11/11/20 lec 17

Consider rv's $X_1$ $X_2$ ... $X_n$ iid
but the pmf / pdf is unknown but we know
it has expectation mu and variance
sigma-squared.

Let $T_n = X_1 + X_2 + \dots X_n$ when the rv are realized you get the sample

let $\overline{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \dots + X_n}{n}$

from 241, we know $E[\overline{X}_n] = \mu$ independent of n, $Var[X_n] = \frac{\sigma^2}{n}$ not a constant

If $n \to \infty$ theres no wiggle room
... law of large numbers, later in class

Let $Z_n := \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \underbrace{\frac{\sqrt{n}}{\sigma}}_{a} \overline{X}_n + \underbrace{\frac{-\sqrt{n}}{\sigma}\mu}_{b}$

" $\overline{X}$ standardized"

var of a constant = 1

$E[Z_n] = 0$
$Var[Z_n] = 1 = SE[Z_n] = sd[Z_n]$

characteristic function of t

$\phi_{T_n}(t) = \phi_{X_1}(t)\phi_{X_2}(t) \dots \phi_{X_n}(t)$ prop 3

b/c iid use prop 1

$\phi_{T_n}(t) \overset{P1}{=} \phi_X(t)^n$

shift and scale

$\phi_{\overline{X}_n}(t) \overset{P2}{=} \phi_T\left(\frac{1}{n}t\right) \overset{P1}{=} \phi_X\left(\frac{t}{n}\right)^n$

$\phi_{Z_n}(t) \overset{P2}{=} e^{itb}\phi_{\overline{X}_n}(at) = e^{\frac{-it\mu\sqrt{n}}{\sigma}} \phi_X\left(\frac{\sqrt{n}}{\sigma}\frac{t}{n}\right)^n$

$\hookrightarrow ax+b$

$$\dots = e^{\frac{-it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}}\phi_x\left(\frac{\cancel{\sqrt{n}}}{\sigma}\frac{t}{\cancel{n}_{\sqrt{n}}}\right)^n$$

fact from pre calc $a = e^{\ln(a)}$

$$e^{-\frac{it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{-\frac{it\mu \cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\frac{1}{\cancel{n}}}$$

factor out a $\frac{n}{n}$

$$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}}\cdot\frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)} = \phi_{Z_n}(t)$$

we did this because we want to investigate now

$$\lim_{n\to\infty}\phi_{Z_n}(t) = ?$$

$$= e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^n}{\sigma^2 n}}}$$

let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n\to\infty \Rightarrow u\to 0$

$$= e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_x(n)) - i\mu u}{u^2}}$$

L'hop rule

$$= e^{2\frac{t^2}{\sigma^2}\lim_{n\to 0}\frac{\frac{\phi_x'(u)}{\phi_x(u)} - i\mu}{n}}$$

do it again...

L'hop $e^{\frac{t^2}{2\sigma^2}} \lim_{n\to 0} \frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}$

sub in $u=0$

$= e^{\frac{t^2}{2\sigma^2}} \frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)}$

$\overset{(P_6)}{=} e^{\frac{t^2}{2\sigma^2}} \underbrace{\left(\phi_X''(0) - \phi_X'(0)^2\right)}$

we have a prop for that too.

its the mgf...

$(P4)$ $= e^{\frac{t}{2\sigma^2}} \left( i^2 E[X^2] - (i\,E[X])^2 \right)$

$= e^{-\frac{t^2}{2\sigma^2}} \underbrace{\left( E[X^2] - E[X]^2 \right)}_{\sigma^2}$ from 241

$= e^{-\frac{t^2}{2}}$

the goal $= \phi_Z(t)$

(P8) $Z_n \xrightarrow{d} Z$ where $Z$ has chf $\phi_Z(t) = e^{-\frac{t^2}{2}}$

how is $Z$ distributed (almost there)

$Z \sim f_Z(z)$ ?

use (P6) to find pdf check $\phi_Z(t) \overset{?}{\in} L^1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dt = \sqrt{2} < \infty$

Yes!

$f_Z(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \phi_Z(t)\, dt$

$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} e^{-\frac{t^2}{2}} dt$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)} \, dt$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2 - \left(\frac{\sqrt{2}\, iz}{2}\right)^2$$

$$= \frac{t^2}{2} + \cancel{2} \frac{\cancel{\sqrt{2}}\, iz}{\cancel{2}} \frac{t}{\cancel{\sqrt{2}}} + \frac{i^2 \cancel{z^2}}{\cancel{2}} - \frac{i^2 z^2}{2}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2} e^{\frac{-z^2}{2}} \, dt$$

$$= \frac{1}{2\pi} e^{\frac{-z^2}{2}} \int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2} \, dt$$

let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}}$$

$t \to \infty$ $y \to \infty$ $t \to -\infty$ $y \to -\infty$

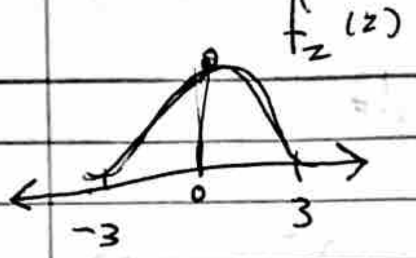
$$\frac{1}{2\pi} e^{\frac{-z^2}{2}} \int_{\mathbb{R}} e^{-y^2} \sqrt{2} \, dy$$

"Standard Normal"

Gaussian Integral

$$= \frac{1}{2\pi} e^{-\frac{z^2}{2}} \sqrt{2} \sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = N(0,1)$$

This completes the proof of the "central limit thm" (CLT) the crown jewl of a basic probability class, one of the most useful results that a probability has given the world at large.




AKA Laplace's second Error Distribution. It is the most famous and widely used error distribution on earth

$$\boxed{\begin{array}{l}\text{CLT: } X_1 \dots X_n \text{ iid mean } \mu \text{ Var } \sigma^2 \\ \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)\end{array}}$$

let $\sigma > 0$

$Z \sim N(0,1)$, $X = \mu + \sigma Z \sim f_X(x) = ?$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\frac{1}{\sigma} \frac{1}{\sqrt{2\pi}}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$= N(\mu, \sigma^2)$$

$$E[Z] = \frac{\phi'_Z(0)}{i} = 0 \qquad Var[Z] = E[Z^2] - \cancel{E[Z]}^2 = \frac{\phi''_Z(0)}{\underbrace{i^2}_{-1}}$$

0

$$\phi'_Z(t) = \frac{d}{dt}\left[e^{-\frac{t^2}{2}}\right] = -t e^{-\frac{t^2}{2}}$$

$$\phi''_Z(t) = \frac{d}{dt}\left[\underbrace{-t}_{u}\underbrace{e^{-\frac{t^2}{2}}}_{v}\right] =$$

$$= -\left(-t^2 e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}\right) = t^2 e^{-\frac{t^2}{2}} - e^{-t^2/2}$$

$E[X] = E[\mu + \sigma Z] = \mu$

$Var[X] = Var[\mu + \sigma Z] = \sigma^2$

$SD[X] = \sigma$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep of $X_2 \sim N(\mu_2, \sigma_2^2)$

$T = X_1 + X_2 \sim f_T(t) = ?$

$\phi_T(t) \overset{(P3)}{=} \phi_{X_1}(t) \; \phi_{X_2}(t)$

$$\boxed{\begin{aligned}\phi_X(t) &\overset{P2}{=} e^{it\mu}\, \phi_Z(\sigma t) \\ &= e^{itm - \sigma^2 t^2/2}\end{aligned}}$$

$\rightarrow e^{it\mu_1 - \sigma_2^2 t^2/2}$

$= e^{it(\mu_1 + \mu_2) - (\sigma_1^2 + \sigma_2^2)\frac{t^2}{2}} \overset{(P_1)}{\Longrightarrow}$

happen w gamma, normal

$T \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$

$g^{-1}(y) = \ln(y) \quad \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \frac{1}{|y|}$

$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2} \frac{1}{|y|} =$

$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2} = \log N(\mu, \sigma^2)$

log normal distribution