Lecture 07 MATH 621 09/29/2020

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \quad D = X_1 + (-X_2) \sim ?$

$$P_D(d) = \sum_{x \in Supp[X]} p^{old}_{x}(x)\, p^{old}_{*}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{(\underbrace{-(d-x)}_{x-d})!} \mathbb{1}_{\overbrace{d-x \in \{\ldots,-1,0\}}^{x-d \in \{0,1,\ldots\}}} \Rightarrow x \in \{d, d+1, \ldots\}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x}{x!} \frac{\lambda^{x-d}}{(x-d)!} \mathbb{1}_{x \in \{d,d+1,\ldots\}} = e^{-2\lambda} \begin{cases} d>0 & \sum_{x \in \{d,d+1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \\ d \le 0 & \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \end{cases}$$

let $x' = x + d$ then $x = x' + d$

$$= e^{-2\lambda} \begin{cases} d>0 & \sum \frac{\lambda^{2x'+d}}{(x'+d)!\,x'!} \\ d \le 0 & \sum \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} \end{cases} \Longrightarrow I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d \in \mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \Rightarrow T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

What is $P_{X_1|T}(x,t)$

$$P_{X_1|T}(x,t) = \frac{P_{X,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x, t-x)}{P_T(t)} = \frac{P_{X_1}(x)\, P_{X_2}(t-x)}{P_T(t)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \frac{t!}{x!(t-x)!}\left(\frac{1}{2}\right)^t = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}(t, \tfrac{1}{2})$$

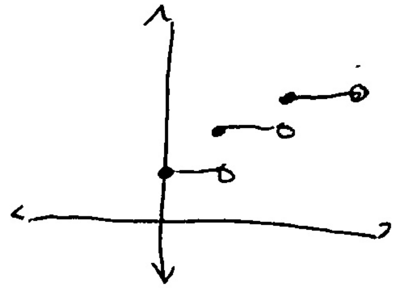
$B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$ $\quad X_1 \sim Geom(p): (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\ldots\}}$

$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = 1-(1-p)^{x+1}$

Let there be n experiments in each second (time unit)

$P_{X_n}(x) = (1-p)^{nx} p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots\}}$

$F_{X_n}(x) = 1-(1-p)^{nx+1}$



Lets put infinite experiments into every second
this is the limit as n goes to ∞ and $p \to 0$ and $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$P_{X_\infty}(x) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}} = \underbrace{\lim_{n\to\infty}\left(\left(1-\frac{\lambda}{n}\right)^{n}\right)}_{e^{-\lambda}}^{x} \lim \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$

$= e^{-\lambda x}(0)\mathbb{1}_{x \in [0,\infty)} = 0 \; \forall x$ Not a pmf.

$F_{X_\infty}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx+1} = 1 - \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{e^{-\lambda x}} \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{1}}_{1} = (1-e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)}$

Is this limiting CDF a legal CDF?
Does it?

1) limit as x goes to neg infnty is zero
2) limit as x goes to pos infnty is one
3) increasing fun derivative is >=0

1) $\lim (1-e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)} = 0$

2) $\lim (1-e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)} = 1 - \lim_{x\to\infty} \frac{1}{e^{\lambda x}} = 1$

3) $\frac{d}{dx}\left[(1-e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)}\right] = \lambda e^{-\lambda x}\mathbb{1}_{x \in [0,\infty)}$

$\Rightarrow F_{X_\infty}$ is a valid CDF.. a continous r.v

A continous r.v X has $Supp[X] \subseteq \mathbb{R}$ but $|Supp[X]| = |\mathbb{R}|$

this is known as "Uncountable infinity". They also have no PMF, the $P(X=x)$ is always zero. But have a CDF ~~(cont~~.

Derivative of CDF is the pdf. (density)

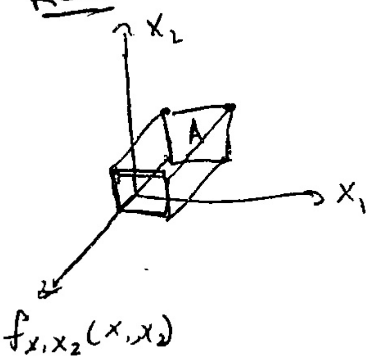
$$f(x) := F'(x), \quad P(X \in (a,b)) = P(X \leq b) - P(X \leq a) = \int_a^b f(x)\,dx$$

$\hookrightarrow f(x) \geq 0 \ \forall x \iff Supp[X] = \{x : f(x) > 0\}$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \overset{1}{F(\infty)} - \overset{0}{F(-\infty)} = 1$$

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim \overset{(jdf)}{f_{\vec{X}}(\vec{x})} \underset{\text{indep.}}{=} f_{X_1}(x_1) \cdots f_{X_k}(x_k) \underset{\text{Inden distr.}}{=} f(x_1) \cdots f(x_k)$$

<u>K=2</u>



$$P(\vec{X} \in A) = \iint_A f_{\vec{X}}(\vec{x})\,dx_1\,dx_2$$

$$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}$$