Math 368 / 621 Fall 2020
Final Examination

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Wednesday, December 14, 2020

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 105 minutes (variable time per question) and closed-book. You are allowed **three** pages (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. No food is allowed, only drinks.

Problem 1 [14min] (and 14min will have elapsed) Let $X_1, X_2, \ldots$ be a sequence of independent rv's distributed as $\text{Deg}(\mu/n^2)$. Let $T_n := X_1 + X_2 + \ldots + X_n$.

- [18 pt / 18 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) X_1 has no PMF

(b) X_1 has no CDF

(c) X_1 is zero with probability one

(d) The convolution of X_1 and X_2 is also degenerate

(e) The expectation of X_1 is μ

(f) The variance of X_1 is μ^2

(g) $\mathbb{P}(X_1 \geq \mu) \leq 0$

(h) The chf of X_1 is μ

(i) The chf of X_1 is $e^{it\mu}$

(j) The chf of X_1 is $\in \mathbb{L}^1$

(k) The chf of X_1 evaluated at 0 is 1

(l) The mgf of X_1 does not exist for some values of t or μ

(m) $X_n \xrightarrow{d} 0$

(n) $X_n \xrightarrow{p} 0$

(o) $X_n \xrightarrow{\mathbb{L}^1} 0$

(p) X_n converges to zero in mean square

(q) $T_n \xrightarrow{d} 0$

(r) T_n does not converge in probability.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

deikmnop

Problem 2 [7min] (and 21min will have elapsed) Let $\boldsymbol{X} \sim \text{Multinomial}(n, \boldsymbol{p})$ and $\phi_{\boldsymbol{X}}(\boldsymbol{t}) = (p_1 e^{it_1} + p_2 e^{it_2} + \ldots + p_K e^{it_K})^n$.

- [9 pt / 27 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

  (a) $\mathbf{1}^\top \boldsymbol{p} = 1$

  (b) $\dim[\boldsymbol{X}] = K$

  (c) $X_1, \ldots, X_n$ are independent

  (d) $X_1, \ldots, X_n$ are identically distributed

  (e) $X_3 \sim \text{Binomial}(n,\, p_3)$

  (f) The chf for X_3 is $\phi_{\boldsymbol{X}}\left([t_1\ t_2\ 0\ t_4\ \ \ldots\ \ t_K]^\top\right)$

  (g) The chf for X_3 is $\phi_{\boldsymbol{X}}\left([\ 0\ 0\ t\ 0\ \ \ldots\ \ 0]^\top\right)$

  (h) The chf for X_3 is $(p_3 e^{it} + p_4 e^{it} \ldots + p_K e^{it_K})^n$

  (i) The chf for X_3 is $(1 + p_3 e^{it} - p_3)^n$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abegi

Problem 3 [9min] (and 30min will have elapsed) Let $X_1, X_2, \ldots$ be a sequence of iid rv's with mean μ and variance σ^2 finite.

- [13 pt / 40 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) $\bar{X}_n$ is exactly normally distributed
(b) $\bar{X}_n$ is approximately normally distributed
(c) The expectation of $\bar{X}_n$ is exactly μ
(d) The expectation of $\bar{X}_n$ is approximately μ
(e) The variance of $\bar{X}_n$ is exactly σ^2/n
(f) The variance of $\bar{X}_n$ is approximately σ^2/n
(g) $\bar{X}_n \overset{d}{\to} \mu$
(h) $\bar{X}_n \overset{p}{\to} \mu$
(i) $Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \overset{d}{\to} \mu$
(j) $Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \overset{p}{\to} \mu$
(k) $Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \overset{d}{\to} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$
(l) Regardless of whether (h) is true or not, it is the main result of the "central limit theorem"
(m) Regardless of whether (k) is true or not, it is the main result of the "central limit theorem"

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

aceghkm

Problem 4 [19min] (and 49min will have elapsed) Let $X \sim \chi^2_k$ and $Y \mid X = x \sim \mathrm{U}(0,\, x)$. For this problem, you may need the following fact from Math 241: for $U \sim \mathrm{U}(a,\, b)$, $\mathbb{V}\mathrm{ar}[U] = (b-a)^2/12$ and the following fact from this class: for $G \sim \mathrm{Gamma}(\alpha,\, \beta)$, $\mathbb{V}\mathrm{ar}[G] = \alpha/\beta^2$.

- [19 pt / 59 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) X and Y are independent

(b) $X \sim \mathrm{Gamma}(k/2,\, 1/2)$

(c) The rv Y is a compound distribution

(d) The rv Y has one parameter

(e) $f^{old}_{X,Y}(x, y) = \frac{1}{2^k\Gamma(k/2)} x^{k/2-2} e^{-x/2}$

(f) $f^{old}_{X,Y}(x, y) = \frac{1}{2^k\Gamma(k/2)} x^{k/2-1} y^{-1} e^{-x/2}$

(g) $f_{X,Y}(x, y)$ is always defined

(h) $f_{X|Y}(x, y)$ is always defined

(i) $f_{Y|X}(y, x)$ cannot be computed given the information you have

(j) The support of the rv Y is all real numbers

(k) The support of the rv Y is all positive real numbers

(l) The expectation of Y can be computed via $\int_{\mathbb{R}} y \int_{\mathbb{R}} f_{X,Y}(x, y) dx dy$

(m) The expectation of Y is $x/2$

(n) The expectation of Y is $k/2$

(o) The expectation of Y is $\sqrt{\pi}$

(p) The variance of Y is $k^2/12$

(q) The variance of $Y \mid X$ is $X^2/12$

(r) The variance of Y is $(8k + k^2)/12$

(s) The variance of Y cannot be computed given the information you have

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

$\boxed{\text{bcdegklnr}}$

Problem 5 [18min] (and 67min will have elapsed) Let $X \sim T_k$, $U = X^2$, $Y = \mu + \sigma X$ and $V = \mu + \sigma U$ where $\mu \in \mathbb{R}$, $\sigma > 0$.

- [16 pt / 75 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) X and V are independent

(b) $f_X^{old}(x) = f_X(x) = \frac{\Gamma((k+1)/2)}{\sqrt{k\pi}\Gamma(k/2)} \left(1 + x^2/k\right)^{-(k+1)/2}$

(c) $\mathbb{S}\text{upp}\left[U\right] = [0, \infty)$

(d) $\mathbb{S}\text{upp}\left[V\right] = [0, \infty)$

(e) $\mathbb{E}\left[X\right] = \mu$

(f) $\mathbb{E}\left[Y\right] = \mu$

(g) $\mathbb{E}\left[U\right] = \mu$

(h) $\mathbb{E}\left[V\right] = \mu$

(i) $Y/V \sim \text{Cauchy}(\mu, \sigma)$

(j) $\mathbb{V}\text{ar}\left[Y\right] = \sigma^2$

(k) $f_Y^{old}(y) = f_Y(y) = \frac{\Gamma((k+1)/2)}{\sigma\sqrt{k\pi}\Gamma(k/2)} \left(1 + (y-\mu)^2/(k\sigma^2)\right)^{-(k+1)/2}$

(l) $f_Y^{old}(y) = f_Y(y) = \mu + \sigma\frac{\Gamma((k+1)/2)}{\sqrt{k\pi}\Gamma(k/2)} \left(1 + y^2/k\right)^{-(k+1)/2}$

(m) $U \sim F_{1,k}$

(n) $V \sim F_{\sigma,k}$ if $\mu = 0$

(o) $f_U^{old}(u) = \frac{(1+u/k)^{-(k+1)/2}}{B(1/2,k/2)\sqrt{ku}}$

(p) $f_V^{old}(v) = \frac{(1+v/(k\sigma))^{-(k+1)/2}}{B(1/2,k/2)\sqrt{k\sigma v}}$ if $\mu = 0$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

$\boxed{\text{bcfkmo}}$

Problem 6 [9min] (and 76min will have elapsed) Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where μ and σ^2 are finite and let $\bar{X} = \frac{1}{n}\sum X_i$ and $S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$. Let $Z_1 = (X_1 - \mu)/\sigma, Z_2 = (X_2 - \mu)/\sigma, \ldots, Z_n = (X_n - \mu)/\sigma$. Let $\boldsymbol{Z} = [Z_1 \; Z_2 \; \ldots \; Z_n]^\top$.

- [11 pt / 86 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$

(b) $\bar{X} \sim \mathcal{N}(\mu, \sigma^2)$

(c) $S^2 \sim \chi^2_{n-1}$

(d) $S^2 \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$

(e) $\bar{X}$ and S^2 are independent

(f) $\bar{Z}$ and S^2 are independent

(g) There exists a matrix A where $S^2 = \boldsymbol{Z} A \boldsymbol{Z}^\top$.

(h) There exists a matrix A where $S^2 = \boldsymbol{Z}^\top A \boldsymbol{Z}$.

(i) If $\boldsymbol{Z}^\top \boldsymbol{Z} = \boldsymbol{Z}^\top B_1 \boldsymbol{Z} + \boldsymbol{Z}^\top B_2 \boldsymbol{Z} + \ldots + \boldsymbol{Z}^\top B_k \boldsymbol{Z}$, then $\boldsymbol{Z}^\top B_1 \boldsymbol{Z}$ is chi-squared distributed

(j) If (e) is assumed, then Cochran's theorem can be proven

(k) If $a > 0$, then $\mathbb{P}(Z_1 > a) < \frac{1}{2}$.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

adefhk

Problem 7 [19min] (and 95min will have elapsed) Let $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ where Σ is full rank, $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{Y} = A\boldsymbol{X} + \boldsymbol{b}$. Let μ_i denote the ith entry in $\boldsymbol{\mu}$ and let $\Sigma_{i,j}$ denote the entry on the ith row and jth column of matrix Σ.

- [17 pt / 103 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

(a) $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu}$

(b) $\boldsymbol{X} - \boldsymbol{\mu}$ is a standard multivariate normal rv

(c) The kernel of the PDF of $\boldsymbol{X}$ is $e^{\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{\mu} - \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}/2}$

(d) $\underset{\boldsymbol{x} \in \mathbb{R}^n}{\arg\max} \; \{f_{\boldsymbol{X}}(\boldsymbol{x})\} = \boldsymbol{\mu}$

(e) $\phi_{\boldsymbol{X}}(\boldsymbol{t}) = e^{i\boldsymbol{t}^\top \boldsymbol{\mu} - \boldsymbol{t}^\top \Sigma \boldsymbol{t}/2}$

(f) $\phi_{\boldsymbol{Y}}(\boldsymbol{t}) = e^{it\boldsymbol{b}} \phi_{\boldsymbol{X}}(A^\top \boldsymbol{t})$

(g) $\int_0^\infty \int_0^\infty f_{\boldsymbol{X}}(x_1, x_2) dx_1 dx_2 = 1$ if $n = 2$

(h) X_1 and X_2 would be independent if $\boldsymbol{\mu} = \boldsymbol{0}$

(i) X_1 and X_2 could be independent regardless of the value of $\boldsymbol{\mu}$

(j) $\mathbb{V}\text{ar}[\boldsymbol{X} + \boldsymbol{b}] = \Sigma \boldsymbol{b}$ if $n = m$

(k) $A\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, A\Sigma)$

(l) $X_3 + X_7$ is normally distributed

(m) X_3 / X_7 is Cauchy distributed

(n) $\begin{bmatrix} X_3 \\ X_7 \end{bmatrix} \sim \mathcal{N}_2\left(\begin{bmatrix} \mu_3 \\ \mu_7 \end{bmatrix}, \begin{bmatrix} \Sigma_{3,3} & \Sigma_{3,7} \\ \Sigma_{3,7} & \Sigma_{7,7} \end{bmatrix}\right)$

(o) $\mathbb{V}\text{ar}[X_1 + X_2] = \Sigma_{1,1} + \Sigma_{2,2} + \Sigma_{1,2} + \Sigma_{2,1}$

(p) $\boldsymbol{X}^\top \boldsymbol{X} \sim \chi^2_n$

(q) $(\boldsymbol{Y} - A\boldsymbol{\mu} - \boldsymbol{b})^\top \left(A\Sigma A^\top\right)^{-1} (\boldsymbol{Y} - A\boldsymbol{\mu} - \boldsymbol{b})$ is always chi-squared distributed

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

acdefilno

Problem 8 [10min] (and 105min will have elapsed) Let $X, Y \overset{iid}{\sim} \chi^2_k$ where μ and σ^2 are finite and $a > 0$.

- [10 pt / 113 pts] Record the letter(s) of all the following that are always **true**. At least one will be true.

  (a) $\mathbb{P}(X > a) \leq k/a$

  (b) $\mathbb{P}\left(2^X > a\right) \leq \mathbb{E}\left[2^X\right]/a$

  (c) $\mathbb{P}\left(2^X > a\right) \leq 2^k/a$

  (d) $Q[X, 1/e] \leq ek$

  (e) $\mathbb{P}(X > b) \leq \mathbb{V}\text{ar}[X]/(b-k)^2$ if $b \geq 2k$

  (f) $\mathbb{E}[\ln(X)] \geq \ln(\mathbb{E}[X])$

  (g) $\mathbb{E}[\ln(X)] \leq \ln(\mathbb{E}[X])$

  (h) If a was large, the Chernoff bound for $\mathbb{P}(X > a)$ would be tighter (smaller) than the Markov bound for $\mathbb{P}(X > a)$

  (i) $\text{Corr}[X, Y] \in [-1, 1]$

  (j) Computing the Cauchy-Schwartz upper bound is the most information you can provide about $\mathbb{E}[XY]$.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abdeghi