Lecture 10 MATH 621 — 10/07/2020

$T_k \sim \text{Erlang}(k, \lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k, \lambda)$

$N \sim \text{Poisson}(\lambda)$

$F_N(x) = Q(x+1, \lambda)$

Say $X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$k=5$

$\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\} \cup \{X_1 + X_2 + X_3 < 1\} \cup \{X_1 + X_2 < 1\} \cup \{X_1 < 1\} \cup \{X_1 > 1\}$

Let N: # events before 1 sec

$\{T_5 > 1\} = \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}$

$P(T_5 > 1) = P(N \le 4) = F_N(4)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$

Why is this Poisson distr? This is the Poisson Process.

Let $T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$ where $k \in \mathbb{N}$, $\lambda \in (0, \infty)$

Let $T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$ where $k \in \mathbb{N}$, $p \in (0,1)$

$\text{Erlang}(k, \lambda) = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$

$\text{NegBin}(k, p) = \left(\frac{\Gamma(k+t)}{\Gamma(k)\, t!}\right) (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

For both, what if $k \in (0, \infty)$? Are both rv's still legal?

We can show the both:

$$\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t}\, dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1$$

We just derived two new r.v's

$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \ge 0}$ → waiting time for exponentials, β is like λ

$X \sim \text{ExtNegBin}(k, p) = \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$ — waiting time for k geometric

Transformations of Discrete r.v's

$X \sim Bern(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \begin{cases} 1 & w.p\ p \\ 0 & w.p\ 1-p \end{cases}$

$Y = X + 3 \sim \begin{cases} 4 & w.p\ p \\ 3 & w.p\ 1-p \end{cases} = p^{y-3} (1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}}$

I want to find the PMF of Y using the PMF of X:

$Y = g(x) \sim P_Y(y) = P_X(g^{-1}(y))$

What assumption did I make when I derived this formula?

I assumed an inverse func. exists i.e is invertible.

$X \sim U(\{1,2,\dots,10\}) = \begin{cases} 1 & w.p\ \frac{1}{10} \\ 2 & w.p\ 1/10 \\ \vdots & \\ 10 & w.p\ 1/10 \end{cases}$, $Y = g(x) = \min\{X, 3\} = \begin{cases} 1 & w.p\ \frac{1}{10} \\ 2 & w.p\ \frac{1}{10} \\ \vdots & \\ 3 & w.p\ P(X=3) + P(X=4) + \dots P(X=10) \end{cases}$

$Y = g(X) \sim P_Y(y) = \sum_{\{x : y = g(x)\}} P_X(x) \overset{\text{if } g \text{ is invertable on } Supp(X)}{=} \sum_{\underbrace{\{x : x = g^{-1}(y)\}}_{\text{one elem only}}} P_X(x) = P_X(g^{-1}(y))$

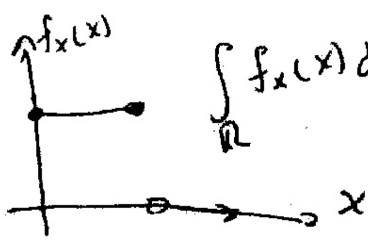
Let $X \sim Binom(n,p)$, $Y = \underbrace{X^3}_{g(x)} \sim P_X(\sqrt[3]{y}) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y} \in \{0,\dots,n\}}$

$Y = X^2 \sim P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$

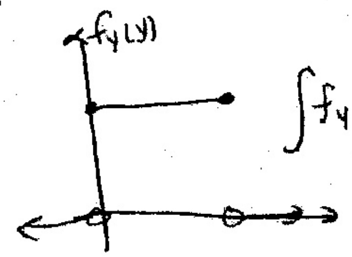
Transformation for continous rv's

For g invertible $f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ NO!

Let $X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$, $Y = 2X \sim f_X(g^{-1}(y)) = f_X\left(\frac{y}{2}\right) = \mathbb{1}_{\frac{y}{2} \in [0,1]} = \mathbb{1}_{y \in [0,2]}$



$\int_{\mathbb{R}} f_X(x)\,dx = 1$



$\int f_Y(y)\,dy = 2$ Not valid pdf.

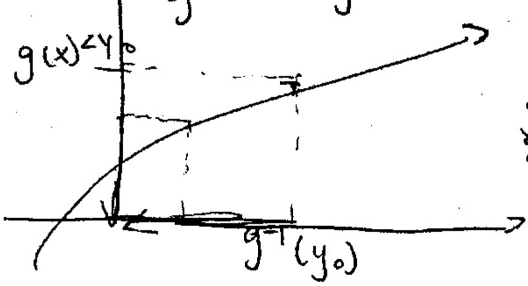
DF are not probabilities. So this fails. However, CDF *are* probabas.

$$F_Y(y) := P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$\Rightarrow \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

$$= f_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

$g' > 0 \Rightarrow g^{-1} > 0$



Decreasing slope case

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

g is invertible, $g' < 0$ $\rightarrow$ $= P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

$$\frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))]$$

$$= f_X(g^{-1}(y))\left(\underbrace{-\frac{d}{dy}[g^{-1}(y)]}_{\text{always neg.}}\right)$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$ (general rule)

We can deriv a less general but usefull corrollary rule:

$Y = \underbrace{aX + c}_{} \sim f_Y(y) = ?$ (shift by c and scale by a)

g(X) is invertible

$g^{-1}(y) = \frac{y-c}{a}$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{a}\right| = \frac{1}{|a|}$

Therefor $f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$

$Y = aX \sim f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$, $Y = X + c \sim f_X(y-c)$

Let $X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x \geq 0}$

$$Y = g(X) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = g(x)$$

$$y = \ln(e^{x}-1) \Rightarrow e^{y} = e^{x}-1 \Rightarrow e^{y}+1 = e^{x} \Rightarrow x = \ln(e^{y}+1) = g^{-1}(y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\ln(e^{y}+1)]\right| = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$$

$$f_Y(y) = f_X(\ln(e^{y}+1)) \cdot \frac{e^{y}}{e^{y}+1} = e^{-\ln(e^{y}+1)} \underset{\ln(e^{y}+1) \geq 0}{\mathbb{1}} \cdot \frac{e^{y}}{e^{y}+1}$$

$e^{y}+1 \geq 1$

$e^{y} \geq 0 \in \mathbb{R}$

$$= \frac{1}{e^{y}+1} \frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{e^{-2y}}{e^{-2y}} = \frac{e^{-y}}{(e^{-y}+1)^2} = \text{Logistic}(0,1) \text{ (Standard Logistic)}$$