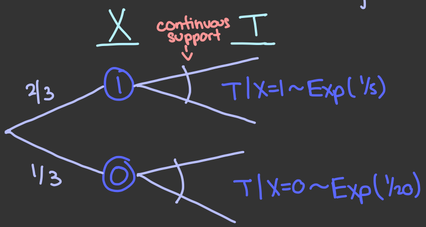
Mixture and compound distributions

Consider a situation where $2/3$ of the time there is fast internet speed so your downloads take $T \sim Exp(1/5) \Rightarrow E[T] = 5s$. and the other $1/3$ of the time, there is Internet traffic, so your downloads take $T \sim Exp(1/20) \Rightarrow E[T] = 20s$. What is the distribution of the "overall T" or "uncoditional on the Internet speed"? Let $X \sim Bern(2/3)$ and $X=1$ corresponds to fast Internet and $X=0$ corresponds to slow internet. Let's draw a tree diagram:



$h(x) = \int_{\mathbb{R}} h(x,y)\,dy$

$h(x) = \sum_{y \in \mathbb{R}} h(x,y)$

$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\, p_X(x)$$
$$= \sum_{x \in \{0,1\}} f_{T|X}(t,x)\, p_X(x) = f_{T|X}(t,0)\,p_X(0) + f_{T|X}(t,1)\,p_X(1)$$
$$= \tfrac{1}{20} e^{-\frac{1}{20}t} \cdot \tfrac{1}{3} + \tfrac{1}{5} e^{-\frac{1}{5}t} \cdot \tfrac{2}{3}$$

If the download speed was $t = 25s$, what is the probability it is a slow internet day, i.e. $x = 0$?

$X|T \sim Bern(?)$

$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, p_X(x)}{f_T(t)} \quad \text{Bayes Rule}$$

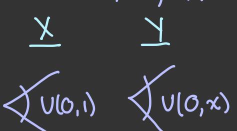
$W \sim Bern(p)$

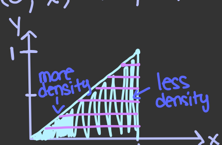
$p = P(W = 1)$

$$\text{Bernoulli param} = p_{X|T}(1,t) = \frac{f_{T|X}(t,1)\, p_X(1)}{f_T(t)}$$
$$= \frac{\tfrac{1}{5} e^{-\frac{1}{5}t} \cdot \tfrac{2}{3}}{\tfrac{1}{20} e^{-\frac{1}{20}t} \cdot \tfrac{1}{3} + \tfrac{1}{5} e^{-\frac{1}{5}t} \cdot \tfrac{2}{3}}$$

$$P_{X|T}(0,25) = 1 - P_{X|T}(1,25) = 1 - \frac{\tfrac{1}{5} e^{-\frac{1}{5}t} \cdot \tfrac{2}{3}}{\tfrac{1}{20} e^{-\frac{1}{20}t} \cdot \tfrac{1}{3} + \tfrac{1}{5} e^{-\frac{1}{5}t} \cdot \tfrac{2}{3}} = 0.842$$

$X \sim U(0,1)$, $Y|X = x \sim U(0,x) \Rightarrow Y \sim ?$

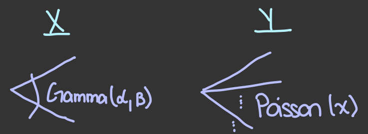




The first example featured T which was continuous (we call that the "model") and X which is discrete (we call that the "mixing distribution". Thus the unconditional distribution T is called a "mixture distribution".

In the second example Y, the model is continuous and X, the mixing distribution is also continuous and we call the unconditional distribution Y a "compound distribution".

page 156 - 157 : Let $Y|X=x \sim \text{Poisson}(x)$, $X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim ?$



$$p_Y(y) = \int_{\text{Supp}[X]} p_{Y|X}(y,x) f_X^{\text{old}}(x)\,dx = \int_0^\infty \frac{e^{-x}x^y}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \underbrace{\int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\,dx}_{\text{lec 9 integral}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}$$

$= \cdots\cdots$ HW

$= \text{ExtNegBin}\left(\alpha, \frac{\beta}{\beta+1}\right)$ → this is a more flexible count model than the Pois.

$Y|X=x \sim \text{Bin}(n, \overset{p}{x})$ where n is known, $X \sim \text{Beta}(\alpha, \beta)$

$$p_Y(y) = \int_{\text{Supp}[X]} p_{Y|X}(y,x) f_X^{\text{old}}(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y\in\{0,\dots,n\}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\,dx$$

$$= \binom{n}{y} \mathbb{1}_{y\in\{0,\dots,n\}} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1}\,dx}_{B(y+\alpha,\, n-y+\beta)}$$

$$= \frac{B(y+\alpha, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y\in\{0,\dots,n\}} \rightarrow \text{BetaBinomial}(n, \alpha, \beta)$$

$Y|X=x \sim \text{Exp}(x)$, $X \sim \text{Gamma}(\alpha, \beta) \overset{HW}{\Rightarrow} Y \sim \text{Lomax}(\beta, \alpha)$

→ Which is a more flexible waiting time than the exponential

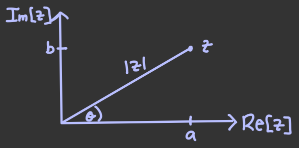
End of Midterm II material.

Moment generating functions (mgf's) and Characteristic functions (chf's). To derive these, we need to review complex/imaginary numbers First define $i := \sqrt{-1}$ "imaginary"

$\rightarrow$ let $a, b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, complex numbers

$Re[z] := a$, $Im[z] := b$, real component and imaginary component of a complex number.

$|z| := \sqrt{a^2 + b^2}$, $Arg[z] := \theta = \arctan\left(\frac{b}{a}\right)$



$i^0 = 1$
$i^1 = i$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i i^2 = -i$
$i^4 = (i^2)^2 = 1$
$i^5 = i^4 i = i$ ... $n \in \mathbb{N}$, $i^n = i^{n \bmod 4}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} - \cdots$$
$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - \cdots$$
$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \cdots$$

$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$

$\overset{tx=\pi}{\Rightarrow} e^{i\pi} = -1$

$\Rightarrow e^{i\pi} + 1 = 0$ Euler's Formula