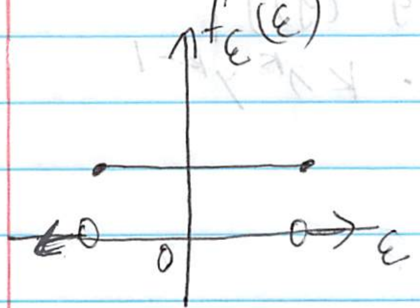
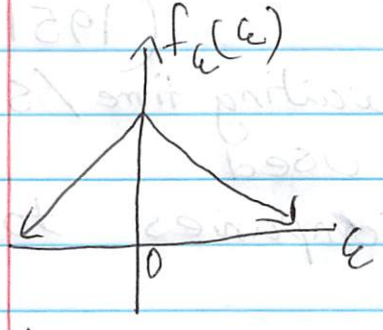
Math 621
Lecture 12
10-19-2020

Laplace first published this in 1774 calling it the "first law of errors".
His context was measurement. When you measure a quantity v, you measure it with error, episilon, so that your measurement is: M = v + epsilon
What makes a good distribution for the error, epsilon?
The expectation should be zero & should be symmetric. How about...



This is not very good. It should have the property that the probability of error should decrease with its magnitude. Also, why should it stop at some maximum magnitude?



Another good property is that the density should be decreasing in magnitude of error.

Laplace assumed for all positive errors that $f_\varepsilon''(\varepsilon) = f_\varepsilon'(\varepsilon)$

$\Rightarrow f(\varepsilon) = c e^{-d\varepsilon}$

$\Rightarrow \varepsilon \sim \text{Laplace}(0,1)$

② $X \sim Exp(1) = e^{-x} \mathbb{1}_{x \geq 0}$

$Y = g(X) = \frac{1}{\lambda} X$

Change it to → $y = g(x) = \frac{1}{\lambda} x^{\frac{1}{k}} \sim f_Y(y) = ?$ Such that $k, \lambda > 0$.

(i) First Step: get inverse function

(ii) Second Step: get abs. inverse derivative

a good test question

$$Y = \frac{1}{\lambda} X^{\frac{1}{k}} \Rightarrow \lambda Y = X^{\frac{1}{k}}$$

$$\Rightarrow X = (\lambda y)^k = \lambda^k y^k = g^{-1}(Y) \quad \cdots (i)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\lambda^k y^k]\right| = |k \lambda^k y^{k-1}|$$

$$= k\lambda^k |y^{k-1}| = k\lambda^k y^{k-1}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$= e^{-(\lambda y)^k} \mathbb{1}_{\underbrace{\underbrace{\lambda^k y^k \geq 0}_{y^k \geq 0}}_{y \geq 0}} \cdot k\lambda^k y^{k-1}$$

$$= k\lambda^k y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}$$

$$= k\lambda (\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0} = \text{Weibull}(k, \lambda)$$

(1951)

this is a very famous waiting time / survival rv model and it's used e.g. in insurance companies to price life insurance.

$$\text{Weibull}(1, \lambda) = (1)\lambda(\lambda y)^{(1)-1} e^{-(\lambda y)^{(1)}} \mathbb{1}_{y \geq 0}$$

$$= \lambda e^{-\lambda y} \mathbb{1}_{y \geq 0} = Exp(\lambda)$$

The k parameter is really "cool".
Here's a property of Weibull rv's under different values of k: Let $c=14$, $y=3$

$k=1$, $P(Y \geq y+c \mid y \geq c) = P(Y \geq y)$

e.g. $P(Y \geq 14+3 \mid y \geq 14) = P(Y \geq 3)$

$P(Y \geq 17 \mid y \geq 14) = P(Y \geq 3)$

this equality is called "memorylessness"

$k>1$, $P(Y \geq y+c \mid y \geq c) < P(Y \geq y)$

ex: (People live pass 97) Prob is low — (people live pass 3 yr) Prob is high

so, memorylessness fails

$k<1$, $P(Y \geq y+c \mid y \geq c) > P(Y \geq y)$

ex: Startup companies

Order Statistics (Pg 160)

Let $X_1, X_2, \ldots, X_n$ be a collection of continuous rv's.

Let the "Order Statistics" be the rv's: $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ defined as:

$X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$

$X_{(k)} = k^{th}$ largest of $X_1, \ldots, X_n$

$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$

$R = X_{(n)} - X_{(1)}$, range

eg. $X_1 = 9$, $X_2 = 2$, $X_3 = 12$, $X_4 = 7$

$X_{(1)} = 2$, $X_{(2)} = 7$, $X_{(3)} = 9$, $X_{(4)} = 12$

$r = \text{range} = 12 - 2 = 10$

(4) We want to find both the CDF & PDF of the k^{th} order statistic. We will build this up in stages. The first thing we'll do is find the CDF & PDF of the maximum.

$$F_{X_{(n)}}(x) = P(\underbrace{X_{(n)} \le x}_{\text{event}}) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x)$$

if $X_1, \ldots, X_n$ independent

$$\hookrightarrow = P(X_1 \le x) \cdot \ldots \cdot P(X_n \le x)$$

$$= \prod_{i=1}^{n} F_{X_i}(x)$$

if iid $= F_X(x)^n$

$$\text{PDF: } f_{X_n}(x) \overset{\text{iid}}{=} \frac{d}{dx}\left[F(x)^n\right]$$

using Chain rule $= n f(x) F(x)^{n-1}$

the next thing we'll do is to find the CDF and PDF of the minimum.

$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$

$$\Rightarrow 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x)$$

if independent $\Rightarrow 1 - \prod_{i=1}^{n}\left(1 - F_{X_i}(x)\right)$

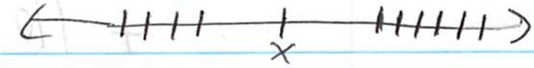
if iid $\Rightarrow 1 - \left(1 - F(x)\right)^n = \text{CDF}$

$$\text{PDF: } f_{X_{(1)}}(x) \overset{\text{iid}}{=} \frac{d}{dx}\left[1 - \left(1 - F(x)\right)^n\right]$$

$$= n f(x)\left(1 - F(x)\right)^{n-1}$$

The next thing we'll do is assuming $n = 10$ and derive the $k = 4^{th}$ order statistic's CDF & PDF:

Before we get there, let's find the probability that the first four numbers are less than x & the last six numbers are greater than x.



$= P(X_1 \le x, \dots, X_4 \le x, X_5 > x, \dots X_{10} > x)$

if independent

$$= \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} (1 - F_{X_i}(x))$$

iid $= F(x)^4 (1 - F(x))^6$

Let's find the probability any 4 of the 10 are below x & the remaining are above x.

Let S be a subset of size 4 of the index set $\{1, 2, \dots, 10\}$.

$$= \sum_{\text{all } S} P(X_{S_1} \le x, \dots, X_{S_4} \le x, X_{S_1^c} > x, \dots, X_{S_6^c} > x)$$

if independent

$$= \sum_{\text{all } S} \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=1}^{6} \left(1 - F_{X_{S_i^c}}(x)\right)$$

if iid

$$= \sum_{\text{all } S} F(x)^4 (1 - F(x))^6 = \binom{10}{4} F(x)^4 (1 - F(x))^6$$

Now, let's derive the CDF for the $k = 4^{th}$ order statistic.

$$F_{X_{(4)}}(x) = P(\underbrace{X_{(4)} \le x}_{\text{event}})$$

$= P\begin{pmatrix}\text{a subset of 4 } X_i\text{'s} \le x \text{ and the} \\ \text{remaining 6 are} > x\end{pmatrix} +$

$P(\text{a subset of 5 } X_i\text{'s} \le x \text{ and the remaining 5 are} > x) + \dots +$

⑥ $P(\text{all } 10\ X_i\text{'s} \le x)$

if iid $= \binom{10}{4} F(x)^4 (1-F(x))^6 + \binom{10}{5} F(x)^5 (1-F(x))^5 +$

$\cdots + \binom{10}{10} F(x)^{10} (1-F(x))^{10-10}$

$= \sum_{j=4}^{10} \binom{10}{j} F(x)^j (1-F(x))^{10-j}$

For iid continous rv's $X_1, \ldots, X_n$, the CDF & PDF for the k^{th} order Statistic is:

CDF: $F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j}$

PDF: $f_{X_{(k)}}(x) = \frac{d}{dx} \left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} \right]$

$= \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx} \left[\underbrace{F(x)^j}_{u} \underbrace{(1-F(x))^{n-j}}_{v} \right]$

Calculus Rule: $\frac{d}{dx} = uv' + u'v$

$u' = j f(x) F(x)^{j-1}$

$v' = -(n-j) f(x) (1-F(x))^{n-j-1}$