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$g: \mathbb{R}^n \to \mathbb{R}^n$ and 1 to 1. $\vec{X}, \vec{Y}$ continuous rv vectors of dim n s.t. $\vec{Y} = g(\vec{X})$

Given $f_{\vec{X}}(x)$, find $f_{\vec{Y}}(y)$

recall what a multidimensional function looks like:

$Y_1 = g_1(X_1, \dots, X_n)$, Since 1 to 1, $X_1 = h_1(Y_1, \dots Y_n)$, $\vec{h} = \vec{g}^{-1} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$

$Y_2 = g_2(X_1, \dots, X_n)$, $X_2 = h_2(Y_1, \dots, Y_n)$,

$\vdots$

$Y_n = g_n(X_1, \dots, X_n)$ $X_n = h_n(Y_1, \dots, Y_n)$

using multivariable calculus, you can show that $f_{\vec{Y}}(y) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h(\vec{y})|$ jacobian determinant

$$J: \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

lets verify the convolution formula via this general change of variables formula

$T = X_1 + X_2 \sim f_T(t)$

Recipe:

1) find a "clever" g so that
2) we can compute an h
3) Compute J_h
4) Compute the multidimensional change of variables formula
5) integrate out the "nuisance dimension"

① $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, let $Y_2 = X_2 = g_2(X_1, X_2)$ a nuisance dimension

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h| = f_{\vec{X}}(Y_1 - Y_2, y_2)\,|1|$

⑤ $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\,dy_2 = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\,du$ general convolution if X_1, X_2 indep $\qquad\qquad\qquad\qquad$ if iid

$$= \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du = \int_{\mathbb{R}} f(t-u) f(u)\,du$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_{Supp[X_2]} f^{old}_{X_1}(t-u) \mathbb{1}_{t-u \in Supp[X_1]} f^{old}_{X_2}(t)\,du \;\Big|\; \int_{Supp[X]} f(t-u) \mathbb{1}_{t-u \in Supp[X]} f(t)\,du$$

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$$\underset{\text{(ratio)}}{R} = \frac{X_1}{X_2} \sim f_R(y) = ?$$

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2) \qquad Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)\ , \quad X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(ru, u)\,|u|\,du$

if X_1, X_2 indep $\qquad\qquad\qquad\qquad$ if iid

$$= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)\,|u|\,du = \int_{\mathbb{R}} f(ru) f(u)\,|u|\,du$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_{Supp[X_2]} f^{old}_{X_1}(ru) \mathbb{1}_{ru \in Supp[X_1]} f^{old}_{X_2}(u)\,|u|\,du \;\Big|\; \int_{Supp[X]} f^{old}(ru) \mathbb{1}_{ru \in Supp[X]} f^{old}(u)\,|u|\,du$$

$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_1+X_2} = g_1(X_1, X_2)$ $\quad Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1(X_1+X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix} = y_2(1-y_1) + y_1 y_2 = y_2 - y_1 y_2 + y_1 y_2 = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|\, dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)|u|\, du$

X_1, X_2 indep $\searrow$ $= \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u-ru)|u|\, du \overset{iid}{=} \int_{\mathbb{R}} f(ru) f(u-ru)|u|\, du$

$\|$

$\int_{\mathbb{R}} f^{old}_{X_1}(ru)\mathbb{1}_{ru \in Supp[X_1]} \; f^{old}_{X_2}(u-ru)\mathbb{1}_{u-ru \in Supp[X_2]} \; |u|\, du$

$\sim$

$X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$

$e^{-\beta u} / e^{\beta r u}$ $\quad u$

$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = \int_{[0,\infty)/r = [0,\infty)} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1-1} e^{-\beta r u} \cdot \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(u-ru)^{\alpha_2-1} e^{-\beta(u-ru)} \mathbb{1}_{u-ru \in [0,\infty)} |u|\, du$

$1 > r > 0$

$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty (ru)^{\alpha_1-1}(u-ru)^{\alpha_2-1} e^{-\beta u} \mathbb{1}_{u-ru \in [0,\infty)} u\, du$

$r^{\alpha_1-1} u^{\alpha_1-1} \quad (u(1-r))^{\alpha_2-1} \quad u(1-r) \in [0,\infty)$

$u^{\alpha_2-1}(1-r)^{\alpha_2-1} \quad u \in [0,\infty)$

$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta u}}_{j(\alpha_1,\alpha_2)} \underset{Kernel}{\propto} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \propto Beta(\alpha_1, \alpha_2)$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, $R = \overset{r>0}{\frac{X_1}{X_2}} \sim f_R(r) = ?$

$$f_R(r) = \int_{Supp[X_2]} f_{X_1}^{old}(ru)\,\mathbb{1}_{ru \in Supp[X_1]}\, f_{X_2}^{old}(u)\,|u|\,du = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1-1} e^{-\beta r u}\,\mathbb{1}_{\substack{ru \in [0,\infty) \\ u \in [0,\infty)}}\, \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2-1} e^{-\beta u}\, u\,du$$

$$= \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}}_{\frac{1}{B(\alpha_1,\alpha_2)}} r^{\alpha_1-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\overset{c}{\widetilde{\beta(r+1)}}u}\,du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{\underbrace{(\beta(r+1))^{\alpha_1+\alpha_2}}_{\beta^{\alpha_1+\alpha_2}(r+1)^{\alpha_1+\alpha_2}}}} = \frac{1}{B(\alpha_1+\alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0}$$

$\vdots$ BetaPrime(α_1, α_2)

$$R \sim \text{BetaPrime}(\alpha_1, \alpha_2) := \underbrace{\frac{1}{B(\alpha_1,\alpha_2)}}_{c} \underbrace{\frac{r^{\alpha-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0}}_{k(r)}$$