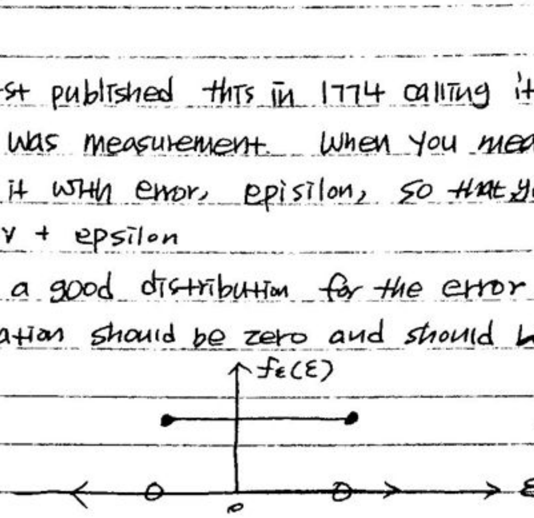
Meeting 12

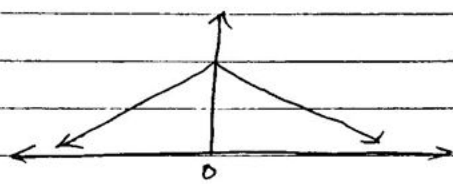
Laplace first published this in 1774 calling it the "first law of errors"
His context was measurement. When you measure a quantity v,
You measure it with error, episilon, so that your measurement is:

$$M = v + \text{epsilon}$$

What makes a good distribution for the error, epsilon?
The expectation should be zero and should be symmetric.
How about



→ This is not very good.
It should have the property that the probability of error should decrease with its magnitude.



Also, why should it stop at some maximum magnitude?

Another good property is that the density should be decreasing in magnitude of error

Laplace assumed for all positive errors that $f_\varepsilon''(\varepsilon) = f_\varepsilon'(\varepsilon)$

$$\Rightarrow f(\varepsilon) = ce^{-d\varepsilon} \Rightarrow \varepsilon \sim \text{Laplace}(0,1)$$

$X \sim \text{Exp}(1) = e^{-x}\mathbb{1}_{x\geq 0}$. $\quad Y = g(X) = \frac{1}{\lambda}X^{\frac{1}{k}} \sim f_Y(y) = ?$ s.t. $k, \lambda > 0$

$$Y = \frac{1}{\lambda}x^{\frac{1}{k}} \Rightarrow \lambda y = x^{\frac{1}{k}} \Rightarrow x = (\lambda y)^k = \lambda^k y^k = g^{-1}(y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\lambda^k y^k]\right| = k\lambda^k y^{k-1} = k\lambda^k |y^{k-1}| = k\lambda^k y^{k-1}$$

$$f_Y(y) = f_X(g^{-1}(x))\left|\frac{d}{dy}[g^{-1}(y)]\right| = e^{-(\lambda y)^k}\mathbb{1}_{\lambda^k y^k \geq 0} \cdot k\lambda^k y^{k-1}$$

$$= k\lambda^k y^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y\geq 0} \qquad y^k \geq 0 \text{ and } y \geq 0$$

$$= k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y\geq 0} = \text{Weibull}(k, \lambda) \qquad \underline{1951}$$

this is a very famous waiting time / survival rv model and it's used e.g. in insurance companies to price life insurance

$\text{Weibull}(1,\lambda) = (1)\lambda(\lambda y)^{(1)-1} e^{-(\lambda y)^{(1)}} \mathbb{1}_{y\geq 0}$

$= \lambda e^{-\lambda y} \mathbb{1}_{y\geq 0} = \text{Exp}(\lambda)$

Property of Weibull rv's under different values of k:

$k=1$, $P(Y \geq y+c \mid Y \geq c) = P(Y \geq y)$

this equality is called "memorylessness"

$k>1$, $P(Y \geq y+c \mid Y \geq c) < P(Y \geq y)$ e.g. [old lifespan of humans, waiting for bus

$k<1$, $P(Y \geq y+c \mid Y \geq c) > P(Y \geq y)$ e.g startup company lifespans

Order Statistics (p.160) Let $X_1, X_2, \dots, X_n$ be a collection of continuous rv.'s.
Let the "order statistics" be the rv's:

$X_{(1)} := \min\{X_1, X_2, \dots, X_n\}$

$\vdots$

$X_{(k)} := k^{th}$ largest of $X_1, \dots, X_n$

$\vdots$

$X_{(n)} := \max\{X_1, \dots, X_n\}$

$R := X_{(n)} - X_{(1)}$ range

e.g. $X_1 = 9, X_2 = 2, X_3 = 12, X_4 = 7$

$X_{(1)} = 2, X_{(2)} = 7, X_{(3)} = 9,$

$X_{(4)} = 12$

$r = 12 - 2 = 10$

We want to find both the CDF and PDF of the kth order statistic.
We'll build this up in stages. The $\boxed{\text{first}}$ thing we'll do is find the CDF and PDF of the maximum.

$$F_{X_{(n)}}(x) = P(\underbrace{X_{(n)} \leq x}_{\text{event}}) = P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x)$$

$$\stackrel{\text{ind}}{=} P(X_1 \leq x) \cdots P(X_n \leq x)$$

$$= \prod_{i=1}^{n} F_{X_i}(x) \stackrel{\text{iid}}{=} [F_X(x)]^n = F(x)^n$$

$$f_{X_n}(x) \stackrel{\text{iid}}{=} \frac{d}{dx}[F_X(x)] = \frac{d}{dx}[F(x)^n] = nf(x)F(x)^{n-1}$$

$\boxed{\text{Next}}$ thing we'll do is to find the CDF and PDF of the minimum

$$F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$\stackrel{\text{ind}}{=} 1 - \prod_{i=1}^{n}(1 - F_{X_i}(x)) \stackrel{\text{iid}}{=} 1 - (1 - F(x))^n$$

$$f_{X_{(1)}}(x) \stackrel{\text{iid}}{=} \frac{d}{dx}[1-(1-F(x))^n] = nf(x)(1-F(x))^{n-1}$$

$\boxed{\text{Next}}$ we'll do is assume $n=10$ and derive the $k = 4^{th}$ order statistic's CDF and PDF. First, let's find the prob. that the first four #s are less than x and the last 6 numbers are ~~greater than~~ x.

$$= P(X_1 \leq x, \dots, X_4 \leq x, X_5 > x, \dots, X_{10} > x)$$

$$\stackrel{\text{ind}}{=} \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{6} 1 - F_{X_i}(x) \stackrel{\text{iid}}{=} F(x)^4 (1 - F(x))^6$$

Let's find the probability any 4 of the 10 are below x and the remaining are below x. Let S be a subset of size 4 of the index set $\{1, 2, \cdots, 10\}$ $= \sum_{all\ S} P(X_{S_1} \le x, \cdots, X_{S_4} \le x, X_{S_1^c} > x, \cdots, X_{S_6^c} > x) \overset{ind}{=} \sum_{all\ S} \prod_{i=1}^{4} F_{X_{S_i}} \prod_{i=1}^{6} (1 - F_{X_{S_i^c}}(x))$

$\overset{iid}{=} \sum_{all\ S} F(x)^4 (1 - F(x))^6$

$= \binom{10}{4} F(x)^4 \cdot (1 - F(x))^6$

Now let's derive the CDF for the $k = 4$th order statistic.

$F_{X_{(4)}}(x) = P(\underbrace{X_{(4)} \le x}_{event}) = P(\text{a subset of } 4\ X_i\text{'s} \le x \text{ and the remaining } 6 \text{ are} > x)$

$+ P(\text{a subset of } 5\ X_i\text{'s} \le x \text{ and the remaining } 5 \text{ are} > x)$

$+ \cdots + P(\text{all } 10\ X_i\text{'s are less than/equal to } x)$

$\overset{iid}{=} \sum_{j=4}^{10} \binom{10}{4} F(x)^4 (1 - F(x))^{10-4}$

For iid continuous rv's $X_1, \cdots, X_n$, the CDF and PDF for the kth order stat. is: $F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}$

$f_{X_{(k)}}(x) = \frac{d}{dx} \left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \right]$

$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx} \left[\underbrace{F(x)^j}_{u} \underbrace{(1 - F(x))^{n-j}}_{v} \right]$

$u' = j f(x) F(x)^{j-1}$, $v' = -(n-j) f(x) (1 - F(x))^{n-j-1}$

$\frac{d}{dx}[uv] = uv' + u'v$.