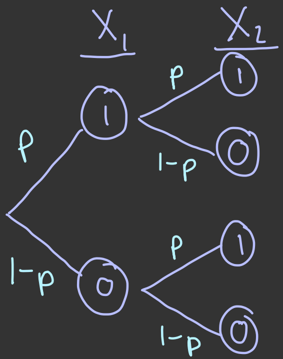
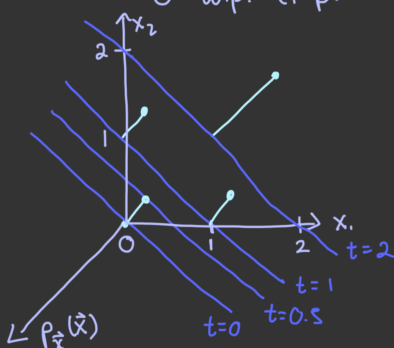


$p_{X_1,X_2}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
p^2	0
1	

mutually exclusive, collectively exhaustive events

$$p_T(t) = P(T=t) = \begin{cases} 2 & \text{w.p. } p^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 0 & \text{w.p. } (1-p)^2 \end{cases}$$



Slope = -1

$t = x_1 + x_2 \Rightarrow x_2 = t - x_1$

$P(\underbrace{T=0})$

$\downarrow$

$x_2 = 0 - x_1 = -x_1$

select events

$$p_T(t) = \sum_{x_1 \in \mathbb{R}} \underbrace{\sum_{x_2 \in \mathbb{R}}}_{\text{search through } \mathbb{R}^2} \underbrace{p_{X_1,X_2}(x_1,x_2)}_{\text{add up all probs}} \boxed{\mathbb{1}_{\substack{x_1+x_2=t \\ x_2=t-x_1}}} = \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$ → general convolution formula

if X_1, X_2 are independent

$$\stackrel{\downarrow}{=} \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x) = \sum_{x \in \mathbb{R}} p^{old}_{X_1}(x) \boxed{\mathbb{1}_{x \in supp[X_1]}} p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in supp[X_2]}$$

$$= \sum_{x \in supp[X]} p^{old}_{X_1}(x)\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in supp[X_2]}$$ → convolution formula for independent r.v.s

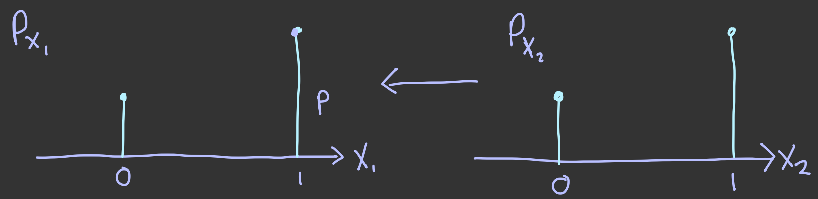
if X_1, X_2 iid

$$\stackrel{\downarrow}{=} \sum_{x \in \mathbb{R}} p(x)p(t-x) = \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in supp[X]}$$

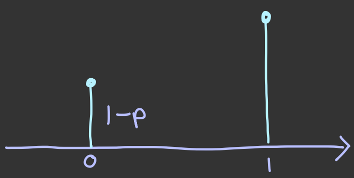
$$= \sum_{x \in supp[x]} p(x)p(t-x)\, \mathbb{1}_{t-x \in supp[X]}$$ → convolution formula for iid r.v.s

"convolve" means to "roll, coil or entwine together"

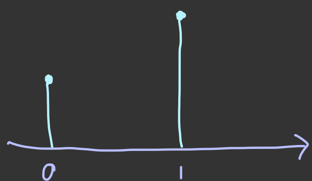
$$p_T = p_{X_1} \underset{\text{convolution}}{*} p_{X_2}$$



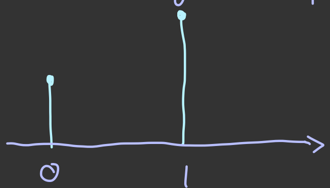
Sum-product.



$\Rightarrow$ $t=1$ w.p $p(1-p)$



$\Rightarrow$ $t=0$ w.p. $(1-p)^2$,
$t=2$ w.p p^2



$\Rightarrow$ $t=1$ w.p. $(1-p)p$

$$p_{T_2}(t) = \sum_{x\in\{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{\underbrace{t-x\in\{0,1\}}_{t\in\{x,\,x+1\}}} = p^t (1-p)^{2-t} \sum_{x\in\{0,1\}} \mathbb{1}_{t\in\{x,x+1\}}$$

$$= \boxed{p^t (1-p)^{2-t} \left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}} \right)} \longrightarrow = \binom{2}{t} p^t (1-p)^{2-t}$$
$$= \text{Binom}(2, p)$$

$$T_2 \sim \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 2 & \text{w.p. } p^2 \end{cases} \qquad = \binom{2}{t}$$

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!} \mathbb{1}_{n\in\mathbb{N}} \mathbb{1}_{k\in\{0,1,\dots,n\}}$$

$$\operatorname{supp}[T] = \operatorname{supp}[X_1] + \operatorname{supp}[X_2]$$

$$A + B := \{a+b \mid a \in A \text{ \& } b \in B\}$$

$$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$$

$$= \binom{1}{x} p^x (1-p)^{1-x} \qquad \binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}$$

$$p_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x} = p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \overset{\overset{\mathbb{1}_{x\in\{0,1\}}}{\parallel}}{\binom{1}{x}} \binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x} = p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

Pascals identity
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p), \quad T_3 = \overbrace{X_1 + X_2}^{T_3} + X_3 = X_3 + T_2 \sim p_{T_3}(t) = ?$$

$$p_{T_3}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x} = p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right) = \binom{3}{t} p^t (1-p)^{3-t} = \text{Binom}(3, p)$$

Hw: Find PMF of $\text{Binom}(n,p)$ via induction.

$$X_1, X_2 \overset{iid}{\sim} \text{Binom}(n,p), \quad T = X_1 + X_2 \sim ?$$

$$p_T(t) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x} = p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x} \binom{n}{t-x}$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t}$$

$$= \text{Binom}(2n, p)$$

Vandermonde's Identity.