Lecture-22

12/02/2020

Consider r.v's X and Y with finite means and Variances $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and let, $W = (X - cY)^2$, where c is a real constant.

Note: W is non negative

$$\Rightarrow E[W] \geq 0 \Rightarrow E[X^2 - 2cXY + c^2Y^2] \geq 0$$

Unique:

$$\frac{E[XY]}{E[Y^2]} \in \mathbb{R}$$

$$\Rightarrow E[X^2] - 2cE[XY] + c^2E[Y^2] \geq 0$$

$$\Rightarrow E[X^2] - 2\frac{E[XY]}{E[Y^2]}E[XY] + \frac{E[XY]^2}{E[Y^2]^2}E[Y^2] \geq 0$$

multiply by $E[Y^2]$

$$\Rightarrow E[X^2]E[Y^2] - 2E[XY]^2 + E[XY] \geq 0$$

$$\Rightarrow E[XY]^2 \leq E[X^2]E[Y^2] \quad \xrightarrow[\text{Non-neg}]{\text{If } XY} E[XY] \leq E[X^2]E[Y^2]$$

$$\Rightarrow |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

$$\text{cov}[X,Y] := E[XY] - E[X]E[Y]$$

$$\text{Cov}[X,Y] := \frac{\text{cov}[X,Y]}{SD[X]\,SD[Y]}$$, this unitless metric is called the "correlation between X and Y"

let, $Z_x = \frac{x - \mu_x}{\sigma_x}$ and $Z_y = \frac{y - \mu_y}{\sigma_y} \Rightarrow E[Z_x] = E[Z_y]$,

$SD[Z_x] = SD[Z_y] = E[Z_x^2] = E[Z_y^2]$

$$\left|E[Z_x Z_y]\right| \le \sqrt{E[Z_x^2]\,E[Z_y^2]} = \sqrt{1 \cdot 1} = 1 \Rightarrow E[Z_x Z_y] \in [-1, 1]$$

$$\mathrm{cov}[X,Y] = \frac{E[XY] - \mu_x \mu_y}{\sigma_x \sigma_y} = \frac{E(\sigma_x Z_x + \mu_x)(\sigma_y Z_y + \mu_y) - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$= \frac{\sigma_x \sigma_y E[Z_x Z_y] + \sigma_x \mu_y E[Z_x] + \sigma_y \sigma_y E[Z_y] + \mu_x \mu_y - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$= E[Z_x Z_y] \in [-1, 1]$$

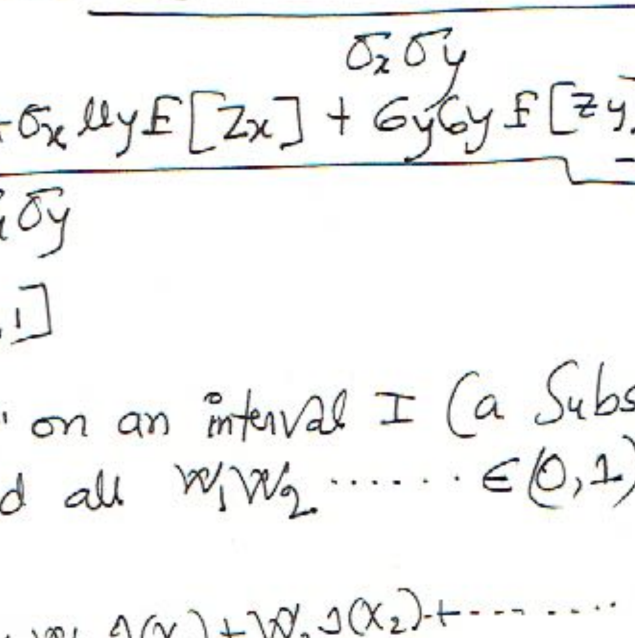
Definations:

g is a "convex function" on an interval I (a subset of reals) If for all $x_1, x_2 \ldots\ldots \in I$ and all $w_1 w_2 \ldots\ldots \in (0,1)$ s.t $\sum w_i = 1$ AKA the "weights",

$$g(w_1 x_1 + w_2 x_2 + \cdots) \le w_1 g(x_1) + w_2 g(x_2) + \cdots\cdots$$

In some notation,

$$g\left(\sum w_i x_i\right) \le \sum w_i g(x_i)$$



$w_1 + w_2 = 1$

$\left\{\sum w_i x_1 + w_2 x_2 : w_1 + w_2 = 1,\ w_1 > 0,\ w_2 > 0\right\}$

$\{ w_1 g(x_1) + w_2 g(x_2) : \cdots\cdots \}$

Theorem: g is convex on I If $\forall x \in I$, $g''(x) > 0$.

let, g be a convex function and x be a discrete r.v. If discrete we know $Supp[X] = \{x_1, x_2 \cdots\}$ and sum $P(x_i) = 1$ (PMF). Thus we can call the PMF values, the weights i.e $w_i = P(x_i)$

$$E[X] = \sum x_i P(x_i) = \sum w_i x_i$$

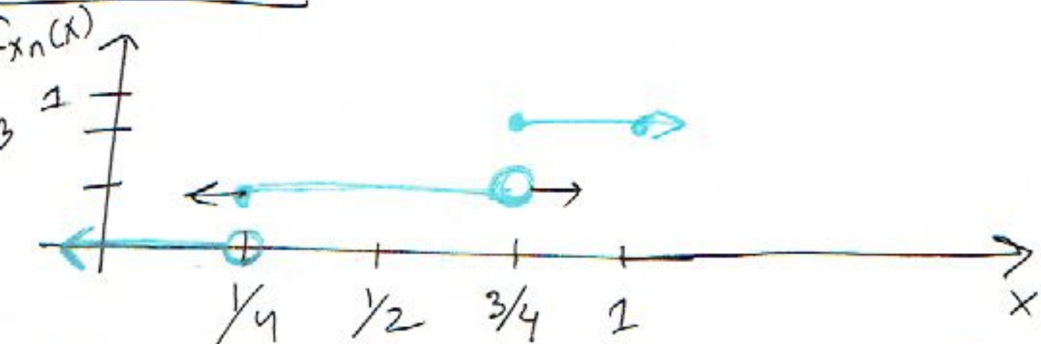
$$g(E[X]) \le \sum w_i g(x_i) = \sum g(x_i) P(x_i) = E[g(X)]$$, Jensen's inequality

Convergence of rv, we will study three different types first let's review " convergence in distribution" we say a sequence of rv's $X_1, X_2 \cdots$ denoted X_n, converges in distribution to X denoted $X_n \xrightarrow{d} X$ means by defination that the limiting CDF is X's CDF:

$$\lim_{x \to \infty} F_{X_n}(x) = F_X(x) \ \forall x$$

Consider, $X_n \sim \begin{cases} \frac{1}{n+1} & wp \ \frac{1}{3} \\ 1 - \frac{1}{n+1} & wp \ \frac{2}{3} \end{cases}$ e.g: $X \sim \begin{cases} \frac{1}{4} & wp \ 1/3 \\ \frac{3}{4} & wp \ 2/3 \end{cases}$

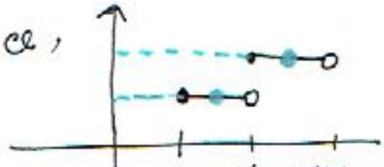
(1) Drawing CDF:



$\Rightarrow X \sim \begin{cases} 0 & wp \ \frac{1}{3} \\ 1 & wp \ \frac{2}{3} \end{cases}$

Conjecture: PMF convergence and CDF convergence are equivalent. This is not true in general. But there's a situation where it is true. let Supp $[X_n]$ be a subset of $\mathbb{Z}$, the integers and let Supp $[X]$ also be a subset of $\mathbb{Z}$, the integers, let's prove it.

Pf: CDF convergence implies the PMF convergence,



$$P_{X_n}(x) = F_{X_n}\left(x + \tfrac{1}{2}\right) - F_{X_n}\left(x - \tfrac{1}{2}\right)$$

$$\lim P_{X_n}(x) = \lim F_{X_n}\left(x + \tfrac{1}{2}\right) - \lim F_{X_n}\left(x - \tfrac{1}{2}\right) = F_X\left(x + \tfrac{1}{2}\right) - F_X\left(x - \tfrac{1}{2}\right) = P_X(x)$$

Pf: PMF convergence implies CDF convergence:

$$F_{X_n}(x) = P(X_n \le x) = \sum_{y=-\infty}^{x} P_{X_n}(y)$$

$$\lim F_{X_n}(x) = \lim \sum_{y=-\infty}^{x} P_{X_n}(y) = \sum_{y=-\infty}^{x} \lim P_{X_n}(y) = \sum_{y=-\infty}^{x} P_X(y) = P(X \le x) = F_X(x)$$

How about the continious r.v's? Is PDF convergence equivalent to CDF convergence?

Not always so. PDF convergence always implies CDF convergence but not vice versa. Here is a counter example.

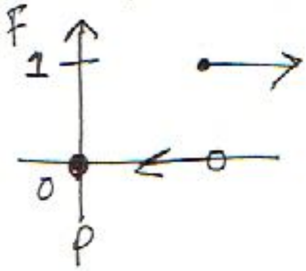
$X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right) = \frac{n}{2} \mathbb{1}_{x \in \left[-\frac{1}{n}, \frac{1}{n}\right]} = f_{X_n}(x)$

$\lim f_{X_n}(x) = \infty$! Not a PDF!

<u>HW:</u> $X_n \sim Bin\left(n, \frac{\lambda}{n}\right), \lambda > 0$, Pm: $X_n \xrightarrow{d} X \sim Poisson(\lambda)$

Define $X_n \xrightarrow{d} c$, $c \in \mathbb{R}$ as $X_n \xrightarrow{d} X \sim Deg(c)$

$\lim F_{X_n} = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases}$



Convergence in Probablity to a constant.

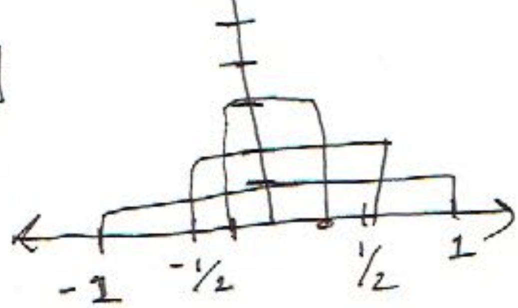
for sequence of random variables $X_1, X_2 \cdots X_n$, X_n convergens in probablity to a constant c.

$X_n \xrightarrow{P} c$ is defined to be:

$\forall \epsilon > 0 \quad P(|X_n - c| \geq \epsilon) = 0$, or $\forall \epsilon > 0 \; P(|X_n - c| < \epsilon) = 1$

here, X_n is uniform

$X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n} \mathbb{1}_{x \in \left[\frac{1}{n}, \frac{1}{n}\right]}$



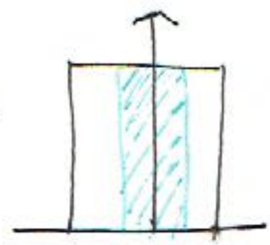
let's say,

$\varepsilon = 0.0001$

$n = 100$, $X_n \sim U(-.01, .01)$

$= P([X_n - 0] \leq 0.0001)$

$= P\left(X_n \in [-0.0001, 0.0001]\right) = \frac{2}{100}, \frac{2}{1000} \neq 1$



$n = 1000$, $X_n \sim U(-0.0001, .0001)$

$P(X_n \in [-.0001, .0001]) = 1$