MA368

08/31/2020

Lecture 02

x_1	x_2	$P_X(x_1, x_2)$	T
1	1	p^2	2
1	0	$p(1-p)$	1
0	1	$(1-p)p$	1
0	0	$(1-p)^2$	0

mutually exclusive, collectively exhaustive events

$$T \sim \begin{cases} 2 & \text{w.p. } p^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 0 & \text{w.p. } (1-p)^2 \end{cases}$$

Sum up all possible x_1, x_2 pair.

$$\sum_{x \in \mathbb{R}} P_{X_1,X_2}(x, t-x)$$

selecting out all the events I care about.

$P_T(t)$

$$p(T) = p(T=t) = \underbrace{\sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}}}_{\text{search through } \mathbb{R}^2} \underbrace{P_{X_1 X_2}(x_1, x_2)}_{\text{add up all possibilities}} \mathbb{1}_{x_1 + x_2 = t}$$

$$= P_{X_1}(t) * P_{X_2}(t)$$

$$x_1 + x_2 = t \Rightarrow x_2 = t - x_1$$

Ex

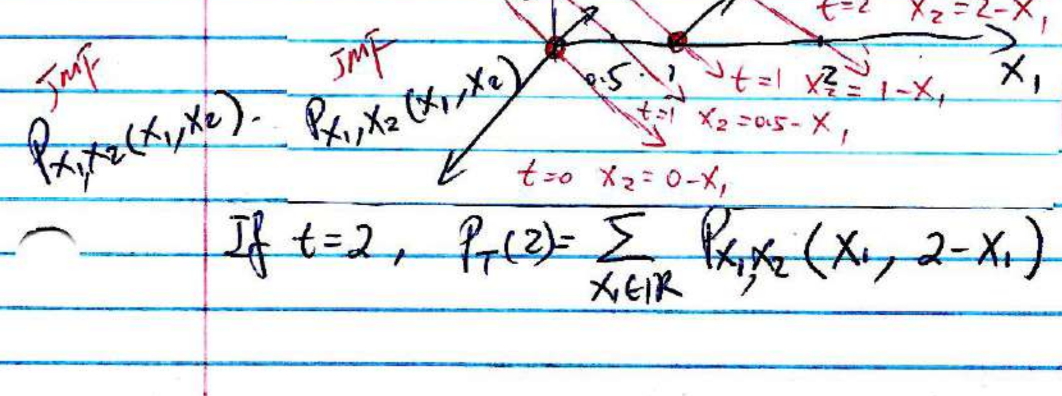
$$P(T=1) = P_{X_1,X_2}(1,0) + P_{X_1,X_2}(0,1)$$

$t = x_1 + x_2$

$\Rightarrow x_2 = t - x_1$

$P(T=0) \quad x_2 = 0 - x_1 = -x_1$

One way to look at.



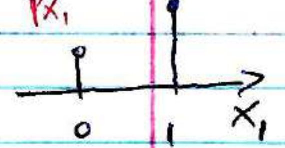
$$\text{If } t=2, \quad P_T(2) = \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1, 2-x_1)$$

Look at tree diagram on lecture ①

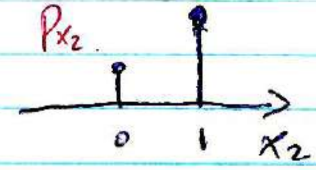
Another way to look at:

"Convolve" means to "roll or coil together / entwine"

pmf. P_{X_1}



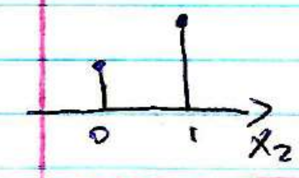
P_{X_2}



convolution

$$P_T = P_{X_1} * P_{X_2}$$

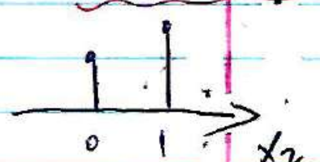
Roll up → Sum-product Hmm.



$t=1$ w.p. $p(1-p)$

$t=0$ w.p. $(1-p)^2$

$t=2$ w.p. p^2



$t=1$ w.p. $(1-p)p$

P_T.

$$P_T(t) = \sum_{X_1 \in \mathbb{R}} \sum_{X_2 \in \mathbb{R}} P_{X_1,X_2}(X_1,X_2)\, \mathbb{1}_{X_2 = t - X_1}$$

$$= \sum_{X \in \mathbb{R}} P_{X_1,X_2}(X, t-X)$$

$\boxed{\text{General Convolution Formula}}$

If X_1, X_2 independent.

Apply Multiplication Rule.

$$= \sum_{X \in \mathbb{R}} P_{X_1}(X)\, P_{X_2}(t-X)$$

$\boxed{\text{Convolution Formular For Independent rvs.}}$

ultimate indicator = TURN OFF.

$$= \sum_{X \in \mathbb{R}} P^{dd}_{X_1}(X)\, \boxed{\mathbb{1}_{X \in \text{Supp}[X_1]}}\, P^{dd}_{X_2}(t-X)\, \mathbb{1}_{t-X \in \text{Supp}[X_2]}$$

Con't

$$= \sum_{x \in Supp[X_1]} p^{old}_{X_1}(x)\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}.$$

If X_1, X_2 iid

$$= \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in Supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$\boxed{\text{Convolution Formula for iid rvs}}$

$$p_{T_2}(t) = T_2 \sim \sum_{x \in \{0,1\}} \overbrace{p^x (1-p)^{1-x} p^{t-x}}^{p^t} (1-p)^{\overset{1-(t-x)}{1-t+x}}\, \underbrace{\mathbb{1}_{t-x \in \{0,1\}}}_{t \in \{x, x+1\}}.$$

$(1-p)^{2-t}$

Add x to both.

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}.$$

Just plug in $\{0,1\}$.

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right).$$

$$T_2 \sim \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 2 & \text{w.p. } p^2 \end{cases} \quad \begin{cases} t=0 & \Rightarrow 1 = \binom{2}{0} \\ t=1 & \Rightarrow 2 = \binom{2}{1} \\ t=2 & \Rightarrow 1 = \binom{2}{2} \end{cases} = \binom{2}{t}$$

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$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0,1,\dots,n\}}$$

Cont → 1st parameter tell you: How many you add.

$$= \binom{2}{t} p^t (1-p)^{2-t} = \text{Binom}(2, p), \quad \text{Supp}[T_2] = \{0,1,2\}$$

It's a set

Generally, $\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2]$.

Define, $A+B := \{a+b : a \in A, b \in B\}$.

$$P_{T_2}(t) = \sum_{x \in \mathbb{R}} p(x)p(t-x) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$= p^x(1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$. $\quad \left\{ \binom{1}{x} = \mathbb{1}_{x \in \{0,1\}} \right\}$ use $\binom{n}{k}$

$$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) = \binom{1}{x} p^x (1-p)^{1-x}.$$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x} \binom{1}{t-x}$$

Recall Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= p^t (1-p)^{2-t} \underbrace{\left(\binom{1}{0}\binom{1}{t} + \binom{1}{1}\binom{1}{t-1} \right)}_{=\binom{2}{t}}$$

$$= \boxed{\binom{2}{t} p^t (1-p)^{2-t}.} = \text{Binom}(2,p).$$

Side Note

$X_1, X_2 \overset{iid}{\sim} Bern(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

$$\binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}$$

$X_1, X_2 \overset{iid}{\sim} Bern(p) = \binom{1}{x} p^x (1-p)^{1-x}.$

$$P_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x}$$

plug in $\{0,1\}$ for x.

$$= p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t}.$$

Recall Pascal's Identity.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= Binom(2, p)$$

$$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$$

Convolution for: Not iid but independent b/c T_2 Binom T_3 Bern.

$$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$$

$$P_{T_3}(t) = \sum_{x \in \text{Supp}[X_3]} P_{X_3}(x) P_{T_3}(t-x)$$

$$= \sum_{\substack{x \in \{0,1\} \\ \text{b/c it's Bern}}} \left(p^x(1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x}(1-p)^{2-t+x}\right)$$

$$= p^t(1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x}$$

$$= p^t(1-p)^{3-t}\left(\binom{2}{t} + \binom{2}{t-1}\right)$$

$$\boxed{= \binom{3}{t} p^t (1-p)^{3-t}} = \text{Binom}(3, p).$$

HW Find PMF of Binom(n,p) via Induction.

$X_1, X_2 \overset{iid}{\sim} Binom(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$

$T = X_1 + X_2 \sim ?$

$P_T(t) = \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$

$= p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t-x}$

Vandermonde's Identity gives us.

$= \binom{2n}{x} p^t (1-p)^{2n-t} = Binom(2n, p)$