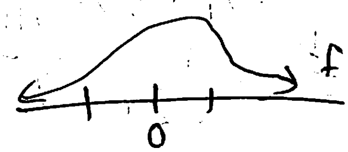
Math 621
Lecture 18
11-16-2020



$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$

$Z \sim N(0,1)$, $Y = Z^2 \sim f_Y(y) = ?$ Not 1:1

$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [\sqrt{y}, \sqrt{y}])$
$= 2P(Z \in [0, \sqrt{y}])$
$= 2(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{\frac{1}{2}}) = 2F_Z(\sqrt{y}) - 1$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right]$$
$$= 2\left(\tfrac{1}{2} y^{-\frac{1}{2}}\right) f_Z(\sqrt{y})$$
$$= y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \underbrace{\mathbb{1}_{\sqrt{y} \in \mathbb{R}}}_{y \ge 0}$$

$$\propto y^{\overbrace{-\frac{1}{2}}^{\frac{1}{2}-1}} e^{-\frac{1}{2}y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$Z_1, Z_2, \ldots, Z_k \overset{iid}{\sim} N(0,1)$, $Y = \underbrace{Z_1^2}_{\text{gamma}(\frac{1}{2},\frac{1}{2})} + \underbrace{Z_2^2}_{\text{gamma}(\frac{1}{2},\frac{1}{2})} + \cdots + \underbrace{Z_k^2}_{\text{gamma}(\frac{1}{2},\frac{1}{2})}$

So, $Y \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$

Note: the beta is always $\frac{1}{2}$, and the alpha is always $\frac{k}{2}$, so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi^2_k$, The "chi squared distribution with k degrees of freedom", $k \in \mathbb{N}$.

$E[Y] = k\,E[Z_1^2] = k.$

$$\chi^2_k = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0}$$

(2)

$k=1, \Gamma(\tfrac{1}{2})=\sqrt{\pi}$

$$\stackrel{\downarrow}{=} \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} \mathbb{1}_{y\ge 0} = \chi^2_1$$

$X \sim \chi^2_k$, $Y=\sqrt{X} \Rightarrow X = Y^2 = g^{-1}(Y)$,

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = |2y| = 2y$$

$$f_Y(y) = f_X(y^2)2y = \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{k-2} e^{-y^2/2} (2y) \mathbb{1}_{\underbrace{y^2\ge 0}_{y\ge 0}}$$

$$= \frac{(\frac{1}{2})^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2})} y^{k-1} e^{-y^2/2} \mathbb{1}_{y\ge 0} = \chi_k$$

the chi distribution with k degrees of freedom.

$$Z \sim N(0,1), \; |Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \mathbb{1}_{y\ge 0}$$

$$= 2\underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)}_{f_Z} \mathbb{1}_{y\ge 0}$$

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{y}{c}\right)^{\alpha-1}}_{\frac{y^{\alpha-1}}{c^{\alpha-1}}} e^{-\underbrace{\beta\frac{y}{c}}_{(\frac{\beta}{c})y}} \mathbb{1}_{\underbrace{\frac{y}{c}>0}_{y>0}}$$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\beta/c)y} \mathbb{1}_{y\ge 0}$$

$$= \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

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③

$X \sim \chi^2_k$, $Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{\frac{1}{2}}{\frac{1}{k}}\right)$

$= \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

Let $X_1 \sim \chi^2_{k_1}$, independent of $X_2 \sim \chi^2_{k_2}$

Let $U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\underbrace{\frac{k_1}{2}}_{a}, \underbrace{\frac{k_1}{2}}_{a}\right)$ indep of

$V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\underbrace{\frac{k_2}{2}}_{b}, \underbrace{\frac{k_2}{2}}_{b}\right)$

$R = \frac{U}{V} \sim f_R(r) = \int_{\text{supp}[V]} f_U(rt) \mathbb{1}_{rt \in \text{supp}[U]} f_V(t) |t| \, dt$

$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \underbrace{\mathbb{1}_{rt \in [0,\infty)}}_{r \in (0,\infty)} \cdot$

$\frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} \, t \, dt$

$= \frac{a^a}{\Gamma(a)} \cdot \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r \geq 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} dt$

$= a^a b^b r^{a-1} \mathbb{1}_{r \geq 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)} \Gamma(a+b)}_{\frac{1}{B(a,b)}} \frac{1}{(ar+b)^{a+b}}$

$= \frac{a^a \cancel{b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\underbrace{b^{-(a+b)}}_{b^{-a}\cancel{b^{-b}}} \left(1 + \frac{a}{b} r\right)^{-(a+b)}} \mathbb{1}_{r \geq 0}$

$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)}$

Now substitute:

$$= \frac{\left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}}}{B\left(\frac{k_1}{2},\frac{k_2}{2}\right)}\, r^{\frac{k_1}{2}-1}\left(1+\frac{k_1}{k_2}r\right)^{-\frac{k_1+k_2}{2}}\mathbb{1}_{r\geq 0}$$

$= F_{k_1,k_2}$ this is the "F distribution" or "Fisher-Snecedor Distribution" with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom. $k_1, k_2 \in \mathbb{N}$.

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let $Z \sim N(0,1)$, $X \sim \underset{(\text{chi})}{\chi^2_k}$, $W = \dfrac{Z}{\sqrt{X/k}} \sim f_W(w) = ?$

Consider $W^2 = \dfrac{Z^2/1}{X/k} \sim F_{1,k}$

Tsymmetric around 0

$$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w])$$
$$= F_W(w) - F_W(-w)$$

Take derivatives...

$$\frac{d}{dw}\left[F_{W^2}(w^2)\right] = \frac{d}{dw}\left[F_W(w)\right] - \frac{d}{dw}\left[F_W(-w)\right]$$

$$2w\, f_{W^2}(w^2) = f_W(w) - -f_W(-w) = 2 f_W(w)$$

$$f_W(w) = \cancel{w}\,\frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{\underbrace{B\left(\frac{1}{2},\frac{k}{2}\right)}_{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2}\right)}}}\underbrace{(w^2)^{\frac{1}{2}-1}}_{\cancel{w^{-1}}}\left(1+\frac{w^2}{k}\right)^{-\frac{1+k}{2}}\mathbb{1}_{\substack{w^2\geq 0\\ w\in\mathbb{R}}}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\;\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}}$$

$= T_k$ Student's T distribution with k degrees of freedom,

discovered by 1908 by William Gosset while he was working at a beer factory.

If $k \to \infty$, $T_k \to Z$

Student's T distribution has the $N(0,1)$ shape but just thicker tails.

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$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim ?$

NO indicator functions

$R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f_r |u| \, du$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u| \, du$

$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} |u| \, du + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} |u| \, du \right)$

$= \frac{1}{2\pi} \left(\int_{0}^{-\infty} e^{-\frac{r^2+1}{2}u^2} |u| \, du + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} |u| \, du \right)$

$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} u \, du$

Let $t = u^2 \Rightarrow \frac{dt}{du} = 2u$

$\Rightarrow du = \frac{1}{2u} dt$, $u = 0 \Rightarrow t = 0$,

$u \to \infty \Rightarrow t \to \infty$

$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\frac{r^2+1}{2}t} \cancel{u} \cdot \frac{1}{2\cancel{u}} \, dt$

$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-\frac{r^2+1}{2}t} \, dt$

$= \frac{1}{\cancel{2}\pi} \frac{1}{\frac{r^2+1}{\cancel{2}}} \int_{0}^{\infty} \underbrace{\frac{r^2+1}{2} e^{-\frac{r^2+1}{2}t} \, dt}$ . . . .

⑥

$$= \frac{1}{\pi} \cdot \frac{1}{1+r^2}$$

$$\underbrace{\int_0^1 \lambda e^{-\lambda x}}_{\text{PDF}} dx = 1$$

exponential rv

$= \text{Cauchy}(0,1)$

Let $X = c + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma \pi} \quad \frac{1}{1+\left(\frac{x-c}{\sigma}\right)^2}$$