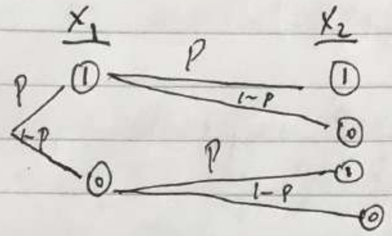
Monday August 31st 2020

Lecture 2

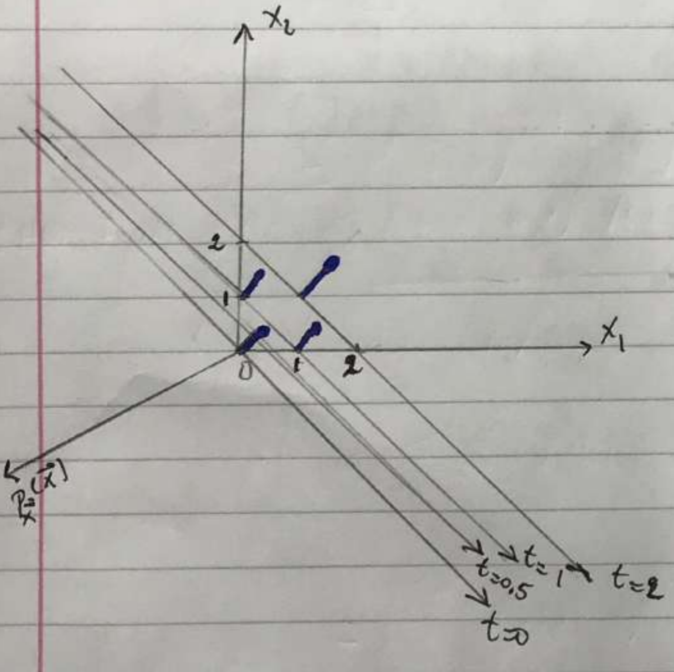


$P_{X_1,X_2}(x_1,x_2)$	T
p^2	1
$p(1-p)$	1
$(1-p)p$	1
p^2	0

$\Sigma = 1$

mutually exclusive Collectively exhaustive events

$$P_T(t) = P(T_2 = t) = \begin{cases} 2 & \text{w.p } p^2 \\ 1 & \text{w.p } 2p(1-p) \\ 0 & \text{w.p } (1-p)^2 \end{cases}$$



slope = -1

$t = x_1 + x_2 \Rightarrow x_2 = t - x_1$

$P(T=0) \rightarrow x_2 = 0 - x_1 = x_1$ select events

$$P_T(t) = \underbrace{\sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}}}_{\text{Surch through } \mathbb{R}^2} \underbrace{P_{X_1,X_2}(x_1,x_2)}_{\text{add all Prob}} \underbrace{\mathbb{1}_{x_1+x_2=t}}_{x_2 = t - x_1}$$

$$= \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1, t - x_1)$$

$$= \sum_{x \in \mathbb{R}} P_{X_1,X_2}(x, t-x)$$

If X_1, X_2 independent

$$= \sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x) = \sum_{x \in \mathbb{R}} P^{old}_{X_1}(x) \mathbb{1}_{x \in Supp[X_1]} P^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$

$$= \sum_{x \in Supp[X_1]} P^{old}_{X_1}(x) P^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$

Convolution formula for independent r.v.s.

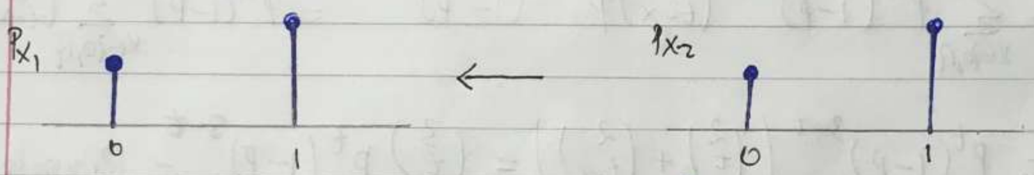
If $X_1, X_2 \overset{iid}{\sim}$

$$= \sum_{x \in \mathbb{R}} P(x)\, P(t-x) = \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in Supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in Supp[X]} P(x)\, P(t-x)\, \mathbb{1}_{t-x \in supp[X]}$$ Convolution formula for iid r.vs

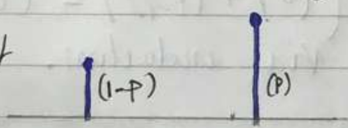
"Convolve" means to "roll, coil or entwine together"

$$P_T = P_{X_1} * P_{X_2}$$
↑ convolution operator

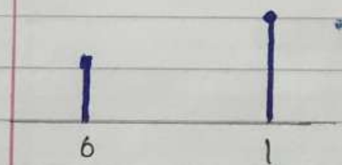


Sum-product

roll them together we get

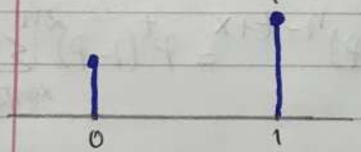


$\Rightarrow t=1$ w.p $p(1-p)$



$\Rightarrow t=0$ w.p $(1-p)^2$

$t=2$ w.p p^2



$\Rightarrow t=1$ w.p $(1-p)\,p$

$$P_{T_2}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x}\, p^{t-x} (1-p)^{1-t+x}\, \mathbb{1}_{\underbrace{t-x \in \{0,1\}}_{t \in \{x, x+1\}}} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \big(\underbrace{\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}}}_{= \binom{2}{t}} \big)$$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0,1,\ldots,n\}}$$

$$T_2 \sim \begin{cases} 0 & \text{w.p } (1-p)^2 \\ 1 & \text{w.p } 2p(1-p) \\ 2 & \text{w.p } p^2 \end{cases}$$

$$= \binom{2}{t} p^t (1-p)^{2-t} = Binom(2, p)$$

$$\mathbb{1}_{x\in\{0,1\}}$$

$$P_{T_2}(t) = \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x} = p^t (1-p)^{2-t} \sum_{x\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \sum_{x\in\{0,1\}} \binom{1}{t-x} = p^t (1-p)^{2-t}\left(\binom{1}{t} + \binom{1}{t-1}\right) = \binom{2}{t} p^t (1-p)^{2-t}$$

Pascal identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(P)$ $T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$

We use the independent formula

$$P_{T_3}(t) = \sum_{x\in\{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x} = p^t (1-p)^{3-t} \sum_{x\in\{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t}\left(\binom{2}{t} + \binom{2}{t-1}\right) = \binom{3}{t} p^t (1-p)^{3-t} = Binom(3,P)$$

HW Find PMF of Binom(n,P) via induction.

$X_1\ X_2 \overset{iid}{\sim} Binomial(n,P)$ $T = X_1 + X_2 \sim ?$

$$P_T(t) = \sum_{x\in\mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x} = p^t (1-p)^{2n-t} \sum_{x\in\mathbb{R}} \binom{n}{x}\binom{n}{t-x}$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t} = Binom(2n, P)$$

↑ Vandermonde's Identity