Wednesday September 2nd 2020

Lecture 3

Let $B_1, B_2 \dots \overset{iid}{\sim} Bern(p)$
Possible an infinite sequence of iid rv's

Let $X :=$ number of zero realization before the first realization of one. Also $X := \min\{t : B_t = 1\} - 1$

$$P(0) = P(X=0) = P(\{1\}) = p \qquad Supp[X] = \{0,1,2,\dots\}$$
$$P(1) = P(X=1) = P(\{0,1\}) = (1-p)\,p$$
$$\vdots$$
$$p(x) = P(X=x) = P(\{\underbrace{0,0,\dots,0}_{x},1\}) = (1-p)^x\,p$$

$X \sim Geom(p) := (1-p)^x\, p\, \mathbb{1}_{x\in\{0,1,2,\dots,n\}}$ "Geometric r.v"

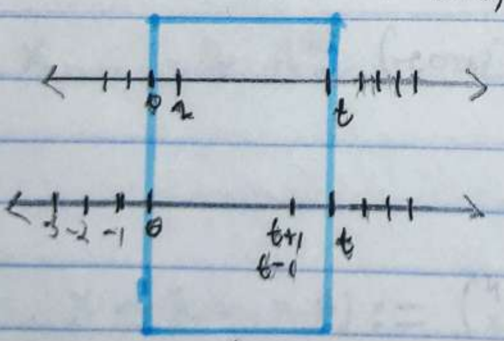
$X_1, X_2 \overset{iid}{\sim} Geom(p)$ $T_2 = X_1 + X_2 \sim P_{T_2}(t) = ?$

$$P_T(t) = \sum_{x\in Supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x\, Supp[X]} = \sum_{x\in\{0,1,\dots\}} (1-p)^x p (1-p)^{t-x}\, \mathbb{1}_{t-x\in\{0,1,\dots\}}$$

$x-t \in \{0,-1,-2,\dots\}$
$x \in \{t, t-1, t-2, \dots\}$

$$= (1-p)^t p^2 \sum_{x\in\{0,1,\dots\}} \mathbb{1}_{x\in\{t,t-1,\dots\}} = (1-p)^t p^2 \sum_{x\in\{0,\dots,t\}} 1$$

$\{0,1,\dots\} \cap \{\dots,t-1,t\} = \{0,1,\dots,t\}$



intersection of two set

$$= (t+1)(1-p)^t p^2$$
$$= Neg\,Bin(2,p)$$
Negative Binomial r.v

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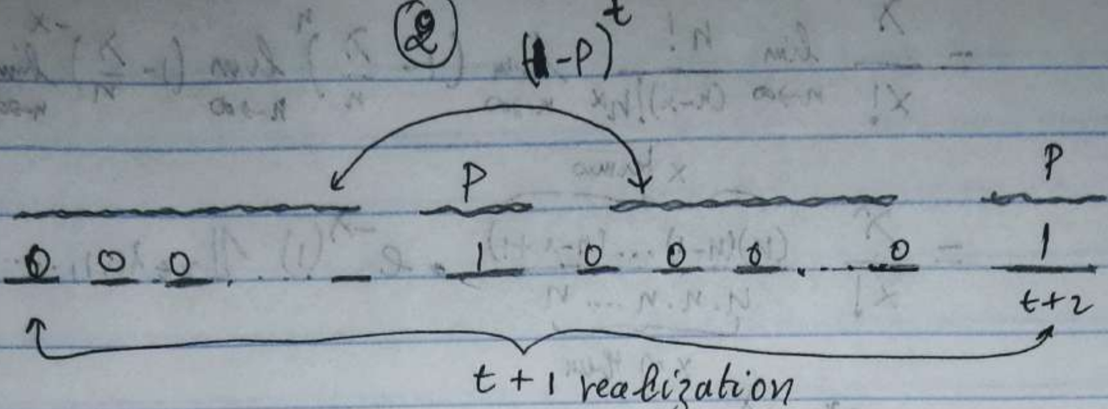

$Supp[T_2] = \{0,1,\dots\}$

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p) \qquad T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$

$$P_{T_3}(t) = \sum_{x\in Supp[X_3]} P^{old}_{X_3}(x)\, P^{old}_{T_2}(t-x)\, \mathbb{1}_{t-x \in Supp[T_2]} = \sum_{x\in\{0,1,\dots\}} (1-p)^x p (t-x+1)(1-p)^{t-x} p^2 \,\mathbb{1}_{x\in\{0,1,2,\dots\}}$$

$x \in \{\dots, t-1, t\}$

$$= (1-p)^t p^3 \left((t+1)\sum_{x\in\{0,1,\dots t\}}(1) - \sum_{x\in\{0,\dots,t\}} x\right) = (1-p)^t p^3\left((t+1)^2 - \frac{t(t+1)}{2}\right)$$

$$\frac{t^2+3t+2}{2} = \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$\boxed{\sum_{x\in S} a+bx = \sum_{x\in S} a + \sum_{x\in S} bx = a\sum_{x\in S} 1 + b\sum_{x\in S} x}$$

So $P_{T_3}(t) = \binom{t+2}{2}(1-p)^t p^3$

$= Neg\,Bin(3, p)$

$X_1, \dots, X_r \overset{iid}{\sim} Geom(p)$, $T_r = X_1 + X_2 + \dots X_r \sim Neg\,Bin(r,p) := \binom{t+r-1}{r-1}(1-p)^t p^r$

$X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x\in\{0,1,\dots,n\}}$

Let $n\to\infty$, $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ let $n\to\infty$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x\in\{0,1,\dots,n\}} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x\in\{1,\dots n\}}$$

③

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x} \lim_{n\to\infty} \mathbb{1}_{x\in\{0,1,\dots,n\}}$$

$$= \frac{\lambda^x}{x!} \frac{\overbrace{(n)(n-1)\dots(n-x+1)}^{x \text{ terms}}}{\underbrace{n.n.n\dots n}_{x \text{ of them}}} \cdot e^{-\lambda}(1)\, \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \text{Poisson}(\lambda)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \quad T = X_1 + X_2 \sim P_T(t) = ?$

$$P_T(t) = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x\in\{0,1,\dots\}}$$

$$= \lambda^t e^{-2\lambda} \boxed{\left(\frac{t!}{t!}\right)} \sum_{x\in\{0,1,\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x\in\{0,\dots,t-1,t\}}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x\in\{0,\dots,t\}} \binom{t}{x} = \frac{(2\lambda)^t e^{-2\lambda}}{t!}$$

$$= \text{Poisson}(2\lambda)$$