Monday November 23rd 2020

Lecture 20

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \frac{\sigma^2}{n-1} \frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z\sim(0,1)}}{\underbrace{\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}}_{U\sim\chi^2_{n-1}}} \sim T_{n-1}$$

Due to cochran's thm, if X_i's are iid $N(\mu, \sigma^2)$ <=> $\bar{X}$ and S^2 are independent and thus numerator and denominator here are independent

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THE Multivariate Normal rv (MVN)

$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ s.t $Z_1, \dots Z_n \overset{iid}{\sim} N(0,1)$, $E[\vec{Z}] = \vec{0}_n$, $Var[\vec{Z}] = I_n$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}, I)$$

Let $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} = \vec{Z} + \vec{\mu} = \begin{bmatrix} Z_1 + \mu_1 \\ \vdots \\ Z_n + \mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1, 1) \\ \\ \sim N(\mu_n, 1) \end{matrix} \sim N_n(\vec{\mu}, I)$

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 \dots & 0 \\ 1 & 1 & \dots & & & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}$, $\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1 + Z_2 \\ Z_1 + Z_2 + Z_3 \\ \vdots \\ Z_1 + Z_2 + \dots + Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \vdots \\ N(0,n) \end{matrix} \sim N_n(\vec{0}, ?)$

$\sigma_{1,2} = Cov[X_1, X_2] = Cov[Z_1, Z_1 + Z_2] = Cov[Z_1, Z_1] + Cov[Z_1, Z_2] = 1 + 0 = 1$

General Rule to figure out variance-covariance matrix of a matrix A times rv vector $\vec{X}$;

②

$$\begin{aligned}
Var[A\vec{x}] &= E[(A\vec{x})(A\vec{x})^T] - E[A\vec{x}]E[A\vec{x}]^T \\
&= E[A\vec{x}\vec{x}^TA^T] - E[A\vec{x}]E[A\vec{x}]^T \\
&= AE[xx^T]A^T - AE[\vec{x}](AE[\vec{x}])^T \\
&= AE[xx^T]A^T - AE[x]E[x]^TA^T \\
&= A(E[xx^T] - E[x]E[\vec{x}]^T)A^T \\
&= A\Sigma A^T
\end{aligned}$$

$$\vec{x} = A\vec{z} + \underset{\overset{\parallel}{g(\vec{z})}}{\vec{\mu}} \sim N_n(\vec{\mu}, AA^T) = f_{\vec{x}}(\vec{x}) = ?$$

$\Downarrow$

$\vec{z} = \underset{B}{\underline{A^{-1}}}(\vec{x} - \vec{\mu}) = h(\vec{y})$ In order for g to be 1:1, the matrix A must be invertible.

$= B\vec{x} - B\vec{\mu} \Rightarrow$ $h(\vec{x}) = \vec{b}_{1\cdot} \cdot \vec{x} - \vec{b}_{1\cdot} \cdot \vec{\mu}$

$\vdots$

$h_n(\vec{x}) = \vec{b}_{n\cdot} \cdot \vec{x} - \vec{b}_{n\cdot} \cdot \vec{\mu}$

$$J_n = \det\begin{bmatrix} \partial h_1/\partial x_1 & \cdots & \partial h_1/\partial x_n \\ \vdots & & \\ \partial h_n/\partial x_1 & \cdots & \partial h_n/\partial x_n \end{bmatrix} = \det\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \det[A^{-1}] = \frac{1}{\det A}$$

Note: $AA^{-1} = I \Rightarrow \det[AA^{-1}] = 1 \rightarrow \det[A]\det[A^{-1}] = 1$

$$f_{\vec{x}}(\vec{x}) = f_{\vec{z}}(h(\vec{x}))|J_n| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-\vec{\mu}))^T(A^{-1}(\vec{x}-\vec{\mu}))} \frac{1}{|\det(A)|}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(A)^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1} (\vec{x}-\vec{\mu})} = \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{m})^T \Sigma^{-1}(\vec{x}-\vec{m})}$$

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$$\det[\Sigma] = \det[AA^T] = \det(A^T) = \det[A]^2$$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{m})^T \Sigma^{-1} (\vec{x}-\vec{\mu})} = f_{\vec{X}}(\vec{x}) = N_n(\vec{\mu}, \Sigma)$$

Does this work is A is $m \times n$? The Answer is no, .. but we will solve that another way.

Multivariate chf's

$$\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right] = E\left[e^{i(t_1 x_1 + \cdots + t_n x_n)}\right] = E\left[e^{it_1 x_1} e^{it_2 x_2} \cdots e^{it_n x_n}\right]$$

if $x_1, \ldots, x_n$ inde

$$\overset{\perp}{=} E\left[e^{it_1 x_1}\right] E\left[e^{it_n x_2}\right] \cdots E\left[e^{it_n, x_n}\right] = \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) \cdots \phi_{X_n}(t_n)$$

(P0) $\phi_{\vec{X}}(\vec{0}) = 1$. (P1) = Yes!

(P2) $\vec{Y} = A\vec{X} + \vec{b} \Rightarrow \phi_{\vec{Y}}(t) = E\left[e^{i\vec{t}^T(A\vec{X}+\vec{b})}\right] = E\left[e^{it^T Ax} e^{i\vec{t}^T\vec{b}}\right]$

$$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{X}}(\vec{t}') = e^{i\vec{t}^T\vec{b}} \phi_X(A^T\vec{t})$$

Let's find the chf of the standard ~~MVN~~ N, $\vec{Z} \sim N_n(\vec{0}, I)$

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^n \phi_Z(t_i) = \prod_{i=1}^n e^{-t_i^2/2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

$\vec{X} = A\vec{z} + \vec{u} \sim N(\vec{m}, \underset{AA^T}{\underset{\parallel}{\Sigma}})$, $\phi_X(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{u}} \phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}\vec{\mu}} e^{-\frac{1}{2}(A^T\vec{t})^T A^T\vec{t}}$

$$= e^{i\vec{t}\vec{m} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{X} \sim N(\vec{m}, \Sigma)$, $\vec{Y} = B\vec{X} + \vec{c}$, $B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^{m}$

$$\Phi_{\vec{Y}}(\vec{t}) \overset{(B)}{=} e^{i\vec{t}^T\vec{c}}\, e^{i(B^T\vec{t})^T\vec{m} - \frac{1}{2}(B^T\vec{t})^T\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(B\vec{m}+\vec{c}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{(F)}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{m}+\vec{c}, B\Sigma B^T)$$

$\vec{X} \sim N_n(\vec{m}, \Sigma)$, $(\vec{X}-\vec{m})^T\Sigma^{-1}(\vec{X}-\vec{m}) \sim ?$

Assume: $\Sigma = AA^T$

$$(AA^T)^{-1} = (A^{-1})^T A^{-1} \qquad (\vec{X}-\vec{m})^T(A^{-1})^T A^{-1}(\vec{X}-\vec{m})$$

$$= (A^{-1}(\vec{X}-\vec{m}))^T(A^{-1}(\vec{X}-\vec{m}))$$

$\vec{Z} \sim N_n(\vec{0}, I)$ $\quad = \vec{Z}^T\vec{Z} \sim \chi^2_n$ Mahalanobis Distance (1932)