Lecture 19

$$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi} \cdot \frac{1}{x^2+1}$$

$$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi} \frac{1}{x^2+1} dx = \infty \rightarrow \text{the expecation does not exist}$$

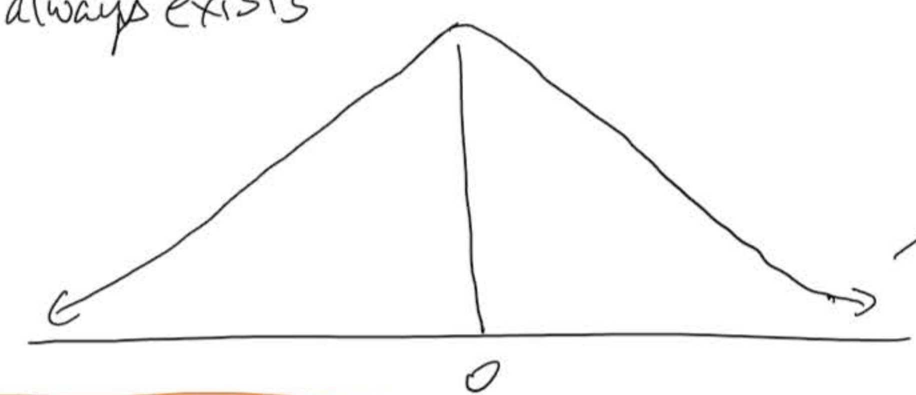
(mgf)

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\pi} \frac{1}{x^2+1} dx = \infty \rightarrow \text{D.N.E.}$$

$$\phi_X(t) = \int e^{itx} \frac{1}{\pi} \frac{1}{x^2+1} dx = \ldots = e^{-|t|}, \quad \phi_X'(t) = \frac{-t}{|t|} e^{-|t|}$$

↳ always exists

$$\phi_X'(0) = \text{DNE}$$



tails are fat enough so that the Integral of this curve weighted by $x \in \mathbb{R}$ does not converge.

Let's derive the Cauchy distribution like the physicists found it.

$$\theta \sim U\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\pi/2, \pi/2]}$$

$x = g(\theta)$

$\theta = g^{-1}(x) = \arctan x$ – Tangent is invertible between $-\pi/2$ and $\pi/2$

$$f_X(x) = f_\theta(g^{-1}(x)) \left| \frac{d}{dx}[g^{-1}(x)] \right| = \frac{1}{\pi} \mathbb{1}_{\underbrace{\arctan(x) \in [-\pi/2, \pi/2]}_{x \in \mathbb{R}}} \cdot \frac{1}{x^2+1}$$

$$= \underline{\text{Cauchy}(0,1)}$$

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \frac{X_i - \mu}{\sigma} = Z_i \sim N(0,1)$, Sample Var.

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow \frac{X_i - \mu}{\sigma} = Z_i \sim N(0,1)$ Sample Var.

$T_n \sim N(n\mu, n\sigma^2)$, $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$, $S_n^2 = \frac{1}{n-1}\sum(X_i - \bar{x})^2 \sim f_{S^2}(s^2) = ?$

$\Rightarrow Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$

$$\vec{Z}^T\vec{Z} = \sum_{i=1}^n Z_i^2 = \chi_n^2 = \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} =$$

$$\sum(X_i - \mu)^2 = \sum\left((X_i - \bar{X}) + (\bar{X} - \mu)\right)^2 = \sum(X_i - \mu)^2 + \underbrace{\sum 2(X_i - \bar{x})(\bar{X} - \mu)}_{0} + \sum(\bar{x} + \mu)^2$$

$$= \sum(X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$$

$$\Rightarrow \frac{\sum(X_i - \mu)^2}{\sigma^2} = \underbrace{\frac{n-1}{\sigma^2}}_{\text{Then maybe } \chi_{n-1}^2} S^2 + \underbrace{\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{Z^2 \sim \chi_1^2} \sim \chi_n^2$$

recall from last class:
$U_1 \sim \chi_{k_1}^2$ indep. of $U_2 \sim \chi_{k_2}^2$
$\Rightarrow U_1 + U_2 \sim \chi_{k_1+k_2}^2$

In order for this "maybe" to be true, we need independence of those two terms i.e. we need S^2 and $\bar{X}$ to be independent.

We need Cochran's Theorem to prove this.

$Z^T Z = \vec{Z}^T I \vec{Z} \sim \chi_n^2$ This scalar is called a "quadratic form"

Consider $\vec{Z}^T \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix} \vec{Z} = Z_1^2 \sim \chi_1^2$

$$\vec{Z}^T I \vec{Z} = \vec{Z}^T(B_1 + \dots + B_n)\vec{Z} = \vec{Z}^T B_1 \vec{Z} + \dots + \vec{Z}^T B_n \vec{Z} \sim \chi_n^2$$

$\chi_1^2 + \dots \chi_1^2 \dots + \chi_1^2$

rank$[B_i] = 1$

$\sum^n$ ...

$$\vec{z}^T I \vec{z} = \vec{z}^T (B_1 + \dots + B_n) \vec{z} = \vec{z}^T B_1 \vec{z} + \dots + \vec{z}^T B_n \vec{z} \sim \chi^2_n$$

$$\chi^2_1 + \dots \chi^2_1 \dots + \chi^2_1$$

$$\boxed{\begin{gathered} \text{rank}[B_i] = 1 \\ \sum_{i=1}^{n} \text{rank}[B_i] = n \end{gathered}}$$

Conjecture: Each of these quadratic forms is independent.

Cochran's thm: If $B_1 + \dots + B_k = I$, $k \le n$ and the sum of their ranks is n then you have two powerful results:

(a) $\vec{z}^T B_j \vec{z} \sim \chi^2_{\text{rank}[B_j]}$ and (b) $\vec{z}^T B_{j_1} \vec{z}$ is indep. of $\vec{z} B_{j_2} \vec{z}$ $\forall j_1 \neq j_2$

Consider $\sum (z_i - \bar{z})^2 = \sum z_i^2 - 2 \underbrace{\sum z_i}_{n\bar{z}} \bar{z} + \sum \bar{z}^2 = \sum z_i^2 - 2n\bar{z}^2 + n\bar{z}^2$

$$= \sum z_i^2 - n\bar{z}^2$$

let $\vec{1}_n$ = n-dim column vector of all ones $\quad \bar{z} = \frac{1}{n} \vec{1}^T \vec{z} = \frac{1}{n} \vec{z}^T \vec{1}$

$$n\bar{z}^2 = n\bar{z}\bar{z} = \cancel{n} \frac{1}{\cancel{n}} \vec{z}^T \vec{1} \frac{1}{n} \vec{1}^T \vec{z} = \frac{1}{n} \vec{z}^T \vec{1} \vec{1}^T \vec{z} = \vec{z}^T \underbrace{\left(\frac{1}{n} J_n\right)}_{B_2} \vec{z}$$

let $J_n = \vec{1}\vec{1}^T$, which is an $n \times n$ matrix of all ones.

$$\sum (z_i - \bar{z})^2 = \vec{z}^T I z - \vec{z}^T \left(\frac{1}{n} J_n\right) \vec{z} = \vec{z}^T \left(I - \frac{1}{n} J_n\right) \vec{z}$$

$$\vec{z}^T \vec{z} = \sum (z_i - \bar{z})^2 + n\bar{z}^2 = \vec{z}^T B_1 \vec{z} + \vec{z}^T B_2 \vec{z}$$

I want to use Cochran's thm on the above expression. So I need to make sure $B_1 + B_2 = I$ and $\text{rank}[B_1] + \text{rank}[B_2] = n$

$$B_1 + B_2 = \left(I - \frac{1}{n} J_n\right) + \frac{1}{n} J_n = I \quad ✓$$

$$\text{rank}[B_2] = \text{rank}\left[\frac{1}{n} J_n\right] = \text{rank}[J_n] = 1$$

$\text{rank}[B_1] = \text{rank}[I - $ [illegible] $\Rightarrow \sum \text{rank}[B_i] = 1 + n - 1 = n$ ✓

$B_1 + B_2 = (I - \frac{1}{n} J_n) + \frac{1}{n} J_n = I$ ✓

$\text{rank}[B_2] = \text{rank}\left[\frac{1}{n} J_n\right] = \text{rank}\left[J_n\right] = 1$

$\text{rank}\left[B_1\right] = \text{rank}\left[I - \frac{1}{n} J_n\right] = n-1$ $\Rightarrow \sum \text{rank}[B_j] = 1 + n - 1 = n$ ✓

need a Thm for this: If A is symmetric and idempotent (ie. $AA = A$) then $\text{rank}[A] = \text{tr}[A] =$ sum of A's diagonal entries.

$(I - \frac{1}{n} J)^T = I^T - \frac{1}{n} J^T = I - \frac{1}{n} J$ ✓

$$(I - \tfrac{1}{n} J)(I - \tfrac{1}{n} J) = I I - \tfrac{1}{n} J I - \tfrac{1}{n} I J + \tfrac{1}{n^2} J J$$

$$= I - \frac{\cancel{2}^{\,1}}{n} J + \cancel{\frac{1}{n^2}\, n}\, J = I - \tfrac{1}{n} J$$

$\text{tr}\left[I - \frac{1}{n} J\right] = (1 - \frac{1}{n} 1) + (1 - \frac{1}{n} 1) + \dots + (1 - \frac{1}{n} 1) = n - 1 = \text{rank}\left[B_1\right]$

Since the two conditions of Cochran's Thm are satisfied, we can apply it to get the two results:

$\Rightarrow \vec{Z}^T B_1 \vec{Z} = \sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1}$ indep. of $\vec{Z}^T B_2 \vec{Z} = n \bar{Z}^2 \sim \chi^2_1$

What does this have to do with our goal? Well, it's the same thing:

$$\bar{Z} = \frac{Z_1 + \dots + Z_n}{n} = \frac{\frac{X_1 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}}{n} = \frac{\sum X_i - n\mu}{\sigma n} = \frac{\bar{X} - \mu}{\sigma}$$

$$\sum (Z_i - \bar{Z})^2 = \sum \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma} \right)^2 = \sum \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum (X_i - \bar{X})^2$$

$$= \frac{n-1}{\sigma^2} S^2$$

$$n \bar{Z}^2 = n \left(\frac{\bar{X} - \mu}{\sigma} \right)^2 = \left(\frac{\sqrt{n}\, (\bar{X} - \mu)^2}{\sigma} \right)^2 = \left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)^2$$

Since the two conditions of Cochran's Thm are satisfied, we can apply it to get the two results:

$$\Longrightarrow \vec{Z}^T B_1 \vec{Z} = \sum (Z_i - \bar{Z})^2 \sim \chi^2_{n-1} \quad \text{indep. of} \quad \vec{Z}^T B_2 \vec{Z} = n\bar{Z}^2 \sim \chi^2_1$$

What does this have to do with our goal? Well, it's the same thing:

$$\bar{Z} = \frac{Z_1 + \dots + Z_n}{n} = \frac{\frac{X_1 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}}{n} = \frac{\sum X_i - n\mu}{\sigma n} = \frac{\bar{X} - \mu}{\sigma}$$

$$\sum (Z_i - \bar{Z})^2 = \sum \left(\frac{X_i - \mu}{\sigma} - \frac{\bar{X} - \mu}{\sigma} \right)^2 = \sum \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum (X_i - \bar{X})^2$$

$$= \frac{n-1}{\sigma^2} S^2$$

$$n\bar{Z}^2 = n\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 = \left(\frac{\sqrt{n}\,(\bar{X} - \mu)^2}{\sigma}\right)^2 = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$\underbrace{\frac{n-1}{\sigma^2} S^2}_{\sim \chi^2_{n-1}} + \underbrace{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2}_{\sim \chi^2_1} \sim \chi^2_n$$

independent

Fisher proved this without Cochran's thm in 1925 and Geary proved in 1936 that this decomposition is exclusive to the iid normal r.v. model.

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1), \quad \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim ? \quad \text{Not } N(0,1)$$