Lecture - 05 09/14/2020

$\vec{X} \sim \text{Multi}_K(n, \vec{p})$

K ↑ dimension $\quad K=2 \quad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

$$\text{Deg}(n-x_2) = P_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) \overset{*}{=} \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$$

$$* = \frac{\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} \underbrace{(1-p_2)}_{p_1}^{n-x_2}}$$

Define $J_n := \{0, 1, \ldots, n\}$

$$= \frac{\frac{\cancel{n!}}{x_1!\,\cancel{x_2!}} \mathbb{1}_{\underbrace{x_1 = n - x_2}_{x_1 + x_2 = n}} \mathbb{1}_{x_1 \in J_n} \cancel{\mathbb{1}_{x_2 \in J_n}} p_1^{x_1} \cancel{p_2^{x_2}}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \cancel{\mathbb{1}_{x_2 \in J_n}} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

$\mathbb{1}^n_{x_2 \in J_n}$

Define a ratio of indicators

Define: $\mathbb{1}^u_A = \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \text{undefined} & \text{if } A^c \end{cases}$

$P(A|B) = \frac{P(A,B)}{P(B)}$ which is undefined if $P(B) = 0$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{=1 \text{ when } x_1 = n - x_2} \mathbb{1}_{x_1 = n - x_2} \underbrace{\mathbb{1}_{x_1 \in J_n}}_{=1 \text{ when } x_1 = n-x_2 \text{ \& } x_2 \in J_n} \mathbb{1}^u_{x_2 \in J_n} = \text{Deg}(n-x_2)\mathbb{1}^u_{x_2 \in J_n}$$

Let's generalize this result a little bit.

$$\vec{X} \sim \text{Multin}_K(n, \vec{p}) \qquad \vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$$

$$P_{\underbrace{\vec{X}_{-j}}|X_j}(\vec{x}_{-j}, x_j) = \frac{P_{\vec{X}}(\vec{x})}{P_{X_j}(x_j)} \sim \text{Multin}_{K-1}(n - x_j, ?)$$

All elements of vector rv X except the jth component.

$$= \frac{\text{Multin}_K(n, \vec{p})}{\text{Bin}(n, p_j)} = \frac{\binom{n}{x_1, \dots, x_K} p_1^{x_1} \dots p_{j-1}^{x_{j-1}} p_j^{x_j} p_{j+1}^{x_{j+1}} \dots p_K^{x_K}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-p_j}}$$

$$= \frac{\frac{n!}{x_1! \dots x_j! \dots x_K!} \mathbb{1}_{x_1 + \dots + x_j + \dots + x_K = n} \mathbb{1}_{x_1 \in \mathcal{J}_n} \dots \mathbb{1}_{x_j \in \mathcal{J}_n} \dots}{\frac{n!}{x_j!(n-x_j)!} \mathbb{1}_{x_j \in \mathcal{J}_n} (1-p_j)^{n-x_j}}$$

cont.

$$\frac{\mathbb{1}_{x_K \in \mathcal{J}_n} p_1^{x_1} \dots p_{j-1}^{x_{j-1}} \dots p_{j+1}^{x_{j+1}} \dots p_K^{x_K}.}{1}$$

Let $n' = n - x_j$ Note $x_1 + \dots + x_K = n$

$$\Rightarrow n - x_j = x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_K.$$

Note $p_1 + \dots + p_K = 1$

$$\Rightarrow p_1 + \dots + p_{j-1} + p_j + p_{j+1} + \dots + p_K = 1 - p_j.$$

divide both sides by $(1-p_j)$ $\Rightarrow \underbrace{\frac{p_1}{(1-p_j)}} + \dots + \frac{p_{j-1}}{(1-p_j)} + \frac{p_j}{(1-p_j)} + \frac{p_{j+1}}{(1-p_j)} + \dots + \frac{p_K}{(1-p_j)} = 1$

$$p_1' + \dots + p_{j-1}' + p_j' + p_{j+1}' + \dots + p_K' = 1$$

$$\binom{n'}{x_1!\cdots x_{j-1}!\, x_{j+1}!\cdots x_k!}$$

$$= \frac{n'!}{x_1!\cdot\ldots\cdot x_{j-1}!\cdot x_{j+1}!\cdots x_k!}\,\mathbb{1}_{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_k=n'}\,\cdot$$

$$\mathbb{1}_{x_1\in\mathbb{J}_n}\cdot\ldots\cdot\mathbb{1}_{x_{j-1}\in\mathbb{J}_n}\cdot\ldots\cdot\mathbb{1}_{x_k\in\mathbb{J}_n}\cdot\mathbb{1}^{u}_{x_j\in\mathbb{J}_n}\;\frac{\dfrac{p_1^{x_1}\cdots p_{j-1}^{x_{j-1}}\,p_{j+1}^{x_{j+1}}\cdots p_k^{x_k}}{(1-p_j)^{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_k}}}{p_1^{x_1}\cdots p_{j-1}^{x_{j-1}}\,p_{j+1}^{x_{j+1}}\cdots p_k^{x_k}}$$

$$= \text{Multin}_{k-1}(n', \vec{p}\,')\,\mathbb{1}^{n}_{x_j\in\mathbb{J}_n}$$

$$\vec{p}\,' = \begin{bmatrix} p_1' \\ \vdots \\ p_k' \end{bmatrix}$$

$\vec{X} \sim \text{Multi}_k(n, \vec{p})$, What is $E[\vec{X}]$? $Var[\vec{X}]$?

We need definitions for expectation and variance for vector rv's ; Review from 241: Let $X_1, \ldots, X_n$ be r.v.'s & $a, c \in \mathbb{R}$

$$E[aX + c] = aE[X] + c$$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\mu \quad \text{if they're identically distributed}$$

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i] \quad \text{if they're independent}$$

$$\sigma^2 := Var[X] := E[(X-\mu)^2] \overset{\text{discrete}}{=} \sum_{x\in\mathbb{R}} (x-\mu)^2 p(x) \;,\quad \overset{\text{continuous}}{=} \int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx$$

$$= E[X^2] - \mu^2$$

$\sigma := \sqrt{Var[X]} = SD[X]$ standard deviation

$$Var[X_1 + X_2] = E[((X_1+X_2) - (\mu_1 + \mu_2))^2]$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2\mu_1 X_1 - 2\mu_1 X_2 - 2\mu_2 X_1 - 2\mu_2 X_2 + 2X_1X_2 + 2\mu_1\mu_2]$$

$$= E[X_1^2] + E[X_2^2] + \cancel{\mu_1^2} + \cancel{\mu_2^2} - \cancel{2}\mu_1^2 - \cancel{2\mu_1\mu_2} - 2\mu_2\mu_1 - \cancel{2}\mu_2^2 + 2E[X_1X_2] + \cancel{2\mu_1\mu_2}$$

$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E[X_1X_2] - \mu_1\mu_2}_{Cov[X_1,X_2]})$$

$\rightarrow$ Cov$[X_1, X_2]$. covariance of X_1 with X_2

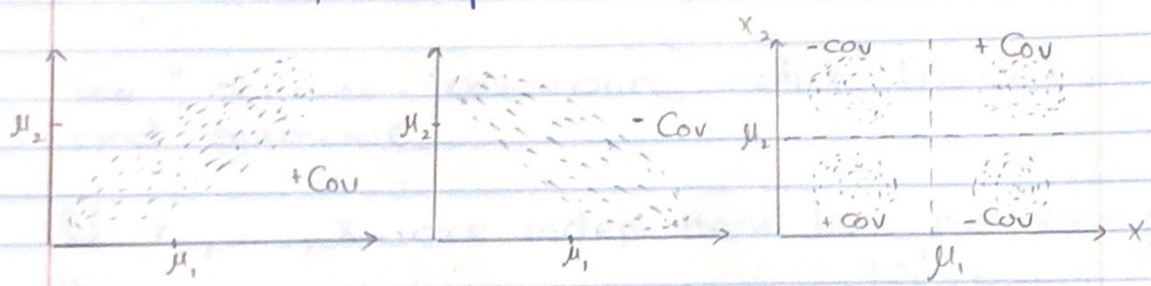
$$= \sigma_1^2 + \sigma_2^2 + 2\underbrace{COV[X_1, X_2]}_{\sigma_{12}} \qquad = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$$

$$= \sigma_1^2 + \sigma_2^2$$

$\hookrightarrow$ if X_1, X_2 independent.

$$\sigma_{12} := Cov[X_1, X_2]$$
$$:= E[X_1 \cdot X_2] - \mu_1\mu_2$$
$$= E(X_1 - \mu_1)(X_2 - \mu_2)$$

If X_1, X_2 independent $\Rightarrow Cov[X_1, X_2] = \mu_1\mu_2 - \mu_1\mu_2 = 0$.



Rules for Covariances $a_1, a_2 \in \mathbb{R}$

1. $Cov[X, X] = \sigma^2$
2. $Cov[X_1, X_2] = Cov[X_2, X_1]$
3. $Cov[X_1 + X_2, X_3] = \sigma_{13} + \sigma_{23}$
4. $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{12}$
5. $Var[X_1 + \dots + X_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j]$

M is a matrix of rv's

$$\vec{\mu} := E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}, \quad \text{let } M = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ X_{21} & \cdots & X_{2m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix},$$

$$E[M] = \begin{bmatrix} E[X_{11}] & \cdots & E[X_{1m}] \\ \vdots & & \vdots \\ E[X_{n1}] & \cdots & E[X_{nm}] \end{bmatrix}$$

$$\Sigma \coloneqq Var[\vec{X}] := E[\underbrace{\vec{X}\vec{X}^T}_{\text{Outer product}}] - \overbrace{\underbrace{\vec{\mu}\vec{\mu}^T}}^{\substack{k \times k \\ (k\times 1)(1\times k)}} = \begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \cdots & Cov[X_1,X_k] \\ Cov[X_2,X_1] & Var[X_2] & \cdots & Cov[X_2,X_k] \\ \vdots & & \ddots & \vdots \\ Cov[X_k,X_1] & Cov[X_k,X_2] & \cdots & Var[X_k] \end{bmatrix}$$

The "Variance-covariance matrix." It's square $k \times k$ and symmetric.

If $X_1, \ldots, X_k$ are independent then the varcov matrix is:

$$Var[\vec{X}] = \text{diag}\{\sigma_1^2, \ldots, \sigma_k^2\} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 \ddots & \\ 0 & & \sigma_k^2 \end{bmatrix}$$

Rules for expectations of vector rv's. Let $\vec{a} \in \mathbb{R}^k$

① $E[\vec{X} + \vec{a}] = \vec{\mu} + \vec{a}$

② $E[\vec{a}^T \vec{X}] = E[a_1 X_1 + a_2 X_2 + \cdots + a_k X_k]$
$= a_1 \mu_1 + \cdots + a_k \mu_k = \vec{a}^T \vec{\mu}$