$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$

$$f_{T_3}(t) = \int_{Supp[T_2]} f^{old}_{T_2}(x) f^{old}_{X_3}(t-x) \mathbb{1}_{t-x \in Supp[X_3]} dx$$
$$= \int_0^\infty x\lambda^2 e^{-\lambda x} \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in [0,\infty)} dx$$
$$= \lambda^3 e^{-\lambda t} \int_0^\infty x \mathbb{1}_{x \le t} dx$$
$$= \lambda^3 e^{-\lambda t} \int_0^t x dx \, \mathbb{1}_{t \in [0,\infty)} = \frac{t^2}{2}\lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)}$$
$$= \text{Erlang}(3, \lambda)$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x) f^{old}_{X_4}(t-x) \mathbb{1}_{t-x \in [0,\infty)} dx$$
$$= \int_0^\infty \frac{x^2}{2}\lambda^3 e^{-\lambda x} \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x \in [0,\infty)} dx$$
$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t x^2 dx \, \mathbb{1}_{t \in [0,\infty)} = \frac{t^3}{3 \cdot 2} \lambda^4 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$$

$$\sum_{i=1}^K X_i = T_K \sim \text{Erlang}(K, \lambda) := \frac{t^{K-1}\lambda^K e^{-\lambda t}}{(K-1)!} \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_K] = [0, \infty)$

param. space $\lambda \in (0,\infty)$, $K \in \mathbb{N}$

$$\text{Exp}(\lambda) = \text{Erlang}(1, \lambda) \qquad \sum_{i=1}^K \text{Exp}(\lambda) = \text{Erlang}(K, \lambda)$$
$$\updownarrow \qquad\qquad \updownarrow$$
$$\text{Geom}(p) = \text{NegBin}(1, p) \qquad \sum_{i=1}^K \text{Geom}(p) = \text{NegBin}(K, p)$$

We will just do some pure math def'n. We'll introduce the gamma family of functions. The "gamma function" is:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

We're only going to care about x being positive in this class.

$$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t} dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} dt}_{\Gamma(x,a)}$$

lower incomplete gamma function — upper incomplete gamma function.

$Q(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1]$ → not including 1

↳ proportion of the gamma function below a.

Lower regularized incomplete gamma function

$P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1]$ → proportion of the gamma function above a.

$Q(x,a) + P(x,a) = 1$

$\Gamma(1) := \int_0^\infty e^{-t}\,dt = 1$ This is the integral of the PDF for Exp(1) over its support.

$\Gamma(x+1) = x\Gamma(x)$ proved on HW via integration by parts.

$\Rightarrow \Gamma(2) = 1\Gamma(1) = 1\cdot 1, \quad \Gamma(3) = 2\Gamma(2) = 2\cdot 1 = 2 \cdots$

for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

The gamma function is an "extension" if the $\overset{\text{factorial}}{\text{function}}$ valid for all positive numbers.

$X \sim \text{Erlang}(\lambda) := \frac{x^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!} \mathbb{1}_{x\in[0,\infty)}$

$F_X(x) := P(X \le x) = \int_0^x \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!}\,dt$

$= \frac{\lambda^k}{(k-1)!}\int_0^x t^{k-1} e^{-\lambda t}\,dt = \frac{\lambda^k}{\Gamma(k)}\frac{\gamma(k,\lambda_x)}{\lambda^k} = \frac{\gamma(k,\lambda_x)}{\Gamma(k)}$

$= P(k, \lambda_x)$

Let's do some calulus

For $c > 0$, $\int_0^\infty t^{x-1}e^{-ct}\,dt = \frac{1}{c^x}\int_0^\infty u^{x-1}e^{-u}\,du = \frac{\Gamma(x)}{c^x}$

$\left[\text{let } u = ct \to t = \frac{u}{c} \to dt = \frac{1}{c}du,\ t=0 \to u=0,\ t\to\infty \to u\to\infty,\ t=a \to u \to ac\right]$

$\int_0^a t^{x-1}e^{-ct}\,dt = \int_0^{ac}\frac{u^{x-1}}{c^{x-1}}e^{-u}\frac{1}{c}\,du = \frac{1}{c^x}\int_0^{ac}u^{x-1}e^{-u}\,du$

$= \frac{\gamma(x,ac)}{c^x}$

$\int_a^\infty t^{x-1}e^{-ct}\,dt = \int_0^\infty \cdots\,dt - \int_0^a \cdots\,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x}$

$= \frac{\Gamma(x) - \gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$

If $n \in \mathbb{N}$ ...

$$\Gamma(n,a) = \int_a^{\infty} \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}dt}_{dv} = [uv]_a^{\infty} - \int_a^{\infty} v\,du$$

$$\left\{ \begin{array}{l} du = (n-1)t^{n-2}\,dt \\ v = -e^{-t} \end{array} \right\}$$

$$\downarrow$$

$$= \left[-t^{n-1}e^{-t}\right]_a^{\infty} + \int_a^{\infty} e^{-t}(n-1)t^{n-2}dt$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^{\infty} t^{n-2}e^{-t}dt$$

$$= a^{n-1}e^{-a} + (n-1)\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\Gamma(n-2,a)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \cdots + (n-1)!\,\Gamma(1,a)\right)$$

$$\left\{ \begin{array}{l} * \ \Gamma(1,a) = \int_a^{\infty} e^{-t}dt = \left[-e^{-t}\right]_a^{\infty} \\ \qquad = e^{-a} \end{array} \right\}$$

$$\leftarrow \quad e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \cdots + \frac{a^0}{0!}\right)$$

$$= e^{-a}(n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$$

$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

$$F_X(x) := P(X \le x) = \sum_{t=0}^{x} \frac{e^{-\lambda}\lambda^t}{t!} = e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^t}{t!}$$

$$= x!\,\frac{1}{x!}e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^t}{t!} = \left(\frac{1}{\Gamma(x+1)}\right)\left(\Gamma(x+1,\lambda)\right)$$

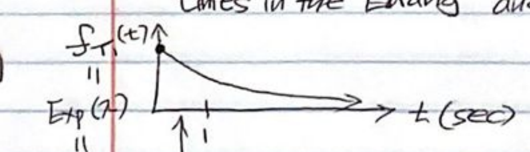
$$= Q(x+1,\lambda)$$

$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1,\lambda) \Rightarrow F_{T_1}(t) = P(1,\lambda t)$

$P(T_1 > 1) = 1 - F_T(1) = 1 - P(1,\lambda) = Q(1,\lambda)$

$N \sim \text{Poisson}(\lambda)$, $P(N=0) = F_N(0) = Q(1,\lambda)$

the first example of the "poisson process", the link between waiting times in the Erlang and the probability of events in a Poisson



Erlang(1, λ) # of events in time btw 0, 1 seconds is Poisson (lambda) distributed.