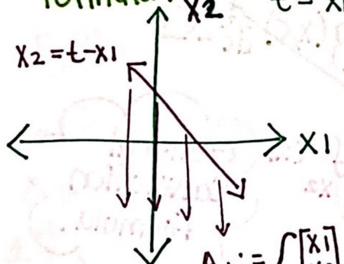
$\vec{X}$ cts. r.v. $\quad P(\vec{X}\in A) = \underset{A}{\int \cdots \int} f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_k$

Lecture 8

Let $T = X_1 + X_2 \sim f_T(t) = ?$ First note, $f_T(t) = F'(t)$.
CDF Method.

- Note: Usually it is difficult to find the CDF of cts. rv.'s, so this is not the usual method. The usual method is to use the convolution formula.

$t = x_1 + x_2 \Leftrightarrow x_2 \le t - x_1$



$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \le t - x_1 \right\}$

$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\, dv\, dx$$

Let $x_1 = x$
$x_2 = v - x$
$\rightarrow v = x_2 + x$
$\rightarrow dx_2 = dv$

$\rightarrow x_2 = -\infty$
$\rightarrow v = -\infty$
$\rightarrow x_2 = t - x$
$\rightarrow v = t$

$$= \int_{-\infty}^{t}\left(\int_{-\infty}^{\infty} f_{X_1,X_2}(x, v-x)\, dx\right) dv$$

$$f_T(t) = \frac{d}{dt}\left[\quad \right]$$

Leibnitz's Rule

$$\frac{d}{dx}\left[\int_{a(x)}^{(b(x))} g(x,y)\,dy\right] = g(x,b(x))\,b'(x) + g(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy.$$

If the outer derivative is a 3rd variable, then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)dy\right] = g(x,b(t))b'(t) + g(x,a(t))a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}(g(x,y))dy}$$

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)dy\right] = g(x,b(t))\,b'(t)^{0} + \cancel{g(x,c)\frac{d}{dc}[c]}^{0}$$

$$\rightarrow = \int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dx = f_T(t) = f_{X_1}(x) * f_{X_2}(x)$$

General Convolution Formula.

if X_1, X_2 independent.

$$\downarrow = \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f_{X_1}^{old}(x)\, f_{X_2}^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

X_1, X_2 iid

$$\hookrightarrow = \int_{\mathbb{R}} f(x)f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\,dx.$$

Continous uniform r.v.

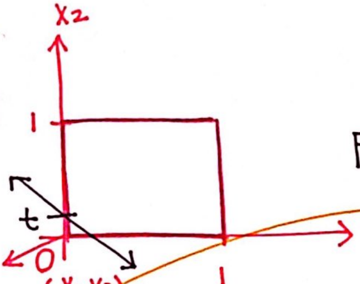
$$X \sim U(a,b) = \underbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x\in[a,b]}}_{f(x)}$$

The standard uniform r.v. is when $a=0, b=1$

$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$

we want to compute CDF which means we want to find volumes in region under the diagonal line.



$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2/2 & \text{if } t \in (0,1] \\ & \text{if } t \in (1,2) \\ 1 & \text{if } t \geq 2 \end{cases}$$

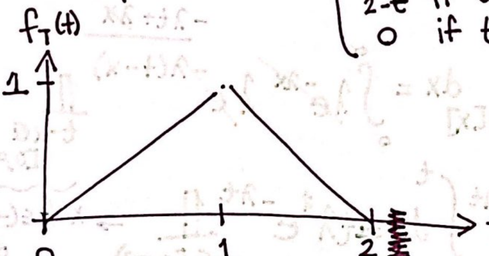
$f_{X_1,X_2}(x_1,x_2)$

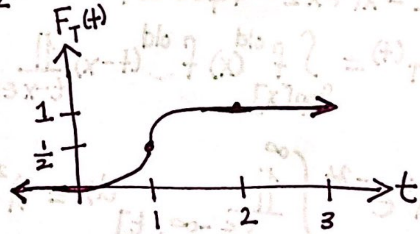
$$= f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ \& } x_2 \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

if $t \in (1,2)$

$$F_T(t) = \frac{t^2}{2} - \frac{2(t-1)^2}{2} = \frac{t^2}{2} - (t^2 - 2t + 1) = -\frac{t^2}{2} + 2t - 1$$

$$\rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$





We just derived the PDF of the convolution by finding it's CDF and taking the derivative. Why can't we just use our fancy formula?

iid old version

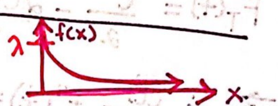
$$f_T(t) = \int_{Supp[X]} f(x) f(t-x) \mathbb{1}_{t-x \in Supp[X]} \, dx = \int_0^1 (1)(1) \, \mathbb{1}_{t-x \in [0,1]} \overbrace{}^{\substack{x \in [t-1,t] \\ x-t \in [-1,0]}} dx =$$

$$\int_0^1 \mathbb{1}_{x \in [t-1,t]} \, dx$$

Let's do some examples. $t = -37$?

$$\int_0^1 \mathbb{1}_{x\in[-38,-37]}\,dx =$$

$$X_1, X_2 \dots \overset{iid}{\sim} Exp(\lambda) := \lambda e^{-\lambda x}\,\mathbb{1}_{x\in[0,\infty)}$$



$$T_2 = X_1 + X_2 \sim f_T(t) = ?$$

$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\,\mathbb{1}_{t-x\in Supp[X]}\,dx = \int_0^\infty \lambda \cancel{e^{-\lambda x}}\, \lambda \cancel{e}^{-\lambda(t-x)}\, \underbrace{\mathbb{1}_{t-x\in[0,\infty)}}_{\to\, x-t\in(-\infty,0] \;\to\, x\in(-\infty,t]}$$

$$-\lambda t + \cancel{\lambda x}$$

$$= \lambda^2 e^{-\lambda t}\int_0^\infty \mathbb{1}_{x\in -\infty,\, t]}\,dx = \lambda^2 e^{-\lambda t}\int_0^t dx = t\lambda^2 e^{-\lambda t}\,\mathbb{1}_{t\in[0,\infty)}$$

$$= \text{Erlang}(2,\lambda)$$