$$= \sum_{j=k}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

Note: $j = n \Rightarrow 0$

$$= \sum_{j=k}^{n} \frac{n!}{\cancel{j!}(n-j)!}_{(j-1)!} \cancel{j} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{\cancel{n}\, n-1} \frac{n!}{j!\cancel{(n-j)!}} \cancel{(n-j)} f(x) F(x)^{j} (1-F_x)^{n-j-1}$$

$(n-j-1)! = (n-(j+1))!$ if $j = n-1$, $\ell - 1 = n-1 \Rightarrow \ell = n$; $n-(j+1)$

reindexing: let $\ell = j+1 \Rightarrow j = \ell - 1$ only

$$= \sum_{j=k}^{n} \underbrace{\frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j}} - \sum_{\ell=k+1}^{n} \underbrace{\frac{n!}{(\ell-1)!(n-\ell)!} f(x) F(x)^{\ell-1} (1-F(x))^{n-\ell}}$$

these are equal now

$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x)$$ DONE for iid case

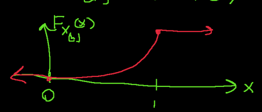
Check min, max:

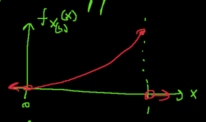
$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1} \checkmark$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1} \checkmark$$

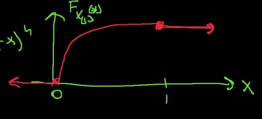
$$X_1, \ldots, X_n \overset{iid}{\sim} U(0,1) = \underbrace{1\, \mathbb{1}_{x \in [0,1]}}_{f(x)} \Rightarrow F(x) = x \text{ in the support}$$

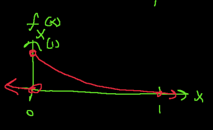
$$F_{X_{(n)}} = F(x)^n = x^n$$





$$F_{X_{(1)}}(x) = 1 - (1-F(x))^n = 1 - (1-x)^n$$





$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]} \Rightarrow f_{X_{(n)}}(x) = n x^{n-1} \mathbb{1}_{x \in [0,1]},\ f_{X_{(1)}}(x) = n(1-x)^{n-1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]} = Beta(k, n-k+1)$$

we will see the general beta distribution later

$X \sim Gamma(\alpha_1, \beta)$ indep of $Y \sim Gamma(\alpha_2, \beta) \Rightarrow X + Y \sim Gamma(\alpha_1 + \alpha_2, \beta)$ this seems right

The easiest proof of this is to employ "kernels". What's a kernel?

$$\underbrace{p(x) = c\, k(x), \quad f(x) = c\, k(x)}_{\text{Kernel}}$$ normalizing constant

$$\Rightarrow p(x) \propto k(x), \quad f(x) \propto k(x)$$

If you know k(x), you can resolve c via the following:

$$1 = \sum p(x) = \sum c\, k(x) \Rightarrow \sum k(x) = \frac{1}{c} \Rightarrow c = \left(\sum k(x) \right)^{-1}$$

$$1 = \int f(x) dx = \int c\, k(x) dx \Rightarrow \int k(x) dx = \frac{1}{c} \Rightarrow c = \left(\int k(x) dx \right)^{-1}$$

this means that k(x) identifies a PMF or PDF. So if you "see" a k(x) in the wild, you immediately know the rv because you know that you can solve for c if you had a computer.

$$X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\ldots,n\}}$$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p} \right)^x \mathbb{1}_{x \in \{0,1,\ldots,n\}}}_{k(x)}$$

$$Y \sim Weibull(k, \lambda) = \underset{c}{k} \lambda (\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = \overbrace{k \lambda^k}^{c} \overbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}}^{k(x)}$$

$$X \sim Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \overbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}}^{k(x)} \propto x^a e^{-bx}$$

Let's add the gammas

$$f_{X+Y}(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} \cancel{e^{-\beta x}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \overbrace{e^{-\beta(t-x)}}^{e^{-\beta t} \cancel{e^{+\beta x}}} \mathbb{1}_{t-x \in [0,\infty)} dx$$

$x - t \in (-\infty, 0]$, $x \in (-\infty, t]$

Let's find this density's kernel, k(t)

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} dx\, \mathbb{1}_{t \geq 0} \propto e^{-\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} dx\, \mathbb{1}_{t \geq 0}$$

$$= e^{-\beta t} t^{\alpha_1 - 1} t^{\alpha_2 - 1} \int_0^t \left(\frac{x}{t} \right)^{\alpha_1 - 1} \left(1 - \frac{x}{t} \right)^{\alpha_2 - 1} dx\, \mathbb{1}_{t \geq 0}$$

this integral is proven to be impossible

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du \Rightarrow x = 0 \Rightarrow u = 0,\ x = t \Rightarrow u = 1$

$$= e^{-\beta t} t^{\alpha_1 + \alpha_2 - 2} \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} t\, du\, \mathbb{1}_{t \geq 0} = e^{-\beta t} t^{\alpha_1 + \alpha_2 - 1} \mathbb{1}_{t \geq 0} \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du$$

even if the integral is impossible, what will the result be a function of? alpha_1 and alpha_2

$$\propto t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t} \mathbb{1}_{t \geq 0} \propto Gamma(\alpha_1 + \alpha_2, \beta) = \overbrace{\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du}^{c} \cdot \underbrace{t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t} \mathbb{1}_{t \geq 0}}_{k(t)}$$

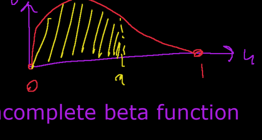
that integral is quite famous and it's called the "beta function"

$$B(\alpha_1, \alpha_2) := \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du$$

We can use probability theory to get an integral identity:

$$X \sim Gamma(\alpha_1 + \alpha_2, \beta) = \frac{\cancel{\beta^{\alpha_1 + \alpha_2}}}{\Gamma(\alpha_1 + \alpha_2)} \cancel{t^{\alpha_1 + \alpha_2 - 1}} \cancel{e^{-\beta x}} \cancel{\mathbb{1}_{x \geq 0}} = \frac{\cancel{\beta^{\alpha_1 + \alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1, \alpha_2) \cancel{t^{\alpha_1 + \alpha_2 - 1}} \cancel{e^{-\beta x}} \cancel{\mathbb{1}_{x \geq 0}}$$

$$\Rightarrow B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$



$$B(a, \alpha_1, \alpha_2) := \int_0^a \underbrace{u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1}}_{g(u)} du$$ incomplete beta function

$$I_a(\alpha_1, \alpha_2) = \frac{B(a, \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \in [0,1]$$ regularized incompleted beta function

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$$X \sim Beta(\alpha, \beta) := \underbrace{\frac{1}{B(\alpha, \beta)} x^{\alpha - 1} (1-x)^{\beta - 1} \mathbb{1}_{x \in [0,1]}}_{f(x)}$$ where $\alpha, \beta > 0$

$$F(x) = \int_0^x \frac{1}{B(\alpha, \beta)} y^{\alpha - 1} (1-y)^{\beta - 1} dy = \frac{\int_0^x y^{\alpha - 1} (1-y)^{\beta - 1} dy}{B(\alpha, \beta)} = \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta)$$

$$1 = \int_{Supp[X]} f(x) dx = \int_0^1 \frac{1}{B(\alpha, \beta)} x^{\alpha - 1} (1-x)^{\beta - 1} dx = \frac{1}{B(\alpha, \beta)} \underbrace{\int_0^1 x^{\alpha - 1} (1-x)^{\beta - 1} dx}_{beta} = \frac{\cancel{B(\alpha, \beta)}}{\cancel{B(\alpha, \beta)}} = 1$$