Lecture 03
09/02/2020
Math 621

Consider $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$
possibly infinite sequence of iid Bernoullis

Let X = # of zeroes before the first one occurs $= \min\{t : b_t = 1\} - 1$

$p(0) = P(X=0) = P(\{1\}) = p$
$p(1) = P(X=1) = P(\{0,1\}) = (1-p)p$
$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$

$p(x) = P(X=x) = P(\{0,0,\ldots,0,1\}) = (1-p)^x p$

$\text{Supp}[X] = \{0,1,2,\ldots\}$
$X \sim \text{Geom}(p) = (1-p)^x p \, \mathbb{1}_{x \in \{0,1,\ldots\}}$
(geometric r.v)

$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p) \qquad T_2 = X_1 + X_2 \sim p_{T_2}(t) = ?$ which formula

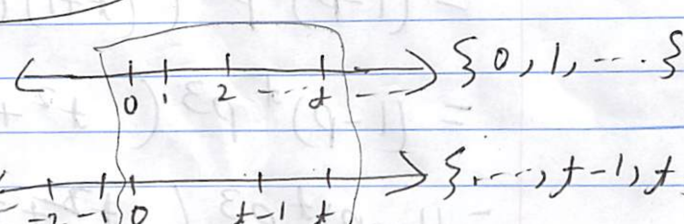
$$p_{T_2}(t) = \sum_{x \in \text{Supp}[X]} p^{odd}(x)\, p^{odd}(t-x)\, \mathbb{1}_{t-x \in \text{Supp}[X]}$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p \, (1-p)^{t-x} p \, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$x - t \in \{\ldots,-1,0\}$
$x \in \{\ldots, t-1, t\}$

$$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{\ldots,t-1,t\}}$$

$$= (1-p)^t p^2 \sum_{x \in \{0,\ldots,t\}} 1$$



$= (t+1)(1-p)^t p^2$
$= \underline{\text{Neg Bin}(2,p)}$, Negative binomial r.v

$$\overset{``\binom{t+1}{1}"}{(t+1)}(1+p)^t p^2$$

$$\underbrace{\underline{0}\ \underline{0}\ \dots\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{0}\ \dots\ \underline{0}}_{t+1 \text{ realization}}\ \underline{1}$$

$\to$ $t+1$ possible locations for the first 1

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$, $T_3 = X_1 + X_2 + X_3 \sim p_{T_3}(t) = ?$

$$= X_3 + T_2$$

$$p_{T_3}(t) = \sum_{x \in Supp[X_3]} p_{X_3}^{odd}(x) p_{T_2}^{odd}(t-x) \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum_{x \in \{0,1,\dots\}} (1-p)^x p\,(t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \{0,1,\dots\}}$$

$$= (1-p)^t p^3 \sum_{x \in \{0,1,\dots\}} (t+1-x) \mathbb{1}_{x \in \{\cancel{0},\dots, t-1, t\}}$$

$$\boxed{\begin{aligned} &\text{Precalculus Review:} \\ &\sum_{x \in S} a + bx = \sum_{x \in S} a + \sum_{x \in S} bx \\ &\qquad = a\sum_{x \in S} 1 + b\sum_{x \in S} x \end{aligned}}$$

$$\to = (1-p)^t p^3 \sum_{x \in \{0,1,\dots t\}} (t+1) + (\cancel{-}x)$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\dots t\}} 1 - \sum_{x \in \{0,1,\dots t\}} x \right)$$

$$= (1-p)^t p^3 \left((t+1)(t+1) - \frac{t(t+1)}{2} \right)$$

$$= (1-p)^t p^3 \left(t^2 + 2t + 1 - \frac{t^2}{2} - \frac{t}{2} \right)$$

$$= (1-p)^t p^3 \left(\frac{\cancel{t^2} + 3t + 2}{2} \right)$$

$$= (1-p)^t p^3 \left[\frac{(t+2)(t+1)}{2} \right]$$

$$= (1-p)^t p^3 \left(\frac{(t+2)!}{t!\,2!} \right) = (1-p)^t p^3 \binom{t+2}{2}$$

$$= \binom{t+2}{2} (1-p)^t p^3 = \text{NegBin}(3, p)$$

$$\underbrace{0\,0 \cdots 0\;1\;0\;0 \cdots 0\;1\;0\;0 \cdots 0}_{(t+2)\text{ realization}}\;\underset{t+3}{1}$$

$t+2$ locations to put 2 ones in
$\Rightarrow$ $t+2$ choose 2

In General,

$X_1, X_2, \cdots, X_r \overset{iid}{\sim} \text{Geom}(p)$,

$$T_r = X_1 + X_2 + \cdots + X_3 \sim \text{NegBin}(r, p) = P_{T_r}(t)$$

$$P_{T_r}(t) = \binom{t+r-1}{r-1} (1-p)^t p^r \, \mathbb{1}_{t \in \{0, 1, \cdots\}}$$

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$X \sim \text{Bin}(n, p)$ Let n be really large & p be really small, $n \to +\infty$, $p \to 0$, but lambda $= np$.
(λ)

Our goal is to get the PMF of X under this limit.

$$\lambda = np \Rightarrow p = \frac{\lambda}{n}$$

$$\lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} \, \mathbb{1}_{x \in \{0, \cdots, n\}}$$

$$= \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x \in \cdots}$$

$$= \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \mathbb{1}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}$$

$$\left.\begin{matrix} n=10 \\ x=4 \end{matrix}\right\} \frac{10!}{(10-4)!} = \frac{10!}{6!} = \overbrace{10\cdot 9\cdot 8\cdot 7}^{4 \text{ terms} \Rightarrow \text{ same as } x}$$

$\frac{\lambda}{n} \Rightarrow$ approaches to 0

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x \text{ terms}}}{\underbrace{n\cdot n\cdot \cdots \cdot n}_{x \text{ terms}}} e^{-\lambda} (1) \mathbb{1}_{x\in\{0,\dots,n\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\{0,\dots,n\}} \lim_{n\to\infty} \cancelto{1}{\frac{n}{n}} \lim_{n\to\infty} \cancelto{1}{\frac{n-1}{n}} \cdots \lim_{n\to\infty} \cancelto{1}{\frac{n-x+1}{n}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\{0,\dots,n\}}$$

$= \text{Poisson}(\lambda)$, $\lambda \in (0,\infty)$ Parameter Space

$$\lim \left(1+\frac{1}{n}\right)^n = e$$

$$\lim \left(1+\frac{a}{n}\right)^n = e^a \quad \text{or} \quad \lim \left(1+\frac{-a}{n}\right)^n = e^{-a}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim p_T(t) = ?$

$$p_T(t) = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \underbrace{\mathbb{1}_{t-x\in\{0,1,\dots\}}}_{x\in\{\dots,t-1,t\}}$$

$$= \lambda^t e^{-2\lambda} \sum_{x\in\{0,\dots,t\}} \frac{1}{x!\,(t-x)!}$$

$$= \lambda^t e^{-2\lambda} \sum_{x\in\{0,\dots,t\}} \frac{t!}{t!} \cdot \frac{1}{x!\,(t-x)!}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \sum_{x \in \{0, \dots t\}} \binom{t}{x}$$

$$\rightarrow \binom{t}{0} + \binom{t}{1} + \dots + \binom{t}{t} = 2^t$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t$$

$$= \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$