Consider X_1, X_2, ..., X_n iid rv's of unknown PMF/PDF but we know it has expectation mu and variance sigsq (both finite).

let $T_n := X_1 + X_2 + \dots + X_n$, $E[T_n] = n\mu$, $Var[T_n] = n\sigma^2$

let $\bar{X}_n := \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$, $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \frac{\sigma^2}{n}$

let $Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \underbrace{\frac{\sqrt{n}}{\sigma}}_{a}\bar{X}_n + \underbrace{-\frac{\sqrt{n}}{\sigma}\mu}_{b}$, $E[Z_n] = 0$, $Var[Z_n] = 1 = SD[Z_n]$

"$\bar{X}_n$ standardized"

ident distr + (P1)

$c = e^{\ln(c)}$

$\phi_{T_n}(t) \overset{(P3)}{=} \phi_{X_1}(t) \cdot \dots \cdot \phi_{X_n}(t) = \phi_X(t)^n$

$\phi_{\bar{X}_n}(t) \overset{(P2)}{=} \phi_{T_n}\left(\frac{t}{n}\right) = \phi_X\left(\frac{t}{n}\right)^n$

$\phi_{Z_n}(t) \overset{(P2)}{=} e^{-\frac{it\mu\sqrt{n}}{\sigma}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right) = e^{-\frac{it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n = e^{-\frac{it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$

$= e^{-\frac{it\mu \cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\frac{\frac{1}{\cancel{n}}}{\frac{1}{n}}} = e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$

$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}\right)} = \phi_{Z_n}(t)$

We want to examine $\lim_{n\to\infty} \phi_{Z_n}(t)$ and if we find its limiting chf, $\phi_Z(t)$, we can use P8 to show that $Z_n \xrightarrow{d} Z \Rightarrow Z_n \overset{d}{\approx} Z$.

$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}} = e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$

let $u = \frac{t}{\sqrt{n}\sigma}$. If $n \to \infty \Rightarrow u \to 0$

L'Hôpital

$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}} = e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}}$

$= e^{\frac{t^2}{2\sigma^2}\frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2}} \overset{(P0)}{=} e^{\frac{t^2}{2\sigma^2}\left(\phi_X''(0) - \phi_X'(0)^2\right)}$

$\overset{(P4)}{=} e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - (iE[X])^2\right)} = e^{-\frac{t^2}{2\cancel{\sigma^2}}\overbrace{\left(E[X^2] - E[X]^2\right)}^{\cancel{\sigma^2}}} = e^{-\frac{t^2}{2}} = \phi_Z(t)$

Is $e^{-t^2/2} \in L_1$? What is $\int_{\mathbb{R}} e^{-t^2/2}\,dt = \sqrt{2\pi} < \infty$ YES!

Gaussian Integral

Now we can use P6 to invert the chf of Z to get the PDF of Z.

$f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\phi_Z(t)\,dt = \frac{1}{2\pi}\int e^{-\left(itz + \frac{t^2}{2}\right)}\,dz$

$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 - \underbrace{\left(\frac{\sqrt{2}iz}{2}\right)^2}_{\frac{2i^2z^2}{4} = -\frac{z^2}{2}}$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} e^{-z^2/2}\,dt = \frac{1}{2\pi}e^{-\frac{z^2}{2}}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2}\,dt$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,dy$, $t \to -\infty \Rightarrow y \to -\infty$, $t \to \infty, y \to \infty$

gaussian integral

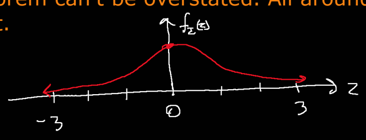
$= \frac{1}{2\pi}e^{-\frac{z^2}{2}}\int_{\mathbb{R}} e^{-y^2}\sqrt{2}\,dy = \frac{1}{2\pi}e^{-\frac{z^2}{2}}\sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ $Z \sim N(0,1)$

"Normal"

$\Longrightarrow X_1, \dots X_n \overset{iid}{\sim}$ mean μ, Variance $\sigma^2 < \infty$ $\Rightarrow \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$

This fact is called the "Central Limit Theorem"
and it is the crown jewel of an intermediate probability class.
The importance of this theorem can't be overstated. All around us we have devices that use it.

$Z \sim f_Z(z) = N(0,1)$



It's called the "Gaussian distribution" but really Laplace discovered it and called it his "second law of errors". It's actually the most common error distribution in the world. A lot the field of statistics is derived by assuming Gaussian / normal iid errors.

$E[Z] \overset{(P4)}{=} i\phi_Z'(0) = 0$ ✓

$\phi_Z'(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -te^{-t^2/2}$, $\phi_Z''(t) = -\frac{d}{dt}\left[te^{-t^2/2}\right] = -\left(-t^2e^{-t^2/2} + e^{-t^2/2}\right)$

$Var[Z] = E[Z^2] - \cancel{E[Z]^2}^{0} \overset{(P4)}{=} i^2\phi_Z''(0) = --1 = 1 = SD[Z]$

for $\sigma > 0$

$X = \mu + \sigma Z \sim f_X(x) = ?$

$f_X(x) = \frac{1}{\sigma}f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$

$E[X] = \mu + \sigma E[Z] = \mu$, $Var[X] = Var[\mu + \sigma Z] = \sigma^2$

$\phi_X(t) \overset{(P2)}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu - \sigma^2t^2/2}$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep. of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim ?$

$\phi_T(t) \overset{(P3)}{=} \phi_{X_1}(t)\phi_{X_2}(t) = e^{it\mu_1 - \sigma_1^2t^2/2}e^{it\mu_2 - \sigma_2^2t^2/2}$

$= e^{it(\mu_1+\mu_2) - (\sigma_1^2 + \sigma_2^2)t^2/2} \overset{(P1)}{\Longrightarrow} X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$ $g^{-1}(y) = \ln(y)$, $\left|\frac{d}{dy}[\,\cdot\,]\right| = \frac{1}{|y|}$

$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}\frac{1}{y}$

$= \frac{1}{\sqrt{2\pi\sigma^2y^2}}e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} = LogN(\mu, \sigma^2)$ Log-Normal model

Not on final....

e.g. you being with I_0 amount of money and each year it goes up/down by a random percentage X_i $X_i \sim N(7\%, 11\%^2)$

$I_f = I_0e^{X_1}e^{X_2}e^{X_3}\cdot\dots\cdot e^{X_n} = I_0e^{\overbrace{X_1 + \dots + X_n}^{\sim N(n\mu, n\sigma^2)}} = I_0\,LogN(\mu, \sigma^2)$

$P(I_f < I_0) = \dots$