Lecture 07
Math 621
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From Last Class:

$\text{Supp}[D] = \text{Supp}[X+Y] = \text{Supp}[X] + \text{Supp}[Y] = \mathbb{Z}$

Convolution formula for independent discrete rv's:

$$P_D(d) = \sum_{x \in \text{Supp}[X]} p_X^{odd}(x)\, p_Y^{odd}(d-x)\, \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{e^{-\lambda}\lambda^{x}}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{\underbrace{(-(d-x))!}_{(x-d)!}} \mathbb{1}_{\underbrace{\underbrace{d-x \in \{\ldots,-1,0\}}_{x-d \in \{0,1,\ldots\}}}_{x \in \{d,d+1,\ldots\}}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,\ldots\}} \frac{\lambda^{x}}{x!} \frac{\lambda^{x-d}}{(x-d)!} \mathbb{1}_{x \in \{d,d+1,\ldots\}}$$

$$= e^{-2\lambda} \begin{cases} d > 0 & \displaystyle\sum_{x \in \{d,\ldots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} \quad (1) \\ d \le 0 & \displaystyle\sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} \quad (2) \end{cases}$$

(1) Let $x' = x - d \Rightarrow x = x' + d$, also $d = |d|$

so, $= \displaystyle\sum_{x' \in \{0,1,\ldots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,(x'+d-d)!} = \sum_{x' \in \{0,\ldots\}} \frac{\lambda^{2x'+d}}{(x'+d)!\,x'!}$

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(2) Let $d' = -d = |d|$

so, $= \displaystyle\sum_{x \in \{0,\ldots\}} \frac{\lambda^{2x+d'}}{x!\,(x+d')!} = \sum_{x \in \{0,\ldots\}} \frac{\lambda^{2x+|d|}}{x!\,(x+|d|)!}$

Note: $|d|$, think of $d \le 0$

$$I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}$$

Modified Bessel function of the first kind (comes up in diff eq's)

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d \in \mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$

Skellam$(\lambda,\lambda)$ discovered in 1946.

It's used to model point spreads in sports games, photo noise, etc.

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$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \Rightarrow T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$$

$$p_{X_1|T}(x,t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} \leftarrow \text{JMF}$$

$$= \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)}$$

Using iid $= \dfrac{p_{X_1}(x)\, p_{X_2}(t-x)}{p_T(t)}$

$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\,\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} \qquad \lambda^t\lambda^{-x}$$

$$= \frac{t!}{x!(t-x)!}\,\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}\left(t,\tfrac{1}{2}\right)$$

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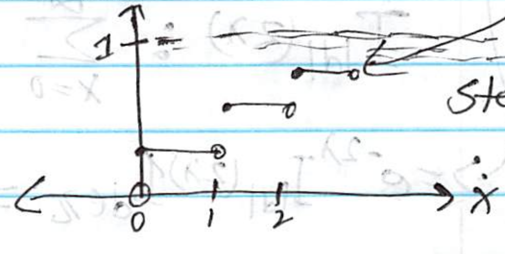
$B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

$$X_1 \sim \text{Geom}(p) = \underbrace{\overbrace{(1-p)^x\, p}^{p(x)}}_{p^{odd}(x)}\, \mathbb{1}_{x\in\{0,1,\ldots\}}$$

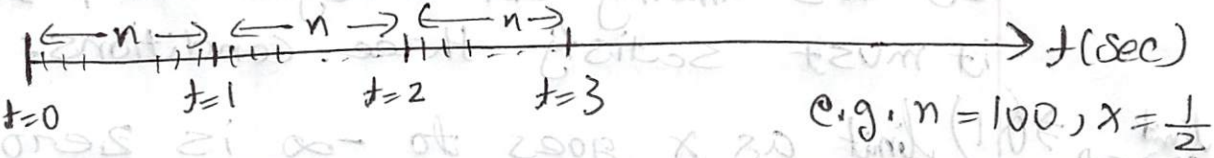
$F_{X_1}(x) = P(X_1 \le x) = 1 - P(X_1 > x) = 1-(1-p)^x$

If $x=7$, then $X_1 > 7 \Rightarrow X_1 \in \{8, 9, \ldots\}$

↪ $b_1, \ldots, b_7 = 0$ means failed 7 times...




Step function




e.g. $n = 100$, $x = \frac{1}{2}$

Let there be $n$ experiments in each second (time unit). $X$ is in seconds...

$$p_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots\}}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx+1}$$

Let's put infinite experiments into every second (time unit), this is the limit as $n$ goes to positive infinity. $X_\infty$ And $p \to 0$.

$p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ a la the Poisson.

$$p_{X_\infty}(x) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$$

$$= \Big(\underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\Big)^x \lim_{n\to\infty} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$$

$$= e^{-\lambda x}\,(0)\,\mathbb{1}_{x\in[0,\infty)} = 0 \;\forall x$$

This is not a PMF because,

$\mathrm{Supp}[X_\infty] = [0,\infty)$

$$\sum_{x \in \mathrm{Supp}[X_\infty]} p_{X_\infty}(x) = 0 \neq 1$$

$$F_{X_\infty}(x) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{n\infty+1}$$

$$= 1 - \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{e^{-\lambda x}} \cdot \cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{1}} \to 1$$

$$= \left(1 - e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty)}$$

intuitive proof

Is this limiting CDF a legal CDF? If so, it must satisfy three conditions:

$\lim_{x \to -\infty} \Rightarrow 0$ 1) limit as $x$ goes to $-\infty$ is zero.

$\lim_{x \to \infty} \Rightarrow 1$ 2) limit as $x$ goes to $+\infty$ is one.

3) increasing function.
i.e. its derivative is $\geq 0$

① $\lim_{x \to -\infty} \left(1 - e^{-\lambda x}\right) \mathbb{1}_{x \in [0,\infty)} = 0$ ✓

② $\lim_{x \to \infty} \left(1 - e^{-\lambda x}\right) \mathbb{1}_{x \in [0,\infty)} = 1 - \lim_{n \to \infty} \frac{1}{e^{\lambda x}} = 1$ ✓
since $\lambda > 0$

③ $\frac{d}{dx}\left[\left(1 - e^{-\lambda x}\right) \mathbb{1}_{x \in [0,\infty)}\right]$
$= \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)} > 0$ ✓


$\Rightarrow F_{X_\infty}$ is a valid CDF! of what rv??
Answer:
A continuous rv.

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A continuous rv $X$ has $Supp[X] \subseteq \mathbb{R}$
but $|Supp[X]| = |\mathbb{R}|$,
this size is known as "uncountable infinity" or the "size of the continuum".
They also have no PMF, the $P(X = x)$ is always 0 for every $x$. But they have a CDF (continuous for the purposes of this class). And the derivative of the CDF is a very useful function, so it gets a

special name which is the "probability density function" or just "density"
(PDF) denoted $f$:

$f_X = F'(X)$ , $P(X \in (a,b)) = \underbrace{P(X \leq b)}_{F(b)} - \underbrace{P(X \leq a)}_{F(a)}$

$\underset{\text{(Fundamental Thm of Calculus)}}{FTC} \Rightarrow = \int_a^b f(x)\,dx$

$\Rightarrow f(x) \geq 0 \quad \forall x$ (Property of the CDF)

$\Leftrightarrow Supp[X] = \{X : f(x) > 0\}$

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$\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \overbrace{F(\infty)}^{1} - \overbrace{F(-\infty)}^{0} = 1$

(Property of CDF)

Joint density function (JDF)

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_K}(x_K) = f(x_1) \cdot \ldots \cdot f(x_K)$

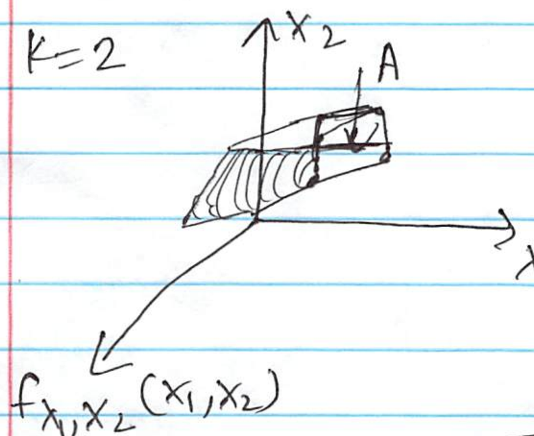
all components of continuous

if $X_1, \ldots, X_K$ independent

identically distributed

$\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{K} f_{\vec{X}}(\vec{x})\,dx_1 \cdot \ldots \cdot dx_K = 1$

$K=2$




$P(\vec{X} \in A) = \iint_A f_{\vec{X}}(\vec{x})\,dx_1\,dx_2$

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$$X \sim Exp(\lambda) = \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{odd}(x)} \mathbb{1}_{x\in[0,\infty)}}_{f(x)}$$

Support

Exponantial RV