Lecture 21

$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{X}}]$ for any vector rv X of dim. n.

consider: $\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{i[t\ 0 \dots 0]\vec{X}}\right] = E\left[e^{itX_1}\right] = \phi_{X_1}(t)$

P1/P8 $\Longrightarrow X_1 \sim f_{X_1}(x)$

would need to solve:

$f_{X_1}(x) = \int \cdots \int f_{X_1, X_2, \dots, X_n}(x, u_1, u_2, \dots, u_{n-1})\, du_1 \dots du_{n-1}$

but instead

e.g. $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$, $X_1 \sim f_{X_1}(x) = ?$

$\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = e^{i[t\ 0 \dots 0]\vec{\mu} - \frac{1}{2}[t\ 0 \dots 0]\Sigma\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = e^{it\mu_1 - \frac{t}{2}[t\ 0 \dots 0]\begin{bmatrix} \sigma_{11} \\ \vdots \\ \sigma_{1n} \end{bmatrix}}$

$= e^{it\mu_1 - \frac{t^2\sigma_1^2}{2}} = \phi_{X_1}(t) \overset{\text{P1}}{\Longrightarrow} X_1 \sim N(\mu, \sigma^2)$

We're essentially done with distribution Theory.
Rest of course will be different.

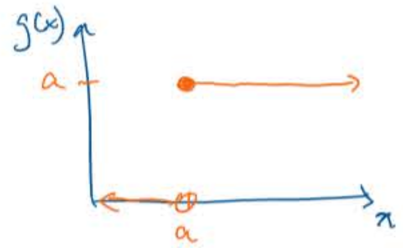
HW 6 ↑

HW 7 ↓

Assume X is a rv with nonnegative support i.e. $\text{Supp}[X] \geq 0$ and has finite expectation. Let $a > 0$, a constant.

Consider the following function:

$g(x) = a\mathbb{1}_{x \geq a}$



Is $a\mathbb{1}_{x \geq a} \leq x$? (Yes!) Two cases:

- if $x < a$ $\quad a\mathbb{1}_{x \geq a} = a \cdot 0 = 0 \leq x$ becomes $\text{Supp}[X] \geq 0$
- if $x \geq a$ $\quad a\mathbb{1}_{x \geq a} = a \cdot 1 = a \leq x$ true by case assumption.

$\Longrightarrow a\mathbb{1}_{x \geq a} \leq X$

Let's take the expectation of both sides:

$E[a\mathbb{1}_{x \geq a}] \leq E[X]$

$\mathbb{1}_{x \geq a} \sim \begin{cases} 1 & \text{w.p. } P(X \geq a) \\ 0 & \text{otherwise} \end{cases} = \text{Bern}(P(X \geq a))$

$\Longrightarrow aE[\mathbb{1}_{x \geq a}] \leq \mu \Rightarrow aP(X \geq a) \leq \mu$

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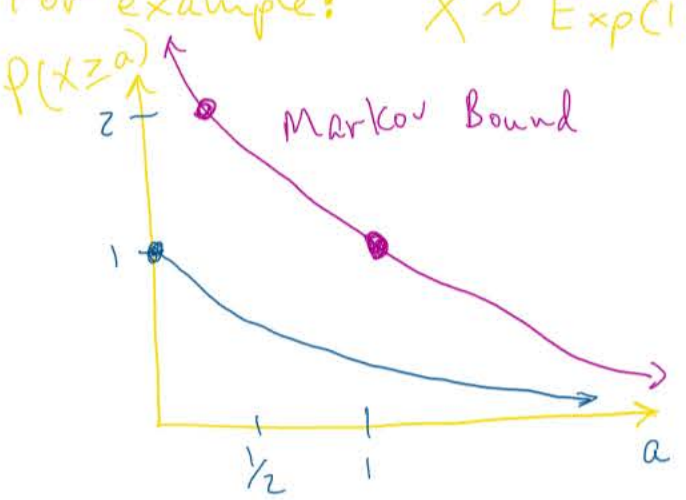
$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$ this is called "Markov's Inequality" and it's very famous.

For example: $X \sim Exp(1) = e^{-x} \Rightarrow P(X \geq a) = 1 - F_X(a) = e^{-a}$

$\Rightarrow \mu = 1$



Markov Bound is very crude.

a	$P(X \geq a)$	Markov Bound	Chebyshev bd.	Chernoff Bd.
2	0.1353	0.5	1	0.73571
5	0.0067	0.2	0.0635	0.09158
10	0.00004	0.1	0.0123	0.00123

We will now prove many corollaries of the Markov Inequality:

- let $b = a\mu$ $\quad P(X \geq b) \leq \frac{\mu}{b} \Rightarrow P(X \geq a\mu) \leq \frac{1}{a}$

- Let h be a monotonically increasing function, $Y = h(x)$

  $P(Y \geq h(a)) \leq \frac{E[Y]}{h(a)} \Rightarrow P(h(X) \geq h(a)) \leq \frac{E[h(x)]}{h(a)} \Rightarrow P(X \geq a) \leq \frac{E[h(x)]}{h(a)}$

- Let X be continuous in addition to nonnegative

  let $a = \text{Quantile}[X, p] = F_X^{-1}(p)$

  $P(X \geq F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow 1 - F_X(F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$

  $\Rightarrow 1 - p \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow \underbrace{F_X^{-1}(p)}_{Q[X,p]} \leq \frac{\mu}{1-p}$ $\quad$ e.g. $\text{Med}[X] \leq 2\mu$

- Let X be any rv $\Rightarrow |X|$ is a nonnegative r.v. $\quad P(|X| \geq a) \leq \frac{E[|X|]}{a}$

- Let X be any rv with finite variance, σ^2. Let $Y = (X-\mu)^2 \Rightarrow Y \geq 0$

• Let X be any rv with finite variance, σ^2. Let $Y=(X-\mu)^2 \Rightarrow Y \geq 0$

$$P(Y\geq b) \leq \frac{E[Y]}{b} \Rightarrow P((X-\mu)^2 \geq b) \leq \frac{E[(X-\mu)^2]}{b}$$ ← Definition of variance

Let $b=a^2$

$$\Rightarrow P((X-\mu)^2 \geq b) \leq \frac{\sigma^2}{b} \Rightarrow P((X-\mu)^2 \geq a^2) \leq \sigma^2/a^2$$

$$\Rightarrow P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$ This is called Chebyshev's Inequality

Let's manipulate this to get it into a more "user friendly" form.

Assume X is nonnegative:

disjoint errors

$$P(|X-\mu|) \geq a = P(X-\mu \geq a \cup -(X-\mu) \geq a) = P(X-\mu \geq a) + P(-(X-\mu) \geq a)$$

$$= P(X \geq \mu + a) + P(X \leq \mu - a) \overset{a \geq \mu}{=} P(X \geq \mu + a) + \underbrace{P(X \leq \text{negative } \#)}_{0}$$

let $b=\mu+a$

$$= P(X \geq b) \leq \frac{\sigma^2}{(b-\mu)^2}$$

• Let X be an rv. Let $Y=e^{tX} \Rightarrow Y$ is a nonnegative rv for all t.

$$P(Y\geq b) \leq \frac{E[Y]}{b} \Rightarrow P(e^{tX} \geq b) \leq \frac{E[e^{tX}]}{b}$$ → moment generating function for X, $M_X(t)$.

$$\Rightarrow P(e^{tX} \geq b) \leq \frac{M_X(t)}{b} \overset{\text{let } b=e^{ta}}{\Longrightarrow} P(e^{tX} \geq e^{ta}) \leq e^{-ta} M_X(t)$$

let $b=e^{ta}$

$$\Rightarrow P(tX \geq ta) \leq e^{-ta} M_X(t) \Rightarrow P(e^{tX} \geq e^{ta}) \leq e^{-ta} M_X(t)$$

$$\Rightarrow P(tX \geq ta) \leq e^{-ta} M_X(t)$$

$\Downarrow$ if $t>0$ 　　　　 ↘ if $t<0$

$$P(X \geq a) \leq e^{-ta} M_X(t) \qquad P(X \leq a) \leq e^{-ta} M_X(t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$P(X \geq a) \leq \min_{t>0} \{e^{-ta} M_X(t)\} \qquad P(X \leq a) \leq \min_{t<0} \{e^{-ta} M_X(t)\}$$

If these inequalities are valid for all t, why not choose the "best" t to get the "sharpest" (lowest) bound?

↳ This is called Chernoff's Inequality.

Let's calculate it for $X \sim Exp(\lambda)$. Warning: It's a lot of work.

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We first need to find the mgf for the exponential rv:

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^\infty = \frac{\lambda}{t-\lambda} \begin{cases} \infty - 1 & \text{if } t > \lambda \\ 0 - 1 & \text{if } t \le \lambda \end{cases} = \frac{\lambda}{\lambda - t} \text{ only for } t < \lambda.$$

If $t > \lambda$, the mgf DNE. This is why you shouldn't be using them. Chf's always exist!

$$X \sim Exp(1) \Rightarrow M_X(t) = \frac{1}{1-t} \text{ for } t < 1$$

$$P(X > a) \le \min_{t>0} \left\{ e^{-ta} \frac{1}{1-t} \right\} \text{ for } t < 1$$

$$\Rightarrow P(X > a) \le \min_{t \in (0,1)} \left\{ \overbrace{e^{-ta} \frac{1}{1-t}}^{h(t)} \right\} = e^{-(1-\frac{1}{a})a} \frac{1}{1-(1-1/a)} = \frac{e^{-a} e}{\frac{1}{a}} = \frac{ae}{e^a}$$

$$h'(t) = \ldots = \frac{e^{-ta}(ta - a + 1)}{(1-t)^2} \overset{\text{set}}{=} 0 \Rightarrow t = \frac{a-1}{a} = 1 - 1/a$$

But Chernoff bd. is sometimes useless. Why? Because it requires the mgf. To get the mgf, you need to know the PDF or PMF. If I know the PDF or PMF, then I know analytically or can numerically compute the CDF which means I know the tail exactly or within small numerical error! So it really is only useful if you're in a situation where you only have the MGF and not the PDF/PMF.