Lecture 08 MATH621 09/30/2020

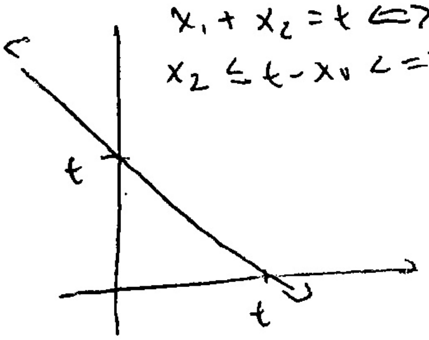
$T = X_1 + X_2 \sim f_T(t) = ?$ It also has a CDF $f_T(t) = F_T'(t)$

$F_T(t) = P(T \le t) = P(\bar{X} \in A_t)$

$\longleftarrow$ CDF method to compute the convolution.

$x_1 + x_2 = t \iff x_2 = t - x_1$

$x_2 \le t - x_1 \iff x_1 + x_2 \le t$



$$\iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1 = \int\int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

let $x_1 = x$, $x_2 = v - x \Rightarrow dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$, $x_2 = t - x \Rightarrow v = t$

$$= \int_{\mathbb{R}}\int_{-\infty}^{t} f_{X_1X_2}(x, v-x)\,dv\,dx = \int_{-\infty}^{t}\int_{\mathbb{R}} f_{X_1X_2}(x,v-x)\,dx\,dv$$

$$\Rightarrow f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\int_{\mathbb{R}} f_{X_1,X_2}(x,v-x)\,dx\,dv\right]$$

Leibnitz's Rule for derivatives of integral functions:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x,b(x))\,b'(x) + g(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy$$

If the derivative is wrt a third var, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} \cdots\right] = g(x,b(t))\,b'(t) + g(x,a(t))\,a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}[g(x,y)]\,dy}^{\,0}$$

if one of the bounds is constant then...

$$\frac{d}{dt}\left[\int_{c}^{b} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + \cancel{g(x,c)\frac{d}{dt}[c]}^{\,0}$$

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\int_{\mathbb{R}} f_{X_1X_2}(x,v-x)\,dx\,dv\right] = \int f_{X_1X_2}(x,t-x)\,dx$$

General convolution formulas:

$\underline{X_1, X_2 \text{ indep}}$

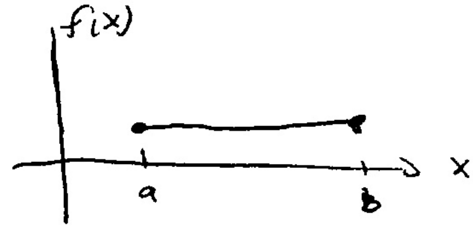
$$= \int f_{X_1}(x) f_{X_2}(t-x)\,dx = \int f_{X_1}^{old}(x)\, f_{X_2}^{old}(t-x)\,\mathbb{I}_{t-x \in Supp(X_2)}\,dx$$

$X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(x) f(t-x)\, dx = \int_{supp[X]} f^{old}(x) f^{old}(t-x) \mathbb{I}_{t-x \in supp[X]}\, dx$$

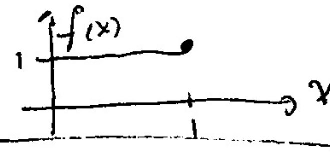
Continuous uniform r.v

$$X \sim U(a,b) = \underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{I}_{x \in [a,b]}$$



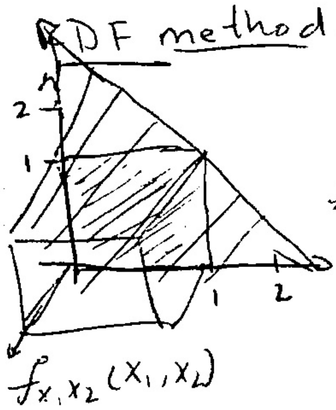
The standard uniform r.v is when $a=0$, $b=1$

$$X \sim U(0,1) = \frac{1}{1-0} \mathbb{I}_{x \in [0,1]}$$



$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$

<u>CDF method:</u>

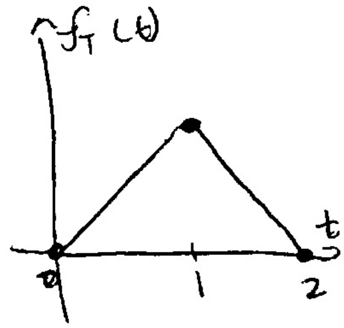


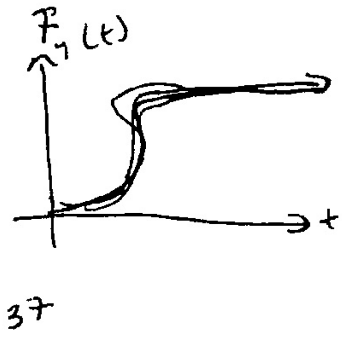
$P(T \leq 2.1) = 1$

$$F_T(t) = P(T \leq t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t^2}{2} & \text{if } t \in [0,1] \\ 1 & \text{if } t \geq 2 \\ \frac{t^2}{2} - (t^2 - 2t +) & \text{if } t \in (1,2) \end{cases}$$

$Supp[T] = [0,2]$

$$f_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in [1,2] \\ 0 & \text{if } t \geq 2 \end{cases}$$



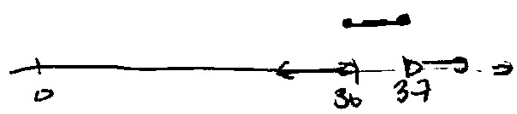


<u>using the convolution formula:</u>

$$f_T(t) = \int f^{old}(x) f^{old}(t-x) \mathbb{I}_{t-x \in supp[X]}\, dx$$

$$= \int_0^1 \mathbb{I}_{t-x \in [0,1]}\, dx = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } \leq 2 \end{cases}$$

let $t = 37$



$$X_1, X_2, \ldots \overset{iid}{\sim} \mathrm{Exp}(\lambda) = \underbrace{\lambda e^{-\lambda x}}_{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)} , \quad T_2 = X_1 + X_2$$

$$f_{T_2}(t) = \int_{supp(X)} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in supp(X)}\, dx = \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} \underbrace{\mathbb{1}_{t-x \in supp(X)}}_{\substack{x-t \in (-\infty, 0) \\ x \in (-\infty, t)}} dx$$

$$= \lambda^2 e^{\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t)}\, dx = \lambda^2 e^{-\lambda t} \int_x^t dx$$

$$= t \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)} = \mathrm{Erlang}(2, \lambda)$$