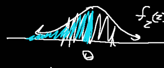
$Z \sim N(0,1)$, $Y = Z^2 \sim f_Y(y) = ?$ Not 1:1



$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2P(Z \in [0, \sqrt{y}])$$

$$= 2(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{\frac{1}{2}}) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{d}{dy}[2F_Z(\sqrt{y}) - 1] = \cancel{2}\left(\frac{1}{\cancel{2}} y^{-\frac{1}{2}}\right) f_Z(\sqrt{y}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \ge 0}}$$

$$\propto y^{\overbrace{-\frac{1}{2}}^{\frac{1}{2}-1}} e^{-\frac{1}{2}y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$Z_1, Z_2, \ldots, Z_K \overset{iid}{\sim} N(0,1), \quad Y = \underbrace{Z_1^2}_{\sim G(\frac{1}{2},\frac{1}{2})} + \underbrace{Z_2^2}_{\sim G(\frac{1}{2},\frac{1}{2})} + \ldots + \underbrace{Z_K^2}_{\sim G(\frac{1}{2},\frac{1}{2})} \sim \text{Gamma}\left(\frac{K}{2}, \frac{1}{2}\right)$$

Note the beta is always 1/2 and the alpha is always k/2 so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}\left(\frac{K}{2}, \frac{1}{2}\right) = \chi^2_K$ The "chi squared distribution with k degrees of freedom" $K \in \mathbb{N}$.

$E[Y] = K\,E[Z^2] = K$

$K = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\chi^2_K = \frac{\left(\frac{1}{2}\right)^{K/2}}{\Gamma\left(\frac{K}{2}\right)} y^{\frac{K}{2}-1} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} \overset{\downarrow}{=} \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} = \chi^2_1$$

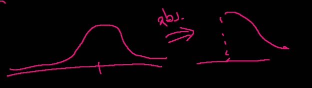
$X \sim \chi^2_K$, $Y = \sqrt{X} \Rightarrow x = y^2 = g^{-1}(y)$, $\left|\frac{d}{dy}[g^{-1}(y)]\right| = |2y| = 2y$

$$f_Y(y) = f_X(y^2)\,2y = \frac{\left(\frac{1}{2}\right)^{K/2}}{\Gamma\left(\frac{K}{2}\right)} y^{K-2} e^{-y^2/2} (2y)\, \mathbb{1}_{\overbrace{y^2 \ge 0}^{y \ge 0}}$$

$$= \frac{\left(\frac{1}{2}\right)^{K/2-1}}{\Gamma\left(\frac{K}{2}\right)} y^{K-1} e^{-y^2/2} \mathbb{1}_{y \ge 0} = \chi_K$$

the chi distribution with k degrees of freedom

$$Z \sim N(0,1), \quad |Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \mathbb{1}_{y \ge 0}$$



abs.

$$= 2\underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)}_{f_Z} \underbrace{\mathbb{1}_{y \ge 0}}$$

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{y}{c}\right)^{\alpha-1}}_{\frac{y^{\alpha-1}}{c^{\alpha-1}}} e^{-\beta\frac{y}{c}} \mathbb{1}_{\overbrace{\frac{y}{c} > 0}^{y > 0}}$$

c^α $\quad$ $(\beta/c)y$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\beta/c)y} \mathbb{1}_{y \ge 0} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$$X \sim \chi^2_K, \quad Y = \frac{X}{K} \sim \text{Gamma}\left(\frac{K}{2}, \frac{\frac{1}{2}}{\frac{1}{K}}\right) = \text{Gamma}\left(\frac{K}{2}, \frac{K}{2}\right)$$

let $X_1 \sim \chi^2_{K_1}$ indep. of $X_2 \sim \chi^2_{K_2}$

let $U = \frac{X_1}{K_1} \sim \text{Gamma}\left(\underbrace{\frac{K_1}{2}}_{a}, \underbrace{\frac{K_1}{2}}_{a}\right)$ indep of $V = \frac{X_2}{K_2} \sim \text{Gamma}\left(\underbrace{\frac{K_2}{2}}_{b}, \underbrace{\frac{K_2}{2}}_{b}\right)$

$$R = \frac{U}{V} \sim f_R(r) = \int_{Supp[V]} f_U(rt)\, \mathbb{1}_{rt \in Supp[U]}\, f_V(t)\, |t|\, dt$$

$$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \mathbb{1}_{\underbrace{rt \in [0,\infty)}_{r \in (0,\infty)}} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\, t\, dt$$

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r \ge 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\, dt$$

$$= a^a b^b r^{a-1} \mathbb{1}_{r \ge 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)} \Gamma(a+b)}_{\frac{1}{B(a,b)}} \cdot \frac{1}{(ar+b)^{a+b}} = \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{b^{-(a+b)}\left(1+\frac{a}{b}r\right)^{-(a+b)}} \mathbb{1}_{r \ge 0}$$

$b^{-a} \cancel{b^{-b}}$

$$= \frac{(a/b)^a}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)} \mathbb{1}_{r > 0} = \frac{(K_1/K_2)^{K_1/2}}{B\left(\frac{K_1}{2}, \frac{K_2}{2}\right)} r^{\frac{K_1}{2}-1} \left(1 + \frac{K_1}{K_2} r\right)^{-\frac{K_1+K_2}{2}} \mathbb{1}_{r \ge 0}$$

$= F_{K_1, K_2}$ this is the "F distribution" or "Fisher-Snecedor Distribution" with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom. $K_1, K_2 \in \mathbb{N}$

Let $Z \sim N(0,1)$, $X \sim \chi^2_K$, $W = \frac{Z}{\sqrt{X/K}} \sim f_W(w) = ?$

Consider $W^2 = \frac{Z^2/1}{X/K} \sim F_{1,K}$ $\qquad$ T symmetric around 0

$$F_{W^2}(w^2) = P(W^2 \le w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$$

Take derivatives...

$$\frac{d}{dw}[F_{W^2}(w^2)] = \frac{d}{dw}[F_W(w)] - \frac{d}{dw}[F_W(-w)]$$

$$\cancel{2}w\, f_{W^2}(w^2) = f_W(w) - -f_W(-w) = \cancel{2} f_W(w)$$

$$f_W(w) = \cancel{w} \underbrace{\frac{\left(\frac{1}{K}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{K}{2}\right)}}_{\frac{\Gamma\left(\frac{K+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{K}{2}\right)}} \underbrace{(w^2)^{\frac{1}{2}-1}}_{\cancel{w^{-1}}} \left(1 + \frac{w^2}{K}\right)^{-\frac{1+K}{2}} \mathbb{1}_{\underbrace{w^2 \ge 0}_{w \in \mathbb{R}}}$$

$$= \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)} \left(1 + \frac{w^2}{K}\right)^{-\frac{K+1}{2}} = T_K$$

Student's T distribution with k degrees of freedom, discovered in 1908 by William Gosset while he was working at a beer factory

If $K \to \infty$ $T_K \to Z$

Student's T distribution has the N(0,1) shape but just thicker tails.

$$Z_1, Z_2 \overset{iid}{\sim} N(0,1), \quad R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f(u) |u|\, du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\, du = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} -|u|\, du + \int_0^\infty e^{-\frac{r^2+1}{2}u^2} |u|\, du\right)$$

$$= \frac{1}{2\pi}\left(\underbrace{\int_0^{-\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du}_{\int_0^\infty \text{———} du} + \int_0^\infty e^{-\frac{r^2+1}{2}u^2} |u|\, du\right) = \frac{1}{\pi} \int_0^\infty e^{-\frac{r^2+1}{2}u^2} u\, du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u} dt$, $u = 0 \Rightarrow t = 0$, $u \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{r^2+1}{2}t} \cancel{u} \frac{1}{2\cancel{u}} dt = \frac{1}{\cancel{2}\pi} \frac{1}{\frac{r^2+1}{\cancel{2}}} \underbrace{\int_0^\infty \frac{r^2+1}{2} e^{-\frac{r^2+1}{2}t} dt}$$

$$= \frac{1}{\pi} \frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

$\int_0 \underbrace{\lambda e^{-\lambda x}}_{\text{PDF exponential rv}} dx = 1$

let $X = c + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi} \frac{1}{1 + \left(\frac{x-c}{\sigma}\right)^2}$$