Consider $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

Let $X := \#$ zeros before the first one occurs.

$$p(0) = P(X=0) = P(\{1\}) = p$$

$$p(1) = P(X=1) = P(\{0,1\}) = (1-p)p$$

$$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$$

$$\vdots$$

$$p(x) = P(X=x) = P(\{0,0,\ldots,1\}) = (1-p)^x p$$

Then we know:

$$X \sim \text{Geom}(p) := (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\ldots\}}$$

Let:

$$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p) \quad T_2 = X_1 + X_2 \sim P_{T_2}(t) = ?$$

$$P_{T_2}(t) = \sum_{x \in Supp[X]} p^{old}(x) \, p^{old}(t-x) \, \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum (1-p)^x p (1-p)^{t-x} p \, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= (1-p)^t p^2 \sum_{x \in \{0,\ldots\}} \mathbb{1}_{x \in \{\ldots, t-1, t\}} = (1-p)^t p^2 \sum_{x \in \{0,1,\ldots,t\}} 1$$

$$= \underbrace{(t+1)}_{} (1-p)^t p^2 = \text{NegBin}(2, p), \quad Supp[T_2] = \{0,1,\ldots\}$$

$t+1$ possible locations for first 1

$$\underbrace{0 \; 0 \; \ldots \; 0 \; 1 \; 0 \; 0 \; \ldots \; 0}_{t+1 \text{ realizations}} \; 1 \quad t+2$$

Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$, $T_3 = X_1 + X_2 + X_3 \sim P_{T_3}(t) = X_3 + T_2$

$$P_{T_3}(t) = \sum P^{old}_{X_3}(x) \, P_{T_2}(t-x) \, \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum (1-p)^x p \, (t-x-1)(1-p)^{t-x} p^2 \, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= (1-p)^t p^3 \sum_{x \in \{0,1,\ldots\}} (t+1-x) \, \mathbb{1}_{x \in \{\ldots, t-1, t\}}$$

$$= (1-p)^t p^3 \sum_{x \in \{0,\ldots,t\}} (t+1) + (-x)$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\ldots,t\}} 1 - \sum_{x \in \{0,\ldots,t\}} x \right)$$

$$= (1-p)^t\, t^3 \left((t+1)(t+1) - \frac{t(t-1)}{2} \right)$$

$$= \binom{t+2}{2} (1-p)^t p^3 = \text{Neg Bin}(3, p)$$

$$\underbrace{0\,0 \dots 0\,1\;0\,0 \dots 0\,1 \dots 0}_{t+2 \text{ realization}}\,1 \quad t+3$$

$t+2$ locations to put 2 ones in

To Generalize let $X_1, X_2, \dots X_r \overset{iid}{\sim} \text{Geom}(p)$

$T_r := X_1 + X_2 + \dots + X_r \sim \text{Neg Bin}(r, p) = P_{T_r}(t)$

$$P_{T_r}(t) = \binom{t+r-1}{r-1} (1-p)^t p^r \, \mathbb{1}_{t \in \{0,1,\dots\}}$$

Now Let $X \sim \text{Bin}(n,p)$

Let n be really large and p be really small,

$n \to +\infty$, $p \to 0$ but $\lambda = np$

Our goal is to get the pmf of X under this limit.

if $\lambda = np$ $\quad p = \frac{\lambda}{n}$ and $X \sim \text{Bin}(n,p)$

$$\lim \binom{n}{x} p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,\dots,n\}} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x \in \{0,\dots,n\}}$$

$$= \lim \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x \in \{0,\dots,n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!\, n^x} \lim \left(1 - \frac{\lambda}{n}\right)^n \lim \left(1 - \frac{\lambda}{n}\right)^{-x} \lim \mathbb{1}_{x \in \{0,\dots,n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{n(n-1)\dots(n-x+1)}{n \cdot n \cdot n \dots n}\, e^{-\lambda}\, \mathbb{1}_{x \in \{0,1,\dots\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,\dots\}} \quad = \text{Poisson}(\lambda) \text{ where } \lambda \in (0, \infty)$$

Therefore: $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ $\quad T = X_1 + X_2 \sim P_T(t) = ?$

$$P_T(t) = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$= e^{-2\lambda} \lambda^t \sum_{x \in \{0,\dots,t\}} \frac{1}{x!\,(t-x)!} = \lambda^t e^{-2\lambda} \sum_{x \in \{0,\dots,t\}} \frac{t!}{t!} \frac{1}{x!\,(t-x)!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0, \dots t\}} \binom{t}{x} = \frac{\lambda^t e^{-2\lambda}}{t!} 2^t =$$

$$= \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$