let $A \in \mathbb{R}^{L \times K}$ matrix of constants

$\vec{a}_i$. row i of matrix A

$$\underbrace{E[A\vec{X}]}_{L \times 1} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \dots + a_{1K}X_K] \\ E[a_{21}X_1 + a_{22}X_2 + \dots + a_{2K}X_K] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \dots + a_{LK}X_K] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{X}] \\ E[\vec{a}_2 \cdot \vec{X}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vec{a}_2 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_L \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$$Var[\vec{a}^T\vec{X}] = Var[\underbrace{a_1X_1}_{Y_1} + \dots \underbrace{a_KX_K}_{Y_K}] = \sum_{i=1}^{K}\sum_{j=1}^{K} Cov[Y_i, Y_j] = \sum\sum Cov[a_iX_i, a_jX_j]$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j \sigma_{ij} = \vec{a}^T \Sigma \vec{a}$$

ij component of Σ; $\to Var[\vec{X}]$

$(1 \times K)(K \times K)(K \times 1) =$ scalar

this is called a "quadratic form"

let $V \in \mathbb{R}^{K \times K}$, $\vec{a} \in \mathbb{R}^{K \times 1}$ $\quad \vec{a} \cdot (V\vec{a})$

$$\vec{a}^T V \vec{a} = \vec{a}^T \begin{bmatrix} a_1V_{11} + \dots + a_KV_{1K} \\ a_1V_{21} + \dots + a_KV_{2K} \\ \vdots \\ a_1V_{K1} + \dots + a_KV_{KK} \end{bmatrix} = \underbrace{a_1a_1V_{11} + \dots + a_1a_KV_{1K}}_{i=1} + \underbrace{a_2a_1V_{21} + \dots + a_2a_KV_{2K}}_{i=2} + \dots + \underbrace{a_Ka_1V_{K1} + \dots + a_Ka_KV_{KK}}_{i=K} = \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j V_{ij}$$

$[a_1 \ \dots \ a_K]$

This is an application in finance. Imagine X_1, ..., X_K are financial assets (e.g. different stocks). Each has mean return mu_i. And each pair have covariance sigma_ij. Let w-vector be a vector of "weights" where each component is the percentage you put into each of these assets. Thus the entries of w sum to 1. Your portolio F is w-vector^T X-vector:

$$F = \vec{w}^T\vec{X}, \quad \vec{w}^T\vec{1} = 1, \quad E[\vec{X}] = \mu, \quad Var[\vec{X}] = \Sigma.$$

$$E[F] = E[\vec{w}^T\vec{X}] = \vec{w}^T\vec{\mu} = \mu_F, \quad Var[F] = Var[\vec{w}^T\vec{X}] = \vec{w}^T\Sigma\vec{w}$$

Goal is to pick mu_F and minimize it's variance by computing the w-vector optimally.

$$\min \vec{w}^T\Sigma\vec{w} \quad \text{subject to} \quad \vec{w}^T\vec{\mu} = \mu_F, \quad \vec{w}^T\vec{1} = 1,$$

Markowitz optimal portfolio design

$$\vec{X} \sim Multin_K(n, \vec{p}), \quad E[\vec{X}] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_K] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_K \end{bmatrix} = n\vec{p}$$

$X_j \sim Bin(n, p_j)$

$$\Sigma = Var[\vec{X}] = \begin{bmatrix} Var[X_1] & & Cov[X_i, X_j] \\ & \ddots & \\ Cov[X_i, X_j] & & Var[X_K] \end{bmatrix} \qquad Var[X_j] = np_j(1-p_j)$$

for $i \neq j$

$$Cov[X_i, X_j] = E[X_iX_j] - \mu_i\mu_j = \sum_{x_i \in \{0,\dots,n\}}\sum_{x_j \in \{0,\dots,n\}} x_ix_j \underbrace{p_{X_i,X_j}}_{\text{JMF of } X_i \text{ and } X_j}(x_i, x_j) - n^2p_ip_j$$

difficult to obtain...

$X_i \sim Bin(n, p_i)$ $\quad X_i = X_{1i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \stackrel{iid}{\sim} Bern(p_i)$

$X_j \sim Bin(n, p_j)$ $\quad X_j = X_{1j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \stackrel{iid}{\sim} Bern(p_j)$

$\vec{X} \sim Multin_K(n, \vec{p})$ $\quad \vec{X} = \vec{X}_1 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \stackrel{iid}{\sim} Multin_K(1, \vec{p})$

$$Cov[X_i, X_j] = Cov[X_{1i} + \dots + X_{ni}, X_{1j} + \dots + X_{nj}]$$

$$= \sum_{l=1}^{n}\sum_{m=1}^{n} Cov[X_{li}, X_{mj}] \quad \text{all pairs} = \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$$

$$= \sum_{l=1}^{n} E[X_{li}X_{lj}] - \underbrace{E[X_{li}]}_{p_i}\underbrace{E[X_{lj}]}_{p_j} = \boxed{-np_ip_j}$$

if $l \neq m$ then is X_{li} independent of X_{mj}? Yes!

only nonzero if $x_{li} = x_{lj} = 1$

$$E[X_{li}, X_{lj}] = \sum_{x_{li} \in \{0,1\}}\sum_{x_{lj} \in \{0,1\}} x_{li}x_{lj}\, p_{X_{li}X_{lj}}(x_{li}, x_{lj}) = p_{X_{li},X_{lj}}(1,1) = 0$$

X_li = 1 means you get an apple, X_lj means you get a banana and both being 1 means you get both an apple and banana at the same time (on one draw). Impossible. Probability 0.

$$\Sigma = Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \dots & -np_1p_K \\ -np_1p_2 & np_2(1-p_2) & & \vdots \\ \vdots & & \ddots & \\ -np_1p_K & \dots & & np_K(1-p_K) \end{bmatrix}$$

uniform discrete

$$X_1, X_2 \stackrel{iid}{\sim} U(\{0,1,2,3\}) = \begin{cases} 0 & \text{wp } 1/4 \\ 1 & \text{wp } 1/4 \\ 2 & \text{wp } 1/4 \\ 3 & \text{wp } 1/4 \end{cases} = \frac{1}{4}\mathbb{1}_{x \in \{0,1,2,3\}}$$

generally $X \sim U(A) = \frac{1}{|A|}\mathbb{1}_{x \in A}$ Param space $A \subset \mathbb{R}$ and $|A| < \infty$.

$T = X_1 + X_2 \sim p_T(t) =$ I meant this as a HW example

Midterm I

Midterm II

$$\underbrace{Y = -X}_{\text{PMF}} = g(X)$$

Y is a function of the rv X (a very simple function).

$Y \sim p_Y(y)$

$X = 0 \Rightarrow Y = 0$
$X = 1 \Rightarrow Y = -1$
$X = 2 \Rightarrow Y = -2$
$X = 3 \Rightarrow Y = -3$

$Supp[Y] = -Supp[X]$

$$p_Y(y) := P(Y = y) = P(-X = y) = P(X = -y) = p_X(-y)$$

This is for all discrete rv's.

let $z' = -z$

$$Supp[Y] := \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} = \{-z' : p_X(z') > 0\}$$

$$= -\{z' : p_X(z') > 0\} = -Supp[X].$$

$X_1, X_2 \stackrel{iid}{\sim} Poisson(\lambda)$ From previous class $X_1 + X_2 \sim Poisson(2\lambda)$

$X_1 - X_2 \sim ?$

(difference)

$$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$$

$$p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$$

$Supp[X] = \{0, 1, 2, \dots\}$

$Supp[Y] = \{\dots, -2, -1, 0\}$

$Supp[X+Y] = Supp(X) + Supp[Y] = \mathbb{Z}$ all integers

$$p_T(t) = \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(d - x)\, \mathbb{1}_{d-x \in Supp[Y]}$$

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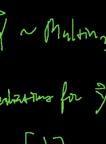
office hours work:

$X_1 = $ # Apples, $X_2 = $ # Bananas, $X_3 = $ # Cantaloupes



$\vec{X} \sim Multin_3(\overset{n}{1}, \vec{p})$ All Realizations:

$K = 3$ $\quad \vec{p} = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{x}^T\vec{1} = 1 = n$

$\vec{Y} \sim Multin_3(\overset{n}{2}, \vec{p})$, $\vec{Y} = \vec{X}_1 + \vec{X}_2$ s.t. $\vec{X}_1, \vec{X}_2 \stackrel{iid}{\sim} Multin_3(1, \vec{p})$

Realizations for $\vec{y}$: two ways

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_1} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$p_Y(\vec{y}) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \binom{2}{x_1, x_2, x_3} 0.4^{x_1}\, 0.4^{x_2}\, 0.2^{x_3}$$

$$= \binom{2}{1, 1, 0} 0.4^1\, 0.4^1\, 0.2^0$$

$$= \frac{2!}{1!\,1!\,0!} \cdot 0.4 \cdot 0.4 = 2 \cdot 0.4 \cdot 0.4$$

1 always

$$\vec{X} \sim Multin_K(1, \vec{p}) = \binom{1}{x_1, x_2, \dots, x_K} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$$

$x_j = 1 \quad x_k = 0 \ k \neq j$

$$\binom{1}{0, 0, \dots, 0, 1, 0, \dots 0} = \frac{1!}{(0!)^{K-1}(1!)^1} = \frac{1}{1^{K-1}1} = 1$$