Monday September 14th 2020

Lecture 5

$X \sim \text{Multin}_2(n, \vec{p})$
$\uparrow$
$k=2$

$$\text{Deg}(n-x_2) = P_{X_1|X_2}(x_1, x_2) := P(X_1 = | X_2 = x_2) = \frac{P(x_1, x_2)}{p(x_2)}$$

last time $p(x_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

$$= \frac{\binom{n}{x_1\, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} (1-\underbrace{p_2}_{p_1})^{n-x_2}} \quad x_1 = n - x_2$$

Define $J_n := \{0, 1, \dots, n\}$

$$= \frac{\cancel{\frac{n!}{x_1! x_2!}} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} \cancel{p_2^{x_2}}}{\cancel{\frac{n!}{x_2!(n-x_2)!}} \cdot \mathbb{1}_{x_2 \in J_n} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

$$\mathbb{1}_A = \frac{\mathbb{1}_A}{\mathbb{1}_A} := \begin{cases} 1 & \text{if } A \\ \text{undefined} & \text{if } A^c \end{cases}$$

$$= \underbrace{\frac{(n-x_2)!}{x!}}_{=1 \text{ if } x_1 = n-x_2} \mathbb{1}_{x_1 = n-x_2} \underbrace{\mathbb{1}_{x_1 \in J_n}}_{=1 \text{ if } x_1 = n-x_2} \underbrace{p_1^{x_1+x_2-n}}_{=1 \text{ if } x_1 = n-x_2} \mathbb{1}_{x_2 \in J_n}$$

$$P(A|B) = \frac{P(A,B)}{P(B)} \quad \text{if } P(B) = 0 \Rightarrow P(A|B) \text{ undefined}$$

Let's generalize this conditional probability a little bit:

$$P_{\vec{X}_{-j}|X_j}(\vec{x}_{-j}, x_j) = \text{Multin}_{k-1}(n - x_j, ?)$$

$\uparrow$ this is the vector without the jth component

$$= \frac{\text{Multin}_k(n, \vec{p})}{\text{Bin}(n, p_j)} = \frac{\binom{n}{x_1, \dots x_j, \dots, x_k} p_1^{x_1} \cdots p_j^{x_j} \cdots p_k^{x_k}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-x_j}}$$

$$= \frac{\frac{\cancel{n}!}{x_1! \dots, x_j! \dots x_k!} \mathbb{1}_{\underbrace{x_1 + x_j + \dots + x_j + \dots + x_k = n}_{x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k = n - x_j}} \mathbb{1}_{x_j \in J} \cdots \mathbb{1}_{x_k \in J_n} p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} \cdots p_k^{x_k}}{\cancel{\frac{n!}{x_j!(n-x_j)!}} \mathbb{1}_{x_j \in J_n} (1-p_j)^{n-x_j}}$$

Note: $p_1 + \dots + p_k = 1 \Rightarrow p_1 + \dots +$

$$\Rightarrow \underbrace{\frac{p_1}{1-p_j}}_{p_1'} + \dots + \underbrace{\frac{p_{j-1}}{1-p_j}}_{p_{j-1}'} + \underbrace{\frac{p_{j+1}}{1-p_j}}_{p_{j+1}'} + \dots + \underbrace{\frac{p_k}{1-p_j}}_{p_k'} = 1$$ divide both side by $1-p_j$

Let $n' := n - x_j$

Note: $n - x_j = x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k$ ∴ Prob zero ↗ $\binom{n'}{x_1,\dots,x_{j-1},x_{j+1},\dots,x_k}$

$$= \frac{n!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_k!} \mathbb{1}_{x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k} = \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_{j-1} \in J_n} \mathbb{1}_{x_{j+1} \in J_n} \cdots \mathbb{1}_{x_k \in J_n}$$

$$p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}}\, p_{j+1}'^{x_{j+1}} \cdots p_k'^{x_k} \qquad \frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}}\, p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{x_1 + \dots + x_{j-1} + x_{j+1} + \dots + x_k}} \mathbb{1}^n_{x_j \in J_n}$$

$$= \text{Multin}_{k-1}(n', \vec{p}\,') \, \mathbb{1}^n_{x_j \in J_n}$$

$\vec{X} \sim \text{Multin}_k(n, \vec{p})$ what is $E[\vec{X}]$? $Var[\vec{X}]$?

Review from Math 241. Let $X_1, \dots, X_n$ be rv's and $a, c \in \mathbb{R}$

$E[aX + c] = a E[X] + c$ identi:

$E[\sum X_i] = \sum E[X_i] = n\mu$

$E[\prod X_i] = \prod E[X_i]$

↑ if independent

← standard deviation

$\sigma^2 := Var[X] := E[(X-\mu)^2] = E(X^2) - \mu^2$, $\sigma := SD[X] := \sqrt{Var[X]}$

$$Var[X_1 + X_2] = E[((X_1+X_2) - (\mu_1+\mu_2))^2] = E[X_1^2 + X_2^2 + \mu_1^2 + \mu_1^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2\mu_1\mu_2]$$

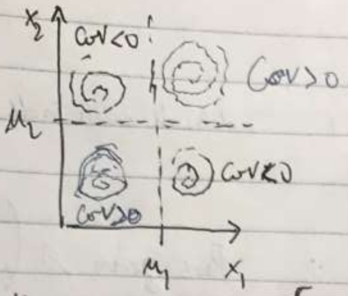
$$= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 + 2E[X_1X_2] - 2\mu_1^2 - \cancel{2\mu_1\mu_2} - 2\mu_1\mu_2 - 2\mu_2^2$$

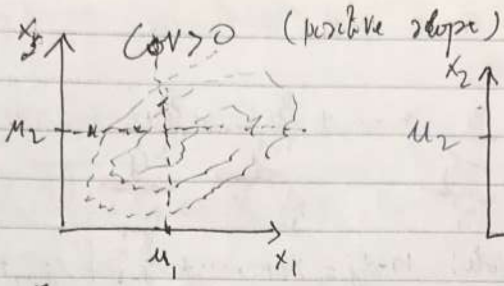
$$= \sigma_1^2 + \mu_1^2 + \sigma_2^2 + \cancel{\mu_2^2} + \cancel{\mu_1^2} + \cancel{\mu_2^2} + 2E[X_1X_2] - 2\mu_1^2 + 2\mu_1\mu_2 \quad - 2\mu_1\mu_2 - \cancel{2\mu_1^2}$$

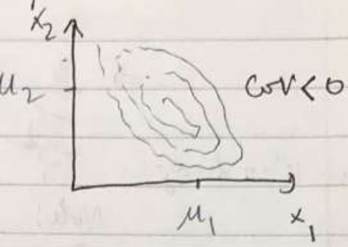
$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E(X_1X_2) - \mu_1\mu_2}_{Cov(X_1,X_2)})$$

$$= \sigma_1^2 + \sigma_2^2 + 2\overset{\sigma_{12}}{Cov[X_1, X_2]}$$ covariance of X_1 with X_2

$= \sigma_1^2 + \sigma_2^2$ (if X_1, X_2 are independent







HW: $cov[x_1, x_2] = E[(x_1-\mu_1)(x_2-\mu_2)]$ "use the above formula"

Covariance Rules:

1. $Cov[X, X] = \sigma^2$
2. $Cov[X_1, X_2] = Cov[X_2, X_1]$
3. $Cov[X_1+X_2, X_3] = Cov[X_1, X_3] + Cov[X_2, X_3]$
4. $Cov[a_1 X_1, a_2 X_2] = a_1 a_2 \sigma_{1,2}$
5. $Var[X_1 + \cdots + X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j]$

$$E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}$$

let $m = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}$

$$E[m] := \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ & & \vdots \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

$$Var[\vec{X}] := E[\underbrace{\vec{X}\vec{X}^T}_{\text{outer product}}] - \vec{\mu}\vec{\mu}^T = \begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \cdots & Cov[X_1,X_k] \\ Cov[X_2,X_1] & Var[X_2] & \cdots & Cov[X_2,X_k] \\ \vdots & & \ddots & \\ Cov[X_k,X_1] & Cov[X_k,X_2] & \cdots & Var[X_k] \end{bmatrix}$$

$(k\times 1)(1\times k) = k\times k$

Variance-Covariance matrix and it is symmetric.