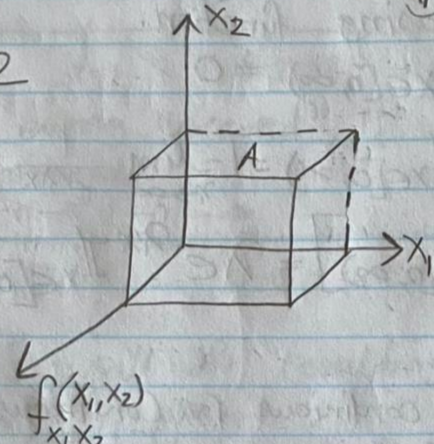
$X_1, \ldots X_n$ indepdt $\Rightarrow f_{X_1 \ldots X_n}(x_1 \ldots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

If $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$, $\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{k} f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k = 1$

$k=2$



$$P(A) = \int \cdots \int_A f_{X_1, X_2}(x_1, x_2)\, dx_1 dx_2$$

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$\vec{X}$ continuum rv,

$$P(\vec{X} \in A) = \int \cdots \int_A f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k$$

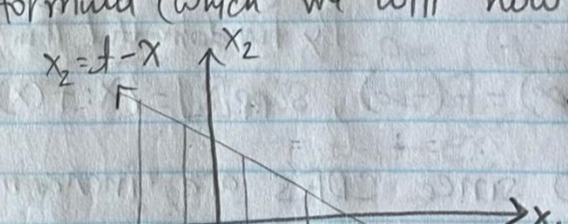
Let $T = X_1 + X_2 \sim f_T(t) = ?$

First note, $f_T(t) = F'(t)$ CDF method.

usually it's difficult to find the CDF of continuous rv's so this isn't the usual method. The usual method is to use the convolution formula (which we will now derive.

$t = X_1 + X_2$

$\Leftrightarrow X_2 \leq t - X_1$



$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \leq t - x_1 \right\}$

$$F_T(t) = P(T \leq t) = P(\vec{x} \in A_t) = \iint_{A_t} f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1, x_2)\, dx_2 dx_1 = \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, t-x)\, dv\, dx$$

let $x_1 = x$; $x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv$

$\Rightarrow x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$= \int_{-\infty}^{t} \left(\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x) \right) dv$$

$$f_T(t) = \frac{d}{dt}\left[\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\, dv \right] =$$

Recall, Leibnitz's Rule

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\, dy \right] = g(x, b(x))\, b'(x) + g(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\, dy$$

If the outer derivative is a third variable, then

$$\frac{d}{dt}\left[\int_{a(x)}^{b(x)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + \cancel{g(x, a(t))\, a'(t)}^{\,0} + \int_{a(t)}^{b(t)} \cancel{\frac{\partial}{\partial t}[g(x,y)]}^{\,0}\, dy$$

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + \cancel{g(x,c) \frac{d}{dt}[c]}^{\,0}$$

$$= \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\, dx = f_T(t) = f_{X_1}(x) * f_{X_2}(x)$$

$*$ general convolution formula.

If X_1, X_2 independent,

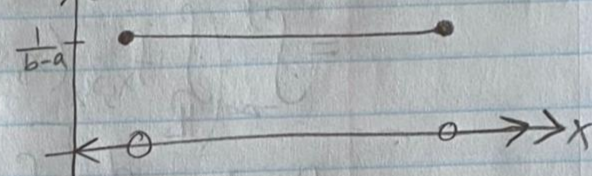
$$\stackrel{\swarrow}{=} \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f^{old}_{X_1}(x) f^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

If $X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(x) f(t-x)\,dx = \int_{Supp[X]} f^{old}(x) f^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}\,dx$$

Continuous uniform rv

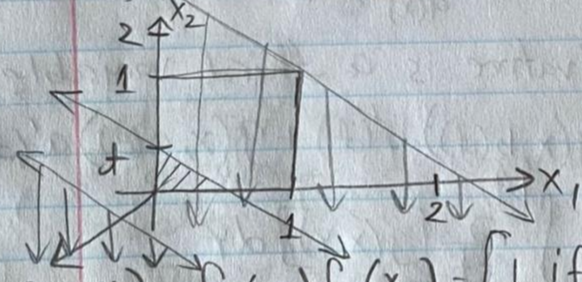
$$X \sim U(a,b) = \underbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x\in[a,b]}}_{f(x)}$$



Standard uniform rv is when $a=0, b=1$

$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$$

$$X_1, X_2 \overset{iid}{\sim} U(0,1), \quad T = X_1 + X_2 \sim f_T(t) = ?$$



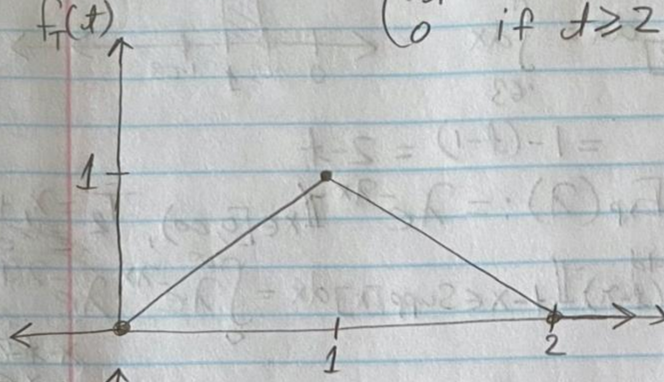
$$f_{X_1X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1 & \text{if } x_1 \in [0,1] \,\&\, x_2 \in [0,1] \\ 0 & \text{o/t} \end{cases}$$

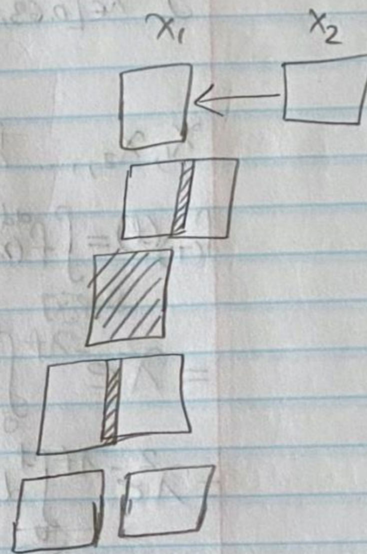
We want to compute CDF which means we want to find volumes in regions under the diagonal line.

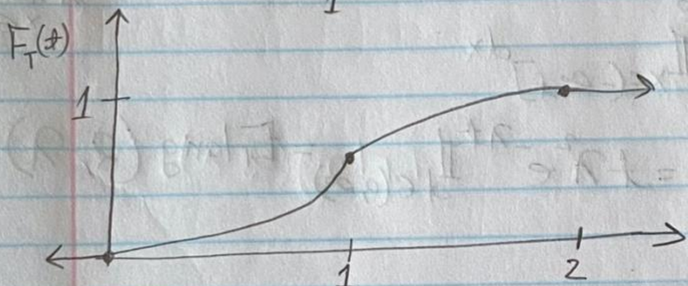
$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2/2 & \text{if } t \in (0,1] \\ \frac{-t^2}{2} + 2t - 1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \geq 2 \end{cases}$$

if $t \in (1,2)$ $F_T(t) = \frac{t^2}{2} - 2\,\frac{(t-1)^2}{2} = \frac{t^2}{2} - (t^2 - 2t + 1) = \frac{-t^2}{2} + 2t - 1$

$$\Rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$







We just derived the PDF of the convolution by finding it's CDF and taking the derivative. why can't we just use our fancy formula?

iid old version $\left\{ f_T(t) = \int_{Supp[X]} f^{old}(x) f^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]} dx = \int_0^1 (1)(1)\, \mathbb{1}_{t-x\in[0,1]} dx \right.$

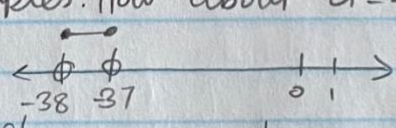
$x - t \in [-1, 0]$

$x \in [t-1, t]$

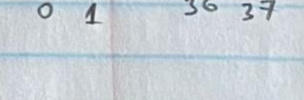
$$= \int_0^1 \mathbb{1}_{x \in [t-1, x]} dx$$

Let's do some examples. How about $t = -37$?

$\int_0^1 \mathbb{1}_{x\in[-38,37]} dx = 0$



How about $t = 37$? $\int_0^1 \mathbb{1}_{x\in[36,37]} dx = 0$



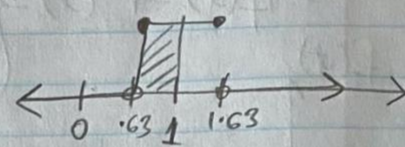
$t \in (0,1)$. eg, 0.7

$$\int_0^1 \mathbb{1}_{x\in[-0.3,\,0.7]} dx = \int_0^{.7} dx = 0.7$$

$$= \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \geq 2 \end{cases}$$

$t \in (1,2)$ e.g. $t = 1.63$

$$\int_0^1 \mathbb{1}_{x \in [0.63, 1.63]} \, dx = \int_{.63}^1 dx$$



$$= 1 - (t-1) = 2 - t$$

$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}$, $T_2 = X_1 + X_2 \sim f_T(t) = ?$

$\cancel{e^{\lambda x}} \, e^{-\lambda t}$

$$f_{(T)}(t) = \int_{Supp[X]} f^{old}(x) f^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]} \, dx = \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in [0,\infty)} \, dx$$

$x - t \in (-\infty, 0]$

$x \in (-\infty, t]$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t]} \, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)} = \text{Erlang}(2, \lambda).$$