M368

11/9

Define $L^1 := \{f : \int_{\mathbb{R}} |f(x)|dx < \infty\}$ "L^1 integrable" or "absolutely integrable" functions.

are all pdfs in L^1? try $\int_{-\infty}^{\infty} |\lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}| dx = \lambda \int_0^{\infty} e^{-\lambda x} dx = 1$
Yes.

if $f \in L^1 \Rightarrow \exists \hat{f}$, the "Fourier Transform" of f: $\hat{f}(\omega) := \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x)dx$
$= \mathcal{F}[f]$

"forward fourier transform operator" aka "Fourier analysis"

If $\hat{f} \in L^1 \Rightarrow$ then we can invert/reverse the Fourier transform via the "inverse/reverse Fourier transform operator" to get the original f back aka "Fourier synthesis":

$f(x) = \int_{\mathbb{R}} e^{i2\pi x\omega} \hat{f}(\omega)d\omega$

Fourier inversion thm: if $f, \hat{f}$ are in L^1, then f and $\hat{f}$ are 1:1, $f(x)$ is known as the "time domain" and $\hat{f}(\omega)$ is known as the "frequency domain". $f(x)$ can be decomposed into a sum of sines and cosines w/frequencies ω, amplitudes given by $|\hat{f}(\omega)|$ and phase shifts given by $Arg[\hat{f}(\omega)]$.

Let X be a r.v. Define the characteristic function chf:

$$\phi_X(t) := E[e^{itX}] = \begin{cases} \int_{\mathbb{R}} e^{itx} f_X(x)dx & \text{if continuous} \\ \sum_{x \in \mathbb{R}} e^{itx} p_X(x) & \text{if discrete} \end{cases}$$

the chf is the Fourier transformation in a different unit $t = -2\pi\omega$.

Properties of the chf:

- P0: $\phi_X(0) := E[e^{i(0)X}] = E[e^0] = 1$ for all rv's
- P1: $\phi_X(t) = \phi_Y(t) \Leftrightarrow X \overset{d}{=} Y$
- P2: $Y = aX + b$ for $a, b \in \mathbb{R}$

  $\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{iatX} e^{itb}] = e^{itb} E[e^{i\overbrace{at}^{t'}X}] = e^{itb}\phi_X(t') = e^{itb}\phi_X(at)$
- P3: $X_1, X_2 \overset{iid}{\sim}$ and $T = X_1 + X_2$

  $\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}] = E[e^{itX_1}]E[e^{itX_2}] = \phi_{X_1}(t)\phi_{X_2}(t)$
- P4: "moment generation" ↙ conditions are satisfied to interchange differentiation and integration

  $\phi_X'(t) = \frac{d}{dt}[E[e^{itX}]] = E[\frac{d}{dt}[e^{itX}]] = E[iXe^{itX}]$

  $\phi_X'(0) = E[iXe^{i(0)X}] = iE[X] \Rightarrow E[X] = \phi_X'(0)/i$

  $\phi_X''(t) = \frac{d}{dt}[E[iXe^{itX}]] = E[iX\frac{d}{dt}[e^{itX}]] = E[i^2X^2e^{itX}] \Rightarrow E[X^2] = \frac{\phi_X''(0)}{i^2}$

  $\Rightarrow E[X^n] = \frac{\phi^{(n)}(0)}{i^n}$ if the moment exists.
- P5: $\phi_X(t) \in [-1,1]$ for all X, t hence it always exists

  $\Updownarrow\ |\phi_X(t)| \in [0,1]$

  Proof: $= |E[e^{itX}]| = |\int_{\mathbb{R}} e^{itx} f(x)dx| \le \int_{\mathbb{R}} |e^{itX} f(x)| dx \le \int_{\mathbb{R}} |e^{itx}||f(x)|dx$

  $bi+a$

  $= \int_{\mathbb{R}} |i\sin(tx) + \cos(tx)| f(x)dx = \int_{\mathbb{R}} \sqrt{\sin^2(tx) + \cos^2(tx)}\, f(x)dx = 1$ Continuous

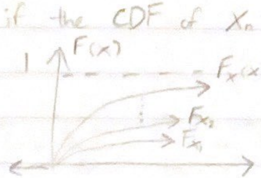
  (discrete is same proof)
- P6: Inversion. If $\phi_X(t) \in L^1$, then

  pdf: $f_X(x) = \int_{\mathbb{R}} e^{-itx}\phi_X(t)dt$
- P7: Levy's CDF theorem. (Works even if $\phi_X \notin L^1$)

  $P(X \in [a,b]) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}\phi_X(t)dt$
- P8: Levy's Continuity Theorem.

  Consider a sequence of r.v.'s $X_1, X_2, \dots, X_n$. We define "X_n converges in distribution to X", and denote it $X_n \xrightarrow{d} X$ if the CDF of X_n converges pointwise to the CDF of X,

  $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)\ \forall x$



P8: if $\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t)\ \forall_t \Rightarrow X_n \xrightarrow{d} X$

the distribution on the left (X_n) is becoming more and more like the distribution on the right (X).

Define $M_X(t) = E[e^{tX}]$, the moment generating function (mgf)

P0 $M_X(0) = E[e^{(0)X}] = 1$

P2 $Y = aX + b \Rightarrow M_Y(t) = e^{tb} M_X(at)$

P1 $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

P3 $X_1, X_2 \overset{ind}{\sim}$, $T = X_1 + X_2$ then $M_T(t) = M_{X_1}(t) M_{X_2}(t)$

P4 $E[X^n] = M_X^{(n)}(0)$ ← derivs

but ... mgf's sometimes don't exist, and sometimes don't exist for all t.

We care about chf's more than mgfs, they're better.

$X \sim \text{Gamma}(\alpha, \beta)$, $\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$

$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

$\phi_{X_1+X_2}(t) \underset{P3}{=} \phi_{X_1}(t)\phi_{X_2}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right)^{\alpha_2} = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2}$

P1 $\Downarrow$

$X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$