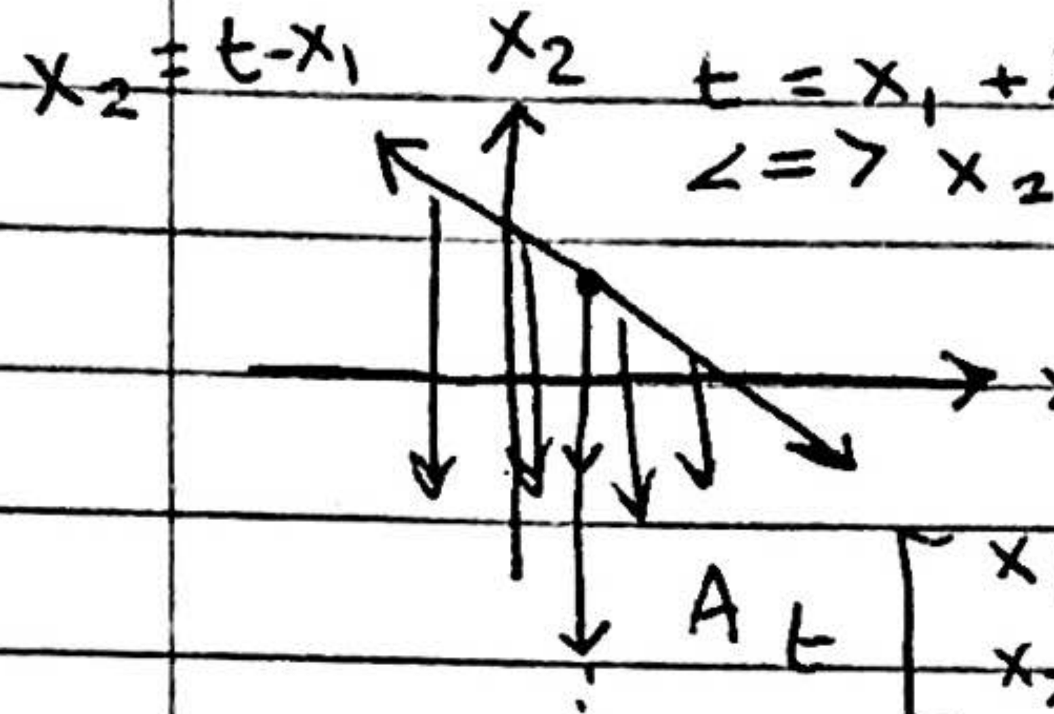
$\vec{X}$ cont rv $P(\vec{X} \in A) = \int \cdots \int_A f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k$

let $T = X_1 + X_2 \sim f_T(t) = ?$

first note $f_T(t) = F'(t)$ cdf method   9/30/20

usually it is difficult to find the cdf of a continous rv's so this is not the usual method, the usual method uses the convolution formula



$t = x_1 + x_2$
$\iff x_2 \le t - x_1$

$F_T(t) = P(T \le t)$
$= P(\vec{X} \in A_t)$

$A_t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x \le t - x_1$

$= \iint_{A_t} f_{x_1 x_2}(x_1, x_2)\, dx_1 dx_2$

condition first on x_1

$$= \int_{-\infty}^{\infty} \int_{\infty}^{t-x_1} f_{x_1 x_2}(x_1, x_2)\, dx_2\, dx_1$$

treat as a constant

do subsitutions:

let $x_1 = x$

$x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv$

if $x_2 = -\infty \Rightarrow ?\ v = -\infty$

if $x_2 = t - x \Rightarrow ?\ v = t$

$$= \int_{\mathbb{R}} \int_{-\infty}^{t} f_{x_1 x_2}(x, v-x)\, dv\, dx$$

you can always change the order of the inter

$g(x)$ of v

$$= \int_{-\infty}^{t} \left(\int_{\mathbb{R}} f_{x_1 x_2}(x, v-x)\, dx \right) dv \qquad = \text{cdf}$$

to get the pdf

$$f_T(t) = \frac{d}{dt} \left[\quad \right]$$

Leibnitz's Rule

calc fact

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x,b(x))\,b'(x) + g(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{dx}[g\,x,y]\,dy$$

If the outer derivative is a third variable then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + g(x,a(t))\,a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\left[g(x,y)\right] dy} \rightarrow 0$$

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + \cancel{g(x,c)\,\frac{d}{dt}[c]} \rightarrow 0$$

$$= \int_{\mathbb{R}}\left(f_{x_1,x_2}^{(x_1,\,t-x)}\,dx\right)\cdot 1 + 0$$

$$= f_T^{(t)} = f_{x_1}^{(x)} * f_{x_2}^{(x)} \rightarrow \text{general convol formula}$$

if x_1, x_2 are independent

old style move the indicator fn up

$$= \int_{\mathbb{R}} f_{x_1}^{(x)}\, f_{x_2}^{(t-x)}\,dx = \int_{supp[X_1]} f_{x_1}^{old(x)}\, f_{x_2}^{old(t-x)}\, \mathbb{1}_{t-x \in supp[X_2]}\,dx$$

iid

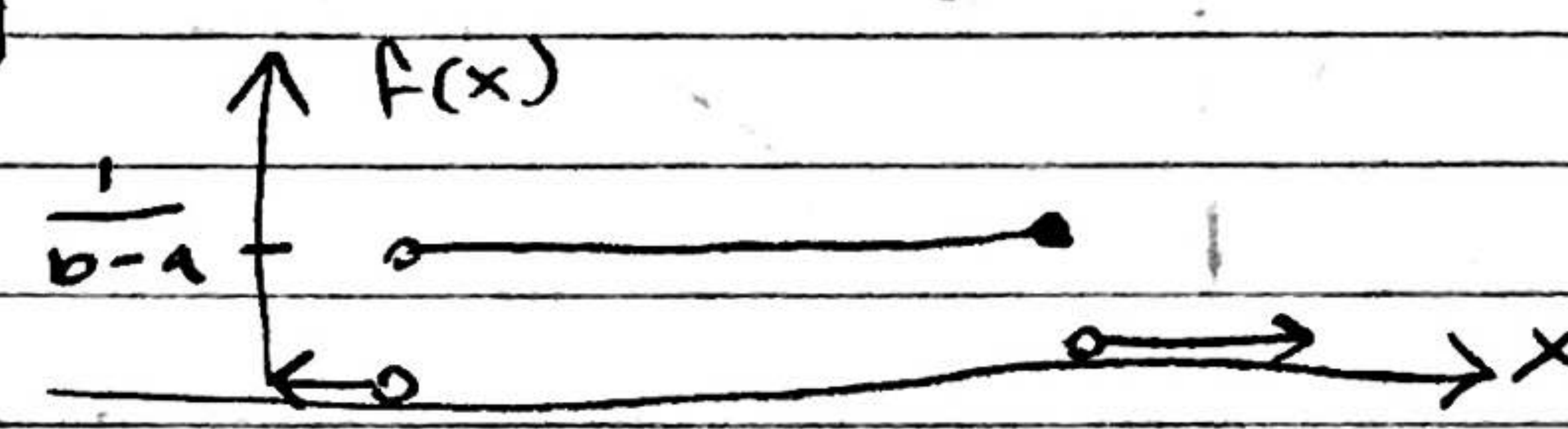
$$= \int_{\mathbb{R}} f(x)\, f(t-x)\, dx$$

old style

$$= \int_{supp[X_1]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in supp[X]}\, dx$$

$X \sim U(a,b)$ continuous uniform rv

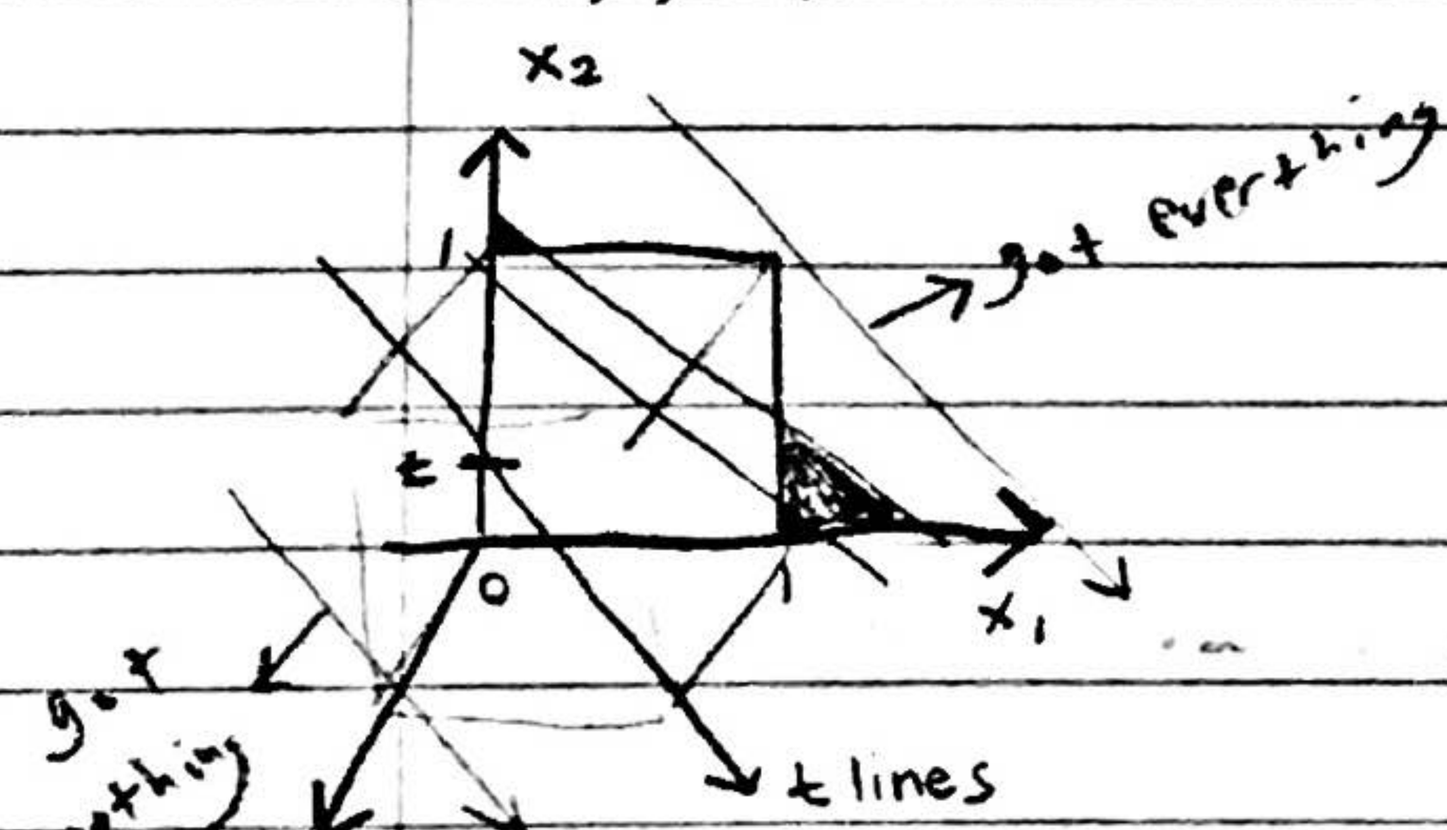
$$= \underbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x \in [a,b]}}_{f(x)}$$



standard uniform rv is when $a=0$ $b=1$

$$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$$

$$X_1, X_2 \overset{iid}{\sim} U(0,1),\ T = X_1 + X_2 \sim f_T(t) = ?$$



we want to compute cdf which means we need to find volumes in regions under the diagonal lines

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = \begin{cases} 1 & \text{if } x_1 \in [0,1],\ x_2 \in [0,1] \\ 0 & o/t \end{cases}$$

$\frac{bh}{2}$

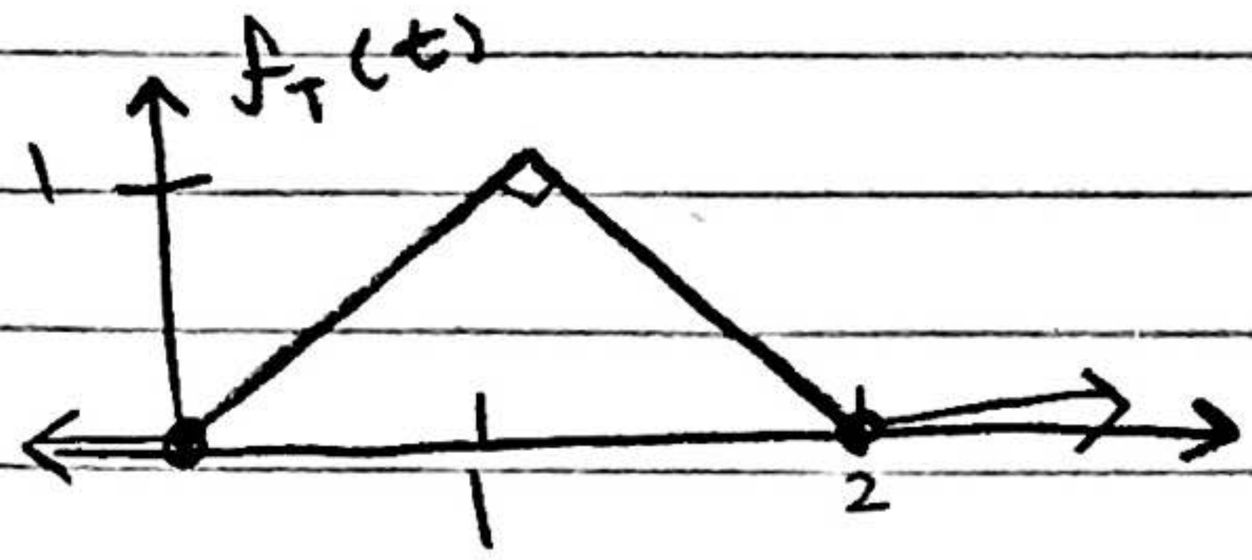
$$F_T(t) = ? \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t^2}{2} & \text{if } t \in [0,1] \\ \frac{t^2}{2} - 2\frac{(t-1)^2}{2} & \text{if } t \in (1,2) \\ \frac{t^2}{2} - (t^2 - 2t + 1) = -\frac{t^2}{2} + 2t - 1 & \text{if } t \in [1,2] \\ 1 & \text{if } t \geq 2 \end{cases}$$

$$\Rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$

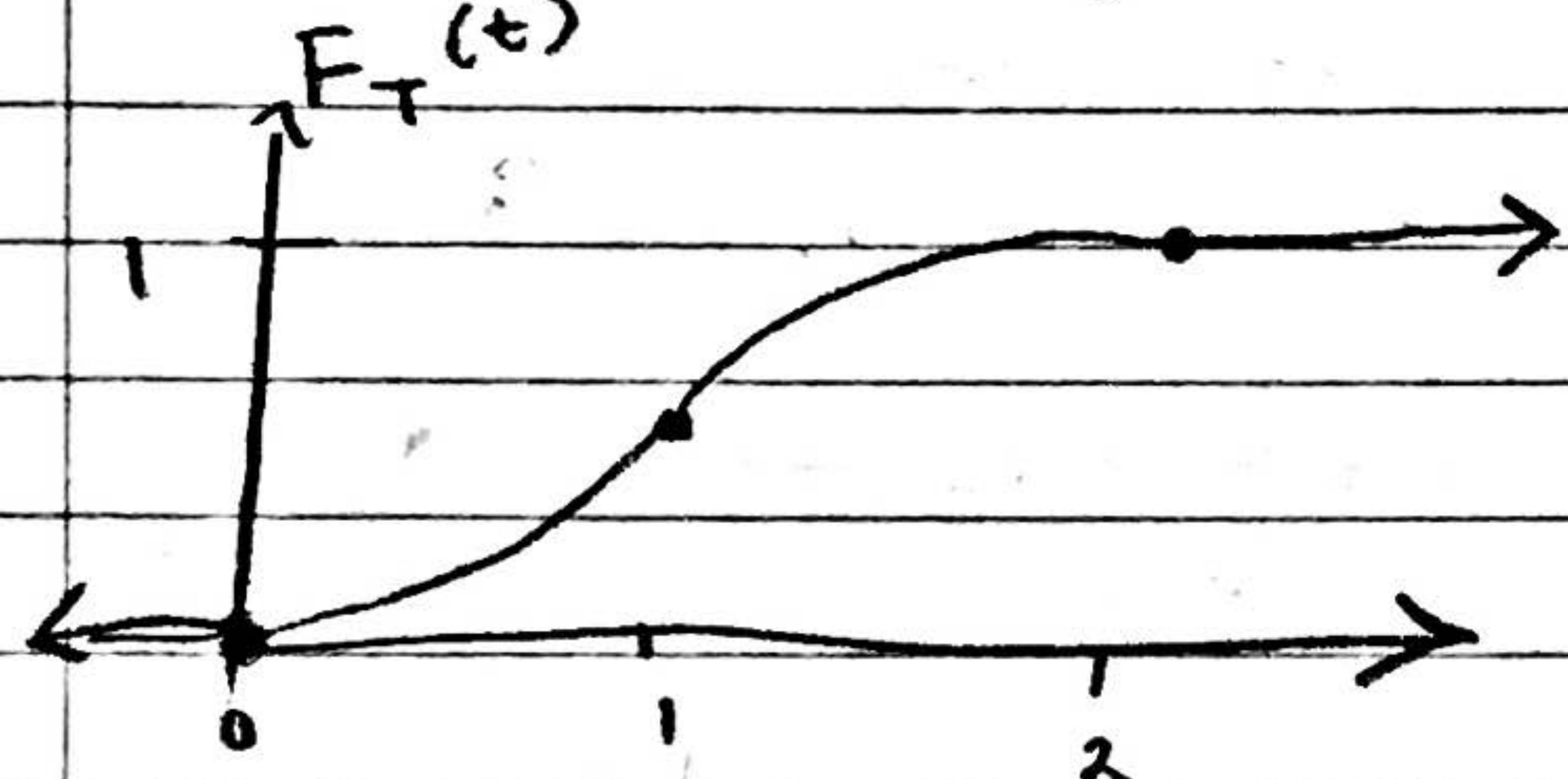
adding two standard uniforms

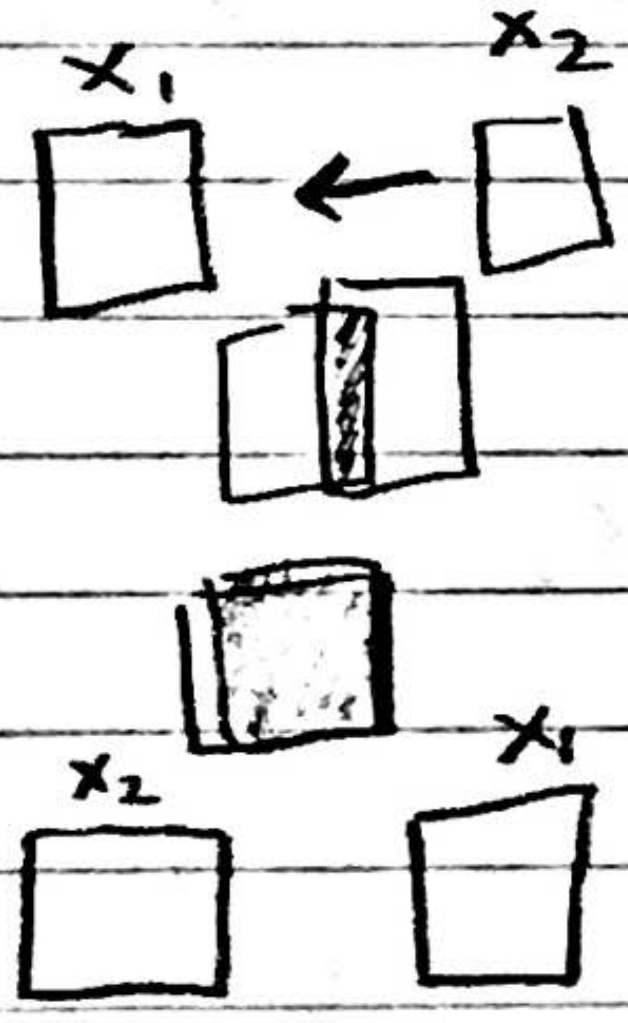
min 0

max 2



two small # low density, two large also low density





"Convolving"

why can't we just use the formulas?

we just derived the pdf of the convolution by finding its cdf why can't we use the formula??

(iid) old

$$f_T(t) = \int_{\text{Supp}[X]} f(x) \overset{\text{old}}{f}(t-x) \overset{\text{old}}{\mathbb{1}}_{t-x \in \text{Supp}[X]} \, dx$$

$\text{Supp}[X] = [0,1]$ density 1's

$x \in [t-1, t]$

$x-t \in [-1, 0]$

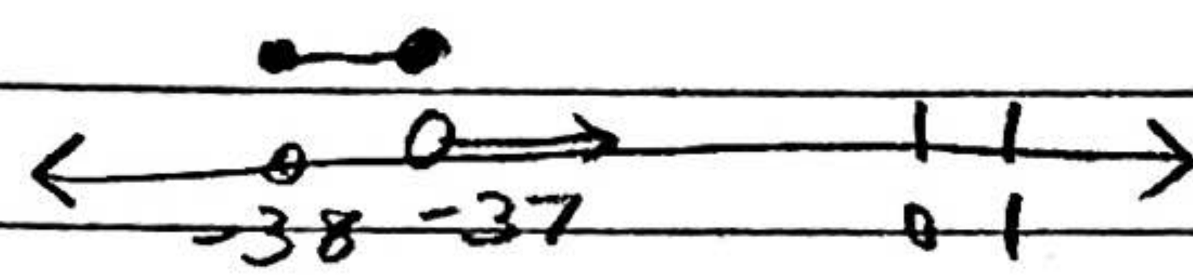
$$= \int_0^1 (1)(1) \mathbb{1}_{t-x \in [0,1]} \, dx = \int_0^1 \mathbb{1}_{x \in [t-1,t]} \, dx$$

$\leftarrow$?
in trouble

$$= \int_0^1 \mathbb{1}_{x \in [t-1, t]} \, dx$$

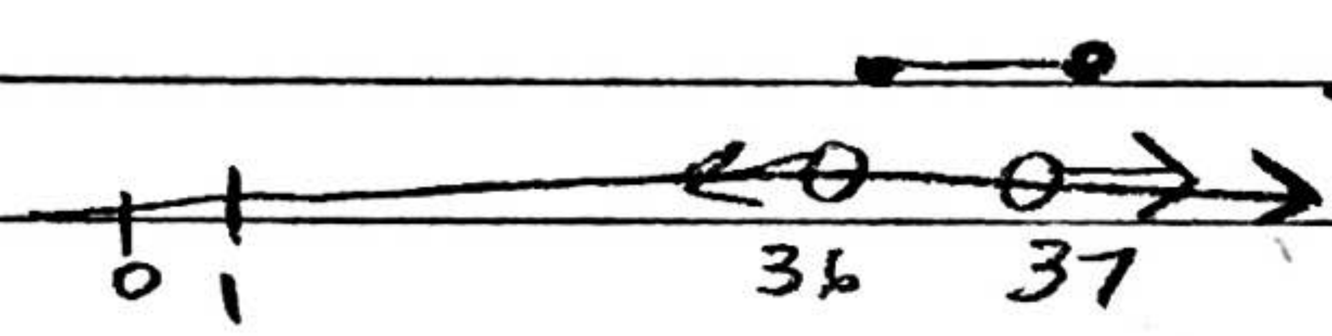
let's do some examples how about $t = -37$

$$\int_0^1 \mathbb{1}_{x \in [-38, -37]} \, dx = ?$$



how about $+37$ $\int_0^1 \mathbb{1}_{x \in [36, 37]}$

get nothing!



$t \in (0,1)$

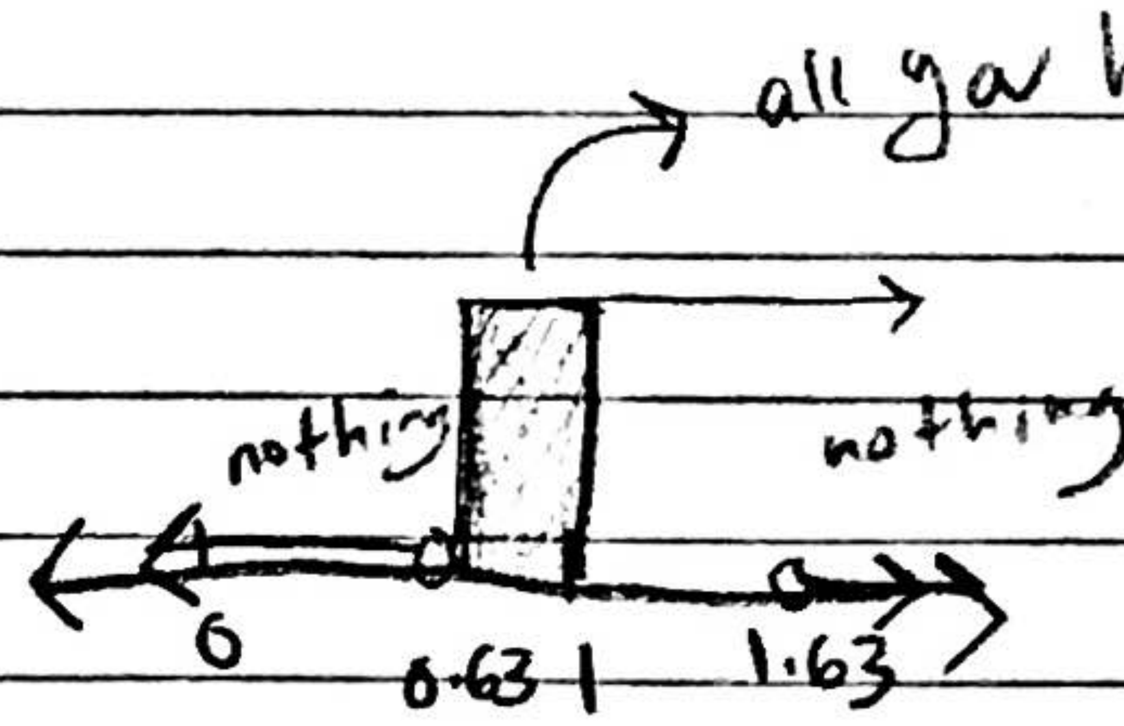
$\int_0^1 \mathbb{1}_{x \in}$ ex $= 0.7$

$$\int_0^1 \mathbb{1}_{x \in [-0.3, 0.7]} \, dx = \int_0^{0.7} dx$$

gonna get cases

$$= \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$

"get wrecked from both sides"



ex $= t = 1.63$

$$\int_0^1 \mathbb{1}_{x \in [0.63, 1.63]} \, dx = \int_{0.63}^1 dx$$

$= 0.37$

as a formula $\quad$ bottom

$$1 - (t-1) = 2 - t$$

it's a geometric but squished

has a waiting time

$$X_1, X_2, \ldots \overset{idd}{\sim} Exp(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}$$

$$T_2 = X_1 + X_2 \sim f_T(t) = ?$$

$$f_T(t) = \int_{supp[X]} f^{old}(x) f^{old}(t-x) \mathbb{1}_{t-x \, supp[X]} \, dx$$

what is supp[X]

$$= \int_0^\infty \underbrace{\lambda e^{-\lambda x}}_{f^{old}(x)} \underbrace{\lambda e^{-\lambda(t-x)}}_{f^{old}(t-x)} \mathbb{1}_{\boxed{t-x \in [0,\infty)}} \, dx$$

gonna look like the [illegible]

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t]} \, dx$$

$$x - t \in (-\infty, 0]$$

$$x \in (-\infty, t]$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx$$

$$= t\left(\lambda^2 e^{-\lambda t}\right) \cdot \mathbb{1}_{t \in [0,\infty)} = Erlang(2, \lambda)$$

$$T_3 = X_1 + X_2 + X_3 = Erlang(3, \lambda)$$

$$T_k = X_1 + \cdots + X_k = Erlang(k, \lambda)$$