$T_k \sim \text{Erlang}(k,\lambda)$ $\quad N \sim \text{Poisson}(\lambda)$ $\quad X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k,\lambda)$ $\quad F_N(x) = Q(x+1,\lambda)$

N: # events before 1 sec → Why is this Poisson distr?

$k=5$

$$\{T_5 > 1\} = \{X_1+X_2+X_3+X_4<1\} \cup \{X_1+X_2+X_3<1\} \cup \{X_1+X_2<1\} \cup \{X_1<1\} \cup \{X_1>1\}$$

$$1-F_{T_5}(1) \;\; = \{N=4\} \cup \{N=3\} \cup \{N=2\} \cup \{N=1\} \cup \{N=0\}$$

$$\Rightarrow P(T_5>1) = P(N \le 4) = F_N(4)$$

$$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k,\lambda) \quad \text{"Poisson Process"}$$

$$T \sim \text{Erlang}(k,\lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0} = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t\ge 0}$$
$$k \in \mathbb{N},\ \lambda \in (0,\infty)$$

$$T \sim \text{NegBin}(k,p) = \binom{k+t-1}{k-1}(1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} = \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$
$$k \in \mathbb{N},\ p \in (0,1)$$

For both, what if $k \in (0,\infty)$? Are both rv's still "legal"? Yes, we can show that both:

$$\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t}\, dt = 1 \quad \& \quad \sum_{t=0}^\infty \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k = 1$$

we just derived two new famous rv's:

$$X \sim \text{Gamma}(\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\ge 0}$$

$$X \sim \text{ExtNegBin}(k,p) = \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$ the "extended negative binomial"

Transformations of Discrete rv's

$$X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$Y = X + 3 \overset{=g(X)}{} \sim \begin{cases} 4 & \text{w.p. } p \\ 3 & \text{w.p. } 1-p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}} \quad (= p_X(y-3))$$

$X = Y - 3 = g^{-1}(Y)$

I want to find the PMF of Y using the PMF of X:

$$Y = g(X) \sim p_Y(y) = p_X(g^{-1}(y))$$

What assumption did I make when I "derived" this formula?
I assumed an inverse function exists i.e. g is invertible. If not...

$$X \sim U(\{1,2,\ldots,10\}) = \begin{cases} 1 & \text{w.p. } \frac{1}{10} \\ 2 & \text{w.p. } \frac{1}{10} \\ \vdots \\ 10 & \text{w.p. } \frac{1}{10}\end{cases}, \quad Y = g(X) = \min\{X,3\} \sim \begin{cases} 1 & \text{w.p. } \frac{1}{10} \\ 2 & \text{w.p. } \frac{1}{10} \\ 3 & \text{w.p. } P(X=3)+P(X=4)+\ldots+P(X=10) = 8/10 \end{cases}$$

if g is invertible on $\text{Supp}[X]$

$$Y = g(X) \sim p_Y(y) = \sum_{\{x:\, y = g(x)\}} p_X(x) = \sum_{\{x:\, x = g^{-1}(y)\}} p_X(x) = p_X(g^{-1}(y))$$

one element only

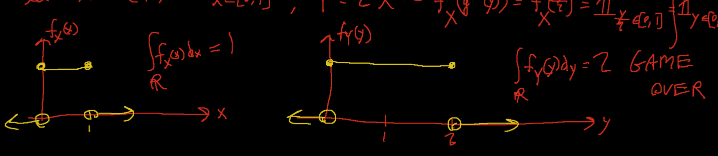
$$X \sim \text{Binom}(n,p),\quad Y = X^3 \;(= g(X)) \sim p_X(\sqrt[3]{y}) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y} \in \{0,1,\ldots,n\}}$$
$\Rightarrow g^{-1}(y) = \sqrt[3]{y}$

$$Y = X^2 \sim p_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}}(1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y}\in\{0,1,\ldots,n\}}$$

Transformations for continuous rv's.

for g invertible $\quad f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ NO!

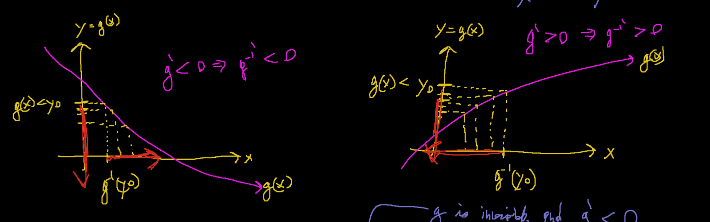
$$\text{let } X \sim U(0,1) = \mathbb{1}_{x\in[0,1]} \;(= f_X(x)),\quad Y = 2X \overset{?}{\sim} f_X(g^{-1}(y)) = f_X\!\left(\tfrac{y}{2}\right) = \mathbb{1}_{\frac{y}{2}\in[0,1]} = \mathbb{1}_{y\in[0,2]}$$



PDF's are not probabilities!! So this was bound to fail because we used them as probabilities. However, CDF's *are* probabilities.

$$F_Y(y) := P(Y \le y) = P(g(X) \le y) \overset{g \text{ is invertible and } g'>0}{=} P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$\Rightarrow \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)] = f_X(g^{-1}(y))\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{always pos.}} \;(= f_Y(y))$$



$$F_Y(y) = P(Y \le y) = P(g(X) \le y) \overset{g \text{ is invertible and } g'<0}{=} P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = f_X(g^{-1}(y))\Big(-\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{always neg}}\Big)$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| \quad \text{general rule.}$$

We can derive a less general but very useful corrollary rule:

$$Y = \underbrace{aX + c}_{g(X)\text{ is invertible}} \;(a, c \in \mathbb{R}) \sim f_Y(y) = ?$$ shift and scale (shift by c, scale by a)

$$\Rightarrow g^{-1}(y) = \frac{y-c}{a} \quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{a}\right| = \frac{1}{|a|}$$

$$f_Y(y) = f_X\!\left(\frac{y-c}{a}\right)\frac{1}{|a|}$$

$$Y = aX \sim f_X\!\left(\frac{y}{a}\right)\frac{1}{|a|}, \quad Y = X + c \sim f_X(y-c)$$

$$X \sim \text{Exp}(1) = e^{-x}\mathbb{1}_{x\ge 0}$$

$$Y = g(X) = -\ln\left(\frac{e^{-X}}{1-e^{-X}}\right) = \ln\left(\frac{1-e^{-X}}{e^{-X}}\right) = \ln(e^X - 1) = g(X) \sim f_Y(y)$$

invertible

$$y = \ln(e^x - 1) \Rightarrow e^y = e^x - 1 \Rightarrow e^y + 1 = e^x \Rightarrow x = \ln(e^y+1) = g^{-1}(y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\ln(e^y+1)]\right| = \left|\frac{e^y}{e^y+1}\right| = \frac{e^y}{e^y+1} \quad \underbrace{y \in \mathbb{R}}\;\; e^y \ge 0$$

$$f_Y(y) = f_X\left(\ln(e^y+1)\right)\cdot\frac{e^y}{e^y+1} = e^{-\ln(e^y+1)}\,\mathbb{1}_{\ln(e^y+1)\ge 0}\cdot\frac{e^y}{e^y+1} \quad (e^y + 1 \ge 1)$$

$$= \frac{1}{e^y+1}\frac{e^y}{e^y+1} = \frac{e^y}{(e^y+1)^2}\cdot\frac{e^{-2y}}{e^{-2y}} = \frac{e^{-y}}{(e^{-y}+1)^2} = \text{Logistic}(0,1)$$

standard logistic.