Lecture - 06 09/16/2020

Let $A \in \mathbb{R}^{L \times K}$ matrix of constant

$\vec{a}_i$ row i if matrix A

$$\underset{\substack{(L\times K)(K\times 1)\\(L\times 1)}}{E[A\vec{x}]} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \ldots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \ldots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \ldots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{x}] \\ E[\vec{a}_2 \cdot \vec{x}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{x}] \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 \vec{\mu} \\ \vec{a}_2 \vec{\mu} \\ \vdots \\ \vec{a}_L \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$\vec{a} \in \mathbb{R}^K$

$$\underset{\substack{\text{scalar}\\(1\times K)\ (K\times 1)}}{Var[\vec{a}^T \vec{X}]} = Var[\underbrace{a_1X_1}_{Y_1} + \ldots + \underbrace{a_KX_K}_{Y_K}] = Var[Y_1 + \ldots + Y_K]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} Cov[Y_i, Y_j] = \sum\sum Cov[a_iX_i, a_jX_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j \sigma_{ij} = \underset{\substack{(1\times K)(K\times K)(K\times 1)\\ = \text{scalar}}}{\vec{a}^T \overset{Var[\vec{X}]}{\Sigma} \vec{a}}$$

this is called a "quadratic form"

Let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^{K+1}$

$$\underbrace{\vec{a}^T V \vec{a}}_{\substack{\text{Quadratic forms with V being the}\\ \text{determining matrix}}} = \vec{a}\cdot(V\vec{a}) = \vec{a} \cdot \begin{bmatrix} a_1V_{11} + \ldots + a_KV_{1K} \\ a_1V_{21} + \ldots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \ldots + a_KV_{KK} \end{bmatrix} = \begin{array}{l} a_1a_1V_{11} + \ldots + a_1a_KV_{1K} + \\ a_2a_1V_{21} + \ldots + a_2a_KV_{2K} + \\ \vdots \\ a_Ka_1V_{K1} + \ldots + a_Ka_KV_{KK} \end{array}$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j V_{ij}$$

Application in finance. Let $X_1, X_2, \ldots, X_k$ be financial assets (e.g. stocks). So let $w_1, w_2, \ldots, w_k$ be the proportion allocated to each of these assets.
Let $\vec{\mu} = E[\vec{X}]$, $\Sigma = Var[\vec{X}]$

$F = \vec{w}^T \vec{X}$ a rv modeling your portfolio.

$\mu_F = E[F] = \vec{w}^T \vec{\mu}$, $Var[F] = \vec{w}^T \Sigma \vec{w}$

It's possible to pick w-vector to optimize the portfolio by minimizing the variance of returns, $Var[F]$, conditional on μ_F. This is called "Markowitz optimal portfolio theory".

min $Var[F]$ subject to μ_F being constant and $\vec{w}^T \vec{1} = 1$.

$\vec{X} \sim \text{multin}_K(n, \vec{p})$ $\quad E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$

$X_j \sim \text{Bin}(n, p_j)$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & & \sigma_{ij} \\ & np_2(1-p_2) & & \\ & & \ddots & \\ \sigma_{ij} & & & np_K(1-p_K) \end{bmatrix}$$

$i \neq j$

$Cov[X_i, X_j] = E[X_i X_j] - E[X_i]E[X_j]$ → both complecated so it's Rail.

$$= \sum_{X_i \in R} \sum_{X_j \in R} X_i X_j \, P_{X_i, X_j}(x_i, x_j) - n^2 p_i p_j$$

$\vec{X} = \begin{bmatrix} \vdots \\ X_i \sim Bin(n, p_i) \\ \vdots \\ X_j \sim Bin(n, p_j) \\ \vdots \end{bmatrix}$

i = Apples $\quad j$ = Orange

$X_i = X_{1i} + X_{2i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} Ber(p_j)$

We've expressed the multinomial rv with $n \times k$ Bernoulli's

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \overset{iid}{\sim} Multin_k(1, \vec{p})$

$$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}] = \sum_{l=1}^{n} \sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

A lot of these covariances are zero due to independence. Which ones? If l is different than m, the covariance is zero.

$$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}] = \sum_{l=1}^{n} \Big(\underbrace{E[X_{li} X_{lj}]} - \underbrace{E[X_{li}]}_{p_i} \underbrace{E[X_{lj}]}_{p_j} \Big) = \boxed{-n p_i p_j}$$

the only term that's nonzero is ... $(X_{li}, X_{lj}) \stackrel{\downarrow}{=}$

$$= \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \, P_{X_{li} X_{lj}}$$

$P_{X_{li}, X_{lj}}(1,1) = 0$

↑ You can't get an apple AND a banana on one grab.

Midterm II

Uniform discrete

$$X \sim U(\{0,1,2,3\}) = \begin{cases} 0 & \text{wp } 1/4 \\ 1 & \text{wp } 1/4 \\ 2 & \text{wp } 1/4 \\ 3 & \text{wp } 1/4 \end{cases}$$

P(x)

Generally,

$$X \sim U(A)$$

$Supp[X] = A, \; A \subset \mathbb{R} \text{ s.t } |A| < \infty \;\&\; A \neq \phi$

$Supp[X] = \{0,1,2,3\}$

Create a new rv $Y = -X = g(X)$, a very simple function

$Supp[Y] = \{-3,-2,-1,0\}$

$$P(y) = \begin{cases} -3 & \text{wp } 1/4 \\ -2 & \text{wp } 1/4 \\ -1 & \text{wp } 1/4 \\ 0 & \text{wp } 1/4 \end{cases}$$

Generally, for discrete rv X, is there a pattern?

$$P_Y(y) \stackrel{:}{=} P(Y=y) = P(-X=y) = P(X=-y) = P_X(-y)$$

$$Supp[Y] = \{2 : P_Y(2) > 0\} = \{2 : P_X(-2) > 0\}$$

$$= \{-2 : P_X(2) > 0\} = -\{2 : P_X(2) > 0\} = -Supp[X]$$

$$X_1, X_2 \stackrel{iid}{\sim} \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1,...\}}$$

In class we should: $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

difference

Let $D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{-X_2}_{Y} = X + Y$,

$$Y \sim P_Y(y) = \frac{e^{-\lambda} \lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\dots, -2, -1, 0\}}$$

$$P_D(d) = \sum_{x \in Supp[X]} P_X^{old}(x) \, P_Y^{old}(d-x) \, \mathbb{1}_{d-x \in Supp[Y]}$$

$Supp[D] = Supp[X] + Supp[Y]$
$= \{ \dots, -1, 0, 1 \dots \} = \mathbb{Z}$
all integers.