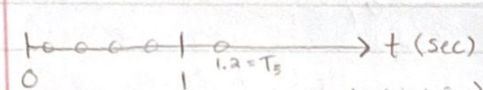
M368

$T_K \sim \text{Erlang}(K, \lambda)$, $N \sim \text{Poisson}(\lambda)$ $\qquad X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_K > 1) = 1 - F_{T_K}(1) = Q(K, \lambda)$ $\quad F_N(x) = Q(x+1, \lambda)$

$\Rightarrow 1 - F_{T_K}(1) = F_N(K-1)$ "Poisson Process"



Let N = # of events by t = 1

(4 dots before)

$K = 5$, $\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\} \cup \{X_1 + X_2 + X_3 < 1\} \cup \{X_1 + X_2 < 1\} \cup \{X_1 < 1\} \cup \{X_1 > 1\}$

$= \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}$

waiting for 5 exponential processes

$1 - F_{T_5}(1) \qquad F_N(4)$

$P(T_5 > 1) = P(N \leq 4)$

$T \sim \text{Erlang}(K, \lambda) = \dfrac{\lambda^K e^{-\lambda t} t^{K-1}}{(K-1)!} \mathbb{1}_{t \geq 0} = \dfrac{\lambda^K e^{-\lambda t} t^{K-1}}{\Gamma(K)} \mathbb{1}_{t \geq 0}$ "Gamma"

$K \in \mathbb{N}$, $\lambda \in (0, \infty)$

$T \sim \text{NegBin}(K, p) = \binom{K+t-1}{K-1} (1-p)^t p^K \mathbb{1}_{t \in \mathbb{N}_0} = \dfrac{\Gamma(K+t)}{\Gamma(K) t!} (1-p)^t p^K \mathbb{1}_{t \in \mathbb{N}_0}$ "extended NegBin"

$K \in \mathbb{N}$, $p \in (0,1)$

What if $K \in (0, \infty)$? is the top PDF legal and the bottom PMF legal? Yes!

$$\int_0^\infty \frac{\lambda^K e^{-\lambda t} t^{K-1}}{\Gamma(K)} dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(K+t)}{\Gamma(K) t!} (1-p)^t p^K = 1$$

which means... these are rv's.

$X \sim \text{Gamma}(\alpha, \beta) := \dfrac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha - 1} e^{-\beta t} \mathbb{1}_{t \geq 0}$

$X \sim \text{ExtNegBin}(K, p)$ .....

Transformations for Discrete rv's

$X \sim \text{Bern}(p)$, $Y = X+3 \sim \begin{cases} 3 \text{ wp } 1-p \\ 4 \text{ wp } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y-3 \in \{0,1\}}$

$= p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$

how do I express the transformed PMF using original PMF?

if $Y = g(X) \sim P_Y(y) = P_X(g^{-1}(y))$

$\Updownarrow$

$g^{-1}(y) = x$

is this formula general? No. Only if g invertible

if non-invertible...

$X \sim U(\{1, 2, \ldots, 10\}) = \frac{1}{10} \mathbb{1}_{x \in \{1,2,\ldots,10\}}$

$Y = \min\{X, 3\} \sim \begin{cases} 1 \text{ wp } \frac{1}{10} \\ 2 \text{ wp } \frac{1}{10} \\ 3 \text{ wp } \frac{8}{10} \quad (P(X=3) + P(X=4) + \ldots P(X=10)) \end{cases}$

$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x) \overset{\text{if } g \text{ invertible for } Supp[X]}{=} \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$

$X \sim \text{Bin}(n, p)$, $Y = X^2 \sim P_Y(y) = P_X(g^{-1}(y)) = P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\ldots,n\}}$

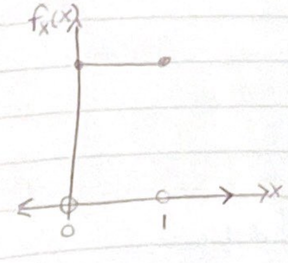
$\Downarrow$

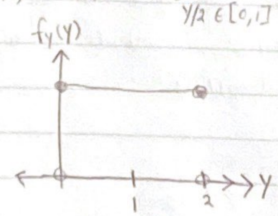
$x = \sqrt{y} = g^{-1}(y)$

Transformations for Continuous rv's, $Y = g(X)$, X is continuous.

for invertible g, $f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ incorrect!

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$, $Y = 2X \sim f_Y(y) = f_X(y/2) = \mathbb{1}_{y/2 \in [0,1]} = \mathbb{1}_{y \in [0,2]}$





$\int_0^2 f_Y(y) = 2 - 0 = 2 \neq 1$

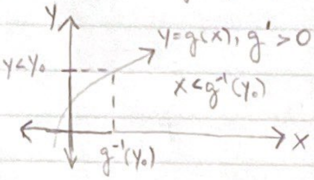
illegal!

Where'd we go wrong? The above formula was derived using probabilities. Densities are not probabilities! But... CDF's are! Strategy: let's derive the CDF of Y using the CDF of X. Then, like w/convolutions, take the derivative to get Y density

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) \overset{\text{if } g' > 0}{=} P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

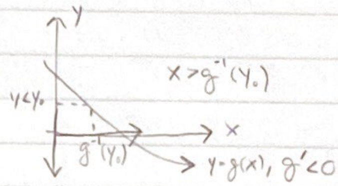
$$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right] = \frac{d}{dy}\left[F_X(g^{-1}(y))\right] = F_X'(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

$$= f_X(g^{-1}(y))\underbrace{\frac{d}{dy}\left[g^{-1}(y)\right]}_{\text{Stretching}}$$



$$= f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$$

$g' > 0 \Rightarrow g^{-1\prime} > 0$



if $g' < 0$

$$= P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}\left[F_X(g^{-1}(y))\right] = \frac{-d}{dy}\left[F_X(g^{-1}(y))\right] = f_X(g^{-1}(y))\left(-\frac{d}{dy}\left[g^{-1}(y)\right]\right)$$

$$\frac{d}{dy}\left[g^{-1}(y)\right] < 0 \longrightarrow \; = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| \quad \text{for all } g \text{ invertible}$$

Let's derive some more rules! The most common invertible function is... the straight line! $Y = aX + c$! $\Rightarrow X = g^{-1}(Y) = \frac{Y-c}{a}$, $\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \frac{1}{|a|}$

s.t. $a, c \in \mathbb{R}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$ "Shift and Scale"

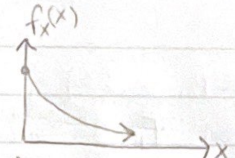
if $c = 0$ just a Scale... $Y = aX$

$f_Y(y) = f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$

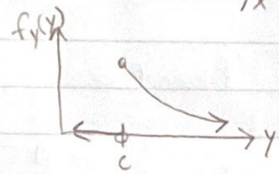
if $a = 1$ just a shift... $Y = X + c$

$f_Y(y) = f_X(y - c)$

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \ge 0}$



$Y = X + c = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y \ge c}$



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$X \sim \text{Exp}(1) = e^{-x} \mathbb{1}_{x \ge 0}$

$Y = g(x) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = y$

$\Rightarrow e^{y} = e^{x} - 1 \Rightarrow e^{y}+1 = e^{x} \Rightarrow x = \ln(e^{y}+1) = g^{-1}(y)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$

$f_Y(y) = f_X(\ln(e^{y}+1))\frac{e^{y}}{e^{y}+1} = \underbrace{e^{-\ln(e^{y}+1)}}_{e^{\ln\left(\frac{1}{e^{y}+1}\right)}} \underbrace{\mathbb{1}_{\ln(e^{y}+1) \ge 0}}_{e^{y}+1 \ge 1} \frac{e^{y}}{e^{y}+1}$

$e^{y} \ge 0$

$y \in \mathbb{R}$

$= \frac{1}{e^{y}+1} \cdot \frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2} = \text{Logistic}(0,1)$

$\cdot \left(\frac{e^{-2y}}{e^{-2y}}\right)$

$= \frac{e^{-y}}{(e^{-y}+1)^2}$