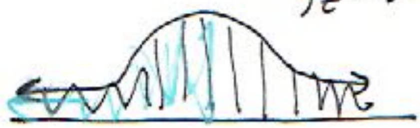
Lecture - ~~#~~ 18

$Z \sim N(0,1)$, $y = z^2 \sim f_y(y) = ?$ Not 1:1 $\qquad f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$



$F_y(y) = P(Y \le y) = P(Z^2 \le y)$

$= P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2P(Z \in [0, \sqrt{y}])$

$= 2(F_Z(\sqrt{y}) - \overset{\frac{1}{2}}{F_Z(0)}) = 2F_Z(\sqrt{y}) - 1$

$f_y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = 2\left(\frac{1}{2} y^{\overset{\frac{1}{2}-1}{-\frac{1}{2}}}\right) f_Z(\sqrt{y}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \ge 0}}$

$\propto y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

let's say,

$Z_1, Z_2 \cdots Z_k \overset{iid}{\sim} N(0,1)$, $y = \overset{\sim G(\frac{1}{2},\frac{1}{2})}{Z_1^2} + \overset{\sim G(\frac{1}{2},\frac{1}{2})}{Z_2^2} \cdots \overset{\sim G(\frac{1}{2},\frac{1}{2})}{Z_k^2} \sim \boxed{\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)}$

<u>Note</u>: The beta is always 1/2 and the alpha is always $k/2$ so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi^2_k$, the "chi squared distribution with k degrees of freedom" $k \in \mathbb{N}$

$E[y] = kE[z^2] = k \qquad k = 1,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\chi^2_k = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{k/2 - 1} e^{-y/2} \mathbb{1}_{y \ge 0} \overset{?}{=} \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-y/2} \mathbb{1}_{y \ge 0} = \chi^2_1$

If, $X \sim \chi^2_k$, $y = \sqrt{x} \Rightarrow x = y^2 = g^{-1}(y)$, $\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = |2y| = 2y$

$f_y(y) = f_x(y^2) \cdot 2y = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{k-2} e^{-y^2/2} (2y) \mathbb{1}_{\overset{y \ge 0}{y^2 \ge 0}}$

$= \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}-1}}{\Gamma\frac{k}{2}} y^{k-1} e^{-y^2/2} \mathbb{1}_{y \ge 0} = \chi_k$, the chi distribution with k degrees of freedom.

$Z \sim N(0,1)$, $|Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\, \mathbb{1}_{y \geq 0}$

$$= 2\underbrace{\left(\frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\right)}_{f_Z} \mathbb{1}_{y \geq 0}$$

Let's, $X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$, when $\underbrace{c>0}$

$$f_Y(y) = \frac{1}{c} f_X\left(y/c\right) = \frac{1}{c}\,\frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{y}{c}\right)^{\alpha-1}}_{\frac{y^{\alpha-1}}{c^{\alpha-1}}} e^{-\underbrace{\beta\frac{y}{c}}_{(\beta/c)y}}\, \mathbb{1}_{\overbrace{\frac{y}{c}>0}^{y>0}}$$

c^α

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\beta/c)y}\, \mathbb{1}_{y \geq 0} = \boxed{\text{Gamma}\left(\alpha, \frac{\beta}{c}\right)}$$

Here, $X \sim \chi^2_k$, $Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{1/2}{1/k}\right) = \text{Gamma}(k/2, k/2)$

Let, $X_1 \sim \chi^2_{k_1}$, indep of $X_2 \sim \chi^2_{k_2}$

Let, $U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\overbrace{\frac{k_1}{2}}^{a}, \overbrace{\frac{k_1}{2}}^{a}\right)$ Independent $V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\overbrace{\frac{k_2}{2}}^{b}, \overbrace{\frac{k_2}{2}}^{b}\right)$

Formula (last note) G

Let, $R = \frac{U}{V} \sim f_R(r) = \int_{Supp[V]} f_U(rt)\, \mathbb{1}_{rt \in Supp[U]}\, f_V(t)\, |t|\, dt$

$$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\, \mathbb{1}_{\overbrace{rt \in [0,\infty)}^{r \in (0,\infty)}} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\, t\, dt$$

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1}\, \mathbb{1}_{r \geq 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\, dt$$

$$= a^a b^b r^{a-1}\, \mathbb{1}_{r \geq 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)}\, \Gamma(a+b)}_{\frac{1}{B(a,b)}} \cdot \frac{1}{(ar+b)^{a+b}} = \frac{a^a b^b}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)}$$

$$\frac{b^{-(a+b)}}{b^{-a} b^{-b}} \left(1 + \frac{a}{b} r\right)^{-(a+b)} \mathbb{1}_{r \geq 0}$$

$$= \frac{(a/b)^a}{B(a,b)} p^{a-1}\left(1+\frac{a}{b}p\right)^{-(a+b)} \mathbb{1}_{p\geq 0}$$

$$= \frac{(k_1/k_2)^{k_1/2}}{B(k_1/2, k_2/2)} p^{k_1/2-1}\left(1+\frac{k_1}{k_2}p\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{p\geq 0}$$

$= F_{k_1,k_2}$. this is the "F distribution" ~~or~~ "Fisher-Snecedor Distribution" with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom $k_1, k_2 \in \mathbb{N}$

let, $Z \sim N(0,1)$, $X \sim \chi^2_K$, $Y = \frac{Z}{\sqrt{X/K}} \sim f_Y(y) = ?$

Symmetric around 0

consider,

$$Y^2 = \frac{Z^2}{X/K} \sim F_{1,K}$$

Consider, $W^2 = \frac{Z^2/1}{X/K} \sim F_{1,K}$

$$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-\omega, \omega]) = F_W(\omega) - F_W(-\omega)$$

Take derivatives ~~e~~

$$\frac{d}{dw}\left[F_{W^2}(\omega^2)\right] = \frac{d}{d\omega}\left[F_\omega(\omega)\right] - \frac{d}{d\omega}\left[F_\omega(-\omega)\right]$$

$$\cancel{2}w f_{\omega^2}(\omega^2) = f_W(w) - f_W(-w) = \cancel{2} f_W(w)$$

$$f_W(w) = w \frac{\left(\frac{1}{K}\right)^{\frac{1}{2}-1}}{B\left(\frac{1}{2}, \frac{K}{2}\right)} (w^2)^{\frac{1}{2}-1}\left(1+\frac{w^2}{K}\right)^{-\frac{1+K}{2}} \mathbb{1}_{\omega^2 \geq 0}$$

$\omega \in \mathbb{R}$

$$\frac{\frac{\Gamma(k+1)}{2}}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)}$$

$$\boxed{\frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)}\left(1+\frac{w^2}{K}\right)^{-\frac{K+1}{2}} = T_K}$$

Students T distribution with K degrees of freedom, discovered in 1908 by william Gosset while he was working at a beer factory

If, $k \to \infty$, $T_k \to Z$

Students T distribution has the $N(0,1)$ shape but just thicker tails.

Let, $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f(r)|u|\,du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\,du$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2}(-u)\,du + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{-\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2}|u|\,du\right)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} u\,du$$

u Substitution:

Let, $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u}dt$, $u = 0 \Rightarrow t = 0$,
$u = \infty \Rightarrow t = \infty$

Plug in,

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-\frac{r^2+1}{2}t}\, u \frac{1}{2u}\,dt = \frac{1}{2\pi}\frac{1}{\frac{r^2+1}{2}}\underbrace{\int_{0}^{\infty}\frac{r^2+1}{2}e^{-\frac{r^2+1}{2}t}\,dt}$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

Let, $X = c + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$, $\underbrace{\int_0^{\infty} \lambda e^{-\lambda x}\,dx = 1}_{\text{Pdf experiments}}$

$X \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi}\frac{1}{1-\left(\frac{x-c}{\sigma}\right)^2}$

srv