Wednesday September 9th 2020

Lecture 4

$X, Y \overset{iid}{\sim} \text{Geom}(P)$

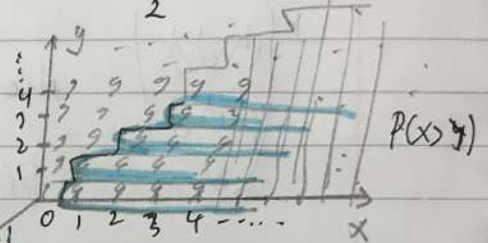
$P(X>Y) = \frac{1}{2}$ "good guess"

$P(X>Y) = P(Y>X)$

$P(X>Y) + P(Y>X) + P(X=Y)$

$2P(X>Y) = 1 - P(X=Y)$

$\Rightarrow P(X>Y) = \frac{1-P(X=Y)}{2} < \frac{1}{2}$



X, Y (independent)

$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} P_X(x) P_Y(y) \mathbb{1}_{x>y}$

$= \sum_{x\in\{0,1,\dots\}} \sum_{y\in\{0,1,\dots\}} (1-P)^x P (1-P)^y P \mathbb{1}_{x>y}$

$\Leftrightarrow x \geq y+1 \Rightarrow x \in \{y+1, y+2, \dots\}$

$P_{X,Y}(x,y)$

$= P^2 \sum_{y\in\{0,1,\dots\}} \sum_{x\in\{0,1,\dots\}} (1-P)^x (1-P)^y \mathbb{1}_{x\in\{y+1, y+2, \dots\}}$

$= P^2 \sum_{y\in\{0,1,\dots\}} \sum_{x\in\{y+1,y+2,\dots\}} (1-P)^x (1-P)^y$

$= P^2 \sum_{y\in\{0,1,\dots\}} (1-P)^y \sum_{x\in\{y+1,\dots\}} (1-P)^x$

"Reindexing Trick"

$x' = x-(y+1) \in \{0,1,2,\dots\}$

$\Rightarrow x = x'+y+1$ (similar to u substitution).

$= P^2 \sum_{y\in\{0,1,2,\dots\}} \sum_{x\in\{0,1,\dots\}} (1-P)^x (1-P)^y (1-P)$

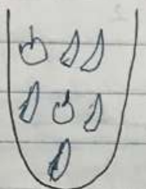
$= P^2(1-P) \sum_{y\in\{0,1,\dots\}} (1-P)^{2y} \sum_{x\in\{0,1,\dots\}} (1-P)^{x'}$

$\frac{1}{1-(1-P)^2} = \frac{1}{1-(1-2P+P^2)} = \frac{1}{2P-P^2} = \frac{1}{P(2-P)}$

$= \frac{1}{1-(1-P)} = \frac{1}{P}$

$= \frac{P(1-P)}{P(2-P)} = \frac{1-P}{2-P} < \frac{1}{2}$ "if P is really small"

Consider a bag of fruit that has apples and bananas. You now draw with replacement n samples from this bag and you count how many are apples and how many are bananas. Let X_1 be the r.v that counts the number of apples and let X_2 be the r.v that count the number of bananas. Let P_1 be the probability of picking an apple and P_2 be the probability of picking a banana.



$X_1 \sim Bin(n, P_1)$, $X_2 \sim Bin(n, P_2)$

Are X_1 and X_2 independent? No $\Rightarrow$ dependent

$$P(X_1 = x_1 \mid X_2 = x_2) \overset{?}{=} (P_{X_1} = x_1)$$

$$n=10 \quad P(X_1 = 6 / X_2 = 4) = 1 \neq \binom{10}{6} P_1^6 (1-P)^4$$
$$P(X_1 = 3 / X_2 = 4) = 0$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{X} \sim P_{\vec{X}}(x_1, x_2) = \underbrace{\frac{n!}{x_1! x_2!}} P_1^{x_1} P_2^{x_2} \mathbb{1}_{\underbrace{x_1+x_2=n}} \mathbb{1}_{\underbrace{x \in \{0,1,\ldots,n\}}} \mathbb{1}_{\underbrace{x_2 \in \{0,1,\ldots,n\}}}$$

$$\binom{n}{x_1, x_2}$$ multichoose notation

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \vec{p}$$

$$\vec{X} \sim Multin(n, \vec{p}) = \binom{n}{x_1, x_2} P_1^{x_1} P_2^{x_2}$$

multinomial rv of dim=2

Let add Cantaloupe to the bag. Let X_3 count the number of Cantaloupes and P_3 be the probability of drawing a Cantaloupe.

The general multinomial r.v of dim $= K$ has PMF

$$\vec{X} \sim Multin(n, \vec{p}) := \binom{n}{x_1, x_2, \ldots, x_K} \prod_{k=1}^{K} P_k^{x_k}$$

Param space

$n \in \mathbb{N}$

$\vec{p} \in \{\vec{v} : \vec{v} \cdot \vec{1} = 1, v_1 \in (0,1), \ldots, v_K \in (0,1)\}$

$$Supp[\vec{X}] = \{\vec{x} : \vec{x} \cdot \vec{1} = n, x_1 \in \{0,1,\ldots,n\}, \ldots, x_K \in \{0,1,\ldots,n\}\}$$

Let derive the condition PMF and the marginal PMF's in the case of $k = 2$ (apples and bananas).

$$Deg(n - x_2) = P_{X_1/X_2}(x_1,x_2) = \frac{P_{X_1,X_2}(x_1,x_2)}{P_{X_2}(x_2)}$$

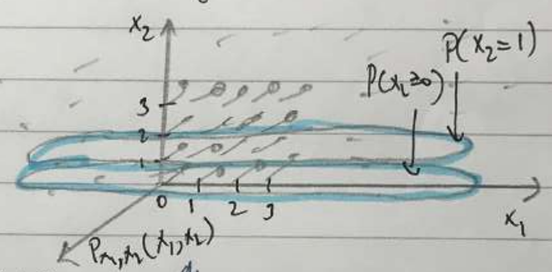
← JMF

← marginal PMF

How to prove that the marginal PMF is Binomial?

How do we compute the marginal PMF from the JMF

$$P_{X_2}(x_2) = \sum_{x_1\in\Omega} P_{X_1,X_2}(x_1,x_2)$$



$$= \sum_{x\in\Omega} \binom{n}{x_1,x_2} p_1^{x_1} p_2^{x_2}$$

$$= p_2^{x_2}\sum_{x\in\Omega}\frac{n!}{x_1!\,x_2!}\, p_1^{x_1}\, \mathbb{1}_{\underbrace{x_1+x_2=n}_{x_1=n-x_2}}\, \mathbb{1}_{x_1\in\{0,1,\dots,n\}}\, \mathbb{1}_{x_2\in\{0,1,\dots,n\}}$$

$$= p_2^{x_2}\,\frac{n!}{x_2!}\,\mathbb{1}_{x_2\in\{0,1,\dots,n\}}\sum_{x\in\Omega}\frac{p_1^{x}}{x!}\,\mathbb{1}_{x_1=x_2}\,\mathbb{1}_{x_1\in\{0,1,\dots,n\}}$$

$$= p_2^{x_2}\,\underbrace{\frac{n!}{x_2!}\,\mathbb{1}_{x_2\in\{0,1,\dots,n\}}\,\frac{p_1^{n-x_2}}{(n-x_2)!}}_{\binom{n}{x_2}} = \binom{n}{x_2}\, p_2^{x_2}\, p_1^{n-x_2}$$

$$= Bin(n, p_2)$$

Since $p_1 + p_2 = 1 \Rightarrow p_1 = (1-p_2)$