Lecture 14 Math 621 10/26/2020

Today Arbitrary multivariate transformations of vars.

$g: \mathbb{R}^n \to \mathbb{R}^n$ and invertible. Let $\vec{X}, \vec{Y}$ be vector r.v w/ dim n and $\vec{Y} = \vec{g}(x)$

Given the jdf of $\vec{X}$, find the jdf of $\vec{Y}$ r.v.

This generalizes what we did with univariate change of variable. Lets recall what a multivariate function looks like:

$Y_1 = g_1(X_1, \ldots, X_n)$ therefor $\vec{g} = [g_1, \ldots, g_n]^T$

$Y_2 = g_2(X_1, \ldots X_n)$ $\vec{g}^{-1} = \vec{h} = [h_1, \ldots, h_n]^T \to$ $X_1 = h_1(Y_1, \ldots Y_n)$

$X_2 = h_2(Y_1, \ldots Y_n)$

$\vdots$

$X_n = h_n(Y_1, \ldots Y_n)$

$\vdots$

$Y_n = g_n(X_1, \ldots, X_n)$

For mult. var calc, you can show the multivariate change of var formula is:

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))|J_n(\vec{y})| \quad , \quad J_n(\vec{y}) = \det \begin{bmatrix} \partial h_1/\partial y_1 & \cdots & \partial h_1/\partial y_n \\ \vdots & \ddots & \\ \partial h_n/\partial y_1 & \cdots & \partial h_n/\partial y_n \end{bmatrix}$$

"Jacobian determinant"

Let's use this formula to prove the convolution formula for $T = X_1 + X_2 = f_T(t) = ?$

This is the recipe:

1) Find g (set of first dim Y_1 = your target)
2) find the h
3) Compute the Jacobian Determinant.
4) Substitute 1-3 into the multivariate change of var formular
5) Integrate the "nuisance dimension(s)".

1) $T = Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$ — nuisance dim.

2) $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

3) $J_n = \det\begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial x_2 \end{bmatrix} = \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = (1)(1) - (-1)\cdot(0) = 1$

4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 - y_2, y_2)\,|1| =$

5) $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(t-u, u)\,du$

$\overset{X_1 X_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(t-u) f(u)\,du$

$\Downarrow$

$\int_{Supp[Y_2]} f_{X_1}^{old}(t-u)\,\mathbb{I}_{t-u \in Supp[X_1]} f_{X_2}^{old}(u)$

$\int_{Supp[X]} f^{old}(t-u)\,\mathbb{I}_{t-u \in Supp[X]} f^{old}(u)\,du$

Let $R = \frac{X_1}{X_2} \sim f_R(y) = ?$

1) $R = Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

2) $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

3) $J_n = \det\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

4) $f_Y(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|$

5) $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|\,dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u)\,|u|\,du \overset{X_1, X_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)\,|u|\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(ru) f(u)\,|u|\,du$

$\int f_{X_1}^{old}(ru)\,\mathbb{I}_{ru \in Supp[X]} f_{X_2}^{old}(u)\,|u|\,du$

$\int_{Supp[X]} f^{old}(ru)\,\mathbb{I}_{ru \in Supp[X]} f^{old}(u)\,|u|\,du$

$$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ?$$

1) $R = Y_1 = \frac{X_1}{X_1+X_2} = g_1(X_1,X_2),\ Y_2 = X_2 + X_2 = g_2(X_1,X_2)$

2) $X_1 = Y_1(X_1+X_2) = Y_1Y_2 = h_1(y_1,y_2),\ X_2 = Y_2 - X_1 = Y_1 - Y_1Y_2 = h_2(Y_1,Y_2)$

3) $J_h = \det\begin{bmatrix} Y_2 & Y_1 \\ -Y_2 & 1-Y_1 \end{bmatrix} = Y_2(1-Y_1) - (-Y_1Y_2) = Y_2$

4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1y_2, y_2 - y_1y_2)|y_2|$

5) $f_R(y) = f_{Y_1}(y) = \int_{\mathbb{R}} f_Y(y_1,y_2)\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(Y_1Y_2, Y_2-Y_1Y_2)|Y_2|\,dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(ru, u-ru)|u|\,du$

$$\overset{\text{indep}}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u-ru)|u|\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(ru) f(u-ru)|u|\,du$$

$\Downarrow \qquad\qquad \Downarrow$

$$\int_{\mathbb{R}} f_{X_1}^{old}(ru)\mathbb{1}_{ru\in Supp[X_1]} f_{X_2}^{old}(u-ru)\mathbb{1}_{u-ru\in Supp[X_2]}|u|\,du = \int_{\mathbb{R}} f^{old}(ru)\mathbb{1}_{ru\in Supp[X]} f^{old}(u-ru)\mathbb{1}_{u-ru\in Supp[X]}|u|\,du$$

$X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2,\beta)$; $R = \frac{X_1}{X_1+X_2} \sim f_R(y) = ?$

R is the proportion of the waiting time for the first gamma and thus $Supp[R] = [0,1]$

$$f_R(y) = \int_{\mathbb{R}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1-1}}_{r^{\alpha_1-1}u^{\alpha_1-1}} \overbrace{e^{-\beta ru}}^{e^{-\beta(u-ru)} e^{-\beta u} e^{\beta ru}} \underbrace{\mathbb{1}_{ru\in[0,\infty)}}_{u\in[0,\infty)} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} \underbrace{(u-ru)^{\alpha_2-1}}_{u^{\alpha_2-1}(1-r)^{\alpha_2-1}} \underbrace{\mathbb{1}_{u-ru\in[0,\infty)}}_{u\in[0,\infty)} |u|\,du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta u}\,du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{\beta^{\alpha_1+\alpha_2}}} \mathbb{1}_{r\in[0,1]}$$

$$= \frac{1}{B(\alpha_1,\alpha_2)} r^{\alpha_1-1}(1-r)^{\alpha_2-1} \mathbb{1}_{r\in[0,1]} = Beta(\alpha_1,\alpha_2)$$

p152 $X_1 \sim \text{Gamma}(\alpha, \beta)$ indep of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, $R = \frac{X}{X_2} \sim f_R(y) = ?$

$$f_R(y) = \int_{\text{Supp}[X_2]} f_{X_1}^{\text{old}}(ru) \cdot \mathbb{1}_{ru \in \text{Supp}[X_1]} f_{X_2}^{\text{old}}(u)|u|\,du = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1-1} e^{-\beta ru} \underbrace{\mathbb{1}_{ru \in [0,\infty)}}_{u \in [0,\infty)} \cdot \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2-1} e^{-\beta u} |u| \cdot du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \mathbb{1}_{r>0} \underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)u}\,du}_{\frac{\Gamma(\alpha_1+\alpha_2)}{(\beta(r+1))^{\alpha_1+\alpha}} = \frac{\Gamma(\alpha_1+\alpha_2)}{\beta^{\alpha_1+\alpha_2}(r+1)^{\alpha_1+\alpha_2}}}$$

$$= \frac{1}{B(\alpha_1, \alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0} = \text{BetaPrime}(\alpha_1, \alpha_2)$$