Lecture - 15

Mixture and compound distributions :-

Consider a situation where 2/3 of the time there is fast internet speed. So your downloads take $T \sim Exp(1/5) \Rightarrow E(T) = 5$ sec
and the other 1/3 of the time, there is internet trafic, so your downloads take $T \sim Exp(1/20) \Rightarrow E(T) = 20$ sec.
What is the distribution of the "overall T" or "unconditional on the internet speed" ??

X, T = are two Variables

Soln: Let $X \sim Bern(2/3)$

$X = 1$ corresponds to fast internet

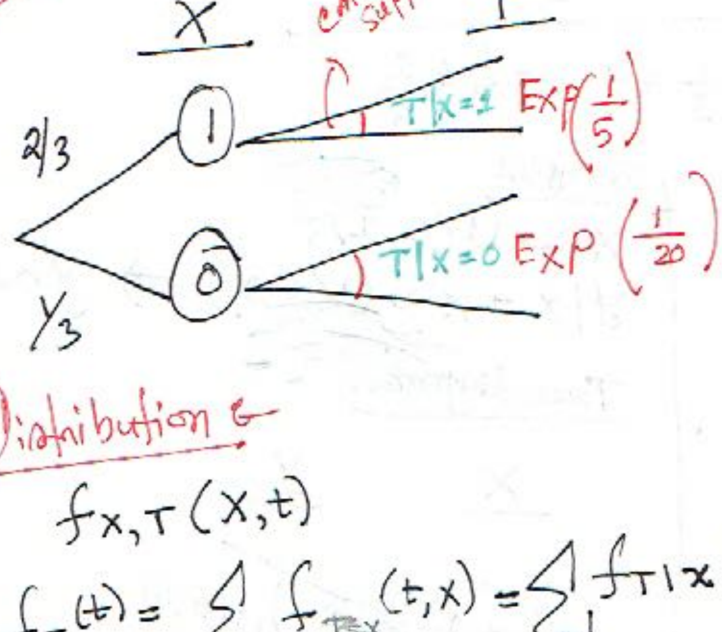
$X = 0$ corresponds to slow internet.

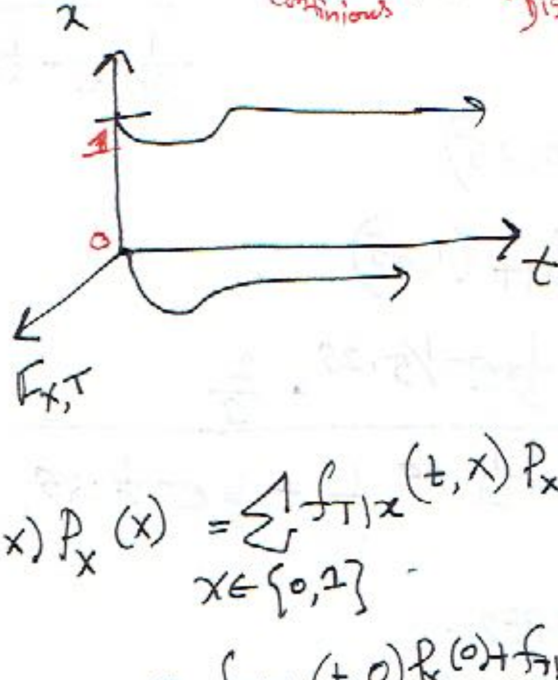
Last class

$\int_{\mathbb{R}} h(x,y)\,dy \qquad \sum_{y \in \mathbb{R}} h(x,y)$

Continious $\quad h(x) \quad$ Discrete

Table-①





Distribution :-

$f_{X,T}(x,t)$

$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp[X]} f_{T|X}(t,x)\,P_X(x) = \sum_{x \in \{0,1\}} f_{T|X}(t,x)\,P_X(x)$$

$$= f_{T|X}(t,0)\,P_X(0) + f_{T|X}(t,1)\,P_X(1)$$

$$\boxed{f_T(t) = \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t}\,\frac{2}{3}}$$

[from table ①]

Equation (A)

Q)(2) If the download Speed was $t = 25s$, what is the Probablity it is a slow internet day. i.e $X=0$?

If we got slow speed ~~that~~ means we in a slow day.

Supp $[0, 1]$, $X|T \sim$ Bernoulli (?)

Slow day, fast day

Since Supp $[0, 1]$

$$P_{X|T}(x,t) = \frac{f_{T|X}(t,x) \, P_X(x)}{f_T(t)}$$ "Bayes rule"

Get Bernoulli Parameter $= P_{X|T}(1,t)$

$W \sim \text{Bern}(P)$
$P = P(W=1)$

$$= \frac{f_{T|X}(t,1)\, P_X(1)}{f_T(x)}$$

$$= \frac{\frac{1}{5}e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20}e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5}e^{-\frac{1}{5}t}\frac{2}{3}}$$

$P_{X|T}(0,25)$

$= 1 - P_{X|T}(1,25)$

$$= 1 - \frac{\frac{1}{5}e^{-1/5 \cdot 25} \cdot \frac{2}{3}}{\frac{1}{20}e^{-\frac{1}{20}\cdot 25} \cdot \frac{1}{3} + \frac{1}{5}e^{-\frac{1}{5}\cdot 25} \cdot \frac{2}{3}}$$

$= 0.842$

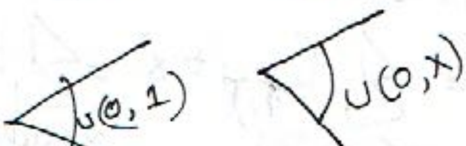
$= 84\%$ chance

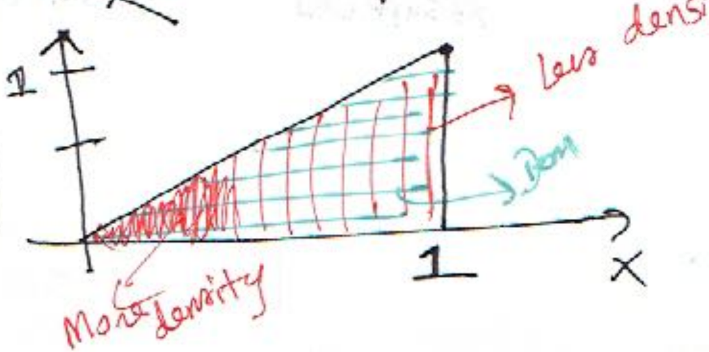
Example:

$X \sim U(0,1)$

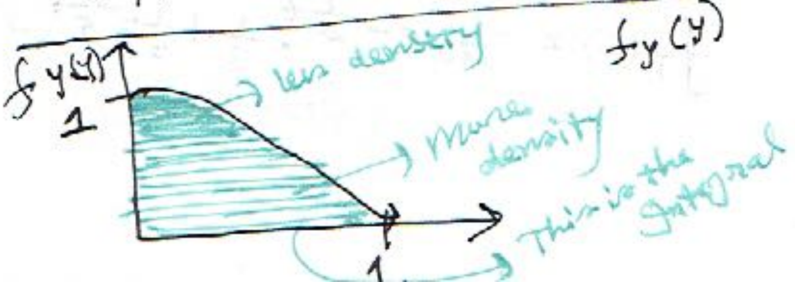
$Y|X = x \sim U(0,x) \Rightarrow Y \sim ?$

Tree diagram:





$f_Y(y)$



The first example featured:

T which was continious (we call that the "model") and X which is discrete (we call that the "mixing distribution". Thus the unconditional distribution T is called a "mixture distribution"

In the Second Example:

Y, the model is continious and X, the mixing distribution is also continious and we call the unconditional distribution y a "compound distribution".

$\boxed{\text{P 156-157 Page Book.}}$

Ex: $Y|X = x \sim \text{Poisson}(x)$, $X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim ?$

X		Y	Replacements
Gamma(α, β)	$\longleftarrow$	Poisson(λ)	$\lambda = X$, $x = y$

$$P_Y(y) = \int_{Supp[X]} P_{Y|X}(y,x)\, f_X^{old}(x)\, dx = \int_0^\infty \frac{e^{-x}x^y}{y!}\, \mathbb{1}_{y \in \mathbb{N}_{0,1,2,\dots}}\, \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \underbrace{\int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x} dx}_{\text{Lec, 9 Intg}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(x+\alpha)}{(\beta+1)^{x+\alpha}}$$

$$= \overset{H.W}{\cdots\cdots\cdots} = \boxed{\text{Extended Neg Bin}\left(\alpha, \frac{\beta}{\beta+1}\right)}$$

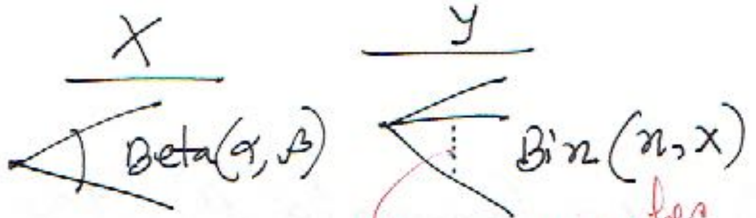
Poisson ↳ one Parameter less flexible

wait time, two Parameter more flexible Poisson →

This is a more flexible count model than the Poisson.

Ex: 2

$Y \mid X = x \sim Bin(n, \overset{p}{x})$, where n is known, $X \sim Beta(\alpha, \beta)$



means n Branches

$p = X, \ X = y$

$$P_Y(y) = P_{Y|X}(y,x)\, f_X^{old}(x)\, dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in 0 \dots n} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx$$

Discrete Discrete continious

$$= \binom{n}{y} \mathbb{1}_{y \in \{0, \dots, n\}} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1} dx}_{B(y+\alpha,\, n-y+\beta)}$$

↳ Beta

$$= \frac{B(y+\alpha, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y \in \{0, \dots, n\}}$$

Beta Binomial (n, α, β)

Ex: (3) $y|x = x \sim Exp(x)$

H.W

$x \sim Gamma(\alpha, \beta) \Rightarrow y \sim$ lomax | y = waiting time y.

which is a more flexible waiting time than the exponential.

END of Midterm (2)

Final Material

Moment generating function (mgf's) and characteristic functions (chf's). To derive these, we need to review complex / Imaginary #.

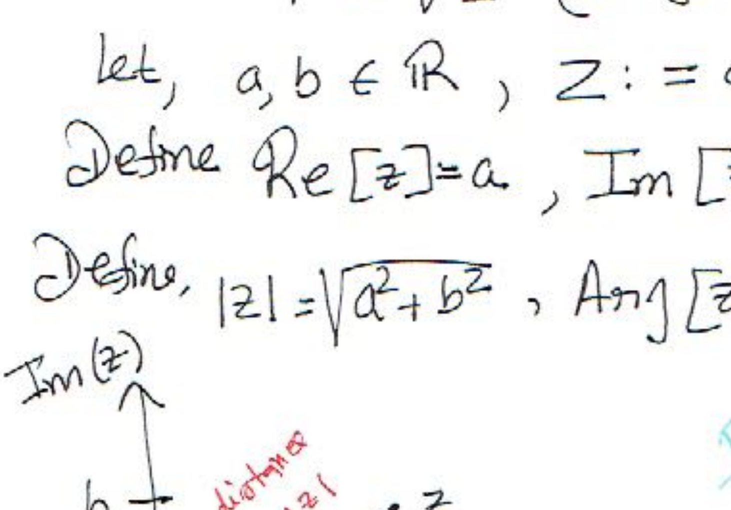
Define:

$i := \sqrt{1}$ "(Imaginary)"

let, $a, b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, Complex #'s

Define $Re[z] = a$, $Im[z] = b$, Real component and Imaginary component of complex #.

Define, $|z| = \sqrt{a^2 + b^2}$, $Arg[z] = \theta = Arc\tan\left(\frac{b}{a}\right) = \tan^{-1}\frac{b}{a}$



Review

$i^0 = 1$

$i^1 = 1$

$i^2 = (\sqrt{-1})^2 = -1$

$i^3 = i \cdot i^2 = -i$

$i^4 = 1$

$i^5 = i$ ------ $n \in \mathbb{N}$; $i^n = i^{n \bmod 4}$

clock cycle

Maclaurence Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \cdots$$

$$i\sin(tx) = \quad itx \quad - \frac{it^3x^3}{3!} \quad + \frac{it^5x^5}{5!} - \cdots$$

$$\cos(tx) = 1 \quad - \frac{t^2x^2}{2!} \quad + \frac{t^4x^4}{4!} \quad \cdots$$

Euler formula:

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \xRightarrow{tx=\pi} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$

Formula:

$$\boxed{e^{itx} = i\sin(tx) + \cos(tx)}$$

Monday → Reveiw for both

Wed → Exam (8:00 - 9:00)