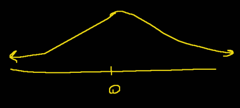
$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi}\frac{1}{x^2+1}$

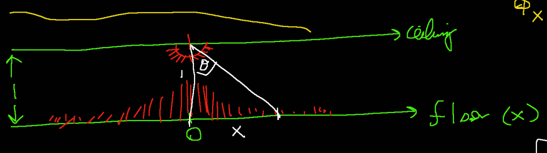


$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi}\frac{1}{x^2+1}\,dx = \infty$

$M_X(t) = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = \infty$ mgf doesn't exist

$\phi_X(t) = \int_{\mathbb{R}} e^{itx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx \overset{\text{Aryeh}}{=} \cdots = e^{-|t|}$ $\quad \phi_X'(t) = -\frac{t}{|t|}e^{-|t|}$

$\phi_X'(0)$ dne



$\theta \sim U\left(-\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{1}{\pi}\mathbb{1}_{\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}$, $\quad X = \tan(\theta) \Rightarrow \theta = \arctan(x)$ ← because it is 1:1 in support

$\left|\frac{d}{dx}\left[\arctan(x)\right]\right| = \frac{1}{x^2+1}$

$f_X(x) = f_\theta(\arctan(x))\frac{1}{x^2+1} = \frac{1}{\pi}\mathbb{1}_{\underbrace{\arctan(x)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}_{x\in\mathbb{R}}}\frac{1}{x^2+1} = \text{Cauchy}(0,1)$

Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$

$T_n \sim N(n\mu, n\sigma^2)$

$\bar{X}_n \sim N\left(\mu,\frac{\sigma^2}{n}\right)$ "sample mean" or "average"

$S_n^2 = \frac{1}{n-1}\sum(X_i-\bar{X})^2 \sim f_{S_n^2}(s^2) = ?$ "sample variance"

$Z_1,\ldots,Z_n \overset{iid}{\sim} N(0,1)$ $\quad \sum Z_i^2 \sim \chi^2_n$ $\quad \vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $\vec{Z}^T\vec{Z} \sim \chi^2_n$

$Z_1 = \frac{X_1-\mu}{\sigma},\ldots,Z_n = \frac{X_n-\mu}{\sigma} \Rightarrow \sum\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$

$(X_i-\mu)^2 = \left((X_i-\bar{X})+(\bar{X}-\mu)\right)^2 = (X_i-\bar{X})^2 + 2\underbrace{(X_i-\bar{X})(\bar{X}-\mu)}_{X_i\bar{X}-\bar{X}^2-X_i\mu+\bar{X}\mu} + (\bar{X}-\mu)^2$

$\sum(X_i-\mu)^2 = \underbrace{\sum(X_i-\bar{X})^2}_{(n-1)S^2} + 2\left(\cancel{\bar{X}\underbrace{\sum X_i}_{n\bar{X}}} - \cancel{n\bar{X}^2} - \cancel{\mu\underbrace{\sum X_i}_{n\bar{X}}} + \cancel{n\bar{X}\mu}\right) + n(\bar{X}-\mu)^2$

$\sum\frac{(X_i-\mu)^2}{\sigma^2} = \underbrace{\frac{n-1}{\sigma^2}S^2}_{\sim\chi^2_{n-1}} + \underbrace{\Big(\underbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}_{Z}\Big)^2}_{\sim\chi^2_1} \sim \chi^2_n$

conjecture:

$U_1 \sim \chi^2_{k_1}$ indep of

$U_2 \sim \chi^2_{k_2}$

$\Rightarrow U_1+U_2 \sim \chi^2_{k_1+k_2}$

this would be true if $\bar{X}$ is independent of S^2

"quadratic form"

$\vec{Z}^T\vec{Z} = \vec{Z}^T I \vec{Z} \sim \chi^2_n$

Consider: $\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = Z_1^2 \sim \chi^2_1$

$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 1_i & \\ 0 & & & \ddots\ 0 \end{bmatrix}}_{B_i} \vec{Z} = Z_i^2 \sim \chi^2_1$ $\quad \text{rank}[B_i] = 1$

$\vec{Z}^T I\vec{Z} = \vec{Z}^T(B_1+B_2+\ldots+B_n)\vec{Z} = \vec{Z}^T B_1\vec{Z} + \ldots + \vec{Z}^T B_n\vec{Z} \sim \chi^2_n$

Cochran's thm: If $B_1+B_2+\ldots+B_K = I$ s.t $\sum_{j=1}^{K}\text{rank}[B_j] = n$

then (a) $\vec{Z}^T B_j\vec{Z} \sim \chi^2_{\text{rank}[B_j]}$ and (b) $\vec{Z}^T B_{j_1}\vec{Z}$ is independent of $\vec{Z}^T B_{j_2}\vec{Z}$ $\forall j_1 \neq j_2$

$\vec{Z}^T\vec{Z} = \sum Z_i^2 = \sum\left((Z_i-\bar{Z})+\bar{Z}\right)^2$

$= \sum(Z_i-\bar{Z})^2 + 2\sum(Z_i-\bar{Z})\bar{Z} + n\bar{Z}^2 = \sum(Z_i-\bar{Z})^2 + n\bar{Z}^2$ $\quad \sum(Z_i-\bar{Z})\bar{Z} = n\bar{Z}^2 - n\bar{Z}^2 = 0$

let $\vec{1}_n$ is a column vector of all ones $\Rightarrow \bar{Z} = \frac{1}{n}\vec{Z}^T\vec{1}$

$n\bar{Z}^2 = n\left(\frac{1}{n}\vec{Z}^T\vec{1}\right)^2 = \cancel{n}\frac{1}{\cancel{n}}\vec{Z}^T\vec{1}\frac{1}{n}\vec{Z}^T\vec{1} = \vec{Z}^T\underbrace{\left(\frac{1}{n}\vec{1}\vec{1}^T\right)}_{B_2 = \frac{1}{n}J_n}\vec{Z}$ $\quad \text{rank}[B_2] = 1$

let $J_n = \vec{1}_n\vec{1}_n^T$ which is an n x n matrix of all entries = 1

$\sum(Z_i-\bar{Z})^2 = \sum Z_i^2 - 2n\bar{Z}^2 + n\bar{Z}^2 = \sum Z_i^2 - n\bar{Z}^2$ $\quad B_1+B_2 =$

$= \vec{Z}^T I\vec{Z} - \vec{Z}^T\left(\frac{1}{n}J_n\right)\vec{Z} = \vec{Z}^T\underbrace{\left(I-\frac{1}{n}J\right)}_{B_1}\vec{Z}$ $\quad \left(I-\frac{1}{n}J\right)+\frac{1}{n}J = I$

Thm from Math 231: if A is symmetric matrix and idempotent which means AA = A then rank[A] = tr[A] = sum of the diagonal of A.

$B_1^T = \left(I-\frac{1}{n}J\right)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J = B_1$ ✓ $\quad \begin{bmatrix} \vdots\vdots\vdots \end{bmatrix}\begin{bmatrix} \vdots\vdots\vdots \end{bmatrix} = \underbrace{\begin{bmatrix} 3&3&3 \\ 3&3&3 \\ 3&3&3 \end{bmatrix}}_{3J}$

$B_1B_1 = \left(I-\frac{1}{n}J\right)\left(I-\frac{1}{n}J\right) = II - \frac{1}{n}JI - \frac{1}{n}IJ + \frac{1}{n^2}JJ$

$= I - \frac{2}{n}J + \frac{1}{n}J = I - \frac{1}{n}J = B_1$ ✓ $\Rightarrow \text{rank}[B_1] = \text{tr}[B_1]$

$B_{1_{i,i}} = 1 - \frac{1}{n}1 = 1 - \frac{1}{n}$ $\quad = \sum_{i=1}^{n} 1 - \frac{1}{n} = n-1$

Putting it all together, we can use Cochran's thm:

$\sum(Z_i-\bar{Z})^2 \sim \chi^2_{n-1}$ indep of $n\bar{Z}^2 \sim \chi^2_1$

$\bar{Z} = \frac{Z_1+\ldots+Z_n}{n} = \frac{\frac{X_1-\mu}{\sigma}+\ldots+\frac{X_n-\mu}{\sigma}}{n} = \frac{\sum X_i - n\mu}{n\sigma} = \frac{\bar{X}-\mu}{\sigma}$

$\Rightarrow n\bar{Z}^2 = (\sqrt{n}\bar{Z})^2 = \left(\frac{\bar{Z}}{\frac{1}{\sqrt{n}}}\right)^2 = \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2_1$ independent of

$\sum(Z_i-\bar{Z})^2 = \sum\left(\frac{X_i-\cancel{\mu}}{\sigma} - \frac{\bar{X}-\cancel{\mu}}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum(X_i-\bar{X})^2 = \frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$

I think the first to prove this was Fisher in 1925 and then in 1936, Geary proved the iid normal rv is the *only* distribution that has the independent of Xbar and S^2.

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ but what about $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim ?$ Not $N(0,1)$!!