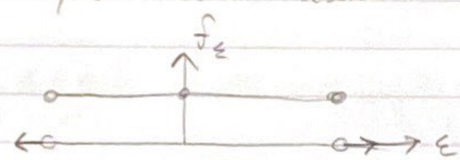
1774. First "law of errors". Imagine you're trying to measure something, a (constant) quantity v, but your measurements have (random) error, epsilon, so your measurement M is a rv looking like: M = v + epsilon. So what is a good model for the error (epsilon)? It makes sense for E[epsilon] = 0, Med[epsilon] = 0, and symmetric. how about...



it also makes sense for larger errors (in magnitude) to be less probable than smaller errors. $\Rightarrow \forall_\varepsilon > 0 \quad f'(\varepsilon) < 0$

$\forall_\varepsilon > 0 \quad f''(\varepsilon) = f'(\varepsilon) \underset{\text{solve}}{\Rightarrow} f(\varepsilon) = ce^{-d\varepsilon} \Rightarrow$ Laplace(0,1)

$X \sim Exp(1) = e^{-x}\mathbb{1}_{x\geq 0}$. let $Y = \frac{1}{\lambda}X^{1/k} = g(X)$ s.t $\lambda, k > 0$

$\Downarrow$

$Y \sim f_Y(y) = ?$ Inverse function $\lambda y = x^{1/k} \Rightarrow x = \lambda^k y^k = g^{-1}(y)$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\lambda^k y^k]\right| = \left|k\lambda^k y^{k-1}\right| = k\lambda^k y^{k-1} \quad (\lambda, k > 0)$$

$$f_Y(y) = f_x(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| = e^{-(\lambda y)^k}\mathbb{1}_{\lambda^k y^k \geq 0} \cdot k\lambda^k y^{k-1}$$

$y^k \geq 0$

$y \geq 0$

$$= k\lambda^k y^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y\geq 0} = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y\geq 0} = \text{Weibull}(k,\lambda) \qquad (1951.)$$

Note $\text{Weibull}(1,\lambda) = (1)\lambda(1)y^{k-1}e^{-(\lambda y)^1}\mathbb{1}_{y\geq 0} = \lambda e^{-\lambda y}\mathbb{1}_{y\geq 0} = Exp(\lambda)$

- k is the main property.

$k=1$, $P(Y \geq y+c \mid Y \geq c) = P(Y \geq y)$ $(P(Y \geq 17 \mid Y \geq 14) = P(Y \geq 3))$

"memorylessness"

$k>1$, $P(Y \geq y+c \mid Y \geq c) < P(Y \geq y)$ Survival less likely as time goes on

$k<1$, $P(Y \geq y+c \mid Y \geq c) > P(Y \geq y)$ Survival more likely as time goes on

Order Statistics (pg 160 in textbook)

Let $X_1, X_2, \dots, X_n$ be a collection of continuous rv's and let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be their "order statistics" defined as:

$X_{(1)} = \min\{X_1, \dots, X_n\}$ $\qquad\qquad$ $x_1 = 9,\ x_2 = 2,\ x_3 = 12,\ x_4 = 7$

$X_n = \max\{X_1, \dots, X_n\}$ $\qquad\qquad$ $x_{(1)} = 2,\ x_{(2)} = 7,\ x_{(3)} = 9,\ x_{(4)} = 12$

$\vdots$ $\qquad\qquad$ $r = 12 - 2 = 10$

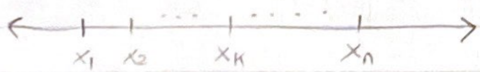
$X_{(K)} = K^{th}$ largest $\{X_1, \dots, X_n\}$

$R := X_{(n)} - X_{(1)}$ "range"

We want to find the CDF and PDF of the order statistics. We'll start by looking at the CDF of the maximum.

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x \ \&\ X_2 \le x \ \&\ \dots \ \&\ X_n \le x)$$

$$\overset{\text{if indep}}{=} \prod_{i=1}^{n} P(X_i \le x) = \prod_{i=1}^{n} F_{X_i}(x) \underset{\text{if iid}}{=} F_X(x)^n$$



$$f_{X_{(n)}}(x) \overset{\text{if iid}}{=} \frac{d}{dx}\left[F_{X_{(n)}}(x)\right] = \frac{d}{dx}\left[F_X(x)^n\right] = n f_X(x) F_X(x)^{n-1}$$

Let's now find the CDF/PDF of the minimum.

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x \ \&\ X_2 > x \ \&\ \dots \ \&\ X_n > x)$$

if indep

$$= 1 - P(X_1 > x) P(X_2 > x) \cdots P(X_n > x) = 1 - \prod_{i=1}^{n} (1 - F_{X_i}(x))$$

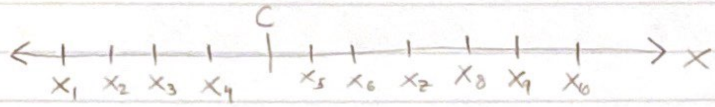
if iid

$$= 1 - (1 - F_X(x))^n$$

$$f_{X_{(1)}}(x) \overset{\text{if iid}}{=} \frac{d}{dx}\left[1 - (1 - F_X(x))^n\right] = n f_X(x)(1 - F_X(x))^{n-1}$$

Let's now find the CDF/PDF for the K^{th} order statistic, $X_{(K)}$.

let $n=10$, $K=4$.



$P(X_1 \le c \;\&\ldots\&\; X_4 \le c \;\&\; X_5 > c \;\&\ldots\&\; X_{10} > c)$

if indep, if iid

$$\overset{\text{if indep}}{=} \prod_{i=1}^{4} P(X_i \le c) \prod_{i=5}^{10} P(X_i > c) = \prod_{i=1}^{4} F_{X_i}(c) \prod_{i=5}^{10} (1 - F_{X_i}(c)) \overset{\text{if iid}}{=} F_X(c)^4 (1 - F_X(c))^6$$

$F_{X_{(4)}}(x) = P(\text{any } 4\ X_i < x \;\&\; \text{other } 6\ X_i > x)$

$$= \sum_{\substack{\text{over all subsets}\\ S \text{ of size } 4,\\ S^c \text{ of size } 6}} P(X_{S_1} \le x, \ldots, X_{S_4} \le x, X_{S_1^c} > x, \ldots X_{S_6^c} > x)$$

$$\overset{\text{if indep}}{=} \sum_{\text{"same"}} \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=1}^{6} F_{X_{S_i^c}}(x) \overset{\text{if iid}}{=} \sum_{\text{same}} F_X(x)^4 (1 - F_X(x))^6$$

$$= \binom{10}{4} F_X(x)^4 (1 - F_X(x))^6$$

$$F_{X_{(4)}}(x) = P(X_{(4)} \le x) = P(4\ X_i\text{'s} \le x,\ 6\ X_i\text{'s} > x) + P(5\ X_i\text{'s} \le x,\ 5\ X_i\text{'s} > x) + \ldots + P(10\ X_i\text{'s} \le x,\ 0\ X_i\text{'s} > x)$$

$$\overset{\text{if iid}}{=} \sum_{j=4}^{10} \binom{10}{j} F_X(x)^j (1 - F_X(x))^{10-j}$$

Binomial Thm: $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$

general case: K, n

$$* \Rightarrow F_{X_{(K)}} = \sum_{j=K}^{n} \binom{n}{j} F_X(x)^j (1 - F_X(x))^{n-j}$$

$$F_{X_{(n)}}(x) = \sum_{j=n}^{n} \binom{n}{j} F_X(x)^j (1 - F_X(x))^{n-j} = F_X(x)^n$$

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F_X(x)^j (1 - F_X(x))^{n-j} = \left(\sum_{j=0}^{n} \binom{n}{j} F_X(x)^j (1 - F_X(x))^{n-j}\right) - (1 - F_X(x))^n$$

$$= (F_X(x) + 1 - F_X(x))^n - (1 - F_X(x))^n$$

$$= 1 - (1 - F_X(x))^n$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[F_{X_{(k)}}(x)\right] = \frac{d}{dx}\left[\sum_{j=k}^{n}\binom{n}{j}F_X(x)^j(1-F_X(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n}\binom{n}{j}\underbrace{\frac{d}{dx}[\overbrace{F_X(x)^j}^{u}\overbrace{(1-F_X(x))^{n-j}}^{v}]}, \quad u' = jf(x)F(x)^{j-1}$$

$$v' = -(n-j)f(x)(1-F(x))^{n-j-1}$$

$d/dx[uv] = uv' + u'v =$