Lecture 16

Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)| dx < \infty \}$ all the functions in this set are called L^1 integrable or absolutely integrable

Are PDFs in L^1? $\int_{\mathbb{R}} |f(x)| dx = \int_{-\infty}^{\infty} |f(x)| dx = F(\infty) - F(-\infty) = 1 - 0 = 1$ ✓ yes.

If $f(x) = x^2$, then $f(x) \notin L^1$ since $\int_{\mathbb{R}} x^2 dx = \infty$.

If $f \in L^1 \Rightarrow \exists \hat{f}$, the Fourier Transform of f:

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x) dx = \mathcal{F}[f]$$

This is called the forward Fourier transform or "fourier analysis". x is called the "time domain" and omega is called the "frequency domain". One of Fourier's ideas is that functions in L^1 can be decomposed into a sum of sines and cosines with different frequencies, omega, and amplitudes, $|f(\text{omega})|$, and phase shifts, $Arg[f(\text{omega})]$.

Further, if $\hat{f} \in L^1$, then we can do a "reverse/inverse Fourier transform" to restore our original func. f:

$$f(x) = \int_{\mathbb{R}} e^{-i2\pi\omega x} \hat{f}(\omega) d\omega = \mathcal{F}^{-1}(\hat{f}).$$

This is called the "inverse fourier transform" or "fourier synthesis".

Fourier Inversion thm: if f and $\hat{f}$ are in L^1, then f and $\hat{f}$ are 1:1.

We define the characteristic function (chf) for rv X as:

$$\phi_X(t) := E[e^{itX}] = \int_{\mathbb{R}} e^{itx} f_X(x) dx = \sum_{x \in \mathbb{R}} e^{itx} p_X(x)$$

this is the Fourier tranform with a different frequency unit $t = -2\pi\omega$

main properties:

0) $\phi_X(0) = E[e^0] = E[1] = 1 \quad \forall x, \forall t.$

1) $\phi_X(t) = \phi_Y(t) \iff X \overset{dist.}{=} Y$ "Uniqueness"

2) if $Y = aX + b$ where $a, b \in \mathbb{R}$ $\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{itaX} e^{itb}]$

$$= e^{itb} E[e^{itaX}] \overset{t' = ta}{=} e^{itb} E[e^{it'X}] = e^{itb} \phi_X(t') = e^{itb} \phi_X(at).$$

3) X_1, X_2 are indep. , $T = X_1 + X_2$

$$\phi_T(t) = E\left[e^{it(X_1+X_2)}\right] = E\left[e^{itX_1} e^{itX_2}\right] = E\left[e^{itX_1}\right] E\left[e^{itX_2}\right] = \phi_{X_1}(t)\,\phi_{X_2}(t)$$

4) Moment Generating: $E[X^n] = \dfrac{\phi_X^{(n)}(0)}{i^n}$

5) $\phi_X(t) \in [-1,1]$ i.e. the chf exists $\forall x, \forall t$

$\Downarrow$

$|\phi_X(t)| \leq 1$ Proof

$$\left|E\left[e^{itX}\right]\right| = \left|\int_{\mathbb{R}} e^{itx} f_X(x)\,dx\right| \leq \int_{\mathbb{R}} |e^{itx} f_X(x)|\,dx$$

$$\leq \int_{\mathbb{R}} |e^{itx}|\,|f_X(x)|\,dx = \int_{\mathbb{R}} |\cos(tx) + i\sin(tx)|\,f_X(x)\,dx =$$

$$= \int_{\mathbb{R}} \sqrt{\cos^2(tx) + \sin^2(tx)}\, f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = 1. \quad \square$$

6) Inversion: If $\phi_X(t) \in L^1$ then $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{-itx}\phi_X(t)\,dt$

7) Levy's CDF formula

$$P(X \in [a,b]) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{itb}}{it}\,\phi_X(t)\,dt$$

8) Levy's Continuity Thm

But first need another concept: Consider a sequence of r.v.'s $X_1, X_2, \ldots, X_n$.

We define "X_n converges in distribution to X" (denoted $X_n \xrightarrow{d} X$) as:

$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x$ "pointwise convergence"

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \implies X_n \xrightarrow{d} X.$$

If n large $\phi_{X_n}(t) \approx \phi_X(t) \Rightarrow X_n \overset{d}{\approx} X$

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chfs can do anything mgfs can do and more!
so emphasis will be on chfs

ex) $X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0} \, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x} dx = \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta-it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$ (chf of Gamma(α,β))

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep. of $X_2 \sim \text{Gamma}(\alpha_1, \beta)$, $T = X_1 + X_2$

$$\phi_T(t) \overset{(P3)}{=} \phi_{X_1}(t)\phi_{X_2}(t) = \left(\frac{\beta}{\beta-it}\right)^{\alpha_1}\left(\frac{\beta}{\beta-it}\right)^{\alpha_2} = \left(\frac{\beta}{\beta-it}\right)^{\alpha_1+\alpha_2}$$

$$\overset{(P1)}{\Longrightarrow} T \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta).$$

Lecture 17 consider $X_1, \dots, X_n$ iid rvs of unknown PMF/PDF but we know it has expectation mu and variance σ^2 (both finite).

let $T_n := X_1 + \dots + X_n$, $E[T_n] = n\mu$, $Var[T_n] = n\sigma^2$

let $\bar{X}_n := \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$, $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = Var\left[\frac{T_n}{n}\right] = \frac{1}{n^2} Var[T_n] = \frac{\sigma^2}{n}$

let $Z_n := \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu$, $E[Z_n] = 0$, $Var(Z_n) = Var\left(\frac{\sqrt{n}}{\sigma}\bar{X}_n\right)$

$$= \frac{n}{\sigma^2} \cdot \frac{\sigma^2}{n} = 1 = SD[Z_n]$$

↓

"X_n standardized"

$$\phi_{T_n}(t) \overset{P3}{=} \phi_{X_1}(t) \cdots \phi_{X_n}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \dots + \alpha_n} \overset{\text{indep.}}{\Longrightarrow} T_n \sim \text{Gamma}(\alpha_1 + \dots + \alpha_n, \beta)$$

$$\overset{\text{iid}}{\Longrightarrow} \left(\phi_X(t)\right)^n$$

$$\phi_{\bar{X}_n}(t) \overset{P2}{=} \phi_{T_n}(t/n) \overset{\text{iid}}{=} \phi_X(t/n)^n$$

$$\phi_{Z_n}(t) \overset{P2}{=} e^{\frac{-it\mu\sqrt{n}}{\sigma}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma} t\right) \overset{\text{iid}}{=} e^{\frac{-it\mu\sqrt{n}}{\sigma} \cdot \frac{\sqrt{n}}{\sqrt{n}}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n = e^{\frac{-it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{\left(-\frac{it\mu \cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\right)\frac{1/n}{1/n}} = e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{1/n} \cdot \frac{t^2/\sigma^2}{t^2/\sigma^2}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}\right)} = \phi_{Z_n}(t)$$

We want to examine $\lim \phi(t)$ and if we find its li...