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$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\left.\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right\} Z\sim N(0,1)}{\sqrt{\frac{\frac{n-1}{\sigma^2}S^2\}\ U\sim\chi^2_{n-1}}{n-1}}} \sim T_{n-1}$$

due to Cochran's theorem, if $X_i$'s are iid $N(\mu,\sigma^2) \Rightarrow \bar{X}$ and $S^2$ are independent and thus numerator and denominator here are independent

The Multivariate Normal rv (MVN)

$$\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \text{ s.t. } Z_1,\dots,Z_n \overset{iid}{\sim} N(0,1),\ E[\vec{Z}] = \vec{0}_n,\ Var[\vec{Z}] = I_n$$

$$\vec{Z} \sim f_Z(\vec{z}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0},I)$$

let $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} = \vec{Z} + \vec{\mu} = \begin{bmatrix} Z_1+\mu_1 \\ \vdots \\ Z_n+\mu_n \end{bmatrix} \begin{matrix} \sim N(\mu_1,1) \\ \\ \sim N(\mu_n,1) \end{matrix} \sim N_n(\vec{\mu},I)$

let $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$, $\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1+Z_2 \\ Z_1+Z_2+Z_3 \\ \vdots \\ Z_1+\cdots+Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \sim N(0,3) \\ \\ \sim N(0,n) \end{matrix} \sim N_n(\vec{0},?)$

$\sigma_{1,2} = Cov[X_1,X_2] = Cov[Z_1, Z_1+Z_2] = Cov[Z_1,Z_1] + Cov[Z_1,Z_2] = 1+0 = 1$

General rule to figure out variance-covariance matrix of matrix A times rv vector X:

$$\begin{aligned} Var[A\vec{X}] &= E[(A\vec{X})(A\vec{X})^T] - E[A\vec{X}]E[A\vec{X}]^T \\ &= E[A\vec{X}\vec{X}^TA^T] - E[A\vec{X}]E[A\vec{X}]^T \\ &= AE[\vec{X}\vec{X}^T]A^T - AE[\vec{X}](AE[\vec{X}])^T \\ &= AE[\vec{X}\vec{X}^T]A^T - AE[\vec{X}]E[\vec{X}]^TA^T \\ &= A(\underbrace{E[\vec{X}\vec{X}^T] - E[\vec{X}]E[\vec{X}]^T}_{\Sigma})A^T \\ &= A\Sigma A^T \end{aligned}$$

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, AA^T) = f_{\vec{X}}(\vec{x}) = ?$$
$$\overset{\shortparallel}{g(\vec{z})}$$

$\Downarrow$

$\vec{Z} = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{\mu}) = h(\vec{X})$, in order for $g$ to be 1:1, the matrix $A$ must be invertible.

$$= B\vec{X} - B\vec{\mu} \Rightarrow \begin{aligned} h_1(\vec{x}) &= \vec{b}_1 \cdot \vec{x} - b_1 \cdot \vec{\mu} \\ &\vdots \\ h_n(\vec{x}) &= \vec{b}_n \cdot \vec{x} - \vec{b}_n \cdot \vec{\mu} \end{aligned}$$

$$J_n = \det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \det \overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B} = \det[A^{-1}] = \frac{1}{\det[A]}$$

Note: $AA^{-1} = I \Rightarrow \det[AA^{-1}] = 1 \Rightarrow \det[A]\det[A^{-1}] = 1$, $(AA^{-1})^T = I^T = I \Rightarrow (A^{-1})^T A^T = I$
$\Rightarrow (A^{-1})^T = (A^T)^{-1}$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))|J_n| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-\vec{\mu}))^T(A^{-1}(\vec{x}-\vec{\mu}))} \frac{1}{|\det[A]|}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})} = \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$$\Sigma = Var[\vec{X}] = AA^T \Rightarrow \Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1}$$
$$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T] = \det[A]^2$$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})} = f_{\vec{X}}(\vec{x}) = N_n(\vec{\mu}, \Sigma)$$

Does this work if $A$ is $m \times n$? No... but there's another way

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Multivariate chf's:

$$\phi_{\vec{X}}(\vec{t}) := E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1 + \cdots + t_nX_n)}] = E[e^{it_1X_1} e^{it_2X_2} \cdots e^{it_nX_n}]$$

indep $= E[e^{it_1X_1}]E[e^{it_2X_2}] \cdot \ldots \cdot E[e^{it_nX_n}] = \phi_{X_1}(t_1)\phi_{X_2}(t_2) \cdot \ldots \cdot \phi_{X_n}(t_n)$

(P0) $\phi_{\vec{X}}(\vec{0}) = 1$

(P1) Yes!

(P2) $\vec{Y} = A\vec{X} + \vec{b} \Rightarrow \phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{i\overbrace{\vec{t}^T A}^{\vec{t}'^T = (A^T\vec{t})^T}\vec{X}} e^{i\vec{t}^T\vec{b}}]$

$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{X}}(\vec{t}') = e^{i\vec{t}^T\vec{b}} \phi_{\vec{X}}(A^T\vec{t})$

Let's find the chf of the standard MVN $Z \sim N_n(\vec{0}, I)$

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n} \phi_Z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

$$X = A\vec{Z} + \vec{\mu} \sim N(\vec{\mu}, \overset{AA^T}{\Sigma}), \quad \phi_{\vec{X}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}(A^T\vec{t})^T A^T\vec{t}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

~

$$\vec{X} \sim N(\vec{\mu}, \Sigma), \quad \vec{Y} = B\vec{X} + \vec{c}, \quad B \in \mathbb{R}^{m \times n}, \quad \vec{c} \in \mathbb{R}^m$$

$$\phi_{\vec{Y}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{c}} \left( e^{i\overbrace{(B^T\vec{t})^T}^{\vec{t}^T B}\vec{\mu} - \frac{1}{2}(B^T\vec{t})^T\Sigma(B^T\vec{t})} \right)$$

$$= e^{i\vec{t}^T(B\vec{\mu} + \vec{c}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{(P1)}{\Rightarrow} \vec{Y} \sim N_m(B\vec{\mu} + \vec{c}, B\Sigma B^T)$$

~

$$\vec{X} \sim N_n(\vec{\mu}, \overset{AA^T}{\Sigma}), \quad \underbrace{(\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu})}_{\text{Scalar r.v.}} \sim ?$$

$(AA^T)^{-1} = (A^{-1})^T A^{-1}$ $\qquad$ $\vec{Z} \sim N_n(\vec{0}, I)$

$$= (\vec{X} - \vec{\mu})^T A^T (A^{-1})^T A^{-1} = (A^{-1}(\vec{X} - \vec{\mu}))^T (A^{-1}(\vec{X} - \vec{\mu})) = \vec{Z}^T\vec{Z} \sim \chi^2_n$$

Mahalanobis Distance (1936)

if $n = 1$, $(x-\mu)\frac{1}{\sigma^2}(x-\mu) = \left(\frac{x-\mu}{\sigma}\right)^2$ Squared z-score