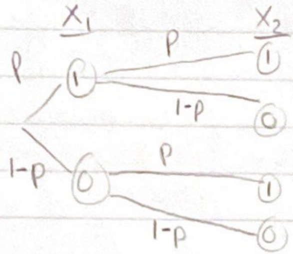
M368 8/31



$p_{\vec{X}}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

$P(T=1) = P_{X_1,X_2}(1,0) + P_{X_1,X_2}(0,1)$

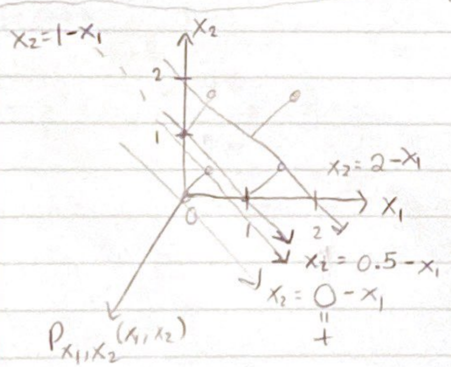
$$T \sim \begin{cases} 2 & \text{w.p } p^2 \\ 1 & \text{w.p } 2p(1-p) \\ 0 & \text{w.p } (1-p)^2 \end{cases}$$

selector function

$$p(T) = P(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_1+x_2=t}$$

$= P_{X_1}(x) * P_{X_2}(x)$

$x_1+x_2=t \iff x_2 = t - x_1$

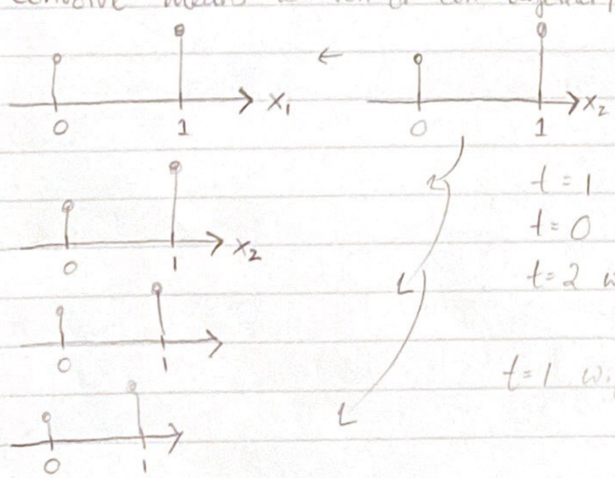


if $t=2$

$$P_T(2) = \sum_{x_1 \in \mathbb{R}} P_{X_1X_2}(x_1, 2-x_1)$$

"Convolve" means to 'roll or coil together/entwine'

pmfs



$t=1$ w.p $p(1-p)$
$t=0$ w.p $(1-p)^2$
$t=2$ w.p p^2
$t=1$ w.p $(1-p)p$
$\Rightarrow P_T$

general convolution formula

$$P_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2)\, \mathbb{1}_{x_2=t-x_1} = \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1, t-x_1)$$

if X_1, X_2 indep. $= \sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x)$ convolution formula for indep. r.v.s

$$\sum_{x\in\mathbb{R}} P_{X_1}(x)\, P_{X_2}(t-x)$$ Convolution formula for indep r.v.s

$$= \sum_{x\in\mathbb{R}} P^{old}_{X_1}(x)\, \mathbb{1}_{x\in Supp[X_1]}\, P^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x\in supp[X_2]}$$

$*$ $$= \sum_{x\in Supp[X_1]} P^{old}_{X_1}(x)\, P^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x\in Supp[X_2]}$$

$X_1, X_2 \overset{iid}{\sim}$

$$= \sum_{x\in\mathbb{R}} p(x)p(t-x) = \sum_{x\in\mathbb{R}} p^{old}(x)\, \mathbb{1}_{x\in Supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}$$

$*$ $$= \sum_{x\in Supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}$$ Convolution formula for iid r.vs

$$T_2 \sim \overset{p_T(t)}{\overbrace{\sum_{x\in\{0,1\}} p^x(1-p)^{1-x}\, p^{t-x}(1-p)^{1-t+x}\, \underbrace{\mathbb{1}_{t-x\in\{0,1\}}}_{t\in\{x,x+1\}}}} = p^t(1-p)^{2-t}\sum_{x\in\{0,1\}} \mathbb{1}_{t\in\{x,x+1\}}$$

$$= p^t(1-p)^{2-t}\Big(\underbrace{\mathbb{1}_{t\in\{0,1\}} + \mathbb{1}_{t\in\{1,2\}}}_{\begin{cases} t=0 \Rightarrow 1 \\ t=1 \Rightarrow 2 \\ t=2 \Rightarrow 1\end{cases} = \binom{2}{t}}\Big)$$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}\, \mathbb{1}_{n\in\mathbb{N}}\, \mathbb{1}_{k\in\{0,1,\dots,n\}}$$

$$= \binom{2}{t} p^t(1-p)^{2-t} = Binom(2,p), \quad Supp[T_2] = \{0,1,2\}$$

Generally $Supp[T] = Supp[X_1] + Supp[X_2]$, $A+B := \{a+b : a\in A,\ b\in B\}$

$$P_{T_2}(t) = \sum_{x\in\mathbb{R}} p(x)p(t-x) = \sum \binom{1}{x} p^x(1-p)^{1-x}\binom{1}{t-x} p^{t-x}(1-p)^{1-t+x}$$

$$= p^t(1-p)^{2-t}\sum_{x\in\mathbb{R}}\binom{1}{x}\binom{1}{t-x} = p^t(1-p)^{2-t}\left(\binom{1}{0}\binom{1}{t} + \binom{1}{1}\binom{1}{t-1}\right)$$

$$= \binom{2}{t} p^t(1-p)^{2-t}$$

$X_1, X_2 \overset{iid}{\sim} Bern(p) = \binom{1}{x} p^x(1-p)^{1-x}$

Recall Pascal's identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p) \quad T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 = X_3 + T_2$

can't use iid formula, use indep.

$$p_{T_3}(t) = \sum_{x \in \text{Supp}[X_3]} p^{old}_{X_3}(x)\, p^{old}_{T_2}(t-x) = \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x}\right)\left(\binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}\right)$$

$$= p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x} = p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1}\right) = \binom{3}{t} p^t (1-p)^{3-t} = \text{Binom}(3,p)$$

HW

$X_1, X_2 \overset{iid}{\sim} \text{Binom}(n,p) \quad T = X_1 + X_2 \sim ? \quad := \binom{n}{x} p^x (1-p)^{n-x}$

$$P_T(t) = \sum_{x \in \mathbb{R}} p(x)p(t-x) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t-x}} = \binom{2n}{x} p^t (1-p)^{2n-t} = \text{Binom}(2n,p)$$

Vandermonde's identity gives us