Lecture 13

$\frac{d}{du}(uv) = uv' + u'v$

$$f_{X_k}(x) = \sum_{j=k}^{n} \binom{n}{j}\Big(j f(x) F(x)^{j-1}(1-F(x))^{n-j} - \underbrace{(n-j)}_{\text{IF } j=n \;\Rightarrow\; 0} f(x) F(x)^{j}(1-F(x))^{n-j-1}\Big)$$

$$= \sum_{j=k}^{n} \frac{n!}{\cancel{j!}_{(j-1)!}(n-j)!}\cancel{j} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n \to n-1} \frac{n!}{j!\,\cancel{(n-j)!}_{(n-j-1)!}} \cancel{(n-j)} f(x) F(x)^{j} (1-F(x))^{n-j-1}$$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!(n-(j+1))!} f(x)F(x)^{j}(1-F(x))^{n-j-1}$$

$\boxed{\text{Reindexing trick}}$ for → let $l = j+1 \Rightarrow j = l-1$ & $j=k \Rightarrow l = k+1$; $j = n-1 \Rightarrow l = n$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{l=k+1}^{n} \frac{n!}{(l-1)!(n-l)!} f(x) F(x)^{l-1}(1-F(x))^{n-l}$$

note that Both sum expression are exactly same, so when We subtract we're left with just expression when $j=k$

$$= \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1}(1-F(x))^{n-k} = f_{X_{(n)}}(x) \text{ PDF}$$

Let make sure we can uncover min/max formulas

$$f_{X_{(1)}}(x) = \frac{n!}{(1-1)!(n-1)!} f(x) F(x)^{1-1}(1-F(x))^{n-1} = n f(x)(1-F(x))^{n-1}$$

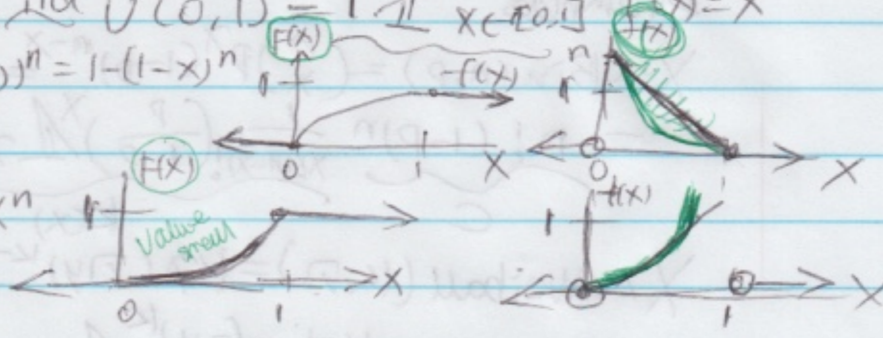
$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!(n-n)!} f(x) F(x)^{n-1}(1-F(x))^{n-n} = n f(x) F(x)^{n-1}$$

example: $X_1 \dots X_n$ iid $U(0,1) = 1 \cdot \mathbb{1}_{x\in[0,1]}$ (density) $F(x) = x$

$F_{X_{(1)}}(x) = 1-(1-F(x))^n = 1-(1-x)^n$

PDF Density $f_{X_{(1)}}(x) = n(1-x)^{n-1}$

$F_{X_{(n)}}(x) = F(x)^n = x^n$



$f_{X_{(n)}}(x) = n x^{n-1}$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \overset{CDF}{x^{k-1}} (1-\overset{CDF}{x})^{n-k}\, \mathbb{1}_{x\in[0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k+1-1}\, \mathbb{1}_{x\in[0,1]}$$

Erlang + Erlang = Erlang

$$= \text{Beta}(k, n-k+1)$$

$\mathbb{Z}_{\geq 0}$

$$X \sim \underbrace{\text{Gamma}(\alpha_1, \beta)}_{\text{Erlang}(\alpha_1,\beta)} \text{ independency of } Y \sim \underbrace{\text{Gamma}(\alpha_2, \beta)}_{\text{Erlang}(\alpha_2,\beta)}$$

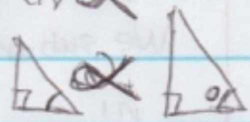
$$T = X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

Reasonable

To prove this, we develop a new tool that makes it easier for us. That's "kernels" $k(x)$. For any PDF or PMF, we can decompose it into a normalization constant c and a kernel $k(x)$,

$$p(x) = c\,k(x) \text{ and } f(x) = c\,k(x) \Rightarrow f(x) \propto k(x). \quad f(x) \propto k(x)$$

c not a function of x



$$1 = \sum_{\text{supp}} p(x) = \sum c\,k(x) \Rightarrow \frac{1}{c} = \sum k(x) \Rightarrow c = \left(\sum k(x)\right)^{-1}$$

$$1 = \int_{\text{supp}} f(x)\,dx = \int c\,k(x)\,dx \Rightarrow \frac{1}{c} = \int k(x)\,dx \Rightarrow c = \left(\int k(x)\,dx\right)^{-1}$$

This means that $k(x)$ is 1:1 with PMF or PDF. If you know $k(x)$, you know the distribution of rv. Let see some examples

$$X \sim \text{Bin}(n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$$

$$= \underbrace{n!\,(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1}_{x \in \{0,1,\dots,n\}}}_{k(x)}$$

$$X \sim \text{Weibull}(k, \lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}$$

$$= \underbrace{k\lambda^k}_{c} \underbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}}_{k(x)}$$

$$X \sim \text{Gamma}(\alpha,\beta) = \underbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}_{C} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\geq 0}}_{k(x)} \propto x^{a} e^{-bx} \mathbb{1}_{x\geq 0}$$

$X \sim \text{Gamma}(\alpha_1,\beta)$ independent $Y \sim \text{Gamma}(\alpha_2,\beta)$ $T = X+Y$

$$f_T(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-\beta x} \underbrace{\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} e^{-\beta(t-x)} \mathbb{1}_{t-x\in[0,\infty)}}_{\sim f_Y(t-x)} dx$$

$\cancel{e^{-\beta t} e^{\beta x}}$

$x-t \in (-\infty,0)$

$x \in (-\infty,t]$

$$= \int_0^t$$

$x \in [-\infty, t]$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1-1} (t-x)^{\alpha_2-1} dx \, \mathbb{1}_{t\geq 0}$$

$$\propto e^{-\beta t} \int_0^t x^{\alpha-1} (t-x)^{\alpha_2-1} dx \, \mathbb{1}_{t\geq 0}$$

multiple

$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1-1} \left(1-\frac{x}{t}\right)^{\alpha_2-1} dx \, \mathbb{1}_{t\geq 0}$$

let $u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$ $\quad x=0 \Rightarrow u=0$, $x=t \Rightarrow u=1$

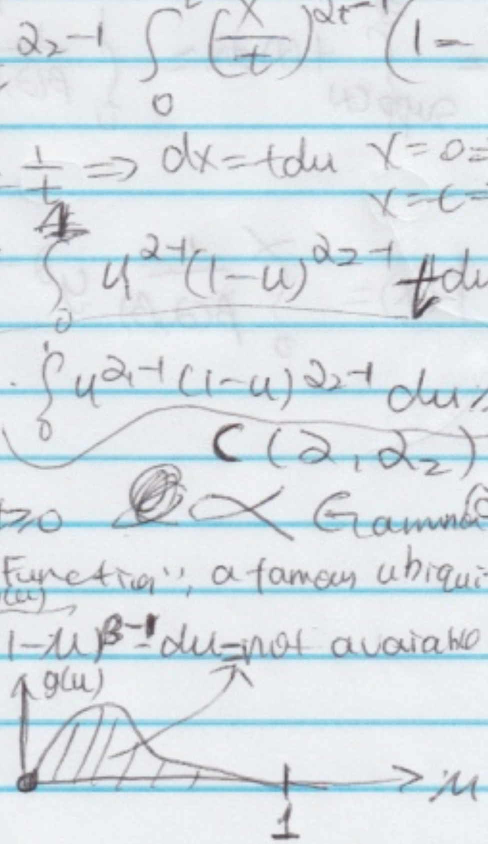
$$= e^{-\beta t} t^{\alpha_1-1} t^{\alpha_2-1} \int_0^1 u^{\alpha-1} (1-u)^{\alpha_2-1} t\, du \, \mathbb{1}_{t\geq 0}$$

$$= e^{-\beta t} t^{\alpha_1+\alpha_2-1} \cdot \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1} du}_{C(\alpha_1,\alpha_2)} \mathbb{1}_{t\geq 0}$$

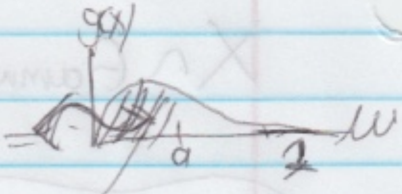
$$\propto e^{-\beta t} t^{\alpha_1+\alpha_2-1} \mathbb{1}_{t\geq 0} \propto \text{Gamma}(\alpha_1+\alpha_2, \beta)$$

Let's talk about "Beta Function", a famous ubiquitous function

$$B(\alpha,\beta) := \int_0^1 \overbrace{u^{\alpha-1}(1-u)^{\beta-1}}^{g(u)} du$$ = not available in closed form



The "incomplete Beta Function" is:

$$B(a, \alpha, \beta) = \int_0^a u^{\alpha-1}(1-u)^{\beta-1}\,du =$$ 

The requalized incomplete Beta Function is:

$$I_a(\alpha,\beta) = \frac{B(a,\alpha,\beta)}{B(\alpha,\beta)} = \underset{\text{Percentage}}{\% \in [0,1]}$$ NOT Bessel Function!

Let's derive a Beta ~~function~~ - gamma Function identity.

$$T \sim \text{Gamma}(\alpha_1+\alpha_2,\beta) = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)} \cancel{t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\geq 0}}$$

$$= \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2) \cancel{t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\geq 0}}$$

$$\Rightarrow B(\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$ Cool idensity!

$$X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]} \quad \alpha,\beta>0$$

$$1 = \int_{supp(X)} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{\cancel{B(\alpha,\beta)}} \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}_{\cancel{B(\alpha,\beta)}} = 1$$

$$\text{CDF } F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-u)^{\beta-1}\,dy = \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^x y^{\alpha-1}(1-y)^{\beta-1}\,dy}_{\text{icomplete } \beta \text{ Funct}}$$

$$= \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$