We'll be doing "arbitrary multivariate transformations" of variables.

$\vec{g}: \mathbb{R}^n \to \mathbb{R}^n$ and invertible. Let $\vec{X}, \vec{Y}$ be cont. vector rv's both with dimension n and $\vec{Y} = \vec{g}(\vec{X})$

Given the jdf of the vector X rv, find the jdf of the vector Y rv. This generalizes what we did previously with univariate change of variables. Let's recall what this multivariate function looks like.

$$Y_1 = g_1(X_1, \ldots, X_n) \qquad \vec{g} = [g_1 \cdots g_n]^T \qquad X_1 = h_1(Y_1, \ldots, Y_n)$$

$$Y_2 = g_2(X_1, \ldots, X_n) \qquad \vec{g}^{-1} = \vec{h} = [h_1 \cdots h_n]^T \qquad X_2 = h_2(Y_1, \ldots, Y_n)$$

$$\vdots \qquad\qquad \vdots$$

$$Y_n = g_n(X_1, \ldots, X_n) \qquad\qquad X_n = h_n(Y_1, \ldots, Y_n)$$

From multivariable calculus, you can show that the multivariate change of variables formula is:

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h(\vec{y})|$$

where $J_h := \det \begin{bmatrix} \partial h_1/\partial y_1 & \cdots & \partial h_1/\partial y_n \\ \vdots & & \\ \partial h_n/\partial y_1 & \cdots & \partial h_n/\partial y_n \end{bmatrix}$, this is called the "Jacobian determinant"

Let's use this formula to prove the convolution formula for $T = X_1 + X_2 \sim f_T(t) = ?$

There's a recipe for these types of problems
(1) Find a g (set the first dimension Y_1 = your target) so that ...
(2) you can find the h.
(3) Compute the Jacobian Determinant J_h
(4) Substitute 1-3 into the multivariate change of variables formula
(5) Integrate the "nuisance dimension(s)".

"nuisance"

(1) $T = Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - -1 \cdot 0 = 1$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 - y_2, y_2)\,|1|$

(5) $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(t-u, u)\,du$

$\overset{X_1, X_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(t-u) f(u)\,du$

$= \int_{Supp[X_2]} f^{old}_{X_1}(t-u)\,\mathbb{1}_{t-u \in Supp[X_1]}\, f^{old}_{X_2}(u)\,du$

$= \int_{Supp[X]} f^{old}(t-u)\,\mathbb{1}_{t-u \in Supp[X]}\, f^{old}(u)\,du$

$R = \frac{X_1}{X_2} \sim f_R(r) = ?$

(1) $R = Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|$

(5) $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|\,dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u)\,|u|\,du$ general formula!

$\overset{X_1, X_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)\,|u|\,du \overset{\text{if iid}}{=} \int_{\mathbb{R}} f(ru) f(u)\,|u|\,du$

$= \int_{Supp[X_2]} f^{old}_{X_1}(ru)\,\mathbb{1}_{ru \in Supp[X_1]}\, f^{old}_{X_2}(u)\,|u|\,du$

$= \int_{Supp[X]} f^{old}(ru)\,\mathbb{1}_{ru \in Supp[X]}\, f^{old}(u)\,|u|\,du$

$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

(1) $R = Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$, $Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1(X_1 + X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{bmatrix} = y_2(1 - y_1) - -y_1 y_2 = y_2 - y_1 y_2 + y_1 y_2 = y_2$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|$

(5) $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)\,|u|\,du$

$\overset{\text{if } X_1, X_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u - ru)\,|u|\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(ru) f(u - ru)\,|u|\,du$

$= \int_{\mathbb{R}} f^{old}(ru)\,\mathbb{1}_{ru \in Supp[X]}\, f^{old}(u - ru)\,\mathbb{1}_{u - ru \in Supp[X]}\,|u|\,du$

$\to \int_{\mathbb{R}} f^{old}_{X_1}(ru)\,\mathbb{1}_{ru \in Supp[X_1]}\, f^{old}_{X_2}(u - ru)\,\mathbb{1}_{u - ru \in Supp[X_2]}\,|u|\,du$

$X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$, $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

R is the proportion of the waiting time for the first gamma and thus Supp[R] = [0,1]

$$f_R(r) = \int_{\mathbb{R}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} \cancel{e^{-\beta r u}}\, \mathbb{1}_{\underbrace{ru \in [0, \infty)}_{u \in [0,\infty)}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} \underbrace{(u - ru)}_{u(1-r)}^{\alpha_2 - 1} \underbrace{e^{-\beta(u - ru)}}_{e^{-\beta u} \cancel{e^{\beta r u}}}\, \mathbb{1}_{\underbrace{u - ru \in [0,\infty)}_{\underbrace{u(1-r) \in [0,\infty)}_{u \in [0,\infty)}}} |u|\,du$$

$u^{\alpha_2 - 1}(1-r)^{\alpha_2 - 1}$

$$= \frac{\cancel{\beta^{\alpha_1 + \alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u}\,du}_{\frac{\Gamma(\alpha_1 + \alpha_2)}{\cancel{\beta^{\alpha_1 + \alpha_2}}}}\, \mathbb{1}_{r \in [0,1]}$$

$\frac{1}{B(\alpha_1, \alpha_2)}$

$$= \frac{1}{B(\alpha_1, \alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1}\, \mathbb{1}_{r \in [0,1]} = Beta(\alpha_1, \alpha_2)$$

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p152 $X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$, $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

$$f_R(r) = \int_{Supp[X_2]} f^{old}_{X_1}(ru)\,\mathbb{1}_{ru \in Supp[X_1]}\, f^{old}_{X_2}(u)\,|u|\,du$$

$$= \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} e^{-\beta r u}\, \mathbb{1}_{\underbrace{ru \in [0,\infty)}_{u \in [0,\infty)}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u}\, u\,du$$

$\mathbb{1}_{r>0}$

$$= \frac{\cancel{\beta^{\alpha_1 + \alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1}\, \mathbb{1}_{r>0} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta(r+1)u}\,du}_{\frac{\Gamma(\alpha_1 + \alpha_2)}{(\beta(r+1))^{\alpha_1 + \alpha}}}$$

$\frac{1}{B(\alpha_1, \alpha_2)}$

(see lec 9)

$\cancel{\beta^{\alpha_1 + \alpha_2}}(r+1)^{\alpha_1 + \alpha_2}$

$$= \frac{1}{B(\alpha_1, \alpha_2)} \frac{r^{\alpha_1 - 1}}{(r+1)^{\alpha_1 + \alpha_2}}\, \mathbb{1}_{r>0} = BetaPrime(\alpha_1, \alpha_2)$$