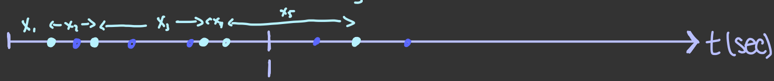
$T_k \sim \text{Erlang}(k, \lambda)$ $\quad$ $N \sim \text{Poisson}(\lambda)$ $\quad$ $X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k, \lambda)$ $\quad$ $F_N(x) = Q(x+1, \lambda)$

N = # events before 1 sec → Why is this Poisson distributed?



$k = 5$

$\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\} \cup \{X_1 + X_2 + X_3\} \cup \{X_1 + X_2 < 1\} \cup \{X_1 < 1\} \cup \{X_1 > 1\}$

$\quad = \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}$

$\Rightarrow \underbrace{P(T_5 > 1)}_{= 1 - F_{T_5}(1)} = P(N \leq 4) = F_N(4)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$ "Poisson Process"

$$T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0} = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

$k \in \mathbb{N}, \lambda \in (0, \infty)$

$$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} = \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

$k \in \mathbb{N}, p \in (0, 1)$

For both, what if $k \in (0, \infty)$? Are both r.v's still "legal"? Yes, we can show that both:

$$\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} dt = 1 \ \& \ \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1$$

we just derived two new famous r.v.'s.

-> $X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x} \mathbb{1}_{x \geq 0}$

-> $X \sim \text{ExtNegBin}(k, p) = \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$ the extended negative binomial

Transformations of Discrete r.v.'s:

$X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$Y = X + 3 \sim \begin{cases} 4 & \text{w.p. } p \\ 3 & \text{w.p. } 1-p \end{cases} = p^{y-3} (1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}}$

$Y = g(X) \sim p_Y(y)$ [want to find the PMF of Y using the PMF of X]

-> $p_Y(y) = p_X(g^{-1}(y))$

What assumption was made when we derived this formula?

I assumed an inverse function exists, i.e., g is invertible. If not ..

$X \sim U(\{1, 2, .., 10\}) = \begin{cases} 1 & \text{w.p. } \frac{1}{10} \\ 2 & \text{w.p. } \frac{1}{10} \\ \vdots \\ 10 & \text{w.p. } \frac{1}{10} \end{cases}$, $\quad Y = g(x) = \min\{X, 3\} \sim \begin{cases} 1 & \text{w.p. } \frac{1}{10} \\ 2 & \text{w.p. } \frac{1}{10} \\ 3 & \text{w.p. } P(X=3) + P(X=4) + \ldots + P(X=10) = \frac{8}{10} \end{cases}$

$$Y = g(X) \sim p_Y(y) = \sum_{\{x \mid y = g(x)\}} p_X(x) \underset{\substack{\downarrow \\ \text{if } g \\ \text{invertible}}}{=} \sum_{\{x \mid x = g^{-1}(y)\}} p_X(x) = p_X(g^{-1}(x))$$

$\hookrightarrow$ one element only

$$X \sim Bin(n,p), \quad Y = \underset{\substack{\| \\ g(x)}}{X^3} \sim p_X(\sqrt[3]{y}) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \mathbb{1}_{\sqrt[3]{y} \in \{0,1,\dots,n\}}$$

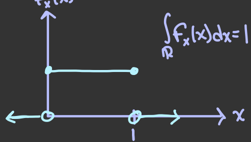
$g(x) \Rightarrow g^{-1}(y) = \sqrt[3]{y}$

$$Y = X^2 \sim p_x(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$$

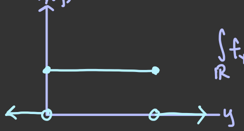
Transformation for continuous r.v.'s:

for g invertible $f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$

Let $X \sim U(0,1) = \underset{\substack{f_X(x) \\ \|}}{\mathbb{1}_{x \in [0,1]}}$, $Y = 2X \overset{?}{\sim} f_X(g^{-1}(y)) = f_X\left(\frac{y}{2}\right) = \mathbb{1}_{\frac{y}{2} \in [0,1]} = \mathbb{1}_{y \in [0,2]}$



$\int_{\mathbb{R}} f_X(x)dx = 1$



$\int_{\mathbb{R}} f_Y(y)dy = 2$ GAME OVER

PDF'S are not probabilities! So this was bound to fail b/c we used them as probabilities.
However, CDF's <u>are</u> probabilities

$$F_Y(y) := P(Y \le y) = P(g(X) \le y) \underset{\substack{\downarrow \\ g \text{ invertible } \& \, g^{-1} > 0}}{=} P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

$$\Rightarrow \underset{\substack{\| \\ f_Y(y)}}{\frac{d}{dy}[F_Y(y)]} = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)] = f_X(g^{-1}(y)) \overbrace{\frac{d}{dy}[g^{-1}(y)]}^{\text{always positive}}$$

$$F_Y(y) = P(Y \le y) = P(g(x) \le y) = P(X \ge g^{-1}(y)) \underset{\substack{\downarrow \\ g \text{ invertible } \& \, g' < 0}}{=} 1 - F_X(g^{-1}(y))$$

$$\Rightarrow \underset{\substack{\| \\ f_Y(y)}}{\frac{d}{dy}[F_Y(y)]} = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = f_X(g^{-1}(y)) \underbrace{\left(-\frac{d}{dy}[g^{-1}(y)]\right)}_{\text{always negative}}$$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|$ $\rightarrow$ general rule

We can derive a less general but very useful corollary rule:

$Y = \underline{aX + c} \sim f_Y(y) = ?$ (shift and scale (shift by c, scale by a))

$a, c \in \mathbb{R}$, $\hookrightarrow g(x)$ invertible $\Rightarrow g^{-1}(y) = \frac{y-c}{a}$ $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{a}\right| = \frac{1}{|a|}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right) \frac{1}{|a|}$

$Y = aX \sim f_X\left(\frac{y}{a}\right) \frac{1}{|a|}$, $Y = X + c \sim f_X(y-c)$

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \ge 0}$

$Y = g(X) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^x - 1) = \underset{\substack{\uparrow \\ \text{invertible}}}{g(x)}$

$$y=\ln(e^x-1) \Rightarrow e^y=e^x-1$$
$$\Rightarrow e^y+1=e^x$$
$$\Rightarrow x=\ln(e^y+1) = g^{-1}(y)$$

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \left|\frac{d}{dy}\left[\ln(e^y+1)\right]\right| = \left|\frac{e^y}{e^y+1}\right| = \frac{e^y}{e^y+1}$$

$$f_Y(y) = f_X(\ln(e^y+1)) \cdot \frac{e^y}{e^y+1} = e^{-\ln(e^y+1)}\mathbb{1}_{\overbrace{\ln(e^y+1)\geq 0}^{\overbrace{e^y+1\geq 1}^{\overbrace{e^y\geq 0}^{y\in\mathbb{R}}}}} \cdot \frac{e^y}{e^y+1}$$
$$= \frac{1}{e^y+1} \cdot \frac{e^y}{e^y+1}$$
$$= \frac{e^y}{(e^y+1)^2} \cdot \frac{e^{-2y}}{e^{-2y}}$$
$$= \frac{e^{-y}}{(e^{-y}+1)^2} = \text{Logistic}(0,1)$$

standard logistic