lec - 20

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}}$$

$$= \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{\chi^2_{n-1}}}{n-1}}} \sim T_{n-1}$$

Due to cochran's theorem, we know $\bar{X}$ and S^2 are Independant.

Multivariate Normal Distribution (MVN)

$Z_1 \cdots\cdots Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}$,

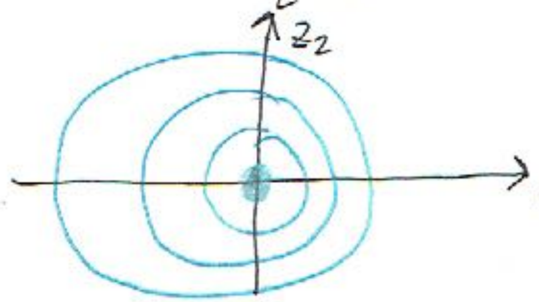
$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i)$ $\quad$ $Var[\vec{Z}] = I_n$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_i z_i^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}}$$

Now,
$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$, $\quad E[\vec{X}] = \vec{\mu}$, $Var[\vec{X}] = I_n$

$\vec{X} \sim N_n(\vec{\mu}, I)$



, $N_2(\vec{0}, I)$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1 + Z_2 \\ \vdots \\ Z_1 + \cdots + Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$$

but components are dependant e.g

$$\mathrm{Cov}[X_1, X_2] = \mathrm{Cov}[Z_1, Z_1+Z_2]$$
$$= \mathrm{Cov}[Z_1, Z_1] + \mathrm{cov}[Z_1, Z_2] = 1$$

that means, X_1, X_2 dependant.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

Let's derive a general (mxn) formula for the Variance - Covariance matrix of A (an nxn matrix of Scalars) times a random Vector X of dim n:

$$\mathrm{Var}[A\vec{x}] = E\left[(A\vec{x})(A\vec{x})^T\right] - E[A\vec{x}]\,E[A\vec{x}]^T$$

$$= A\,E[XX^T]A^T - A\,E[\vec{x}]\,(A\,E[\vec{x}])^T$$

$$= A\Big(\overbrace{E[XX^T] - E[\vec{x}]\,E[\vec{x}^T]}^{\Sigma = \mathrm{Var}[\vec{x}]}\Big)A^T = A\Sigma A^T$$

Now,

$\vec{X} = A\vec{Z}$, $\mathrm{Var}[\vec{x}] = A I_n A^T = AA^T$, conjugence: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{X} = A\vec{z} + \vec{\mu}$, $A \in \mathbb{R}^{n\times n}$, $\vec{\mu} \in \mathbb{R}^n$, $\vec{x} \sim f_{\vec{x}}(\vec{x}) = ?$

$\rightarrow g(\vec{z}),\ h(\vec{x}) = \vec{z}$, where hopefully g and h are inverses

$h(\vec{x}) = \underbrace{\vec{A}}_{B}(\vec{x} - \vec{\mu})$ $\longrightarrow$ In order for the Inverse to exist... A has to be invertible.

$$= B\vec{x} - B\vec{\mu} = \begin{bmatrix} \vec{b}_1\vec{x} - \vec{b}_1, \vec{\mu} \\ \vec{b}_2\vec{x} - \vec{b}_2, \vec{n} \\ \vdots \\ \vec{b}_n\vec{x} - \vec{b}_n, \vec{n} \end{bmatrix} \begin{matrix} = h_1 \\ = h_2 \\ \vdots \\ = h_n \end{matrix}$$

$$J_n = \det\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \det\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \det\left[A^{-1}\right] = \frac{1}{\det[A]}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_n|$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\underbrace{(A^{-1}(\vec{x}-\vec{\mu}))^T A^{-1}(\vec{x}-\vec{\mu})}_{(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})}} \frac{1}{|\det[A]|}$$

Now,

$$DD^{-1} = I \Rightarrow (DD^{-1})^T = I^T = I \Rightarrow (D^{-1})^T D^T = I \Rightarrow (D^{-1})^T = (D^T)^{-1}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \underbrace{(A^T)^{-1} A^{-1}}_{(AA^T)^{-1}} (\vec{x}-\vec{\mu})}$$

, let,

$$\Sigma = AA^{-1} = Var[\vec{X}]$$

$$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T] = \det[A]^2$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-1/2(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})}$$

Multivariate characteristic function :-

$$\phi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{x}}\right] = E\left[e^{i(t_1x_1 + \cdots + t_nx_n)}\right] = E\left[e^{it_1x_1} \cdots e^{it_nx_n}\right]$$

If $X_1, X_2, \ldots X_n$ are independant,

$$= E\left[e^{it_1x_1}\right] \cdots E\left[e^{it_nx_n}\right] = \phi_{x_1}(t_1)\phi_{x_2}(t_2) \cdots \phi_{x_n}(t_n)$$

(P0) $\phi_{\vec{X}}(\vec{0}) = E\left[e^{i\vec{0}^T\vec{x}}\right] = 1$

(P1) If two ch's are equal $\Rightarrow$ the two rv's are equal in distribution.

(P3) $\vec{y} = A\vec{x} + \vec{b}$, $A \in \mathbb{R}^{m\times n}$, $\vec{b} \in \mathbb{R}^m$, $\vec{x}$ in dim $n \Rightarrow \vec{y}$ is dim m

$$\phi_y(\vec{t}) := E\left[e^{i\vec{t}^T(A\vec{x}+\vec{b})}\right] = E\left[e^{\underbrace{i\vec{t}^T A\vec{x}}_{(A^T\vec{t})^T}} e^{i\vec{t}^T\vec{b}}\right]$$

$$= e^{i\vec{t}^T\vec{b}} E\left[e^{i(A^T\vec{t})^T\vec{x}}\right]$$

$$= e^{i\vec{t}^T\vec{b}} \phi_{\vec{x}}(A^T\vec{t})$$

Let's derive the ch's of the standard normal MVN

$$\phi_{\vec{z}}(\vec{t}) = \prod_{i=1}^{n} \phi_z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{z}}$$

let's derive the ch's of the general MVN:

$$\vec{x} = A\vec{z} + \vec{\mu} \sim N(\vec{\mu}, \overbrace{AA^T}^{\Sigma})$$

$$\phi_{\vec{x}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{\mu}} \phi_{\vec{z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} e^{-\frac{1}{2}\underbrace{(A^T\vec{t})^T A^T\vec{t}}_{\vec{t}^T \underbrace{AA^T}_{\Sigma} \vec{t}}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{y} = B\vec{x} + \vec{c} \sim ?$, $B \in \mathbb{R}^{m\times n}$, $\vec{c} \in \mathbb{R}^n$

$$\phi_{\vec{y}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{c}} \phi_{\vec{x}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}} \cdot e^{i\overbrace{(B^T\vec{t})^T}^{t^TB}\mu - \frac{1}{2}\overbrace{(B^T\vec{t})^T}^{t^TB}\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(\vec{c}+B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{(P1)}{\Rightarrow}$$

$$\vec{y} \sim N_\mu(B\vec{\mu}+c, B\Sigma B^T)$$

Let, $\vec{x} \sim N_n(\vec{\mu}, \Sigma)$, consider If ($B\Sigma B^T$ is invertible) $(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu}) \sim ?$

Recall :- $\vec{z} = A^{-1}(\vec{x}-\vec{\mu})$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$$

Recall :- $\vec{z} = A^{-1}(\vec{x}-\vec{\mu})$

$$\Sigma^{-1} = (AA^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$$

$$= (\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})$$

$$= (A^{-1}(\vec{x}-\vec{\mu}))^T A^{-1}(\vec{x}-\vec{\mu})$$

$$= \vec{z}^T\vec{z} \sim \chi^2_n$$

PC Mahalanobis discovered this in 1936. He was indian's founding father of statistics and founded the indian institute of statistics.

This kind of like distance in R^n adjusted for all dependencies among the dimensions like a multivariate "Z-Score"

In one dimensions, $$(X-\mu)(\sigma^2)^{-1}(X-\mu) = \frac{(X-\mu)^2}{\sigma^2} = \left(\frac{X-\mu}{\sigma}\right)^2 = z^2$$

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