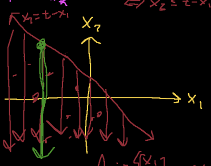
$\vec{X}$ continuous rv $\qquad P(\vec{X} \in A) = \int \cdots \int_A f_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_k$

$t = x_1 + x_2 \Leftrightarrow x_2 \le t - x_1$



$A_t := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_2 \le t - x_1 \right\}$

let $T = X_1 + X_2 \sim f_T(t) = ?$

First note $f_T(t) = F'(t)$ CDF method

usually it is difficult to find the CDF of continuous rv's, so this is not the usual method. The usual method is to use the convolution formula (which we will now derive)

$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1, x_2)\, dx_2\, dx_1 = \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\, dv\, dx = \int_{-\infty}^{t} \left(\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\, dx \right) dv$$

let $x_1 = x$

$x_2 = v - x \Rightarrow v = x_2 + x \Rightarrow dx_2 = dv$

$\Rightarrow x_2 = -\infty \Rightarrow v = -\infty$

$x_2 = t - x \Rightarrow v = t$

$$f_T(t) = \frac{d}{dt}\left[\quad \right] =$$

Leibnitz's Rule

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\, dy \right] = g(x, b(x))\, b'(x) + g(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[g(x,y) \right] dy$$

If the outer derivative is a third variable, then

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + g(x, a(t))\, a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\left[g(x,y) \right] dy}$$

0

$$\frac{d}{dt}\left[\int_{c}^{b(t)} g(x,y)\, dy \right] = g(x, b(t))\, b'(t) + \cancel{g(x, c)\, \frac{d}{dt}[c]}$$

0

$$= \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\, dx = f_T(t) = f_{X_1}(x) * f_{X_2}(x)$$ general convolution formula

if X_1, X_2 independent

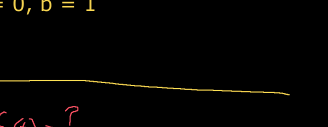
$$= \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx = \int_{Supp[X_1]} f_{X_1}^{old}(x)\, f_{X_2}^{old}(t-x)\, \mathbb{1}_{t-x \in Supp(X_2)}\, dx$$

$X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(x) f(t-x)\, dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

continuous uniform rv

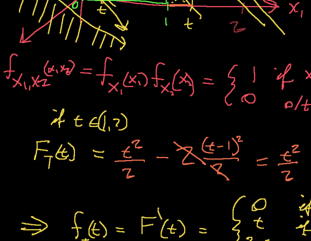
$$X \sim U(a,b) = \underbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)} \mathbb{1}_{x \in [a,b]}}_{f(x)}$$



standard uniform rv is when a = 0, b = 1

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $\quad T = X_1 + X_2 \sim f_T(t) = ?$



we want to compute CDF which means we want to find volumes in regions under the diagonal line

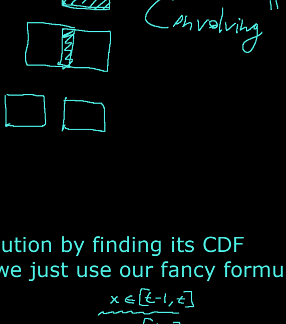
$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^2/2 & \text{if } t \in [0,1] \\ -\frac{t^2}{2} + 2t - 1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \ge 2 \end{cases}$$

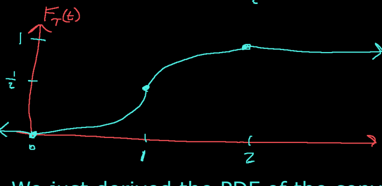
$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} 1 & \text{if } x_1 \in [0,1] \;\&\; x_2 \in [0,1] \\ 0 & o/t \end{cases}$$

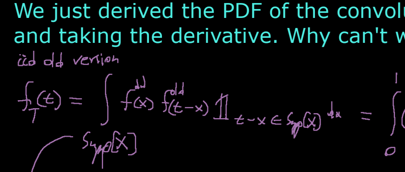
if $t \in (1,2)$

$$F_T(t) = \frac{t^2}{2} - \cancel{2}\frac{(t-1)^2}{\cancel{2}} = \frac{t^2}{2} - (t^2 - 2t + 1) = -\frac{t^2}{2} + 2t - 1$$

$$\Rightarrow f_T(t) = F'(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$







We just derived the PDF of the convolution by finding its CDF and taking the derivative. Why can't we just use our fancy formula?

iid old version

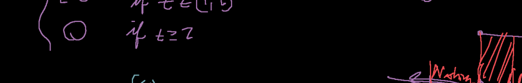
$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp(X)}\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{t-x \in [0,1]}\, dx = \int_0^1 \mathbb{1}_{x \in [t-1,t]}\, dx$$

$x - t \in [-1, 0]$

$x \in [t-1, t]$

Let's do some examples. How about t = -37? $\int_0^1 \mathbb{1}_{x \in [-38,-37]}\, dx = 0$

How about +37? $\int_0^1 \mathbb{1}_{x \in [36,37]}\, dx = 0$

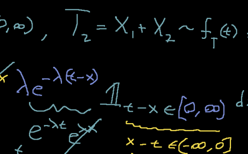


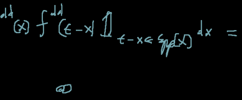
$$= \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$

$t \in (1,2)$ e.g. $t = 1.63$

$$\int_0^1 \mathbb{1}_{x \in [0.63, 1.63]}\, dx = \int_{0.63}^{1} dx = 1 - (t-1) = 2 - t$$

$|2-t|$





$$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) := \lambda e^{-\lambda x}\, \mathbb{1}_{x \in [0,\infty)}, \quad T_2 = X_1 + X_2 \sim f_T(t) = ?$$

$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx = \int_0^{\infty} \cancel{\lambda e^{-\lambda x}}\, \underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t}\, \cancel{e^{\lambda x}}}\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$x - t \in (-\infty, 0]$

$x \in (-\infty, t]$

$$= \lambda^2 e^{-\lambda t} \int_0^{\infty} \mathbb{1}_{x \in (-\infty, t]}\, dx = \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = Erlang(2, \lambda)$$