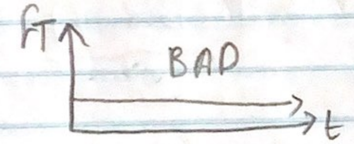
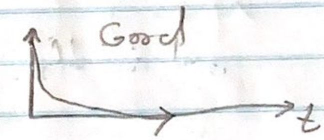
15

10/28

Mixture and compound distributions

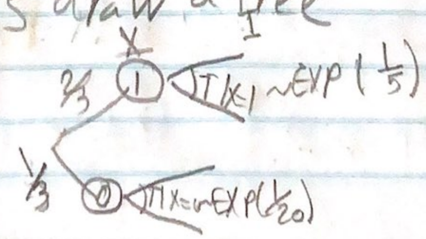
eg. $1/3$ of the time, you get bad traffic and your download speed are $T \sim EXP(1/20)$ i.e. $E(T) = 20s$ and $2/3$ of the time you have good Interent traffic and your download speeds are $T \sim EXP(1/5)$ i.e. $E(T) = 5s$. what is the distribution of T "overall"





Let $X \sim Bern(2/3)$, a rv modeling traffic.

If $X=1$, then we have good traffic and if $X=0$, we have bad traffic. So now we have $T|X=1 \sim EXP(1/5)$ and $T|X=0 \sim EXP(1/20)$. Now we essentailly use marginalization to get T "unconditional" (meaning overall). first Let's draw a tree



Marginalization:

$$h(x) = \int_{R} h(x,y)\,dy \quad \text{or} \quad d(x) = \sum_{y} d(x,y)$$

$$F_T(t) = \sum_{X \in Supp[X]} f_{T,X}(t,x) = \sum_{X \in Supp[X]} f_{T|X}(t,x)\,P_X(x) = \sum_{X \in \{0,1\}} f_{T|X}(t,x)\,P_X(x)$$

$$= f_{T|X}(t,0)\,P_X(0) + f_{T|X}(t,1)\,P_X(1) = \frac{1}{20} e^{-\frac{1}{20}t}\,\frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t}\,\frac{2}{3}$$

This was a our first "mixture model" where generally $Y|X$ is the model and X is the mixing distribution.

If the download took $t = 25$ s, what is the probability you had bad traffic? Let's find the distribution of traffic conditional on $t = 25$ s.

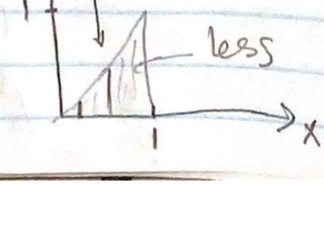
$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x)\,P_X(x)}{f_T(t)} = \overset{X|T\sim}{Bern(?)}$$

$$P_{X|T}(1,t) = \frac{f_{T|X}(t,1)\,P_X(1)}{f_T(t)} = \frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}$$

$$P_{X|T}(0,25) = 1 - P_{X|T}(1,25) = 1 - 0.581 = 0.842.$$

$X \sim U(0,1)$, $Y|X = x \sim U(0,x)$

X	Y
$U(0,1)$	$U(0,1)$



$$f_y(y) = \int_{supp[X]} f_{y|X}(y,x)\, f_X(x)\, dx$$

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p156-157, Let $Y|X \sim$ Poisson$[X]$ and $X \sim$ Gamma(α,β), $Y \sim$?

X	Y
~~X~~ Gamma(α,β)	~~X~~ Poisson(X)

$$P_y(y) = \int_{supp[X]} P_{y,X}(y,x) = \int_{supp[X]} P_{y|X}(y,x) f_X(x)\, dx = \int_0^\infty \frac{x^y e^{-x}}{y!} 1_{y\in\{0,1,\dots\}}$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} 1_{y\in N_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} dx$$ see q.

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}} 1_{y\in N_0} \overset{HW}{=} \dots \propto \text{Ex+ NegBin}\left(\alpha, \frac{\beta}{1+\beta}\right)$$

This is a "more flexible" count distribution than the Poisson.

$Y|X=x \sim$ Bin(n, x), n fixed, $X \sim$ Beta(α,β). $Y \sim$?

This is analogue to the problem above because binomial is also a count distribution with a fixed upper bound n.

$$P_y(y) = \int_{supp[X]} P_{y,X}(y,x)\, dx = \int_{supp[X]} P_{y|X}(y,x) f_X(x)\, dx = \int_0 \binom{n}{y} x^y (1-x)^{n-y} 1_{y\in\{0,\dots,n\}} \cdot$$

$$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} 1_{y\in\{0,\dots,n\}} \overbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{\beta+n-y-1}\,dx}^{B(y+\alpha,\,\beta+n-y)}$$

$$= \frac{B(y+\alpha,\beta+n-y)}{B(\alpha,\beta)} \binom{n}{y} 1_{y\in\{0,\dots,n\}} = \text{BetaBinomial}(n,\alpha,\beta)$$

H.W.: $Y|X \sim X \sim EXP(X)$, $X\sim \text{Gamma}(\alpha,\beta) \Rightarrow Y \sim \text{Lomax}(\beta,\alpha)$

Midterm II ↑

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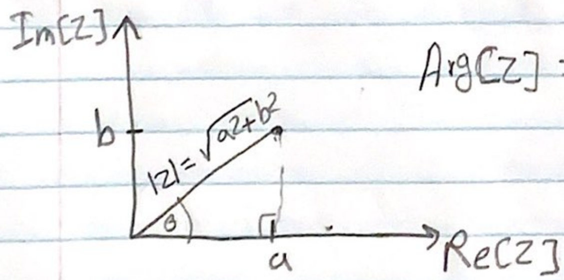
final ↓

Moment Generating functions (mgf's) and characteristic functions (chf's). We first need to reivew imaginary #'s and high school trig.

$a,b \in \mathbb{R}$, $Z := a+bi \in \mathbb{C}$, the complex #'s, $i = \text{sqrt}(-1)$

Let $Re[Z] := a$, the "real" component of the imaginary # Z.

Let $Im[Z] : b$, the "imaginary" component of the imaginary # Z.



$$Arg[Z] = \theta \underset{=}{\text{usually}} \arctan\left(\frac{b}{a}\right)$$

$i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = 1$, $i^c = i^{c \bmod 4}$ --- clock 4 system

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$\sin(x) = \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - -$$

$$\cos(x) = \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - -$$

$$i\sin(tx) = \quad = itx - i\frac{t^3x^3}{3!} + i\frac{t^5x^5}{5!} - - -$$

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \Rightarrow e^{i\theta} = i\sin(\theta) + \cos(\theta)$$

$$\overset{\theta=\pi}{\Rightarrow} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$ Euler's Formula