Lecture 06
09/16/20
Math 621

a_i = i^{th} row vector of A

matrix $A \in \mathbb{R}^{L\times K}$

$$E[A\vec{x}] = \begin{bmatrix} E[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1K}x_K] \\ E[a_{21}x_1 + a_{22}x_2 + \cdots + a_{2K}x_K] \\ \vdots \\ E[a_{L1}x_1 + a_{L2}x_2 + \cdots + a_{LK}x_K] \end{bmatrix}$$

$(L\times K)(K\times 1)$ = $(L\times 1)$

$$= \begin{bmatrix} E[\vec{a}_1 \vec{x}] \\ E[\vec{a}_2 \vec{x}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{x}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_L \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$\vec{a} \in \mathbb{R}^K$

$$Var[\vec{a}^T\vec{x}] = Var[\underbrace{a_1x_1}_{Y_1} + \cdots + \underbrace{a_Kx_K}_{Y_K}]$$

$(1\times K)(K\times 1)$ = scalar

$$= Var[Y_1 + \cdots + Y_K]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} Cov[Y_i, Y_j]$$

$$= \sum\sum cov[a_ix_i, a_jx_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_ia_j\sigma_{ij} = \vec{a}^T\Sigma\vec{a}$$

$(1\times K)(K\times K)(K\times 1)$ = scalar

$\Sigma = Var[\vec{x}]$

Let $V \in \mathbb{R}^{K\times K}$, $\vec{a} \in \mathbb{R}^K$

$$\vec{a}^TV\vec{a} = \vec{a}\cdot(V\vec{a}) = \vec{a}\cdot\begin{bmatrix} a_1v_{11} + \cdots + a_Kv_{1K} \\ a_1v_{21} + \cdots + a_Kv_{2K} \\ \vdots \\ a_1v_{K1} + \cdots + a_Kv_{KK} \end{bmatrix}$$

Quadratic Forms with V being the "determining matrix"

$$= a_1 a_1 v_{11} + \cdots + a_1 a_k v_{1k} +$$
$$a_2 a_1 v_{21} + \cdots + a_2 a_k v_{2k} +$$
$$\vdots$$
$$a_k a_1 v_{k1} + \cdots + a_k a_k v_{kk}$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j v_{ij}$$

Application in finance. Let $X_1, X_2, \ldots, X_k$ be returns of financial assets (e.g. stocks).
So, let $w_1, w_2, \ldots, w_k$ be the proportio allocated to each of these assets.
Let $\vec{\mu} = E[\vec{X}]$, $\Sigma = Var[\vec{X}]$. sigma
Let $F = \vec{w}^T \vec{X}$ $\Leftarrow$ a rv modeling your portfolio

$$\mu_F = E[F] = \vec{w}^T \vec{\mu}, \quad Var[F] = \vec{w}^T \Sigma \vec{w}$$

It's possible to pick w-vector to optimize the portfolio by minimizing the variance of returns, $Var[F]$, conditional on μ_F.

This is called "Markowitz optimal portfolio theory".

$\min_{\vec{w}} Var[F]$ subject to μ_F being constant & $\vec{w}^T \vec{1} = 1$.

$\vec{X} \sim Multin_k(n, \vec{p})$. $E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix}$

$$E[\vec{x}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$$

$X_j \sim Bin(n, p_j)$

$$Var[\vec{x}] = \begin{bmatrix} np_1(1-p_1) & & \sigma_{ij} & \\ & np_2(1-p_2) & & \\ & & \ddots & \\ \sigma_{ij} & & & np_K(1-p_K) \end{bmatrix}$$

$i \neq j$

$$Cov[X_i, X_j] = E[X_iX_j] - E[X_i]E[X_j]$$

$$= \underbrace{\sum_{x_i \in \mathbb{R}} \sum_{x_j \in \mathbb{R}} x_i x_j \, p_{X_i, X_j}(x_i, x_j)}_{\text{Complicated}} - n^2 p_i p_j$$

Complicated $\Rightarrow$ fail!!

$$\vec{x} = \begin{bmatrix} X_i \sim Bin(n, p_i) \\ \vdots \\ X_j \sim Bin(n, p_j) \\ \vdots \end{bmatrix}$$

$X_i = X_{1i} + X_{2i} + \cdots + X_{ni}$
where $X_{1i}, \ldots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \cdots + X_{nj}$
where $X_{1j}, \ldots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

Assume
i = Apple, j = Banana

We've expressed the multinomial RV with
$n \times K$ Bernoulli's.

$X_{1i}, X_{2i}, \ldots, X_{1j}, X_{2j}, \ldots$ [did we get an apple / a banana in each grab]

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \cdots + \vec{X}_n$ where

$\vec{X}_1, \ldots, \vec{X}_n \overset{iid}{\sim} \text{Multin}_K(1, \vec{P})$

$\text{cov}[X_i, X_j] = \text{cov}[X_{1i} + \cdots + X_{ni}, X_{1j} + \cdots]$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} \text{cov}[X_{li}, X_{mj}]$$

A lot of these covariances are zero due to independence.
Which ones?
If l doesn't eqival m, then the covariance is zero.
(horizontal)

$$= \sum_{l=1}^{n} \text{cov}[X_{li}, X_{lj}]$$

$$= \sum_{l=1}^{n} \left(E[X_{li} X_{lj}] - \underbrace{E[X_{li}]}_{P_i} \underbrace{E[X_{lj}]}_{P_j} \right)$$

$$= \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} P_{X_{li}, X_{lj}}(x_{li}, x_{lj})$$

→ whether you get or

~~the~~ only term that's nonzero is

$= P_{X_{li}, X_{lj}}(1,1) = 0$

bcuz it's impossib get both apple & banana in one grab.

So back together:

$= -n P_i P_j$

end of Midterm I materials

Midterm II materials

$X \sim \mathcal{U}(\{0,1,2,3\})$ → Uniform discrete $= \begin{cases} 0 & \text{wp } 1/4 \\ 1 & \text{wp } 1/4 \\ 2 & \text{wp } 1/4 \\ 3 & \text{wp } 1/4 \end{cases}$ ← $p(x)$

Suppose $[X] = \{0,1,2,3\}$

Create a new rv

$Y = -X = g(X)$, a very simple function

Generally, $X \sim \mathcal{U}(A)$

$Supp[X] = A$, $A \subset \mathbb{R}$ such that $|A| < \infty$ and $|A| \geq 1$

$\Updownarrow$

$A \neq \emptyset$ ← null set

$Supp[Y] = \{-3,-2,-1,0\}$

$$p(y) = \begin{cases} -3 & \text{wp } 1/4 \\ -2 & \text{wp } 1/4 \\ -1 & \text{wp } 1/4 \\ 0 & \text{wp } 1/4 \end{cases}$$

Generally, for discrete rv X, is there a pattern?

$$p_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) = p_X(-y)$$

$$Supp[Y] = \{z : p_Y(z) > 0\}$$
$$= \{z : p_X(-z) > 0\}$$
$$= \{-z : p_X(z) > 0\} \implies \text{let } z' = -z$$
$$= -\{z : p_X(z) > 0\} = -Supp[X]$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1,\ldots,\}}$

difference:

let $D = X_1 - X_2$

$= \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y}$

In class, we showed:

$T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$$\boxed{Supp[D] = Supp[X] + Supp[Y]}$$
$= \{\ldots,-1,0,1,\ldots\} = \mathbb{Z}$

$$= X + Y, \quad Y \sim p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\ldots,-1,0\}}$$

$$p_D(d) = \sum_{x \in Supp[X]} p_X^{\text{odd}}(x)\, p_Y^{\text{odd}}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$