9/2/2020
Lecture 03
Math 621

Let $B_1, B_2, \ldots \overset{iid}{\sim} Bern(P)$ possibly an infinite sequence of iid rv's.

Let $X :=$ # of zeros realizations before the first realization of one, also

$$X := \min\{t : B_t = 1\} - 1$$

$$P(0) = P(X=0) = P(\{\text{no } 0\text{'s, just a } 1\}) = P$$

$$P(1) = P(X=1) = P(\{0, \text{ then a } 1\}) = (1-P)P$$

$$P(2) = P(X=2) = P(\{0,0,1\}) = (1-P)^2$$

$$\vdots$$

$$P(x) = P(X=x) = P(\{\overbrace{0,0,\ldots 0}^{x}, 1\}) = (1-P)^x P$$

$$X \sim \underset{\text{geometric}}{\underset{\parallel}{Geom(P)}} := \underbrace{(1-P)^x P}_{P^{old}} \mathbb{1}_{x \in \{0,1,2,\ldots\}}$$

$$X_1, X_2 \overset{iid}{\sim} Geom(P), \quad T_2 = X_1 + X_2 \sim P_T(t) = ?$$

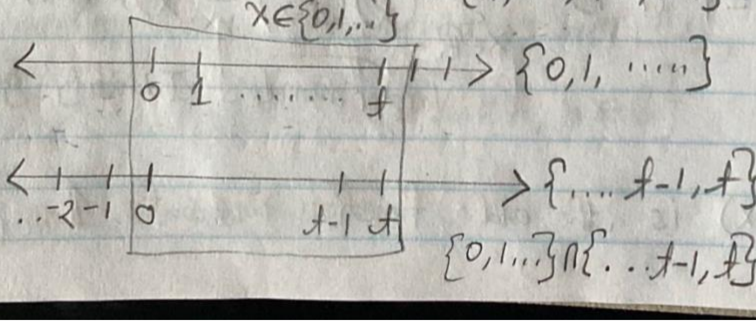
$$P_T(t) = \sum_{x \in Supp[X]} P^{old}(x) P^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in \{0,1,\ldots\}} (1-P)^x P (1-P)^{t-x} P \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$\Rightarrow x - t \in \{0,-1,-2,\ldots\}$$

$$\Rightarrow x \in \{t, t-1, t-2, \ldots\}$$

$$= (1-P)^t P^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{t, t-1, t-2, \ldots\}}$$



$$\{0,1,\ldots\} \cap \{\ldots t-1, t\} = \underline{\{0,1,\ldots,t\}}$$

$Supp[T_2] = \{0, 1, \dots\}$

$$= (1-p)^t p^2 \underbrace{\sum_{x \in \{0, \dots, t\}} 1} = (t+1)(1-p)^t p^2 = NegBin(2, p)$$

Negative Bionomial r.v.

$$\underbrace{0\ 0\ \dots\ 0\ \overset{p}{1}\ 0\ 0\ 0\ \dots\ 0}_{t+1 \text{ realization}}\ \underbrace{\overset{p}{1}}_{t+2} \qquad (1-p)^t$$

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p)$ $\quad T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$

$$P_{T_3}(t) = \sum_{x \in Supp[X_3]} P^{old}_{X_3}(x)\, P^{old}_{T_3}(t-x)\, \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum_{x \in \{0,1,\dots\}} (1-p)^x p \underbrace{(t-x+1)(1-p)^t p^2}\ \mathbb{1}_{\substack{t-x \in \{0,1,\dots\} \\ x \in \{\dots, t-1, t\}}}$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0, \dots, t\}} (1) - \sum_{x \in \{0, \dots, t\}} x \right)$$

$$= (1-p)^t p^3 \underbrace{\left((t+1)^2 - \frac{t(t+1)}{2} \right)}$$

$$t^2 + 2t + 1 - \frac{t^2 + t}{2} = \frac{2t^2 + 4t + 2 - t^2 - t}{2} = \frac{t^2 + 3t + 2}{2}$$

$$= \frac{(t+2)(t+1)}{2}$$

$$= \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$= \binom{t+2}{2} (1-p)^t p^3 = NegBin(3, p)$$

Spliting the $\Sum$:

$$\sum_{x \in S} (\overset{a}{t+1} - \overset{b}{x}) = \sum_{x \in S} (t+1) - \sum_{x \in S} s = (t+1) \sum 1 - \sum x$$

Not constant

$$\underbrace{0\ 0\ \dots\ 0\ 1\ 0\ 0\ \dots\ 0\ 1\ 0\ 0\ \dots\ 0}_{t+2 \text{ realization}}\ \underbrace{1}_{t+3}$$

Pick 2 position for the first two 1's.

$X_1, \dots X_r \overset{iid}{\sim} Gmp(P)$, $T_r = X_1 + X_2 + \dots + X_r \sim NgBin(r, P) :=$

$$NgBin(r,P) := \binom{t+r-1}{r-1}(1-p)^t p^r$$

$$\underbrace{\underbrace{\overbrace{0\ 0\ 1\quad 0\ 0\ 0\ 1\quad 0\ 0\ 0\ 1\dots}^{t\ 0's}}_{r-1\ 1's}}_{t+r-1}\ \underbrace{1}_{t+r}$$

$$X \sim Bin(r,P) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots n\}}$$

Let $n \to \infty$, $P \to 0$ but $\lambda := nP \Rightarrow P = \frac{\lambda}{n}$, let $n \to \infty$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \mathbb{1}_{x\in\{0,1,\dots n\}}$$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x\in\{0,1,\dots n\}}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} \lim_{n\to\infty} \mathbb{1}_{x\in\{0,1\dots}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{\overbrace{(n)(n-1)\dots(n-x+1)}^{x\text{-terms}\ \to 1}}{\underbrace{n\ n\ \dots\ n}_{x\ \text{terms}}} \cdot e^{-\lambda} \cdot (1) \cdot \mathbb{1}_{x\in\{0,1,\dots\}}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda} \mathbb{1}_{x\in\{0,1,\dots\}} = Poisson(\lambda).$$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$ $T = X_1 + X_2 \sim P_T(t) = ?$

$$P_T(t) = \sum_{x\in\{0,1\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{t-x\in\{0,1,\dots\}}$$

$$= \lambda^t e^{-2\lambda} \frac{t!}{t!} \sum_{x\in\{0,1\dots\}} \frac{1}{x!(t-x)!} \mathbb{1}_{x\in\{\dots, t-1, t\}}$$

$$= \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum_{x\in\{0,\dots t\}} \binom{t}{x}}_{2^t} = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = Poisson(2\lambda).$$