Monday August 31 (Lecture 02)

(cont.)

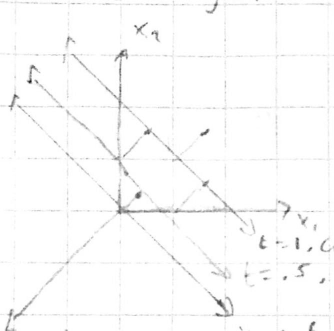
Has to be a better way of expressing the pmf of T

$$p(T) = p(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1, X_2}(x_1, x_2) \, \mathbb{1}_{x_1 + x_2 = t} = p_{X_1}(x) * p_{X_2}(x)$$

ex: $p(T=1) = p_{X_1, X_2}(1,0) + p_{X_1, X_2}(0,1)$

Even if we put $(\frac{1}{2}, \frac{1}{2})$ in, the jmf will nuke it

Another way of looking at the convolusion formula:



Everything in $\mathbb{R}^2$ is nuked except for these 4 positions/hairs

we are going to graph this

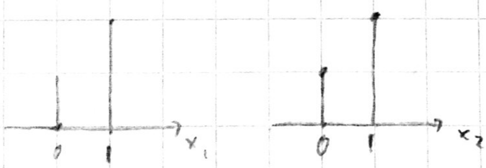
$x_2 = t - x_1$, $t = 0$. $\mathbb{1}$ is summing across whole line.

IF $t = 2$, $p_T(2) = \sum_{x_1 \in \mathbb{R}} p_{X_1 X_2}(x, 2 - x_1)$ not summing over x_2 since we "pinned" $x_2 = t - x_1$ (x_2 isn't a free variable)

Convolve: Roll/coil together/entwine

$$p_T = \begin{cases} t = 0 & \text{w.p } (1-p)^2 \\ t = 1 & \text{w.p } p(1-p)\,2 \\ t = 2 & \text{w.p } p^2 \end{cases}$$

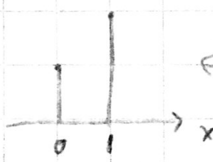
pmF(x_1)



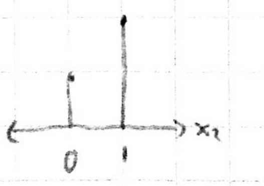
roll one more time:

$t = 1$ w.p $p(1-p)$

IF none of the hairs intersect, multiply by zero



$t = 1$ w.p $p(1-p)$



$t = 0$ w.p $(1-p)^2$, $t = 2$ w.p p^2

$$P_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{x_1, x_2}(x_1, x_2) \, \mathbb{1}_{x_2 = t - x_1}$$

But we don't need to sum over x_2 since the indicator function handles it

$$= \sum_{x \in \mathbb{R}} P_{x_1, x_2}(x, t - x)$$

x_2 must equal $t - x_1$

$\boxed{\text{General Convolution Formula}}$

IF x_1, x_2 are independent,

$$= \sum_{x \in \mathbb{R}} P_{x_1}(x) \, P_{x_2}(t - x) \quad \boxed{\text{Convolution Formula for independent variables}}$$

$$= \sum_{x \in \mathbb{R}} P^{old}_{x_1}(x) \, \mathbb{1}_{x \in Supp[x_1]} \, P^{old}_{x_2}(t - x) \, \mathbb{1}_{(t-x) \in Supp[x_2]}$$

$$= \sum_{x \in Supp[x_1]} P^{old}_{x_1}(x) \, P^{old}_{x_2}(t - x) \, \mathbb{1}_{t-x \in Supp[x_2]}$$

IF $x_1 \overset{iid}{\sim} x_2$

$$= \sum_{x \in \mathbb{R}} P(x) \, P(t - x) = \sum_{x \in \mathbb{R}} P^{old}(x) \, \mathbb{1}_{x \in Supp[x]} \, P^{old}(t - x) \, \mathbb{1}_{t-x \in Supp[x]}$$

$\boxed{\text{Convolution Formula for iid}}$

$$= \sum_{x \in Supp[x]} P^{old}(x) \, P^{old}(t - x) \, \mathbb{1}_{t-x \in Supp[x]}$$

$P_T(t)$

$\shortparallel$

$$T_2 \sim \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \, p^{(t-x)} \, p^{(1-t+x)} \, \underbrace{\mathbb{1}_{t-x \in \{0,1\}}}$$

(plug in from convolution formula for iid)

$\hookrightarrow$ $P_T(2)$ parameters on 2 iid Bern(x) with support $\{0,1\}$

$= \mathbb{1}_{t \in \{x, x+1\}}$ (add x to both sides)

$$P_T(t) = p^{(t)} (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^{(t)} (1-p)^{2-t} \Big(\underbrace{\mathbb{1}_{t \in \{0,1\}}}_{(x=0)} + \underbrace{\mathbb{1}_{t \in \{1,2\}}}_{(x=1)} \Big)$$

Recall:

$$\binom{n}{k} := \frac{n!}{(n-k)! \, k!} \, \mathbb{1}_{n \in \mathbb{N}} \cdot \mathbb{1}_{k \in \{0,1,\ldots,n\}}$$

$$p \begin{cases} t = 0 \Rightarrow 1 \\ t = 1 \Rightarrow 2 \\ t = 2 \Rightarrow 1 \end{cases} = \binom{2}{t}$$

In General:

$Supp[T] = Supp[x_1] + Supp[x_2]$,

$A + B := \{a + b \mid a \in A, b \in B\}$

$$= \binom{2}{t} p^t (1-p)^{2-t} = Binom(2, p)$$

$Supp[T_2] = \{0, 1, 2\}$

$X_1, X_2 \overset{iid}{\sim} Bern[p] = \binom{1}{x} p^x (1-p)^{1-x}$

$$p_{T_2}(t) = \sum_{x\in\mathbb{R}} p(x)\, p(t-x) = \sum_{x\in\mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$

Recall Pascal's Identity:

$$= p^t (1-p)^{2-t} \left(\overset{=1}{\cancel{\binom{1}{0}}}\binom{1}{t} + \overset{=1}{\cancel{\binom{1}{1}}}\binom{1}{t-1}\right)$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n+1}{k+1}$$

$$= p^t (1-p)^{2-t} \binom{2}{t}$$

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$ $\quad T_3 = X_1 + X_2 + X_3$

$$T_3 = X_3 + \binom{2}{t} p^t (1-p)^{2-t} = X_3 + T_2 \sim p_{T_3}(t) = ?$$

Can't use 3rd convolution formula since T_2 and X_3 are not identical, but they are independent

$$p_{T_3}(t) = \sum_{x\in supp[X_3]} \overset{old}{p_{X_3}(x)}\, \overset{old}{p_{T_2}(t-x)} = \sum_{x\in\{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}$$

$$= p^t (1-p)^{3-t} \sum_{x\in\{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1}\right)$$

$$= p^t (1-p)^{3-t} \binom{3}{t}$$

$X_1, X_2 \sim Binom(n,p)$ $\quad T = X_1 + X_2 \sim ?$ $\quad \Rightarrow$ Basically doing 2 consecutive binomial trials (2n Bernoulli trials)

$Binom(n,p) := \binom{n}{t} p^t (1-p)^{n-t}$

$$p_T(t) = \sum_{x\in\mathbb{R}} p(x)\, p(t-x)$$

$$= \sum_{x\in\mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \sum_{x\in\mathbb{R}} \binom{n}{t-x}\binom{n}{x}$$

Vandermonde's identity gives us:

$$= p^t (1-p)^{2n-t} \binom{2n}{x} = Binom(2n, p)$$