Lecture 7

convolution formula for independent discrete rv's

$$p_D(d) = \sum_{x \in \text{Supp}[X]} p_X^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum_{x\in\{0,1,\dots\}} \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{-(d-x)}}{(-(d-x))!} \mathbb{1}_{d-x \in \{\dots,-2,-1,0\}}$$

$-(d-x) = x-d$; $x-d \in \{0,1,\dots\} \Leftrightarrow x \in \{d, d+1, d+2, \dots\}$

$$= e^{-2\lambda} \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x}{x!} \frac{\lambda^{x-d}}{(x-d)!} \mathbb{1}_{x \in \{d, d+1, \dots\}}$$

let $x' = x-d \Rightarrow x = x'+d$

$$= e^{-2\lambda} \begin{cases} d > 0 & \displaystyle\sum_{x\in\{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'\in\{0,1,\dots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,x'!} = \sum_{x'\in\{0,1,\dots\}} \frac{\lambda^{2x'+d}}{x'!(x'+d)!} \\ d \le 0 & \displaystyle\sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} \end{cases}$$

let $d' = -d = |d|$, $d \le 0$

$$I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$ Modified Bessel Function of First kind (comes up in differ eq's)

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d\in\mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$ discovered in 1946

It used to model point spreads in sports games, photo noise, etc

Yankee λ_1 Mets λ_2

$X_1, X_2 \overset{iid}{\sim} \text{Poission}(\lambda) \Rightarrow T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

success came from X_1: $\text{Bern}(\frac{1}{2})$ $\underline{\ \ }_1\ \underline{\ \ }_2\ \underline{\ \ }_3 \ \dots\ \underline{\ \ }_t$

$$p_{X_1|T}(x,t) \overset{\text{definition}}{=} \frac{p_{X_1,T}(x,t)}{p_T(t)} \;(\text{jmf}/\text{PMF}) = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} \overset{\text{iid}}{=} \frac{p_{X_1}(x)\,p_{X_2}(t-x)}{p_T(t)}$$

X_1 given T; p_T = Poisson(2λ)

T success

How come from X_1, How come from X_2, no Poission

$$\frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!}\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}(t, \tfrac{1}{2})$$

$\frac{t!}{x!(t-x)!} = \binom{t}{x}$, $\frac{\lambda^t}{(2\lambda)^t} = \left(\frac{\lambda}{2\lambda}\right)^t$

half come X_1, half come X_2

Review Geometric

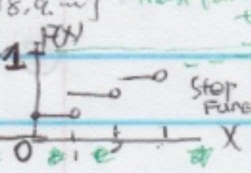
$B_1, B_2, \dots \overset{iid}{\sim} \text{Bern}(p)$

$X_1 \sim \text{Geo}(p) := \underbrace{(1-p)^x p\, \mathbb{1}_{x\in\{0,1,\dots\}}}_{p(x)}$ — $(1-p)^x$: odds, failer

$X=7$ $X_1 > 7 \Leftrightarrow X_1 \in \{8, 9, \dots\}$; All $b_1 \dots b_7 = 0$

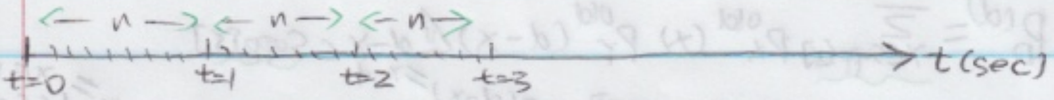
get 1 Fail 8 time, success must fail 7 times

CDF: $F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = 1 - (1-p)^{x+1}$



Step Function

Graph CDF



there be n experiments in each second (time unit)

X is in second, eg $n=100$, $x=\frac{1}{2}$

$$P_{X_n}(x) = (1-p)^{nx} \, p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots\}}$$ possible second in $\frac{1}{n}$

CDF: $F_{X_n}(x) = 1-(1-p)^{nx+1}$

Let's put infinite experiments into every second (time unit)

this is the limit as n goes to positive infinity X_∞, And $p \to 0$

but $\lambda = np$ ⟹ $p = \frac{\lambda}{n}$ a (a = the Poisson (French Fish) style of

$$P_{X_\infty}(x) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \dots\}}$$ constant

$$= \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x \lim_{n\to\infty} \frac{\lambda}{n} \lim_{n\to\infty} \mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}$$

$(e^{-\lambda})^x$, $\operatorname{Supp}[X_\infty] = [0,\infty)$

$$= e^{-\lambda x}\,(0)\,\lim_{x\in[0,\infty)} = 0 \quad \forall_{\text{all}}\, x$$

not PMF

This is not PMF, because $\sum_{x\in\operatorname{Supp}[X_\infty]} P_{X_\infty}(x) = 0 \neq 1$

CDF. $F_{X_\infty}(x) := P(X_\infty \le x)$

$$F_{X_\infty}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx+1} = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^1$$

$e^{-\lambda x}$, $=1$

$$= \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty)}$$ ⟸ Intuition Proof

Is this limiting CDF a legal CDF? If so, it must satisfy

From 241: three conditions

1) limit as x goes to negative infinity is 0

2) limit as x goes to positive infinity is one

3) Increasing function ie its derivative is ≥ 0

Proof. ① $\lim_{x\to-\infty} (1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 0$ ✓ since $\lambda > 0$

② $\lim_{x\to\infty} (1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 1 - \lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 1$ ✓ since $\lambda>0$

$\mathbb{1} = 1$, $\lim = 0$

$\frac{d}{dx}(1-e^{-8x}) = 8e^{-8x}$

$\frac{d}{dx}(e^{-x}) = e^{-x}\frac{d}{dx}[-x]$

$\frac{d}{dx}x^n = nx^{n-1}$

{x is negative 𝟙 is 0, x is possive

$f_{old}(x)$

③ $\frac{d}{dx}[(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}] = \underbrace{\lambda e^{-\lambda x}\,\mathbb{1}_{x\in[0,\infty)}}_{f(x)} > 0$

So: $= F_{X}(x)$ is a valid CDF! of where rv?? Answer "A continues rv"

Scalar & vector

Lecture 1 size of supprt is uncountable

A continuous rv X has $Supp[X] \subseteq \mathbb{R}$ but $|Sup[X]| = |\mathbb{R}|$ (Alwy True) this size is known as "uncountable infinity" or the size of the continuum. They also have no PMF, the $P(X=x)$ is always zero 0 For every x. But they have a CDF (Continuous For the purposes of this class). And derivative of CDF is Very useful Function. So it gets a special name which is the probability ~~distribution~~ density Function (OR just density) (PDF denoted f)

$f(x) := F'(x)$ $\quad P(X\in(a,b]) = P(X\le b) - P(X\le a)$ ($F(b)$, $F(a)$)

Fundamental theorem of Calculus $\underline{\underline{FTC}}$ $\int_a^b f(x)dx$

$f(x)\ge 0\ \forall x$ (for all x) (property of CDF) $\Leftrightarrow Supp[X] = \{x : f(x)>0\}$ (f(x) is possive)

Another property

$\int_{\mathbb{R}} f(x)\,dx = 1$ proof: $\int_{-\infty}^{\infty} f(x)dx = F(\infty) - F(-\infty) = 1$ (properties of CDF)

Joint density Function (JDF)

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1)\cdots f_{X_k}(x_k) = f(x_1)\cdots f(x_k)$

Multification Rules

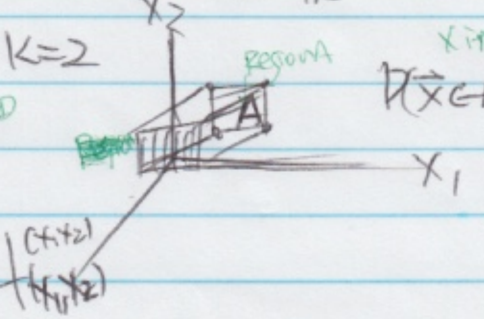
if $X_1 \dots X_k$ are independent; Identical distribue

all components continuous

$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f_{\vec{X}}(\vec{x})\,dx_1\cdots dx_k = 1$ (k)

Summ all k = 1

example: K=2 3D



X in A

$P(\vec{X}\in A) = \iint_A f_{\vec{X}}(\vec{x})\,dx_1 dx_2$

Background of Continuous rv

rv:

exponti~~all~~onal rv

$$X \sim Exp(\lambda) := \underbrace{\overbrace{\lambda e^{-\lambda x}}^{F\,a(dx)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)} \text{ support}$$

$\lambda \in (0,\infty)$