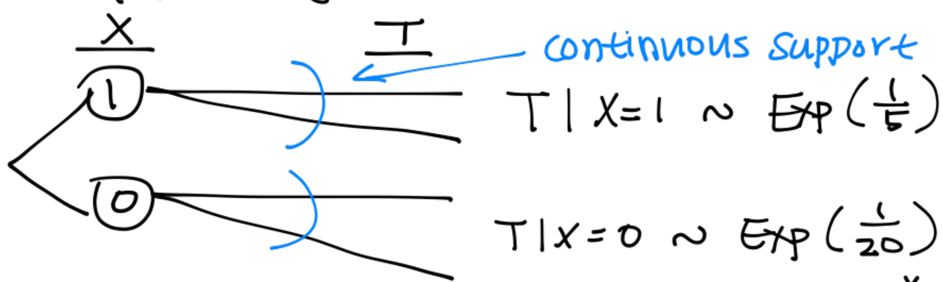
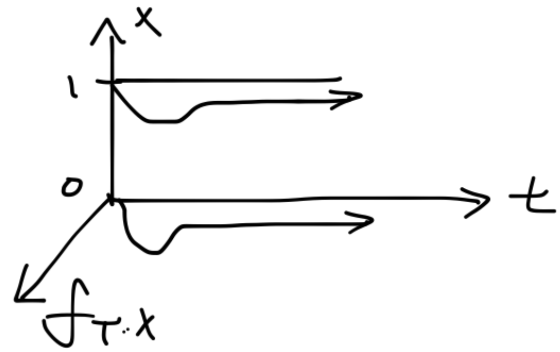
Mixture and compound distributions

Consider a situation where $\frac{2}{3}$ of the time there is fast internet speed so your downloads take $T \sim Exp(\frac{1}{5}) \Rightarrow E[T] = 5s$ and the other $\frac{1}{3}$ of the time, there is Internet traffic, so your downloads $T \sim Exp(\frac{1}{20}) \Rightarrow E[T] = 20s$. What is the dist. of the "overall T" or "unconditional on the Internet speed"? Let $X \sim Bern(\frac{2}{3})$ and $X=1$ corresponds to fast Internet and $X=0$ corresponds to slow Internet.

Let's draw a tree diagram



$h(x) = \int_{\mathbb{R}} h(x,y)\,dy$ " $\int_{y \in \mathbb{R}} h(x,y)$ "



$$f_T(t) = \int_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\, P_X(x)$$

$$= \sum_{x \in \{0,1\}} f_{T|X}(t,x) P_X(x)$$

$$= f_{T|X}(t,0) P_X(0) + f_{T|X}(t,1) P_X(1)$$

$$= \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}$$

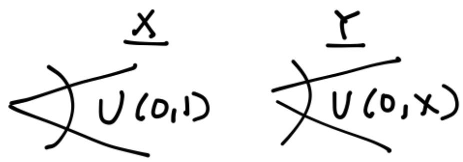
If the download speed was $t = 25s$, what is the probability it is a slow Internet day, i.e. $x = 0$?

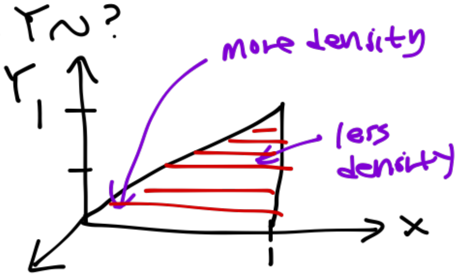
$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, P_X(x)}{f_T(t)} \quad \text{"Bayes Rule"}$$

$$\text{Bernoulli param} = P_{X|T}(1,t) = \frac{f_{T|X}(t,1) P_X(1)}{f_T(t)} = \frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}$$

$$P_{X|T}(0,25) = 1 - P_{X|T}(1,25) = 1 - \frac{\frac{1}{5} e^{-\frac{1}{5} \cdot 25} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{80} \cdot 25} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5} \cdot 25} \cdot \frac{2}{3}} = 0,842$$

$X \sim U(0,1)$, $Y \mid X = x \sim U(0,x) \Rightarrow Y \sim ?$

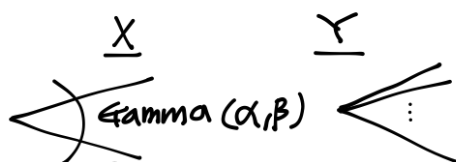




The first example featured Y which was continuous (we call that the "model") and X which is discrete (we call that the "mixing distribution". Thus the unconditional distribution Y is called a "mixture distribution"

In the second example Y, the model is continuous and X, the mixing dist. is also continuous and we call the unconditional distribution Y a "compound distribution."

(p.156-157) Let $Y \mid X = x \sim \text{Poisson}(x)$, $X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim ?$

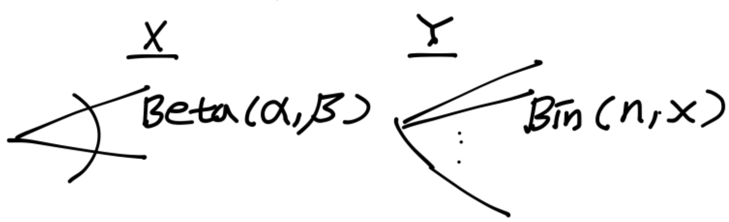


$$P_Y(y) = \int_{Supp[X]} P_{Y|X}(y,x) f_X(x)\, dx = \int_0^\infty \frac{e^{-x} x^y}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}$$

$\overset{HW}{=} \cdots = \text{ExpNegBin}\left(\alpha, \frac{\beta}{\beta+1}\right)$ This is a more flexible count model than the Poisson

$Y \mid X = x \sim \text{Bin}(n, x)$ where n is known, $X \sim \text{Beta}(\alpha, \beta)$. $Y \sim ?$



$$P_Y(y) = \int_{Supp[X]} P_{Y|X}(y,x) f_X^{old}(x)\, dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,\ldots,n\}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \binom{n}{y} \mathbb{1}_{y \in \{0,\ldots,n\}} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1} dx}_{B(y+\alpha,\, n-y+\beta)} = \frac{B(y+\alpha, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y \in \{0,\ldots,n\}}$$

$= \text{BetaBinomial}(n, \alpha, \beta)$

$Y|X=x \sim \text{Exp}(x), \quad X \sim \text{Gamma}(\alpha,\beta) \Rightarrow Y \sim \text{Lomax}(\beta,\alpha)$

which is a more flexible waiting time than the exponential

MT2

Moment generating functions (mgf's) and characteristic functions (chf's). To derive these, we need to review complex / imaginary numbers.

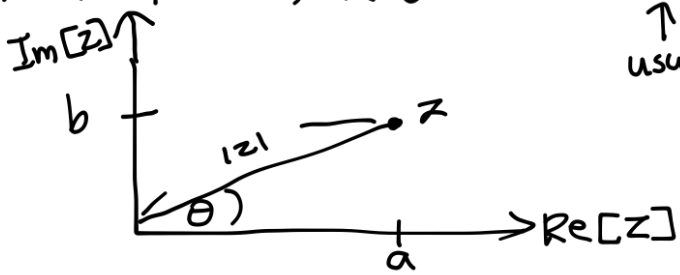
First define $i := \sqrt{-1}$ "imaginary"

let $a,b \in \mathbb{R}$, $z := a+bi \in \mathbb{C}$, complex #'s

$\text{Re}[z] := a$, $\text{Im}[z] := b$,

real component and imaginary component of a complex #

$|z| := \sqrt{a^2+b^2}$, $\text{Arg}[z] := \theta \underset{\uparrow \text{ usually}}{=} \arctan\left(\frac{b}{a}\right)$



$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5} - \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!}$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \cdots$$

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \xRightarrow{tx=\pi} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$

Euler's Formula