Monday october 5th 2020

Lecture 9

$$T_3 = X_1 + X_2 + X_3 \sim f_{T_3}(t) = ?$$

$$= \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx$$

$$= \int_0^{\infty} x\, \lambda^2 e^{-\lambda x}\, \overbrace{\lambda e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{\lambda x}}\, \overbrace{\mathbb{1}_{t-x \in [0,\infty)}}^{x \leq t}\, dx = \lambda^3 e^{-\lambda t} \int_0^{\infty} x\, \mathbb{1}_{x \leq t}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$$

$$T_4 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4 \sim f_{T_4}(t) = ?$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in Supp[X_4]}\, dx$$

$$= \int_0^{\infty} \frac{x^2}{2} \lambda^3 e^{-\lambda x}\, \overbrace{\lambda e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{\lambda x}}\, \overbrace{\mathbb{1}_{t-x \in [0,\infty)}}^{x \leq t}\, dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^{\infty} x^2\, \mathbb{1}_{x \leq t}\, dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$$

$$T_k = \sum_{i=1}^{k} X_i \sim \text{Erlang}(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!}\, \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_k] = [0, \infty)$, $\lambda \in (0, \infty)$ $k \in \mathbb{N}$.

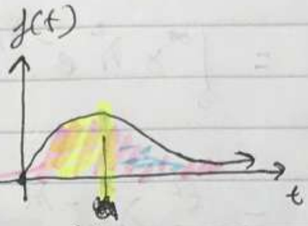
$\text{Exp}(\lambda) \overset{\text{add}}{\Longrightarrow} \text{Erlang}(k, \lambda)$

$\Updownarrow \qquad\qquad \Updownarrow$ conceptually analogous.

$\text{Geom}(p) \overset{\text{add}}{\Longrightarrow} \text{NegBin}(k, p)$

Let's do some pure math. I want to define the gamma family of functions. Beginning with the gamma function for x non-neg:

$$\Gamma(x) := \int_0^\infty \underbrace{t^{x-1} e^{-t}}_{f(t)} \, dt$$



e.g. $\Gamma(3) = \int_0^\infty t^2 e^{-t} \, dt = 2$

$a > 0$

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, dt = \underbrace{\int_0^a t^{x-1} e^{-t} \, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} \, dt}_{\Gamma(x,a)}$$

Lower incomplete gamma function

Upper incomplete gamma function

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\gamma(x,a) + \Gamma(x,a)}{\Gamma(x)} = \overbrace{\frac{\gamma(x,a)}{\Gamma(x)}}^{P(x,a)} + \overbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}^{Q(x,a)} = P(x,a) + Q(x,a) = 1$$

$P(x,a)$ is called the "lower regularized gamma function"

$Q(x,a)$ is called the "upper regularized gamma function"

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1$$ $\leftarrow X \sim \exp(1) = e^{-x} \; \forall t \in [0, \infty)$

Hw: $\Gamma(x+1) = x\Gamma(x)$

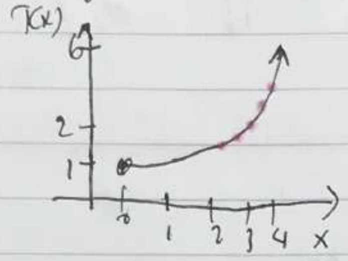
let $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots(3)(2)(1) = (n-1)!$$

Let $x \in (0, \infty)$

$\Gamma(x) = (x-1)\Gamma(x-1) = \cdots = (x-1)(x-2)\cdots \Gamma(c)$ where $c \in (0,1)$

The gamma function "extends" the factorial function to all positive #'s



$c>0$

- $\int_0^\infty t^{x-1} e^{-ct}\, dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\, du = \frac{1}{c^x}\int_0^\infty u^{x-1} e^{-u}\, du = \frac{\Gamma(x)}{c^x}$

Let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow \frac{du}{dt} = c \Rightarrow dt = \frac{1}{c}du$, $t=0 \Rightarrow u$, $t\to\infty \Rightarrow u \to \infty$

- $\int_0^a t^{x-1} e^{-ct}\, dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u}\frac{1}{c}\, du = \frac{1}{c^x}\int_0^{ac} u^{x-1} e^{-u}\, du = \frac{\gamma(x, ac)}{c^x}$
- $\int_a^\infty t^{x-1} e^{-ct}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, ac)}{c^x} = \frac{\Gamma(x, ac)}{c^x}$

If $n \in \mathbb{N}$

$\Gamma(n,a) := \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}}_{dv}\, dt = [uv]_a^\infty - \int_a^\infty v\,du$ (Integration by part)

$V = \int dv = \int e^{-t}\, dt = -e^{-t}$

$\frac{du}{dt} = (n-1)t^{n-2} \Rightarrow du = (n-1)t^{n-2}dt$

$= \left[t^{n-1}(-e^{-t})\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}dt$

← Repeat

$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2} e^{-t}\, dt = e^{-a} + (n-1)\Gamma(n-1, a) =$

$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\Gamma(n-2,a)\right) = a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\left(a^{n-3}e^{-a} + (n-3)\Gamma(n-3,a)\right)\right)$

$= e^{-a}\left(a^{n-1} + (n-1)\left(a^{n-2} + (n-2)\left(a^{n-3} + (n-3)\Gamma(n-3,a)\right)\right)\right)$

$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + (n-1)(n-2)(n-3)\Gamma(n-3,a)\right)$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!}+\frac{a^{n-2}}{(n-2)!}+\frac{a^{n-3}}{(n-3)!}+\frac{1}{(n-4)!}\Gamma(n-3,a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!}+\frac{a^{n-2}}{(n-2)!}+\dots+\frac{a!}{1!}+\frac{a^0}{0!}\right)=e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

$$\Gamma(1,a)=\int_a^\infty e^{-t}\,dt=\left[-e^{-t}\right]_a^\infty=e^{-a}$$

Back to probability land...

$$X\sim \text{Erlang}(k,\lambda):=\frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}\quad \forall x\geq 0$$

CDF:

$$F_X(x):=P(X\leq x)=\int_0^x \frac{\lambda^t e^{-\lambda t} t^{k-1}}{(k-1)!}\,dt\ \forall x\geq 0=\frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}}\overbrace{\int_0^x t^{k-1}e^{-\lambda t}\,dt}^{\frac{\gamma(k,\lambda x)}{\lambda^k}}$$

$$=\frac{\lambda^k}{\Gamma(k)}\frac{\gamma(k,\lambda x)}{\lambda}=P(k,\lambda x)$$

$$=1-F_X(x)=1-P(k,\lambda x)=Q(k,\lambda x)$$

$$X\sim \text{Poisson}(\lambda):=\frac{\lambda^x e^{-\lambda}}{x!}\quad \forall x\in\mathbb{N}_0$$

CDF

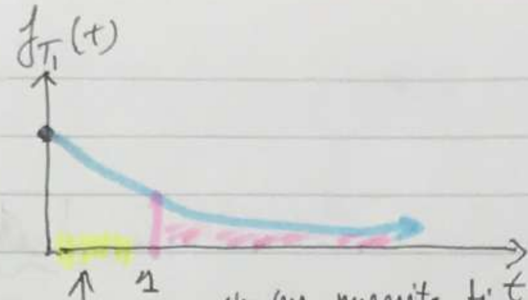
$$F_X(x):=P(X\leq x)=\sum_{t=0}^{x}\frac{\lambda^t e^{-\lambda}}{t!}=e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^t}{t!}=\overbrace{x!\,e^{-\lambda}\frac{1}{\underbrace{x!}_{\Gamma(x+1)}}\sum_{t=0}^{x}\frac{\lambda^t}{t!}}^{\Gamma(x+1,\lambda)}$$

$$=\frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)}=Q(x+1,\lambda)$$

The relationship between the Erlang and Poisson is known as the "poisson process"

$T_1 \sim Exp(\lambda) = Erlang(1, \lambda)$

$P(T_1 > 1) = Q(1, \lambda)$



$N \sim Poisson(\lambda)$

$F_N(0) = P(N \le 0) = P(N = 0) = Q(1, \lambda)$

events ~~of~~ in one minute time is Poisson (λ)