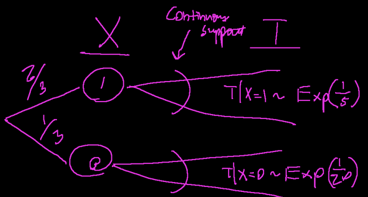
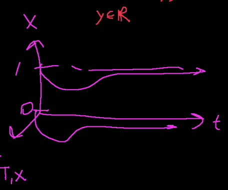
Mixture and compound distributions.

Consider a situation where 2/3 of the time there is fast internet speed so your downloads take T ~ Exp(1/5) => E[T] = 5s and the other 1/3 of the time, there is Internet traffic, so your downloads take T ~ Exp(1/20) => E[T] = 20s. What is the distribution of the "overall T" or "unconditional on the Internet speed"? Let X ~ Bern(2/3) and X = 1 corresponds to fast internet and X = 0 corresponds to slow internet. Let's draw a tree diagram:



$h(x) = \int_{\mathbb{R}} h(x,y)\,dy$

$\quad = \sum_{y \in \mathbb{R}} h(x,y)$



$$f_T(t) = \sum_{x \in Supp[X]} f_{T,X}(t,x) = \sum_{x \in Supp(X)} f_{T|X}(t,x)\, p_X(x)$$

$$= \sum_{x \in \{0,1\}} f_{T|X}(t,x)\, p_X(x) = f_{T|X}(t,0)\, p_X(0) + f_{T|X}(t,1)\, p_X(1)$$

$$= \frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}$$

If the download speed was t = 25s, what is the probability it is a slow internet day, i.e. x = 0? $X|T \sim Bern(?)$

$W \sim Bern(p)$

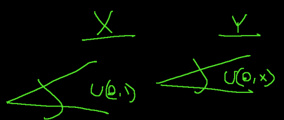
$p = P(W=1)$

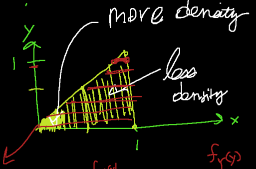
$$p_{X|T}(x,t) = \frac{f_{T|X}(t,x)\, p_X(x)}{f_T(t)} \quad \text{"Bayes Rule"}$$

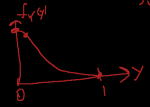
$$\text{Bernoulli param} = p_{X|T}(1,t) = \frac{f_{T|X}(t,1)\, p_X(1)}{f_T(t)} = \frac{\frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}t} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}t} \cdot \frac{2}{3}}$$

$$p_{X|T}(0,25) = 1 - p_{X|T}(1,25) = 1 - \frac{\frac{1}{5} e^{-\frac{1}{5}\cdot 25} \cdot \frac{2}{3}}{\frac{1}{20} e^{-\frac{1}{20}\cdot 25} \cdot \frac{1}{3} + \frac{1}{5} e^{-\frac{1}{5}\cdot 25} \cdot \frac{2}{3}} = 0.842$$

$X \sim U(0,1),\ Y|X=x \sim U(0,x) \Rightarrow Y \sim ?$



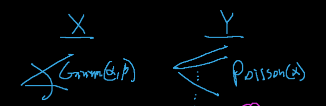




The first example featured
T which was continuous (we call that the "model") and X which is discrete (we call that the "mixing distribution". Thus the unconditional distribution T is called a "mixture distribution".

In the second example Y, the model is continuous and X, the mixing distribution is also continouous and we call the unconditional distribution Y a "compound distribution".

p156-157 Let $Y|X=x \sim Poisson(x)$, $X \sim Gamma(\alpha,\beta)$, $Y \sim ?$

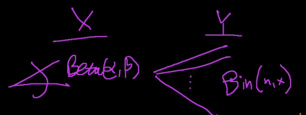


$$p_Y(y) = \int_{Supp[X]} p_{Y|X}(y,x)\, f_X^{old}(x)\,dx = \int_0^\infty \frac{e^{-x} x^y}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \underbrace{\int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\,dx}_{\text{lee 9 integral}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y \in \mathbb{N}_0} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}}$$

$= \ldots \overset{HW}{=} ExtNegBin\left(\alpha, \frac{\beta}{\beta+1}\right)$ this is a more flexible count model than the Poisson

$Y|X=x \sim Bin(n, \overset{p}{x})$ where n is known, $X \sim Beta(\alpha,\beta)$, $Y \sim ?$



$$p_Y(y) = \int_{Supp[X]} p_{Y|X}(y,x)\, f_X^{old}(x)\,dx = \int_0^1 \binom{n}{y} x^y (1-x)^{n-y} \mathbb{1}_{y \in \{0,\ldots,n\}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}\,dx$$

$$= \binom{n}{y} \mathbb{1}_{y \in \{0,\ldots,n\}} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 x^{y+\alpha-1} (1-x)^{n-y+\beta-1}\,dx}_{B(y+\alpha,\, n-y+\beta)} = \frac{B(y+\alpha, n-y+\beta)\binom{n}{y}}{B(\alpha,\beta)} \mathbb{1}_{y \in \{0,\ldots,n\}}$$

$= BetaBinomial(n,\alpha,\beta)$

$Y|X=x \sim Exp(x),\ X \sim Gamma(\alpha,\beta) \overset{HW}{\Rightarrow} Y \sim Lomax(\beta,\alpha)$

which is a more flexible waiting time than the exponential

Midterm II ↑

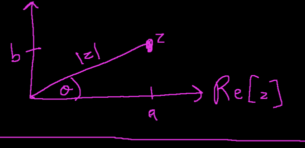
Final ↓

Moment generating functions (mgf's) and characteristic functions (chf's). To derive these, we need to review complex / imaginary numbers. First define $i := \sqrt{-1}$ "imaginary"

let $a,b \in \mathbb{R}$, $z := a + bi \in \mathbb{C}$, complex #'s

$Re[z] := a$, $Im[z] := b$, real component and imaginary component of a complex #

$|z| := \sqrt{a^2+b^2}$, $Arg[z] := \theta \overset{usually}{=} \arctan\left(\frac{b}{a}\right)$



$i^0 = 1$
$i^1 = i$
$i^2 = (\sqrt{-1})^2 = -1$
$i^3 = i^2 i = -i$
$i^4 = (i^2)^2 = 1$
$i^5 = i^4 i = i$ $\ldots n \in \mathbb{N},\ i^n = i^{n \bmod 4}$

clock cycle

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$e^{itx} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} + \ldots$$

$$i\sin(tx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - \ldots$$

$$\cos(tx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - \ldots$$

Euler's Formula

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx) \overset{tx=\pi}{\Longrightarrow} e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$