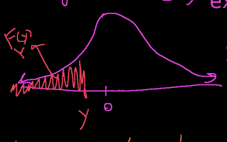
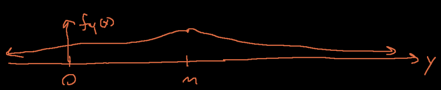
$X \sim \text{Logistic}(0,1) \approx N(0,1)$ except it has thicker tails



Standard logistic $E[X] = 0,\ SD[X] = \frac{\pi}{\sqrt{3}}$

$$Y = \mu + \sigma X \sim \text{Logistic}(\mu, \sigma) := f_Y(y) = \frac{1}{\sigma} \frac{e^{\frac{y-\mu}{\sigma}}}{\left(e^{\frac{y-\mu}{\sigma}} + 1\right)^2}$$

$\mu \in \mathbb{R},\ \sigma > 0$



$\sigma > 1$

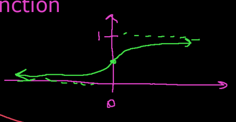
Why is this called the "logistic distribution"?

There's a function call ed the "logistic function" and it has 3 parameters: L (maximum value), k (steepness), mu (center)

$$\ell(x) := \frac{L}{1 + e^{-k(x-\mu)}} = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$

the standard logistic function

if $L=1,\ k=1,\ \mu=0$



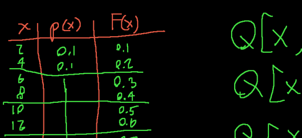
$$Y \sim \text{Logistic}(0,1) = \frac{e^y}{(1+e^y)^2}$$

$$F_Y(y) := P(Y \le y) = \int_{-\infty}^{y} \frac{e^t}{(1+e^t)^2}\,dt = \int_1^{1+e^y} \frac{\cancel{u-1}}{u^2}\frac{1}{\cancel{u-1}}\,du = \left[-\frac{1}{u}\right]_1^{1+e^y} = 1 - \frac{1}{1+e^y} = \frac{e^y}{1+e^y}$$

$e^t = u-1$

let $u = 1+e^t \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du = \frac{1}{u-1}du,\ t=-\infty \Rightarrow u=1,\ t=y \Rightarrow u = 1+e^y$

The qth "quantile" or 100q "percentile" of a rv X. Definition: solve for x where $q \ge P(X \le x) = F_X(x)$. denoted Q[X, q].t



I want the 33rd %ile

If q = 0.5, that quantile is called the "median", Med[X].t

$X \sim U(\{2, 4, \ldots, 20\})$

x	p(x)	F(x)
2	0.1	0.1
4	0.1	0.2
6		0.3
8		0.4
10		0.5
12		0.6
14		0.7
16		0.8
18		0.9
20		1.0

$Q[X, 0.3] = 6$

$Q[X, 0.9] = 18$

$Q[X, 0.85] = 16 \quad \nexists x \quad F(x) = 0.85$

If X is a continuous rv with "contiguous support" i.e. one interval with no gaps e.g. [0, 10], the real numbers but not e.g. [0,1] union [2,3] where there is a gap between 1 and 2, then F(x) is strictly increasing thus invertible and the minimum x s.t. q >= F(x) would be $F^{-1}(q) \ge x \Rightarrow x = \underbrace{F^{-1}}_{\text{quantile function}}(q) = Q[X, q]$.

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \Rightarrow F(x) = 1 - e^{-\lambda x}$. Find the quantile function, $F^{-1}(q)$

$$q = 1 - e^{-\lambda x} \Rightarrow 1 - q = e^{-\lambda x} \Rightarrow \ln(1-q) = -\lambda x \Rightarrow x = -\frac{1}{\lambda}\ln(1-q)$$

$$\Rightarrow x = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right) = F^{-1}(q).$$

$X \sim \text{Exp}(1) \Rightarrow \text{Med}[X] = F^{-1}(0.5) = \ln(2).$

$Q[X, 0.8] = \ln(5).$

It's actually rare to have a quantile function in closed form since it's rare to even have a CDF in closed form e.g.

$X \sim \text{Erlang}(k, \lambda)$, $F(x) = P(k, \lambda x)$. $Q[X, q]$ can be found by solving for x in the following equation: q = $P(k, \lambda x)$.

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x}$, $Y = ke^X = g(X)$, $k > 0$. Find $f_Y(y)$.

$$\frac{y}{k} = e^x \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k) = g^{-1}(y), \quad \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \frac{1}{|y|} = \frac{1}{y}$$

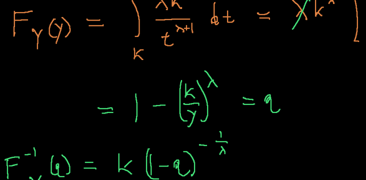
since y is always pos.

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right| = \lambda e^{-\lambda \ln\left(\frac{y}{k}\right)} \frac{1}{y} \mathbb{1}_{\ln(y) - \ln(k) \in [0,\infty)}$$

$$= \frac{\lambda}{y} e^{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)} \mathbb{1}_{\ln(y) \in [\ln(k), \infty)} = \frac{\lambda}{y}\left(\frac{y}{k}\right)^{-\lambda} \mathbb{1}_{y \in [k,\infty)}$$

$= \text{ParetoI}(k, \lambda)$

$k \in (0,\infty),\ \lambda \in (0,\infty)$



$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{t^{\lambda+1}}\,dt = \cancel{\lambda} k^\lambda \left[-\frac{1}{\cancel{\lambda} t^\lambda}\right]_k^y = k^\lambda\left(\frac{1}{k^\lambda} - \frac{1}{y^\lambda}\right)$$

$$= 1 - \left(\frac{k}{y}\right)^\lambda = q$$

$$F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}$$

Remember, the exponential is a survival / waiting time rv.
So is the Pareto. The Pareto is used to model population spread, hard drive time-to-failure. There is also the "Pareto Principle". In 1896, Vilfredo Pareto noticed that 80% of the land in Italy was owned by 20% of the people. That is a property of a specific Pareto distribution, ParetoI(1, 1.161).
In previous years, we spent another 30min proving that...
but this year we won't.

$X, Y \overset{iid}{\sim} \text{Exp}(1) = e^{-x}\mathbb{1}_{x \in [0,\infty)}$, let $D = X - Y = X + \underbrace{(-Y)}_{Z}$

$$Z \sim f_Z(z) = e^z \mathbb{1}_{z \in (-\infty, 0]}$$

$$f_D(d) = \int_{\text{Supp}[X]} f_X^{old}(x) f_Z^{old}(d-x) \mathbb{1}_{d-x \in \text{Supp}[Z]}\,dx$$

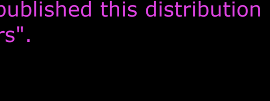
$$= \int_0^\infty e^{-x} e^{d-x} \mathbb{1}_{\underbrace{\underbrace{d-x \in (-\infty, 0]}_{x-d \in [0,\infty)}}_{x \in [d,\infty)}}\,dx = e^d \int_0^\infty e^{-2x} \mathbb{1}_{x \in [d,\infty)}\,dx$$

$$= e^d \begin{cases} \int_d^\infty e^{-2x}\,dx & \text{if } d \ge 0 \\ \int_0^\infty e^{-2x}\,dx & \text{if } d < 0 \end{cases} = e^d \begin{cases} \left[-\frac{1}{2}e^{-2x}\right]_d^\infty & \text{if } d \ge 0 \\ \left[-\frac{1}{2}e^{-2x}\right]_0^\infty & \text{if } d < 0 \end{cases}$$

$$= \frac{1}{2}e^d \begin{cases} e^{-2d} & \text{if } d \ge 0 \\ 1 & \text{if } d < 0 \end{cases} = \begin{cases} \frac{1}{2}e^{-d} & \text{if } d \ge 0 \\ \frac{1}{2}e^d & \text{if } d < 0 \end{cases} = \frac{1}{2}e^{-|d|}$$

$= \text{Laplace}(0,1)$

Standard Laplace



$\mu \in \mathbb{R},\ \sigma > 0,$

$$X = \mu + \sigma D \sim \text{Laplace}(\mu, \sigma) := \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

this is also a famous rv and it has another name: the "double exponential". Laplace published this distribution in 1774 calling it the "first law of errors".