Lecture 17 consider $X_1, \dots, X_n$ iid rvs of unknown PMF/PDF but we know it has expectation mu and variance σ^2 (both finite).

let $T_n := X_1 + \dots + X_n$, $E[T_n] = n\mu$, $Var[T_n] = n\sigma^2$

let $\bar{X}_n := \frac{X_1 + \dots + X_n}{n} = \frac{T_n}{n}$, $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = Var\left[\frac{T_n}{n}\right] = \frac{1}{n^2} Var[T_n] = \frac{\sigma^2}{n}$

let $Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu$, $E[Z_n] = 0$, $Var(Z_n) = Var\left(\frac{\sqrt{n}}{\sigma}\bar{X}_n\right)$

$= \frac{n}{\sigma^2} \cdot \frac{\sigma^2}{n} = 1 = SD[Z_n]$

"$\bar{X}_n$ standardized"

$$\phi_{T_n}(t) \overset{P3}{=} \phi_{X_1}(t) \cdots \phi_{X_n}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \dots + \alpha_n} \text{ indep.} \Rightarrow T_n \sim Gamma(\alpha_1 + \dots + \alpha_n, \beta)$$

$$\overset{iid}{\Rightarrow} \left(\phi_X(t)\right)^n$$

$$\phi_{\bar{X}_n}(t) \overset{P2}{=} \phi_{T_n}(t/n) \overset{iid}{=} \phi_X(t/n)^n$$

$$\phi_{Z_n}(t) \overset{P2}{=} e^{\frac{-it\mu\sqrt{n}}{\sigma}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right) \overset{iid}{=} e^{\frac{-it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n = e^{\frac{-it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)}$$

$$= e^{\left(-\frac{it\mu \cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\right)\frac{\cancel{1/n}}{1/n}} = e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{1/n} \cdot \frac{t^2/\sigma^2}{t^2/\sigma^2}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{t^2/n\sigma^2}\right)} = \phi_{Z_n}(t)$$

We want to examine $\lim_{n\to\infty} \phi_{Z_n}(t)$ and if we find its limiting chf, $\phi_Z(t)$, we can use P8 to show that $Z_n \xrightarrow{d} Z \Rightarrow Z_n \overset{d}{\approx} Z$.

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{t^2/\sigma^2 \lim_{n\to\infty} \frac{\ln(\phi_X(t/\sigma\sqrt{n})) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}} = e^{\frac{t^2}{\sigma^2}\lim_{u\to 0} \frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

let $u = \frac{t}{\sqrt{n}\sigma}$. If $n \to \infty \Rightarrow u \to 0$

$$\text{L'Hopital} \overset{\Downarrow}{=} e^{t^2/2\sigma^2 \lim_{u\to 0} \frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}} = e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0} \frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}}$$

$$= e^{t^2/2\sigma^2 \frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2}} \overset{P0}{=} e^{\frac{t^2}{2\sigma^2}(\phi_X''(0) - \phi_X'(0)^2)}$$

$$= e^{t^2/2\sigma^2\left(i^2 E[X^2] - (i\,E[X])^2\right)} = e^{-t^2/2\cancel{\sigma^2}\overbrace{\left(E[X^2] - E[X]^2\right)}^{\sigma^2}} = e^{-t^2/2} = \phi_Z(t)$$

L'Hopital $\overset{\downarrow}{=}$ $e^{t^2/2\sigma^2} \lim_{u\to 0} \dfrac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u} = e^{\frac{t^2}{2\sigma^2}} \lim_{u\to 0} \dfrac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}$

$= e^{t^2/2\sigma^2} \dfrac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)^2} \overset{P0}{=} e^{\frac{t^2}{2\sigma^2}} (\phi_X''(0) - \phi_X'(0)^2)$

$\overset{P4}{=} e^{t^2/2\sigma^2}\left(i^2 E[X^2] - (i E[X])^2\right) = e^{-t^2/2\sigma^2}\underbrace{\left(E[X^2] - E[X]^2\right)}_{\sigma^2} = e^{-t^2/2} = \phi_Z(t)$

Is $e^{-t^2/2}$ in L^1? $\int_{\mathbb{R}} e^{-t^2/2} dt = \sqrt{\pi} < \infty$. ✓ ← Gaussian Integral

Now we can use P6 to invert the chf of Z to get the pdf of Z.

$f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz} \phi_Z(t) dt = \frac{1}{2\pi}\int e^{-(itz + t^2/2)} dz$

$\dfrac{t^2}{2} + itz = \left(\dfrac{t}{\sqrt{2}} + \dfrac{\sqrt{2}\, i z}{2}\right)^2 - \left(\dfrac{\sqrt{2}\, i z}{2}\right)^2$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(t/\sqrt{2} + \frac{\sqrt{2} i z}{2}\right)^2} e^{-z^2/2} dt = \frac{1}{2\pi} e^{-z^2/2} \int_{\mathbb{R}} e^{-\left(t/\sqrt{2} + \frac{\sqrt{2} i z}{2}\right)^2} dt$

let $y = t/\sqrt{2} + \frac{\sqrt{2} i z}{2} \Rightarrow dy/dt = 1/\sqrt{2} \Rightarrow dt = \sqrt{2}\, dy$, $t \to -\infty \Rightarrow y \to -\infty$
$t \to \infty, y \to \infty$

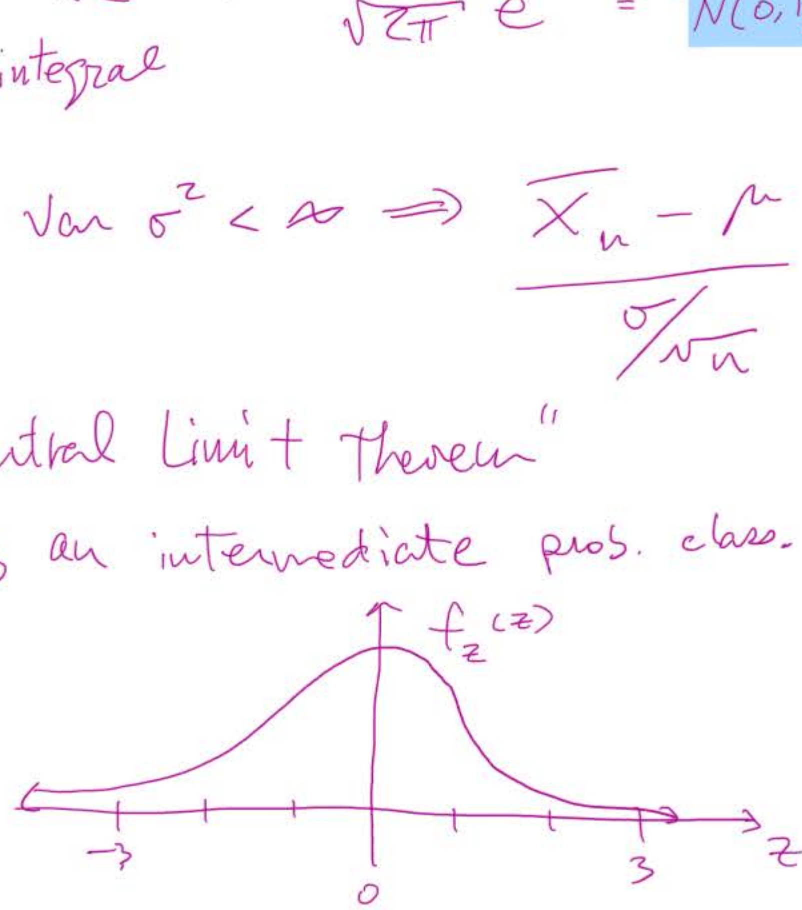
$= \frac{1}{2\pi} e^{-z^2/2} \int_{\mathbb{R}} e^{-y^2} \sqrt{2}\, dy = \frac{1}{2\pi} e^{-z^2/2} \sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} =$ $Z \sim N(0,1)$

gaussian integral

$\Rightarrow X_1, \dots, X_n \overset{iid}{\sim}$ mean μ, Var $\sigma^2 < \infty \Rightarrow \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$

This fact is called the "Central Limit Theorem"
and it's the crown jewel of an intermediate prob. class.

$Z \sim f_Z(z) = N(0,1)$



It's called the "Gaussian dist." → but Laplace discovered it and
[illegible]

It's called the "Gaussian dist." $\rightarrow$ but Laplace discovered it and called it the "second law of errors". It's actually the most common error dist. in the world. $\rightarrow$ This makes sense!

$$E[Z] \overset{P4}{=} i\,\phi_Z'(0) = 0 \checkmark$$

$$\phi_Z'(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -t e^{-\frac{t^2}{2}}, \quad \phi_Z''(t) = -\frac{d}{dt}\left[t e^{-t^2/2}\right]$$

$$= -\left(-t^2 e^{-t^2/2} + e^{-t^2/2}\right)$$

$$Var[Z] = E[Z^2] - \cancel{E[Z]}^{\,0} = 0 \overset{P4}{=} i^2 \phi_Z''(0) = --1 = 1 = SD(Z).$$

$X = \mu + \sigma Z \sim f_X(x) = ?\ (\sigma > 0)$

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{(-1/2\sigma^2)(x-\mu)^2} = N(\mu, \sigma^2)$$

$$E[X] = \mu + \sigma E[Z] = \mu, \quad Var[X] = Var[\mu + \sigma Z] = \sigma^2$$

$$\phi_X(t) \overset{P2}{=} e^{it\mu} \phi_Z(\sigma t) = e^{it\mu - \sigma^2 t^2/2}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim ?$

$$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 - \sigma_1^2 t^2/2}\, e^{it\mu_2 - \sigma_2^2 t^2/2}$$

$$= e^{it(\mu_1 + \mu_2) - \frac{t^2}{2}(\sigma_1^2 + \sigma_2^2)} \overset{P1}{\Longrightarrow} X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$ $\quad g^{-1}(y) = \ln(y)$, $\left|\frac{d}{dy}\left(g^{-1}(y)\right)\right| = \frac{1}{|y|}$

$$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2} \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-1/2\sigma^2(\ln(y) - \mu)^2} = \text{LogN}(\mu, \sigma^2)$$

Log-Normal model