Lecture - 17 11/11/2020

Consider rv's $X_1, X_2, \ldots, X_n$ iid but PMF/PDF is unknown but we know it has expectation μ and variance σ^2;

Let $T_n = X_1 + X_2 + \ldots + X_n$.

Let $\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n}$.

From 241, we know $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \sigma^2/n$

Let $Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \underbrace{\frac{\sqrt{n}}{\sigma}}_{a} \bar{X}_n + \underbrace{\frac{-\sqrt{n}}{\sigma}}_{b} \mu$

"$\bar{X}$ standardized"

$E[Z_n] = 0$
$Var[Z_n] = 1 = SD[Z_n]$

$$\phi_{T_n}(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) \ldots\ldots \phi_{X_n}(t) \overset{P1}{=} \phi_X(t)^n$$

$a = e^{\ln(a)}$

$$\phi_{\bar{X}_n}(t) \overset{P2}{=} \phi_T(1/n\, t) = \phi_X(t/n)^n$$

$$\phi_{Z_n}(t) = e^{itb}\,\phi_{\bar{X}_n}(at) = e^{-\frac{it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}}\,\phi_X\left(\frac{\sqrt{\cancel{n}}}{\sigma}\cdot\frac{t}{\cancel{n}_{\sqrt{n}}}\right)^n$$

$$= e^{-\frac{it\mu n}{\sigma\sqrt{n}}}\, e^{\ln\left(\phi_X(t/\sigma\sqrt{n})^n\right)}$$

$$= e^{\left(\frac{-it\mu\cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(t/\sigma\sqrt{n}\right)\right)\right)\frac{1/\cancel{n}}{1/n}}$$

$$= e^{\frac{\frac{-it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(t/\sigma\sqrt{n}\right)\right)}{1/n} \cdot \frac{t^2/\sigma^2}{t^2/\sigma^2}}$$

$$= e^{t^2/\sigma^2 \left(\frac{\ln\left(\phi_X\left(t/\sigma\sqrt{n}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{t^2/\sigma^2 n} \right)} = \phi_{Z_n}(t)$$

We want to investigate now $\lim_{n\to\infty} \phi_{Z_n}(t) = ?$

$$= e^{t^2/\sigma^2 \lim_{n\to\infty} \frac{\ln\left(\phi_X\left(t/\sigma\sqrt{n}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{t^2/\sigma^2 n}}$$

$$= e^{t^2/\sigma^2 \lim_{u\to 0} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

let $u = t/\sigma\sqrt{n}$ $\Rightarrow n \to \infty$ $\Rightarrow u \to 0$

$$= e^{t^2/2\sigma^2 \frac{\cancel{\phi_X(0)}^{1}\, \phi_X''(0) - \phi_X'(0)^2}{\cancel{\phi_X(0)}^{1}}}$$

P0

$$= e^{t^2/2\sigma^2 \left(\phi_X''(0) - \phi_X'(0)^2\right)}$$

P4

$$= e^{t^2/2\sigma^2 \left(i\, E[X^2] - (i\, E[X])^2\right)}$$

$$= e^{-t^2/2\sigma^2 \left(\overbrace{E[X^2] - E[X]^2}^{\sigma^2}\right)} = e^{-t^2/2} = \phi_Z(t)$$

P8

$\Longrightarrow Z_n \xrightarrow{d} Z$ where Z has chf $\phi_Z(t) = e^{-t^2/2}$.

$Z \sim f_Z(z) = ?$

Use P6 to find PDF of Z. Check $\phi_Z(t) \overset{?}{\in} L^1 \Rightarrow \underbrace{\int_{\mathbb{R}} e^{-t^2/2}\, dt = \sqrt{\pi} <}_{\text{Gaussian Integral}}$ YE

$$f_Z(z) = 1/2\pi \int_{\mathbb{R}} e^{-itz}\, \phi_Z(t)\, dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\, \underbrace{e^{-t^2/2}}_{i^2z^2/2}\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(itz + t^2/2)}\, dt$$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}\, iz}{2}\right)^2 - \left(\frac{\sqrt{2}\, iz}{2}\right)^2 = \frac{t^2}{2} + \frac{2\sqrt{2}\, iz}{2}\,\frac{t}{\sqrt{2}} + \cancel{\frac{i^2z^2}{2}} - \cancel{\frac{i^2z^2}{2}}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(t/\sqrt{2} + \sqrt{2}\, iz/2\right)^2} e^{-z^2/2}\, dt.$$

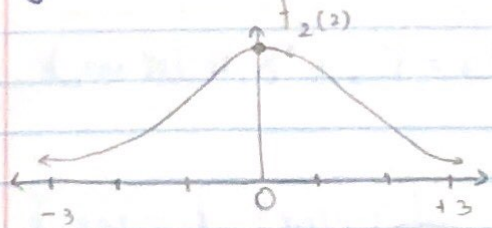
$$= \frac{1}{2\pi} e^{-z^2/2}\int_{\mathbb{R}} e^{-\left(t/\sqrt{2} + \sqrt{2}\, iz/2\right)^2}\, dt.$$

let $y = t/\sqrt{2} + \sqrt{2}\, iz/2 \Rightarrow dy/dt = 1/\sqrt{2}$, $t \to \infty \Rightarrow y \to \infty$, $t \to -\infty \Rightarrow y \to -\infty$

Gaussian Integral — "standard Normal rv"

$$= \frac{1}{2\pi} e^{-z^2/2}\int_{\mathbb{R}} e^{-y^2}\sqrt{2}\, dy = \frac{1}{2\pi} e^{-z^2/2}\sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = N(0,1)$$

This completes the proof of the "central limit theorem" (CLT), the crown jewel of a basic probability class, one of the most useful results that probability has given to the world at large.



AKA Laplace's Second Error Distribution. It is the most famous and widely-used error distribution on Earth.

$$\boxed{\text{CLT: } X_1, \dots, X_n \text{ iid mean } \mu, \text{ Variance } \sigma^2, \ \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)}$$

Let $\sigma > 0$

$Z \sim N(0,1)$, $X = \mu + \sigma Z \sim f_X(x) = ?$

$$\phi_X(t) \overset{(P2)}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu - \sigma^2 t^2/2}.$$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

$$E[Z] = \frac{\phi_Z'(0)}{i} = 0, \quad Var[Z] = E[Z^2] - \cancel{E[Z]}^2 = \frac{\phi_Z''(0)}{i^2} = 1 \checkmark$$

$$\phi_Z'(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -t e^{-t^2/2}, \quad \phi_Z''(t) = \frac{d}{dt}\left[-t e^{-t^2/2}\right] = -\left(-t^2 e^{-t^2/2} + e^{-t^2/2}\right)$$

$$= t^2 e^{-t^2/2} - e^{-t^2/2}$$

$$E[X] = E[\mu + \sigma Z] = \mu, \quad Var[X] = Var[\mu + \sigma Z] = \sigma^2,$$

$$SD[X] = \sigma$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep. of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim f_T(t) = ?$

$$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 - \sigma_1^2 t^2/2}\, e^{it\mu_2 - \sigma_2^2 t^2/2}$$

$$= e^{it(\mu_1+\mu_2) - (\sigma_1^2+\sigma_2^2)t^2/2} \overset{P1}{\Longrightarrow} T \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$ $\quad g^{-1}(y) = \ln(y)$ $\quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|}$

$$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \frac{1}{|y|}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} = \text{Log}\,N(\mu, \sigma^2)$$

Log-Normal distribution.