From previous class:

$$\star\; p_D(d) = \sum_{x\in Supp[X]} p_X^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{d-x\in Supp[Y]}$$

$$= \sum_{x\in\{0,1,..\}} \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{\underbrace{(-(d-x))!}_{(x-d)!}} \mathbb{1}_{\underbrace{d-x\in\{..,-1,0\}}_{\substack{x-d\in\{0,1,..\} \\ x\in\{d,d+1,d+2,...\}}}}$$

$$= e^{-2\lambda}\sum_{x\in\{0,1,..\}} \frac{\lambda^x}{x!}\frac{\lambda^{x-d}}{(x-d)!}\mathbb{1}_{x\in\{d,d+1,..\}}$$

let $x' := x-d \Rightarrow x = x'+d$

$$= e^{-2\lambda}\begin{cases} d>0 & \sum_{x\in\{d,d+1,..\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'\in\{0,1,..\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!(x'+d-d)!} = \sum_{x'\in\{0,1,..\}}\frac{\lambda^{2x'+d}}{(x'+d)!\,x'!} \\ d\le 0 & \sum_{x\in\{0,1,..\}}\frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x\in\{0,1,..\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} = \sum_{x\in\{0,1,..\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}\end{cases} \quad " \rightarrow d=|d|$$

let $d' := -d = |d|$

$I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{(\frac{2\lambda}{2})^{2x+|d|}}{x!(x+|d|)!}$ -> Modified Bessel Function of the first kind.

$\star = e^{-2\lambda} I_{|d|}(2\lambda)\mathbb{1}_{d\in\mathbb{Z}}$ = Skellam(λ, λ) discovered in 1946

It's used to model point spreads in sports games, photo noise, etc..

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \Rightarrow T = X_1+X_2 \sim \text{Poisson}(2\lambda)$

$$P_{X_1|T}(x,t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} = \frac{p_{X_1}(x)\,p_{X_2}(t-x)}{p_T(t)}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!}\overbrace{\frac{\lambda^t}{(2\lambda)^t}}^{(\frac{\lambda}{2\lambda})^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = Bin(t,\tfrac{1}{2})$$

$B_1, B_2, .. \overset{iid}{\sim} \text{Bern}(p)$

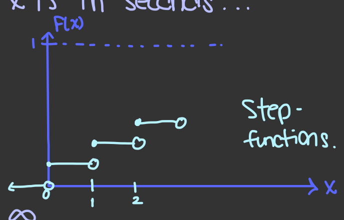
$X_1 \sim \text{Geom}(p) := (1-p)^x p\, \mathbb{1}_{x\in\{0,1,..\}}$ if $x=7$, $X_1>7 \Rightarrow X_1\in\{8,9,10,...\}$

$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1>x) = 1-(1-p)^{x+1}$

Let there be n experiments in each second (time unit). x is in seconds...

$p_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},...,1,1+\frac{1}{n},..\}}$

$F_X(x) = 1-(1-p)^{nx+1}$



Let's put infinite experiments into every second (time unit). this is the limit as n goes to positive ∞. X_∞

And $p\to 0$ but $\lambda=np \Rightarrow p=\frac{\lambda}{n}$ a la Poisson.

$$p_{X_\infty}(x) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \mathbb{1}_{x\in\{0,\frac{1}{n},\ldots\}}$$
$$= \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x \lim_{n\to\infty}\frac{\lambda}{n} \lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\ldots\}}$$
$$= e^{-\lambda x}(0)\mathbb{1}_{x\in[0,\infty)} \to \text{Supp}[X_\infty]=[0,\infty)$$
$$= 0 \quad \forall x$$

This is not a PMF b/c $\sum_{x\in\text{Supp}[X_\infty]} p_{X_\infty}(x) = 0 \neq 1$

$$F_{X_\infty}(x) \stackrel{:= P(X_\infty \le x)}{=} \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx+1} = 1 - \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{e^{-\lambda x}} \cdot \underbrace{\lim_{n\to\infty}\cancel{\left(1-\frac{\lambda}{n}\right)^1}}_{1} = (1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}$$

Is this limiting CDF a legal CDF? If so, it must satisfy three conditions:

(1) limit as x goes to negative infinity is zero.

(2) limit as x goes to positive infinity is one.

(3) increasing function i.e. its derivative is >=0

-> (1) $\lim_{x\to-\infty}(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 0$

(2) $\lim_{x\to\infty}(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)} = 1 - \lim_{x\to\infty}\frac{1}{e^{\lambda x}} \stackrel{\text{size } \lambda>0}{=} 1$

(3) $\frac{d}{dx}\left[(1-e^{-\lambda x})\mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)}$

=> F_{X_∞} is a valid CDF... but of what? A continuous r.v.

A continuous r.v. X has $\text{supp}[X]\subseteq\mathbb{R}$ but $|\text{supp}[X]| = |\mathbb{R}|$, this size is known as "uncountable infinity" or the "size of the continuum". They also have no PMF, the $P(X=x)$ is always zero for every x. But they have a CDF And the derivative of the CDF is a very useful function, so it gets a special name which is the "probability density function", or just "density" (PDF) denoted f:

$$f(x) := F'(x), \quad P(X\in(a,b)) = \underbrace{P(X\le b)}_{F(b)} - \underbrace{P(X\le a)}_{F(a)} = \int_a^b f(x)dx$$

Fundamental Thm of Calc.

$f(x) \ge 0 \ \forall x$ (property of CDF) <=> $\text{Supp}[X] = \{x \mid f(x)>0\}$

$\int_{\mathbb{R}} f(x)dx = 1$, $\int_{-\infty}^{\infty} f(x)dx = \underbrace{F(\infty)}_{1} - \underbrace{F(-\infty)}_{0} = 1$ (properties of CDF)

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(x) \stackrel{\text{if } X_1,\ldots,X_K \text{ independent}}{=} f_{X_1}(x_1)\cdot\ldots\cdot f_{X_K}(x_K) \stackrel{\text{identically dist.}}{=} f(x_1)\cdot\ldots\cdot f(x_K)$$

all componets continuous

$$\overbrace{\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}}^{K} f_{\vec{x}}(x)\,dx_1\cdot\ldots\cdot dx_K = 1$$

If $K=2$ -> $P(\vec{X}\in A) = \iint_A f_{\vec{x}}(\vec{x})\,dx_1 dx_2$

$$X \sim \text{Exp}(\lambda) := \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)} \to \text{supp}$$

$\to$ Exponential R.V.

$\lambda \in (0, \infty)$