Lecture -09 10/05/2020

$$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$$

$$f_{T_3}(t) = \int_{supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in supp[X_3]}\, dx.$$

$$= \int_0^{\infty} x \lambda^2 \cancel{e^{-\lambda x}}\, \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}}\, \mathbb{1}_{\overbrace{t-x \in [0,\infty)}^{x \le t}}\, dx.$$

$$= \lambda^3 e^{-\lambda t} \int_0^{\infty} x\, \mathbb{1}_{x \le t}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^{t} x\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$$

$$f_{T_4}(t) = \int_{supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in [0,\infty)}\, dx.$$

$$= \int_0^{\infty} \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}}\, \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t}\cancel{e^{\lambda x}}}\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \lambda^4 e^{-\lambda t}\, \frac{1}{2} \int_0^{t} x^2\, dx\, \mathbb{1}_{t \in (0,\infty)}$$

$$= \frac{t^3}{3.2} \lambda^4 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \text{Erlang}(4, \lambda)$$

$$\sum_{i=1}^{k} X_i = T_k \sim \text{Erlang}(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in [0, \infty)}$$

$\text{Supp}[T_k] = [0, \infty)$

param. space $\lambda \in (0, \infty)$, $k \in \mathbb{N}$

$\text{Exp}(\lambda) = \text{Erlang}(1, \lambda)$ $\quad \sum_{i=1}^{k} \text{Exp}(\lambda) = \text{Erlang}(k, \lambda)$

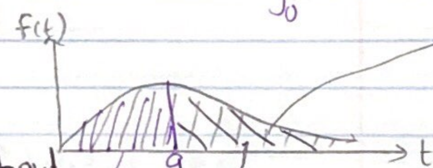
$\Updownarrow \qquad \Updownarrow$

$\text{Geom}(p) = \text{NegBin}(1, p)$ $\quad \sum_{i=1}^{k} \text{Geom}(p) = \text{NegBin}(k, p)$

We will just do some pure math definitions. We will introduce the gamma family of functions. The "gamma function" is:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, dt$$

e.g $\Gamma(3) = \int_0^\infty \overbrace{t^2 e^{-t}}^{f(t)} \, dt = 2$



We're only going to care about x being positive in this class

$$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t} \, dt}_{\gamma(x, a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t} \, dt}_{\Gamma(x, a)} \qquad a \in [0, \infty)$$

Lower incomplete gamma function $\qquad$ Upper incomplete gamma function

$$Q(x, a) := \frac{\gamma(x, a)}{\Gamma(x)} \in [0, 1)$$ proportion of the gamma function below a.

Lower regularized incomplete gamma function

$P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1)$ proportion of the gamma function above a.

Qs $Q(x,a) + P(x,a) = 1$

$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$ This is the integral of the PDF for Exp(1) over its support

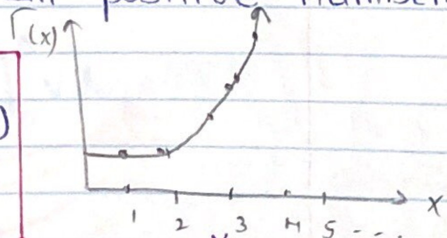
$\Gamma(x+1) = x\Gamma(x)$ proved on the HW via integration by parts

$\Rightarrow \Gamma(2) = 1\Gamma(1) = 1\cdot 1$, $\Gamma(3) = 2\Gamma(2) = 2\cdot 1 = 2$,
$\Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1 \dots$

for $n \in \mathbb{N}$, $\boxed{\Gamma(n) = (n-1)!}$

$$\Gamma(4.5) = 3.5\,\Gamma(3.5) = 3.5\cdot 2.5\,\Gamma(2.5)$$
$$= 3.5\cdot 2.5\cdot 1.5\,\Gamma(1.5)$$
$$= 3.5\cdot 2.5\cdot 1.5\cdot 0.5\,\Gamma(0.5)$$

The gamma function is an "extension" if the factorial function valid for all positive numbers.



$X \sim \text{Erlang}(\lambda) : \frac{x^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!} \mathbb{1}_{x\in[0,\infty)}$

$$F_X(x) := P(X \le x) = \int_0^x \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!}\,dt = \frac{\lambda^k}{(k-1)!}\int_0^x \overbrace{t^{k-1}e^{-\lambda t}\,dt}^{\gamma(k,\lambda x)/\lambda^k}$$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)}\,\frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = P(k,\lambda x)$$

Let's do some more calculus.... for $c > 0$,

$$\int_0^\infty t^{x-1} e^{-ct}\,dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du = \frac{1}{c^x}\int_0^\infty u^{x-1} e^{-u}\,du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = u/c \Rightarrow dt = 1/c\,du$, $t = 0 \Rightarrow u = 0$, $t \to \infty \Rightarrow u \to \infty$, $t = a \Rightarrow u = ac$

$$\int_0^a t^{x-1} e^{-ct}\,dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du$$

$$= \frac{1}{c^x}\int_0^{ac} u^{x-1} e^{-u}\,du = \frac{\gamma(x, ac)}{c^x}$$

$$\int_a^\infty t^{x-1} e^{-ct}\,dt = \int_0^\infty \dots\,dt - \int_0^a \dots\,dt$$

$$= \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, ac)}{c^x} = \frac{\Gamma(x) - \gamma(x, ac)}{c^x} = \frac{\Gamma(x, ac)}{c^x}$$

If $n \in \mathbb{N}$...

$$\Gamma(n, a) = \int_a^\infty \underbrace{t^{n-1}}_{u}\, \underbrace{e^{-t}\,dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du = \left[-t^{n-1} e^{-t}\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}\,dt$$

$du = (n-1)\,t^{n-2}\,dt.$
$v = -e^{-t}$

$$= a^{n-1} e^{-a} + (n-1) \int_a^{\infty} t^{n-2} e^{-t} \, dt = a^{n-1} e^{-a} + (n-1) \underbrace{\Gamma(n-1, a)}_{\text{iterate}}$$

$$= a^{n-1} e^{-a} + (n-1) \left(a^{n-2} e^{-a} + (n-2) \Gamma(n-2, a) \right)$$

$$= e^{-a} \left(a^{n-1} + (n-1) a^{n-2} + (n-1)(n-2) a^{n-3} + \dots + (n-1)! \, \Gamma(1, a) \right)$$

$$\Gamma(1,a) = \int_a^{\infty} e^{-t} \, dt = [-e^{-t}]_a^{\infty} = e^{-a}$$

$$= e^{-a} (n-1)! \left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \dots + \frac{a^0}{0!} \right)$$

$$= e^{-a} (n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$$

$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda} \lambda^x}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

$$F_X(x) := P(X \leq x) = \sum_{t=0}^{x} \frac{e^{-\lambda} \lambda^t}{t!} = e^{-\lambda} \sum_{t=0}^{x} \frac{\lambda^t}{t!}$$

$$= \underbrace{\frac{1}{x!}}_{\Gamma(x+1)} \overbrace{e^{-\lambda} \, x! \sum_{t=0}^{x} \lambda^t / t!}^{\Gamma(x+1, \lambda)}$$

$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = Q(x+1, \lambda)$$

$$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1, \lambda) \Longrightarrow F_{T_1}(t) = P(1, \lambda t)$$

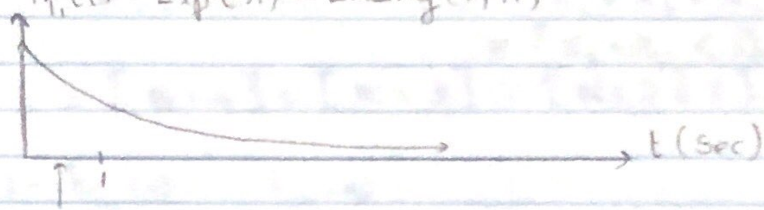
$$P(T_1 > 1) = 1 - F_{T_1}(1) = 1 - P(1, \lambda) = Q(1, \lambda)$$

$$\|$$

$$N \sim \text{Poisson}(\lambda), \quad P(N = 0) = F_N(0) = Q(1, \lambda)$$

the first example of the "poisson process", the link between waiting times in the Erlang and the probability of events in a poisson.

$f_{T_1}(t) = \text{Exp}(\lambda) = \text{Erlang}(1, \lambda)$



# of events in time between 0, 1 seconds is Poisson (λ) distributed.