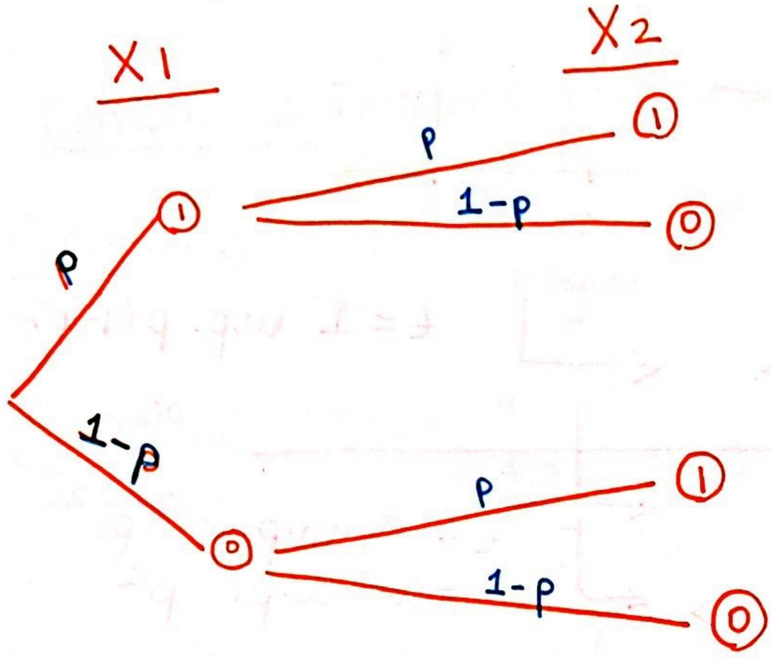
Lecture 2

8/31/20



$p(\vec{x})^{(x_1,x_2)}$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

$$\boxed{p(T=1) = p_{x_1,x_1}\langle 1,0\rangle + p_{x_1,x_2}(0,1)}$$

$$T \sim \begin{cases} 2 \text{ w.p. } p^2 \\ 1 \text{ w.p. } 2p(1-p) \\ 0 \text{ w.p. } (1-p)^2 \end{cases}$$

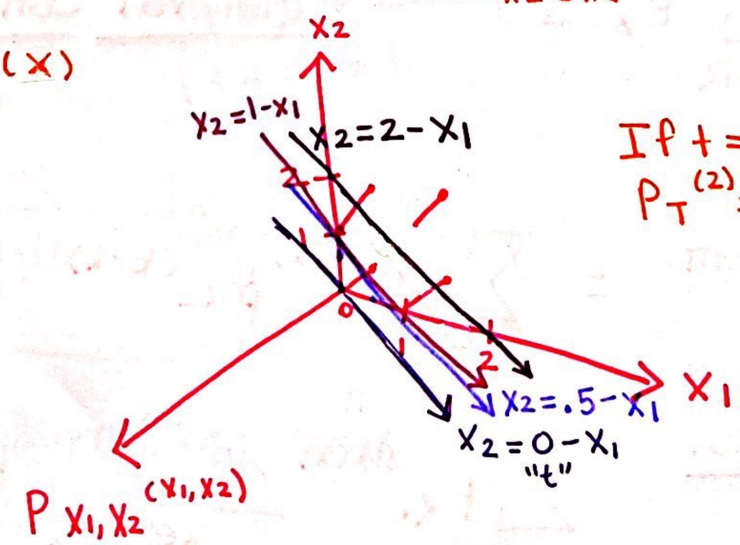
$$p(T) = p(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{x_1,x_2}^{(x_1,x_2)} \mathbb{1}_{x_1+x_2=t}$$

$$\parallel$$

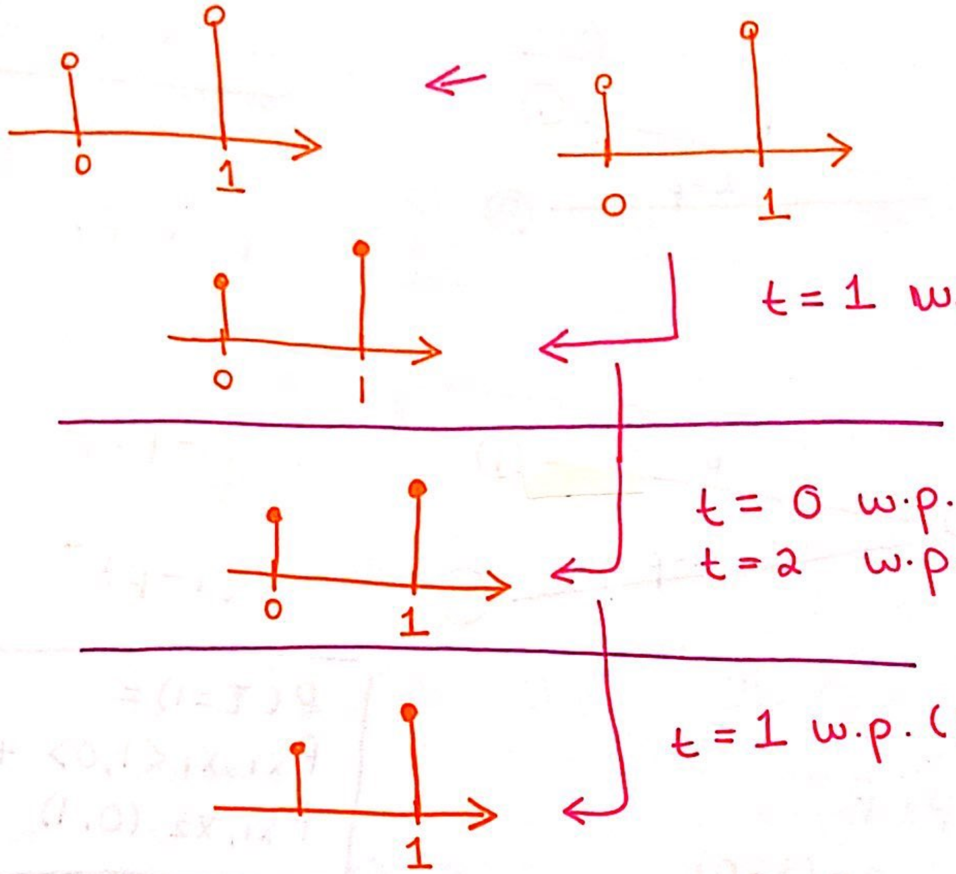
$$p_{x_1}(x) * p_{x_2}(x)$$

$x_1 + x_2 = t$
$\downarrow$
$x_2 = t - x_1$

If $t = 2$,
$$P_T^{(2)} = \sum_{x_1 \in \mathbb{R}} P_{x_1 x_2}^{(x_1, 2-x_1)}$$



"Convolve" means to "roll or coil together / entwine"



$t = 1$ w.p. $p(1-p)$

$t = 0$ w.p. $(1-p)^2$
$t = 2$ w.p. p^2

$t = 1$ w.p. $(1-p)p$

(adding them all together) P_T

$$P_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1, x_2) \, \mathbb{1}_{x_2 = t - x_1}$$

$$= \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1, t-x_1)$$

<u>General Convolution Formula</u>

If X_1, X_2 are independent.

$$= \sum_{x_1 \in \mathbb{R}} P_{X_1}(x) \, P_{X_2}(t-x)$$

<u>Convolution formula for independent r.v.s.</u>

$$\parallel$$

$$\sum_{x=\mathbb{R}} P_{X_1}^{old}(x) \, \mathbb{1}_{x \in Supp[X_1]} \, P_{X_2}^{old}(t-x) \, \mathbb{1}_{t-x \in Supp[X_2]}$$

$$\parallel$$

$$\overset{\shortparallel}{\sum_{x \in Supp[X_1]}} p_{X_1}^{old}(x)\, p_{X_2}^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}$$

$X_1, X_2 \overset{iid}{\sim}$

Convolusion Formulas for iid's rvs.

$$= \sum_{x\in\mathbb{R}} p(x)p(t-x) = \sum_{x\in\mathbb{R}} p^{old}(x)\,\mathbb{1}_{x\in Supp[x]}\, p^{old}(t-x)\,\mathbb{1}_{t-x\in Supp[x]} =$$

$$= \sum_{x\in Supp[x]} p^{old}(x)\, p^{old}(t-x)\,\mathbb{1}_{t-x\in Supp[x]}$$

$p_T(t)$

$$T_2 \sim \overset{\shortparallel}{\sum_{x\in\{0,1\}}} p^x(1-p)^{1-x} p^{t-x}(1-p)^{1-t+x}\, \mathbb{1}_{t-x\in\{0,1\}} = p^t \cancel{(1-p)}^{2-t}$$

" $+x$ $+x$ $+x$ "

$t \in \{x, x+1\}$

$$= p^t(1-p)^{2-t} \sum_{x\in\{0,1\}} \mathbb{1}_{t\in\{x,x+1\}}$$

$$= p^t(1-p)^{2-t}\left(\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}}\right)$$

$$\begin{cases} t=0 \Rightarrow 1 \\ t=1 \Rightarrow 2 \\ t=2 \Rightarrow 1 \end{cases} = \binom{2}{t}$$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}\,\mathbb{1}_{n\in\mathbb{N}}\,\mathbb{1}_{k\in\{0,1,\dots,n\}}$$

$$= \binom{2}{t} p^t(1-p)^{2-t} = Binom(2,p),\quad Supp[T_2] = \{0,1,2\}$$

In general, $Supp[T] = Supp[X_1] + Supp[X_2]$,

$A+B := \{a+b : a \in A, b \in B\}$

$$P_{T_n}(t) = \sum_{x \in \mathbb{R}} P(x)\,P(t-x) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x} =$$

$$X_1, X_2 \overset{iid}{\sim} Bern(p) = \binom{1}{x} p^x (1-p)^{1-x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x}\binom{1}{t-x} = p^t (1-p)^{2-t} \left(\cancel{\binom{1}{0}}^{1} \binom{1}{t} + \cancel{\binom{1}{1}}^{1} \binom{1}{t-1} \right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t} \quad ✓ = Binom(2,p)$$

Recall Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$ $\quad T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$

$$P_{T_3}(t) = \sum_{x \in Supp[X_3]} p^{old}_{(X_3)}(x)\, P_{T_2}(t-x) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}\right)$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x} = p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right) =$$

$$= \binom{3}{t} p^t (1-p)^{3-t} = Binom(3,p)$$

$\Rightarrow$

$X_1, X_2 \overset{iid}{\sim} \text{Binom}(n,p)$ $\qquad T = X_1 + X_2 \sim ?$

↳ on the h.w. $:= \binom{n}{x} p^{x}(1-p)^{n-x}$

$$P_T(t) = \sum_{x \in \mathbb{R}} p(x)\,p(t-x) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^{x}(1-p)^{n-x} \binom{n}{t-x} p^{t-x}(1-p)^{n-t+x}$$

$$= p^{t}(1-p)^{2n-t} \underbrace{\sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t=x}}_{\text{Vandermonde's identity gives us}} = \binom{2n}{x} p^{t}(1-p)^{2n-t} = \text{Binom}(2n,p)$$