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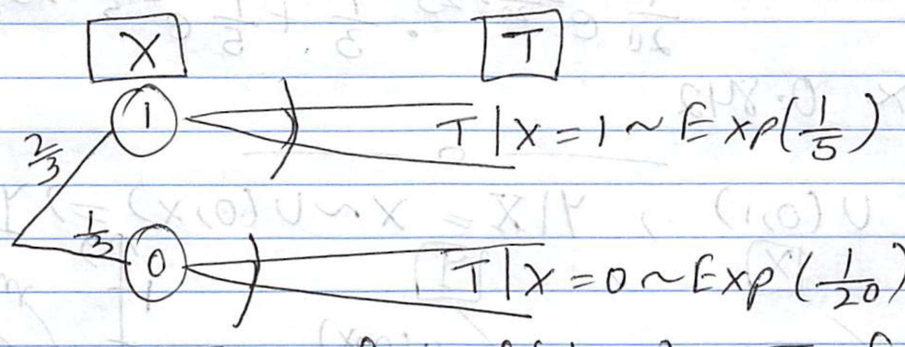
Math 621 : 10-28-2020

Lecture 15

Mixture and compound distributions

Consider a situation where $\frac{2}{3}$ of the time there is fast internet speed so your downloads take $T \sim Exp(\frac{1}{5})$
$\Rightarrow E(T) = 5s$
and the other $\frac{1}{3}$ of the time, there is Internet traffic, so your downloads take $T \sim Exp(\frac{1}{20}) \Rightarrow E(T) = 20s$.
What is the distribution of the "overall T" or "unconditional on the internet speed"?
Let $X \sim Bern(\frac{2}{3})$ and $x=1$ corresponds to fast internet and $x=0$ corresponds to slow internet.
Let's draw a tree diagram:

$$h(x) = \int_{\mathbb{R}} h(x,y)\,dy = \sum_{y \in \mathbb{R}} h(x,y)$$



$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}\, f(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\, p_X(x)$$

$$= \sum_{x \in \{0,1\}} f_{T|X}(t,x)\, p_X(x)$$

$$= f_{T|X}(t,0)\, p_X(0) + f_{T|X}(t,1)\, p_X(1)$$

$$= \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}$$

②

If the download speed was $t = 25$ s, what is the probability it is a slow internet day, i.e. $X = 0$?

$X \mid T \sim \text{Bern}(?)$

$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, P_X(x)}{f_T(t)}$$ "Bayes Rule"

Bernoulli Parameter $= P_{X|T}(1,t)$

If $W \sim \text{Bern}(p)$
$p = P(W=1)$

$$= \frac{f_{T|X}(t,1)\, P_X(1)}{f_T(t)}$$

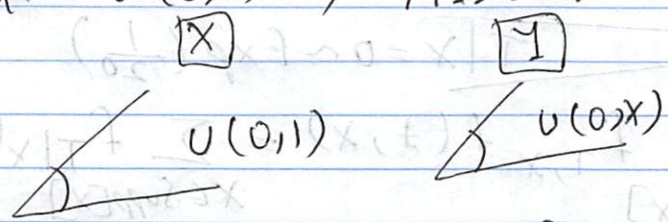
$$= \frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}$$

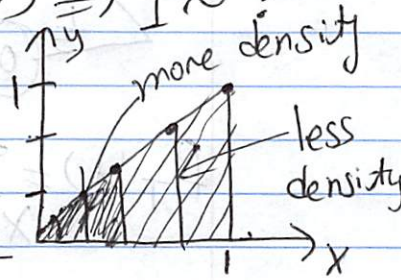
$$P_{X|T}(0,25) = 1 - P_{X|T}(1,25)$$

$$= 1 - \frac{\frac{1}{5} e^{-\frac{1}{5} \cdot 25} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20} \cdot 25} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5} \cdot 25} \cdot \frac{2}{3}}$$

≈ 0.842

$X \sim U(0,1)$, $Y \mid X = x \sim U(0,x) \Rightarrow Y \sim ?$





The first example featured T which was continuous (we call that the "model") and X which is discrete (we call that the "mixing distribution".) Thus the unconditional distribution T is called a "mixture distribution".

In the second example Y, the model is

continuous and X, the mixing distribution is also continuous and we call the unconditional distribution Y a "Compound distribution".

Pg 156 - 157

Let $Y \mid X = x \sim \text{Poisson}(x)$,
$X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim ?$

$\boxed{X}$ $\qquad\qquad\qquad$ $\boxed{Y}$

$\text{Gamma}(\alpha, \beta) \longleftarrow \text{Poisson}(X)$

$$p_Y(y) = \int_{Supp[X]} \underbrace{p_{Y|X}(y, x)}_{\text{discrete part}} \underbrace{f_X^{old}(x)}_{\text{cont. part}} \, dx$$

discrete

$$= \int_0^\infty \frac{e^{-x} x^y}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x} \, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \int_0^\infty x^{y + \alpha - 1} e^{-(\beta+1)x} \, dx$$

Lecture 09

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(x + \alpha)}{(\beta + 1)^{x + \alpha}}$$

$$= \ldots \text{HW} \ldots = \text{ExtNegBin}\left(\alpha, \frac{\beta}{\beta + 1}\right)$$

this is a more flexible count model than the Poisson

$Y \mid X = x \sim \text{Bin}(n, x)$ when n is known,
$X \sim \text{Beta}(\alpha, \beta)$

$\boxed{X}$ $\qquad\qquad\qquad$ $\boxed{Y}$

$\text{Beta}(\alpha, \beta) \longleftarrow \text{Bin}(n, x)$

④ $P_Y(y) = \int_{\text{Supp}[X]} P_{Y|X}(y,x) f_X^{old}(x)\, dx$

$= \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,\dots,n\}} \cdot \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$

$= \binom{n}{y} \mathbb{1}_{y\in\{0,\dots,n\}} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1} dx}_{B(y+\alpha,\, n-y+\beta)}$

$= \frac{B(y+\alpha,\, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y\in\{0,\dots,n\}}$

$= \text{Beta Binomial}(n,\alpha,\beta)$

o

$Y|X=x \sim \text{Exp}(x),\quad X \sim \text{Gamma}(\alpha,\beta) \xRightarrow{(HW)} Y \sim \text{Lomax}(\beta,\alpha)$

which is a more flexible waiting time than the exponential.

$\Uparrow$ end of Midterm II $\Uparrow$

$\Downarrow$ Final $\Downarrow$

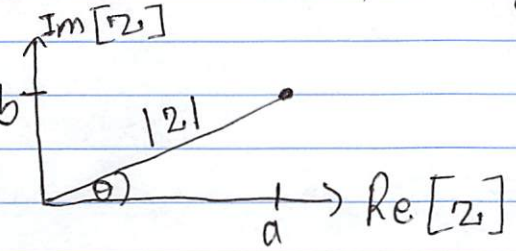
Moment Generating functions (mgf's) and characteristic functions (chf's).

To derive these, we need to review Complex / imaginary numbers.
first define $i = \sqrt{-1}$ "imaginary"

let $a,b \in \mathbb{R}$, $z = a+bi \in \mathbb{C}$, complex #'s

$Re[z] = a$, $Im[z] = b$,
real component and imaginary component of a complex #

$|z| = \sqrt{a^2+b^2}$, $Arg[z] = \theta \underset{=}{\overset{\text{Usually}}{=}} \text{Arctan}\left(\frac{b}{a}\right)$



$i^0 = 1$ $\quad i^5 = i^4 \cdot i = i$
$i^1 = i$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i^2 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = 1$

$\boxed{\text{Clock Cycle!!}}$

$\ldots n \in \mathbb{N},\ i^n = i^{n \bmod 4} \ldots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^3}{4!} + \frac{x^5}{5!} + \ldots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \ldots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - \ldots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \ldots$$

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$$

let $tx = \pi \Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$

Euler's Formula

Next Class: Monday (11/02): Review for Midterm II

Wednesday (11/04) Exam from 8PM

(5)

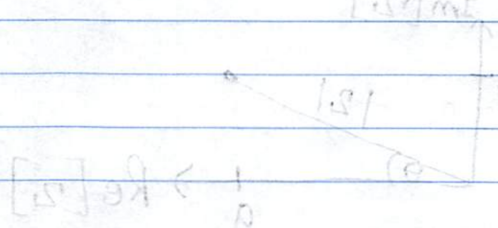
$Re[z] = a$, $Im[z] = b$,
real component and imaginary component of a complex #

$|z| = \sqrt{a^2+b^2}$, $Arg[z] = \theta \underset{\text{usually}}{=} \text{Arctan}\left(\frac{b}{a}\right)$

$i^0 = 1$
$i^1 = i$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i^2 i = -i$
$i^4 = i^2 i^2 = 1$
$i^5 = i^4 i = i$
⋮

clock cycle!!

i^n, $n \in \mathbb{N}$: $n \bmod 4$



$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

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$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} + \cdots$$

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$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$$

let $tx = \pi \Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$
Euler's formula

Next class: Monday (11/02): Review for Midterm II
Wednesday (11/04) Exam from 6PM