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let $\underset{L\times 1}{A} \in \mathbb{R}^{L\times K}$ matrix of constants

$$E[A\vec{X}] = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \dots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \dots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \dots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{X}] \\ E[a_2 \cdot \vec{X}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu}_1 \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_L \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$$Var[\vec{a}^T \vec{X}] = Var[\overset{Y_1}{a_1X_1} + \dots + \overset{Y_K}{a_KX_K}] = \sum_{i=1}^{K}\sum_{j=1}^{K} Cov[Y_i, Y_j] = \sum\sum Cov[a_iX_i, a_jX_j]$$
$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j \sigma_{ij} = \underbrace{\vec{a}^T \overset{Var[\vec{X}]}{\Sigma} \vec{a}}_{\substack{(1\times K)(K\times K)(K\times 1) \\ = \text{scalar}}} \text{ "quadratic form"}$$

let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^{K+1}$

$$\vec{a}^T V \vec{a} = \vec{a}^T \begin{bmatrix} a_1V_{11} + \dots + a_K V_{1K} \\ a_1 V_{21} + \dots + a_K V_{2K} \\ \vdots \\ a_1 V_{K1} + \dots + a_K V_{KK} \end{bmatrix} = \overbrace{a_1a_1V_{11} + \dots + a_1a_KV_{1K}}^{i=1} + \overbrace{a_2a_1V_{21} + \dots + a_2a_KV_{2K}}^{i=2} + \dots +$$
$$a_Ka_1V_{K1} + \dots + a_Ka_KV_{KK} = \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_jV_{ij}$$

$$\vec{a}^T = [a_1, \dots, a_K]$$

Application in finance. Imagine $X_1, \dots, X_K$ are financial assets (e.g. different stocks), Each has mean return μ_i. And each pair have covariance Sigma-ij. Let ω-vector be a vector of "weights" where each component is the percentage you put into each of these assets. Thus the entries of ω sum to 1. Your portfolio F is $\vec{\omega}^T$-vector $\vec{X}$:

$F = \vec{\omega}^T \vec{X}$, $\vec{\omega}^T \cdot \vec{1} = 1$, $E[\vec{X}] = \mu$, $Var[\vec{X}] = \Sigma$.

$E[F] = E[\vec{\omega}^T \vec{X}] = \vec{\omega}^T \vec{\mu} = \mu_F$, $Var[F] = Var[\vec{\omega}^T \vec{X}] = \vec{\omega}^T \Sigma \vec{\omega}$

Goal is to pick μ_F and minimize it's variance by computing the $\vec{\omega}$ optimally.

$\min \vec{\omega}^T \Sigma \vec{\omega}$ subject to $\vec{\omega}^T \vec{\mu} = \mu_F$, $\vec{\omega}^T \vec{1} = 1$

Markowitz optimal portfolio design

$\vec{X} \sim \text{Multin}_K(n, \vec{p})$, $E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$

$X_j \sim \text{Bin}(n, p_j)$

$\Sigma = \text{Var}[\vec{X}] = \begin{bmatrix} \text{Var}[X_1] & & \text{Cov}[X_i, X_j] \\ & \ddots & \\ \text{Cov}[X_i, X_j] & & \text{Var}[X_K] \end{bmatrix}$ $\quad \text{Var}[X_j] = np_j(1-p_j)$

$i \neq j$

difficult to obtain... (sum out all other dim)

$\text{Cov}[X_i, X_j] = E[X_i X_j] - \mu_i \mu_j = \sum_{x_i \in \{0,\dots,n\}} \sum_{x_j \in \{0,\dots,n\}} x_i x_j \overbrace{p_{X_i X_j}}^{\text{Jnt } X_i, X_j}(x_i, x_j) - n^2 p_i p_j$

$X_i \sim \text{Bin}(n, p_i)$ $\quad X_i = X_{1i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} \text{Bernoulli}(p_i)$

$X_j \sim \text{Bin}(n, p_j)$ $\quad X_j = X_{1j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} \text{Bern}(p_j)$

$\vdots$

$\vec{X} \sim \text{Multin}_K(n, \vec{p})$ $\quad \vec{X} = \vec{X}_1 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \overset{iid}{\sim} \text{Multin}_K(1, \vec{p})$

$\text{Cov}[X_i, X_j] = \text{Cov}[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$

$= \sum_{l=1}^{n} \sum_{m=1}^{n} \text{Cov}[X_{li}, X_{mj}]$ all pairs $\overset{\text{if } l \neq m}{=} \sum_{l=1}^{n} \text{Cov}[X_{li}, X_{lj}]$

$= \sum_{l=1}^{n} E[X_{li} X_{mj}] - \overbrace{E[X_{li}]}^{p_i} \overbrace{E[X_{lj}]}^{p_j} = -n p_i p_j$ (iid Bern)

if $l \neq m$ then is X_{li} independent of X_{mj}? Yes!

only nonzero if $x_{li} = x_{lj} = 1$

$E[X_{li}, X_{lj}] = \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \, p_{X_{li} X_{lj}}(x_{li}, x_{lj}) = p_{X_{li}, X_{lj}}(1,1) = 0$

$X_{li} = 1$ means apple, $X_{lj} = 1$ means got banana. Both $= 1$ means you draw an apple and banana on one draw (impossible).

$$\Sigma = Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_K \\ -np_1p_2 & np_2(1-p_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ -np_1p_K & \cdots & \cdots & np_K(1-p_K) \end{bmatrix}$$

$$X_1, X_2 \overset{iid}{\sim} \overset{\text{uniform discrete}}{U(\{0,1,2,3\})} = \begin{cases} 0 & wp\ 1/4 \\ 1 & wp\ 1/4 \\ 2 & wp\ 1/4 \\ 3 & wp\ 1/4 \end{cases} = \frac{1}{4}\mathbb{1}_{x \in \{0,1,2,3\}}$$

generally $X \sim U(A) = \frac{1}{|A|}\mathbb{1}_{x \in A}$ Param space $A \subset \mathbb{R}$ and $|A| < \infty$.

mid 1

$T = X_1 + X_2 \sim p_T(t) =$ meant as a hw example

mid 2

$Y = -X = g(x)$ Y is a func. of rv X

$Y \sim p_Y(y)$ pmf

$x = 0 \Rightarrow Y = 0$ $\quad Supp[Y] = -Supp[X]$

$x = 1 \Rightarrow Y = -1$

$x = 2 \Rightarrow Y = -2$

$x = 3 \Rightarrow Y = -3$

$p_Y(y) := P(Y = y) = P(-X = y) = P(X = -y) = p_X(-y)$

this is for all discrete r.v.'s

let $z' = -z$

$Supp[Y] = \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z' : p_X(z') > 0\}$

$= -\{z' : p_X(z') > 0\} = -Supp[X]$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$ from prev. class $X_1 + X_2 \sim Poisson(2\lambda)$

$X_1 - X_2 \sim ?$

(difference) $D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$ $\quad p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$ $\quad Supp[X] = \{0, 1, 2, \ldots\}$

$Supp[Y] = \{\ldots, -2, -1, 0\}$

$Supp[X+Y] = \mathbb{Z}$ all integers

$$p_D(d) = \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$