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$$P_D(d) = \sum_{x \in \mathrm{Supp}[X]} P_X^{\mathrm{old}}(x)\, P_Y^{\mathrm{old}}(d-x)\, \mathbb{1}_{d-x \in \mathrm{Supp}[X]}$$

$$= \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\lambda^{\overset{=(x-d)}{-(d-x)}} e^{-\lambda}}{\underset{=(x-d)!}{(-(d-x))!}}\, \mathbb{1}_{d-x \in \{\dots,-1,0\}}$$

$x-d \in \{0,1,\dots\}$

$x \in \{d, d+1, \dots\}$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\, \mathbb{1}_{x \in \{d,d+1,\dots\}}$$

let $d' = -d = |d'|$ ($|d|$?) but $d \in \mathbb{Z}\dots$

$$= e^{-2\lambda} \begin{cases} \text{if } d \le 0 & \displaystyle\sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} \\ \text{if } d > 0 & \displaystyle\sum_{x \in \{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,x'!} \end{cases}$$

$d = |d|$

let $x' = x - d \Leftrightarrow x = x' + d$

$$I_{|d|}(2\lambda) := \sum_{x \in \{0,1,\dots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$

Modified Bessel Function of the First Kind

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d \in \mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$, discovered 1946.

Used to model point spreads in sports games, photon noise, ...

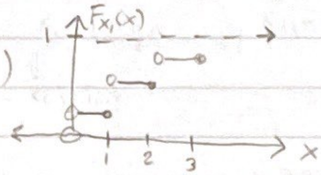
$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

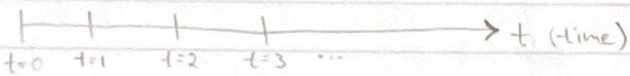
$$P_{X_1|T}(x,t) = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x, t-x)}{P_T(t)} \underset{(iid)}{=} \frac{P_{X_1}(x) P_{X_2}(t-x)}{P_T(t)} = \frac{\frac{e^{-\lambda}\lambda^{\cancel{x}}}{x!} \cdot \frac{e^{-\lambda}\lambda^{\cancel{t-x}}}{(t-x)!}}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}}$$

$$= \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}(t, 1/2)$$

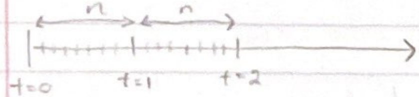
$$X_1 \sim \text{Geom}(p) := \overbrace{(1-p)^x p}^{p(x)} \mathbb{1}_{x \in \{0,1,\dots\}}, \quad \text{Supp}[X_1] = \{0,1,\dots\}$$

$$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = \left(1 - (1-p)^x\right)$$





in every "second", let's do n iid Bernoulli(p) experiments.



let's call the resulting geometric rv X_n and its units of realization is t.

$$p_{X_n}(x) = (1-p)^{nx} p \, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1, 1+\frac{1}{n}, \dots, 2, \dots\}}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx}$$

let $n \to \infty$, $p \to 0$ but $\lambda = np$ where $\lambda \in (0,\infty) \Rightarrow p = \frac{\lambda}{n}$ same as Poisson

$$p_X(x) := \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \dots\}}$$

$$= \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda x}} \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0} \underbrace{\lim_{n\to\infty} \mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\mathbb{1}_{x\in[0,\infty)}} = 0 \quad \forall x!$$

Not a valid pmf! $\Rightarrow \sum p_{X_\infty}(x) = 0$

$\text{Supp}[X_\infty] = [0, \infty)$

$$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nx} = \left(1 - e^{-\lambda x}\right) \mathbb{1}_{x\in[0,\infty)}$$

the pmf wasn't valid. Is the cdf valid? If so, need to check three properties.

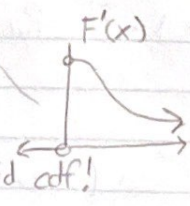
1) it's 0 as I go to $-\infty$, 2) it's 1 as I go to $+\infty$, 3) it's an increasing function ($\frac{d}{dx} \ge 0$)

(1) $\lim_{x\to-\infty} (1 - e^{-\lambda x}) \mathbb{1}_{x\in[0,\infty)} = 0$ ✓

We now have a continuous rv X.

(2) $\lim_{x\to\infty} (1 - e^{-\lambda x}) \mathbb{1}_{x\in[0,\infty)} = 1 - \lim_{x\to\infty} \frac{1}{e^{\lambda x}} = 1$ ✓

(3) $\frac{d}{dx}\left[(1 - e^{-\lambda x}) \mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x} \mathbb{1}_{x\in[0,\infty)} \ge 0$ ✓



Valid cdf!

Cont. r.v. properties:

$|Supp[X]| = |\mathbb{R}|$, uncountable infinity (the size of the continuum)

they do not have pmf's (because the probability of the r.v being at any specific number is zero) but they do have cdfs.

The derivative of the cdf is a very useful function, it is called the pdf denoted $f(x)$. (discrete rvs do not have pdf).

$f(x) := F'(x)$, $P(X \in [a,b]) = \overbrace{P(X \leq b)}^{F(b)} - \overbrace{P(X \leq a)}^{F(a)} = \int_a^b f(x)dx$ by fundamental theorem

$\int_{\mathbb{R}} f(x) = 1 =$ "$F(\overset{1}{\infty}) - F(\overset{0}{-\infty})$" $\quad f(x) \geq 0$ since cdfs are increasing functions

$\Rightarrow Supp[X] = \{x : f(x) > 0\}$ (if $X_1, \ldots, X_n$ indep, $\overbrace{f_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}^{\text{joint density func.}} = \prod_{i=1}^{n} f_{X_i}(x_i)$)

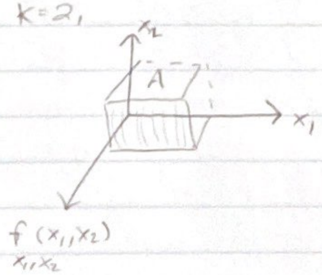
$X \sim Exp(\lambda) := \underbrace{\underbrace{\lambda e^{-\lambda x}}_{f^{\circ H}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}$, $F(x) = (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)}$

exponential r.v.

$\underbrace{\lambda \in (0, \infty)}_{\text{its parameter space}}$

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$. $\quad \underbrace{\int_{R} \cdots \int_{R}}_{K} f_{\vec{X}}(\vec{x}) dx_1 \ldots dx_k = 1$

$k=2$,



$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2) dx_1 dx_2$