Define $L^1 := \left\{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \right\}$ "L1 integrable" or "absolutely integrable" functions.

Are all PDFs in the set L1? YES. $\int_{-\infty}^{\infty} \left|\lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}\right| dx = \lambda \int_0^{\infty} e^{-\lambda x} dx = 1$

If $f \in L^1 \Rightarrow \exists\, \hat{f}$, the "Fourier Transform" of f:

$$\hat{f}(\omega) := \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x)\,dx = \mathcal{F}[f]$$

"forward fourier transform operator" AKA "Fourier analysis"

If $\hat{f} \in L^1 \Rightarrow$ then we can invert / reverse the Fourier transform via the "inverse / reverse Fourier transform operator" to get the original f back AKA "Fourier synthesis":

$$f(x) = \int_{\mathbb{R}} e^{i2\pi x\omega} \hat{f}(\omega)\,d\omega$$

Fourier inversion thm: if f, fhat are in L1, then f and fhat are 1:1.

f(x) is known as the "time domain" and fhat(omega) is known as the "frequency domain". f(x) can be decomposed into a sum of sines and cosines with frequencies omega, amplitides given by |fhat(omega)| and phase shifts given by Arg[fhat(omega)].

Let X be a rv. Define the characteristic function chf:

$$\phi_X(t) := E\left[e^{itX}\right] = \begin{cases} \int_{\mathbb{R}} e^{itx} f_X(x)\,dx & \text{if continuous} \\ \sum_{x \in \mathbb{R}} e^{itx} p_X(x) & \text{if discrete} \end{cases}$$

The chf is the Fourier transform in a different unit t = -2 pi omega.

Properties of the chf:

(P0) $\phi_X(0) := E\left[e^{i(0)X}\right] = E\left[e^0\right] = 1$ for all rv's.

(P1) $\phi_X(t) = \phi_Y(t) \iff X \stackrel{d}{=} Y$

(P2) $Y = aX + b$ for $a, b \in \mathbb{R}$

$$\phi_Y(t) = E\left[e^{it(aX+b)}\right] = E\left[e^{iatX} e^{itb}\right]$$
$$= e^{itb} E\left[e^{i\underbrace{at}_{t'}X}\right] = e^{itb} \phi_X(t') = e^{itb} \phi_X(at).$$

(P3) $X_1, X_2 \stackrel{ind}{\sim}$ and $T = X_1 + X_2$

$$\phi_T(t) = E\left[e^{it(X_1+X_2)}\right] = E\left[e^{itX_1} e^{itX_2}\right] = E\left[e^{itX_1}\right] E\left[e^{itX_2}\right]$$
$$= \phi_{X_1}(t)\,\phi_{X_2}(t)$$

(P4) "Moment generation"

conditions are satisfied to be able to interchange differentiation and integration

$$\phi_X'(t) = \frac{d}{dt}\left[E\left[e^{itX}\right]\right] = E\left[\frac{d}{dt}\left[e^{itX}\right]\right]$$
$$= E\left[iXe^{itX}\right]$$
$$\phi_X'(0) = E\left[iXe^{i(0)X}\right] = iE[X] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$
$$\phi_X''(t) = \frac{d}{dt}\left[E\left[iXe^{itX}\right]\right] = E\left[iX\frac{d}{dt}\left[e^{itX}\right]\right]$$
$$= E\left[i^2X^2e^{itX}\right] \Rightarrow E\left[X^2\right] = \frac{\phi_X''(0)}{i^2}$$
$$\Rightarrow E\left[X^n\right] = \frac{\phi_X^{(n)}(0)}{i^n} \quad \text{if the moment exists}$$

(P5) $\phi_X(t) \in [-1,1]$ for all X, t hence it always exists

$\Updownarrow$

$|\phi_X(t)| \in [0,1]$

Proof

$$|\phi_X(t)| = \left|E\left[e^{itX}\right]\right| = \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| \le \int_{\mathbb{R}} \left|e^{itx} f(x)\right| dx \le \int_{\mathbb{R}} \left|e^{itx}\right| |f(x)|\,dx$$
$$= \int_{\mathbb{R}} \big|\underbrace{i\sin(tx) + \cos(tx)}_{bi+a}\big| f(x)\,dx$$
$$= \int_{\mathbb{R}} \underbrace{\sqrt{\sin^2(tx) + \cos^2(tx)}}_{1}\, f(x)\,dx = 1$$

discrete same proof

(P6) Inversion. If $\phi_X(t) \in L^1$, then

PDF: $f_X(x) = \int_{\mathbb{R}} e^{-itx} \phi_X(t)\,dt$

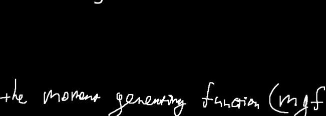
(P7) Levy's CDF thm. (works even if $\phi_X \notin L^1$)

$$P(X \in [a,b]) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\,dt$$

(P8) Levy's Continuity Thm.

consider a sequence of rv's X_1, X_2, ..., X_n. We define "X_n converges in distribution to X" and denote it $X_n \xrightarrow{d} X$ if the CDF of X_n converges pointwise to the CDF of X,

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \quad \forall x.$$



If $\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)\ \forall t \Rightarrow X_n \xrightarrow{d} X$.

the distribution on the left (X_n) is becoming more and more like the distribution on the right (X)

Define $M_X(t) := E\left[e^{tX}\right]$, the moment generating function (mgf).

(P0) $M_X(0) = E\left[e^{(0)X}\right] = 1$ (P1) $M_X(t) = M_Y(t) \Rightarrow X \stackrel{d}{=} Y$

(P2) $Y = aX + b \Rightarrow M_Y(t) = e^{tb} M_X(at)$

(P3) $X_1, X_2 \stackrel{ind}{\sim}$, $T = X_1 + X_2$ then $M_T(t) = M_{X_1}(t)\,M_{X_2}(t)$

(P4) $E\left[X^n\right] = M_X^{(n)}(0)$

but... mgf's sometimes don't exist!! And sometimtes don't exist for all t.

I don't care about mgf's. Why? Because chf's can do everything they can do and much much more!

$X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E\left[e^{itX}\right] = \int_0^{\infty} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\beta - it)x}\,dx = \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ ind. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

$$\phi_{X_1+X_2}(t) \underset{(P3)}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right)^{\alpha_2} = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2}$$

$\Downarrow$ (P1)

$X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$