Math 621
Lecture 17
11-11-2020

Consider $X_1, X_2, \ldots, X_n$ iid rv's of unknown PMF / PDF but we know it has expectation μ and variance sigsq (both finite).

let $T_n = X_1 + X_2 + \cdots + X_n$,

Math 241 $\begin{cases} \Rightarrow E[T_n] = n\mu \\ \Rightarrow Var[T_n] = n\sigma^2 \end{cases}$

let $\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{n} = \dfrac{T_n}{n}$,

$E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \dfrac{\sigma^2}{n}$.

let $Z_n = \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{\sqrt{n}}{\sigma}\bar{X}_n - \dfrac{\sqrt{n}}{\sigma}\mu$,

"$\bar{X}_n$ standarized"

$E[Z_n] = 0$, $Var[Z_n] = 1 = SD$

Using Characteris function:

$\phi_{T_n}(t) \overset{(P3)\ \text{Lec 16}}{=} \phi_{X_1}(t) \cdot \ldots \cdot \phi_{X_n}(t) \overset{\text{identy dist} + (P1)}{=} \phi_X(t)^n$

$\phi_{\bar{X}_n}(t) \overset{(P2)}{=} \phi_{T_n}\left(\frac{t}{n}\right) = \phi_X\left(\frac{t}{n}\right)^n$

$\phi_{Z_n}(t) = e^{\overbrace{\frac{-it\mu\sqrt{n}}{\sigma}}^{\text{cmu}(\mu)}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right) \overset{\vee}{=} e^{\frac{-it\mu\sqrt{n}}{\sigma}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n$

$\boxed{\dfrac{it\mu\sqrt{n}}{\sigma} \cdot \dfrac{\sqrt{n}}{\sqrt{n}}}$

$= e^{\frac{-it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$

$C = e^{\ln(C)}$

$= e^{\left(\frac{-it\mu n}{\sigma\sqrt{n}} + n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) \frac{\frac{1}{n}}{\frac{1}{n}}\right)}$

$$= e^{\left(\frac{\frac{-it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}\right)}$$

$$= e^{\left(\frac{\frac{t^2}{\sigma^2}\left(\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}\right)}{\frac{t^2}{n\sigma^2}}\right)}$$

$$= \phi_{Z_n}(t)$$

We want to examine $\lim_{n\to\infty} \phi_{Z_n}(t)$ and if we find its limiting Chf, $\phi_Z(t)$, we can use P8 to show that $Z_n \xrightarrow{d} Z \Rightarrow Z_n \stackrel{d}{=} Z$.

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}}$$

U-subtitution;
let $u = \frac{t}{\sqrt{n}\,\sigma}$. If $n \to \infty \Rightarrow u \to 0$

So, $e^{\left(\frac{t^2}{\sigma^2} \lim_{u\to 0}\left(\frac{\ln(\phi_X(u)) - i\mu u}{u^2}\right)\right)}$

We get $\frac{0}{0}$, use L'Hopital Rule:

L'Hopital

$$= e^{\frac{1}{2}\cdot\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}}$$

L'Hopital

$$= e^{\frac{t^2}{2\sigma^2} \lim_{u\to 0} \frac{\phi_X(u)\,\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}}$$

$$= e^{\frac{t^2}{2\sigma^2} \frac{\phi_X(0)\,\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2}}$$

$$\overset{P0}{=} e^{\frac{t^2}{2\sigma^2}\left(\phi_X''(0) - \phi_X'(0)^2\right)}$$

(P4) $= e^{\frac{t^2}{2\sigma^2}} \left(i^2 E[X^2] - (i E[X])^2 \right)$ ③

$= e^{-\frac{t^2}{2\sigma^2}} \left(\underbrace{E[X^2 - E[X]^2}_{\sigma^2} \right)$

$= e^{-\frac{t^2 \cdot \sigma^2}{2\sigma^2}} = e^{-\frac{t^2}{2}} = \phi_Z(t).$

IS $e^{-\frac{t^2}{2}} \overset{?}{\in} L_1$. What is $\int_{\mathbb{R}} e^{-t^2/2} dt$

Gaussian Integral $= \sqrt{\pi} < \infty$. Yes!

Now we can use (P6) to invert the chf of Z to get the PDF of Z.

$$f_Z(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \phi_Z(t) dt$$

$$= \frac{1}{2\pi} \int e^{-\left(itz + \frac{t^2}{2}\right)} dz$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2} \right)^2 - \left(\frac{\sqrt{2}\, iz}{2} \right)^2$$

$$\left(\frac{\sqrt{2}\, iz}{2} \right)^2 = \frac{2 i^2 z^2}{4} = -\frac{z^2}{2}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2} \right)^2} e^{-z^2/2} dt$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2} \right)^2} dt$$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}$

$\Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\, dy$,

$t \to -\infty \Rightarrow y \to -\infty$, $t \to \infty$, $y \to \infty$

④

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy$$

gaussian integral

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

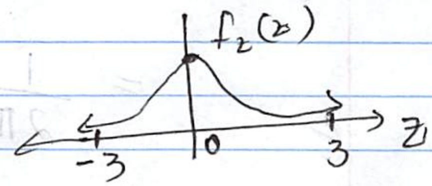
$$= N(0,1)$$

$$\Rightarrow X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mu, \text{ Variance } \sigma^2 < \infty$$

$$\Rightarrow \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

This fact is called the "Central Limit Theorem" and it is the crown jewel of an intermediate probability class.
The importance of this theorem can't be overstated. All around us we have devices that use it.

$Z \sim f_Z(z) = N(0,1)$



It's called the "Gaussian distribution" but really Laplace discovered it & called it his "second law of errors".
It's actually the most common error distribution in the world. A lot the field of Statistics is derived by assuming Gaussian / normal iid errors.

$$E[Z] \overset{(P4)}{=} j\,\phi_Z'(0) = 0 \checkmark$$

$$\phi_Z'(t) = \frac{d}{dt}\left[e^{-\frac{t^2}{2}}\right] = -t e^{-t^2/2},$$

$$\phi_Z''(t) = -\frac{d}{dt}\left[t e^{-t^2/2}\right] = -\left(-t^2 e^{-t^2/2} + e^{-t^2/2}\right)$$

$Var[Z] = E[Z^2] - \cancel{E[Z]^2}^{\to 0} \overset{P4}{=} j^2 \phi_Z''(0)$ ⑤

$= --1 = 1 = SD[Z]$

for $\sigma > 0$ — o —

$X = \mu + \sigma Z \sim f_X(x) = ?$

$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$

$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$ ← "Normal"

$E[X] = \mu + \sigma E[Z] = \mu$

$Var[X] = Var[\mu + \sigma Z] = \sigma^2$

$\phi_X(t) \overset{P2}{=} e^{it\mu} \phi_Z(\sigma t) = e^{it\mu - \sigma^2 t^2/2}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ independent of

$X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim ?$

$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t)$

$= e^{it\mu_1 - \sigma_1^2 t^2/2} e^{it\mu_2 - \sigma_2^2 t^2/2}$

$= e^{it(\mu_1+\mu_2) - (\sigma_1^2 + \sigma_2^2)t^2/2}$

$\overset{P1}{\Rightarrow} X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

— o —

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$

$g^{-1}(y) = \ln(y)$, $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|}$

$f_Y(y) = f_X(\ln(y)) \frac{1}{|y|}$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \frac{1}{y}$

(6)

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}$$

$$= \text{LogN}(\mu, \sigma^2) \implies \text{Log-Normal Model.}$$

e.g. You began with I_0 amount of money and each year it goes up/down by a random percentage X_i:

$$I_f = I_0 e^{X_1} \cdot e^{X_2} \cdot e^{X_3} \cdot \cdots \cdot e^{X_n}$$

$$= I_0 e^{\underbrace{X_1 + \cdots + X_n}_{\approx N(n\mu, n\sigma^2)}} = I_0 \text{LogN}(\mu, \sigma^2)$$