Lecture 17

11/11/20.

Math 621
Prof. Kapelner

Consider r.v.'s $X_1, X_2, \ldots, X_n$ iid but PDF/PMF is unknown but we know $E = \mu$ & $Var = \sigma^2$

Let $T_n = X_1 + X_2 + \cdots + X_n$

Let $\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$

From Math 241, we know $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \frac{\sigma^2}{n}$.

Let $Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma} \bar{X}_n + \frac{-\sqrt{n}}{\sigma} \mu$

"$\bar{X}$ standardized"

$E[Z_n] = 0$, $Var[Z_n] = 1 = SD[Z_n]$.

$$\phi_{T_n}(t) \underset{P_3}{=} \phi_{X_1}(t) \cdot \ldots \cdot \phi_{X_n}(t) \underset{P_1}{=} \phi_X(t)^n$$

$$\phi_{\bar{X}_n}(t) \underset{P_2}{=} \phi_T\left(\frac{1}{n}t\right) = \phi_X\left(\frac{t}{n}\right)^n$$

$$\phi_{Z_n}(t) = e^{itb} \phi_{\bar{X}_n}(at) = e^{\frac{-it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}} \phi_X\left(\frac{\cancel{\sqrt{n}}t}{\sigma \cancel{n}_{\sqrt{n}}}\right)^n$$

$$\to e^{\left(\frac{-it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\right)/\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}} \to$$

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$$\to \quad e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)} = \phi_{Z_n}(t)$$

We want to investigate $\lim_{n\to\infty} \phi_{Z_n}(t) = ?$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}})}{\frac{t^2}{\sigma^2 n}}} = \to$$

$$\to e^{\frac{t^2}{r^2} \lim_{n\to 0} \frac{\ln(\phi_X(u) - i\mu u}{u^2}} = e^{\frac{t^2}{r^2} \lim_{n\to 0} \frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}}$$

$$\to = e^{\frac{t^2}{2\sigma^2} \lim_{n\to 0} \frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}} = \to$$

$$\to = e^{\frac{t^2}{2\sigma^2} \frac{\overset{1}{\cancel{\phi_X(0)}}\,\phi_{(X)}''(0) - \phi_X'(0)^2}{\underset{1}{\cancel{\phi_X(0)}}}} \underset{\text{P0}}{=} e^{\frac{t^2}{2\sigma^2}(\phi_X''(0) - \phi_X'(0)^2)}$$

P4

$$= e^{\frac{t^2}{2\sigma^2}(i^2 E[X^2] - (i E[X])^2)} = e^{\frac{-t^2}{2\cancel{\sigma^2}} \overbrace{(E[X^2] - E[X]^2)}^{\cancel{\sigma^2}}}$$

$$\to = e^{\frac{-t^2}{2}} = \phi_Z(t).$$

P8

$\Rightarrow Z_n \xrightarrow{d} Z$ where Z has chf. $\phi_Z(t) = e^{\frac{-t^2}{2}}$

$Z \sim f_Z(z) = ?$

Use (P6) to find PDF of Z.

$\phi_Z(t) \overset{?}{\in} L_1 \rightarrow \underbrace{\int_{R} e^{-t^2/2}\,dt = \sqrt{\pi} < \infty}_{\text{GAUSSIAN INTEGRAL}}$

$$f_Z(z) = \frac{1}{2\pi}\int_R e^{-itz}\,\phi_Z(t)\,dt.$$

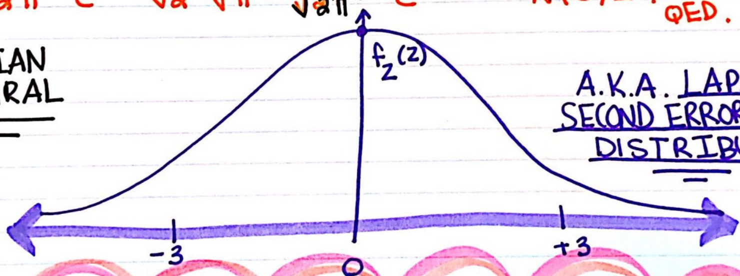
$$= \frac{1}{2\pi}\int e^{-itz}\,e^{\frac{-t^2}{2}}\,dt = \frac{1}{2\pi}\int_R e^{-\left(itz+\frac{t^2}{2}\right)}\,dt.$$

$$= \frac{1}{2\pi}\int_R e^{-\left(\frac{t}{\sqrt{2}}+\frac{\sqrt{2}\,iz}{2}\right)^2}\,e^{\frac{-z^2}{2}}\,dt = \frac{1}{2\pi}\,e^{-z^2/2}\int_R e^{-\left(t/\sqrt{2}+\sqrt{2}\,iz/2\right)^2}\,dt. = \frac{1}{2\pi}\,e^{\frac{-z^2}{2}}\int_R e^{-y^2}\sqrt{2}\,dy$$

$$= \frac{1}{2\pi}\,e^{\frac{-z^2}{2}}\sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}}\,e^{\frac{-z^2}{2}} = N(0,1).$$

QED.

↑ GAUSSIAN INTEGRAL



A.K.A. LAPLACE SECOND ERROR DISTRIBUTION

CLT: $X_1, \ldots, X_n$ iid, mean μ, variance σ^2,

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1).$$

$Z \sim N(0,1)$, $X = \mu + \sigma Z \sim f_X(x) = ?$

$$\phi_X(t) = e^{it\mu} \phi_Z(\sigma t) = e^{it\mu - \sigma^2 \frac{t^2}{2}}$$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\sigma^2 \cdot (x-\mu)^2} = N(\mu, \sigma^2).$$

$*$ $E[Z] = \frac{\phi'_Z(0)}{i} = 0$, $Var[Z] = E[Z^2] - E[Z]^2 = \frac{\phi''_Z(0)}{i^2} = 1 \checkmark$

$$\phi'_{(Z)}(t) = \frac{d}{dt}\left[e^{\frac{-t^2}{2}}\right] = -t e^{-t^2/2}.$$

$$\phi''_{(Z)}(t) = \frac{d}{dt}[\text{"} \quad \text{"}] = t^2 e^{-t^2/2} - e^{-t^2/2} \checkmark$$