Let $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$ {Lecture 3}

possibly an infinite sequence of iid" rv's.

Let $X :=$ # of zero realization before the first realzation of one. Also, $X := \min\{t : B_t = 1\} - 1$

$p(0) = P(X=0) = P(\{\text{no zeros, just a } 1\}) = p$

$p(1) = P(X=1) = P(\{0, \text{then a } 1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0,0,\ldots,0}_{x},1\}) = (1-p)^x p$

$X \sim \text{Geom}(p) := \underbrace{(1-p)^x p}_{p^{old}} \mathbb{1}_{x \in \{0,1,2,\ldots\}}$

"geometric"

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$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p)$, $T_2 = X_1 + X_2 \sim p_T(t) = ?$

$p_T(t) = \sum_{x \in \text{Supp}[X]} p^{old}(x)\, p^{odd}(t-x)\, \mathbb{1}_{t-x \in \text{Supp}[X]} = \sum_{x \in \{0,1,\ldots\}} (1-p)^x$

$p(1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$

$x-t \in \{0,-1,-2,\ldots\}$

$x \in \{t, t-1, t-2, \ldots\}$

$\Rightarrow (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{t, t-1, \ldots\}} = (1-p)^t p^2 \sum_{x \in \{0,\ldots,t\}} 1 = (t+1)(1-p)^t p^2$

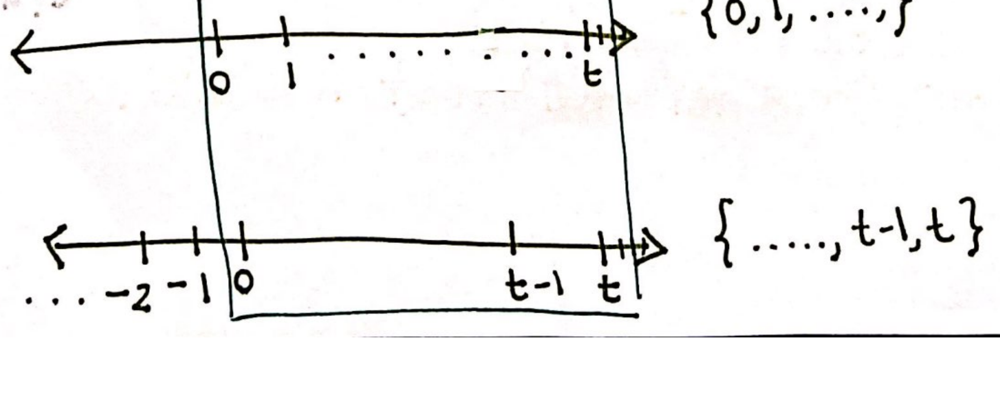
$= \text{NegBin}(2,p)$

"Negative Binominal rv".

$= p_T(t)$

$\text{Supp}[T_2] = \{0,1,\ldots\}$

$\{0,1,\ldots\} \cap \{\ldots, t-1, t\} = \{0,1,\ldots,t\}$



$$\underbrace{\cancel{0}\ \cancel{0}\ \underline{0}\ \underline{0}\ \dots\dots\ \underline{0}\ \overset{p}{\underline{1}}\ \overbrace{\underline{0}\ \underline{0}\ \dots\ \underline{0}}^{(1-p)^t}}_{t+1 \text{ realization}}\ \overset{p}{\underset{t+2}{\underline{1}}}$$

$$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p)$$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p_{T_3}(t) = ?$$

$$p_{T_3}(t) = \sum_{x\in Supp[X_3]} p^{old}_{X_3}(x)\, p^{old}_{T_3}(t-x)\, \mathbb{1}_{t-x\in Supp[T_2]} = \sum_{x\in\{0,1,\dots\}}$$

$$(1-p)^t p \underset{(t-x+1)}{\cancel{\quad}} (1-p)^{t-x} p^2\, \mathbb{1}_{t-x\in\{0,1,\dots\}} =$$

$$x \in \{\cancel{\quad}\dots, t-1, t\}$$

$$= (1-p)^t p^3 \left((t+1)\sum_{x\in\{0,\dots,t\}} (1) - \sum_{x\in\{0,\dots,t\}} x\right) = (1-p)^t p^3 \underbrace{\left((t+1)^2 - \frac{t(t+1)}{2}\right)}$$

$$\Rightarrow t^3+2t+1-\frac{t^2+t}{2}$$

$$\Rightarrow \frac{t^2+3t+2}{2}$$

$$\Rightarrow \frac{(t+2)(t+1)}{2}$$

$$\Rightarrow \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$= \binom{t+2}{2}(1-p)^t p^3 = Negbin.(3-p)$$

$$\underbrace{\underline{0}\ \underline{0}\ \dots\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \dots\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \dots\ \underline{0}}_{t+2 \text{ realization}}\ \underset{t+3}{\underline{1}}$$

pick 2 positions for the first two 1's.

$$X \sim \text{Binom}(\cancel{p}\, n,p) := \binom{n}{x} p^x (1-p)^{n-x}\, \mathbb{1}_{x\in\{0,1,\dots,n\}}$$

Let $n \to \infty$, $p \to 0$ but $\lambda := np \Rightarrow p = \frac{\lambda}{n}$ let $n \to \infty$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x\in\{0,1,\dots,n\}} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\,\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x\in\{0,1,\dots,n\}} =$$

$$= \frac{\lambda^x}{x!}\, \lim_{n\to\infty}\frac{n!}{(n-x)!\,n^x} \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} \cdot \lim_{n\to\infty} \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{\cancel{n}^{1}\cancel{(n-1)}\dots\cancel{(n-x+1)}}^{x\text{ terms}}}{\underbrace{\cancel{n}\cdot\cancel{n}\dots\cdot\cancel{n}}_{x\text{ terms}}} \cdot e^{-\lambda} \cdot (1)\ \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \frac{\lambda^x}{x!}\, e^{-\lambda}\, \mathbb{1}_{x\in\{0,1,2,\dots\}} = \text{Poisson}(\lambda)$$

$$\underbrace{\overbrace{n(n-1)\dots\dots(n-x+1)}^{x}\overbrace{(n-x)\dots(1)}^{n-x}}_{n}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \qquad T = X_1 + X_2 \sim P_T(t) = ?$

$$P_T(t) = \sum_{x\in\{0,1,2,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x\in\{0,1,2,\dots\}}$$

$$= \lambda^t e^{-\lambda 2} \frac{t!}{t!} \sum_{x\in\{0,1,\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x\in\{\dots,t-1,t\}}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x\in\{0,\dots,t\}} \binom{t}{x}}_{2^t} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$