$\vec{x} \sim \text{multin}_K(n, \vec{p})$ of dim K

Marin Azhar 4/14

$K = 2 \quad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

$$P_{x_1|x_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \text{Deg}(n - x_2)$$

$P(x_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$ $\quad J_n = \{0, \dots, n\}$

$$\frac{\binom{n}{x_2, x_1} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} \underbrace{(1-p_2)}_{p_1}^{n-x_2}} = \frac{\frac{n!}{x_1!\,x_2!} \mathbb{1}_{\overbrace{x_1+x_2=n}^{x_1 = n - x_2}} \mathbb{1}_{x_1 \in J_n} \cancel{\mathbb{1}_{x_2 \in J_n}} \cancel{p_1^{x_1} p_2^{x_2}}}{\frac{n!}{x_2!(n-x_2)!} \cancel{\mathbb{1}_{x_2 \in J_n}} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

$\mathbb{1}^u_{x_2 \in J_n}$ Define a ratio of indicators:

$$\mathbb{1}^u_A := \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \text{undef} & \text{if } A^c \end{cases}$$

$$\text{Deg}(n-x_2)\, \mathbb{1}^u_{x_2 \in J_n} = \underbrace{\frac{(n-x_2)!}{x!}}_{\substack{=1 \text{ when} \\ x_1 = n - x_2}} \mathbb{1}_{x_1 = n - x_2} \underbrace{\mathbb{1}_{x \in J_n}}_{\substack{=1 \text{ when} \\ x_1 = n - x_2 \\ \text{\& } x_2 \in J_n}} \mathbb{1}^u_{x_2 \in J_n}$$

hint $P(A|B) = \frac{P(A,B)}{P(B)}$ which is undef if prob b is zero

Call it undef if x_2 is ilig

let's generalize this result a bit $\vec{x} \sim \text{multin}_K(n, \vec{p})$ $= \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$ K dimention vector

$$P_{\vec{x}_{-j}|x_j}(\vec{x}_{j}, \vec{x}_j) = \frac{P_{\vec{x}}(\vec{x})}{P_{x_j}(x_j)} \sim ? \sim \text{Multin}_{K-1}(n - x_j, ?)$$

All elements of vector rv X except the j^{th} component

$$= \frac{\text{Multin}_K(n, \vec{p})}{\text{Bin}(n, p_i)} = \frac{\binom{n}{x_1, x_2 \cdots x_K} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cancel{p_j^{x_j}} \cdot p_{j+1}^{x_{j+1}} \cdots p_K^{x_K}}{\binom{n}{x_j} \cancel{p_j^{x_j}} (1-p_j)^{n-p_j}}$$

$$= \frac{\cancel{n!}}{x_1! \cdots \cancel{x_j!} \cdots x_k!} \mathbb{1}_{\underbrace{x_1 + \cdots + x_j + \cdots + x_k = n}_{= n - x_j}} \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_j \in J_n} \cdots \mathbb{1}_{x_k \in J_n} \frac{(\cdots)}{\ }$$

$$(p_1^{x} \cdot p_{j+1}^{x_{j+1}} \cdot p_k^{x_k})$$

$$\frac{\cancel{n!}}{\cancel{x_j!}\,(n-x_j)!} \mathbb{1}_{x_j \in J_n} (1-p_j)^{n-x_j}$$

$$\mathbb{1}^{u}_{x_j \in J_n}$$

let $n' = n - x_j$ Note $x_1 + \cdots + x_k = n \Rightarrow n - x_j = x_1 + \cdots + x_{j-1} + x_{j+1} \cdots + x_k$

Note $p_1 + \cdots + p_k = 1$

$\Rightarrow p_1 + \cdots + p_{j+1} + \cdots + p_k = 1 - p_j$ its a trick

Divide both sides by $1 - p_j$ $\Rightarrow \frac{p_i}{1-p_j} + \cdots + \frac{p_{j-1}}{1-p_j} + \frac{p_{j+1}}{1-p_j} + \cdots + \frac{p_k}{1-p_j} = 1$

$$p_1' + \cdots + p_{j-1}' + p_{j+1}' + \cdots + p_k' = 1$$

$$= \frac{n'!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_k!} \mathbb{1}_{x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k = n'} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_{j-1} \in J_n} \cdots \mathbb{1}_{x_k \in J_n} (\cdots)$$

$$\left(p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{k} \quad \mathbb{1}^{u}_{x_j \in J_n} \right)$$

$$\underbrace{(1-p_j)^{x_1 + \cdots + x_{j-1} + x_{j+1} \cdots x_k}}_{p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}} p_{j-1}'^{x_j+1} p_k^{x_k}} = \text{multin}_{k-1}(n', \vec{p'}) \mathbb{1}^{n}_{x_j \in J_n}$$

$$\begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix}$$

$\vec{X} \sim \text{multin}(n, \vec{p})$ what is $E[\vec{x}]$? $Var[\vec{X}]$?

x is a scalar r.v $a, c \in \mathbb{R}$

rule * $E[aX + c] = aE[X] + C$ If identically distributed iid

* $E\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} E[x_i] = n \cdot \mu$

* $E\left[\prod_{i=1}^{n} x_i\right] = \prod_{i=1}^{n} E[x_i]$ If they are independent

$\sum_{x \in \mathbb{R}} (x-\mu)^2 p(x)$ discrete

$$\sigma^2 := Var[X] = E[(X-\mu)^2]$$

$\int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx$ continous

$\sigma = \sqrt{Var[X]} = SD[X] \to$ standard d.viation

$$Var[X_1 + X_2] = E\left[\left((X_1+X_2) - (\mu_1 - \mu_2)\right)^2\right]$$

now we foil

$$= E\left[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2\mu_1 X_1 - 2\mu_1 X_2 - 2\mu_2 X_1 - 2\mu_2 X_2 + 2X_1X_2 + 2\mu_1\mu_2\right]$$

$$= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_2\mu_1 - 2\mu_2^2 + 2E[X_1X_2]$$

$$+ 2\mu_1\mu_2$$

$\sigma_{12} = \sigma_{21}$ it's the same

remember $Var[X] = E[X^2] - \mu^2$

σ_{12}

$$Cov[X_1, X_2] := E[X_1X_2] - \mu_1\mu_2$$

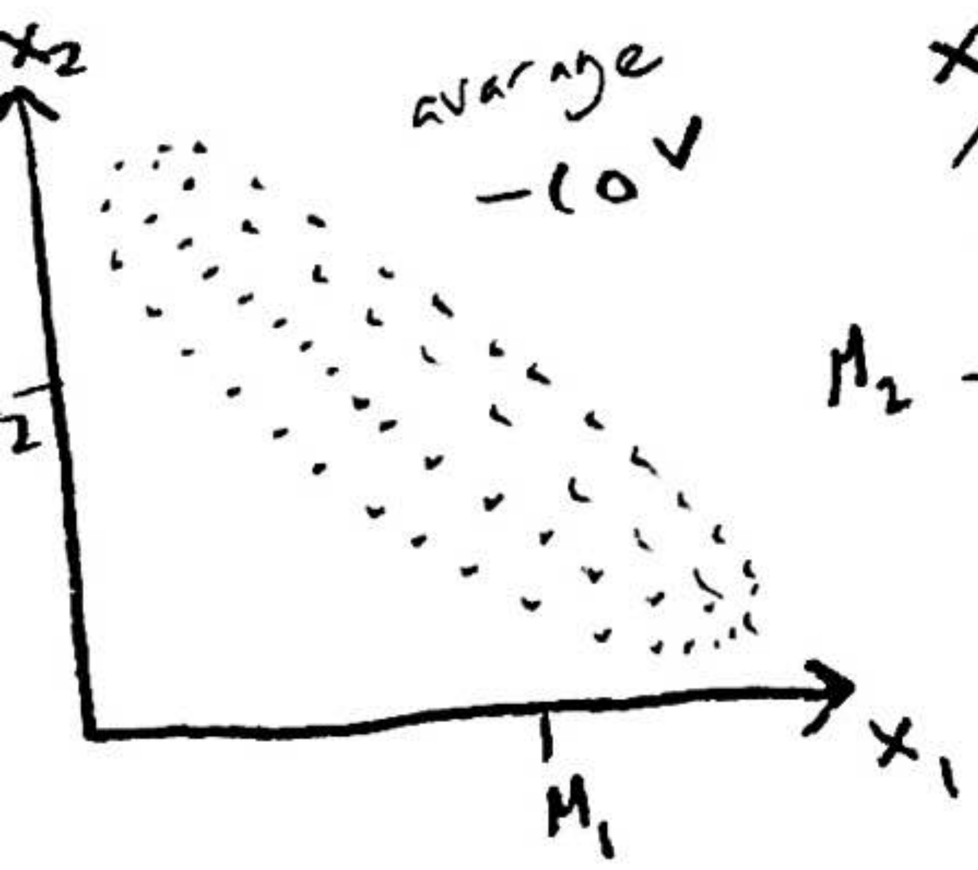
$$= \sigma_1^2 + \sigma_2^2 + 2\left(E[X_1X_2] - \mu_1\mu_2\right)$$

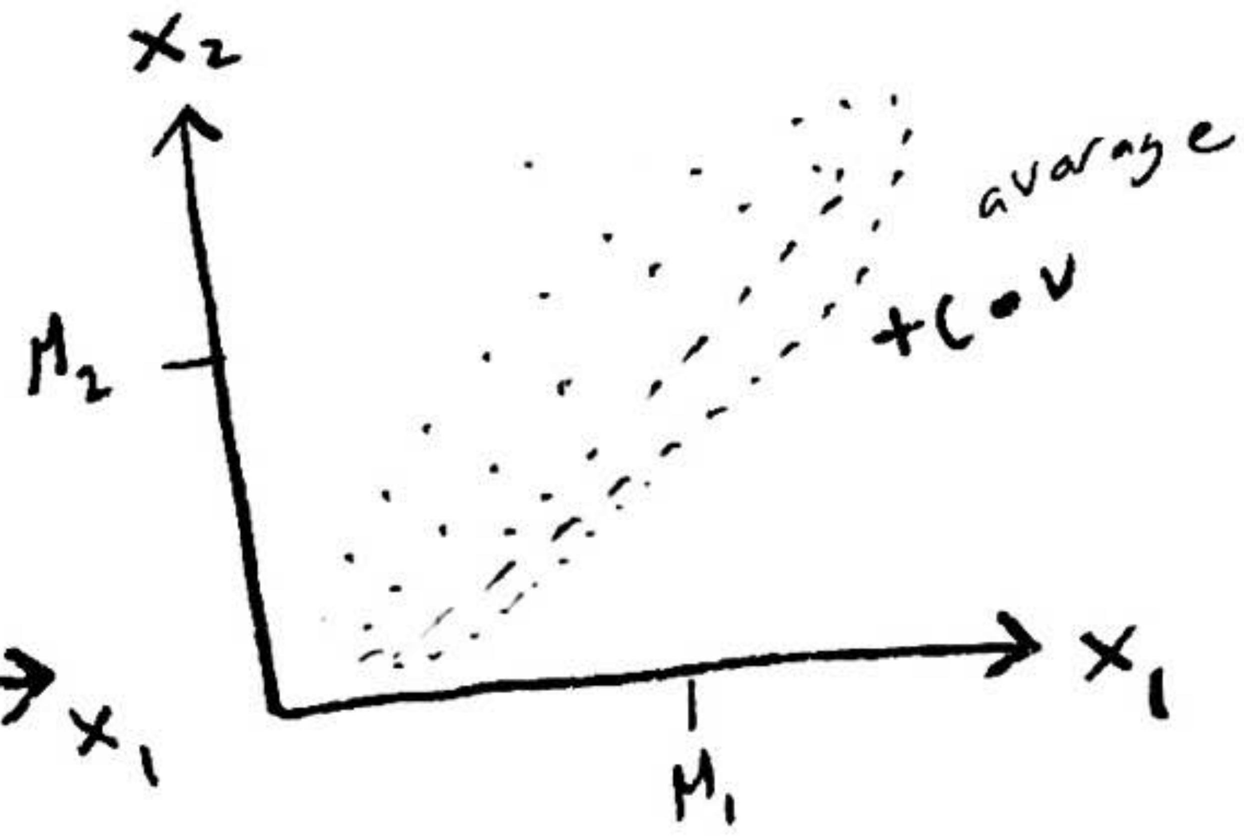
If X_1, X_2 independent

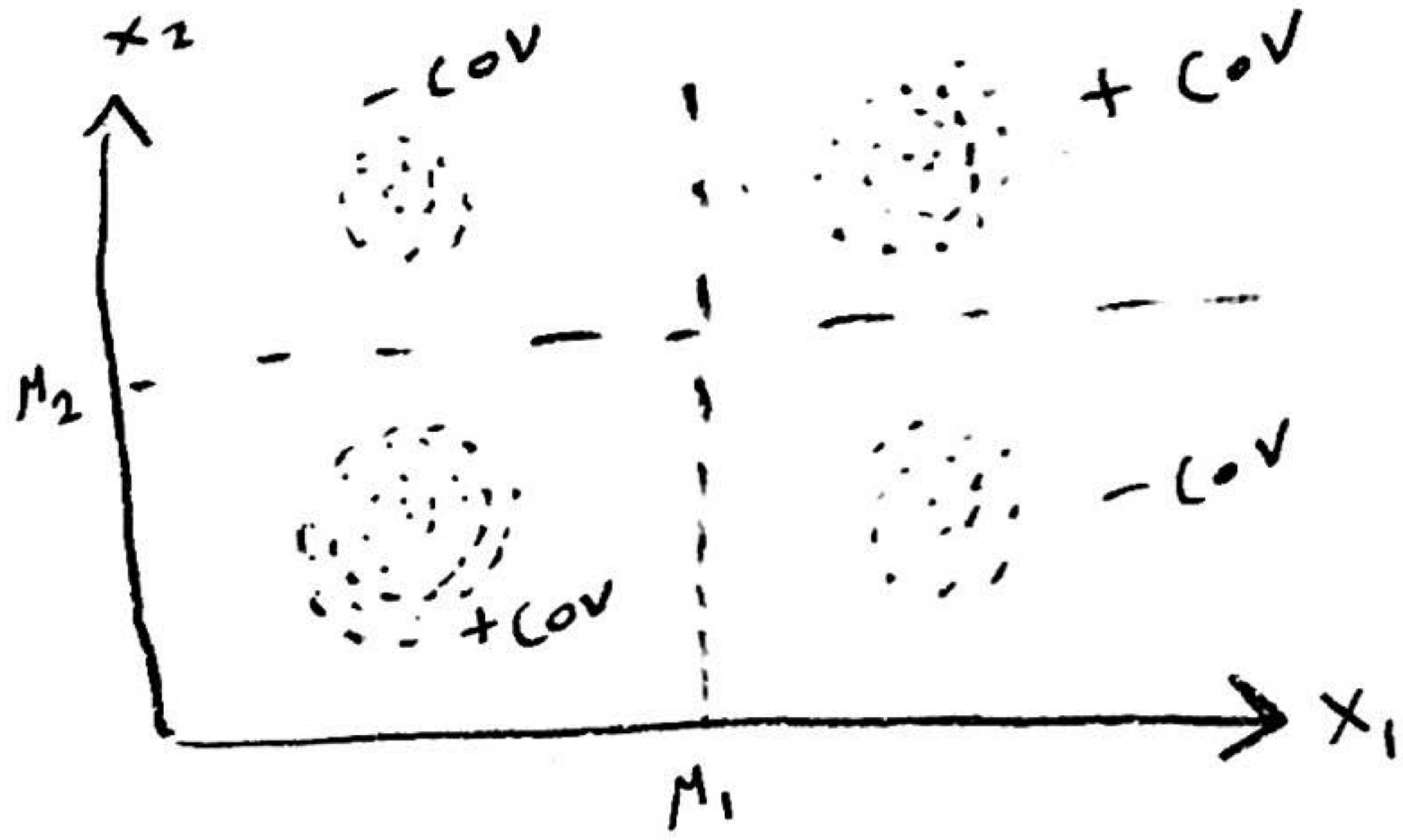
$$\Rightarrow Cov[X_1, X_2] = \mu_1\mu_2 - \mu_1\mu_2 = 0$$

$$= \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

$= \sigma_1^2 + \sigma_2^2 \rightarrow$ if $X_1 X_2$ are independent







rules for covariances $a_1, a_2 \in \mathbb{R}$

1. $Cov[X, X] = \sigma^2$
2. $Cov[X_1, X_2] = Cov[X_2, X_1]$
3. $Cov[X_1 + X_2, X_3] = \sigma_{13} + \sigma_{23}$
4. $Cov[a_1X_1, a_2X_3] = a_1a_2\sigma_{12}$
5. $Var[X_1 + \cdots + X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j]$

$\vec{\mu} := E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}$

expectation of the vector

M is a matrix of rv's

let $M := \begin{bmatrix} X_{11}, & \ldots, & X_{1m} \\ X_{21}, & \ldots, & X_{2m} \\ \vdots & & \\ X_{n1} & \ldots & X_{nm} \end{bmatrix}$, $E[M] = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1m}] \\ \vdots & & \\ \vdots & & \\ E[X_{n1}] & \cdots & E[X_{nm}] \end{bmatrix}$

$$\Sigma = Var[\vec{X}] := E[\underbrace{\vec{X}\vec{X}^T}_{\text{outer product}}] - \underbrace{\vec{\mu}\vec{\mu}^T}_{(K\times 1)(1\times K) = K\times K} \overset{H.W}{=} \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_K] \\ Cov[X_2, X_1] & Var[X_2] & & Cov[X_2, X_K] \\ & & \ddots & \vdots \\ Cov[X_K, X_1] & Cov[X_K, X_2] & \cdots & Var[X_K] \end{bmatrix}$$

Σ → Capital sigma

The "Variance - Covariance matrix" It's square $K\times K$ and Symmetric

If $X_1, \ldots, X_K$ are independent then the Var cov matrix is:

$$Var[\vec{X}] = diag\{\sigma_1^2, \ldots, \sigma_K^2\} = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_K^2 \end{bmatrix}$$

Rules for expectation of vector r.v's

let $\vec{a} \in \mathbb{R}^K$

① $E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a}$

② $E[\vec{a}^T \vec{X}] = E[a_1 X_1 + a_2 X_2 + \cdots + a_K X_K] = a_1\mu_1 + \cdots + a_K \mu_K = \vec{a}^T \vec{\mu}$