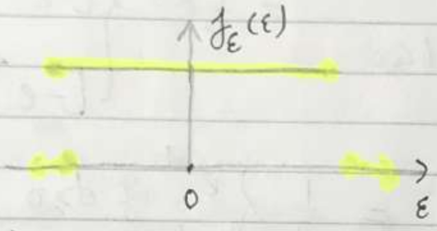
Monday October 19 2020

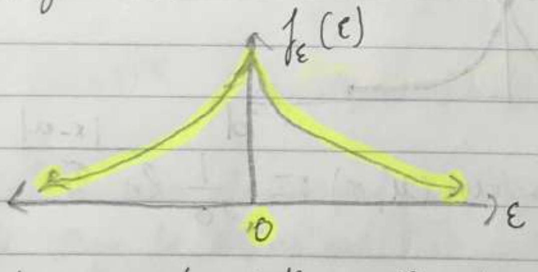
Lecture 12

"Laplace first published this in 1774 calling it the "first law of errors". His context was measurement. When you measure a quantity v, your measurement is: $M = v + \varepsilon$

What makes a good distribution for the error, epsilon? The expectation should be zero and should be symmetric. How about...



This is not very good. It should have the property that the probability of error should decrease with its magnitude. Also, why should it stop at some maximum magnitude?



Another good property is that the density should be decreasing in magnitude of error.

Laplace assumed for all positive errors that $f''_\varepsilon(\varepsilon) = f'_\varepsilon(\varepsilon)$

$$\Rightarrow f(\varepsilon) = C e^{-d\varepsilon} \Rightarrow \varepsilon \sim \text{Laplace}(0,1)$$

2

$X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x \geq 0}$, $Y = g(x) = \frac{1}{\lambda} x^{\frac{1}{k}}$ s.t. $k, \lambda > 0$. and $f_Y(y) = ?$

First step: get inverse function, second step: abs. inverse derivative

$$Y = \frac{1}{\lambda} X^{\frac{1}{k}} \Rightarrow \lambda y = x^{\frac{1}{k}} \Rightarrow x = (\lambda y)^k = \lambda^k y^k = g^{-1}(y)$$

Can be a test question!

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\lambda^k y^k]\right| = |k\lambda^k y^{k-1}| = k\lambda^k |y^{k-1}| = k\lambda^k y^{k-1}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right| \quad e^{-(\lambda y)^k} \underbrace{\mathbb{1}_{\lambda^k y^k \geq 0}}_{\substack{y^k \geq 0 \\ y \geq 0}} \cdot k\lambda^k y^{k-1}$$

$$= k \cdot \lambda^k y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = k\lambda (\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = \text{Weibull}(k, \lambda)$$

1951

This is a very famous waiting time / survival r.v model and it's used e.g in insurance companies to price life insurance (I think).

$$\text{Weibull}(1, \lambda) = (1)\,\lambda(\lambda y)^{(1)-1} e^{-(\lambda y)^{(1)}} \mathbb{1}_{y \geq 0} = \lambda e^{-\lambda y} \mathbb{1}_{y \geq 0} = \text{Exp}(\lambda)$$

The k parameter is really "cool". Here's a property of weibull r.v's under different values of k:

$c = 14, \; y = 3$

$k = 1$, $P(Y \geq y + c \mid Y \geq c) = P(Y \geq y)$ e.g $P(Y \geq 17 \mid Y \geq 14) = P(Y \geq 3)$

this equality is called "memorylessness"

$k > 1$, $P(Y \geq y + c \mid Y \geq c) < P(Y \geq y)$ e.g old lifespan of human, waiting for bus.

$k < 1$, $P(Y \geq y + c \mid Y \geq c) > P(Y \geq y)$ e.g: Startup company lifespan, natal lifespan.

Order Statistics (P160). Let $x_1; x_2, \ldots, x_n$ be a collection of Continuous rv's. Let the "order statistics" be the rv's:

$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ defined as:

$X_{(1)} := \min\{X_1, X_2, \ldots, X_n\}$

$\vdots$

$X_{(k)} := k^{th}$ largest of $X_1, \ldots, X_n$

$\vdots$

$X_{(n)} := \max\{X_1, X_2, \ldots, X_n\}$

$\vdots$

$R := X_{(n)} - X_{(1)}$ range

e.g.: $X_1 = 9,\ X_2 = 2,\ X_3 = 12,\ X_4 = 7$

$X_{(1)} = 2,\ X_{(2)} = 7,\ X_{(3)} = 9$

$X_{(4)} = 12$

$r = 12 - 2 = 10$

We want to find both the CDF and PDF of the kth order statistic. We will build up in stages. The first thing we'll do is find the CDF and PDF of the maximum.

$n = 5$

$$F_{X_{(n)}} = P(\underbrace{X_{(n)} \le x}_{\text{event}}) = P(X_1 \le x,\ X_2 \le x, \ldots, X_n \le x)$$

If $X_1, \ldots, X_n$ independent

$$= P(X_1 \le x) \cdot \ldots \cdot P(X_n \le x) = \prod_{i=1}^{n} F_{X_i}(x) \overset{\text{if iid}}{=} F_X(x)^n$$

assume iid

$$f_{X_n}(x) := \frac{d}{dx}\left[F(x)^n\right] = n f(x) F(x)^{n-1}$$

The next thing we'll do is to find the CDF and PDF of the minimum.

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

if independent

$$= 1 - P(X_1 > x) \cdot \ldots \cdot P(X_n > x) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(x)) \overset{\text{iid}}{=} 1 - (1 - F(x))^n$$

$$f_{X_{(1)}}(x) \overset{\text{if iid}}{=} \frac{d}{dx}\left[1-(1-F(x))^n\right] = n f(x)(1-F(x))^{n-1}$$

The next thing we'll do is assume $n=10$ and derive the $k=4$th order statistic's CDF and PDF. Before we get there, let's find the probability that the first four numbers are less than x and the last six numbers are greater than x.

$$= P(X_1 \le x, \dots, X_4 \le x, X_5 > x, \dots, X_{10} > x)$$

if indep

$$= \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} (1-F_{X_i}(x)) \overset{\text{iid}}{=} F(x)^4 (1-F(x))^6$$

Let's find the probability any 4 of the 10 are below x and the remaining are above x. Let S be a subset of size 4 of the index set $\{1, 2, \dots, 10\}$

$$= \sum_{\text{all } S} P(X_{S_1} \le x, \dots, X_{S_4} \le x, X_{S_1^c} > x, \dots, X_{S_6^c} > x)$$

if indep

$$= \sum_{\text{all } S} \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=1}^{6} (1-F_{X_{S_i^c}}(x)) \overset{\text{if iid}}{=} \sum_{\text{all } S} F(x)^4 (1-F(x))^6$$

$$= \binom{10}{4} F(x)^4 (1-F(x))^6$$

Now let's derive the CDF for $k=4$th order statistic

$$F_{X_{(4)}}(x) = P(\underbrace{X_{(4)} \le x}_{\text{event}}) = P(\text{a subset of } 4\ x_i\text{'s} \le x \text{ and the remaining 6 are} > x)$$

$$P(\text{a subset of } 5 x_i \le x \text{ and remaining 5 are} \ge x)$$

$$\vdots$$

$$+$$

$$P(\text{all } 10\ x_i\text{'s} \le x)$$

if iid

$$= \binom{10}{4}(F(x))^4(1-F(x))^6 + \binom{10}{5}F_x^5(1-F(x))^5 + \cdots + \binom{10}{10}F(x)^{10}(1-F(x))^{10-10}$$

$$= \sum_{j=4}^{10}\binom{10}{j}F(x)^j(1-F(x))^{10-j}$$

For iid continuous rv's $X_1, \dots, X_n$, the CDF and PDF for the kth order statistic is

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j}\right] \qquad \frac{d}{dx}[uv] = uv' + u'v$$

$$= \sum_{j=k}^{n}\binom{n}{j}\frac{d}{dx}\left[\underbrace{F(x)^j}_{u}\underbrace{(1-F(x))^{n-j}}_{v}\right]$$

$$u' = j f(x) F(x)^{j-1}, \quad v' = -(n-j) f(x)(1-F(x))^{n-j-1}$$