lec12 October 19, 2020

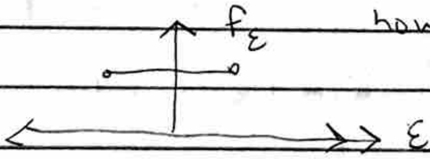
1774 . "First law of errors" Imagine you're trying to measure something, quantity v but your measurements have error, ~~e~~, ε (eps), so your measurement m is a r.v looking like $M = v + \varepsilon$ so what is a good model for the error ε? it makes sense for $E[\varepsilon] = 0$ you could also say $med[\varepsilon] = 0$

$\to$ 50% mark

and symmetric

if $E[\varepsilon] = 0$ & $med[\varepsilon] = 0$ they are symmetric

how about $U(-L, L)$



It also makes sence for larger errors (in magnitude) to be less probable than smaller errors

he was thinking about error distribution

$\Rightarrow \forall \varepsilon > 0 \quad f'(\varepsilon) < 0$

"we made up"

$\forall_\varepsilon > 0 \quad f''(\varepsilon) = f'(\varepsilon) \underset{\text{solve}}{\Rightarrow} f(\varepsilon) = ce^{-d\varepsilon}$

solve for constants

$\Rightarrow$ Laplace $(0, 1)$

"second law of errors" 1778 ...

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \geq 0}$ let $Y = \underbrace{\frac{1}{\lambda}}_{\text{mult scaling}} X^{\frac{1}{k}} = g(x)$

s.t $\lambda, k > 0$

$Y \sim f_Y(x) = ?$

i) find the inverse function first

Generalization of Exp

pre steps $\lambda y = x^{\frac{1}{k}} \Rightarrow x = \lambda^k y^k = g^{-1}(y)$

2) take abs $\frac{d}{dt}$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\lambda^k y^k]\right|$$

always positive

$$= \left|k\lambda^k y^{k-1}\right| = k\lambda^k y^{k-1}$$

$k\lambda^k y^{k-1}$

$$f_y(y) = f_x(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|_{y \geq 0} = e^{-(\lambda y)^k} \mathbb{1}_{\lambda^k y^k \geq 0}$$

$$\left(= e^{-(\lambda y)^k} \left(\mathbb{1}_{\overbrace{\lambda^k y^k \geq 0}^{y^k \geq 0}}\right) \left(k\lambda^k y^{k-1}\right) \right)$$

inverse cdf survival distribution

$$= k\lambda^k y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = k\lambda(\lambda y)^{k-1}$$

1951

$$= k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = \text{Weibol}(\underline{\underline{k}}, \lambda)$$

Note $\text{Weibull}(1, \lambda) = (1)\lambda(1) \cancel{y^{(1)-1}} e^{-(\lambda y)^1} \mathbb{1}_{y \geq 0}$

$$= \lambda e^{-\lambda y} \mathbb{1}_{y \geq 0} \quad \text{Exp}(\lambda).$$

$\underline{\underline{k}}$ is really cool.. this is the main property

$k = 1 \quad P(y \geq y + c \mid Y \geq c) = P(Y \geq y)$

e.g $y = 3$, $c = 14$

$P(y \geq 17 \mid Y \geq 14) = P(Y \geq 3)$

allready waited 14 min what's the prob of waiting 3
is the same thing as waiting 3

Every few thing are memoryless
Geometric is memoryless

It doesn't matter how long you waited the random variable resets, it's weird.

expand
it's memorylessness, its a bad model for lifespan which should get shorter

survival less likely

$k > 1 \quad P(Y \geq y + c \mid Y \geq c) < P(Y \geq y)$

"yeah it's more realistic"

→ "same thing as a baby"

$k < 1 \quad P(Y \geq y + c \mid Y \geq c) > P(Y \geq y)$

Survival more likely as time goes on...

you can have piecewise, this part of the lifespan and the other part of the lifespan

You will prove these facts on the HW

Order Statistic (P 160 in the textbook)

Let $X_1, X_2, \ldots, X_n$ be a collection of continous rv's and let

$X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be their "order statistic" defined as:

$X_{(1)} = \min\{X_1, \ldots, X_n\}$

$X_{(n)} = \max\{ \text{"} \quad \text{"} \}$

$\vdots$

$X_{(k)} = k^{th} \text{ largest} \{ \text{"} \quad \text{"} \}$

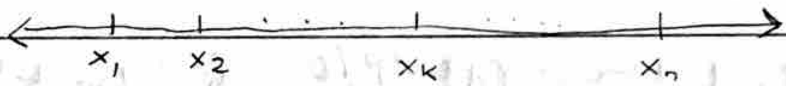
$x_1 = 9 \quad x_2 = 2 \quad x_3 = 12 \quad x_4 = 7$

$x_{(1)} = 2 \quad x_{(2)} = 7 \quad x_{(3)} = 9 \quad x_{(4)} = 12$

$R := X_{(n)} - X_{(1)}$ "range"

We want to find the cdf and pdf of the order statistic we'll start by looking at the CDF of the maximum.

$$F_{X_{(n)}}(x) = P(X_{(n)} \le \overset{\text{max}}{x}) = P(X_1 \le x \;\&\; X_2 \le x \;\&\; X_n \le x)$$



def of cdf

$$\text{if independent} = \prod_{i=1}^{n} P(X_i \le x) = \prod_{i=1}^{n} F_{X_i}(x)$$

take the cdf and rase it to a power

$$\text{if iid} = \left(F_X(x)\right)^n$$

how to get densitys

$$f_{X_{(n)}}(x) \overset{\text{if iid}}{=} \frac{d}{dx}\left[F_{X_{(n)}}(x)\right] = \frac{d}{dx}\left[F_X(x)^n\right] =$$

typesetting

$$= n F_X(x)^{n-1} \qquad \underbrace{n f_X(x)} \; F_X(x)^{n-1}$$

Let's now find the cdf / pdf of the minimum

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) =$$

$$= 1 - P(X_1 > x \;\&\; X_2 > x \;\&\; X_n > x)$$

if independent

$$= 1 - P(X_1 > x)\,P(X_2 > x)\cdots P(X_n > x)$$

$$= 1 - \prod_{i=1}^{n} \left(1 - F_{X_i}(x)\right)$$

if iid

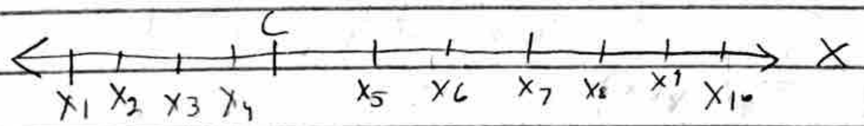
$$= 1 - \left(1 - F_X(x)\right)^n$$

$$f_{X_{(1)}}(x) \overset{\text{if idd}}{=} \frac{d}{dx}\left[1 - \left(1 - F_X(x)\right)^n\right] =$$

$$= n f_X(x) \left(1 - F_X(x)\right)^{n-1}$$

lets now find the cdf / pdf for the k^{th} order statistic, $X_{(k)}$

let's let $(n) = 10$, $k = 4$



$$P\{X_1 \le c \;\&\; \ldots\; X_4 \le c \;\&\; X_5 \ge c \;\&\; X_{10} \ge c\}$$

$$\overset{\text{if indep}}{=} \prod_{i=1}^{4} P(X_i \le c) \prod_{i=5}^{10} P(X_i > c)$$

$F_{X(4)}(x) = \ldots$ showing the picture

$$\overset{\text{cdf notation}}{=} \prod_{i=1}^{4} F_{X_i}(c) \prod_{i=5}^{10} \left(1 - F_{X_i}(c)\right)$$

$$\overset{\text{idd}}{=} F_X(c)^4 \left(1 - F_X(c)\right)^6$$

$F_{X_{(4)}}(x) = P($any 4 x_i's $\le x$ and the other 6 x_i's $> x)$

$$= \sum_{\text{over all subsets}} P(X_{S_1} \le x, \ldots, X_{S_4} \le x, X_{S^c_1} > x, \ldots X_{S^c_6} > x)$$

S size 4, S^c (comp) size 6

$$= \text{if independent } \sum \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=1}^{6} F_{X_{S^c}}(x)$$

if idd

no dep between the rvs

$$= \sum_{\text{same}} F_x(x)^4 (1-F_x(x))^6$$

$$= \binom{10}{4} F_x(x)^4 (1-F_x(x))^6$$

$F_{X_{(4)}}(x) = P(X_{(4)} \le x) = P(4\ x_i\text{'s} \le x;\ 6 x_i\text{'s} > x)$

it allows the 5th to be less than x and the 10th to be less than x

$= P(5\ x_i\text{'s} \le x, 5\ x_i\text{'s} > x) +$

$\vdots$

$+ P(10\ x_i\text{'s} \le x, 0\ x_i\text{'s} > x)$

Subsumed

Knowing the 4th doesn't tell you anything about the 5, ..., last is X

when you say 3 less and 7 greater that is a prob?

3 random variable

can't ~~$P(3 x_i \le x, 7 x_i\text{'s} > x)$~~

add this?

$X_{(4)} > X$

if idd

$$= \sum_{j=4}^{n} \binom{10}{j} F_X(x)^j \left(1-F_X(x)\right)^{10-j}$$

→ should give you the final formula

general case: k, n

$$\Rightarrow F_{X_{(k)}} = \sum_{j=k}^{n} \binom{n}{j} F_X(x)^j \left(1-F_X(x)\right)^{n-j}$$

$$F_{X_{(n)}}{}^{x} = \sum_{j=n}^{n} \binom{n}{j} F_X(x)^j \left(1-F_X(x)\right)^{n-j} =$$

$$= F_X(x)^n$$

use binomial

$$F_{X_{(1)}}(x) = \sum_{j=1}^{n} \binom{n}{j} F_X(x)^j \left(1-F_X(x)\right)^{n-j}$$

$$= \left(\sum_{j=0}^{n} \binom{n}{j} \underbrace{F_X(x)^j}_{a} \underbrace{\left(1-F_X(x)\right)^{n-j}}_{b} \right) - \binom{n}{0} F_X(x)^0 \left(1-F_X(x)\right)^{n-0}$$

what is the point? remember bin is

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} = \text{binomial trm}$$

using that ↓

$$(a+b)^n = \left(F_X(x) + 1 - F_X(x)\right)^n - \left(1-F_X(x)\right)^n$$

$$= 1 - \left(1-F_X(x)\right)^n$$

$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[F_{X_{(k)}}(x)\right] = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F_X(x)^j \left(1-F_X(x)\right)^{n-j}\right]$$

put the $\frac{d}{dx}$ in the sum

$$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx}\left[\overbrace{F_X(x)^j}^{u} \overbrace{\left(1-F_X(x)\right)^{n-j}}^{v}\right]$$

$$\frac{d}{dx}[uv] = uv' + vu'$$

$$u' = j\, f_X(x)\, F(x)^{j-1}$$

$$v' = n-j\, f(x)\left(1-F(x)\right)^{n-j-1}$$

tbc in lec13