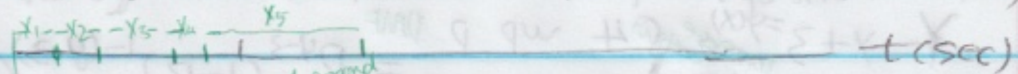
Lecture 10

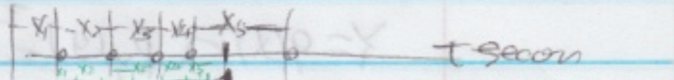
$T_k \sim \text{Erlang}(k, \lambda)$ $\qquad N \sim \text{Poisson}(\lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k, \lambda)$ $\qquad F_N(x) = Q(x+1, \lambda)$



$X_1, X_2 \overset{iid}{\sim} \text{Exp}(\lambda)$ $\quad k = 5$ takes more than 1 second

$\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\}$



waiting time for 5th λt $\quad \cup \{X_1 + X_2 + X_3 < 1\} \cup \{X_1 + X_2 < 1\} \cup \{X_1 < 1\}$

$\cup \{X_1 > 1\} \overset{\text{set also be}}{=} \{N=4\} \cup \{N=3\} \cup \{N=2\} \cup \{N=1\} \cup \{N=0\}$

Let N: number events before 1 sec $\Rightarrow P\{T_5 > 1\} = P(N \le 4) = F_N(4)$

$\Rightarrow 1 - F_{T_5}(1) =$

wait for k things

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$ "Poisson Process"

N: # event before 1 second

Why is this poisson discrete?

Geometric: n experiment $\to \infty$ $\quad$ n experiment $\to \infty$ $\to t$ (sec)

$p \to 0 \quad \lambda = np$ $\qquad p \to 0 \quad \lambda = np$

waiting time $\quad T \sim \text{Erlang}(k, \lambda) = \dfrac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0} \quad k \in \mathbb{N}, \lambda \in (0, \infty)$

$= \dfrac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$

$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} \quad k \in \mathbb{N}, p \in (0,1)$

wait for geometric time $\quad = \dfrac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

For Both, what if $k \in (0, \infty)$? Are Both rv's still "legal"

Yes, we can show that Both $\displaystyle\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t}\, dt = 1$ &

$\displaystyle\sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1$

We just derived two new famous rv's

$X \sim \text{Gamma}(\alpha, \beta) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x} \mathbb{1}_{x \ge 0}$

waiting α number,

$X \sim \text{ExtNegBin}(k, p) = \dfrac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

the "extended Negative Binomial"

Transformations of Discrete rv's

$X \sim Bern(p) = p^x(1-p)^{1-x} \mathbb{1}_{x\in\{0,1\}} = \begin{cases} 1 & wp\ p \\ 0 & wp\ 1-p \end{cases}$

$Y = X+3 = g(X)$ $\begin{cases} 4 & wp\ p \\ 3 & wp\ 1-p \end{cases}$ $= p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y-3\in\{0,1\}}$ PMF, $y \in \{3,4\}$

inverse function: $x = Y-3 = g^{-1}(y)$

I want to find PMF of Y using PMF of X: $P_X(y-3)$

$Y = g(X) \sim P_Y(y) = P_X(g^{-1}(y))$

What assumption did I make when I "derived" this formula

I assumed an inverse function exists. i.e g is invertible

If not...

$X \sim U(\{1,2,\ldots 10\}) = \begin{cases} 1 & wp\ \frac{1}{10} \\ 2 & wp\ \frac{1}{10} \\ 10 & wp\ \frac{1}{10} \end{cases}$ $Y = g(X) = \min\{X,3\} \sim \begin{cases} 1 & wp\ \frac{1}{10} \\ 2 & wp\ \frac{1}{10} \\ 3 & wp\ \frac{8}{10}\ P(X=3)+P(X=4)+P(X=5)\cdots+P(X=10) \end{cases}$

$Y = g(X) \sim P_Y(y) = \sum_{\{x:\, y=g(x)\}} P_X(x) \overset{\text{if } g \text{ invertable on Supp}[X]}{=\!=} \sum_{\{x:\, x=g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$

one element only

$X \sim Binom(n,p)$ $Y = X^3 = g(x) \sim P_X(\sqrt[3]{y}) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y}\in\{0,1,\ldots n\}}$

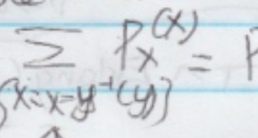
$= g^{-1}(y) = \sqrt[3]{y}$

$Y = X^2 \sim P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y}\in\{0,1,\ldots\}}$

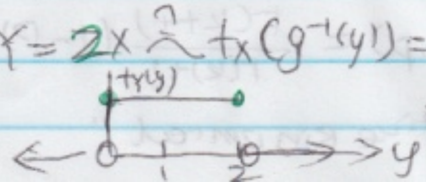
Transformations for continuous rv's

for g invariable $f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ NO!

let $X \sim U(0,1) = \mathbb{1}_{x\in[0,1]} = f_X(x)$ $\int_{\mathbb{R}} f_X(x)dx = 1$



$Y = 2X \overset{?}{\sim} f_X(g^{-1}(y)) = f_X(\frac{y}{2}) = \mathbb{1}_{\frac{y}{2}\in[0,1]} = \mathbb{1}_{y\in[0,2]}$



$\int_{\mathbb{R}} f_Y(y)dy = 2$ Game over

PDF are not probabilities! ($=1$) So this was bound to Fail because we used them as probabilities. However, CDF's are probabilities

CDF F_Y, 1-1 Function

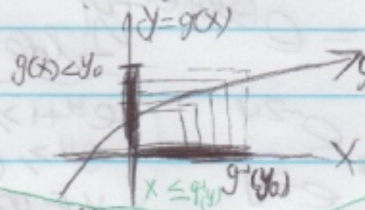
$$F_Y(y) := P(Y \le y) = P(g(X) \le y) \overset{g \text{ is invertible}}{=} P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

1-1 Function

$$\Rightarrow \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

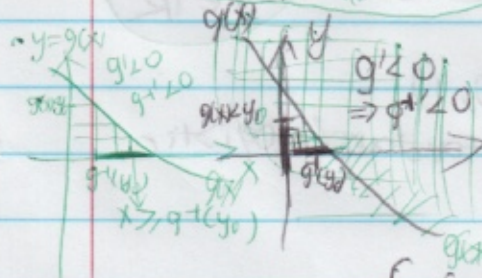
$$\underbrace{= f_X(g^{-1}(y))\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{always positive} \ge 0}}_{= f_Y(y)}$$

Proof



$g' > 0 \Rightarrow g^{-1\prime} > 0$

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) \overset{g \text{ is invertible and } g' < 0}{=} P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$



$g' < 0 \Rightarrow g^{-1\prime} < 0$

$$\underset{f_Y(y)}{\frac{d}{dy}[F_Y(y)]} = \frac{d}{dy}[1 - F_X(g^{-1}(X))]$$

$$= f_X(g^{-1}(y))\underbrace{\left(-\frac{d}{dy}[g^{-1}(y)]\right)}_{\text{alway negative} < 0}$$

Rules $F_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$ absolute value — general Rules

We can derive a less general but very useful corrollary rule:

$Y = aX + c \wedge f_Y(y) = ?$ ($a, c \in \mathbb{R}$) shift and scale (shift by c, scale by a)

$\underbrace{g(x)}_{\text{Straight line}}$ is invertible $\Rightarrow g^{-1}(y) = \frac{y-c}{a}$ $\quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{a}\right| = \frac{1}{|a|}$

PDF $f_Y(y) = f_X\left(\frac{y-c}{a}\right)\cdot\frac{1}{|a|}$

$Y = aX \sim f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$, $Y = X + c \sim f_X(y-c)$

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$X \sim Exp(1) = e^{-x}\mathbb{1}_{x \geq 0}$

invertable

$Y = g(x) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = g(x)$

$\sim f_Y(y)$

$y = \ln(e^{x}-1) \Rightarrow e^{y} = e^{x}-1 \Rightarrow e^{y}+1 = e^{x} = x = \ln(e^{y}+1)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\ln(e^{y}+1)]\right| = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1} \quad (= g^{-1}(y))$

$f_Y(y) = f_X(\ln(e^{y}+1)) \cdot \frac{e^{y}}{e^{y}+1} = e^{-\ln(e^{y}+1)}\mathbb{1}_{\ln(e^{y}+1) \geq 0} \cdot \frac{e^{y}}{e^{y}+1}$

$e^{y}+1 \geq 1$
$e^{y} \geq 0$
$y \in \mathbb{R}$

$= \frac{1}{e^{y}+1}\,\frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{e^{-2y}}{e^{-2y}}$

$= \frac{e^{-y}}{(e^{-y}+1)^2} =$ Logistic $(0,1)$

standard logistic