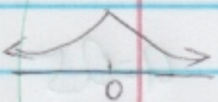
$Y = aX \sim f_X(\frac{y}{a})\frac{1}{|a|}$, $Y = X + c \sim f_X(y-c)$

Lecture 11

normal 0,1

$X \sim \text{logistic}(0,1) = \frac{e^x}{(1+e^x)^2} \approx N(0,1)$ but with thicker tails



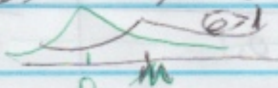
$\underset{\mu}{E[X]=0}$ $\quad SD[X] = \frac{\pi}{\sqrt{3}} \approx 1.8 > 1$

consider the shift and scale where sigma > 0

$Y = \mu + \sigma X \sim f_Y(y)$

Shift and scale: $f_Y(y) = f_X(\frac{y-\mu}{\sigma})\frac{1}{|\sigma|} = \frac{e^{\frac{y-\mu}{\sigma}}}{\sigma(1+e^{\frac{y-\mu}{\sigma}})} = \text{logistic}(\mu, \sigma)$

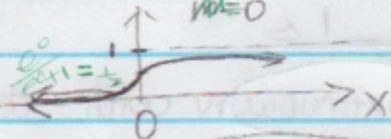
standard



Why is this called logistic distribution? There is a famous function called "the logistic function" it has three parameters L (maximum value) K (steepness) μ (center) and it is

$\ell(x) := \frac{L}{1-e^{-k(x-\mu)}} \underset{\substack{L=1\\k=1\\\mu=0}}{=} \frac{1}{1+e^{-x}} \cdot \boxed{\frac{e^x}{e^x}} = \frac{e^x}{e^x+1}$ $\boxed{\text{Standard logistic Function}}$

Graphic



famous function $X \sim \text{logistic}(0,1)$

calculus

$CDF = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{e^t}{(1+e^t)^2}dt = \int_1^{1+e^x} \frac{1-u}{u^2}\frac{1}{1-u}du = [-u^{-1}]_1^{1+e^x} = 1 - \frac{1}{1+e^x} = \frac{e^x}{1+e^x}$

Let: $u = 1+e^t \Rightarrow e^t = 1-u \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du = \frac{1}{1-u}du \Rightarrow t = -\infty \Rightarrow u = 1$

$t = x \Rightarrow u = 1+e^x$

241 Quantile Function

The "quantile" q or "percentile" 100q ~~is defined as~~ for r.v. X is defined as minimum x such that $q \le P(X \le x) = F(x) \Leftrightarrow F(x) \ge q$ (CDF)

it denoted $Q[X,q]$ ~~if q=0.5~~, it other Q is the "quantile operator" (not the upper incomplete regularized gamma function) when $q = 0.5$ the quantile has a special name the "median". ~~Med[X]=Q~~

$Med[X] = Q[X, q]$

Here is an example:

$X \sim U(\{2,4,6,\ldots,20\}) = \frac{1}{10} \mathbb{1}_{x \in \{2,4,6,\ldots,20\}}$

X	P(X)	F(X)
2	0.1	0.1
4	0.1	0.2
6	0.1	0.3
8	0.1	0.4
10	0.1	0.5
12	0.1	0.6
14	0.1	0.7
16	0.1	0.8
18	0.1	0.9 ←
20	0.1	1 ←

$Q[X, 30\%] = 6$

$Q[X, 80\%] = 16$

$Q[X, 85\%] = 18 = Q[X, 0.9]$

0.9 ← minimum X

1 ←

$q = 0.5$

$Med[X] = 10$

However, if X is a continuous rv with "contiguous support" e.g $[0,10]$, $[0, \text{infinity}]$ (∞), all real number, ect and and not something like $[0,1] \cup [2,3]$ In the latter case, F(X) is flat between $[1,2]$ which mean it's not invertible, in the former case, F(X) is invertible

$Q[X, q] = F_X^{-1}(q)$ and inverse CDF is called appropriately the "quantile Function" quantile Function

example $X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} \Rightarrow F_X(x) = 1 - e^{-\lambda x} = q \Rightarrow$ CDF $1-q = e^{-\lambda x}$

$\Rightarrow \ln(1-q) = -\lambda x = x = -\frac{1}{\lambda} \ln(1-q) = \frac{1}{\lambda} \ln\left(\frac{1}{1-q}\right) = F_X^{-1}(q)$

$F_X^{-1}(0.5) = Med[X] \overset{q=0.5}{=} \frac{\ln(2)}{\lambda} = F_X^{-1}(0.5)$

Quantile function are not usually avaiable in closed form since CDF's aren't even usually avaiable in closed form eg

$X \sim \text{Erlang}(k, \lambda) \Rightarrow F_X(x) = P(k, \lambda x)$ Gamma Function not close form

$Med[X] \overset{q=0.5}{=} x$ such that $P(k, \lambda x) = 0.5$ Need a computer solver

$k>0$

$X \sim Exp(\lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x\geq 0}$, $Y = g(X) = ke^X \sim f_Y(y) = ?$

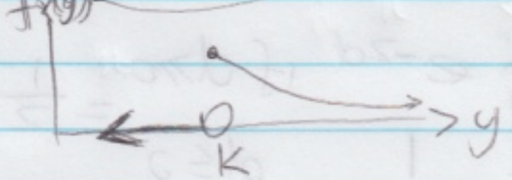
$y = ke^x \Rightarrow \frac{y}{k} = e^x \Rightarrow x = \ln(\frac{y}{k}) = \ln(y) - \ln(k) = g^{-1}(y)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{y}\right| = \frac{1}{|y|}$ $\quad \ln((\frac{y}{k})^{-\lambda})$ $\quad y\in[k,\infty)$ $\quad \ln y \in [\ln(k),\infty)$

$f_Y(y) = f_X(\ln(\frac{y}{k}))\frac{1}{|y|} = \frac{\lambda}{|y|}e^{-\lambda \ln(\frac{y}{k})}\mathbb{1}_{\ln y - \ln k \in [0,\infty)}$

Test $\quad = \frac{\lambda}{y}(\frac{y}{k})^{-\lambda}\mathbb{1}_{y\in[0,\infty)} = Pareto\ I_{one}(k,\lambda)$

$\frac{c}{y^{\lambda+1}}$



$k\in(0,\infty)$ $\quad \lambda\in(0,\infty)$

$\frac{d}{dt}[-\frac{t^{-\lambda}}{\lambda}] = -\lambda(-\frac{t^{-\lambda-1}}{\lambda}) = \frac{1}{t^{\lambda+1}}$

CDF: $F_Y(y) = \int_k^y \frac{\lambda}{k^{-\lambda}}\frac{1}{t^{\lambda+1}}\,dt = \frac{\lambda}{k^{-\lambda}}\left[-\frac{1}{\lambda t^{\lambda}}\right]_k^y = k^{\lambda}\left(\frac{1}{k^{\lambda}} - \frac{1}{y^{\lambda}}\right)$

$= 1-(\frac{k}{y})^{\lambda} \Rightarrow F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}$

$\frac{d}{dt}[-\frac{t^{-\lambda}}{\lambda}] = -\lambda(-\frac{t^{-\lambda-1}}{\lambda}) = \frac{1}{t^{\lambda+1}}$

This distribution is ~~was~~ discovered by Vilfredo Pareto, an Italian economis in 1896. when he observed that 20% of richest Italians owned 80% of land (i.e. the wealth)

This known as Pareto Principle and it corresponds to Pareto I (1, 1.161) distribution

Further, the Pareto distribution is a waiting time / Survial time model. It's used for [wikipedia] wealth music talent, number of patents, ...

$X, Y \overset{iid}{\sim} Exp(1) = e^{-x}\mathbb{1}_{x\in[0,\infty)}$ $\quad D = X - Y$
$= X + (-Y) \sim f_D(d) = ?$

$f_D(d) := \int_{Supp[X]} f_X^{old}(x)\, f_Z^{old}(d-x)\,\mathbb{1}_{d-x\in Supp[Z]}\,dx$ $\quad x \in [d,\infty)$

independent X & Z are not iid

$= \int_0^{\infty} e^{-x} e^{d-x}\mathbb{1}_{d-x\in(-\infty,0]}\,dx$ $\quad x-d\in[0,\infty)$

scale: $-1Z \sim \frac{1}{|-1|}f_X(\frac{z}{-1}) = e^{z}\mathbb{1}_{z\in(-\infty,0]}$

$$= e^{d}\int_0^\infty e^{-2x}\,\mathbb{1}_{x\in[d,\infty)}\,dx = e^{d}\begin{cases}\int_d^\infty e^{-2x}\,dx & \text{if } d>0\\ \int_0^\infty e^{-2x}\,dx & \text{if } d<0\end{cases}$$

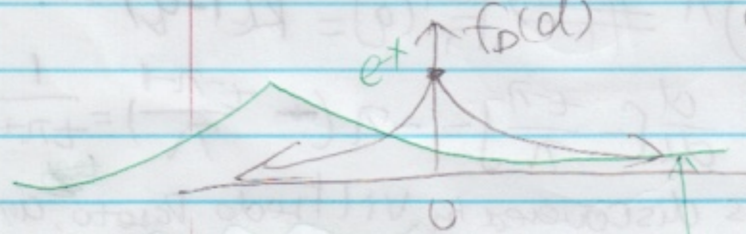
$$= e^{d}\begin{cases}\left[-\tfrac{1}{2}e^{-2x}\right]_d^\infty & \text{if } d\geq 0\\ \left[-\tfrac{1}{2}e^{-2x}\right]_0^\infty & \text{if } d<0\end{cases} = \tfrac{1}{2}e^{d}\begin{cases}\left[-e^{-2x}\right]_d^\infty & d\geq 0\\ \left[-e^{-2x}\right]_0^\infty & d<0\end{cases}$$

$$= \tfrac{1}{2}e^{d}\begin{cases}e^{-2d} & \text{if } d\geq 0\\ 1 & d\leq 0\end{cases} = \tfrac{1}{2}\begin{cases}e^{-d} & d\geq 0\\ e^{d} & d<0\end{cases}$$

$$= \frac{1}{2\times 1}e^{-|d|} = \text{Laplace}(0,1)$$

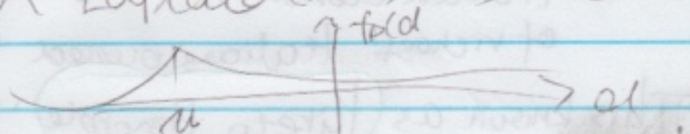
Standard laplace distribution

AKA "double exponential" distribution (Also known as)



$$= \frac{1}{2\sigma}e^{-\frac{|x-\mu|}{\sigma}}$$

$X = \mu + \sigma D \sim \text{Laplace}(\mu, \sigma)$ $\quad \sigma > 0$



Waiting time distribution