define

$:=$

iid: independent and identically distributed rv

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let $B_1, B_2, \dots, \overset{iid}{\sim} \text{Bern}(p)$

possibly an infinite sequence of iid rv's

let $X :=$ # of zero realizations before the first realization of one.

also, $X := \min\{t : B_t = 1\} - 1$

$p(0) = P(X=0) = P(\{\text{no 0's, just a 1}\}) = p$

$p(1) = P(X=1) = P(\{0, \text{then } 1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0,0,\dots,0}_{x \text{ zeros}},1\}) = (1-p)^x p$

$X \sim \text{Geom}(p) := (1-p)^x p \, \mathbb{1}_{x \in \{0,1,2,\dots\}}$ (nonnegative)

$X_1, X_2, \dots \overset{iid}{\sim} \text{geom}(p)$. $T_2 = X_1 + X_2 \sim p_T(t) = ?$

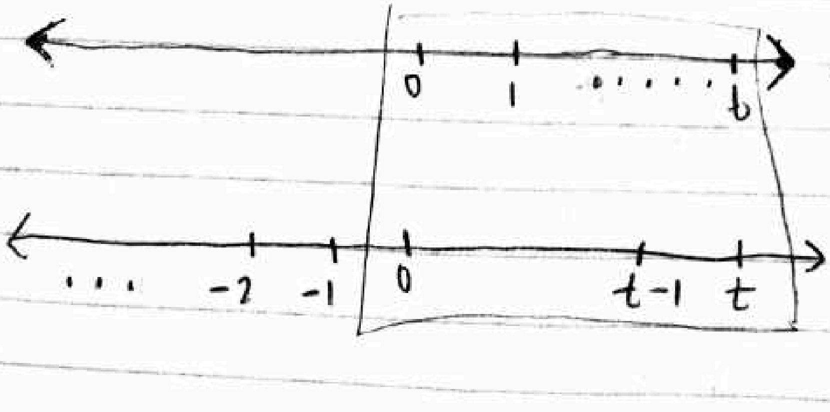
$$p_T(t) = \sum_{x \in \text{Supp}[X]} \overset{\text{old}}{p}(x) \, \overset{\text{old}}{p}(t-x) \, \mathbb{1}_{t-x \in \text{Supp}[X]}$$

$$= \sum_{x \in \{0,1,\dots\}} (1-p)^x (1-p)^{t-x} p \, \mathbb{1}_{\underbrace{x \in \{0,1,\dots\}}}$$

$x - t \in \{0,-1,-2,\dots\}$

$x \in \{t, t-1, t-2, \dots\}$

$$\boxed{= (1-p)^t p^2 \sum_{x \in \{0,1,\dots\}} \mathbb{1}_{x \in \{t,t-1,\dots\}}} = (1-p)^t p \sum_{x \in \{0,\dots,t\}} 1$$



$= (t+1)(1-p)^t p^2$

$= \text{NegBin}(2,p)$

Negative Binomial rv

$$\overbrace{0\ 0 \ldots 0}^{(1-p)^t}\ \overset{p}{1}\ \overbrace{0\ 0 \ldots 0}^{(1-p)^t}\ \overset{p}{\underset{t+2}{1}}$$

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$t+1$ realizations

geometric
$(1-p)^x p$

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p) \quad T_3 = X_1 + X_2 + X_3$

$$P_{T_3}(t) = \sum_{x \in supp[X_3]} p^{old}_{X_3}(x)\, P^{old}_{T_2}(t-x)\, \mathbb{1}_{t-x \in supp[T_2]} \qquad = X_3 + T_2 \sim P_{T_3}(t) = ?$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}\underbrace{_{t-x \in \{0,1,\ldots\}}}_{x \in \{t, t-1, \ldots, t\}}$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\ldots,t\}} (1) - \sum_{x \in \{0,\ldots,t\}} x \right)$$

$$= (1-p)^t p^3 \underbrace{\left((t+1)^2 - \frac{t(t+1)}{2} \right)}_{t^2+2t+1-\frac{t^2+t}{2}}$$

$$= \frac{t^2+3t+2}{2}$$

$$= \frac{(t+2)(t+1)}{2}$$

$$= \frac{(t+2)!}{t!\,2!}$$

$$= \binom{t+2}{2}$$

$$= \binom{t+2}{2} (1-p)^t p^3 = NegBin(3, p)$$

$$\underbrace{\underline{0}\ \underline{0} \ldots \underline{0}\ \underline{1}\ \underline{0}\ \underline{0} \ldots \underline{0}\ \underline{1}\ \underline{0}\ \underline{0} \ldots \underline{0}}_{t+2 \text{ realizations}}\ \underset{t+3}{\underline{1}}$$

pick 2 positions for the first two 1's

$X_1, \ldots, X_r \overset{iid}{\sim} Geom(p)$, $T_r = X_1 + X_2 + \ldots + X_r \sim NegBin(r, p)$
$:= \binom{t+r-1}{r-1}(1-p)^t p^r$

$$\underbrace{\overbrace{\_0\ \ \_0\ \underbrace{1001\ \_0\ 00.1\ \_0}_{r-1\ 1's}}^{t\ 0's}\ 1}_{t+r-1}\ \ \frac{1}{t+r}$$

$X \sim Bin(n, p) := \binom{n}{x} p^x (1-p)^{n-x} \ \forall x \in \{0, 1, \ldots, n\}$

let $n \to \infty$, $p \to 0$ but $\lambda := np \Rightarrow p = \frac{\lambda}{n}$ let $n \to \infty$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \ \forall x \in \{0, 1, \ldots, n\}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} \ \forall x \in \{0, 1, \ldots, n\}$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{n!}{(n-x)!\, n^x} \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \cdot \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} \cdot \lim_{n\to\infty} \ \forall x \in \{0, \ldots$$

$$= \frac{\lambda^x}{n!} \lim_{n\to\infty} \frac{\overbrace{(n)(n-1)\ldots(n-x+1)}^{x\ terms}}{\underbrace{n\ n \ldots\ n}_{x\ terms}}\ e^{-\lambda}\ (1) \ \forall x \in \{0, 1, \ldots, n\}$$

$$= \frac{\lambda^x}{n!} e^{-\lambda} \ \forall x \in \{0, 1, \ldots\} = \text{Poisson}(\lambda)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ $T = X_1 + X_2 \sim p_T(t) = ?$

$$p_T(t) = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \ \forall t-x \in \{0, 1, \ldots\}$$

$$= \lambda^t e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{1}{x!(t-x)!} \ \forall x \in \{\ldots, t-1, t\}$$

$$= \frac{\lambda^t e^{-x}}{t!} \underbrace{\sum_{x \in \{0,\dots,t\}} \binom{t}{x}}_{2^t} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$