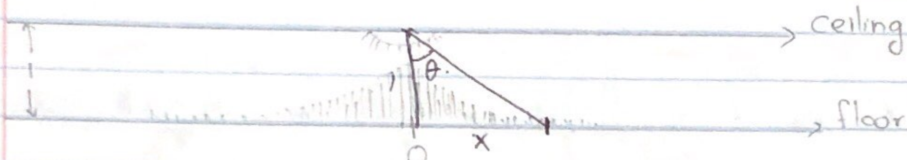
$X \sim \text{Caushy}(0,1) = \frac{1}{\pi}\frac{1}{x^2+1}$

$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi}\frac{1}{x^2+1}\,dx = \infty$



$M_X(t) = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = \infty$ mgf doesn't exist.

$\phi_X(t) = \int_{\mathbb{R}} e^{itx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx = \cdots = e^{-|t|}$



bc it is 1:1 in support

$\theta \sim U(-\pi/2, \pi/2) = 1/\pi\, \mathbb{1}_{\theta \in [-\pi/2, \pi/2]}$, $X = \tan(\theta) \Rightarrow \theta = \arctan(x)$

$\left|\frac{d}{dx}[\arctan(x)]\right| = \frac{1}{x^2+1}$

$f_X(x) = f_\theta(\arctan(x))\frac{1}{x^2+1} = \frac{1}{\pi}\mathbb{1}_{\arctan(x)\in[-\pi/2,\pi/2]}\frac{1}{x^2+1}$

$= \text{Caushy}(0,1)$

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$T_n \sim N(n\mu, n\sigma^2)$

$\bar{X}_n \sim N(\mu, \sigma^2/n)$ "sample mean" or "average"

$S_n^2 = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2 \sim f_{S_n^2}(s^2) = ?$ "sample Variance"

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$ $\quad \Sigma Z_i^2 \sim \chi^2_n$ $\quad \vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}, \vec{Z}^T \vec{Z} \sim \chi^2_n$

$$Z_1 = \frac{X_1 - \mu}{\sigma}, \ldots, Z_n = \frac{X_n - \mu}{\sigma} \Rightarrow \Sigma \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2_n$$

$$(X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2 = (X_i - \bar{X})^2 + \underbrace{2(X_i - \bar{X})(\bar{X} - \mu)}_{X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu} + (\bar{X} - \mu)^2$$

$$\Sigma (X_i - \mu)^2 = \underbrace{\Sigma (X_i - \bar{X})^2}_{(n-1)S^2} + 2(\bar{X} \underbrace{\Sigma X_i}_{n\bar{X}} - n\bar{X}^2 - \mu \underbrace{\Sigma X_i}_{n\bar{X}} + n\bar{X}\mu) + n(\bar{X} - \mu)^2.$$

$$\frac{\Sigma (X_i - \mu)^2}{\sigma^2} = \underbrace{\frac{n-1}{\sigma^2} S^2}_{\text{conjecture: } \sim \chi^2_{n-1}} + \underbrace{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2}_{\sim \chi^2_1} \sim \chi^2_n$$

$U_1 \sim \chi^2_{k_1}$ indep of $U_2 \sim \chi^2_{k_2}$ $\Rightarrow U_1 + U_2 \sim \chi^2_{k_1 + k_2}$

this would be true if $\bar{X}$ is independent of S^2.

"quadratic form"

$$\vec{Z}^T \vec{Z} = \vec{Z}^T \underbrace{I}_{B} \vec{Z} \sim \chi^2_n.$$

consider: $\vec{Z}^T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \vec{Z} = Z_1^2 \sim \chi^2_1$.

rank$[B_i] = 1$

$$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix}}_{B_i} \vec{Z} = Z_i^2 \sim \chi^2_1$$

$$\vec{Z}^T I \vec{Z} = \vec{Z}^T (B_1 + B_2 + \ldots + B_n) \vec{Z} = \vec{Z}^T B_1 \vec{Z} + \ldots + \vec{Z}^T B_n \vec{Z} \sim \chi^2_n.$$

Cochran's thm: let $B_1 + B_2 + \dots + B_k = I$ s.t

$\sum_{j=1}^{k} \text{rank}[B_j] = n$ then (a) $\vec{z}^{\,T} B_j \vec{z} \sim \chi^2_{\text{rank}[B_j]}$

and (b) $\vec{z}^{\,T} B_{j_1} \vec{z}$ is independent of $\vec{z}^{\,T} B_{j_2} \vec{z}$ $\forall j_1 \neq j_2$.

$\sum (z_i - \bar{z})\bar{z} = n\bar{z}^2 - n\bar{z}^2 = 0$

$$\vec{z}^{\,T}\vec{z} = \sum z_i^2 = \sum \left((z_i - \bar{z}) + \bar{z}\right)^2$$

$$= \sum (z_i - \bar{z})^2 + 2\sum (z_i - \bar{z})\bar{z} + n\bar{z}^2 = \sum (z_i - \bar{z})^2 + n\bar{z}^2$$

let $\vec{1}_n$ is a column vector of all ones $\Rightarrow \bar{z} = \frac{1}{n}\vec{z}^{\,T}\vec{1}$

$$n\bar{z}^2 = n\left(\frac{1}{n}\vec{z}^{\,T}\vec{1}\right)^2 = \cancel{n}\frac{1}{\cancel{n}}\vec{z}^{\,T}\vec{1}\,\frac{1}{n}\,\vec{z}^{\,T}\vec{1} = \vec{z}^{\,T}\underbrace{\left(\tfrac{1}{n}\vec{1}\vec{1}^{\,T}\right)}_{B_2 = 1/n\,J_n}\vec{z}$$

rank$[B_2] = 1$

let $J_n = \vec{1}_n\vec{1}_n^{\,T}$ which is an $n \times n$ matrix of all entries =

$$\sum (z_i - \bar{z})^2 = \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2 = \sum z_i^2 - n\bar{z}^2 \cdot \quad B_1 + B_2 =$$

$$= \vec{z}^{\,T} I \vec{z} - \vec{z}^{\,T}(1/n\,J_n)\vec{z} = \vec{z}^{\,T}(\underbrace{I - 1/n\,J}_{B_1})\vec{z}$$

$$(I - 1/n\,J) + 1/n\,J = I$$

Thm from math 231: If A is symmetric matrix and idempotent which means $AA = A$ then rank$[A]$ = tr$[A]$ = sum of the diagonal of A

$$\begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix}\begin{bmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3&3&3 \\ 3&3&3 \\ 3&3&3 \end{bmatrix}}_{3J}$$

$$B_1^T = (I - 1/n\,J)^T = I^T - 1/n\,J^T = I - 1/n\,J = B_1 \checkmark$$

$$B_1 B_1 = (I - 1/n\,J)(I - 1/n\,J) = II - 1/n\,JI - 1/n\,IJ + 1/n^2\,JJ$$

$$= I - 2/n\,J + 1/n\,J = I - 1/n\,J = B_1 \checkmark \Rightarrow \text{rank}[B_1] = \text{tr}[B_1]$$

$$B_{1_{i,i}} = 1 - 1/n1 = 1 - 1/n \quad = \sum_{i=1}^{n} 1 - 1/n = n-1$$

Putting it all together, we can use Cochran's thm!

$\Sigma(Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ indep of $n\bar{Z}^2 \sim \chi^2_1$

$$\bar{Z} = \frac{Z_1 + \cdots + Z_n}{n} = \frac{\frac{X_1-\mu}{\sigma} + \cdots + \frac{X_n-\mu}{\sigma}}{n} = \frac{\Sigma X_i - n\mu}{n\sigma}$$

$$= \frac{\bar{X}-\mu}{\sigma}$$

$$\Rightarrow n\bar{Z}^2 = (\sqrt{n}\,\bar{Z})^2 = \left(\frac{\bar{Z}}{1/\sqrt{n}}\right)^2 \overset{\downarrow}{=} \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_1$$

$$\Sigma(Z_i - \bar{Z})^2 = \Sigma\left(\frac{X_i - \cancel{\mu}}{\sigma} - \frac{\bar{X} - \cancel{\mu}}{\sigma}\right)^2 = \frac{1}{\sigma^2}\Sigma(X_i - \bar{X})^2.$$

independent of

$$= \frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$$

I think the first to prove this was Fisher in 1925 and then in 1936, Geary proved the iid normal rv is the *only* distribution that has the independent of $\bar{X}$ and S^2.

$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ but what about $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim ?$ Not $N(0,1)$