Monday November 23 2020

Lecture 20

$$\frac{\bar{X}}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S\cdot\frac{S^2}{n-1}\cdot\frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\left.\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right\} Z}{\left.\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}\right\} \chi^2_{n-1}} \sim T_{n-1}$$

Due to Cochran's thm we know Xbar and S^2 are independent.

"New Unit" Multivariate Normal Distribution (MVN)

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$. $\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}_n$, $Var[\vec{Z}] = I_n$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

Standard MVN $= N_n(\vec{0}, I)$

$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$. $E[\vec{X}] = \vec{\mu}$, $Var[\vec{X}] = I_n$

$\Rightarrow \vec{X} \sim N_n(\vec{\mu}, I)$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} z_1 \\ z_1 + z_2 \\ \vdots \\ z_1 + \dots + z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$$

but the component are dependent

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$\begin{aligned} Cov[X_1, X_2] &= Cov[Z_1, Z_1 + Z_2] \\ &= Cov[Z_1, Z_1] + Cov[Z_1, Z_2] \\ &= 1 \Rightarrow X_1, X_2 \text{ dependent} \end{aligned}$$

Let's derive a general formula for the variance-covariance matrix of A (an $n \times n$ matrix of scalar) times a random vector x of dim n:

$$Var[A\vec{X}] := E[\underbrace{(A\vec{X})(A\vec{X})^T}_{A\vec{X}\vec{X}^TA^T}] - E[A\vec{X}]E[A\vec{X}]^T$$

$$= A\, E[XX^T] A^T - A E[\vec{x}] \underbrace{(A\, E[\vec{x}])^T}_{E[X]^T A^T}$$

(Home Work)

$$= A \big(\underbrace{E[XX^T] - E[\vec{x}] E[\vec{x}]^T}_{\Sigma = Var[\vec{x}]} \big) A^T = A \Sigma A^T$$

$\vec{X} = A\vec{Z}$, $Var[\vec{x}] = A I_n A^T = AA^T$ Conjucture: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{x} = A\vec{z} + \vec{u}$, $A \in \mathbb{R}^{n \times n}$. $\vec{u} \in \mathbb{R}^n$, $\vec{x} \sim f_{\vec{x}}[\vec{x}] = ?$

$\quad = g(\vec{z})$, $h(\vec{x}) = \vec{z}$ where g, h are inverses

$h(\vec{x}) = A^{-1}(\vec{x} - \vec{u}) \Rightarrow$ In order for the inverse function to exist... A has to be invertible.

$$= B\vec{x} - B\vec{u} = \begin{bmatrix} \vec{b}_1 \cdot \vec{x} - \vec{b}_1 \cdot \vec{u} \\ \vec{b}_2 \cdot \vec{x} - \vec{b}_2 \cdot \vec{u} \\ \vdots \\ \vec{b}_n \cdot \vec{x} - \vec{b}_n \cdot \vec{u} \end{bmatrix} \begin{matrix} = h_1 \\ = h_2 \\ \vdots \\ = h_n \end{matrix}$$

$$J_n = \det \begin{bmatrix} \partial h_1/\partial x_1 & \cdots & \partial h_1/\partial x_n \\ \vdots & & \\ \partial h_n/\partial x_1 & \cdots & \partial h_n/\partial x_n \end{bmatrix} = \det \overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B} = \det[A^{-1}] = \frac{1}{\det[A]}$$

$DD^{-1} = I$

$\Rightarrow \det[DD^{-1}] = \det[I] = 1$

$\Rightarrow \det[D] \det[D^{-1}] = 1$

$$f_{\vec{x}}(\vec{x}) = f_{\vec{z}}(h(\vec{x})) |J_n| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2} \underbrace{(A^{-1}(\vec{x}-\vec{u}))^T A^{-1}(\vec{x}-\vec{u})}_{(\vec{x}-\vec{u})^T (A^{-1})^T A^{-1} (\vec{x}-\vec{u})}} \frac{1}{|\det[A]|}$$

$DD^{-1} = I \Rightarrow (DD^{-1})^T = I^T = I \Rightarrow (D^{-1})^T D^T = I \Rightarrow (D^{-1})^T = (D^T)^{-1}$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{u})^T \underbrace{(A^T)^{-1} A^{-1}}_{(AA^T)^{-1}} (\vec{x}-\vec{u})}, \quad (CD)^{-1} = D^{-1}C^{-1}$$

let $\Sigma = AA^T = Var[\vec{X}]$

$\det[\Sigma] = \det[AA^T]$

$= \det[A] \det[A^T]$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{u})^T \Sigma^{-1} (\vec{x}-\vec{u})} = N_n(\vec{u}, \Sigma)$$

you need Σ to be invertible.

A little bit of multivariate characteristic functions:

$$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{x}}] = E[e^{i(t_1x_1+\cdots+t_nx_n)}] = E[e^{it_1x_1} \cdot \ldots \cdot e^{it_nx_n}]$$

If $x_1, \ldots, x_n$ iid

$$= E[e^{it_1x_1}] \cdot \ldots \cdot E[e^{it_nx_n}] = \phi_{x_1}(t_1)\phi_{x_2}(t_2) \cdot \ldots \cdot \phi_{x_n}(t_n)$$

P_0 $\phi_{\vec{X}}(\vec{0}) = E[e^{i\vec{0}^T\vec{x}}] = 1$

P_1 If two chf's are equal $\Rightarrow$ the two rv's are equal in distribution

P_2 $\vec{Y} = A\vec{X} + \vec{b}$. $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$, $\vec{X}$ is dim n. $\Rightarrow \vec{Y}$ is dim m

$$\phi_{\vec{Y}}(\vec{t}) := E[e^{i\vec{t}^T(A\vec{x}+\vec{b})}] = E[e^{i\underbrace{\vec{t}^T A}_{(A^T\vec{t})^T}\vec{x}} e^{i\vec{t}^T\vec{b}}] = e^{i\vec{t}^T\vec{b}} E[e^{i(A^T\vec{t})^T\vec{x}}]$$

$$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{X}}(A^T\vec{t})$$

Let's derive the chf of the standard MVN

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n} \phi_Z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let's derive the chf of the general MVN $\vec{X} = A\vec{Z} + \vec{\mu} \sim N(\vec{\mu}, \overbrace{AA^T}^{\Sigma})$

$$\phi_{\vec{X}}(\vec{t}) \overset{P_2}{=} e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\underbrace{(A^T\vec{t})^T A^T\vec{t}}_{\vec{t}^T \underbrace{AA^T}_{\Sigma}\vec{t}}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{Y} = B\vec{X} + \vec{c} \sim ?$ $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}$

$$\phi_{\vec{Y}}(\vec{t}) \overset{P_2}{=} e^{i\vec{t}^T\vec{c}} \phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}} e^{i\underbrace{(B^T\vec{t})^T}_{t^TB}\vec{\mu} - \frac{1}{2}\underbrace{(B^T\vec{t})^T}_{t^TB}\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(\vec{c}+B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{P_1}{\Longrightarrow} \vec{Y} \sim N(B\vec{\mu}+\vec{c}, B\Sigma B^T)$$

if $B\Sigma B^T$ is invertible

Let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$. Consider $(\vec{X}-\vec{\mu})^T \Sigma^{-1}(\vec{X}-\vec{\mu}) \sim ?$

$$= (\vec{X}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{X}-\vec{\mu})$$

Recall: $\vec{Z} = A^{-1}(\vec{X}-\vec{\mu})$

$$= (A^{-1}(\vec{X}-\vec{\mu}))^T A^{-1}(\vec{X}-\vec{\mu})$$

$$= \vec{Z}^T\vec{Z} \sim \chi^2_n$$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$$

PC Mahalanobis discovered this in 1936. He was India's founding father of statistics and founded the Indian Institute of Statistics.

This is kind of like distance in $\mathbb{R}^n$ adjusted for all the dependencies among the dimensions like a multivariate "z-score."

In one dimension $(x-\mu)(\sigma^2)^{-1}(x-\mu) = \dfrac{(x-\mu)^2}{\sigma^2} = \left(\dfrac{x-\mu}{\sigma}\right)^2 = z^2$