Math 621
Lecture 19
11-18-2020

$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi} \frac{1}{x^2+1}$

$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi} \frac{1}{x^2+1} \, dx = \infty$ the expectation doesn't exist

$M_X(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\pi} \frac{1}{x^2+1} \, dx$ doesn't exist

$\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{1}{\pi} \frac{1}{x^2+1} \, dx = \cdots = e^{-|t|}$,

$\phi_X'(t) = -\frac{t}{|t|} e^{-|t|}$, $\phi_X'(0) = \text{DNE}$

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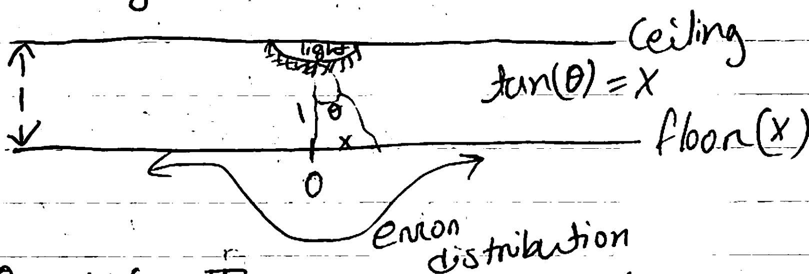
Let's derive the Cauchy distribution like the physicists found it:



$\theta \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$

$X = g(\theta)$ $\quad \theta = g^{-1}(x) = \arctan(x)$

tangent is invertible between $-\pi/2$ and $\pi/2$.

$f_X(x) = f_\theta(g^{-1}(x)) \left| \frac{d}{dx}[g^{-1}(x)] \right|$

$= \frac{1}{\pi} \underbrace{\mathbb{1}_{\arctan(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}}_{x \in \mathbb{R}} \frac{1}{x^2+1}$

$= \text{Cauchy}(0,1)$.

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(2) Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \frac{X_i - \mu}{\sigma} = Z_i \sim N(0,1)$

$T_n \sim N(n\mu, n\sigma^2)$, $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$,

$$S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2 \sim f_{S_n^2}(s^2) = ?$$

sample variance

(will come back...)

o

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$\underline{Z}^T\underline{Z} = \sum_{i=1}^n Z_i^2 \sim \chi_n^2$$

chi

$$= \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{\sum (X_i - \mu)^2}{\sigma^2}$$

$$(X_i - \mu)^2 = \left((X_i - \bar{X}) + (\bar{X} - \mu)\right)^2$$

$$= \sum (X_i - \bar{X})^2 + \underbrace{\sum 2(X_i - \bar{X})(\bar{X} - \mu)}_{\sum X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu} + \sum (\bar{X} - \mu)^2$$

$$\sum X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu = \underbrace{\bar{X}\sum X_i}_{n\bar{X}^2} - \cancel{n\bar{X}^2} - \underbrace{\mu\sum X_i}_{n\bar{X}\mu} + \cancel{n\bar{X}\mu} = 0$$

$$= \sum (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

$$\boxed{\frac{n(\bar{X} - \mu)^2}{\sigma^2} = \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2 = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}$$

$$\Rightarrow \frac{\sum (X_i - \mu)^2}{\sigma^2} = \frac{n-1}{\sigma^2} S^2 + \underbrace{\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{Z^2 \sim \chi_1^2} \sim \chi_n^2$$

$\frac{n-1}{\sigma^2} S^2 \; \| \;$ Maybe χ_{n-1}^2

$$\boxed{\begin{array}{l} U_1 \sim \chi_{k_1}^2 \text{ indep of } U_2 \sim \chi_{k_2}^2 \\ \Rightarrow U_1 + U_2 \sim \chi_{k_1 + k_2}^2 \end{array}}$$

In order for this "maybe" to be true, we need independence of those two terms ie we need S^2 and $\bar{X}$ to be independent.

We need Cochran's Theorem to prove this.

$\vec{z}^T\vec{z} = \underbrace{\vec{z}^T I \vec{z}}_{} \sim \chi^2_n$ this scalar is called a "quadratic form"

Consider $\vec{z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{B_1 \text{ Matrix}} \vec{z} = z_1^2 \sim \chi^2_1$

Consider $\vec{z}^T \underbrace{\begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}}_{B_i} \vec{z} = z_i^2 \sim \chi^2_1$

$\text{rank}[B_i] = 1$

$\sum \text{rank}[B_i] = n$

$$\vec{z}^T I \vec{z} = \vec{z}^T(B_1 + B_2 + \cdots + B_n)\vec{z}$$
$$= \underbrace{\vec{z}^T B_1 \vec{z}}_{\chi^2_1} + \underbrace{\vec{z}^T B_2 \vec{z}}_{\chi^2_1} + \cdots + \underbrace{\vec{z}^T B_n \vec{z}}_{\chi^2_1} \sim \chi^2_n$$

Conjecture: each of these quadratic forms is independent.

Cochran's Theorem: If $B_1 + B_2 + \cdots + B_k = I$, $k \leq n$ and the sum of their ranks is n, then you have two powerful results:

a) $\vec{z}^T B_j \vec{z} \sim \chi^2_{\text{rank}[B_j]}$ and

b) $\vec{z}^T B_{j_1} \vec{z}$ is indep of $\vec{z}^T B_{j_2} \vec{z}$, $\forall j_1 \neq j_2$.

Consider $\sum(z_i - \bar{z})^2 = \sum z_i^2 - 2\underbrace{\sum z_i}_{n\bar{z}}\bar{z} + \sum \bar{z}^2$

$= \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2$

$= \sum z_i^2 - n\bar{z}^2$

let $\vec{1}_n$ = n-dim column vector of all ones

$\bar{z} = \frac{1}{n}\vec{1}^T\vec{z} = \frac{1}{n}\vec{z}^T\vec{1}$ $\boxed{T = \text{transpose}}$

④

$$n\bar{z}^2 = n\bar{z}\bar{z} = \cancel{n}\frac{1}{\cancel{n}}\vec{z}^T\vec{1}\ \frac{1}{n}\vec{1}^T\vec{z}$$
$$= \frac{1}{n}\vec{z}^T\vec{1}\vec{1}^T\vec{z} = \vec{z}^T\underbrace{\left(\frac{1}{n}J_n\right)}_{B_2}\vec{z}$$

let $J_n = \vec{1}\vec{1}^T$, which is an $n \times n$ matrix of all ones.

$$\sum(z_i - \bar{z})^2 = \vec{z}^T I \vec{z} - \vec{z}^T\left(\frac{1}{n}J_n\right)\vec{z}$$
$$= \vec{z}^T\underbrace{\left(I - \frac{1}{n}J_n\right)}_{B_1}\vec{z}$$

$$\vec{z}^T\vec{z} = \sum(z_i - \bar{z})^2 + n\bar{z}^2$$
$$= \vec{z}^T B_1 \vec{z} + \vec{z}^T B_2 \vec{z}$$

I want to use Cochran's thm on the above expression. So I need to make sure $B_1 + B_2 = I$ and $\text{rank}[B_1] + \text{rank}[B_2] = n$.

$$B_1 + B_2 = \left(I - \frac{1}{n}J_n\right) + \frac{1}{n}J_n = I \checkmark$$

$$\text{rank}[B_2] = \text{rank}\left[\frac{1}{n}J_n\right] = \text{rank}[J] = 1$$

$$\text{rank}[B_1] = \text{rank}\left[I - \frac{1}{n}J\right] = ?$$

Theorem from 231 class: If A is symmetric & idempotent (ie. $AA = A$), then $\text{rank}[A] = \text{tr}[A] =$ sum of A's diagonal entries.

$$\left(I - \frac{1}{n}J\right)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J \checkmark$$

$$\left(I - \frac{1}{n}J\right)\left(I - \frac{1}{n}J\right) = II - \frac{1}{n}JI - \frac{1}{n}IJ + \frac{1}{n^2}JJ$$
$$= I - \frac{2}{n}J + \frac{1}{n^2}nJ$$
$$= I - \frac{1}{n}J \checkmark$$

$$J_3J_3 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix} = 3J$$

Continued

$$\mathrm{tr}[I - \tfrac{1}{n}J] = (1 - \tfrac{1}{n}1) + (1 - \tfrac{1}{n}1) + \cdots + (1 - \tfrac{1}{n}1)$$ ⑤

$$= n - 1 = \text{rank}[B_1]$$

So; rank $[B_1] = n-1$ ✓

$\Rightarrow \sum \text{rank}[B_j] = 1 + n - 1 = n$ ✓

Since the two conditions of Cochran's Thm are satisfied, we can apply it to get the two results:

$\vec{Z}^T B_1 \vec{Z} = \sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ indep of
$\vec{Z}^T B_2 \vec{Z} = n\bar{Z}^2 \sim \chi^2_1$

What does this have to do with our goal? Well, it's the same thing:

$$\bar{Z} = \frac{Z_1 + \cdots + Z_n}{n} \quad \frac{\frac{X_1 - \mu}{\sigma} + \cdots + \frac{X_n - \mu}{\sigma}}{n}$$

$$= \frac{\sum X_i - n\mu}{\sigma n} = \frac{\bar{X} - \mu}{\sigma}$$

$$\sum (Z_i - \bar{Z})^2 = \sum \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma}\right)^2$$

$$= \sum \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 = \frac{1}{\sigma^2} \sum (X_i - \bar{X})^2 = \frac{n-1}{\sigma^2} S^2$$

$$n\bar{Z}^2 = n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 = \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2 = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$$

$$\underbrace{\frac{n-1}{\sigma^2} S^2}_{\sim \chi^2_{n-1}} + \underbrace{\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{Z}}}\right)^2}_{\sim \chi^2_1} \sim \chi^2_n$$

independent

Fisher proved this w/o Cochran's thm in 1925 and Geary proved in 1936 that this decomposition is exclusive to the iid normal rv model.

⑥ $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim ?$ Not $N(0,1)$

Next Class

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