Tuesday September 29th 2020

Lecture 7

$$P_D(d) = \sum_{x \in \mathrm{Supp}[X]} P_X^{old}(x)\, P_Y^{old}(d-x)\, \mathbb{1}_{d-x \in \mathrm{Supp}[Y]}$$

$$= \sum_{x \in \{0,1,\dots\}} \frac{e^{-\lambda}\lambda^{x}}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{(x-d)}}}{\underbrace{(-(d-x))!}_{(x-d)}}\, \mathbb{1}_{d-x \in \{\dots,-1,0\}}$$

$x-d \in \{0,1,\dots\}$

$x \in \{d, d+1, d+2, \dots\}$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{x}}{x!} \frac{\lambda^{x-d}}{(x-d)!}\, \mathbb{1}_{x \in \{d, d+1, \dots\}}$$

let $x' = x-d \Rightarrow x = x'+d$

$$= e^{-2\lambda} \begin{cases} d > 0 & \displaystyle\sum_{x \in \{d, d+1, \dots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} = \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,(x'+d-d)!} = \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{2x'+d}}{(x'+d)!\,x'!} \\ d \le 0 & \displaystyle\sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!\,(x+d')!} = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!\,(x+|d|)!} \end{cases}$$

$d = |d|$

let $d' = -d = |d|$

$$I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}$$

Modified Bessel Function of the First kind (Comes up in diff eq's)

$$= e^{-2\lambda} I_{|d|}(2\lambda)^{\mathbb{1}_{d \in \mathbb{Z}}} = \text{Skellam}(\lambda, \lambda)$$

discover in 1946

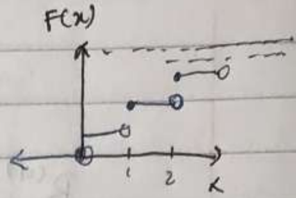
It's used to model point spreads in games, photo noise, etc.

$X_1, X_2 \overset{iid}{\sim}$ Poisson$(\lambda) \Rightarrow T = X_1 + X_2 \sim$ Poisson(2λ)

$$P_{X_1|T}(x,t) = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x, t-x)}{P_T(t)} = \frac{P_{X}(x)\, P_{X_2}(t-x)}{P_T(t)}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^{x}}{x!} \frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^{t}}{t!}} \; (**) = \frac{t!}{x!\,(t-x)!} \underbrace{\frac{\lambda^{t}}{2\lambda^{t}}}_{\left(\frac{\lambda}{2\lambda}\right)^{t}} = \binom{t}{x}\left(\frac{1}{2}\right)^{t} = \mathrm{Bin}\left(t, \tfrac{1}{2}\right)$$

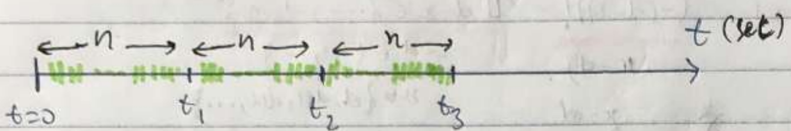
$B_1, B_2, \ldots \overset{iid}{\sim} Ber(P)$



By definition this is $X_1 \sim Geom(P) := \underbrace{\overbrace{(1-P)^x P}^{p^{old}(x)} \mathbb{1}_{x \in \{0,1,\ldots\}}}_{p(x)}$

Review: CDF

$$F_{X_1}(x) := P(X_1 \leq x) = 1 - P(X_1 > x) = 1 - (1-P)^{x+1}$$



eg: $n = 100$, $x = \frac{1}{2}$

Let there be n experiments in each second (time unit). x is in second

$$P_{X_n}(x) = (1-P)^{nx} P \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, +\frac{2}{n}, \ldots\}}$$

My CDF we look like $\quad F_{X_n}(x) = 1 - (1-P)^{nx+1}$

Let's put infinite experiments into every second (time unit), thus is the limit as n goes to positive infinity. X_∞ and $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ a la poisson.

$$P_{X_\infty}(x) = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right) \frac{\lambda}{n} \quad \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$$

$$= \left(\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n\right)^x \lim \frac{\lambda}{n} \lim \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$$= e^{-\lambda x} (0) \mathbb{1}_{x \in [0,\infty)} \quad Supp[X_\infty] = [0, \infty)$$

$$= 0 \quad \text{this is not a PMF because } \sum_{x \in Supp[x,\infty)} P_{X_\infty}(x) = 0 \neq 1$$

③

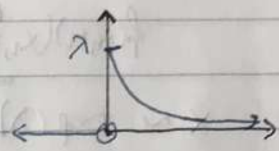
$$F_{X_\infty}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx+1} = 1-\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{e^{-\lambda x}} \cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)}_{=1} = 1-e^{-\lambda x}\,\mathbb{1}_{x\geq 0}$$

Is this limiting CDF a legal CDF? If so, it must satisfy three conditions

(1) Limit as x goes to negative infinity is zero

(2) Limit as x goes to positive infinity is one

(3) Increasing function ie its derivative is ≥ 0



(1) $\lim_{x\to-\infty}(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 0$ ✓

(2) $\lim_{x\to\infty}(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 1-\lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 1$ ✓ (since $\lambda>0$)

(3) $\frac{d}{dx}\left[(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)} > 0$ ✓

$\Rightarrow F_{X_\infty}$ is a valid CDF! of a Continuous Random Variable.

A Continuous r.v X has $supp[X] \subseteq \mathbb{R}$ but $|supp[X]| = |\mathbb{R}|$
This size we know as "uncountable infinity" or the size of the continuous". The also have no PMF the $P(X=x)$ is always zero for every x. But they have a CDF (continuous for the purpose of this class). And the derivative of the CDF is a very useful function, so it gets a special name which is the "probability density function" or just density (PDF) denote f:

$$f(x) := F'(x), \quad P(X\in(a,b)) = P(X\leq b) - P(X\leq a) \overset{\text{fundamental theorem of algebra}}{=} \int_a^b f(x)\,dx$$

$f(x)\geq 0$ $\forall x$ (properties of the CDF) $\Longleftrightarrow$ $Supp[X] = \{x : f(x) > 0\}$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \qquad \int_{-\infty}^{\infty} f(x)\,dx = \underbrace{F(\infty)}_{1} - \underbrace{F(-\infty)}_{0} = 1 \text{ (properties of CDF)}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) \overset{\uparrow}{=} f_{X_1}(x_1)\cdots f_{X_k}(x_k) \overset{\uparrow}{=} f(x_1)\cdots f(x_k)$$

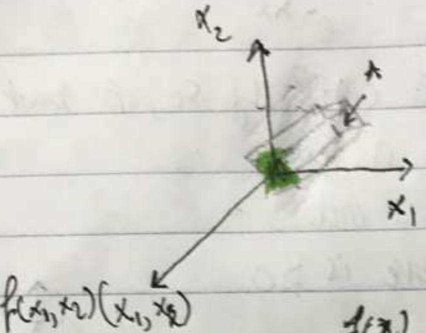
All components continuous

if $X_1, \ldots, X_k$ independent

identically distribution

$$\overbrace{\int_{\mathbb{R}} \cdots\cdots \int_{\mathbb{R}}}^{k} f_{\vec{X}}(\vec{x})\,dx \cdots dx_k = 1$$

$k=2$



$$P(\vec{X} \in A) = \iint_A f_{\vec{X}}(\vec{x})\,dx_1\,dx_2$$

$f(x)$ exponential r.v

$$X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x}\, \mathbb{1}_{x \in [0,\infty)}$$

pdf $f(x)$

supp: $x \in (0, \infty)$