Wednesday November 25 2020

Lecture 21

$\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right]$ for any vector rv X of dimension n.

Consider: $\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{i[t\,0\cdots0]\vec{X}}\right] = E\left[e^{itX_1}\right] = \phi_{X_1}(t)$

$$\overset{P_1/P_2}{\Longrightarrow} X_1 \sim f_{X_1}(x)$$

$$f_{X_1}(x) = \int\cdots\int f_{X_1,X_2,\dots,X_n}(x, u_1, u_2, \dots, u_{n-1})\, du_1 \cdots du_{n-1}$$

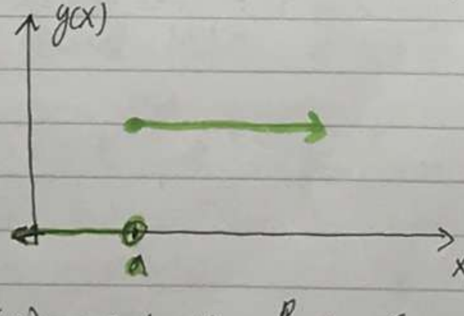
e.g. $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$, $X_1 \sim f_{X_1}(x) = ?$

$$\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = e^{i[t\,0\cdots0]\vec{\mu} - \frac{1}{2}[t\,0\cdots0]\Sigma\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = e^{it\mu_1 - \frac{t}{2}[t\,0\cdots0]\begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \vdots \\ \sigma_{1n} \end{bmatrix}}$$

$$= e^{it\mu_1 - \frac{t^2\sigma_1^2}{2}} = \phi_{X_1}(t) \overset{P_1}{\Longrightarrow} X_1 \sim N(\mu, \sigma_1^2)$$

Assume X is a rv with nonnegative support i.e Supp[X] ≥ 0 and has finite expectation. Let $a > 0$, a constant. Consider the following function:

$$g(x) = a\,\mathbb{1}_{x\geq a}$$



Is $a\,\mathbb{1}_{x\geq a} \leq x$? Two Cases

- If $x < a$: $a\,\mathbb{1}_{x\geq a} = a(0) = 0 \leq x$ have Supp$[X] \geq 0$
- If $x \geq a$: $a\,\mathbb{1}_{x\geq a} = a(1) = a \leq x$ have by case assumption.

$$\Rightarrow a\,\mathbb{1}_{x\geq a} \leq x$$

Let's take the expectation of both sides:

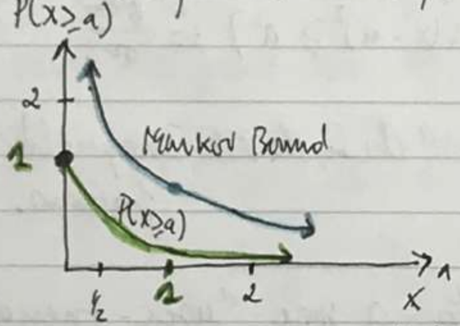
$$E[a\mathbb{1}_{X\geq a}] \leq E[X]$$

$$\mathbb{1}_{X\geq a} \sim \begin{cases} 1 & \text{wp } P(X\geq a) \\ 0 & \text{o/t} \end{cases} = \text{Bern}(P(X\geq a)).$$

$$\Rightarrow aE[\mathbb{1}_{X\geq a}] \leq \mu \Rightarrow aP(X\geq a) \leq \mu$$

$\Rightarrow P(X\geq a) \leq \frac{\mu}{a}$ this is called "Markov's Inequality" and it's very famous.

For example $X \sim Exp(1) = e^{-x} \Rightarrow P(X\geq a) = 1 - F_X(a) = e^{-a}$



We see here is that the Markov Bound is very "crude" meaning very approximate, much bigger than the truth.

a	$P(X\geq a)$	Markov Bound	Chebyshev Bound	Chernoff Bound
2	0.1353	0.5	1	0.73576
5	0.0067	0.2	0.0635	0.05158
10	0.00004	0.1	0.0123	0.00123

We will now prove many, many corollary of the Markov Inequality:

- let $b = a\mu$ $P(X\geq b) \leq \frac{\mu}{a} \Rightarrow P(X\geq a\mu) \leq \frac{1}{a}$
- let h be a monotonically increasing function, $Y = h(X)$
  $$P(Y\geq h(a)) \leq \frac{E[Y]}{h(a)} \Rightarrow P(h(X)\geq h(a)) \leq \frac{E[h(X)]}{h(a)} \Rightarrow P(X\geq a) \leq \frac{E[h(X)]}{h(a)}$$
- let X be continuous in addition to nonnegative

  let $a = \text{Quantile}[X, p] = F_X^{-1}(p)$

  $$P(X\geq F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow 1 - \cancel{F_X}(F_X^{-1}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$$

$\Rightarrow 1 - P \le \frac{\mu}{F_x^{-1}(P)} \Rightarrow \underbrace{F_x^{-1}(P)}_{Q[X,P]} \le \frac{\mu}{1-P}$ e.g. $Med[X] \le 2\mu$

- let X be any rv $\Rightarrow |x|$ is a nonnegative rv. $P(|x| \ge a) \le \frac{E[|x|]}{a}$

- Let x be any rv with finite, σ^2. Let $Y = (x-\mu)^2 \Rightarrow Y$ is a nonnegative rv

$$P(Y \ge b) \le \frac{E[Y]}{b} \Rightarrow P((x-\mu)^2 \ge b) \le \frac{E[(x-\mu)^2]}{b}$$ ← definition of variance

$$\Rightarrow P((x-\mu)^2 \ge b) \le \frac{\sigma^2}{b} \xRightarrow{\text{let } b = a^2} P((x-\mu)^2 \ge a^2) \le \frac{\sigma^2}{a^2}$$

$\Rightarrow P(|x-\mu| \ge a) \le \frac{\sigma^2}{a^2}$ This is "Chebyshev's Inequality" and it's also very famous.

Let's manipulate this to get it into a more "user-friendly" form:

Assume X is nonnegative:

disjoint terms

$$P(|x-\mu| \ge a) = P(x-\mu \ge a \cup -(x-\mu) \ge a) = P(x-\mu \ge a) + P(-(x-\mu) \ge a)$$

$$= P(X \ge \mu + a) + P(X \le \mu - a) \xRightarrow{a \ge \mu} \overset{a \ge \mu}{=} P(X \ge \mu + a) + P(X \le \text{negative } \#)$$

0

$$\overset{\text{let } b = \mu + a}{=} P(X \ge b) \le \frac{\sigma^2}{(b-\mu)^2}$$

- Let x be an rv. Let $Y = e^{tx} \Rightarrow Y$ is a nonnegative rv for all t.

$$P(Y \ge b) \le \frac{E[Y]}{b} \Rightarrow P(e^{tx} \ge b) \le \frac{E[e^{tx}]}{b}$$ ← the moment generating function for x, $M_x(t)$

$$\Rightarrow P(e^{tx} \ge b) \le \frac{M_x(t)}{b} \xRightarrow{\text{let } b = e^{ta}} P(e^{tx} \ge e^{ta}) \le e^{-ta} M_x(t)$$

$$\Rightarrow P(tx \ge ta) \le e^{-ta} M_x(t)$$

If I know the PDF or PMF, then I know analytically or can numerically compute the CDF which means I know the tail exactly or within small numerical error! So it really is only useful if you're in a situation where you only have the MGF and not the PDF/PMF.