Lecture 05
09/14/2020
Math 621

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$X \sim \text{Multin}_2(n, \vec{p})$

$\text{Deg}(n-x_2)$ $p_{X_1|X_2}(x_1, x_2) = P(X_1 = x_1 \mid X_2 = x_2) = \dfrac{P(X_1, X_2)}{P(X_2)}$

Degenerate $(n-x_2)$

Last time: $P(X_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

$$= \frac{\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} \underbrace{(1-p_2)}_{p_1}^{n-x_2}}$$

Define; $J_n = \{0, 1, \ldots, n\}$

$$= \frac{\frac{\cancel{n!}}{x_1! \cancel{x_2!}} \mathbb{1}_{x_1+x_2=n}^{\,x_1 = n-x_2} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} \cancel{p_2^{x_2}}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \mathbb{1}_{x_2 \in J_n} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

Define:

$$\mathbb{1}_A^u = \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \text{undefined} & \text{if } A^c \end{cases}$$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{\substack{= \\ 1 \text{ if} \\ x_1 = n-x_2}} \mathbb{1}_{x_1 = n-x_2} \underbrace{\mathbb{1}_{x_1 \in J_n}}_{\substack{1 \text{ if} \\ x_1 = n-x_2}} \underbrace{p_1^{x_1+x_2-n}}_{\substack{= \\ 1 \text{ if} \\ x_1 = n-x_2}} \mathbb{1}_{x_2 \in J_n}^u$$

(is either 0 or 1) Degenerate

$= \text{Deg}(n-x_2)\, \mathbb{1}_{x_2 \in J_n}^u$

Recall 24)

$P(A|B) = \dfrac{P(A,B)}{P(B)}$ if $P(B) = 0$

then $P(A|B) =$ undefined

Let's generalize this conditional probability a little bit: $\vec{X} \sim \text{Multin}_k(n, \vec{p})$

$$P_{\vec{X}_{-j} | X_j}(\vec{x}_{-j}, x_j) = \frac{P_{\vec{X}}(\vec{x})}{P_{X_j}(x_j)}$$

$$= \text{Multin}_{k-1}(n - x_j, ?)$$

$\vec{X}_{-j}$: this is the vector w/o the jth component

$$= \frac{\text{Multin}_k(n, \vec{p})}{\text{Bin}(n, p_j)} \Rightarrow \frac{\binom{n}{x_1, \ldots x_j, \ldots, x_k} p_1^{x_1} \cdots \cancel{p_j^{x_j}} \cdots p_k^{x_k}}{\binom{n}{x_j} \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}}$$

$$= \frac{\frac{\cancel{n!}}{x_1! \cdots \cancel{x_j!} \cdots x_k!} \mathbb{1}_{\overbrace{x_1 + \cdots x_j + \cdots x_k = n}^{x_1 + \cdots + x_{j+1} + \cdots x_k = n - x_j}} \mathbb{1}_{x_1 \in \mathbb{J}_n} \mathbb{1}_{x_j \in \mathbb{J}_n} \cdots \mathbb{1}_{x_k \in \mathbb{J}_n} \, p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_n}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)!} \mathbb{1}_{x_j \in \mathbb{J}_n} (1-p_j)^{n-x_j}}$$

Note:

$$p_1 + \cdots + p_k = 1$$

$$p_1 + \cdots p_{j-1} + p_{j+1} + \cdots + p_k = 1 - p_j$$

divide both sides by $1 - p_j$

$$\Rightarrow \underbrace{\frac{p_1}{1-p_j}}_{p_1'} + \cdots + \underbrace{\frac{p_{j-1}}{1-p_j}}_{p_{j-1}'} + \underbrace{\frac{p_{j+1}}{1-p_j}}_{p_{j+1}'} + \cdots + \underbrace{\frac{p_k}{1-p_j}}_{p_k'} = 1$$

Note:

$$n - x_j = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k$$

Otherwise probability is zero.

Let $n' = n - x_j$

$$= \frac{n'!}{x_1! \cdots x_{j-1}! \, x_{j+1}! \cdots x_k!} \mathbb{1}_{x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k = n'} \mathbb{1}_{x_1 \in \mathbb{J}_n} \cdots \mathbb{1}_{x_{j-1} \in \mathbb{J}_n} \cdots \mathbb{1}_{x_k \in \mathbb{J}_n} \cdot$$

$$\frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k}} \cdot \mathbb{1}^{u}_{x_j \in \mathbb{J}_n}$$

$$p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}} p_{j+1}'^{x_{j+1}} \cdots p_k'^{x_k}$$

$$\binom{n'}{x_1, \cdots x_{j-1}, x_{j+1}, \cdots x_k}$$

$$= \text{Multin}_{k-1}(n', \vec{p}') \, \mathbb{1}^{u}_{x_j \in \mathbb{J}_n}$$

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Expected value

If $\vec{X} \sim \text{Multin}_k(n, \vec{p})$, what is $E[\vec{X}]$?
$Var[\vec{X}]$?

variance

Review 241:

Let $X_1, \cdots, X_n$ be RV's & $a, c \in \mathbb{R}$ if

$E[aX + c] = aE[X] + c$ identically distributed

$E[\Sigma X_i] = \Sigma E[X_i] = n\mu$ ← mu

if independent

$E[\Pi X_i] = \Pi E[X_i]$

$\sigma^2 = Var[X] = E[(X-\mu)^2]$

$\sigma = \sqrt{Var[X]} \Leftarrow SD[X]$ ← Standard deviation

$= E[X^2] - \mu^2$

$Var[X_1 + X_2] = E[((X_1 + X_2) - (\mu_1 + \mu_2))^2]$

$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2\mu_1\mu_2]$

$= E[X_1^2] + E[X_2]^2 + \mu_1^2 + \mu_2^2 + 2E[X_1 X_2]$
$\quad - 2\mu_1^2 - 2\mu_1\mu_2 - \cancel{2\mu_1\mu_2} - 2\mu_2^2 + \cancel{2\mu_1\mu_2}$

$= \sigma_1^2 + \cancel{\mu_1^2} + \sigma_2^2 + \cancel{\mu_2^2} + \cancel{\mu_1^2} + \cancel{\mu_2^2} +$
$\quad 2E[X_1 X_2] - \cancel{2\mu_1^2} - 2\mu_1\mu_2 - \cancel{2\mu_2^2}$

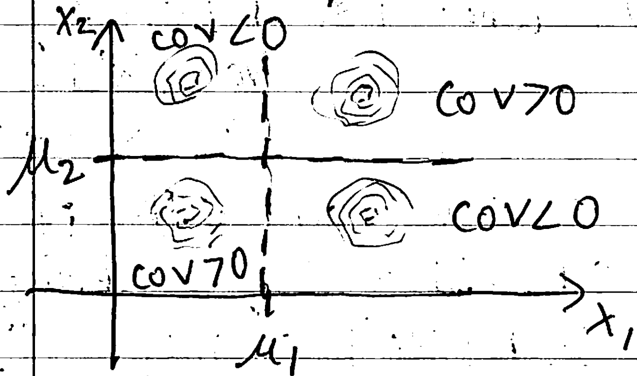
$= \sigma_1^2 + \sigma_2^2 + 2[\underbrace{E[X_1 X_2] - \mu_1\mu_2}_{Cov(X_1, X_2)}]$

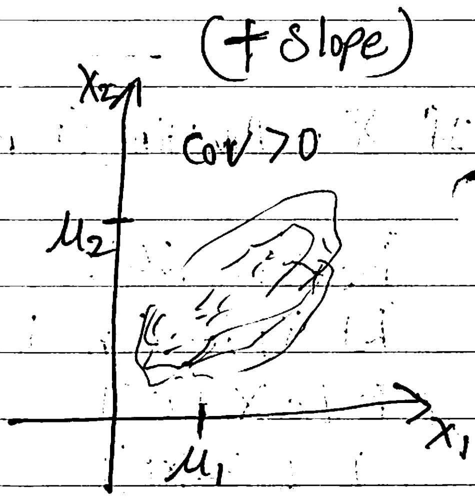
$\Rightarrow$ Convariance of X_1 with X_2.

$= \sigma_1^2 + \sigma_2^2 + 2\underbrace{cov(X_1, X_2)}_{\sigma_{(1,2)}}$

$= \sigma_1^2 + \sigma_2^2$

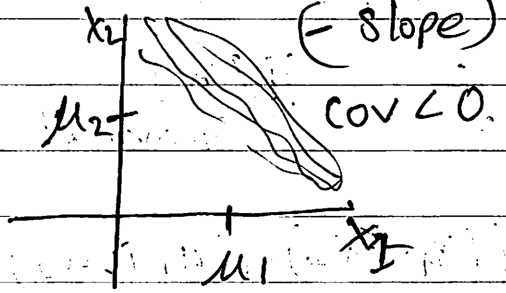
if X_1, X_2 independent





Homework:

$Cov[X_1, X_2] = E[(X_1 - \mu_1)(X_2 - \mu_2)]$



Covariance Rules:

$\text{cov}[x, x] = \sigma^2$

$\text{cov}[x_1, x_2] = \text{cov}[x_2, x_1]$

$\text{cov}[x_1 + x_2 + x_3] = \text{cov}[x_1, x_3] + \text{cov}[x_2, x_3]$

$\text{cov}[a_1 x_1, a_2 x_2] = a_1 a_2 \sigma_{12} \Rightarrow \sigma_{1,2}$

$\text{Var}[x_1 + \cdots + x_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}[x_i + x_j]$

$$\vec{\mu} = E[\vec{x}] = \begin{bmatrix} E[x_1] \\ \vdots \\ E[x_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}$$

$$\text{let } m = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix}$$

$$E[m] = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \vdots \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

$k \times k$ matrix

$(k \times 1)(1 \times k) = k \times k$

outer products

$$\Sigma = \text{Var}[\vec{x}] = E[\vec{x}\vec{x}^T] - \vec{\mu}\vec{\mu}^T$$

$$= \begin{bmatrix} \text{Var}[x_1] & \text{Cov}[x_1, x_2] & \cdots & \text{Cov}[x_1, x_k] \\ \text{Cov}[x_2, x_1] & \text{Var}[x_2] & \cdots & \text{Cov}[x_2, x_k] \\ \vdots & \vdots & & \vdots \\ \text{cov}[x_k, x_1] & \text{cov}[x_k, x_2] & \cdots & \text{Var}[x_k] \end{bmatrix}$$

$\Uparrow$ Capital letter Sigma

$k \times k$ matrix

Variance - covariance matrix / or, (var cov matrix) and it is Symmetric.

If $X_1, \ldots, X_K$ are independent,
What is the varcov matrix?

$$\Sigma = \operatorname{diag}\{\sigma_1^2, \ldots, \sigma_K^2\}$$

$$= \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_K^2 \end{bmatrix}$$

Rules about vector rv expectations:

$$E[aX + \vec{c}] = \begin{bmatrix} a\mu_1 + c_1 \\ a\mu_2 + c_2 \\ \vdots \\ a\mu_K + c_K \end{bmatrix} = a\vec{\mu} + \vec{c}$$

$$E[\vec{a}^T X] = E[a_1 X_1 + \cdots + a_K X_K]$$

$$= a_1\mu_1 + \cdots + a_K\mu_K = \vec{a}^T\vec{\mu}$$

Recall Linear Alg:

$$(\vec{V_1}^T \vec{V_2})^T = \vec{V_2}^T \vec{V_1} = \vec{V_1}^T \vec{V_2}$$