LECTURE 4

Let $X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X>Y)$

$P(X>Y) = P(Y>X)$ but theres a possibility that both would win

$P(X>Y) + P(Y>X) + P(X=Y) = 1$

$2P(X>Y) + P(X=Y) = 1$

$P(X>Y) = \frac{1 - P(X=Y)}{2} < \frac{1}{2}$ since $P(X=Y) > 0$

from Lec 2 convolution

$$P(X>Y) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} P_{X,Y}(x,y) \mathbb{1}_{x>y}$$

This is the event we care about

$P_X(x) P_Y(y)$

$$= \sum_{x \in \mathbb{R}} P_X(x) \sum_{y \in \mathbb{R}} P_Y(y) \mathbb{1}_{x>y}$$

$$= \sum_{x \in \{0,1\ldots\}} P_X^{old}(x) \sum_{y \in \{0,1\ldots\}} P_Y^{old}(y) \mathbb{1}_{x>y}$$

This is a constant

$$\sum_{y \in \mathbb{R}} \sum_{x \in \mathbb{R}} P_X(x) P_Y(y) \mathbb{1}_{x>y}$$

$$\sum_{y \in \mathbb{R}} P_Y(y) \sum_{x \in \mathbb{R}} P_X(x) \mathbb{1}_{x>y}$$

$$\sum_{y \in \{0,1\ldots\}} p(1-p)^y \sum_{x \in \{0,1\ldots\}} p(1-p)^x \mathbb{1}_{x>y}$$

combine and get rid of $\mathbb{1}$

$x \geq y+1$

$$= p^2 \sum_{y \in \{0,1,\ldots\}} (1-p)^y \sum_{x \in \{y+1, y+2\ldots\}} (1-p)^x$$

now we need a trick

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let $x' = x - (y+1) \Rightarrow x' \in \{0, 1 \ldots\}$ Reindexing Trick

$\Rightarrow x = x' + y + 1$

$$p^2 \sum_{y \in \{0,1\ldots\}} (1-p)^y \sum_{x' \in \{0,1\ldots\}} (1-p)^{x'} (1-p)^y (1-p)$$

$$= p^2 (1-p) \sum_{y \in \{0,1\ldots\}} (1-p)^{2y} \boxed{\sum_{x' \in \{0,1\ldots\}} (1-p)^{x'}}$$

Geometric Series

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$$

$a \in (-1,1) \setminus \{0\}$

$$\frac{1}{1-(1-p)} = \frac{1}{p}$$

cancels

$$= p^{2*} (1-p) \sum_{y \in \{0,1\ldots\}} (1-p)^{2y} \left(\frac{1}{p^*}\right)$$

$$= p(1-p) \sum_{y \in \{0,1\ldots\}} \left((1-p)^2\right)^y$$

$$\frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{p(2-p)}$$

$$= \frac{p(1-p)}{p(2-p)} = \boxed{\frac{1-p}{2-p}}$$

which is $< \frac{1}{2}$

Bag of fruit:

Apples & Bannanas

Draw with replacements n times

Let x_1 = # apples $p_1 = p(\text{apples})$ $n = 10$

$\Rightarrow X_1 \sim Bin(n, p_1)$

now n with replacement

X_1 = # apples X_2 = # banana's

$X_1 \sim Bin(n, p_1)$, $X_2 \sim Bin(n, p_2)$

Are X_1 & X_2 independent?

Since $X_1 + X_2 = n \Rightarrow$ not independent cuz if we know one we know the other.

$$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim P_{\bar{X}}(\bar{x}) = P_{\vec{X}}(x_1, x_2) = \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \left[\mathbb{1}_{n \in \mathbb{N}} \right] \mathbb{1}_{x_1 + x_2 = n}$$

$$\mathbb{1}_{x_1 \in \{0, 1 \dots n\}} \mathbb{1}_{x_2 \in \{0, 1 \dots n\}}$$

$\binom{n}{x_1, x_2}$ multichoose notation

$$= \vec{X} \sim \text{multi}(n, \vec{p}) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} \qquad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

multinomial rv of dim = 2

Since x_1, x_2 are dependent, we cannot factor this JMF

now bag of fruit has cantelopes draw it w/ probability p_3 and x_3 is count of contelopes

$$\vec{X} \sim \text{multi}(n, \vec{p}) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

$$= \frac{n!}{x_1!\, x_2!\, x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1+x_2+x_3=n}$$

$$\mathbb{1}_{x_1 \in \{0,\dots n\}}$$

$$\mathbb{1}_{x_2 \in \{0,\dots n\}}$$

$$\mathbb{1}_{x_3 \in \{0,\dots n\}}$$

In general, if there are k types of fruit (# catogories) then the general multinomial rv of dim k is:

$$\vec{X} \sim \text{mult}(n, \underbrace{\vec{p}}_{\begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix}}) = \binom{n}{x_1, x_2 \dots x_k} \prod_{k=1}^{k} p_k^{x_k}$$

Parameter Space: $n \in \mathbb{N}$, $\cancel{\vec{p} \in \{\vec{v} : \vec{p} \cdot \vec{1} = 1, p_1 \in (0,1) \dots}$

$$\vec{p} \in \{\vec{v} : \vec{v} \cdot \vec{1} = 1, v_1 \in (0,1) \dots v_k \in (0,1)\}$$

$$\text{Support}[\vec{X}] = \{\vec{x} : \vec{x} \cdot \vec{1} = n,\ x_1 \in \{0, 1, \dots, n\} \dots x_k \in \{0, \dots n\}\}$$

k=2

$$\vec{X} \sim \text{multi}\left(n, \begin{bmatrix} p \\ 1-p \end{bmatrix}\right) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

Dependent?

$$P(X_1 = x_1 \mid X_2 = x_2) \overset{?}{=} P(X_1 = x_1)$$

$$= \text{Deg}(n - x_2) \qquad = \text{Bin}(n, p_1)$$

Dependent

conditional PMF

NOW WE WILL SHOW ITS DEPENDENT

$$P_{X_1|X_2}(x_1, x_2) := \frac{P_{X_1 X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

← JMF

← Marginal PMF of X_2

want to show $X_2 \sim Bin(n, p_2)$

$$P_{X_2}(x_2) = P(X_2 = x_2) = \sum_{x_1 \in \mathbb{R}} P_{X_1, X_2}(x_1, x_2)$$

margining out x_1

$$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1\, x_2} p^{x_1} (1-p)^{x_2}$$

$$= \sum_{x_1 \in \mathbb{R}} \frac{n!}{x_1!\, x_2!} p^{x_1} (1-p)^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0 \ldots n\}} \mathbb{1}_{x_2 \in \{0 \ldots n\}}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \sum_{x_1 \in \{0, 1 \ldots n\}} \frac{p^{x_1}}{x_1!} \mathbb{1}_{x_1 = n - x_2}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \mathbb{1}_{x_2 \in \{0, 1 \ldots n\}} \sum_{x_1 \in \{0 \ldots n\}} \frac{p^{x_1}}{x_1!} \mathbb{1}_{x_1 = n - x_2}$$

$$= \frac{n!}{x_2!} (1-p)^{x_2} \mathbb{1}_{x_2 \in \{0, 1 \ldots n\}} \frac{p^{n - x_2}}{(n - x_2)!} =$$

$$\binom{n}{x_2} p^{n - x_2} (1-p)^{x_2} = Bin(n, 1-p)$$

marginning a multinomial to yeild 1D is Binomial