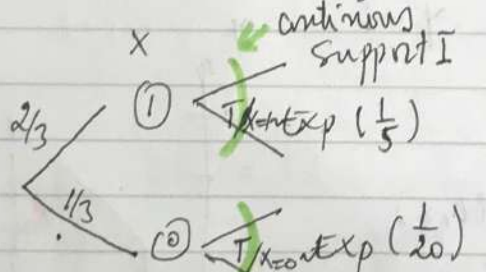
Wednesday October 28, 2020

Lecture 15

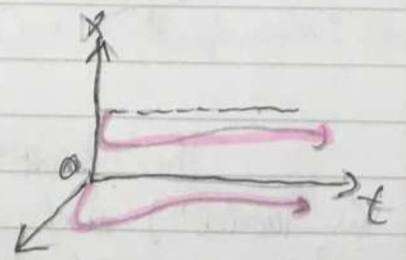
Mixture and compound distributions:

Consider a situation where 2/3 of the time there is fast internet speed so your downloads take $T \sim Exp(1/5) \Rightarrow E[T] = 5s$ and the other 1/3 of the time, there is internet traffic, so your downloads take $T \sim (1/20) \Rightarrow (1/20) \Rightarrow E[T] = 20s$. What is the distribution of the $X \sim Bern(2/3)$ and $X = 1$ corresponds to fast internet and $X = 0$ corresponds to slow internet. Let's draw a tree diagram:



$$h(x) = \int_{\mathbb{R}} h(x,y)\,dy = \sum_{y \in \mathbb{R}} h(x,y)$$

$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\, P_X(x)$$



$$= \sum_{x \in \{0,1\}} f_{T|X}(t,x)\, P_X(x) = f_{T|X}(t,0)\, P_X(0) + f_{T|X}(t,1)\, P_X(1) \quad f_{T,X}$$

$$= \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}$$

If download speed was $t = 25s$, what is the probability it is a slow internet day, i.e. $X = 0$? $X|T \sim Bern(?)$

$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, P_X(x)}{f_T(t)} \quad \text{"Bayes Rule"}$$

Bernoulli param $= P_{X|T}(1,t)$

$X|T \sim Bern(?)$ $\qquad$ $W \sim Bern(p)$, $p = P(W=1)$

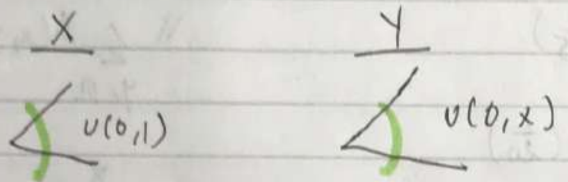
Bernoulli param $= P_{X|T}(1,t) = \dfrac{f_{T|X}(t,1)\,P_X(1)}{f_T(t)}$

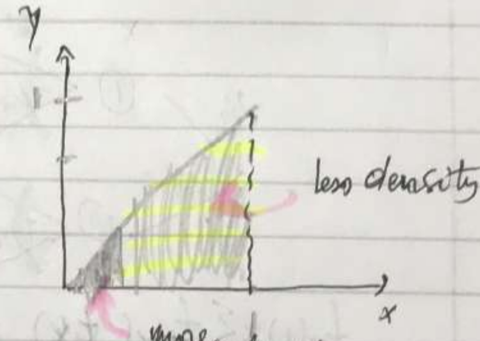
$$= \frac{\frac{1}{5}e^{-\frac{1}{5}t}\cdot\frac{2}{3}}{\frac{1}{20}e^{-\frac{1}{20}t}\cdot\frac{1}{3}+\frac{1}{5}e^{-\frac{1}{5}t}\cdot\frac{2}{3}}$$

$$P_{X|T}(0,25) = 1 - P_{X|T}(1,25) = 1 - \frac{\frac{1}{5}e^{-\frac{1}{5}\cdot 25}\cdot\frac{2}{3}}{\frac{1}{20}e^{-\frac{1}{20}\cdot 25}+\frac{1}{5}e^{-\frac{1}{5}\cdot 25}\cdot\frac{2}{3}} = 0.842$$

~~Example~~

$X \sim U(0,1)$, $Y|X = x \sim U(0,x) \Rightarrow Y \sim ?$

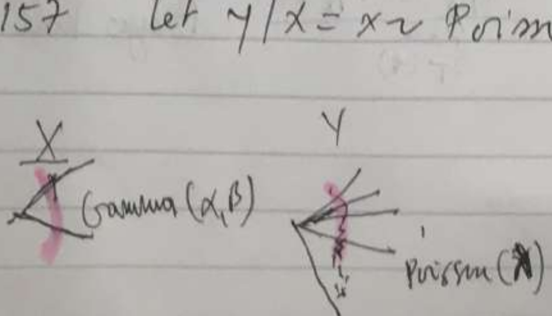




The first example featured T which was continuous (we call that the "model") and X which is descrete (we call that "mixing" distribution".
~~In the second example Y, the model is continuous and~~
Thus the unconditional distribution T is called a mixture "distribution".

In the second Y, model is continuous and X, the mixing distribution is also continuous and we call the unconditional distribution Y a "compound distribution".

P156 – 157 $\quad$ Let $Y|X = x \sim Poisson(x)$, $X \sim Gamma(\alpha,\beta)$, $Y \sim ?$

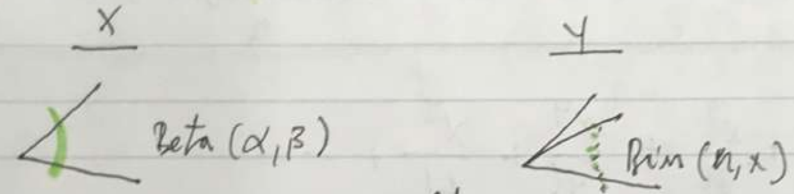


$$P_Y(y) = \int_{Supp[X]} P_{Y|X}(y,x)\, f_X^{old}(x)\,dx = \int_0^\infty \frac{e^{-x}x^y}{y!}\mathbb{1}_{y\in\mathbb{N}_0}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{y!}\mathbb{1}_{y\in\mathbb{N}_0}\underbrace{\int_0^\infty x^{y+\alpha-1}e^{-(\beta+1)x}\,dx}_{\text{lec 9 integral}} = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{y!}\mathbb{1}_{y\in\mathbb{N}_0}\frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}$$

$$\overset{HW}{=} \cdots = \text{Ext Neg Bin}\left(\alpha, \frac{\beta}{\beta+1}\right)$$ This is a more flexible count model than the poisson.

$Y \mid X = x \sim \text{Bin}(n, \underset{p}{x})$ where n is known, $X \sim \text{Beta}(\alpha,\beta)$



$$P_Y(y) = \int_{Supp[X]} p_{Y|X}(y,x)\, f_X^{old}(x)\,dx = \int_0^1 \binom{n}{y}x^y(1-x)^{n-y}\mathbb{1}_{y\in\{0,\dots,n\}}\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\binom{n}{y}\mathbb{1}_{y\in\{0,1,\dots,n\}}}{B(\alpha,\beta)}\underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}}_{B(y+\alpha,\, n-y+\beta)}\,dx = \frac{B(y+\alpha, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)}\mathbb{1}_{y\in\{0,\dots,n\}}$$

$Y \mid X = x \sim \text{Exp}(x)$, $X \sim \text{Gamma}(\alpha,\beta) \overset{HW}{\Rightarrow} Y \sim \text{Lomax}(\beta,\alpha)$

Which is a ~~more~~ flexible waiting time than the exponential

End of Midterm III material

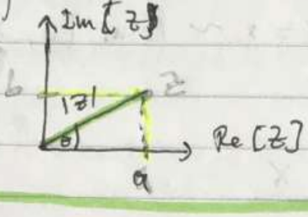
Start "Final"

Moment generating functions (mgf's) and Characteristic functions (chf's). To derive these, we need to review Complex / imaginary numbers. First define $i = \sqrt{-1}$ "imaginary"

Let $a, b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, Complex number

$Re[z] := a$, $Im[z] := b$, real component and imaginary component of a Complex #

$|z| = \sqrt{a^2+b^2}$, $Arc[z] := \theta \overset{\text{usually}}{=} arc(\frac{b}{a})$



$i^0 = 1$

$i^1 = i$ $\quad i^5 = i$

$i^2 = -1$ $\quad n \in \mathbb{N}$

$i^3 = -i$ $\quad$ and $i^{n} = i^{n}$ works

$i^4 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots -$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots -$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$i\sin(tx) = itx - i\frac{t^3x^3}{3!} + \frac{it^5x^5}{5!} - \cdots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} + \cdots - \cdots$$

Euler Formula

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \xRightarrow{tx=\pi} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$