$A \in \mathbb{R}^{L\times K}$ constants

$\vec{a}_{i\cdot} :=$ ith row vector of A

$$\underset{\substack{(L\times K)(K\times 1)\\ \shortparallel \\ L\times 1}}{E[A\vec{X}]} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \dots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \dots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \dots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_{1\cdot}\cdot\vec{X}] \\ E[\vec{a}_{2\cdot}\cdot\vec{X}] \\ \vdots \\ E[\vec{a}_{L\cdot}\cdot\vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot}\cdot\vec{\mu} \\ \vec{a}_{2\cdot}\cdot\vec{\mu} \\ \vdots \\ \vec{a}_{L}\cdot\vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$\vec{a} \in \mathbb{R}^K$

$$\underset{\substack{(1\times K)(K\times 1)\\ \shortparallel \\ \text{scalar}}}{Var[\vec{a}^\top\vec{X}]} = Var[\underbrace{a_1X_1}_{Y_1} + \dots + \underbrace{a_KX_K}_{Y_K}] = Var[Y_1 + \dots + Y_K] = \sum_{i=1}^{K}\sum_{j=1}^{K} Cov[Y_i, Y_j]$$

$$= \sum\sum Cov[a_iX_i, a_jX_j] = \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_j\sigma_{ij} = \underset{(1\times K)(K\times K)(K\times 1) = \text{scalar}}{\vec{a}^\top\Sigma\vec{a}}$$

$\Sigma = Var[\vec{X}]$

let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^K$

$$\vec{a}^\top V\vec{a} = \vec{a}\cdot(V\vec{a}) = \vec{a}\cdot\begin{bmatrix} a_1V_{11} + \dots + a_KV_{1K} \\ a_1V_{21} + \dots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \dots + a_KV_{KK} \end{bmatrix} = \begin{matrix} a_1a_1V_{11} + \dots + a_1a_KV_{1K} + \\ a_2a_1V_{21} + \dots + a_2a_KV_{2K} + \\ \vdots \\ a_Ka_1V_{K1} + \dots + a_Ka_KV_{KK} \end{matrix}$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_jV_{ij}$$

Quadratic Forms with V being the "determining matrix"

Application in finance. Let X_1, X_2, ..., X_K be returns of financial assets (e.g. stocks). So let w_1, w_2, ..., w_K be the proportion allocated to each of these assets. Let $\vec{\mu} = E[\vec{X}]$, $\Sigma = Var[\vec{X}]$

$F = \vec{w}^\top\vec{X}$ a rv modeling your portfolio

$\mu_F = E[F] = \vec{w}^\top\vec{\mu}$, $Var[F] = \vec{w}^\top\Sigma\vec{w}$

It's possible to pick w-vector to optimize the portfolio by minimizing the variance of returns, Var[F], conditional on mu_F. This is called "Markowitz optimal portfolio theory".

$\min_{\vec{w}} Var[F]$ subject to μ_F being constant and $\vec{w}^\top\vec{1} = 1$.

$\vec{X} \sim Multin_K(n, \vec{p})$

$X_j \sim Bin(n, p_j)$

$$E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & \sigma_{ij} & \\ & np_2(1-p_2) & & \\ \sigma_{ij} & & \ddots & \\ & & & np_K(1-p_K) \end{bmatrix}$$

$i \neq j$

complicated

FAIL...

$$Cov[X_i, X_j] = E[X_iX_j] - E[X_i]E[X_j] = \sum_{x_i\in\mathbb{R}}\sum_{x_j\in\mathbb{R}} x_ix_j\, p_{X_i,X_j}(x_i,x_j) - n^2p_ip_j$$

$i =$ Apple, $j =$ Banana

$$\vec{X} = \begin{bmatrix} \vdots \\ X_i \sim Bin(n, p_i) \\ \vdots \\ X_j \sim Bin(n, p_j) \\ \vdots \end{bmatrix}$$

$X_i = X_{1i} + X_{2i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

We've expressed the multinomial rv with n x K Bernoulli's.

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \overset{iid}{\sim} Multin_K(1, \vec{p})$

$$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}] = \sum_{l=1}^{n}\sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

A lot of these covariances are zero due to independence. Which ones? If l is different than m, the covariance is zero.

$$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}] = \sum_{l=1}^{n}\left(E[X_{li}X_{lj}] - \underbrace{E[X_{li}]}_{p_i}\underbrace{E[X_{lj}]}_{p_j}\right) = \boxed{-np_ip_j}$$

the only term that's nonzero is...

$$= \sum_{x_{li}\in\{0,1\}}\sum_{x_{lj}\in\{0,1\}} x_{li}x_{lj}\, p_{X_{li},X_{lj}}(x_{li}, x_{lj}) = p_{X_{li},X_{lj}}(1,1) = 0$$

you can't get an apple AND a banana on one grab

Midterm I ↑

Midterm II ↓

uniform discrete

$$X \sim U(\{0,1,2,3\}) = p(x) = \begin{cases} 0 & \text{wp } 1/4 \\ 1 & \text{wp } 1/4 \\ 2 & \text{wp } 1/4 \\ 3 & \text{wp } 1/4 \end{cases}$$

$Supp[X] = \{0,1,2,3\}$

Generally, $X \sim U(A)$

$Supp[X] = A$, $A \subset \mathbb{R}$ st $|A| < \infty$ and $A \neq \emptyset$

Create a new rv $Y = -X = g(X)$, a very simple function

$Supp[Y] = \{-3, -2, -1, 0\}$

$$p(Y) = \begin{cases} -3 & \text{wp } 1/4 \\ -2 & \text{wp } 1/4 \\ -1 & \text{wp } 1/4 \\ 0 & \text{wp } 1/4 \end{cases}$$

Generally, for discrete rv X, is there a pattern?

let $z' = -z$

$$p_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) =: p_X(-y) \ast$$

$$Supp[Y] := \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z : p_X(z) > 0\}$$

$$= -\{z : p_X(z) > 0\} =: -Supp[X].$$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda) := \dfrac{e^{-\lambda}\lambda^x}{x!}\mathbb{1}_{x\in\{0,1,\dots\}}$

In class we showed:

$T = X_1 + X_2 \sim Poisson(2\lambda)$

difference

let $D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{-X_2}_{Y} = X + Y$, $Y \sim p_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!}\mathbb{1}_{y\in\{\dots,-1,0\}}$

$Supp[D] = Supp[X] + Supp[Y]$

$= \{\dots, -1, 0, 1, \dots\} = \mathbb{Z}$

all integers

$$p_D(d) = \sum_{x\in Supp[X]} p_X^{old}(x)\, p_Y^{old}(d-x)\,\mathbb{1}_{d-x\in Supp[Y]}$$