MATH 368/621 Fall 2020 Homework #4

Professor Adam Kapelner

Due by email 11:59PM Sunday, Nov 1, 2020

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Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review from math 241 about conditional probability, expectation and variance then read on your own about PMF transformations, the family of gamma functions, the negative binomial, poisson, exponential, Erlang, uniform rv's and the Poisson process.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 7 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ______________________________ SECTION: _____ CLASS: 368 | 621

Problem 1

These exercises will give you practice with transformations of discrete r.v.'s.

(a) [easy] Let $X \sim \text{Binomial}\,(n,\, p)$. Find the PMF of $Y = g(X) = \ln(X+1)$.

(b) [harder] Let $X \sim \text{Binomial}\,(n,\, p)$. Find the PMF of $Y = g(X) = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

(c) [difficult] Let $X \sim \text{Binomial}\,(n,\, p)$ where n is an even number. Find the PMF of $Y = g(X) = \text{mod}(X, 2)$ where "mod" denotes modulus division of the first argument by the second argument.

(d) [difficult] [MA] Let $X \sim \text{NegBin}\,(k,\, p)$. Find the PMF of $Y = g(X) = \text{mod}(X, n)$ where $n \in \mathbb{N}$.

Problem 2

These exercises will give you practice with transformations of continuous r.v.'s and the quantile function.

(a) [harder] Let $X \sim \mathrm{U}(0, 1)$. Find the PDF of $Y = g(X) = aX + c$. Make sure you're careful with the indicator function that specifies the support. There are two cases.

(b) [harder] Let $X \sim \mathrm{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.

(c) [E.C.] Let $X \sim \mathrm{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$.

(d) [harder] Let $X \sim \mathrm{U}(0, 1)$. Find the PDF of $Y = g(X) = \ln\left(\frac{X}{1-X}\right)$. If this is a brand name r.v., mark it so and include its parameter values.

(e) [easy] Find the Quantile function of X where $X \sim \text{Logistic}(0, 1)$.

(f) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Logistic}(\mu, \sigma)$ where $X \sim \text{Logistic}(0, 1)$.

(g) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

(h) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^{X}$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.

(i) [easy] Rederive the $X \sim \text{Laplace}(0, 1)$ r.v. model by taking the difference of two standard exponential r.v.'s.

(j) [easy] Let $X \sim \text{Laplace}(0, 1)$. Prove that $\mathbb{E}[X] = 0$ without using the integral definition. There's a trick.

(k) [easy] Find the PDF of $Y = \sigma X + \mu \sim \text{Laplace}(\mu, \sigma)$ where $X \sim \text{Laplace}(0, 1)$.

(l) [difficult] Show that $\mathcal{E} \sim \text{Laplace}(0, \sigma)$ is a reasonable error distribution.

(m) [harder] [MA] Find the Quantile function of X where $X \sim \text{Laplace}(0, 1)$.

(n) [difficult] [MA] Let $X \sim \text{ParetoI}(k, \lambda)$. Show that $Y = X \mid X > c$ where $c > k$ is also a ParetoI r.v. and find its parameter values.

Problem 3

We will now explore a couple of extreme distributions.

(a) [harder] Let $X \sim \text{Exp}\,(1)$ and $Y = -\ln{(X)} \sim \text{Gumbel}(0, 1)$. Find the PDF of this standard Gumbel distribution. Make sure you include the indicator function throughout your proof.

(b) [easy] Find the CDF of Y.

(c) [easy] Let $G = \beta Y + \mu \sim \text{Gumbel}(\mu, \beta)$. Find the PDF of G, the general Gumbel distribution.

(d) [easy] [MA] Show that for any r.v. X, if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

(e) [easy] Using the answer in the previous question, find the CDF of $G \sim \text{Gumbel}(\mu, \beta)$.

Problem 4

These exercises will give you practice with the Weibull distribution.

(a) [easy] If $X \sim \text{Exp}(1)$ then show that $Y = \frac{1}{\lambda} X^{\frac{1}{k}} \sim \text{Weibull}(k, \lambda)$ where $k, \lambda > 0$.

(b) [harder] Find $\text{Med}[Y]$.

(c) [difficult] [MA] Prove that if $k > 1$ then $\mathbb{P}(Y \geq y + c \mid Y \geq c) < \mathbb{P}(Y \geq y)$ for $c > 0$.

(d) [difficult] If $X \sim \text{Exp}(\lambda)$ then show that $Y = X^\beta \sim$ Weibull where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class (i.e. your answer in part a).

(e) [easy] Using Y, the Weibull in terms of the parameterization we learned in class (i.e. your answer in part a), find the PDF of $W = Y + c \sim \text{Weibull}(k, \lambda, c)$ which is known as the "translated Weibull" or "3-parameter Weibull model".

Problem 5

We will practice finding kernels and relating them to known distributions. The gamma function and the beta function will come up as well.

(a) [easy] Find the kernel of the negative binomial PMF.

(b) [easy] Find the kernel of the beta PDF.

(c) [easy] If $k(x) = e^{-\lambda x} x^{k-1} \mathbb{1}_{x>0}$ how would you know if the r.v. X was an $\text{Erlang}(k, \lambda)$ or a $\text{Gamma}(k, \lambda)$?

(d) [harder] If $k(x) = x e^{-x^2} \mathbb{1}_{x>0}$, how is X distributed?

Problem 6

We will now practice using order statistics concepts.

(a) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the CDF of the maximum X_i and express the CDF of the minimum X_i.

(b) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF of the maximum X_i and express the PDF of the minimum X_i.

(c) [easy] If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$ where its CDF is denoted $F(x)$, express the PDF and the CDF of $X_{(k)}$ i.e. the kth smallest X_i.

(d) [difficult] [MA] If discrete $X_1, \ldots, X_n \overset{iid}{\sim} p(x)$, why would the formulas in (a-c) not be accurate?

(e) [harder] If $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{U}(0,\, 1)$, show that $X_{(k)} \sim \mathrm{Beta}(k,\, n-k+1)$.

(f) [harder] Express $\binom{n}{k}$ in terms of the beta function.

(g) [E.C.] If $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{U}(a,\, b)$, show that $X_{(k)}$ is a linear transformation of the beta distribution and find its parameters.

(h) [harder] [MA] Show that $I_x(a, b+1) = I_x(a, b) + \frac{x^a(1-x)^b}{b\mathrm{B}(a,b)}$.

Problem 7

We will now practice multivariate change of variables where $\boldsymbol{Y} = \boldsymbol{g}(\boldsymbol{X})$ where $\boldsymbol{X}$ denotes a vector of k continuous r.v.'s and $\boldsymbol{g} : \mathbb{R}^k \to \mathbb{R}^k$ and is 1:1.

(a) [easy] State the formula for the PDF of $\boldsymbol{Y}$.

(b) [harder] Demonstrate that the formula for the PDF of $\boldsymbol{Y}$ reduces to the univariate change of variables formula if the dimensions of $\boldsymbol{Y}$ and $\boldsymbol{X}$ are 1.

(c) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$.

(d) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent.

(e) [easy] State the formula for the PDF of $R = \frac{X_1}{X_2}$ if X_1 and X_2 are independent and have positive supports.

(f) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$.

(g) [easy] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent.

(h) [harder] State the formula for the PDF of $R = \frac{X_1}{X_1+X_2}$ if X_1 and X_2 are independent and have positive supports. This should be a simpler expression than the previous.

(i) [difficult] Find a formula for the PDF of $E = X_1^{X_2}$ where $X_1, X_2 \stackrel{iid}{\sim} f(x)$.

(j) [difficult] Find the simplest formula you can for the PDF of $Q = \frac{X_1}{X_2} e^{X_3}$ where X_1, X_2, X_3 are dependent r.v.'s.

(k) [difficult] Show that $R = \frac{X_1}{X_2} \sim \beta'(\alpha, \beta)$, the beta prime distribution, if $X_1 \sim \text{Gamma}\,(\alpha, 1)$ independent of $X_2 \sim \text{Gamma}\,(\beta, 1)$.