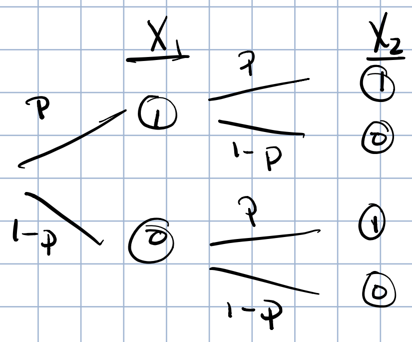
Recall $X_1, X_2 \sim Bern(p)$



$P_{X_1,X_2}(X_1,X_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

Mutually Exclusive.

$$P_T(t) = P(T=t) = \begin{cases} 2 & w.p \ \ p^2 \\ 1 & w.p \ \ 2p(1-p) \\ 0 & w.p \ \ (1-p)^2 \end{cases}$$

Let $t = X_1 + X_2$ then $X_2 = t - X_1$

$$P_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(X_1,X_2) \mathbb{1}_{x_1+x_2=T}$$

$$= \sum_{X_1 \in \mathbb{R}} P_{X_1,X_2}(X_1, t-X_2) = \sum_{x \in \mathbb{R}} P_{X_1,X_2}(X_1, t-X_1)$$

If X_1, X_2 are indep then using the multip. rule.

$$\sum_{x \in \mathbb{R}} P_{X_1,X_2}(X_1, t-X) = \sum_{x \in \mathbb{R}} P_{X_1}(X) P_{X_2}(t-X)$$

$$= \sum_{x \in Supp[X_1]} P_{X_1}^{old}(X_1) \mathbb{1}_{x \in Supp[X_1]} P_{X_2}^{old}(t-X_1) \mathbb{1}_{t \in Supp[X_2]}$$

$$= \sum_{x \in Supp[X_1]} P_{X_1}^{old}(X_1) P_{X_2}^{old}(t-X_1) \mathbb{1}_{t-x \in Supp[X_2]}$$

If X_1, X_2 are iid:

$$\sum_{x \in \mathbb{R}} P_{X_1,X_2}(X_1, t-X_1) = \sum P(X) P(t-x)$$

$$= \sum_{x \in \mathbb{R}} P^{old}(x) \mathbb{1}_{x \in Supp[X]} P^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum P(x) P(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

Convolve: roll, coil or entwine together.

$$P_T(t) = P_{X_1}(X) * P_{X_2}(X)$$

$$P_{T_2}(t) = \sum p^x(1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right)$$

Thus:

$$T_2 \sim \begin{cases} 0 & \text{w.p } (1-p)^2 \\ 1 & \text{w.p } 2p(1-p) \\ 2 & \text{w.p } p^2 \end{cases}$$

Let:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0,1,..n\}}$$

We could rewrite it as:

$$P_{T_2}(t) = p^t (1-p)^{2-t} \binom{2}{t} = \text{Binomial}(2, p)$$

with supp:

$$\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2]$$

We define set addition as:

$$A + B := \{a+b : a \in A, b \in B\}$$

We can rewrite the $X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$ case as:

$$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$$

$$= \binom{1}{x} p^x (1-p)^{1-x}$$

Then:

$$P_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x}\binom{1}{t-x} = \sum_{x \in \{0,1\}} \binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \left(\underbrace{\binom{1}{t} + \binom{1}{t-1}}_{\text{Pascal identity}} \right) = \binom{2}{t} p^t (1-p)^{2-t}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \leftarrow \text{Pascal identity}$$

Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$ $\quad T_3 = X_1 + X_2 + X_3$

$$T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p_{T_3}(t)$$

$$p_{T_3}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x} = p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1}\right)$$

$$= \binom{3}{t} p^t (1-p)^{3-t} = \text{Binom}(3, p)$$

Let $X_1, X_2 \overset{iid}{\sim} \text{Binomial}(n, p)$

$$p_T(t) = \sum \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x} \binom{n}{t-x} = \binom{2n}{t} p^t (1-p)^{2n-t}$$

$$= \text{Binom}(2n, p)$$