After exam 1 368 lec 7 9/29/20

$D = X_1 - X_2 \sim ?$ $X_1 + X_2 \sim \text{poisson}(2\lambda)$

$D = \underbrace{X_1}_{x} + \underbrace{(-X_2)}_{y} \sim ?$ $\text{Sup}[X] = \{0, 1, 2 \ldots\}$

$\text{Sup}[Y] = \{\ldots -2, -1, 0\}$

$\text{Sup}[X+Y] = \text{Supp}[X] + \text{Supp}[Y] = \mathbb{Z}$

convolution formula for independent discreate rv's

$$P_T(t) = \sum_{x \in \text{supp}[X]} \overset{\text{old}(x)}{P_x} \underset{x-d}{\overset{\text{old}}{P_y}} (d-x) \mathbb{1}_{d-x \in \text{Supp}[X]}$$

$\underbrace{x \in \{d, d+1, d+2 \ldots\}}$

$x - d \in \{0, 1, \ldots\}$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{-(d-x)} e^{-\lambda}}{(\underbrace{-(d-x)}_{x-d})!} \mathbb{1}_{d-x \in \{\ldots, -1, 0\}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \in \{d, d+1, \ldots\}}$$

two different cases

let say $d = 17$ $\quad 2x - dd = -17$ $\quad d \geq 0 \quad d \leq 0$

$$= e^{-2\lambda} \begin{cases} \text{if } d \leq 0 & \displaystyle\sum_{x \in \{0,1\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} \\ \text{if } d > 0 & \displaystyle\sum_{x \in \{d, d+1, \ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x' \in \{0,1\ldots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)(x')!} \end{cases}$$

let $d' = -d$ $= |d'|$

$2x' + d$

let $x' = x - d \iff x = x' + d$

$d = |d|$

$d' = -d = |d'|$

both are summing over the same set

$d = |d|$

$$\sum_{x \in \{0,1\ldots} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} \to \sum_{x \in \{0,1\ldots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!} =: I_{|d|}(2\lambda)$$

modified bessel function of the first kind (it's a solution to a famous dif eq.)

$$= e^{-\lambda 2} I_{|d|}(2\lambda) \mathbb{1}_{d \in \mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$

this is used to model point speads in sports games, photon noise, ...

~ · ~

$X_1, X_2 \overset{idd}{\sim} \text{poisson}(\lambda)$, $T = X_1 + X_2 \sim \text{poisson}(2\lambda)$

(x, t)

$P(X_1 | T)$ you can break it up

↳ 37

↳ not poisson anymore?

def ÷ Joint / marginal

$$= \frac{P_{X_1, T}(x, t)}{P_T(t)} = \frac{P_{X_1, X_2}(x, t-x)}{P_T(t)} = \frac{P_{X_1}(x) P_{X_2}(t-x)}{P_T(\cancel{x} t)}$$

$$= \frac{\frac{\cancel{e^{-\lambda}} \lambda^x}{x!} \frac{\cancel{e^{-\lambda}} \lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}} (2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x} \left(\frac{1}{2}\right)^t$$

how many successes come from X_1 → X_2

everything is the same ..? so they have $\frac{1}{2}$

get the most likely scenario?

$$= \text{Bin}\left(t, \frac{1}{2}\right)$$

If they add up to 10,

$$X_1 \sim \text{Geom}(p) := \overbrace{(1-p)^x p}^{P_{X_1}(x)} \mathbb{1}_{x \in \{0, 1, \ldots\}}, \quad \text{Sup}[X] = \{0, 1, \ldots\}$$

$$F(x) := P(X_1 \leq x) = 1 - P(X_1 > x) = (1 - (1-p)^x) \mathbb{1}_{x \in \{0, 1, \ldots}$$

10 total experiments

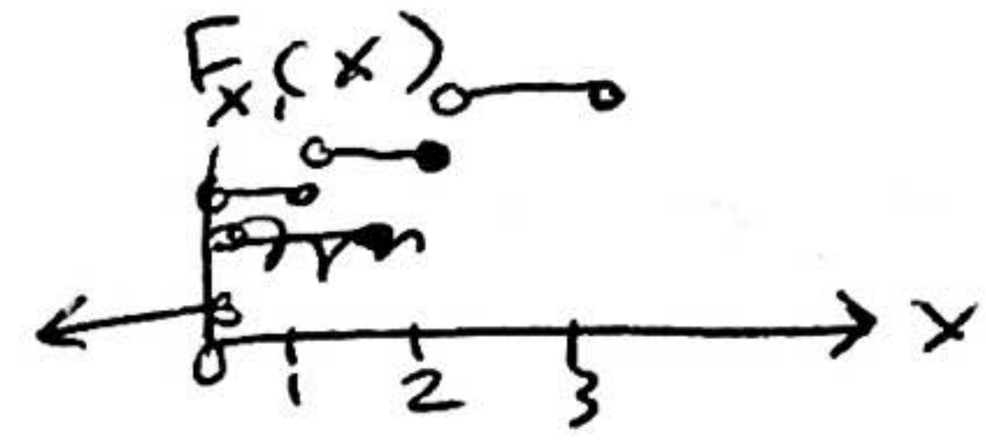
do them in order, 1 if it comes from X_1

0 if it comes from X_2

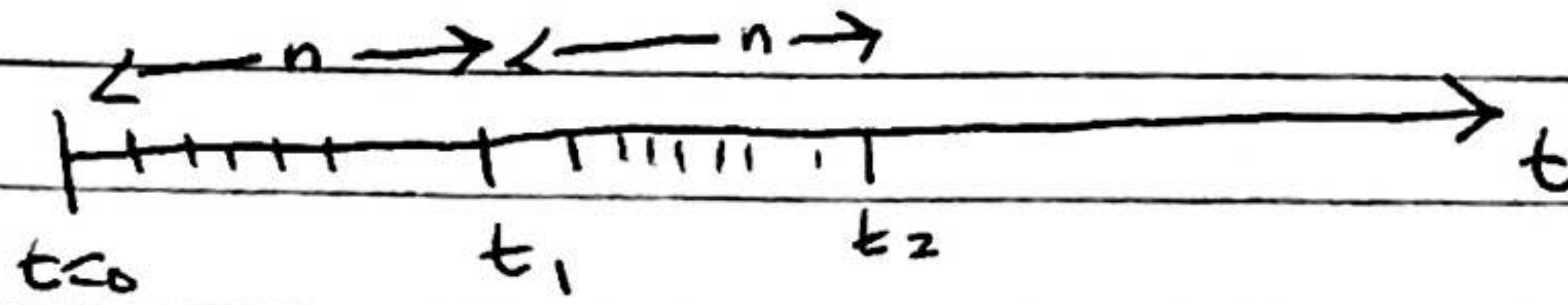
X_1 in seconds, every second you do 1 experiment and stop

$t=0$ ... $t=1$ ... $t=2$ ... $t=3$ .... → x (time)

In every "second", let's do n idd Bernoulli (p) experiments.



$n \to \infty$ you get all $\mathbb{R}$



let's call the resulting geometric rv X_n and its unit of realization is t

$P_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots 2 \dots}$ ← wrong

↳ you now can stop at any fraction of a second.

we don't need it

$F_{X_n}(x) = 1-(1-p)^{nx}$

let $n \to \infty$, $p \to 0$ but $\lambda = np$ where $\lambda \in (0, \infty)$

↳ can't be zero has to be non degenerate

$\Rightarrow p = \frac{\lambda}{n}$ same as poisson

$P_{X_\infty}(x) := \lim_{n\to\infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^{nx}} \cdot \underset{\hookrightarrow \frac{\lambda}{n}}{p} \; \mathbb{1}_{x \in \{0, \frac{1}{n}, \dots\}}$

$\underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{x}} \lim_{n\to\infty}\frac{\lambda}{n} \underbrace{\lim \mathbb{1}_{x \in 0, \frac{1}{n}, \dots}}$

$e^{-\lambda x} \cdot 0 \cdot \mathbb{1}_{x \in [0,\infty)}$

"that's bad"

$= 0 \; \forall x!$

$\operatorname{supp}[X_\infty] = [0, \infty]$

$$\boxed{\text{but } \sum_{x \in \operatorname{supp}[X_\infty]} P_{X_\infty}(x) = 0 \quad \text{not a valid pmf!}}$$

(you don't get 1 you get zero)

for the cdf it's valid.

$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x)$

$= \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx} = 1-e^{-\lambda x}\, \mathbb{1}_{x \in [0,\infty)}$

previously the pmf was not valid. Is the cdf valid?

have to check ~~[illegible]~~ nothing or something

If so, I need to check three properties

1) It's 0 as I go to negative infinity

2) it's 1 as I go to positive ininity

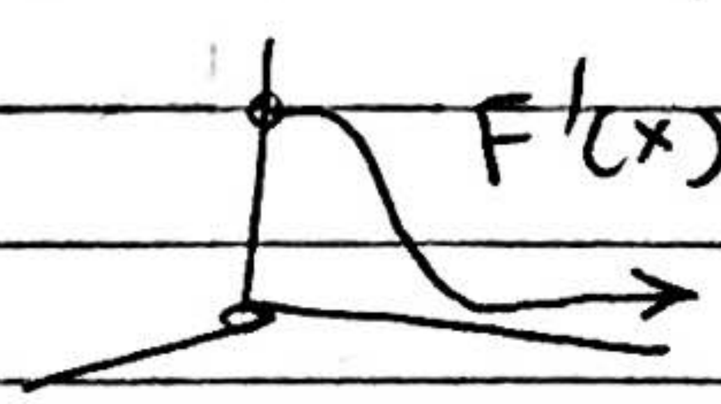
3) its an increasing function.

(1) $\lim_{x \to -\infty} (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)} = 0$

(2) $\lim_{x \to \infty} (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)} = 1 - \lim_{x \to \infty} \frac{1}{e^{\lambda x}} = 1$ ✓

(3) $\frac{d}{dx}\left[(1 - e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)}\right]$

$= \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)} \geq 0$ ✓



$F'(x)$ always positive

$\Rightarrow$ Yes this is a valid cdf!

what is it a cdf of?

we now have a continuous r.v.

Continuous rv's have the following prop

b/c the prob. for any specific number is zero

$|Supp[X]| = |\mathbb{R}|$ uncountable ∞ (the size of the continuum)

They do not have PMF's but they have Cdfs

The derivative of the cdf is a very useful function it is called the probability density function (pdf) denoted $f(x)$

discrete r.v's do not have PDF's

$$f(x) \stackrel{!}{=} F'(x), \quad P(X \in [a,b]) = \underbrace{P(X \le b)}_{F(b)} - \underbrace{P(X \le a)}_{F(a)}$$

by fundimental thrm of Calc

$$= \int_a^b f(x)\,dx$$

kinda like

$$\int_{\mathbb{R}} f(x) = 1 = \text{"}\cancel{F(\infty)}^{1} - \cancel{F(-\infty)}^{0}$$

$f(x) \ge 0$ since cdf's are increasing functions

$\Rightarrow \text{Sup}[X] = \{x : f(x) > 0\}$

Joint density function

$*$ $X_1, \dots X_n$ indep $\Rightarrow f_{X_1, \dots X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

$$X \sim Exp(\lambda) := \underbrace{\underbrace{\lambda e^{-\lambda x}}_{f^{\text{old}}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}$$

$$F(x) = \left(1 - e^{-\lambda x}\right) \mathbb{1}_{x \in [0,\infty)}$$

Exponential r.v

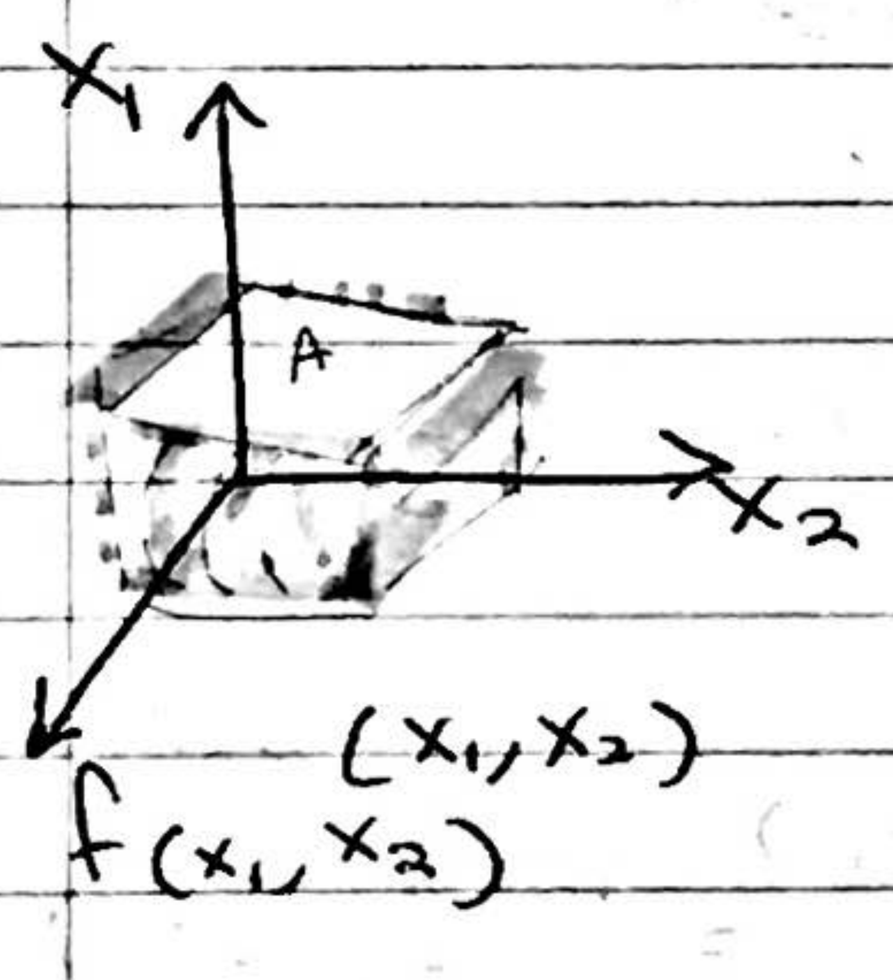
$\lambda \in (0, \infty)$

it's parameter space

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$$

$$\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{K} f_{\vec{X}}(\vec{x}) \, dx_1 \cdots dx_K = 1$$

$k=2$



$$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

volume

if it's more than 2d it's a hyper volume.