Lecture 18

11/16/20.

Math 621
Prof. Kapelner

$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$ not 1-1.

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= 2\int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 2\left(F_Z(\sqrt{y}) - \cancel{F_Z(0)}^{\,1/2}\right) = 2F_Z(\sqrt{y}) - 1.$$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = \cancel{2} \cdot f_Z(\sqrt{y})\, \cancel{\tfrac{1}{2}}\, y^{-1/2} = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\overbrace{\sqrt{y} \in \mathbb{R}}^{y \ge 0.}} \propto y^{-1/2} e^{-1/2\, y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$E[Y] = K\,E[Z^2] = K$

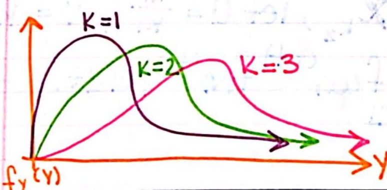
$Z_1, Z_2, \ldots, Z_K \overset{iid}{\sim} N(0,1)$ and $Y = Z_1^2 + Z_2^2 + \ldots + Z_K^2 \sim ?$

$Y \sim \text{Gamma}(K/2, 1/2) = \chi^2_K \leftarrow$ chi-squared distribution w/ parameter K, & this parameter is called "degrees of freedom".

the only parameter here is K.

$$= \frac{\left(\frac{1}{2}\right)^{K/2}}{\Gamma(K/2)} y^{K/2 - 1} e^{-y/2} \mathbb{1}_{y \ge 0}.$$

$$= (K=1)\ 1/\sqrt{2\pi}\ x^{-1/2} e^{-x/2} \mathbb{1}_{x \ge 0}$$



$X \sim \chi^2_K$, $Y = \sqrt{X} \sim f_Y(y)$, $X = y^2 = g^{-1}(y)$, $\left|\frac{d}{dy} g^{-1}(y)\right| = 2y$.

$f_Y(y) = f_X(y^2)\, 2y = \frac{(1/2)^{K/2}}{\Gamma(K/2)}\, y^{K-2} e^{-y^2/2}\, 2y\, \underbrace{\mathbb{1}_{y^2 \geq 0}}_{y \geq 0}$

$= \frac{\left(\frac{1}{2}\right)^{K/2-1}}{\Gamma(K/2)}\, y^{K-1} e^{-y^2/2}\, \mathbb{1}_{y \geq 0} = \chi_K$.

this is the chi distribution w/ K degrees of freedom.

$X \sim N(0,1)$, $|X| \sim ?$ $|X| = \sqrt{X^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\, \mathbb{1}_{y \geq 0}$

$= 2 \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ (2 times PDF of $N(0,1)$.

$X \sim \mathrm{Gamma}(\alpha, \beta)$, for $c > 0$, $Y = cX \sim \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{x}{c}\right)^{\alpha-1}}_{\frac{x^{\alpha-1}}{c^{\alpha-1}}} e^{-\beta\frac{x}{c}}$.

$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta/c)x} = \mathrm{Gamma}(\alpha, \beta/c)$.

$X \sim \chi^2_K$, $Y = \frac{X}{K} \sim \mathrm{Gamma}(K/2, K/2)$.

$X_1 \sim \chi^2_{K_1}$, indep. of $X_2 \sim \chi^2_{K_2}$, let $U = \frac{X_1}{K_1} \sim \mathrm{Gamma}\left(\frac{K_1}{2}, \frac{K_1}{3}\right)$, $V = \frac{X_2}{K_2} \sim \mathrm{Gamma}\left(\frac{K_2}{2}, \frac{K_2}{3}\right)$

$R = \frac{X_1/K_1}{X_2/K_2} = \frac{U}{V} \sim \int_{\mathrm{Supp}[U]} f_U(rt)\, f_V(t)\, |t|\, dt.$ $\rightarrow$

$\mathbb{1}_{rt \in [0,\infty)}$

$$= \int_0^\infty t \frac{a^a}{\Gamma(a)}(rt)^{a-1} e^{-art} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} \, dt \, \mathbb{1}_{r\geq 0}$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r>0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} \, dt.$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r>0} \frac{\Gamma(a+b)}{(ar+b)^{a+b}} = \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\underbrace{b^{-(a+b)}}_{b^{-a}\cancel{b^{-b}}} (1+\frac{a}{b}r)^{-(a+b)}}$$

$\mathbb{1}\, r>0.$

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} (1+a/b)^{-(a+b)} \mathbb{1}_{r>0}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{k_1/2}}{B(k_1/2, k_2/2)} r^{k_1/2-1} (1+k_1/k_2)^{-\frac{k_1+k_2}{2}} \mathbb{1}\, r>0$$

$= F_{k_1,k_2} \leftarrow$ the "F distribution" or the "Fisher-Snecedor" distribution w/ k_1 numerator degrees of freedom & k_2 denominator degrees of freedom, $k_1 \in \mathbb{N}$, $k_2 \in \mathbb{N}$.

$Z \sim N(0,1)$ indep. of $X \sim \chi^2_k$. Let $W = \frac{Z}{\sqrt{X/k}} \sim f_W(w)$

$W^2 = \frac{Z^2}{X/k} \sim F_{1,k}$.

$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$

Take d/dw of both sides: $\longrightarrow$ $\longrightarrow$

$$\frac{d}{dw}\left[F_{W^2}(w^2)\right] = \frac{d}{dw}\left[F_W(w)\right] - \frac{d}{dw}\left[F_W(-w)\right]$$

$$\rightarrow 2w f_{W^2}(w^2) = f_W(w) - -f_W(w) \rightarrow f_W(w) = w \cdot f_{W^2}(w^2)$$

$$\rightarrow f_W(w) = \cancel{w}\,\frac{(1/k)^{1/2}}{B(1/2, k/2)} \underbrace{\cancel{(w^2)^{1/2-1}}}_{\cancel{w}^{-1}} \left(1+\frac{w^2}{k}\right)^{-k+1/2}$$

$$\underbrace{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(k/2)}}_{\sqrt{k\pi}} \rightarrow \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma(k/2)} \left(1+w^2/k\right)^{\frac{-k+1}{2}} = T_k$$

↑ students T distribution w/ k degrees of freedom

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$. $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f(u) |u| \, du.$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \, 1/\sqrt{2\pi}\, e^{-u^2/2} |u| \, du.$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du + \int_{0}^{\infty} e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du \right)$$

$$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\left(\frac{1+r^2}{2}\right)u^2} du.$$

let $t = u^2 \rightarrow \frac{dt}{du} = 2u \rightarrow du = 1/2 \, 1/u \, dt$, $u=0 \rightarrow t=0$,

$u \rightarrow \infty \rightarrow t \rightarrow \infty$.

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{1+r^2}{2} t} \cancel{u}\frac{1}{2}\cancel{\frac{1}{u}}\, du. = \frac{1}{2\pi} \cdot \frac{1}{\frac{1+r^2}{2}} = \frac{1}{\pi} \cdot \frac{1}{1+r^2}$$

$= \text{Cauchy}(0,1)$.

$$X = \sigma R + c \sim \text{Cauchy}(c,\sigma) = \frac{1}{\sigma\pi} \cdot \frac{1}{1+\left(\frac{r-c}{\sigma}\right)^2}$$

$T_1 = \text{Cauchy}(0,1)$.