Lecture - 08 10/05/2020

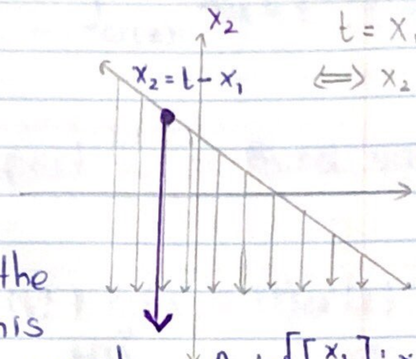
$\vec{X}$ continuous rv $\qquad P(\vec{X} \in A) = \int \cdots \int_A f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k$.

$t = x_1 + x_2$

$\iff x_2 \le t - x_1$

let $T = X_1 + X_2 \sim f_T(t) = ?$

note $f_T(t) = F'(t)$ CDF method



usually it is difficult to find the CDF of continuous rv's, so this is not the usual method. The usual method is to use the convolution formula (which we will now derive).

$A_t \doteq \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \le t - x_1 \right\}$

$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t - x_1} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1 = \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1, X_2}(x, v - x)\, dv\, dx$$

let $x_1 = x$

$x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv$

$\Updownarrow$

$x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$.

$$= \int_{-\infty}^{t} \left(\int_{\mathbb{R}} f_{X_1, X_2}(x, v - x)\, dx \right) dv$$

$$f_T(t) = \frac{d}{dt}\Big[\qquad \Big] = \int_{\mathbb{R}} f_{X_2, X_2}(x, t - x)\, dx = f_T(t)$$

$$= f_{X_1}(x) * f_{X_2}(x)$$

General convolution formula.

Leibnitz's Rule for derivatives of integral functions.

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x,b(x))b'(x) + g(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy.$$

If the derivative is with respect to a third variable, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))b'(t) + g(x,a(t))a'(t) + \underbrace{\cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}[g(x,y)]\,dy}}_{0}$$

If one of the bounds in constant then ....

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + \cancel{g(x,c)\frac{d}{dt}[c]}$$

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\left(\int_{\mathbb{R}} f_{X_1,X_2}(x,v-x)\,dx\right)dv\right]$$

$$= \int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dv$$ General Convolution formula

X_1, X_2 independent

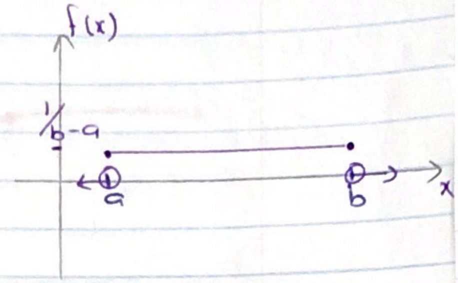
$$\stackrel{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(x)\,f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f^{old}_{X_1}(x)\,f^{old}_{X_2}(t-x)\,\mathbb{1}_{t-x\in Supp[X_2]}\,dx$$

X_1, X_2 iid

$$= \int_{\mathbb{R}} f(x)f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\,f^{old}(t-x)\,\mathbb{1}_{t-x\in Supp[X_2]}\,dx$$

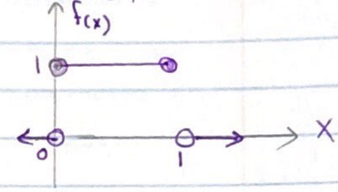
continuous uniform rv

$$X \sim U(a,b) = \underbrace{\overbrace{1/b-a \; \mathbb{1}_{x \in [a,b]}}^{f(x)}}_{f^{old}(x)}$$

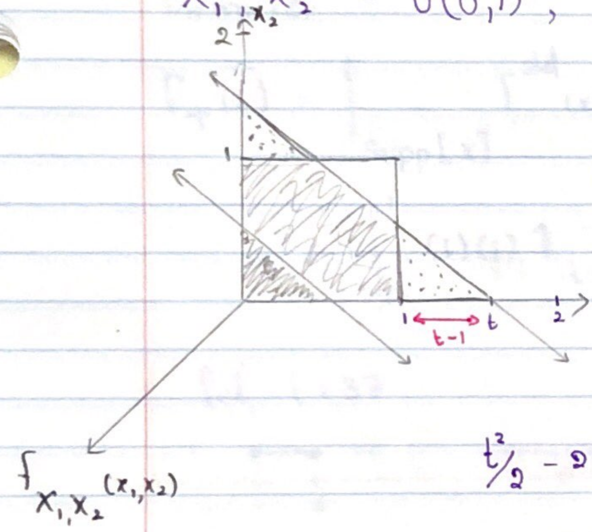


The "standard uniform" rv is when $a=0$, $b=1$

$$X \sim U(0,1) = \overbrace{1/1-0}^{1} \; \mathbb{1}_{x \in [0,1]}$$



$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$ CDF method first..

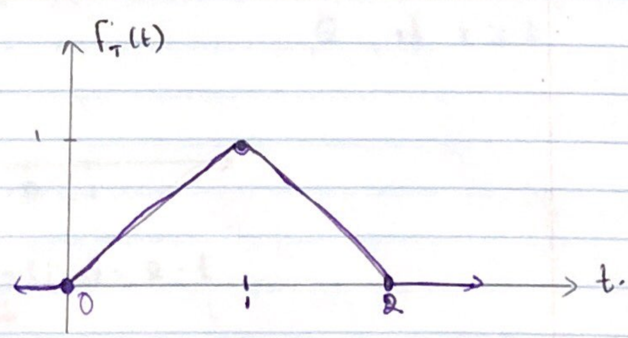


$$F_T(t) = P(T \le t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^2/2 & \text{if } t \in [0,1] \\ -t^2/2 + 2t - 1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \ge 2. \end{cases}$$

$t = 0.3$

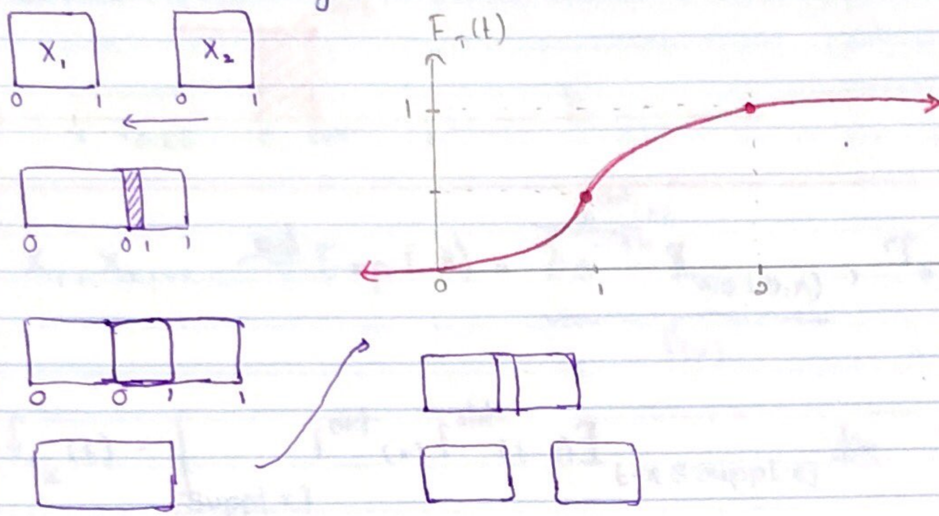
$$t^2/2 - 2\left(\frac{(t-1)^2}{2}\right) = t^2/2 - (t^2 - 2t + 1)$$

$$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2. \end{cases}$$



supp[T] = [0, 2]

Convolving

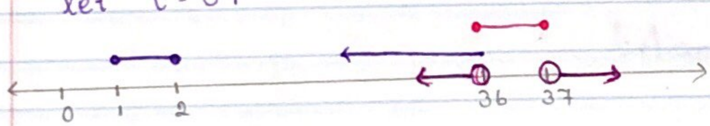


Let's try to derive the PDF of T using the convolution formula.

$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

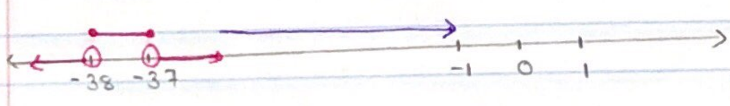
$$= \int_0^1 (1)(1)\, \mathbb{1}_{\underset{\underset{x \in [t-1,t]}{\sim\sim\sim}}{\overset{\overset{x \in [t-1,t]}{\sim\sim\sim}}{\scriptstyle x-t \in [-1,0]}}\ \atop t-x \in [0,1]}\, dx \quad = \int_0^1 \overbrace{\mathbb{1}_{x \in [t-1,t]}}^{f(x)}\, dx.$$

let $t = 37$

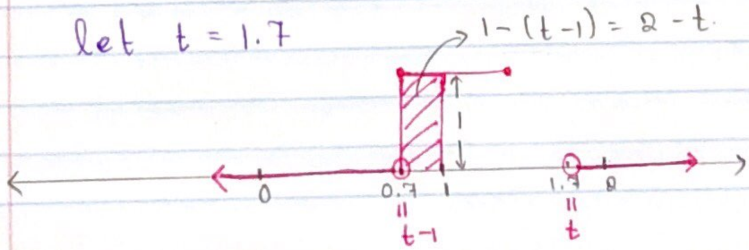


$$\begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in [0,1) \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$

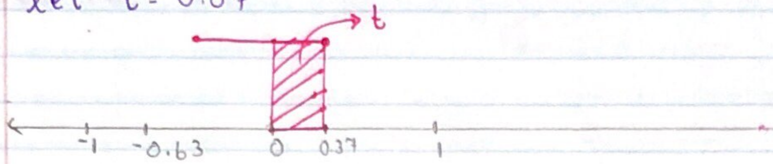
let $t = -37$



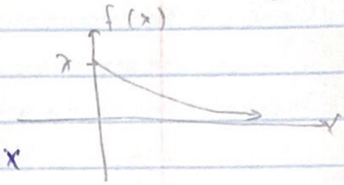
let $t = 1.7$



let $t = 0.37$



$$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) = \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)} \quad , \quad T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$$



$$f_{T_2}(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

$$= \int_0^\infty \lambda \cancel{e^{-\lambda x}}\; \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}}\; \mathbb{1}_{\overbrace{\overbrace{t-x \in [0,\infty)}^{x-t \in [-\infty, 0]}}^{x \in (-\infty, t]}}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t]}\, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = t\lambda^2 e^{-\lambda t}\, \mathbb{1}_{t \in (0,\infty)}$$

$$\| $$

$$Erlang(2, \lambda) = f_{T_2}(t)$$