$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$

$$f_{T_3}(t) = \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx$$

$$= \int_0^\infty x \lambda^2 \cancel{e^{-\lambda x}}\, \lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{\underbrace{t-x \in [0,\infty)}_{x \le t}}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty x\, \mathbb{1}_{x \le t}\, dx = \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = Erlang(3, \lambda)$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \int_0^\infty \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}}\, \lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t x^2 dx\, \mathbb{1}_{t \in [0,\infty)} = \frac{t^3}{3 \cdot 2} \lambda^4 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = Erlang(4, \lambda)$$

$$\sum_{i=1}^{k} X_i = T_k \sim Erlang(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_k] = [0, \infty)$

param. space $\lambda \in (0, \infty)$, $k \in \mathbb{N}$

$$Exp(\lambda) = Erlang(1, \lambda) \qquad \sum_{i=1}^{k} Exp(\lambda) = Erlang(k, \lambda)$$

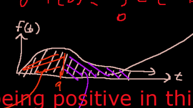
$$\Updownarrow \qquad\qquad \Updownarrow$$

$$Geom(p) = NegBin(1, p) \qquad \sum_{i=1}^{k} Geom(p) = NegBin(k, p)$$

We will just do some pure math definitions. We will introduce the gamma family of functions. The "gamma function" is:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

e.g. $\Gamma(3) = \int_0^\infty \underbrace{t^2 e^{-t}}_{f(t)} dt = 2$



we're only going to care about x being positive in this class

$$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)}$$

$a \in [0, \infty)$

Lower incomplete gamma function

Upper incomplete gamma function

$$Q(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1)$$ proportion of the gamma function below a

Lower regularized incomplete gamma function

$$P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1]$$ proportion of the gamma function above a

Is $Q(x,a) + P(x,a) = 1$

$$\Gamma(1) := \int_0^\infty e^{-t}\, dt = 1$$ This is the integral of the PDF for Exp(1) over its support.

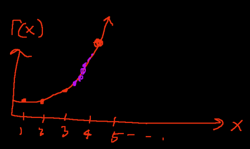
$\Gamma(x+1) = x\Gamma(x)$ Proved on the HW via integration by parts.

$\Rightarrow \Gamma(2) = 1\Gamma(1) = 1 \cdot 1$, $\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2$, $\Gamma(4) = 3\Gamma(3) = 3 \cdot 2 \cdot 1 \ldots$

for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

$\Gamma(4.5) = 3.5\,\Gamma(3.5) = 3.5 \cdot 2.5\,\Gamma(2.5) = 3.5 \cdot 2.5 \cdot 1.5\,\Gamma(1.5) = 3.5 \cdot 2.5 \cdot 1.5 \cdot 0.5\,\Gamma(0.5)$

The gamma function is an "extension" if the factorial function valid for all positive numbers.



$$X \sim Erlang(\lambda) := \frac{x^{k-1} \lambda^k e^{-\lambda x}}{(k-1)!} \mathbb{1}_{x \in [0,\infty)}$$

$$F_X(x) := P(X \le x) = \int_0^x \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} dt = \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}} \underbrace{\int_0^x t^{k-1} e^{-\lambda t}\, dt}_{\frac{\gamma(k,\lambda x)}{\lambda^k}}$$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)} \frac{\gamma(k, \lambda x)}{\cancel{\lambda^k}} = P(k, \lambda x)$$

Let's do some more boring calculus... for $c > 0$,

$$\int_0^\infty t^{x-1} e^{-ct}\, dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c} du = \frac{1}{c^x} \int_0^\infty u^{x-1} e^{-u} du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c} du$, $t = 0 \Rightarrow u = 0$, $t \to \infty \Rightarrow u \to \infty$, $t = a \Rightarrow u = ac$

$$\int_0^a t^{x-1} e^{-ct}\, dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c} du = \frac{1}{c^x} \int_0^{ac} u^{x-1} e^{-u} du = \frac{\gamma(x, ac)}{c^x}$$

$$\int_a^\infty t^{x-1} e^{-ct}\, dt = \int_0^\infty {/\!/\!/\!/}\, dt - \int_0^a {/\!/\!/\!/}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x) - \gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$$

If $n \in \mathbb{N}$... $du = (n-1)t^{n-2} dt$, $v = -e^{-t}$

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t} dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\, du = \left[-t^{n-1} e^{-t}\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2} dt$$

$$= a^{n-1} e^{-a} + (n-1) \int_a^\infty t^{n-2} e^{-t}\, dt = a^{n-1} e^{-a} + (n-1) \underbrace{\Gamma(n-1, a)}_{\text{iterate ....}}$$

$$= a^{n-1} e^{-a} + (n-1)\left(a^{n-2} e^{-a} + (n-2) \Gamma(n-2, a)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \ldots + (n-1)!\, \Gamma(1,a)\right)$$

$$\Gamma(1,a) = \int_a^\infty e^{-t} dt = \left[-e^{-t}\right]_a^\infty = e^{-a}$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \ldots + \frac{a^0}{0!}\right) = e^{-a}(n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$$

$$X \sim Poisson(\lambda) := \frac{e^{-\lambda} \lambda^x}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

$$F_X(x) := P(X \le x) = \sum_{t=0}^{x} \frac{e^{-\lambda} \lambda^t}{t!} = e^{-\lambda} \sum_{t=0}^{x} \frac{\lambda^t}{t!} = \frac{1}{\underbrace{x!}_{\Gamma(x+1)}} \underbrace{e^{-\lambda} x! \sum_{t=0}^{x} \frac{\lambda^t}{t!}}_{\Gamma(x+1,\lambda)}$$

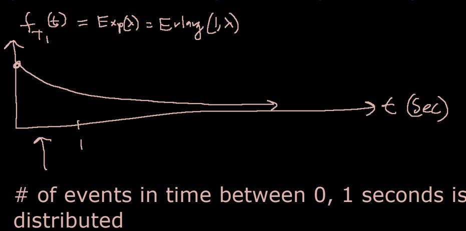
$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = Q(x+1, \lambda)$$

$T_1 \sim Exp(\lambda) = Erlang(1, \lambda) \Rightarrow F_{T_1}(t) = P(1, \lambda t)$

$P(T_1 > 1) = 1 - F_{T_1}(1) = 1 - P(1, \lambda) = Q(1, \lambda)$

$N \sim Poisson(\lambda)$, $P(N = 0) = F_N(0) = Q(1, \lambda)$

the first example of the "poisson process", the link between waiting times in the Erlang and the probabilty of events in a Poisson.



# of events in time between 0, 1 seconds is Poisson(lambda) distributed