Lecture 10

$T_K \sim \text{Erlang}(K, \lambda)$, $N \sim \text{Poisson}(\lambda)$

$P(T_K > 1) = 1 - F_{T_K}(1) = Q(K, \lambda)$ $\quad F_N(x) = Q(x+1, \lambda)$

$\Rightarrow 1 - F_{T_K}(1) = F_N(K-1)$ "Poisson Process"

$N :=$ # of events by $t = 1$.

0 —— 1 ——→ t (seconds)

$$\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\} \cup \{X_1 + X_2 + X_3 < 1\} \cup \{X_1 + X_2 < 1\} \cup$$

$K = 5$

$$\{X_1 < 1\} \cup \{X_1 > 1\} = \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}.$$

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$$\begin{array}{ccc} P(T_5 > 1) & = & P(N \le 4) \\ \| & & \| \\ 1 - F_{T_5}(1) & & F_N(4) \end{array}$$

$T \sim \text{Erlang}(K, \lambda) = \dfrac{\lambda^K e^{-\lambda t} t^{K-1}}{(K-1)!} \mathbb{1}_{t \ge 0} = \dfrac{\lambda^K e^{-\lambda t} t^{K-1}}{\Gamma(K)} \mathbb{1}_{t \ge 0}$ "Gamma"

$K \in \mathbb{N}, \lambda \in (0, \infty)$

$T \sim \text{NegBin}(K, p) = \binom{K+t-1}{K-1} (1-p)^t p^K \mathbb{1}_{t \in \mathbb{N}_0} = \dfrac{\Gamma(K+t)}{t!\,\Gamma(K)} (1-p)^t p^K \mathbb{1}_{t \in \mathbb{N}_0}$

$K \in \mathbb{N}, p \in (0, 1)$ "Extended NegBinom."

What if $K \in (0, \infty)$? Is the top PDF legal and the bottom PMF legal?

$$\int_0^\infty \frac{\lambda^K e^{-\lambda t} t^{K-1}}{\Gamma(K)} dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(K+t)}{\Gamma(K)\, t!} (1-p)^t p^K = 1$$

which means, these are r.v.'s.

$X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \mathbb{1}_{t \geq 0}$.

Ext.

$X \sim \text{Neg Binom}(k, p)$ .....

Transformations for discrete rv's.

$X \sim \text{Bern}(p)$, $Y = X+3 \sim \begin{cases} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y \in \{3,4\}}$

$y-3 \in \{0,1\}$

$\| \; p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

If $Y = g(x) \sim P_Y(y) =$ how do I express the transformation PMF using the original PMF?

$\Updownarrow$

$g^{-1}(y) = X$

$= P_X(g^{-1}(y))$.

Is this formula general?

- No, this is only the formula for g invertible. If g is non-invertible ...

$X \sim U(\{1, 2, \ldots, 10\}) = \frac{1}{10} \mathbb{1}_{x \in \{1,2,\ldots,10\}}$

$Y = \min\{X, 3\} \sim \begin{cases} 1 & \text{w.p. } 1/10 \\ 2 & \text{w.p. } 1/10 \\ 3 & \text{w.p. } 8/10 \end{cases}$

$\left[P(X=3) + P(X=4) + \ldots + P(X=10) = 8/10 \right]$

$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x) \overset{\text{if invertible}}{=} \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$

$X \sim Bin(n,p)$, $Y = X^2 \sim p_Y(y) = p_X(g^{-1}(y)) = p_X(\sqrt{y}) =$

$\Downarrow$

$X = \sqrt{y}$
$= g^{-1}(y)$

$\binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$

Transformations for Continous r.v's

$Y = g(X)$, X is cts. for invertible g,

$f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ ← INCORRECT

$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$, $Y = 2X \sim f_Y(y) = f_X(\frac{y}{2}) = \mathbb{1}_{\frac{y}{2}\in[0,1]} = \mathbb{1}_{y\in[0,2]}$

$\int_0^2 f_Y(y) = 2 - 0 = 2 \neq 1$ ILLEGAL

Where'd we go wrong? The above formula was derived using probabilites. Densities are not probabilites! But.... CDF's are probabilites! **Strategy**: let's derive the CDF of Y using the CDF of X! And then, like when we did convolusions, take the derivative to get the density of Y.

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) \overset{*}{=} P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$

$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$

$= f_X(g^{-1}(y)) \underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{stretching}} =$

if $g' < 0$

$= P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

$f_Y(y) = \frac{d}{dy}[\quad] = -\frac{d}{dy}[F_X(g^{-1}(y))] = f_X(g^{-1}(y))\left(-\frac{d}{dy}[g^{-1}(y)]\right)$

$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$

$\frac{d}{dy}[g^{-1}(y)] < 0$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$ for all g invertible.

Let's derive some rules! The most common invertible function is... the straight line!

$y = ax + c$! $\Rightarrow x = g^{-1}(y) = \frac{y-c}{a}$, $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|a|}$

s.t. $a, c \in \mathbb{R}$.

$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$ "shift and scale".

- if $c = 0$ just a scale ... $Y = aX$

  $f_Y(y) = f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$
- if $a = 1$ just a shift ... $Y = X + c$

  $f_Y(y) = f_X(y - c)$

$X \sim Exp(\lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x \geq 0}$

$Y = X + c = \lambda e^{-\lambda(y-c)}\mathbb{1}_{y \geq c}$

$X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x \geq 0}$

$Y = g(X) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^x - 1) = y$

$\rightarrow e^y = e^x - 1 \Rightarrow e^y + 1 = e^x \Rightarrow \ln(e^y + 1) = x = g^{-1}(y)$ ✓

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \left|\frac{e^y}{e^y+1}\right| = \frac{e^y}{e^y+1}$$

$$f_Y(y) = f_X \cancel{(g^{-1}(y))} \cdot (\ln(e^y+1)) \frac{e^y}{e^y+1} = e^{-\ln(e^y+1)} \underbrace{\mathbb{1}_{\ln(e^y+1) \geq 0}}_{\substack{e^y + 1 \geq 1 \\ e^y \geq 0 \\ y \in \mathbb{R}}} \frac{e^y}{e^y+1}$$

$$= \frac{1}{e^y+1}\left(\frac{e^y}{e^y+1}\right) = \frac{e^y}{(e^y+1)^2} = \text{Logistic}(0,1) \checkmark$$

$$\frac{e^y}{(e^y+1)^2} = \frac{e^{-y}}{(e^{-y}+1)^2}$$