$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \cdot \frac{\sigma^2}{n-1} \cdot \frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{\chi^2_{n-1}}}{n-1}}} \sim T_{n-1}$$

Due to cochran's thm, we know Xbar and S^2 are independent

Multivariate Normal Distribution (MVN)

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}_n$, $Var[\vec{Z}] = I_n$

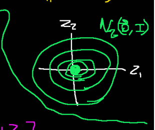
Standard MVN

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}, I)$$

$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$, $E[\vec{X}] = \vec{\mu}$, $Var[\vec{X}] = I_n \Rightarrow \vec{X} \sim N_n(\vec{\mu}, I)$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1 + Z_2 \\ \vdots \\ Z_1 + \ldots + Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$$

but the components are dependent e.g



$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{n \times n}$$

$$Cov[X_1, X_2] = Cov[Z_1, Z_1 + Z_2] = \underbrace{Cov[Z_1, Z_1]}_{1} + \underbrace{Cov[Z_1, Z_2]}_{0} = 1 \Rightarrow X_1, X_2 \text{ dep.}$$

Let's derive a general formula for the variance-covariance matrix of A (an n x n matrix of scalars) times a random vector X of dim n:

$$Var[A\vec{X}] := E[\underbrace{(A\vec{X})(A\vec{X})^T}_{A\vec{X}\vec{X}^TA^T}] - E[A\vec{X}]E[A\vec{X}]^T$$

$$= A\,E[XX^T]A^T - A\,E[\vec{X}]\underbrace{(A\,E[\vec{X}])^T}_{E[\vec{X}]^TA^T}$$

$$= A\Big(\underbrace{E[XX^T] - E[\vec{X}]E[\vec{X}]^T}_{\Sigma = Var[\vec{X}]}\Big)A^T = A\Sigma A^T$$

$\vec{X} = A\vec{Z}$, $Var[\vec{X}] = A I_n A^T = AA^T$ Conjecture: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{X} = A\vec{Z} + \vec{\mu}$, $A \in \mathbb{R}^{n\times n}$, $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} \sim f_{\vec{X}}(\vec{x}) = ?$

$\phantom{\vec{X}=}$ ″ $g(\vec{Z})$, $h(\vec{X}) = \vec{Z}$ where hopefully g, h are inverses

$\vec{Z}$, $h(\vec{X}) = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{\mu})$ $\Rightarrow$ in order for the inverse to exist... A has to be invertible.

$$= B\vec{X} - B\vec{\mu} = \begin{bmatrix} \vec{b}_{1\cdot}\cdot\vec{X} - \vec{b}_{1\cdot}\cdot\vec{\mu} \\ \vec{b}_{2\cdot}\cdot\vec{X} - \vec{b}_{2\cdot}\cdot\vec{\mu} \\ \vdots \\ \vec{b}_{n\cdot}\cdot\vec{X} - \vec{b}_{n\cdot}\cdot\vec{\mu} \end{bmatrix} \begin{matrix} = h_1 \\ = h_2 \\ \vdots \\ = h_n \end{matrix}$$

$DD^{-1} = I$
$\Rightarrow \det[DD^{-1}] = \det[I] = 1$
$\Rightarrow \det[D]\det[D^{-1}] = 1$

$$J_h = \det\begin{bmatrix} \partial h_1/\partial x_1 & \cdots & \partial h_1/\partial x_n \\ \vdots & & \\ \partial h_n/\partial x_1 & \cdots & \partial h_n/\partial x_n \end{bmatrix} = \det\overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B} = \det[A^{-1}] = \frac{1}{\det[A]}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_h| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\underbrace{(A^{-1}(\vec{x}-\vec{\mu}))^T}_{(\vec{x}-\vec{\mu})^T(A^{-1})^T}A^{-1}(\vec{x}-\vec{\mu})} \frac{1}{|\det[A]|}$$

$DD^{-1} = I \Rightarrow (DD^{-1})^T = I^T = I \Rightarrow (D^{-1})^T D^T = I \Rightarrow (D^{-1})^T = (D^T)^{-1}$, $(CD)^{-1} = D^{-1}C^{-1}$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\underbrace{(A^T)^{-1}A^{-1}}_{(AA^T)^{-1}}(\vec{x}-\vec{\mu})}$$

let $\Sigma = AA^T = Var[\vec{X}]$

$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T]$
$= \det[A]^2$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})} = N_n(\vec{\mu}, \Sigma)$$

you need Σ to be invertible

A little bit of multivariate characteristic functions:

$$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1 + \ldots + t_nX_n)}] = E[e^{it_1X_1} \cdot \ldots \cdot e^{it_nX_n}]$$

if $X_1, \ldots, X_n$ indep

$$\downarrow$$

$$= E[e^{it_1X_1}] \cdot \ldots \cdot E[e^{it_nX_n}] = \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) \cdot \ldots \cdot \phi_{X_n}(t_n)$$

(P0) $\phi_{\vec{X}}(\vec{0}) = E[e^{i\vec{0}^T\vec{X}}] = 1$

(P1) If two chf's are equal => the two rv's are equal in distribution

(P2) $\vec{Y} = A\vec{X} + \vec{b}$, $A \in \mathbb{R}^{m\times n}$, $\vec{b} \in \mathbb{R}^m$, $\vec{X}$ is dim. n. $\Rightarrow \vec{Y}$ is dim m

$$\phi_{\vec{Y}}(\vec{t}) := E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{i\underbrace{\vec{t}^TA}_{(A^T\vec{t})^T}\vec{X}} e^{i\vec{t}^T\vec{b}}] = e^{i\vec{t}^T\vec{b}} E[e^{i(A^T\vec{t})^T\vec{X}}]$$

$$= e^{i\vec{t}^T\vec{b}}\,\phi_{\vec{X}}(A^T\vec{t})$$

Let's derive the chf of the standard MVN

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n}\phi_Z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let's derive the chf of the general MVN $\vec{X} = A\vec{Z} + \vec{\mu} \sim N(\vec{\mu}, \overbrace{AA^T}^{\Sigma})$

$$\phi_{\vec{X}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{\mu}}\phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\underbrace{(A^T\vec{t})^TA^T\vec{t}}_{\vec{t}^T\underbrace{AA^T}_{\Sigma}\vec{t}}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{Y} = B\vec{X} + \vec{c} \sim ?$ $B \in \mathbb{R}^{m\times n}$, $\vec{c} \in \mathbb{R}^m$

$$\phi_{\vec{Y}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{c}}\phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}} e^{i\overbrace{(B^T\vec{t})^T}^{\vec{t}^TB}\vec{\mu} - \frac{1}{2}\overbrace{(B^T\vec{t})^T}^{\vec{t}^TB}\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(\vec{c}+B\vec{\mu}) - \frac{1}{2}\vec{t}^TB\Sigma B^T\vec{t}} \overset{(P1)}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{\mu}+\vec{c}, B\Sigma B^T)$$

(if $B\Sigma B^T$ is invertible)

Mahalanobis Distance

let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$. Consider $\underbrace{(\vec{X}-\vec{\mu})^T\Sigma^{-1}(\vec{X}-\vec{\mu})}_{\text{Mahalanobis Distance}} \sim ?$

$$= (\vec{X}-\vec{\mu})^T(A^{-1})^TA^{-1}(\vec{X}-\vec{\mu})$$

$$= (A^{-1}(\vec{X}-\vec{\mu}))^TA^{-1}(\vec{X}-\vec{\mu})$$

Recall: $\vec{Z} = A^{-1}(\vec{X}-\vec{\mu})$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^TA^{-1}$$

$$= \vec{Z}^T\vec{Z} \sim \chi^2_n$$

PC Mahalanobis discovered this in 1936. He was India's founding father of statistics and founded the Indian Institute of Statistics.

This is kind of like distance in R^n adjusted for all the depend-encies among the dimensions like a multivariate "z-score"

In one dimension $(x-\mu)(\sigma^2)^{-1}(x-\mu) = \frac{(x-\mu)^2}{\sigma^2} = \left(\frac{x-\mu}{\sigma}\right)^2 = z^2$