Lecture 20

$$\frac{\bar X - \mu}{\frac{S}{\sqrt n}} = \frac{\bar X-\mu}{\frac{1}{\sqrt n}\sqrt{S^2\frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}}} = \frac{\bar X-\mu}{\frac{\sigma}{\sqrt n}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\frac{\bar x-\mu}{\frac{\sigma}{\sqrt n}}}{\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}}$$

$\sim T_{n-1}$ Due to cochran's thm, we know xbar and S^2 are independent.

Multivariate Normal Distribution (MVN)

$Z_1,\cdots,Z_n \overset{iid}{\sim} N(0,1)$, $\vec Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n\end{bmatrix}$, $E[\vec Z]=\vec 0_n$, $Var[\vec Z] = I_n$

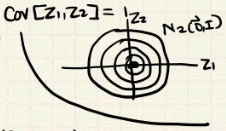
$\vec Z \sim f_{\vec Z}(\vec z) = \prod_{i=1}^n f_Z(z_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2}$

$= \frac{1}{(2\pi)^{n/2}} e^{-\frac12 \sum z_i^2} = \frac{1}{(2\pi)^{\frac n2}} e^{-\frac12 \vec z^T\vec z} = N_n(\vec 0, I)$

$\vec X = \vec Z + \vec\mu$, $\vec\mu\in\mathbb R^n$, $E[\vec X]=\vec\mu$, $Var[\vec X] = I_n \Rightarrow \vec X\sim N_n(\vec\mu, I)$

$\vec X = A\vec Z = \begin{bmatrix} Z_1\\ Z_1+Z_2 \\ \vdots \\ Z_1+\cdots+Z_n\end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n)\end{matrix}$ but the component are dependent

e.g. $Cov[X_1,X_2] = Cov[Z_1, Z_1+Z_2]$
$= Cov[Z_1,Z_1]+Cov[Z_1,Z_2] = 1$
$\Rightarrow X_1, X_2$ dep



Lets derive a general formula for the variance-covariance matrix of A (an nxn matrix of scalars) times a random vector X of dim n: $Var[A\vec X] := E[(A\vec X)(A\vec X)^T] - E[A\vec X]E[A\vec X]^T$

$= A\cdot E[XX^T]A^T - AE[\vec X](AE[\vec X])^T$

$= A(E[XX^T] - E[\vec X]E[\vec X])A^T = A\Sigma A^T$

$\vec X = A\vec Z$, $Var[\vec X] = AI_nA^T = AA^T$ Conjecture: $\vec X\sim N(\vec 0, AA^T)$

$\vec X = A\vec Z+\vec\mu$, $A\in\mathbb R^{n\times n}$, $\vec\mu\in\mathbb R^n$, $\vec X\sim f_{\vec X}(\vec x)=?$

$\overset{\parallel}{g(\vec z)}$, $h(\vec x)=\vec z$ where hopefully g, h are inverses

$\vec z = h(\vec x) = \underbrace{A^{-1}}_{B}(\vec x-\vec\mu) \Rightarrow$ in order for the inverse to exist.. A has to be invertible.

$= B\vec x - B\vec\mu = \begin{bmatrix} \vec b_1\vec x - \vec b_1\vec\mu \\ \vdots \\ \vec b_n \vec x - \vec b_n\vec\mu\end{bmatrix} \begin{matrix} =h_1 \\ \vdots \\ =h_n\end{matrix}$

$DD^{-1}=I$
$\Rightarrow det[DD^{-1}] = det[I]$
$\Rightarrow det[D]det[D^{-1}]=1$

$J_n = det\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n}\end{bmatrix} = det\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{bmatrix} = det[A^{-1}] = \frac{1}{det[A]}$

$f_{\vec X}(\vec x) = f_{\vec Z}(h(\vec x))|J_n| = \frac{1}{(2\pi)^{\frac n2}} e^{-\frac12(A^{-1}(\vec x-\vec\mu))^T A^{-1}(\vec x-\vec\mu)}\frac{1}{|det[A]|}$

$= \frac{1}{\sqrt{(2\pi)^n det[A]^2}} e^{-\frac12(\vec x-\vec\mu)^T(A^T)^{-1}A^{-1}(\vec x-\vec\mu)}\frac{1}{|det[A]|}$

$= \frac{1}{\sqrt{(2\pi)^n det[A]^2}} e^{-\frac12(\vec x-\vec\mu)^T(A^T)^{-1}A^{-1}(\vec x-\vec\mu)}$

$= \frac{1}{\sqrt{(2\pi)^n det[\Sigma]}} e^{-\frac12(\vec x-\vec\mu)^T\Sigma^{-1}(\vec x-\vec\mu)} = N_n(\vec\mu,\Sigma)$

You need Σ to be invertible.

A little bit of multivariate characteristic functions:

$\phi_{\vec X}(\vec t) := E[e^{i\vec t^T\vec X}] = E[e^{i(t_1X_1+\cdots+t_nX_n)}] = E[e^{it_1X_1}\cdot\ldots\cdot e^{it_nX_n}]$

$\overset{ind}{=} E[e^{it_1X_1}]\cdot\ldots\cdot E[e^{it_nX_n}] = \phi_{X_1}(t_1)\phi_{X_2}(t_2)\cdots\phi_{X_n}(t_n)$

(P) $\phi_{\vec X}(\vec 0) = E[e^{i\vec 0^T\vec X}]=1$

(P) If two chf's are equal $\Rightarrow$ the two rv's are equal in dist.

(P) $\vec Y = A\vec X+\vec b$, $A\in\mathbb R^{m\times n}$, $\vec b\in\mathbb R^m$, $\vec X$ in dim n, $\Rightarrow \vec Y$ is dim m

$\phi_{\vec Y}(\vec t) = E[e^{i\vec t^T(A\vec X+\vec b)}] = E[e^{i\vec t^TA\vec X}e^{i\vec t^T\vec b}] = e^{i\vec t^T\vec b}E[e^{i(A^T\vec t)^T\vec X}]$

$= e^{i\vec t^T\vec b}\phi_{\vec X}(A^T\vec t)$

Let's derive the chf of the standard MVN

$\phi_{\vec Z}(\vec t) = \prod_{i=1}^n \phi_Z(t_i) = \prod_{i=1}^n e^{-t_i^2/2} = e^{-\frac12\sum t_i^2} = e^{-\frac12\vec t^T\vec t}$

Let's derive the chf of the general MVN

$\vec X = A\vec Z+\vec\mu \sim N(\vec\mu, \overset{\Sigma}{\widehat{AA^T}})$

$\phi_{\vec X}(\vec t) = e^{i\vec t^T\vec\mu}\phi_{\vec Z}(A^T\vec t) = e^{i\vec t^T\vec\mu}e^{-\frac12(A^T\vec t)^TA^T\vec t}$

$= e^{i\vec t^T\vec\mu - \frac12\vec t^T\Sigma\vec t}$

$\vec Y = B\vec X+\vec c \sim ?$ $B\in\mathbb R^{m\times n}$, $\vec c\in\mathbb R$

$\phi_{\vec Y}(\vec t) = e^{i\vec t^T\vec c}\phi_{\vec X}(B^T\vec t) = e^{i\vec t^T\vec c}e^{i(B^T\vec t)^T\vec\mu - \frac12(B^T\vec t)^T\Sigma(B^T\vec t)}$

$= e^{i\vec t^T(\vec c+B\vec\mu) - \frac12\vec t^TB\Sigma B^T\vec t}$

$= \vec Y\sim N_m(B\vec\mu+\vec c, B\Sigma B^T)$

(if $B\Sigma B^T$ is invertible)

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let $\vec X\sim N_n(\vec\mu,\Sigma)$, consider $\overbrace{(\vec X-\vec\mu)^T\Sigma^{-1}(\vec X-\vec\mu)}^{\text{Mahalanobis distance}} \sim ?$

$= (\vec X-\vec\mu)^T(A^{-1})^TA^{-1}(\vec X-\vec\mu)$

$= (A^{-1}(\vec X-\vec\mu))^TA^{-1}(\vec X-\vec\mu)$

Recall: $\vec Z = A^{-1}(\vec X-\vec\mu) \longrightarrow = \vec Z^T\vec Z\sim\chi^2_n$

$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^TA^{-1}$

* In one dimension $(X-\mu)(\sigma^2)^{-1}(X-\mu) = \frac{(x-\mu)^2}{\sigma^2}$

$= \left(\frac{x-\mu}{\sigma}\right)^2$