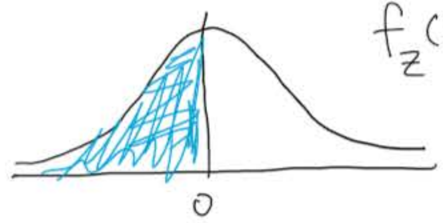
Lecture 18



$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$Z \sim N(0,1)$, $Y = Z^2 \sim f_Y(y) = ?$ Not 1:1

$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2P(Z \in [0, \sqrt{y}])$

$= 2(F_Z(\sqrt{y}) - \cancel{F_Z(0)}^{1/2}) = 2F_Z(y) - 1$

$f_Y(y) = \frac{d}{dy}[2F_Z(\sqrt{y}) - 1] = 2(\frac{1}{2} y^{-1/2}) f_Z(\sqrt{y}) = y^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \ge 0}}$

$\propto y^{1/2 - 1} e^{-1/2 y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}(1/2, 1/2)$

$Z_1, \ldots, Z_k \overset{iid}{\sim} N(0,1)$, $Y = Z_1^2 + \ldots + Z_k^2 \sim \text{Gamma}(\frac{k}{2}, \frac{1}{2})$

Note the beta is always 1/2 and the alpha is always k/2 so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}(\frac{k}{2}, 1/2) = \chi^2_k$ the "chi squared dist. with k degrees of freedom" ($k \in \mathbb{N}$).

$E[Y] = k \underbrace{E[Z^2]}_{1} = k$

PDF

$\hookrightarrow \chi^2_k : \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{k/2 - 1} e^{-y/2} \mathbb{1}_{y \ge 0}$ $\overset{k=1,\ \Gamma(1/2) = \sqrt{\pi}}{\downarrow}$ $= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} = \chi^2_1$

$X \sim \chi^2_k$, $Y = \sqrt{X} \Rightarrow x = y^2 = g^{-1}(y)$, $|\frac{d}{dy}[g^{-1}(y)]| = |2y| = 2y$

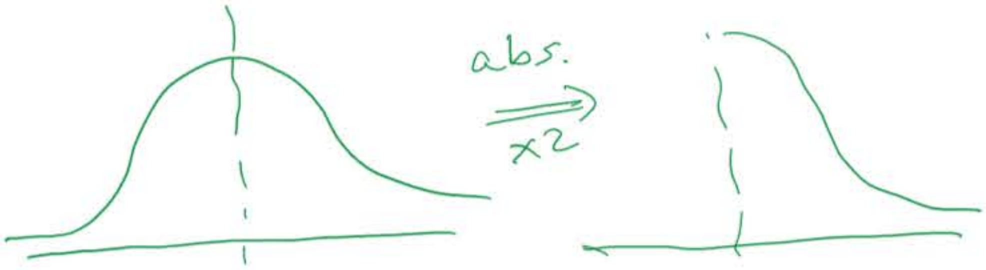
$f_Y(y) = f_X(y^2) \cdot 2y = \frac{(\frac{1}{2})^{k/2}}{\Gamma(k/2)} y^{k-2} e^{-y^2/2} \cdot 2y \, \mathbb{1}_{\underbrace{y^2 \ge 0}_{y \ge 0}}$

$= \frac{(\frac{1}{2})^{k/2 - 1}}{\Gamma(k/2)} y^{k-1} e^{-y^2/2} \mathbb{1}_{y \ge 0} = \chi_k$ The chi distribution with

$$\Gamma(\tfrac{k}{2}) \qquad \underbrace{y^2 \geq 0}_{y \geq 0}$$

$$= \frac{\left(\frac{1}{2}\right)^{k/2-1}}{\Gamma(k/2)} y^{k-1} e^{-y^2/2} \mathbb{1}_{y \geq 0} = \chi_k$$ The chi distribution with k degrees of freedom.

Note: $Z \sim N(0,1)$, $|Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2} \mathbb{1}_{y \geq 0}$



$$= 2\underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)}_{f_Z} \mathbb{1}_{y \geq 0}$$

Let's prove something about the Gamma:

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{y}{c}\right)^{\alpha-1} e^{-\beta \frac{y}{c}} \mathbb{1}_{\frac{y}{c} \geq 0 \Rightarrow y \geq 0}$$

$$= \frac{\left(\frac{\beta}{c}\right)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\left(\frac{\beta}{c}\right)y} \mathbb{1}_{y \geq 0} = \text{Gamma}(\alpha, \beta/c)$$

$$X \sim \chi^2_k, \quad Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{1/2}{1/k}\right) = \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$$

Let $X_1 \sim \chi^2_k$ indep. of $X_2 \sim \chi^2_k$

Let $U = \frac{X_1}{k_1} \sim \text{Gamma}\Big(\underbrace{\frac{k_1}{2}}_{a}, \underbrace{\frac{k_1}{2}}_{a}\Big)$ indep of $V = \frac{X_2}{k_2} \sim \text{Gamma}\Big(\underbrace{\frac{k_2}{2}}_{b}, \underbrace{\frac{k_2}{2}}_{b}\Big)$

$$R = \frac{U}{V} \sim f_R(r) = \int_{\text{Supp}[V]} f_U(rt) \mathbb{1}_{rt \in \text{Supp}[U]} f_V(t) |t| \, dt$$

formula derived previously – check notes.

Now, plug + chug:

$$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \underbrace{\mathbb{1}_{rt \in [0,\infty)}}_{r \in (0,\infty)} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} \, t \, dt$$

↳ The reason why we're doing this is because we want to

↳ The reason why we're doing this is because we want to derive all the distributions in the normal family.

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r=0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} dt$$

$$= a^a b^b r^{a-1} \mathbb{1}_{r\geq 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)} \cdot \Gamma(a+b)}_{\frac{1}{B(a,b)}} \cdot \frac{1}{(ar+b)^{a+b}} = \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\underbrace{b^{-(a+b)}}_{b^{-a}\cancel{b^{-b}}} (1+\frac{a}{b}r)^{-(a+b)}} \mathbb{1}_{r\geq 0}$$

$$= \frac{(a/b)^a}{B(a,b)} r^{a-1} \left(1+\frac{a}{b}r\right)^{-(a+b)} \mathbb{1}_{r\geq 0} = \frac{(k_1/k_2)^{k_1/2}}{B(\frac{k_1}{2},\frac{k_2}{2})} r^{k_1/2-1} \left(1+\frac{k_1}{k_2}r\right)^{-(\frac{k_1+k_2}{2})} \mathbb{1}_{r\geq 0}$$

$= F_{k_1,k_2}$ this is the "F distribution" or Fisher-Snecedor dist. with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom. $k_1, k_2 \in \mathbb{N}$.

Let $Z \sim N(0,1)$, $X \sim \chi^2_k$, $W = \frac{Z}{\sqrt{X/k}} \sim f_W(w) = ?$

↳ symmetric around 0

Consider $W^2 = \frac{Z^2/1}{X/k} \sim F_{1,k}$

$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$

Take derivatives:

$$\frac{d}{dw}\left[F_{W^2}(w^2)\right] = \frac{d}{dw}\left[F_W(w)\right] - \frac{d}{dw}\left[F_W(-w)\right]$$

$$\cancel{2}w f_{W^2}(w^2) = f_W(w) + f_W(-w) = \cancel{2} f_W(w)$$

$$\Longrightarrow f_W(w) = \cancel{w} \underbrace{\frac{(\frac{1}{k})^{1/2}}{B(\frac{1}{2},\frac{k}{2})}}_{\frac{\Gamma(\frac{k+1}{2})}{\Gamma(1/2)\Gamma(k/2)}} \underbrace{(w^2)^{1/2-1}}_{\cancel{w^{-1}}} \left(1+\frac{w^2}{k}\right)^{-\frac{1+k}{2}} \underbrace{\mathbb{1}_{w^2\geq 0}}_{w\in\mathbb{R}}$$

plug into F dist.

$Z \in \mathbb{R}$ divided by nonneg. is any $\mathbb{R}$

$$= \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\,\Gamma(\frac{k}{2})} \left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

student's T dist. with k degrees of freedom.

If $k \to \infty$ $T_k \to Z$

student's T dist. has the $N(0,1)$ shape but [illegible]

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$ student's T dist. with k degrees of freedom.

If $k \to \infty$ $T_k \to Z$

student's T dist. has the $N(0,1)$ shape but just thicker tails.

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru)f(u)|u|\,du =$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\,du = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} -|u|\,du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du\right)$$

$$= \frac{1}{2\pi}\left(\int_0^{-\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du + \int_0^{\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du\right) = \frac{1}{\pi}\int_0^{\infty} e^{-\frac{r^2+1}{2}u^2}\cdot u\,du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u}dt$, $u=0 \to t=0$, $u \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\frac{r^2+1}{2}t}\,\cancel{u}\,\frac{1}{2\cancel{u}}\,dt = \frac{1}{\cancel{2}\pi}\cdot\frac{1}{\frac{r^2+1}{\cancel{2}}}\underbrace{\int_0^{\infty}\frac{r^2+1}{2}e^{-\frac{r^2+1}{2}t}\,dt}_{\int_0^{1}\underbrace{\lambda e^{-\lambda x}}_{\text{PDF of exponential r.v.}}dx = 1}$$

$$= \frac{1}{\pi}\,\frac{1}{1+r^2} = \text{Cauchy}(0,1).$$

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let $X = c + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(c,\sigma) = \frac{1}{\sigma\pi}\cdot\frac{1}{1+\left(\frac{x-c}{\sigma}\right)^2}$$

Lecture 19

$$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi}\cdot\frac{1}{x^2+1}$$

$E[X] = \int_{\mathbb{R}} x\,\frac{1}{\pi}\,\frac{1}{x^2+1}\,dx = \infty \to$ The expecation does not exist

(mgf) $M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx = \infty \to$ D.N.E.

$\phi(t) = \int e^{itx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx = \dots = e^{-|t|}$, $\phi_X'(t) = \frac{-t}{|t|}e^{-|t|}$