Convergence in probability to a constant
$X_n \xrightarrow{P} c$. this means $\forall \varepsilon > 0 \lim P(|X_n - c| \geq \varepsilon) = 0$

Thm: if X_n has expectation mu for all n and variance Sigsq-n which is finite for all n, then lim Sigsq-n = 0 $\Rightarrow X_n \xrightarrow{P} \mu$.

Recall chebyshev's Inequality.

$P(|X_n - \mu| \geq \varepsilon) \leq \frac{\sigma_n^2}{\varepsilon^2}$ (Now take the limit both side)

$\lim P(|X_n - \mu| \geq \varepsilon) \leq \lim \frac{\sigma_n^2}{\varepsilon^2} = \frac{1}{\varepsilon^2} \lim \sigma_n^2 = 0$

because probabilities are between 0 and 1, if you know the probability is <= 0. that means the probability is 0.

$\Rightarrow \lim P(|X_n - \mu| \geq \varepsilon) = 0 \Rightarrow X_n \xrightarrow{P} \mu$

eg $X_n \sim (-\frac{1}{n}, \frac{1}{n})$ Prove $X_n \xrightarrow{P} 0$

e.g: $X_n \sim U(-\frac{1}{n}, \frac{1}{n})$ Prove $X_n \xrightarrow{P} 0$

$E[X_n] = 0 = \mu \ \forall n$, $\sigma_n^2 = \frac{(\frac{1}{n} - -\frac{1}{n})^2}{12} = \frac{4}{12n^2} = \frac{1}{3n^2}$

$\lim \sigma_n^2 = \lim \frac{1}{3n^2} = 0 \xRightarrow{\text{Thm}} X_n \xrightarrow{P} 0$

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e.g: $X_n \sim N(0, \frac{1}{n})$ Prove $X_n \xrightarrow{P} 0$

$E[X_n] = 0 = \mu \ \forall n$, $\sigma_n^2 = \frac{1}{n}$, $\lim \sigma_n^2 = \lim \frac{1}{n} = 0$

$\Rightarrow X_n \xrightarrow{P} 0$ ✓

Let $X_1, X_2, \ldots$ be iid with mean mu and Variance Sigsq < infinity. $\bar{X}_n = \frac{1}{n} \sum x_i$, $E[\bar{X}_n] = \mu \ \forall n$, $Var[\bar{X}_n] = \frac{\sigma^2}{n}$

Prove $\bar{X}_n \xrightarrow{P} \mu$.. $\lim Var\{\bar{X}_n\} = \lim \frac{\sigma^2}{n} = 0 \Rightarrow \bar{X}_n \xrightarrow{P} \mu$ ②

this is a very famous thm. it's called the "weak" weak law of large # (WLLN)

The Last type of Convergence we'll study is called Convergence in law or Convergence in L^r norm to a Constant. where $r \geq 1$ denoted: $X_n \xrightarrow{L^r} c$ which means by definition:

$\lim E[|X_n - c|^r] = 0,$

e.g: $r=1$ $\lim E[|X_n - c|] = 0$ "Convergence in mean"

$r=2$ $\lim E[(X_n - c)^2] = 0$ "mean square Convergence"

e.g $X_n \sim U(0, \frac{1}{n})$

Prove $X_n \xrightarrow{L^r} 0$ $\lim E[|X_n - 0|^r] = \lim E[X_n^r]$

$= \lim \int_{\mathbb{R}} x^r n \mathbb{1}_{x \in [0, \frac{1}{n}]} dx = \lim n \int_0^{\frac{1}{n}} x^r dx$

$= \lim n \left[\frac{x^{r+1}}{r+1}\right]_0^{\frac{1}{n}} = \frac{1}{r+1} \lim n \frac{1}{n^{r+1}} = \frac{1}{r+1} \lim \frac{1}{n^r} = 0$ ✓

Law. proof for $X_n \xrightarrow{L^r} \Rightarrow X_n \xrightarrow{P} c$

$\lim P[|X_n - c| \geq \varepsilon) = \lim P(|X_n - c|^r \geq \varepsilon^r) \leq \lim \frac{E[|X_n - c|^r]}{\varepsilon^r} = 0$