Lecture 11 10/14/2020

$X \sim \text{logistic}(0,1) = \frac{e^x}{(1+e^x)^2} \approx N(0,1)$ but with thicker tails.

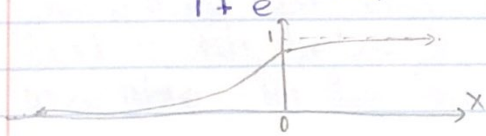

$E[X] = 0, \quad SD[X] = \frac{\pi}{\sqrt{3}} \approx 1.8 > 1$

Consider the shift and scale where sigma > 0.

$$Y = \mu + \sigma X \sim f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{|\sigma|} = \frac{e^{\frac{y-\mu}{\sigma}}}{\sigma\left(1+e^{\frac{y-\mu}{\sigma}}\right)} = \text{Logistic}(\mu,\sigma$$

Why is this called the "logistic distribution?" There's a famous function called the "logistic function." It has three parameters: L (maximum value), k (steepness), μ (center) and it is:

$$l(x) := \frac{L}{1+e^{-k(x-\mu)}} \overset{\text{if } L=1,\, k=1,\, \mu=0}{=} \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{e^x+1}$$ (standard logistic function)




$X \sim \text{logistic}(0,1)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} e^t/(1+e^t)^2\,dt = \int_1^{1+e^x} \frac{\cancel{1-u}}{u^2 \cdot \cancel{1-u}}\,du = \left[-u^{-1}\right]_1^{1+e^x}$$

$$= 1 - 1/(1+e^x) = e^x/(1+e^x)$$

let $u = 1+e^t \Rightarrow e^t = 1-u \Rightarrow du/dt = e^t \Rightarrow dt = e^{-t}du = \frac{1}{1-u}du$ $\Rightarrow t = -\infty \Rightarrow u = 1,$

$t = x \Rightarrow u = 1+$

The "quantile" $q$ or "percentile" $100q$ for a rv $X$ is defined as the minimum $x$ s.t. $q \leq P(X \leq x) = F(x) \iff F(x) \geq q$. It is denoted $Q[X, q]$ where $Q$ is the "quantile operator" (not the upper incomplete regularized gamma function). When $q = 0.5$, the quantile has a special name, the "median", $Med[X] := Q[X, q]$. Here's an example:

$$X \sim U(\{2, 4, 6, \ldots, 20\}) = \tfrac{1}{10} \mathbb{1}_{x \in \ldots}$$

| $x$ | $p(x)$ | $F(x)$ |
|---|---|---|
| 2 | 0.1 | 0.1 |
| 4 | 0.1 | 0.2 |
| 6 | | 0.3 |
| 8 | | 0.4 |
| 10 | | 0.5 |
| 12 | | 0.6 |
| 14 | | 0.7 |
| 16 | | 0.8 |
| 18 | ↓ | 0.9 ← |
| 20 | | 1.0 ← |

$Q[X, 30\%] = 6$

$Q[X, 80\%] = 16$ $\qquad$ $Med[X] = 10$

$Q[X, 85\%] = 18 = Q[X, 0.9]$

However, if $X$ is a continuous rv with "contiguous support" e.g. $[0, 10]$, $[0, \text{infinity})$, all real numbers, etc and not something like $[0,1]$ union $[2,3]$. In the latter case, $F(x)$ is flat between $[1,2]$ which means it's not invertible. In the former case, $F(x)$ is invertible.

$Q[X, q] = F_X^{-1}(q)$, and the inverse CDF is called appropriately, the "quantile function".

$$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} \Rightarrow F_X(x) = 1 - e^{-\lambda x} = q \Rightarrow 1 - q = e^{-\lambda x}$$

$$\Rightarrow \ln(1-q) = -\lambda x \Rightarrow x = -\tfrac{1}{\lambda} \ln(1-q) = \tfrac{1}{\lambda} \ln\left(\tfrac{1}{1-q}\right) = F_X^{-1}(q)$$

$$Med[X] = \ln(2)/\lambda = F_X^{-1}(0.5)$$

Quantile functions are not usually available in closed form since CDF's aren't even usually available in closed form e.g.

$$X \sim \text{Erlang}(k, \lambda) \Rightarrow F_X(x) = P(k, \lambda x)$$

$\text{Med}[X] = x$ s.t. $P(k, \lambda x) = 0.5$ Need a computer solver

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$k > 0 \ldots$

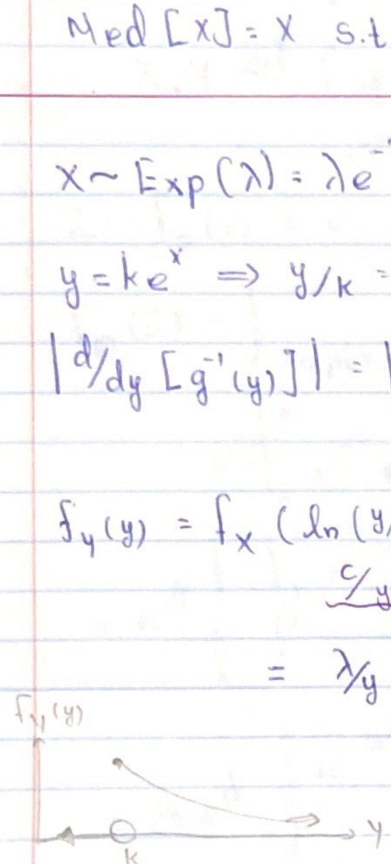
$$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}, \quad Y = g(x) = k e^{x} \sim f_Y(y) = ?$$

$$y = k e^{x} \Rightarrow y/k = e^{x} \Rightarrow x = \ln(y/k) = \ln(y) - \ln(k) = g^{-1}(y)$$

$$\left| \frac{d}{dy}[g^{-1}(y)] \right| = |1/y| = 1/y$$

$$f_Y(y) = f_X(\ln(y/k)) \frac{1}{|y|} = \underbrace{\lambda/|y| \, e^{\overbrace{-\lambda \ln(y/k)}^{\ln((y/k)^{-\lambda})}}}_{c/y^{\lambda+1}} \mathbb{1}_{\overbrace{\ln(y) - \ln(k) \in [0,\infty)}^{\overbrace{\ln(y) \in [\ln(k),\infty)}^{y \in [k,\infty)}}}$$

$$= \lambda/y \, (y/k)^{-\lambda} \mathbb{1}_{y \in [k,\infty)} = \text{Pareto I}(k, \lambda)$$




$k \in (0,\infty), \ \lambda \in (0,\infty)$

$$\frac{d}{dt}\left[-t^{-\lambda}/\lambda\right] = -\lambda\left(-t^{-\lambda-1}/\lambda\right) = 1/t^{\lambda+1}$$

$$F_Y(y) = \int_k^y \frac{\lambda}{k^{-\lambda}} \frac{1}{t^{\lambda+1}} \, dt = \frac{\lambda}{k^{-\lambda}} \left[-\frac{1}{\lambda t^{\lambda}}\right]_k^y = k^{\lambda}\left(\frac{1}{k^{\lambda}} - \frac{1}{y^{\lambda}}\right)$$

$$= 1 - (k/y)^{\lambda} \Rightarrow F_Y^{-1}(q) = k(1-q)^{-1/\lambda}$$

This distribution was discovered by Vilfredo Pareto, an Italian economist in 1896 when he observed that 20% of the richest Italians owned 80% of the land (i.e. the wealth). This is known as the "Pareto Principle" and it corresponds to the Pareto I(1, 1.161) distribution.

Further, the Pareto distribution is a waiting time / survival time model. It's used for [see wikipedia if you're interested]. Wealth, music talent, number of patents, ...

$$Z \sim \frac{1}{|-1|} f_X\left(\tfrac{z}{-1}\right) = e^{z} \mathbb{1}_{z \in (-\infty, 0]}$$

$$X, Y \overset{iid}{\sim} \underset{= e^{-x} \mathbb{1}_{x \in [0,\infty)}}{\text{Exp}(1)}, \text{ let } D = X - Y = X + \underbrace{(-Y)}_{Z}$$

$$Z \sim f_Z(z) = e^{z} \mathbb{1}_{z \in (-\infty, 0]}$$

$$f_D(d) = \int_{Supp[X]} f_X^{old}(x)\, f_Z^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Z]}\, dx$$

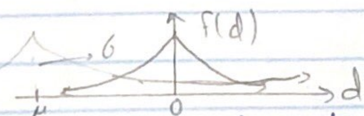
$$= \int_0^\infty e^{-x} e^{d-x} \mathbb{1}_{\substack{x \in [d,\infty) \\ x-d \in [0,\infty) \\ d-x \in (-\infty, 0]}} dx = e^d \int_0^\infty e^{-2x} \mathbb{1}_{x \in [d,\infty)} dx$$

$$= e^d \begin{cases} \int_d^\infty e^{-2x} dx & \text{if } d \geq 0 \\ \int_0^\infty e^{-2x} dx & \text{if } d < 0 \end{cases} = e^d \begin{cases} \left[-\tfrac{1}{2} e^{-2x}\right]_d^\infty & \text{if } d \geq 0 \\ \left[-\tfrac{1}{2} e^{-2x}\right]_0^\infty & \text{if } d < 0 \end{cases}$$

$$= \tfrac{1}{2} e^d \begin{cases} e^{-2d} & \text{if } d \geq 0 \\ 1 & \text{if } d < 0 \end{cases} = \begin{cases} \tfrac{1}{2} e^{-d} & \text{if } d \geq 0 \\ \tfrac{1}{2} e^{d} & \text{if } d < 0 \end{cases} = \tfrac{1}{2} e^{-|d|}$$

$= \text{Laplace}(0,1)$ standard Laplace distr AKA double exponential

$\mu \in \mathbb{R}, \sigma > 0,$




$$X = \mu + \sigma D \sim \text{Laplace}(\mu, \sigma) := \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

$\sigma > 0$

this is also a famous rv and it has another name: the "double exponential". Laplace published this distribution in 1774 calling it the "first law of errors".