Lecture 09
Math 621
10-05-2020

Tama C.

$$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 \sim f_{T_3}(t) = ?$$

$$= \int_{Supp[T_2]} f_{T_2}^{odd}(x) f_{X_3}^{odd}(t-x) \mathbb{1}_{t-x \in Supp[X_3]} \, dx$$

$$= \int_0^\infty x \lambda^2 e^{-\lambda x} \underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{\overbrace{t-x \in [0,\infty)}^{x \le t}} \, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty x \, \mathbb{1}_{x \le t} \, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^t x \, dx \, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$$

$$T_3 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4 \sim f_{T_4}(t) = ?$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f_{T_3}^{odd}(x) f_{X_4}^{odd}(t-x) \mathbb{1}_{t-x \in Supp[X_4]} \, dx$$

$$= \int_0^\infty \frac{x^2}{2} \lambda^3 e^{-\lambda x} \lambda \frac{e^{-\lambda(t-x)}}{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{\underbrace{t-x \in [0,\infty)}_{x \le t}} \, dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^\infty x^2 \mathbb{1}_{x \le t} \, dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2 \, dx \, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$$

$$T_K = \sum_{i=1}^{K} X_i \sim \text{Erlang}(K, \lambda) = \frac{t^{K-1} \lambda^K e^{-\lambda t}}{(K-1)!} \mathbb{1}_{t \in [0,\infty)}$$

$\text{Supp}[T_K] = [0,\infty)$, $\lambda \in (0,\infty)$, $K \in \mathbb{N}$

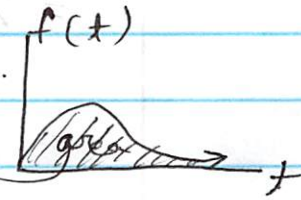
$$\begin{array}{ccc} \text{Exp}(\lambda) & \xRightarrow{\text{add}} & \text{Erlang}(K,\lambda) \\ \Updownarrow & & \Updownarrow \quad \text{Conceptually Analogous} \\ \text{Geom}(p) & \xRightarrow{\text{add}} & \text{NegBin}(K,p) \end{array}$$

Pure Math: define the gamma family of functions. Beginning with the gamma function for x non-neg:

upper-case gamma

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

e.g. $\Gamma(3) = \int_0^\infty \underbrace{t^2 e^{-t}}_{f(t)}\, dt = 2$



$$= \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)}$$

lower-case gamma — upper-case gamma

lower incomplete gamma function — upper incomplete gamma function

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\gamma(x,a) + \Gamma(x,a)}{\Gamma(x)}$$

$$= \overbrace{\frac{\gamma(x,a)}{\Gamma(x)}}^{P(x,a)} + \overbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}^{Q(x,a)} = P(x,a) + Q(x,a) = 1$$

$P(x,a)$ is called the "Lower Regularized gamma function"

$Q(x,a)$ is called the "Upper Regularized Gamma Function"

$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$ ← $X \sim Exp(1) = e^{-t}\mathbb{1}_{t\in[0,\infty)}$

HW: $\Gamma(x+1) = x\,\Gamma(x)$

let $n \in \mathbb{N}$

$\Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2) = \cdots$

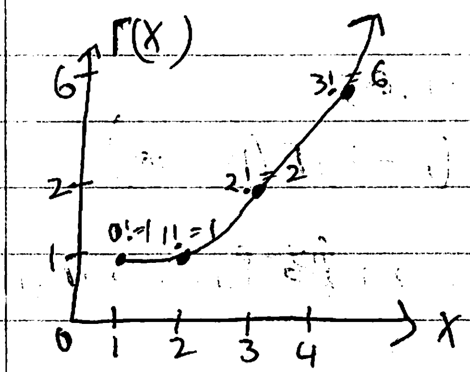
$= (n-1) \cdots (3)(2)(1) = (n-1)!$

let $x \in (0,\infty)$, any positive number

$\Gamma(x) = (x-1)\,\Gamma(x-1) = \cdots$

$= (x-1)(x-2)\cdots\Gamma(c)$ where $c \in (0,1)$

the gamma function "extends" the factorial function to all positive #'s



$c > 0$,

$\int_0^\infty t^{x-1} e^{-ct}\, dt$

u-substitution, let $u = ct$

$\Rightarrow t = \frac{u}{c}$

$\Rightarrow \frac{du}{dt} = c$

$\Rightarrow dt = \frac{1}{c}\, du$, $t=0$

$\Rightarrow u$, $t \to \infty \Rightarrow u \to \infty$, $t=a$

$\Rightarrow u = ac$

$= \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\, du$

$= \frac{1}{c^x}\int_0^\infty u^{x-1} e^{-u}\, du = \frac{\Gamma(x)}{c^x}$

— o —

$\int_0^a t^{x-1} e^{-ct}\, dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\, du$

$= \frac{1}{c^x}\int_0^{ac} u^{x-1} e^{-u}\, du = \frac{\gamma(x, ac)}{c^x}$

$\int_a^\infty t^{x-1} e^{-ct}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$

complete − lower = upper

If $n \in \mathbb{N}$

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}\,dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du$$

$$\frac{du}{dt} = (n-1)t^{n-2} \qquad v = \int dv = \int e^{-t}\,dt = -e^{-t}$$

$$\Rightarrow du = (n-1)t^{n-2}\,dt$$

$$= \left[t^{n-1}(-e^{-t})\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}\,dt$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}\,dt$$

$$= a^{n-1}e^{-a} + (n-1)\,\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-1}e^{-a} + (n-2)\,\Gamma(n-2,a)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)\left(a^{n-2} + (n-2)\left(a^{n-3} + (n-3)\,\Gamma(n-3,a)\right)\right)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + (n-1)(n-2)(n-3)\,\Gamma(n-3,a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \frac{1}{(n-4)!}\Gamma(n-3,a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \cdots + \frac{a^1}{1!} + \underbrace{\frac{a^0}{0!}}_{1}\right)$$

$$= e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

$$\Gamma(1,a) = \int_a^\infty e^{-t}\,dt$$

$$= \left[-e^{-t}\right]_a^\infty = e^{-a}$$

$$X \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!} \mathbb{1}_{x \geq 0}$$

CDF:

$$F_X(x) = P(X = x) = \int_0^x \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \, dt \, \mathbb{1}_{x \geq 0}$$

$$= \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}} \underbrace{\int_0^x t^{k-1} e^{-\lambda t} \, dt}_{\frac{\gamma(k, \lambda x)}{\lambda^k}}$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\gamma(k, \lambda x)}{\lambda^k} = P(k, \lambda x)$$

$$1 - F_X(x) = 1 - P(k, \lambda x) = Q(k, \lambda x)$$

$$X \sim \text{Poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

CDF:

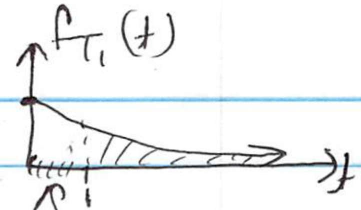
$$F_X(x) = P(X \leq x)$$

$$= \sum_{t=0}^{x} \frac{\lambda^t e^{-\lambda}}{t!} = e^{-\lambda} \sum_{t=0}^{x} \frac{\lambda^t}{t!}$$

$$= \frac{1}{\underbrace{x!}_{\Gamma(x+1)}} \underbrace{e^{-\lambda} x! \sum_{t=0}^{x} \frac{\lambda^t}{t!}}_{\Gamma(x+1, \lambda)}$$

$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = Q(x+1, \lambda)$$

the relationship between the Erlang and the Poisson is known as the "Poisson process"



$T_1 \sim Exp(\lambda) = Erlang(1, \lambda)$

$P(T_1 > 1) = Q(1, \lambda)$ ⟵ equal

$N \sim Poisson(\lambda)$

\# events as Poisson(λ)

$F_N(0) = P(N \leq 0) = P(N = 0) = Q(1, \lambda)$