Math 621

Lecture 10

10-07-2020

$T_K \sim$ Erlang (k, λ) $\qquad$ $N \sim$ Poisson(λ)

$P(T_K > 1) = 1 - F_{T_k}(1) = Q(k, \lambda)$ $\qquad$ $F_N(x) = Q(x+1, \lambda)$

N: # events before 1sec

$x_1, x_2, \ldots \overset{iid}{\sim} Exp(\lambda)$ $\quad k = 5$

$$\{T_5 > 1\} = \{x_1 + x_2 + x_3 + x_4 < 1\} \cup \{x_1 + x_2 + x_3 < 1\}$$
$$\cup \{x_1 + x_2 < 1\} \cup \{x_1 < 1\} \cup \{x_1 > 1\}$$
$$= \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}$$

$$\Rightarrow P(T_5 > 1) = P(n \leq 4) = F_N(4)$$
$$\|$$
$$1 - F_{T_5}(1)$$

$$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$$

"Poisson Process"

Question: Why is this poisson distutied?

Remember the development of $Exp(\lambda)$

$\Rightarrow$

n expenanats $\to \infty$ $\Big\}$ such that $\lambda = np$

$p \to 0$

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$$T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

$$k \in \mathbb{N}, \quad \lambda \in (0, \infty)$$

$$T \sim \text{Neg Bin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

$$k \in \mathbb{N}, \; p \in (0,1)$$

$$\rightarrow = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \geq 0}$$

$$= \frac{\Gamma(k+t)}{\Gamma(k) t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

For both,
What if, $k \in (0, \infty)$? Are both rv's still "legal"?

$\Rightarrow$ Yes, we can show that both

$$\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \, dt = 1 \;\; \forall$$

$$\sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k) t!} (1-p)^t p^k = 1$$

We just derived two new famous rv's:

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}$$

$$X \sim \text{Ext NegBin}(k, p) = \frac{\Gamma(k+t)}{\Gamma(k) t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$$

$\Downarrow$

the "extended negative binomial"

Transformations of Discrete rv's:

$$X \sim Bern(p) = p^x(1-p)^{1-x}\, \mathbb{1}\, x \in \{0,1\} = \begin{cases} 1 & wp\ p \\ 0 & wp\ 1-p \end{cases}$$

$$Y = X + 3 \overset{}{=} g(x) \sim \begin{cases} 4 & wp\ p \\ 3 & wp\ 1-p \end{cases}$$

$$//$$

$$y-3 = x = g^{-1}(y) = p^{y-3}(1-p)^{1-(y-3)}\ \mathbb{1}\, y-3 \in \{0,1\}$$

I want to find the PMF of Y using the PMF of X:

$$Y = g(x) \sim P_y(y) = P_x\left(g^{-1}(y)\right)$$ inverse function

What assumption did I make when I "derived" this formula?

⇒ I assumed an inverse function exists i.e. g is invertible. If not....

$$X \sim U(\{1, 2, \dots, 10\}) = \begin{cases} 1 & wp\ \frac{1}{10} \\ 2 & wp\ \frac{1}{10} \\ \vdots & \\ 10 & wp\ \frac{1}{10} \end{cases},$$

$$y = g(x) = \min(x, 3) \sim \begin{cases} 1 & wp\ 1/10 \\ 2 & wp\ 1/10 \\ 3 & wp\ P(X+3) + P(X+4) + \dots + P(X=10) = \frac{8}{10} \end{cases}$$

$$Y = g(x) \sim P_y(y) = \sum_{\{x;\, y = g(x)\}} P_x(x) = \sum_{\{x;\, x = y^{-1}(y)\}} P_x(x) = P_x(g^{-1}(y))$$

if g is invertable

One element only

$X \sim \text{Binom}(n,p)$, $Y = X^3 = g(X) \sim p_X(\sqrt[3]{y}) =$

$\Rightarrow g^{-1}(y) = \sqrt[3]{y}$

$$= \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}}$$

$$\mathbb{1}_{\sqrt[3]{y} \in \{0,1,\dots,n\}}$$

$$Y = X^2 \sim p_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,\dots,n\}}$$

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Transformations for Continuous rv's:

for g invertible $f_Y(y) \overset{?}{=} f_X(g^{-1}(y)) \leftarrow$ NO?

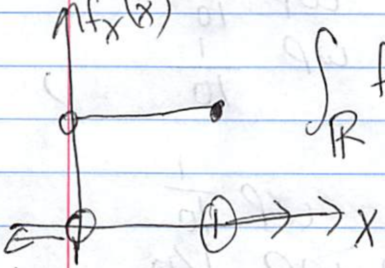
let $X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$ $= f_X(x)$, let $Y = 2X \overset{?}{\sim} f_X(g^{-1}(y))$

$= f_X(\frac{y}{2})$

$= \mathbb{1}_{\frac{y}{2} \in [0,1]}$

$= \mathbb{1}_{y \in [0,2]}$

$$\int_{\mathbb{R}} f_X(x)\,dx = 1$$



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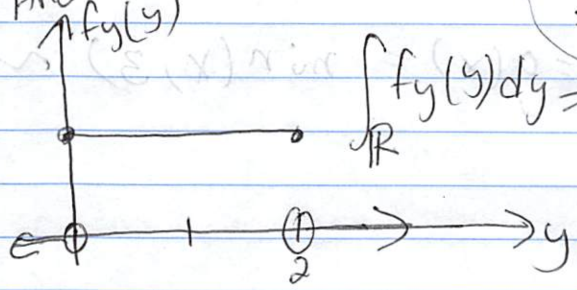


$$\int_{\mathbb{R}} f_Y(y)\,dy = 2$$

$\boxed{\text{Game Over!!}}$

PDF are not probabilities!!
So this was bound to fail because we used them as probabilites.
However, CDF's *are* probabilities.

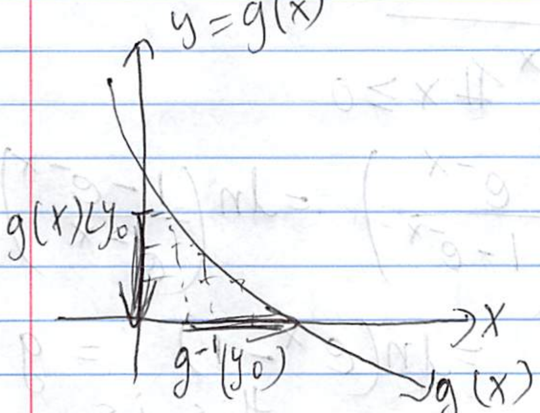
$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) \overset{g \text{ is invertible}}{=} P(X \leq g^{-1}(y))$$

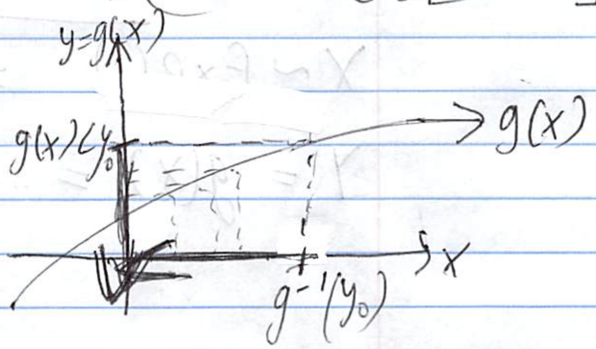
$$= F_X(g^{-1}(y))$$

$$\Rightarrow \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))]$$

Using Chain Rule $= F'_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)] =$

$$f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$





$$F_Y(Y) = P(Y \leq y) = P(g(X) \leq y) \overset{g \text{ is invertible and } g'<0}{=} P(X \geq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

$f_Y(y)$

$$\frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))]$$

$$= f_X(g^{-1}(y)) \underbrace{\left(-\frac{d}{dy}[g^{-1}(y)]\right)}_{\text{always negative}}$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$$

$\Leftarrow$ General Rule

We can also derive a less general but very useful corollary rule:

$Y = aX + c \sim f_Y(y) = ?$ shift and scale (shift by c, scale by a)

$\underbrace{}_{g(x) \text{ is invertible}} \Rightarrow g^{-1}(y) = \frac{y-c}{a} \quad \left| \frac{d}{dy}[g^{-1}(y)] \right| =$

$g^{-1}(y) = \frac{y-c}{a}$ $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{a}\right| = \frac{1}{|a|}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$

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$Y = aX \sim f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$,

$Y = X + c \sim f_X(y-c)$

* * *

$X \sim Exp(1) = e^{-x}\,\mathbb{1}\,x \geq 0$

$Y = g(X) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right)$

$= \ln(e^{x}-1) = g(x) \sim f_Y(y)$

this is invertible

$Y = \ln(e^{x}-1)$

$\Rightarrow e^{Y} = e^{x} - 1$

$\Rightarrow e^{Y} + 1 = e^{x} \Rightarrow x = \ln(e^{Y}+1) = g^{-1}(y)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{d}{dy}[\ln(e^{y}+1)]\right|$

$= \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$

$f_Y(y) = f_X \ln(e^{y}+1) \cdot \frac{e^{y}}{e^{y}+1}$

$= e^{-\ln(e^{y}+1)}\,\mathbb{1}\,\underbrace{\underbrace{\underbrace{\ln(e^{y}+1) \geq 0}_{e^{y}+1 \geq 1}}_{e^{y} \geq 0}}_{y \in \mathbb{R}} \cdot \frac{e^{y}}{e^{y}+1}$

$= \frac{1}{e^{y}+1} \cdot \frac{e^{y}}{e^{y}+1}$

$= \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{e^{-2y}}{e^{-2y}}$ means

$= e^{-y} / (e^{-y}+1)^2 = \text{Logistic}(0,1)$

Standard Logistic