Lecture 02
08/31/2020
Math 621: Prof. Kopelner

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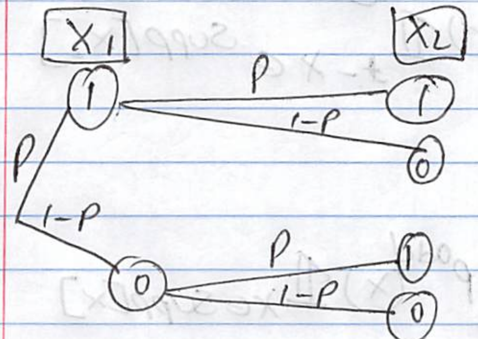
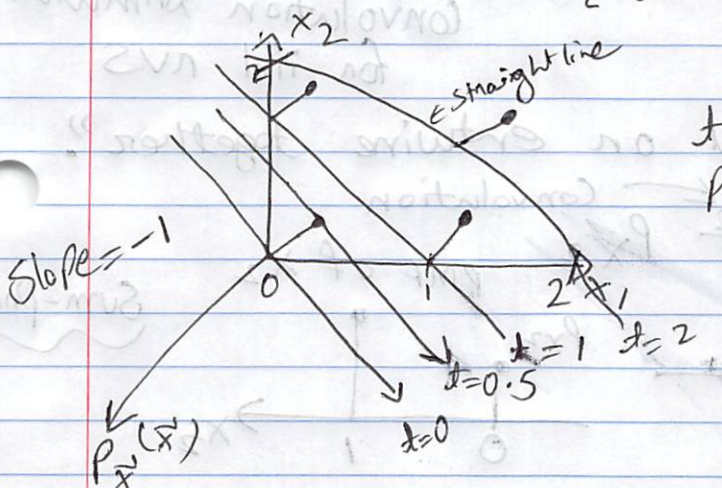


$p_{X_1,X_2}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0
1	

mutually exclusive, collectively exhausive events

$$p_T(t) = P(T=t) = \begin{cases} 2 & wp \quad p^2 \\ 1 & wp \quad 2p(1-p) \\ 0 & wp \quad (1-p)^2 \end{cases}$$



$t = x_1 + x_2 \Rightarrow x_2 = t - x_1$

$P(T=0) \rightsquigarrow x_2 = 0 - x_1 = -x_1$

$\boxed{\text{General Convolution Formula}}$

select events

$$p_T(t) = \underbrace{\sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}}}_{\text{Search through } \mathbb{R}^2} \underbrace{p_{X_1,X_2}(x_1,x_2)}_{\text{add up all probs}} \boxed{\mathbb{1}_{\underbrace{x_1+x_2=t}_{x_2 = t-x_1}}} = \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$

if X_1, X_2 independent → use multiplication rule

$$= \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x) = \sum_{x \in \mathbb{R}} p^{odd}_{X_1}(x)\, \boxed{\mathbb{1}}_{x \in Supp[X_1]}\; p^{odd}_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}$$

iid $\Rightarrow$ independent & identically distributed

convolution formula for independent RVs

$$= \sum_{x \in \mathrm{Supp}[X_1]} p_{X_1}^{odd}(x)\, p_{X_2}^{odd}(t-x)\, \mathbb{1}_{t-x \in \mathrm{Supp}[X_2]}$$

if $X_1, X_2 \overset{iid}{\sim}$

$$= \sum_{x \in \mathbb{R}} p(x)\, p(t-x) = \sum_{x \in \mathbb{R}} p^{odd}(x)\, \mathbb{1}_{x \in \mathrm{Supp}[X]}\, p^{odd}(t-x)\, \mathbb{1}_{t-x \in \mathrm{Supp}[X]}$$

$$= \sum_{x \in \mathrm{Supp}[X]} p(x)\, p(t-x)\, \mathbb{1}_{t-x \in \mathrm{Supp}[X]}$$

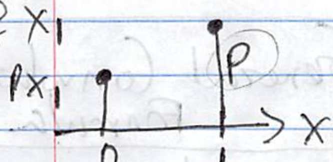
convolution formula for iid RVs

"Convolve" means to "roll, coil or entwine together".

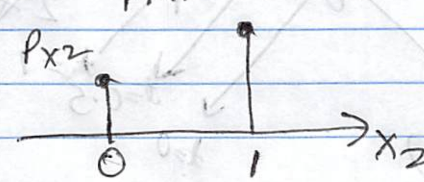
$$p_T = p_{X_1} * p_{X_2} \quad \leftarrow \text{Convolution}$$

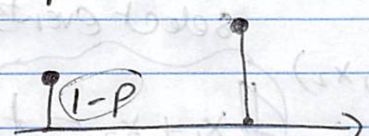
Sum-Product

PMF of X_1

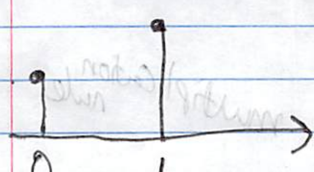


PMF of X_2





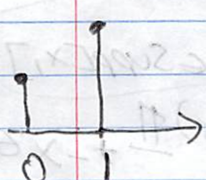
$\Rightarrow t = 1$ wp $p(1-p)$



$\Rightarrow t = 0$ wp $(1-p)^2$

$t = 2$ wp p^2

$\Rightarrow p_T$



$\Rightarrow t = 1$ wp $(1-p)p$

$$p_{T_2}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \, p^{t-x} (1-p)^{1-t+x} \, \mathbb{1}_{\underbrace{t-x \in \{0,1\}}_{t \in \{x, x+1\}}}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right)$$

$$T_2 \sim \begin{cases} 0 & \text{wp} \quad (1-p)^2 \\ 1 & \text{wp} \quad 2p(1-p) \\ 2 & \text{wp} \quad p^2 \end{cases} \qquad = \binom{2}{t}$$

Review 241 $\left\{ \binom{n}{k} = \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0, 1, \ldots, n\}} \right.$

$$= \binom{2}{t} p^t (1-p)^{2-t} = \text{Binomial}(2, p)$$

$$\text{Supp}[T] = \text{Supp}[X_1] + \text{Supp}[X_2]$$

$$A + B = \{ a + b : a \in A, b \in B \}$$

$$X_1, X_2 \overset{iid}{\sim} \text{Bern} = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} \qquad \boxed{\binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}}$$

$$= \binom{1}{x} p^x (1-p)^{1-x}$$

$$p_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x} \binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x} = p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

Pascal identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$ $T_3 = X_1 + X_2 + X_3$

$= X_3 + T_2 \sim P_{T_3}(t) = ?$

$$P_{T_3}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right)$$

$$= \binom{3}{t} p^t (1-p)^{3-t} = \text{Binom}(3, p)$$

HW: find PMF of Binom (n, p) via induction.

$X_1, X_2 \overset{iid}{\sim}$ Binomial (n, p), $T = X_1 + X_2 \sim ?$

$$P_T(t) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x} \binom{n}{t-x}$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t} = \text{Binom}(2n, p)$$

Vandermonde's identity

PMF $\Rightarrow P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$