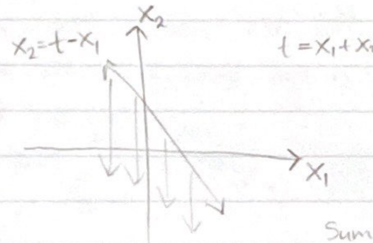
M368 9/30

$\vec{X}$ continuous r.v. $P(\vec{X} \in A) = \int_A \cdots \int f_{\vec{X}}(\vec{x})\,dx_1 \cdots dx_K$

let $T = X_1 + X_2 \sim f_T(t) = ?$

first note $f_T(t) = F'(t)$ CDF method

usually difficult to find the CDF of continuous r.v.'s, so this is not the usual method. The usual method is to use the Convolution formula (which we will derive)



$t = x_1 + x_2 \Rightarrow x_2 \le t - x_1$

Sum of all points under is less than t

$$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \le t - x_1 \right\}$$

$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

let $\begin{cases} x_1 = x \\ x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv \end{cases}$

$\Downarrow$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$= \int_{\mathbb{R}}\int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\,dv\,dx = \int_{-\infty}^{t}\left(\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx\right)dv$$

$$f_T(t) = \frac{d}{dt}\left[\quad\right]$$

Leibnitz's Rule:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x, b(x))b'(x) + g(x, a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy$$

if the outer derivative is a third variable, then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x, b(t))b'(t) + g(x, a(t))a'(t) + \int_{a(t)}^{b(t)} \cancel{\frac{\partial}{\partial t}[g(x,y)]}^{\,0}\,dy$$

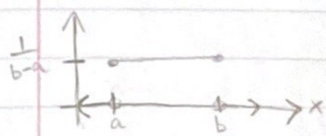
$$\frac{d}{dt}\left[\int_c^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + \cancelto{0}{g(x,c)\frac{d}{dt}[c]} \quad \text{(our case)}$$

- $f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dx = f_{X_1}(x) * f_{X_2}(x)$ general convolution formula

- if X_1, X_2 independent; $= \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f^{old}_{X_1}(x)\, f^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}\,dx$

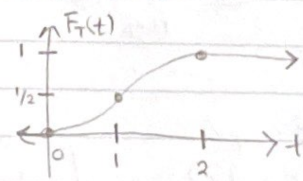
- $X_1, X_2 \overset{iid}{\sim}$; $= \int_{\mathbb{R}} f(x) f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\,dx$

$X \sim U(a,b)$ (Continuous uniform r.v.) $= \underbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x \in [a,b]}}_{f(x)}$

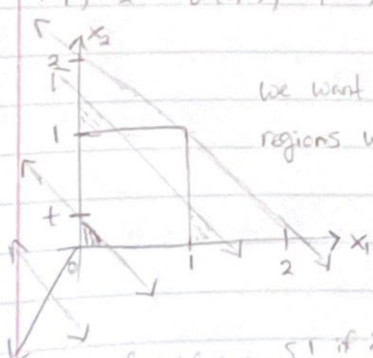


Standard uniform r.v. is when $a=0$, $b=1$.

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$



$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$



We want to compute cdf which means we want to find volumes in regions under the diagonal line.

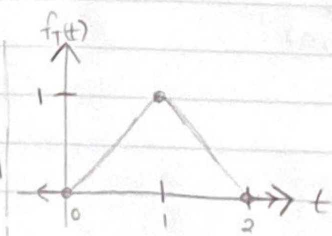
$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^2/2 & \text{if } t \in (0,1] \\ -t^2/2 + 2t - 1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \ge 2 \end{cases}$$

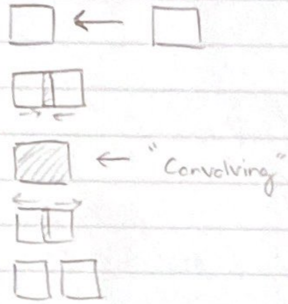
$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1 & \text{if } x_1 \in [0,1] \;\&\; x_2 \in [0,1] \\ 0 & o/w \end{cases}$$

if $t \in (1,2)$

$$F_T(t) = \frac{t^2}{2} - 2\frac{(t-1)^2}{2} = t^2/2 - (t^2 - 2t + 1) = -t^2/2 + 2t - 1$$

$$\Rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$





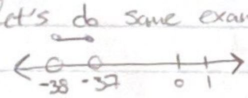
We just derived the pdf of the convolution by finding its cdf, and taking the derivative. Why can't we use our fancy formula?

iid old version:

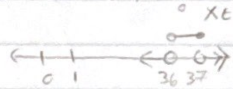
$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{t-x\in[0,1]}\, dx$$

$$x-t \in [-1,0] \qquad x \in [t-1,t] \qquad = \int_0^1 \mathbb{1}_{x\in[t-1,t]}\, dx$$

Let's do some examples. How about $t=-37$ $\int_0^1 \mathbb{1}_{x\in[-38,-37]}\, dx$



$+37 = t$?, $\int_0^1 \mathbb{1}_{x\in[36,37]}\, dx$



$$= \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t\in(1,2) \\ 0 & \text{if } t>2 \end{cases}$$

$X_1, X_2 \ldots \overset{iid}{\sim} Exp(\lambda) := \lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)}$, $T_2 = X_1 + X_2 \sim f_T(t) = ?$

$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}\, dx = \int_0^\infty \lambda e^{\cancel{-\lambda x}}\, \lambda e^{-\lambda(t-x)}\, \mathbb{1}_{t-x\in[0,\infty)}\, dx$$

$e^{-\lambda t} \cancel{e^{\lambda x}}$

$x-t\in(-\infty,0]$

$x\in(-\infty,t]$

$$= \lambda^2 e^{-\lambda t}\int_0^\infty \mathbb{1}_{x\in(-\infty,t]}\, dx = \lambda^2 e^{-\lambda t}\int_0^t dx = t\lambda^2 e^{-\lambda t} = Erlang(2,\lambda) \quad \text{(two exponentials added)}$$

$\underbrace{\mathbb{1}_{t\in[0,\infty)}}_{\text{pdf}}$