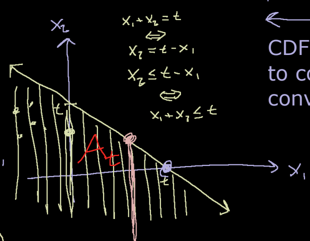
$T = X_1 + X_2 \sim f_T(t) = ?$ It also has a CDF $f_T(t) = F_T'(t)$ ⟵ CDF method to compute the convolution

$F_T(t) = P(T \le t)$
$= P(\vec{X} \in A_t)$
$= \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$



$= \int_{\mathbb{R}} \left(\int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2 \right) dx_1 = \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\,dv\,dx = \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx\,dv$

let $x_1 = x$, $x_2 = v - x \Rightarrow dx_2 = dv$, $x_2 = -\infty \Rightarrow v = -\infty$, $x_2 = t - x \Rightarrow v = t$

$\Rightarrow f_T(t) = \frac{d}{dt}[\;\;]$

Leibnitz's Rule for derivatives of integral functions.

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x, b(x))\,b'(x) + g(x, a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy$$

If the derivative is with respect to a third variable, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x, b(t))\,b'(t) + g(x, a(t))\,a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}[g(x,y)]\,dy}^{\,0}$$

If one of the bounds in constant then...

$$\frac{d}{dt}\left[\int_c^{b(t)} g(x,y)\,dy\right] = g(x, b(t))\,b'(t) + \cancel{g(x,c)\frac{d}{dt}[c]}^{\,0}$$

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\left(\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx\right)dv\right] = \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\,dx$$ general convolution formula

X_1, X_2 independent

$$= \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f_{X_1}^{old}(x)\, f_{X_2}^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

X_1, X_2 iid

$$= \int_{\mathbb{R}} f(x) f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

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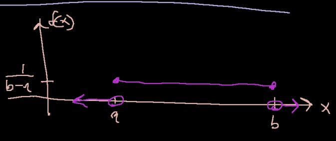
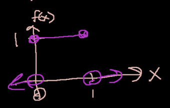
continuous uniform rv

$X \sim U(a,b) = \underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x \in [a,b]}$ $f(x)$

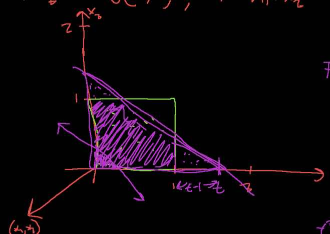


The "standard uniform" rv is when a = 0, b = 1

$X \sim U(0,1) = \frac{1}{1-0}\mathbb{1}_{x \in [0,1]}$



$X_1, X_2 \stackrel{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$ CDF method first...



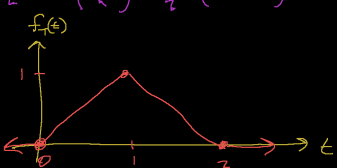
$$F_T(t) = P(T \le t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t^2}{2} & \text{if } t \in (0,1] \\ -\frac{t^2}{2} + 2t - 1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \ge 2 \end{cases}$$

$t = 0.3$

$f_{X_1,X_2}(x_1,x_2)$

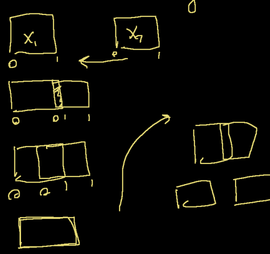
$\frac{t^2}{2} - 2 \times \left(\frac{(t-1)^2}{2}\right) = \frac{t^2}{2} - (t^2 - 2t + 1)$

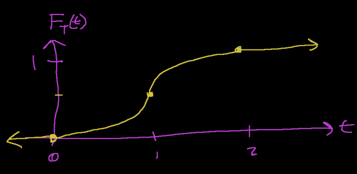
$$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$



$Supp[T] = [0,2]$

Convolving



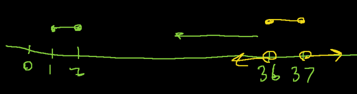


Let's try to derive the PDF of T using the convolution formula.

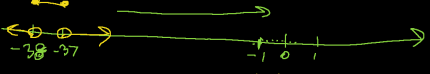
$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\,dx = \int_0^1 (1)(1)\, \underbrace{\mathbb{1}_{t-x \in [0,1]}}_{\substack{x \in [t-1,t] \\ x - t \in [-1,0]}}\,dx = \int_0^1 \underbrace{\mathbb{1}_{x \in [t-1,t]}}_{f(x)}\,dx$$

$$= \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$

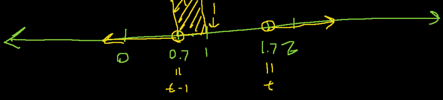
let $t = 37$



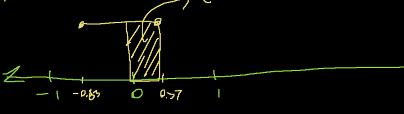
let $t = -37$



let $t = 1.7$



let $t = 0.37$



$X_1, X_2, \ldots \stackrel{iid}{\sim} Exp(\lambda) = \underbrace{\lambda e^{-\lambda x}}_{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)}$, $T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$

$$f_{T_2}(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\,dx = \int_0^\infty \lambda \cancel{e^{-\lambda x}}\, \lambda e^{-\lambda(t-x)} \underbrace{\mathbb{1}_{t-x \in [0,\infty)}}_{\substack{x \in (-\infty, t] \\ x - t \in (-\infty, 0]}}\,dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t]}\,dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t\lambda^2 e^{-\lambda t}\, \mathbb{1}_{t \in (0,\infty)}$$

$= Erlang(2, \lambda) = f_{T_2}(t)$