Lecture - 13 10/21/2020

$$= \sum_{j=k}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

$$= \sum_{j=k}^{n} \frac{n!}{\cancel{j}!\,(n-j)!} \cancel{j} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \cancel{\Sigma}$$

$(j-1)!$

Note $j = n$

$$- \sum_{j=k}^{\cancel{n}\; n-1} \frac{n!}{j!\,\cancel{(n-j)}!} \overbrace{\cancel{(n-j)}}^{0} f(x) F(x)^{j} (1-F_x)^{n-(j+1)}$$

$n-(j+1) = n-j-1$

$(n-j-1)! = (n-(j+1))!$ if $j = n-1$

$l-1 = n-1$

$\Downarrow$

$l = n$

reindexing: let $l = j+1 \Rightarrow j = l-1$ only

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{l=k+1}^{n} \frac{n!}{(l-1)!\,(n-l)!} f(x) F(x)^{l-1} (1-F(x))^{n-l}$$

these are equal now

$$= \frac{n!}{(k-1)!\,(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k}$$

$= f_{X_{(k)}}(x)$ DONE for iid case

Check min, max:

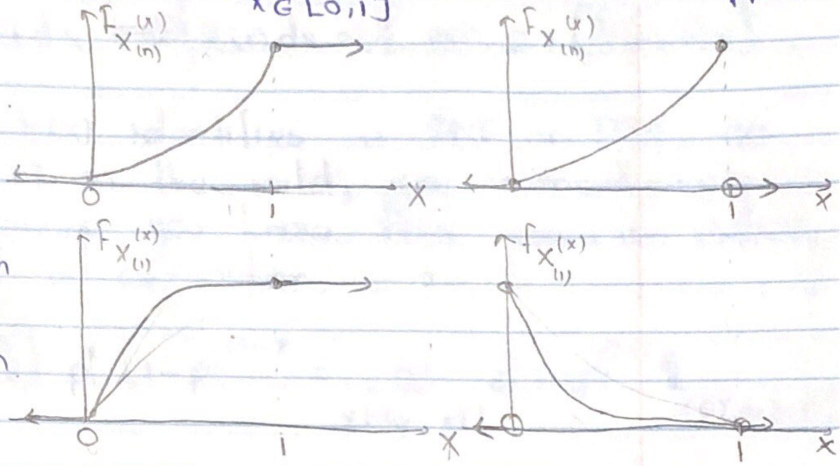
✓ $f_{X_{(1)}}(x) = \frac{n!}{(1-1)!\,(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1}$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x)\, F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1}$$

$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1) = 1\mathbb{1}_{x\in[0,1]} \implies F(x) = x$ in the support

$$F_{X_{(n)}} = F(x)^n = x^n$$

$$F_{X_{(1)}}(x) = 1-(1-F(x))^n = 1-(1-x)^n$$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in[0,1]} \implies f_{X_{(n)}}(x) = n x^{n-1}\mathbb{1}_{x\in[0,1]},$$

$$f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k}\mathbb{1}_{x\in[0,1]} = \text{Beta}(k, n-k+1)$$

We will see the general beta distribution later

$X \sim \text{Gamma}(\alpha, \beta)$ indep of $Y \sim \text{Gamma}(\alpha_2, \beta)$

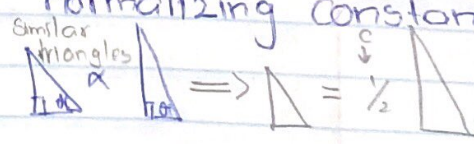
this seems right $\implies X+Y \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

The easiest proof of this is to employ "kernals". What's a kernal?

$$p(x) = c\,k(x), \quad f(x) = c\,k(x) \quad \text{normalizing constant}$$

Kernal

$$\implies p(x) \propto k(x), \quad f(x) \propto k(x)$$



If you know $k(x)$, you can resolve c via the following:

$$1 = \sum p(x) = \sum c k(x) \Rightarrow \sum k(x) = 1/c \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int f(x)\,dx = \int c k(x)\,dx \Rightarrow \int k(x)\,dx = 1/c \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

this means that $k(x)$ identifies a PMF or PDF. So if you "see" a $k(x)$ in the wild, you immediately know the rv because you know that you can solve for c if you had a computer.

$$X \sim \mathrm{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\ldots,n\}}$$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1}_{x \in \{0,1,\ldots,n\}}}_{k(x)}$$

$$Y \sim \mathrm{Weibull}(k,\lambda) = k\lambda(\lambda y)^{k-1} e^{(-\lambda y)^k} \mathbb{1}_{y \geq 0}$$

$$= \underbrace{k\lambda^k}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}}_{k(x)}$$

$$X \sim \mathrm{Gamma}(\alpha,\beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}}_{k(x)} \propto x^a e^{-bx} \mathbb{1}_{x \geq 0}$$

Let's add the gamma's

$$f_{X+Y}(t) = \int_0^\alpha \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \overbrace{e^{-\beta(t-x)}}^{e^{-\beta t}\cancel{e^{+\beta x}}} \mathbb{1}_{\frac{\underbrace{x \in (-\alpha, t]}\quad x - t \in (-\alpha, 0]}{t - x \in [0, \alpha)}} \, d$$

Let's find this density's kernel, $k(t)$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}\,dx\, \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}\,dx\, \mathbb{1}_{t\geq 0}.$$

$$= e^{-\beta t}\, t^{\alpha_1-1}\, t^{\alpha_2-1} \int_0^t \left(x/t\right)^{\alpha_1-1} \left(1-x/t\right)^{\alpha_2-1} dx\, \mathbb{1}_{t\geq 0}.$$

let $u = x/t \Rightarrow du/dx = 1/t \Rightarrow dx = t\,du \Rightarrow x=0$
$\Rightarrow u=0, x=t \Rightarrow u=1$

$$= e^{-\beta t}\, t^{\alpha_1+\alpha_2-2} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, t\, du\, \mathbb{1}_{t\geq 0}$$

this integral is proven to be impossible

$$= e^{-\beta t}\, t^{\alpha_1+\alpha_2-1}\, \mathbb{1}_{t\geq 0} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du.$$

even if the integral is impossible, What will the result be a function of? α_1 & α_2

$$\propto t^{\alpha_1+\alpha_2-1} e^{-\beta t}\, \mathbb{1}_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta)$$

$$= \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du}_{c} \cdot \underbrace{t^{\alpha_1+\alpha_2-1} e^{-\beta t}\, \mathbb{1}_{t\geq 0}}_{k(t)}$$

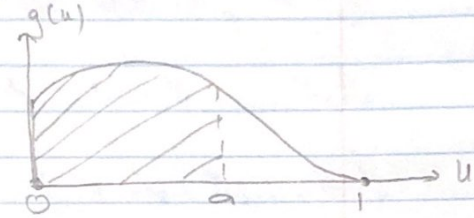
that integral is quite famous and it's called the "beta function"

$$B(\alpha_1, \alpha_2) := \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du$$

We can use probability theory to get an integral identy:

$$X \sim \text{Gamma}(\alpha_1+\alpha_2, \beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2) t^{\alpha_1+\alpha_2-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$$



$$\Rightarrow B(\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

$$B(a, \alpha_1, \alpha_2) := \int_0^a \underbrace{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}_{g(u)} du$$ incomplete beta func.

$$I_a(\alpha_1,\alpha_2) = \frac{B(a,\alpha_1,\alpha_2)}{B(\alpha_1,\alpha_2)} \in [0,1]$$ regularized incomplete beta func.

$$X \sim \text{Beta}(\alpha,\beta) := \underbrace{\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]}}_{f(x)}$$ when $\alpha,\beta > 0$.

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\int_0^x y^{\alpha-1}(1-y)^{\beta-1} dy}{B(\alpha,\beta)} = \frac{\overset{I_x(\alpha,\beta)}{\overset{\|}{B(x,\alpha,\beta)}}}{B(\alpha,\beta)}$$

$$1 = \int_{supp[x]} f(x) du = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx}_{\text{beta}} = \frac{B(\alpha,\beta)}{B(\alpha,\beta)} = 1$$