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consider rv's $X_1, X_2, \ldots X_n$ iid but PMF/PDF is unknown but we know it has expecation mu and variance sigma-squared

Let $T_n = X_1 + X_2 + \cdots X_n$

let $\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \cdots X_n}{n}$

- From 4.21, we know $E[\bar{X}_n] = \mu$, $Var[X_n] = \frac{\sigma^2}{n}$

Let $Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \underbrace{\frac{\sqrt{n}}{\sigma}}_{a} \bar{X}_n + \underbrace{\frac{-\sqrt{n}}{\sigma}}_{b} \mu$

"$\bar{X}$ standardizan"

$E[Z_n] = 0$

$Var[Z_n] = 1 = SD[Z_n]$

$\phi_{T_n}(t) \overset{P_3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) \cdots \phi_{X_n}(t) \overset{P_1}{=} \phi_X(t)^n$

$\phi_{\bar{X}_n}(t) \overset{P_2}{=} \phi_T\left(\frac{1}{n} t\right) = \phi_X\left(\frac{t}{n}\right)^n$

$\phi_{Z_n}(t) \overset{P_2}{=} e^{itb}\phi_{\bar{X}_n}(at) = e^{\frac{-it\mu\sqrt{n}}{\sigma}}\phi_X\left(\frac{\sqrt{n}}{\sigma}\frac{t}{n}\right)^n$

$= e^{\frac{-it\mu n}{\sigma\sqrt{n}}}\, e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)} = e^{-\frac{it\mu n}{\sigma\sqrt{n}} + n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)} \frac{\frac{1}{n}}{\frac{1}{n}}$

$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}} = e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)} = \phi_{Z_n}(t)$

we want to investigate now $\lim_{n\to\infty} \Phi_{Z_n}(t) = ?$

$$= e^{\frac{t^2}{\sigma^2} \lim_{n\to\infty} \frac{\ln\left(\Phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}} = e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\ln(\Phi_X(u)) - i\mu u}{u^2}}$$

let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n\to\infty \Rightarrow u\to 0$

L' Hopitals

$$\stackrel{\downarrow}{=} e^{\frac{t^2}{\sigma^2} \lim_{u\to 0} \frac{\frac{\Phi_X'(u)}{\Phi_X(u)} - i\mu}{u}} = e^{\frac{t^2}{2\sigma^2} \lim_{u\to 0} \frac{\Phi_X(u)\Phi_X''(u) - \Phi_X'(u)^2}{\Phi_X(u)^2}}$$

$$= e^{\frac{t^2}{2\sigma^2} \frac{\Phi_X(0)\Phi_X''(0) - \Phi_X'(0)^2}{\Phi_X(0)}} \stackrel{P_0}{=} e^{\frac{t^2}{2\sigma^2}\left(\Phi_X''(0) - \Phi_X'(0)^2\right)}$$

$$\stackrel{P_4}{=} e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - (iE[X])^2\right)} = e^{\frac{-t^2}{2\sigma^2}\overbrace{\left(E[X^2] - E[X]^2\right)}^{\sigma^2}}$$

$$= e^{\frac{-t^2}{2}} = \Phi_Z(t)$$

$\stackrel{P_8}{\Longrightarrow} Z_n \xrightarrow{d} Z$ where Z has chf $\Phi_Z(t) = e^{-\frac{t^2}{2}}$.

$$Z \sim f_Z(z) = ?$$

use P_6 to find PDF of Z, check $\Phi_Z(t) \stackrel{?}{\in} L^1 \Rightarrow$

$$\Rightarrow \underbrace{\int_{\mathbb{R}} e^{-t^2/2}\, dt = \sqrt{\pi} < \infty}_{\text{Gaussian Intigral}} \quad \text{Yes!}$$

$$f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\,\Phi_Z(t)\, dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz} e^{\frac{-t^2}{2}}\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)}\, dt$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,itz}{2}\right)^2 - \left(\frac{\sqrt{2}\,iz}{2}\right)^2$$

$$= \frac{t^2}{2} + 2\,\frac{\sqrt{2}\,iz}{2}\,\frac{t}{\sqrt{2}} + \cancel{\frac{i^2z^2}{2}} - \frac{i^2z^2}{2}$$

$$= \frac{1}{2\pi}\int_R e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2}\right)^2} e^{-\frac{z^2}{2}}\,dt = \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_R e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2}\right)^2} dt$$

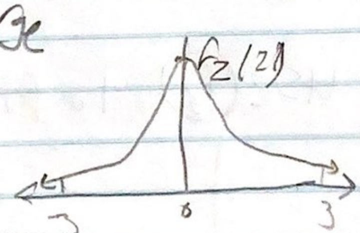
let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\,iz}{2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}}$, $t \to \infty \Rightarrow y \to \infty$,

$t \to -\infty \Rightarrow y \to -\infty$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \int_R e^{-y^2}\sqrt{2}\,dy \overset{\text{Gaussian Integral}}{=} \frac{1}{2\pi} e^{-z^2/2}\sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

standard normal $= N(0,1)$

This completes the proof of the "central limit theorem" (CLT), the crown jewel of a basic probability class, one of the most useful results that probability has given to the world at large



$$\boxed{\text{CLT}: X_1, \ldots X_n \text{ iid } \eta,\ \sigma^2,\ \frac{\bar{X}-\eta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)}$$

let $\sigma > 0$

$Z \sim N(0,1)$, $X = \eta + \sigma Z \sim f_X(x) = ?$

$$\varphi_X(t) \overset{P2}{=} e^{it\eta}\varphi_Z(\sigma t) = e^{it\eta - \sigma^2 t^2/2}$$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\eta}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\eta}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\eta)^2}$$

$$= N(\eta, \sigma^2)$$

$$E[Z] = \frac{\Phi_Z'(0)}{i} = 0 \quad , Var[Z] = E[Z^2] - \cancel{E[Z]}^2$$

$$= \frac{\Phi_Z''(0)}{i^2} = 1 \checkmark$$

$$i^2 = -1$$

$$\Phi_Z'(t) = \frac{d}{dt}[e^{-t^2/2}] = -te^{-t^2/2}$$

$$\Phi_Z''(t) = \frac{d}{dt}[\underbrace{-t}_{u}\underbrace{e^{-\frac{t^2}{2}}}_{v}] = -(-t^2e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}})$$

$$= t^2e^{-\frac{t^2}{2}} - e^{-\frac{t^2}{2}}$$

$$E[X] = E[\eta + \sigma Z] = \eta, \quad Var[X] = Var[\eta + \sigma Z] = \sigma^2, SD[X] = \sigma$$

$X_1 \sim N(\eta, \sigma^2)$ indep of $X_2 \sim N(\eta_2, \sigma_2^2)$, $T = X_1 + X_2 \sim f_T(t) = ?$

$$\Phi_T(t) \overset{P_3}{=} \Phi_{X_1}(t)\,\Phi_{X_2}(t) = e^{it\eta_1 - \sigma_1^2 t^2/2} e^{it\eta_2 - \sigma_2^2 t^2/2}$$

$$= e^{it(\eta_1+\eta_2) - (\sigma_1^2+\sigma_2^2)t^2/2} \overset{P_1}{\Longrightarrow} T \sim N(\eta_1+\eta_2, \sigma_1^2+\sigma_2^2)$$

$X \sim N(\eta, \sigma^2)$, $y = e^X \sim f_Y(y) = ?$ $\quad g^{-1}(y) = \ln(y) \quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|}$

$$f_Y(y) = f_X \ln y \frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\eta)^2} \frac{1}{|y|} =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\eta)^2}$$

$$= \text{Log}\,N(\eta, \sigma^2)$$

Log-Normal distribution