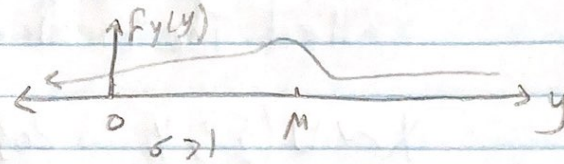
10/14

$X \sim$ Logestic $(0,1) \approx N(0,1)$ except it has thicker tails

standard logistic $E(X)=0$, $SD[X]=\frac{\pi}{\sqrt{3}}$

$$Y=\mu+\sigma X \sim \text{Logistic}(\mu,\sigma) := f_Y(y)=\frac{1}{\sigma}\frac{e^{\frac{y-\mu}{\sigma}}}{\left(e^{\frac{y-\mu}{\sigma}}+1\right)^2}$$

$\mu \in \mathbb{R}$, $\sigma > 0$




Why is this called the "logistic distribution"?

There's a function called the "logistic function" and it has 3 parameters = $L$(Maximum Value), $K$(steepness), mu(center)

$$L(X)=\frac{L}{1-e^{-K(X-\mu)}} = \frac{1}{1+e^{-X}} = \frac{e^X}{1+e^X}$$ the standard logistic function

if $L=1$, $K=1$, $\mu=0$

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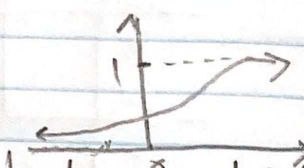
$$X \sim \text{Logistic}(0,1) = \frac{e^y}{(1+e^y)^2}$$




$$F_Y(y)=P(Y\le y)=\int_{-\infty}^{y}\frac{e^t}{(1+e^t)^2}\,dt = \int_1^{1+e^y}\frac{u-1}{u^2}\frac{1}{u-1}\,du = \left[-\frac{1}{u}\right]_1^{1+e^y} =$$

$e^t = u-1$

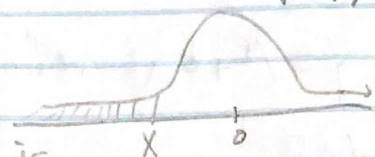
let $u=1+e^t \Rightarrow \frac{du}{dt}=e^t \Rightarrow dt=e^{-t}du=\frac{1}{u-1}du$, $t=-\infty \Rightarrow u=1$, $t=y \Rightarrow u=1+e^y$

$$1-\frac{1}{1+e^y}=\frac{e^y}{1+e^y}$$

The $q$th "quantile" or $100q$ "percentile" of a rv $X$.

distribution. Definition:
Solve for $x$ where $q \geq P(X \leq x) = F_X(x)$ denoted $Q[X, q]$.

I want the 33rd %ile




*If $q = 0.5$, that quantile is called the "median", $Med[X] := t$

$X \sim U(\{2, 4, \ldots 20\})$

| $x$ | $P(x)$ | $F(x)$ |
|---|---|---|
| 2 | 0.1 | 0.1 |
| 4 | 0.1 | 0.2 |
| 6 | | 0.3 |
| 8 | | 0.4 |
| 10 | | 0.5 |
| 12 | | 0.6 |
| 14 | | 0.7 |
| 16 | | 0.8 |
| 18 | | 0.9 |
| 20 | | 1.0 |

$Q[X, 0.3] = 6$

$Q[X, 0.9] = 18$

$Q[X, 0.85] = 16 \quad \nexists F(x) = 0.85$

If $X$ is a continuous rv with "contiguous support"

i.e. one interval with no gaps e.g. $[0, 10]$, the

real numbers but not e.g. $[0,1]$ union $[2,3]$

where there is a gap between 1 and 2, then
$F(x)$ is strictly increasing thus invertible and
the minimum $x$ s.t. $q \geq F(x)$ would be

$F^{-1}(q) \geq X \Rightarrow X = F^{-1}(q) = Q[X, q]$

$X \sim EXP(\lambda) = \lambda e^{-\lambda x} \Rightarrow F(X) = 1 - e^{-\lambda X}$ find the q function

$F^{-1}(q)$

$q = 1 - e^{-\lambda X} \Rightarrow 1 - q = e^{-\lambda X} \Rightarrow \ln(1-q) = -\lambda X \Rightarrow X = -\frac{1}{\lambda}\ln(1-q)$

$\Rightarrow X = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right) = F^{-1}(q)$

$X \sim EXP(1) \Rightarrow Med[X] = F^{-1}(0.5) = \ln(2)$

$Q[X, 0.8] = \ln(5)$

It's actually rare to have a quantile function in closed form since it's rare to even have a CDF in closed from e.g.

$X \sim Erlang(k, \lambda)$, $F(x) = P(k, \lambda x)$ .. $Q[X, q]$ can be found by solving for $X$ in the following equation: $q = P(k, \lambda x)$

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$X \sim EXP(\lambda)$, $Y = \underbrace{ke^{X}}_{= g(x)}$, $k > 0$ find $f_Y(y)$.

$\frac{y}{k} = e^{x} \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k) = g^{-1}(y)$, since $y$ is almos pos.

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|} \stackrel{\downarrow}{=} \frac{1}{y}$

$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| = \lambda e^{-\lambda \ln(y/k)} \frac{1}{y} \mathbb{1}_{\ln(y) - \ln(k) \in [0, \infty)}$

$= \frac{\lambda}{y} e^{\ln(\frac{y}{k})^{-\lambda}} \mathbb{1}_{\ln(y) \in [\ln(k), \infty)} = \frac{\lambda}{y}\left(\frac{y}{k}\right)^{-\lambda} \mathbb{1}_{y \in [k, \infty)}$

$\frac{1}{x^5} \to \frac{-1}{4x^4}$

$\Rightarrow$ Paredo I $(k, \alpha)$
$k \in (0,\infty),\ \alpha \in (0,\infty)$

$$F_Y(y) = \int_k^y \frac{\alpha k^\alpha}{t^{\alpha+1}}\,dt = \alpha k^\alpha \left[-\frac{1}{\alpha t^\alpha}\right]_k^y = k^\alpha\left(\frac{1}{k^\alpha} - \frac{1}{y^\alpha}\right)$$
$$= 1 - \left(\frac{k}{y}\right)^\alpha = q$$

$$F_y^{-1}(q) = k(1-q)^{-\frac{1}{\alpha}}$$

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$X, Y \overset{iid}{\sim} \text{Exp}(1)$, let $D = X - Y = X + (\underbrace{-Y}_{Z})$
$\qquad \| \ e^{-x}\mathbb{1}_{x\in[0,\infty)}$

$$Z \sim f_Z(z) = e^{z}\,\mathbb{1}_{z\in(-\infty,0]}$$

$$f_D(d) = \int_{\text{Supp}[X]} f_X^{old}(x)\, f_Z^{old}(d-x)\,\mathbb{1}_{d-x\in \text{Supp}[Z]}$$

$$= \int_0^\infty e^{-x}e^{d-x}\,\mathbb{1}_{\overset{x\in[d,\infty)}{\overbrace{d-x\in(-\infty,0)}^{x-d\in[0,\infty)}}} = e^d\int_0^\infty e^{-2x}\,\mathbb{1}_{x\in[d,\infty)}\,dx$$

$$= e^d \begin{cases} \int_d^\infty e^{-2x}\,dx & \text{if } d \geq 0 \\ \int_0^\infty e^{-2x}\,dx & \text{if } d < 0 \end{cases} = e^d \begin{cases} \left[-\frac{1}{2}e^{-2x}\right]_d^\infty & \text{if } d \geq 0 \\ \left[\frac{-1}{2}e^{-2x}\right]_0^\infty & d < 0 \end{cases}$$

$$= \frac{1}{2}e^d \begin{cases} e^{-2d} & \text{if } d \geq 0 \\ 1 & \text{if } d < 0 \end{cases} = \begin{cases} \frac{1}{2}e^{-d} & \text{if } d \geq 0 \\ \frac{1}{2}e^{d} & d < 0 \end{cases}$$

$$= \frac{1}{2}e^{-|d|}$$
$\qquad\qquad \|$
Laplace $(0,1)$
Standard Laplace

MGR, 670

$$X \overset{d}{=} \mu + \sigma D \sim \text{Laplace}(\mu, \sigma) := \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

this is also famous rv and it has another name: the "double exponential".

Laplace published this distribution in 1774 calling it the "first law of errors".