Wednesday October 21st 2020

Lecture 13

$$f_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} \left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1} \right)$$

$n-1$ if $j=n$

$$= \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} j f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$\cdot 0$

$(n-j-1)!$ $\quad n-(j+1)$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!(n-j-1)!} f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$(n-(j+1))!$

reindexing trick for

$\Rightarrow j=n-1 \Rightarrow \ell = n$

Let $\ell = j+1 \Rightarrow j = \ell - 1 \Rightarrow j = k \Rightarrow \ell = k+1$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{\ell=k+1}^{n} \frac{n!}{(\ell-1)!(n-\ell)!} f(x) F(x)^{\ell-1} (1-F(x))^{n-\ell}$$

Note that both sum expression are exactly the same, so when we substract we're left with just the expression when $j=k$.

$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x)$$

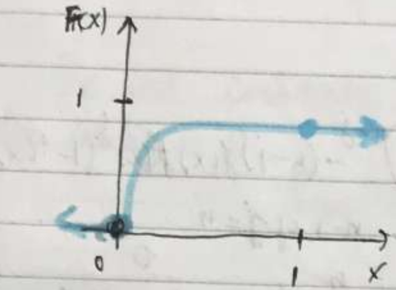
Let's make sure we can recover the min/max formula:

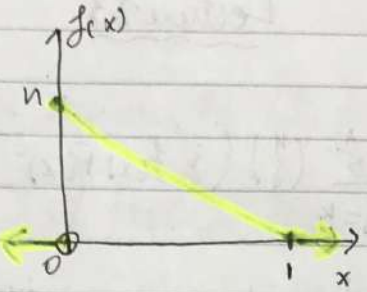
$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1}$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1}$$

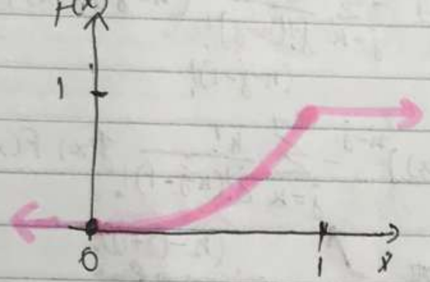
$X_1, \dots, X_n \overset{iid}{\sim} U(0,1) = \overbrace{1\, \mathbb{1}_{x \in [0,1]}}^{f(x)}, \quad F(x) = x$

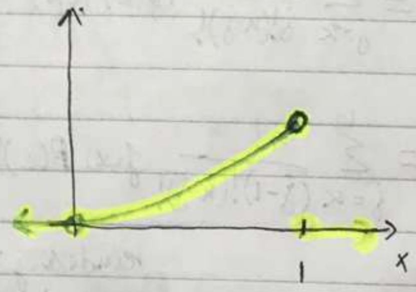
$F_{X_{(1)}}(x) = 1-(1-F(x))^n = 1-(1-x)^n$





$f_{X_{(1)}}(x) = n(1-x)^{n-1}$





$F_{X_{(n)}}(x) = F(x)^n = x^n$

$f_{X_{(n)}}(x) = n\, x^{n-1}$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k+1-1} \mathbb{1}_{x \in [0,1]}$$

$$= Beta(k, n-k+1)$$

$X \sim \underbrace{Gamma(\alpha_1, \beta)}_{\text{"Erlang}(\alpha_1,\beta)\text{"}}$ independent of $Y \sim \underbrace{Gamma(\alpha_2, \beta)}_{\text{Erlang}(\alpha_2,\beta)}$, $T = X+Y \underset{\text{Remarkable}}{\sim} Gamma(\alpha_1+\alpha_2, \beta)$

To Prove this, we develop a new tool that makes it easier for us. That's "kernels", $k(x)$. For any PMF or PDF, we can decompose it into a normalization constant c and a kernel $k(x)$.

$$P(x) = c\,k(x) \text{ and } f(x) = c\,k(x) \Rightarrow P(x) \propto k(x),\ f(x) \propto k(x)$$

not a function of x

$$1 = \sum_{supp} p(x) = \sum c\,k(x) \Rightarrow \frac{1}{c} = \sum k(x) \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int_{supp} f(x)\,dx = \int c\,k(x)\,dx \Rightarrow \frac{1}{c} = \int k(x)\,dx \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

This means that $k(x)$ is 1:1 with the PMF or PDF. If you know $k(x)$, you know the distribution of the r.v. Let's see some examples:

Let $X \sim Bin(n,p) = \binom{n}{x} p^x (1-p)^{1-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x \mathbb{1}_{x\in\{0,1,\dots,n\}}}_{k(x)}$$

$$X \sim Weibull(k,\lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y\geq 0} = \underbrace{k\lambda^k}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y\geq 0}}_{k(x)}$$

$$X \sim Gamma(\alpha,\beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x}\mathbb{1}_{x\geq 0}}_{k(x)} \propto x^{\alpha} e^{-\beta x} \mathbb{1}_{x \geq 0}$$

$X \sim Gamma(\alpha_1,\beta)$ ind of $Y \sim Gamma(\alpha_2,\beta)$; $T = X+Y \sim f_T(t) = ?$

$$f_T(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-\beta x} \frac{\beta^{\alpha_2-1}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t} e^{\beta x}} \underbrace{\mathbb{1}_{t-x\in[0,\infty)}}_{\substack{x-t\in(-\infty,0] \\ x\in(-\infty,t]}} dx$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1} dx\ \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t}\int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}dx\ \mathbb{1}_{t\geq 0} = e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1}\left(1-\frac{x}{t}\right)^{\alpha_2} dx\ \mathbb{1}_{t\geq 0}$$

U-substitution

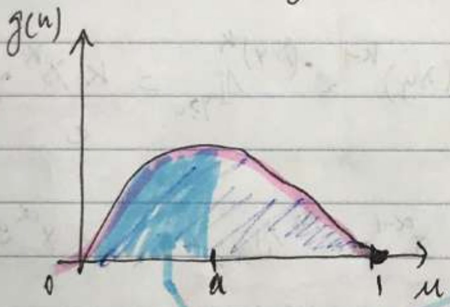
let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$, $x=0 \Rightarrow u=0$, $x=t \Rightarrow u=1$

$$\stackrel{=}{} e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} t\,du\, \mathbb{1}_{t\geq 0} = e^{-\beta t} t^{\alpha_1+\alpha_2-1} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du}_{C(\alpha_1,\alpha_2)} \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t} t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta)$$

Let's talk about the "beta function".

$$B(\alpha, \beta) := \int_0^1 \underbrace{u^{\alpha-1}(1-u)^{\beta-1}}_{g(u)} du \text{ not available in closed form}$$



The "Incomplete beta function" is: $B(a, \alpha, \beta) = \int_0^a u^{\alpha-1}(1-u)^{\beta-1} du =$

The Regularized incomplete beta function is: $I_a(\alpha, \beta) := \frac{B(a,\alpha,\beta)}{B(\alpha,\beta)}$

not Bessel function!

Let's derive a beta function - gamma function identity

$$T \sim \text{Gamma}(\alpha_1+\alpha_2; \beta) = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)} \cancel{t^{\alpha_1+\alpha_2-1}} \cancel{e^{-\beta t}} \cancel{\mathbb{1}_{t\geq 0}} = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2) \cancel{t^{\alpha_1+\alpha_2-1}} \cancel{e^{-\beta t}} \cancel{\mathbb{1}_{t\geq 0}}$$

$$\Rightarrow B(\alpha_1; \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$ [illegible]

$$X \sim Beta(\alpha, \beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]} \qquad \alpha, \beta > 0$$

$$1 = \int_{Supp} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{1}{B(\alpha,\beta)} \overbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}^{B(\alpha,\beta)} = 1$$

CDF:

$$F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)}$$

$$= I_x(\alpha,\beta)$$