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$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$ not 1:1

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 2(F_Z(\sqrt{y}) - \overset{1/2}{F_Z(0)}) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = 2 f_Z(\sqrt{y})\frac{1}{2} y^{-1/2} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\substack{\sqrt{y}\in\mathbb{R} \\ y \ge 0}}$$

$$\propto y^{-1/2} e^{-\frac{1}{2}y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

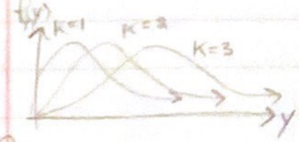
$Z_1, Z_2, \dots, Z_k \overset{iid}{\sim} N(0,1)$ and $Y = Z_1^2 + Z_2^2 + \dots + Z_k^2 \sim ?$

$E[Y] = kE[Z^2] = k(1)$

$$Y \sim \text{Gamma}\left(\tfrac{k}{2}, \tfrac{1}{2}\right) = \chi^2_k = \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{k/2-1} e^{-\frac{y}{2}} \mathbb{1}_{y\ge0} \overset{k=1}{=} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} \mathbb{1}_{y \ge 0}$$

chi-squared distribution with parameter k

the only parameter here is k ← and this parameter is called "degrees of freedom"



$X \sim \chi^2_k$, $Y = \sqrt{X} \sim f_Y(y)$, $x = y^2 = g^{-1}(y)$, $\left|\frac{d}{dy} g^{-1}(y)\right| = 2y$

$$f_Y(y) = f_X(y^2)2y = \frac{(\frac{1}{2})^{k/2}}{\Gamma(\frac{k}{2})} y^{k-2} e^{-\frac{y^2}{2}} 2y\, \mathbb{1}_{\substack{y^2 \ge 0 \\ y \ge 0}} = \frac{(\frac{1}{2})^{k/2-1}}{\Gamma(k/2)} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y \ge 0} = \chi_k$$

this is the chi distribution with k degrees of freedom

$X \sim N(0,1)$, $|X| \sim ?$, $|X| = \sqrt{X^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \mathbb{1}_{y \ge 0} = 2\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

2 times pdf $N(0,1)$

$$X \sim \text{Gamma}(\alpha, \beta),\quad Y \overset{c>0}{=} cX \sim \frac{1}{c}\frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{x}{c}\right)^{\alpha-1}}_{x^{\alpha-1}/c^{\alpha-1}} e^{-\beta x/c}$$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta/c)x} = \text{Gamma}(\alpha, \beta/c)$$

$X \sim \chi^2_k$, $Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

$X_1 \sim \chi^2_K$ indep of $X_2 \sim \chi^2_{K_2}$, let $U = \frac{X_1}{K_1} \sim \text{Gamma}(\overset{a}{\frac{K_1}{2}}, \frac{K_1}{2})$, $V = \frac{X_2}{K_2} \sim \text{Gamma}(\overset{b}{\frac{K_2}{2}}, \frac{K_2}{2})$

$$R = \frac{X_1/K_1}{X_2/K_2} = \frac{U}{V} \sim \int_{\text{Supp}[U]} f_U(rt) f_V(t)|t|\,dt = \int_0^\infty t \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\,dt\, \mathbb{1}_{r \geq 0}$$

$$= \frac{a^a b^a}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r \geq 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\,dt = \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r \geq 0} \frac{\Gamma(a+b)}{(ar+b)^{a+b}} =$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)} \mathbb{1}_{r \geq 0} = \frac{a^a b^b}{B(a,b)} r^{a-1} b^{-a} b^{-b} (1 + \tfrac{a}{b} r)^{-(a+b)} \mathbb{1}_{r \geq 0}$$

$$= \frac{(\frac{a}{b})^a}{B(a,b)} r^{a-1} (1 + \tfrac{a}{b})^{-(a+b)} \mathbb{1}_{r \geq 0} = \frac{(\frac{K_1}{K_2})^{K_1/2}}{B(\frac{K_1}{2}, \frac{K_2}{2})} r^{\frac{K_1}{2}-1} (1 + \tfrac{K_1}{K_2})^{-\frac{K_1+K_2}{2}} \mathbb{1}_{r > 0}$$

$= F_{K_1,K_2}$ the "F distribution" or the "Fischer - Snecedor" distribution with K_1 numerator degrees of freedom and K_2 denominator degrees of freedom $K_1 \in \mathbb{N}$, $K_2 \in \mathbb{N}$

$Z \sim N(0,1)$ indep of $X \sim \chi^2_K$. Let $W = \frac{Z}{\sqrt{X/K}} \sim f_W(w)$

$$W^2 = \frac{Z^2}{X/K} \sim F_{1,K}$$

$$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$$

take d/dw of both sides.

$$\frac{d}{dw}[F_{W^2}(w^2)] = \frac{d}{dw}[F_W(w)] - \frac{d}{dw}[F_W(-w)]$$

$$\Rightarrow 2w f_{W^2}(w^2) = f_W(w) + f_W(w) \Rightarrow f_W(w) = w f_{W^2}(w^2)$$

$$\Rightarrow f_W(w) = \cancel{w} \underbrace{\frac{(\frac{1}{K})^{\frac{1}{2}}}{B(\frac{1}{2}, \frac{K}{2})}}_{= \frac{\Gamma(\frac{K+1}{2})}{\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}\Gamma(\frac{K}{2})}} \underbrace{(w^2)^{\frac{1}{2}-1}}_{\cancel{w^{-1}}} (1 + \tfrac{w^2}{K})^{-\frac{K+1}{2}} = \frac{\Gamma(\frac{K+1}{2})}{\sqrt{K\pi}\,\Gamma(\frac{K}{2})} (1 + \tfrac{w^2}{K^2})^{-\frac{K+1}{2}} = T_K$$

Student's T distribution with K degrees of freedom

$K \to \infty$ $T \sim N(0,1)$

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru)f(u)|u|du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|du = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-\left(\frac{1+r^2}{2}\right)u^2}(-u)du + \int_{0}^{\infty} e^{-\left(\frac{1+r^2}{2}\right)u^2} udu\right]$$

$$= \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} udu = \frac{1}{\pi}\int_0^\infty e^{-\frac{1+r^2}{2}t} \cancel{u}\,\frac{1}{2}\frac{1}{\cancel{u}}\,du$$ ← pdf of exponential

let $t = u^2$, $dt = 2udu$, $u = 0$, $t = 0$, $u \to \infty$, $t \to \infty$

$$= \frac{1}{2\pi} \cdot \frac{1}{\frac{1+r^2}{2}} = \frac{1}{\pi} \cdot \frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

$$X = \overset{\sigma > 0}{\sigma} R + c \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi}\frac{1}{1+\left(\frac{r-c}{\sigma}\right)^2}$$

$T_1 = \text{Cauchy}(0,1)$