Lecture 8

PDF

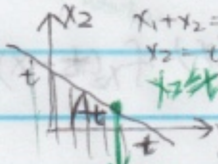
$T = X_1 + X_2 \sim f_T(t) = ?$ It also has a CDF $f_T(t) = F_T'(t)$ ← go this way

all rv has CDF

CDF method to Compute the convolution

CDF: $F_T(t) = P(T \le t)$ Define Region $T \le t$

$= P(\vec{X} \in A_t)$ Look Apply



$x_1 + x_2 = t$

$x_2 = t - x_1$

$x_2 \le t - x_1$

$x_1 + x_2 \le t$

From last class $= \iint_{A_t} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$

constant function of x_1 Integrate line

$= \int_{\mathbb{R}} \left(\int_{-\infty}^{t - x_1} f_{X_1 X_2}(x_1, x_2)\, dx_2 \right) dx_1$

Integrate everything $\mathbb{R}$

two substitution

Replace $\Big\|$ let $x_1 = x$, $x_2 = v - x \Rightarrow dx_2 = dv$ $x_2 = -\infty$

$\Rightarrow v = -\infty$. $x_2 = t - x \Rightarrow v = t$

$= \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1 X_2}(x, v - x)\, dv\, dx \overset{\text{switch}}{=} \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1 X_2}(x, v - x)\, dx\, dv$ // CDF

PDF $\Rightarrow f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1 X_2}(x, v - x)\, dx\, dv \right]$

Leibnitz's Rule for derivatives of integral function

Calculus I

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x, y)\, dy \right] = g(x, b(x))\, b'x + g(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\, dy$$

with respect to

If derivative is ~~w~~ a third variable, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + g(x, a(t))\, a'(t) + \underbrace{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}[g(x,y)]\, dy}_{= 0}$$

If one of the bounds in constant then...

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + g(x, c) \underbrace{\frac{d}{dt}[c]}_{= 0}$$

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} \left(\int_{\mathbb{R}} f_{X_1 X_2}(x, v - x)\, dx \right) dv \right]$$

$$= \int_{\mathbb{R}} f_{X_1 X_2}(x, t - x)\, \underbrace{dx}_{= dx} \cdot 1$$

general convolution formula

$X_1 X_2$ independent

indicator function $\int_{\mathbb{R}}$

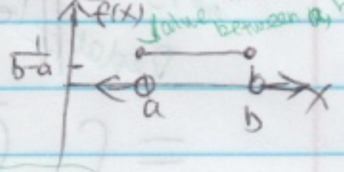
$$= \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f^{old}_{X_1}(x)\, f^{old}_{X_2}(t-x)\,\mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

$X_1 X_2$ iid

$$= \int_{\mathbb{R}} f(x)\, f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\,\mathbb{1}_{t-x \in Supp[X]}\,dx$$

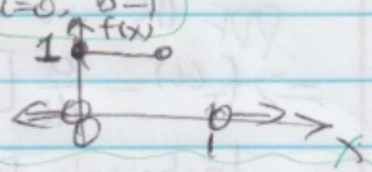
continuous uniform rv

$$X \sim U(a,b) = \underbrace{\frac{1}{b-a}\,\mathbb{1}_{x\in[a,b]}}_{f(x)} \qquad \frac{1}{b-a} = f^{old}(x)$$



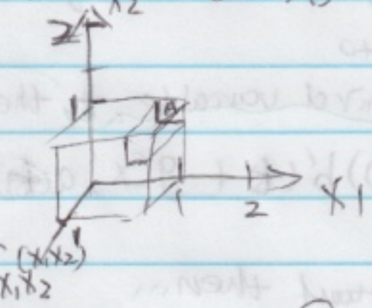
The "Standard uniform" rv is when $a=0,\ b=1$

$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$$



First convolution of continuous rv two standard uniform

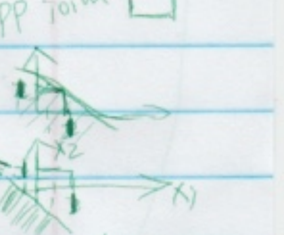
$X_1, X_2 \overset{iid}{\sim} U(0,1)$ $T = X_1 + X_2 \sim f_T(t) = ?$ CDF method First



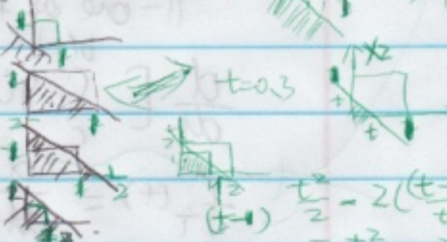
$Supp[X_1] = [0,1]$
$Supp[X_2] = [0,1]$

Supp joint □

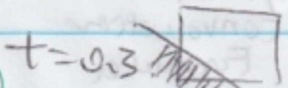
example $P(T \le 2.1) = 1$ whole square
$P(T \le -0.1) = 0$



$$F_T(t) = P(T \le t) = \begin{cases} 0 & \text{IF } t \le 0 \\ \frac{t^2}{2} & \text{IF } t \in [0,1] \\ -\frac{t^2}{2} + 2t - 1 & \text{IF } t \in (1,2) \\ 1 & \text{IF } t \ge 2 \end{cases}$$



$$\frac{t^2}{2} - 2\left(\frac{(t-1)^2}{2}\right) = \frac{t^2}{2} - (t^2 - 2t + 1)$$



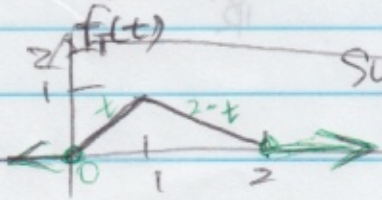
Density

~~CDF~~

Density

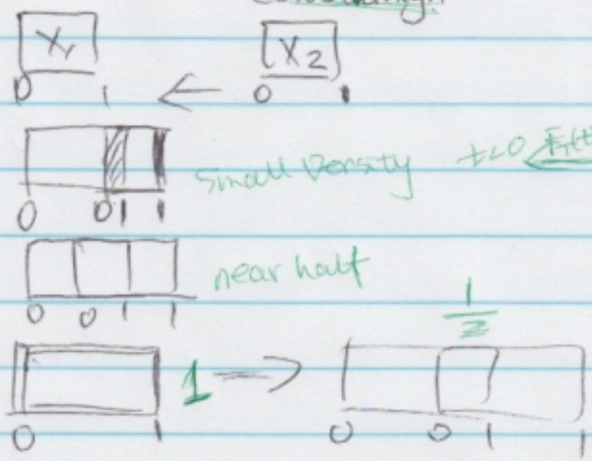
$$f_T(t) = \begin{cases} 0 & \text{IF } t \le 0 \\ t & \text{IF } t \in [0,1] \\ 2-t & \text{IF } t \in (1,2) \\ 0 & \text{IF } t \ge 2 \end{cases}$$

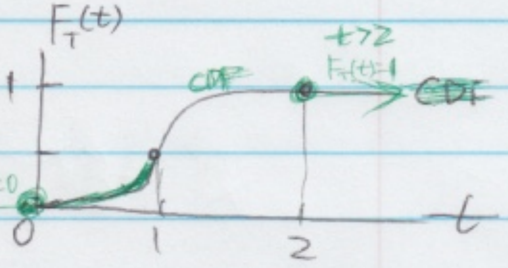
$t + (-t+2)$



$Supp[T] = [0,2]$

Convolving
~~Convolution~~





Let's try to derive the PDF of T using the convolution formula

iid Formula $f_T(t) = \int_{Supp[X]=(0,1)} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}\, dx = \int_0^1 (1)(1)\, \underbrace{\mathbb{1}_{t-x\in[0,1]}}\, dx$

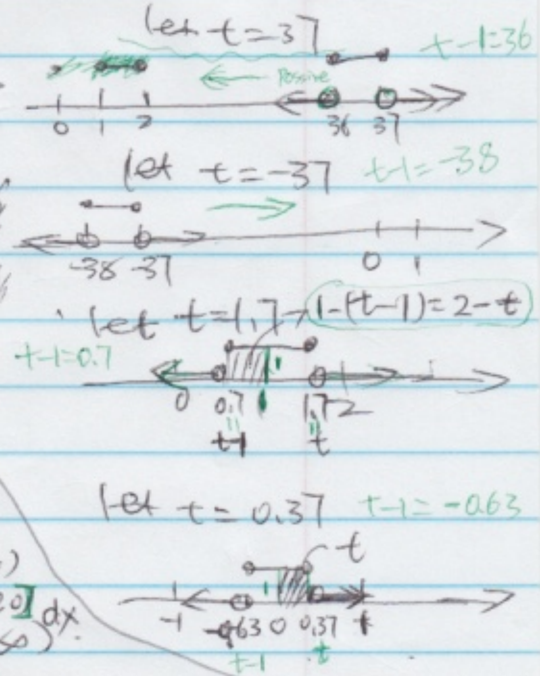
$x-t\in[-1,0]$

add t $\;x\in[t-1,t]$

$= \int_0^1 \mathbb{1}_{x\in[t-1,t]}\, dx$ intergate $f(x)$, $f(t)$

$\begin{cases} 0 & \text{if } t\le 0 \\ t & t\in[0,1] \\ 2-t & t\in(1,2) \\ 0 & \text{if } t\ge 2 \end{cases}$

Standard uniform resolution



$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) = \underbrace{\lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)}}_{f(x)}$ $f^{old}(x)$

$T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$

iid Formula $f_{T_2}(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}\, dx$

$= \int_0^\infty \lambda e^{-\lambda x}\, \lambda e^{-\lambda(t-x)}\, \mathbb{1}_{t-x\in[0,\infty)}\, dx$ $\quad e^{-\lambda t}\, e^{\lambda x}$ $\quad x-t\in(-\infty,0]$ $\quad x\in(-\infty,t)$

$= \lambda^2 e^{-\lambda t}\cdot \int_0^\infty \mathbb{1}_{x\in(-\infty,t)}\, dx = \lambda^2 e^{-\lambda t}\int_0^t dx = t\lambda^2 e^{-\lambda t}\mathbb{1}_{t\in(0,\infty)}$

$= \text{Erlang}(2,\lambda) = f_{T_2}(t)$

Erlang