3

9/02

let $B_1, B_2 \ldots \overset{iid}{\sim} Bern(p)$

Possibly an infinite sequence of iid rv's

let $X := \#$ of zero realization before the first realization of one.

also, $X := \min\{t : B_t = 1\} - 1$

$p(0) = P(X=0) = P(\{\text{no } 0\text{'s, just a } 1\}) = p$

$p(1) = P(X=1)\ P(\{0, \text{then a } 1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0,0,0,1\}) = (1-p)^2 p$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0,0,\ldots}_{x}, 1\}) = (1-p)^x p$

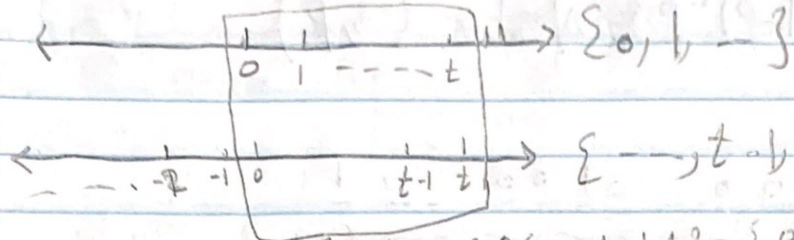
$\underset{\text{geometric}}{X} \sim \text{Geom}(p) := \underbrace{(1-p)^x p}_{p_{old}} \mathbb{1}_{x \in \{0,1,2,\ldots\}}$

$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p),\ T_2 = X_1 + X_2 \sim p_T(t) = ?$

$p_T(t) = \sum_{x \in supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in supp[X]} =$

$= \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (1-p)^{t-x} p\, \mathbb{1}_{\underbrace{t-x \in \{0,1,\ldots\}}_{\underbrace{x-t \in \{0,-1,-2\ldots\}}_{x \in \{t, t-1, t-2, \ldots\}}}}$

$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{t, t-1, \ldots\}}$



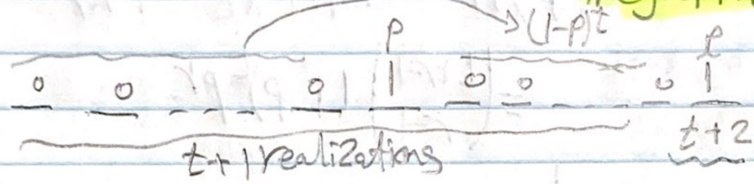
$\Sigma\, a+b = \Sigma a + \Sigma b$

$\{0,1,--\} \cap \{--, t-1, t\} = \{0,1,--t\}$

$$= (1-p)^t p \sum_{x\in\{0,--,t\}} 1 \quad = (t+1)(1-p)^t p^2$$

$$= NegBin(2,p)$$

Negative Binomial rv



$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p) \quad T_3 = X_1+X_2+X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$

$$P_{T_3}(t) = \sum_{x\in supp[X_3]} p_{X_3}^{old}(x)\, P_{T_3}^{old}(t-x)\, \mathbb{1}_{t-x\in supp[T_2]}$$

$$= \sum_{x\in\{0,1,--\}} (1-p)^x p\, (t-x+1)(1-p)^{t-x} p^2\, \mathbb{1}_{\underbrace{t-x\in\{0,1,--\}}_{x\in\{--,\ --t-1,\ t\}}}$$

$$= (1-p)^t p^3 \Big((t+1) \underbrace{\sum_{x\in\{0,--t\}} (1)}_{(t+1)} - \sum_{x\in\{0,--t\}} x$$

$$= (1-p)^t p^3 \Big(\underbrace{(t+1)^2 - \frac{t(t+1)}{2}}_{t^2+2t+1\ -\ \frac{t^2+t}{2}} \Big)$$

$$= \frac{t^2+3t+2}{2} = \frac{(t+2)(t+1)}{2}$$

$$= \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$= \binom{t+2}{2}(1-p)^t p^3 = \text{Neg bin}(3, p)$$

$$\underbrace{\underline{0}\ \underline{0}\ \_\ \_\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \_\ \_\ \_\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \_\ \_\ \_\ \underline{0}}_{t+2 \text{ realization}}\ \underset{t+3}{\underline{1}}$$

Pick 2 position for the first two 1's.

* $X_1, \ldots X_r \overset{iid}{\sim} \text{Geom}(p)$, $T_r = X_1 + X_2 \cdots + X_r \sim \text{Neg bin}(r, p)$

$$= \binom{t+r-1}{r-1}(1-p)^t p^r$$

$$\underbrace{\underbrace{\underline{0}\ \underline{0}\ \_\ \_\ \_\ \underline{0}\ \_\ \_\ \_}_{r-1 \text{ 1's}}\ \underline{0}}_{t+r-1}\ \underset{t+r}{\underline{1}}$$

$$X \sim \text{Bin}(n, p) = \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{\in\{0,1,\ldots n}$$

let $n \to \infty$, $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ let $n \to \infty$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x\in\{0,1,\ldots n\}} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\mathbb{1}_{x\in\{0,1,\ldots n\}}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!\,n^x}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}\lim_{n\to\infty}\mathbb{1}_{x\in\{0,1,\ldots n\}}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{(n)(n-1)\cdots(n-x+1)}^{x\text{ terms}}}{\underbrace{n\,n\cdots n}_{x\text{ terms}}} e^{-\lambda}(1)\, \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x\in\{0,1,\dots\}} = \text{Poisson}(\lambda).$$

$n=10,\ x=4$

$$\frac{10\cdot 9\cdot 8\cdot 7}{10\cdot 10\cdot 10\cdot 10}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \quad T = X_1 + X_2 \sim p_T(t) = ?$

$$p_T(t) = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x\in\{0,1,\dots\}}$$

$$= \lambda^t e^{-2\lambda} \frac{t!}{t!} \sum_{x\in\{0,1,\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x\in\{\dots,t-1,t\}}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x\in\{0,\dots,t\}} \binom{t}{x}}_{2^t} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$$