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Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)|\, dx < \infty \}$ "$L^1$ integrable" or "absolutely integr functions.

Are all PDFs in the set $L1$? yes. $\int_{-\infty}^{\infty} |\lambda e^{-\lambda x} 1_{x \in [0,\infty)}| dx = \lambda \int_0^{\infty} e^{-\lambda x} dx = 1$

If $f \in L^1 \Rightarrow \exists \hat{f}$, the "fourier transform" of $f$:

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-i2\pi wx} f(x)\, dx = \mathcal{F}[f]$$

"forward fourier transform operator" AKA "fourier analysis"

If $\hat{f} \in L^1 \Rightarrow$ then we can invert/reverse the fourier transform via the "inverse/reverse transform operator" to get the original $f$ back AKA "fourier synthesis":

$$f(x) = \int_{\mathbb{R}} e^{i2\pi xw} \hat{f}(w)\, dw$$

Fourier inversion thm: if $f$, fhat are in $L1$, then $f$ and fhat are 1:1.

$f(x)$ is known as the "time domain" and fhat (omega) is known as the "frequency domain". $f(x)$ can be decomposed into a sum of sines and cosines with frequencies and amplitides given by |fhat (omega)| and phase shifts given by Arg[fhat(omega)].

Let $X$ be a rv. Define the charactoric function chf:

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} f_X(x)\,dx \text{ if continous}$$

$$= \sum_{x \in \mathbb{R}} e^{itx} P_X(x) \text{ if discrete}$$

The chf is the fourier transform in a different unit $t = -2$ Pi omega

Properties of the chf:

(P0) $\phi_X(0) = E[e^{i(0)X}] = E[e^0] = 1$ for all rv's

(P1) $\phi_X(t) = \phi_Y(t) \iff X = Y$

(P2) $Y = aX + b$ for $a, b \in \mathbb{R}$

$$\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{iatX} e^{itb}]$$

$$= e^{itb} E[e^{i\overset{t'}{at}X}] = e^{itb}\phi_X(t') = e^{itb}\phi_X(at)$$

(P3) If $X_1, X_2 \overset{ind}{\sim}$ and $T = X_1 + X_2$

$$\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}] = E[e^{itX_1}]E[e^{itX_2}]$$
$$= \phi_{X_1}(t)\,\phi_{X_2}(t)$$

(P4) "moment generation"

conditions are satisfied to be able to interchange differentiation and integration

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] = E\left[\frac{d}{dt}[e^{itX}]\right]$$
$$= E[iXe^{itX}]$$

$$\phi_X'(0) = E[iXe^{i(0)X}] = iE[X] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$

$$\phi_X''(t) = \frac{d}{dt}\left[E[iXe^{iXt}]\right] = E\left[iX\frac{d}{dt}[e^{itX}]\right]$$
$$= E[i^2X^2e^{itX}] \Longrightarrow E[X^2] = \frac{\phi_X''(0)}{i^2}$$

$$\Rightarrow E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}$$ if the moment exists

(P5) $\phi_X(t) \in [-1,1]$ for all $X, t$ hence it always exists

$|\phi_X(t)| \in [0,1]$

Proof

$$|E[e^{itX}]| = \left|\int_R e^{itx} f(x)\,dx\right| \leq \int_R |e^{itx}f(x)|\,dx \leq \int |e^{itx}||f(x)|\,dx$$

discrete same proof

$$= \int |\overbrace{i\sin(tx) + \cos(tx)}^{bi+a}|\,f(x)\,dx$$
$$= \int_R \underbrace{\sqrt{\sin^2(tx)+\cos^2(tx)}}_{1}\,f(x)\,dx = 1$$

(P6) Inversion If $\phi_X(t) \in L^1$, then

PDF: $f_X(x) = \int_R e^{itx}\,\phi_X(t)\,dt$

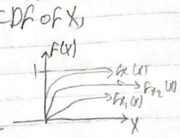
(P7) Levy's CDF thm (works even if $\phi_X \notin L^1$)

$$P(X \in [a,b]) = \frac{1}{2\pi}\int_R \frac{e^{ita} - e^{-itb}}{it}\,\phi_X(t)\,dt$$

(P8) Levy's continuing thm.

Consider a sequence of rv's $X_1, X_2, \dots X_n$. we define "$X_n$ converges in distribution to $X$" and denote it $X_n \xrightarrow{d} X$ if the CDF $X_n$ converges pointwise to the CDF of $X$,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x.$$




If $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)\ \forall t \Rightarrow X_n \xrightarrow{d} X$

The distribution on the left ($X_n$) is becoming more and more like the distribution on the right ($X$)

Define $m_X(t) = E[e^{tX}]$, the moment generation function (mgf)

(P0) $m_X(0) = E[e^{(0)X}] = 1$

(P1) $m_X(t) = m_Y(t) \Rightarrow X \overset{d}{=} Y$

(P2) $Y = aX + b \Rightarrow m_Y(t) = e^{tb}m_X(at)$

(P3) $X_1, X_2 \overset{ind}{\sim}$, $T = X_1 + X_2$ then $M_T(t) = m_{X_1}(t)\,m_{X_2}(t)$

$P_4$ $E[X^n] = M_X^{(n)}(0)$

but mgf's sometimes don't exist!! And sometimes don't exist for all $t$.

I don't care about mgf's. Why? Because chf's can do everything they can do and much much more!

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$X \sim \text{Gamma}(\alpha, B)$

$$\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx} \frac{B^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-Bx}\, dx$$

$$= \frac{B^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(B-it)x}\, dx = \frac{B^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(B-it)^\alpha} = \left(\frac{B}{B-it}\right)^\alpha$$

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$X_1 \sim \text{Gamma}(\alpha_1, B)$ ind of $X_2 \sim \text{Gamma}(\alpha_2, B)$

$$\phi_{X_1+X_2}(t) \underset{P_3}{=} \phi_{X_1}(t)\, \phi_{X_2}(t) = \left(\frac{B}{B-it}\right)^{\alpha_1} \left(\frac{B}{B-it}\right)^{\alpha_2} = \left(\frac{B}{B-it}\right)^{\alpha_1+\alpha_2}$$

$\Downarrow P_1$

$X_1 + Y_2 \sim \text{Gamma}(\alpha_1+\alpha_2, B)$