lec02Claros

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Time (0:00 → 6:03 recap of Lec01)

- $p(T) = P(T=t) = p_{X_1}(x_1) \times p_{X_2}(x_2) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1,x_2) \mathbb{1}_{x_1+x_2=t}$
- Given $t = x_1 + x_2 \Rightarrow x_2 = t - x_1$
- $P(T=0) : x_2 = 0 - x_1 = -x_1$
- Want $P_T(t) = \underbrace{\sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}}}_{\text{(search through } \mathbb{R}^2)} \underbrace{P_{X_1X_2}(x_1,x_2)}_{|\text{(add up all pr in } supp)} \mathbb{1}_{x_1+x_2=t}$
- General convolution formula: $\sum_{x \in \mathbb{R}} P_{X_1X_2}(x, t-x)$

if x_1, x_2 indep (convolutions):

- New: $= \sum_{x \in \mathbb{R}} P_{X_1}(x) P_{X_2}(t-x)$
- Old with indicator: $\sum_{x \in \mathbb{R}} P^{old}_{X_1}(x) \mathbb{1}_{x \in Supp[X_1]} \; P^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$
- Modify equation: $= \sum_{x \in Supp[X_1]} P^{old}_{X_1}(x) P^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$

If $X_1, X_2 \overset{iid}{\sim}$

- $\sum_{x \in \mathbb{R}} p(x) P(t-x) = \sum_{x \in \mathbb{R}} P^{old}(x) \mathbb{1}_{x \in Supp[X]} \; P^{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$
- $= \sum_{x \in Supp[X]} p^{old}(x) p^{old}(t-1) \mathbb{1}_{t-x \in Supp[X]}$
- $T_2 = P_{T_2}(t) = \sum_{x \in [0,1]} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$
- $p^t (1-p)^{2-t} (\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}})$

Binomial

- $p(2) = p^t (1-p)^{2-t} (\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}}) \leftarrow \binom{2}{t}$
- $\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0,1,\ldots,n\}} = \binom{2}{t} \; p^t (1-p)^{2-t} = Binom(2,p)$

Generally

- $Supp[T] = Supp[X_1] + Supp[X_2]$
- $A + B := \{a + b : a \in A, b \in B\}$

Convolve two Bernouli

- $X_1, X_2 \overset{iid}{\sim} Bern(p) := p^x(1-p)^{1-x}\mathbb{1}_{x\in\{0,1\}} = \binom{1}{x}p^x(1-p)^{1-x}$
- $P_{T_3}(t) = \sum_{x\in\mathbb{R}} \binom{1}{x}p^x(1-p)^{1-x}\binom{1}{t-x}p^{t-x}(1-p)^{1-t+x} = p^t(1-p)^{2-t}\sum_{x\in\mathbb{R}} \binom{1}{x}\binom{1}{t-x}$
- $p^t(1-p)^{2-t}\sum_{x\in\{0,1\}} \binom{1}{t-x} = p^t(1-p)^{2-t}(\binom{1}{t} + \binom{1}{t-1}) = \binom{2}{t}p^t(1-p)^{2-t}$
- Pascal's Identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Convolve X_1, X_2, X_3

- $X_1, X_2, X_3 \overset{iid}{\sim} Bern(p), T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \; P_{T_3}(t) = ?$

$X_3 + T_2$ looks like

- $P_{T_3}(t) = \sum_{x\in\{0,1\}} p^x(1-p)^{1-x}\binom{2}{t-x}p^{t-x}(1-p)^{2-t+x} = p^t(1-p)^{3-t}\sum_{x\in\{0,1\}} \binom{2}{t-x}$
- $= p^t(1-p)^{3-t}(\binom{2}{t} + \binom{2}{t-1}) = \binom{3}{t}p^t(1-p)^{3-t} = Binomial(3,p)$

Convolve two $\overset{iid}{\sim}$ Binomial

- Say$X_1, X_2 \overset{iid}{\sim} Binomial(n,p), T = X_1 + X_2 := ?$
- $P_T(t) = \sum_{x\in\mathbb{R}} \binom{n}{x}p^x(1-p)^{n-x}\binom{n}{t-x}p^{t-x}(1-p)^{n-t+x} = p^t(1-p)^{2n-t}\sum_{x\in\mathbb{R}} \binom{n}{x}\binom{n}{t-x}$
- Vandermonde's Identity: $\binom{2n}{t}p^t(1-p)^{2n-t} = Binom(2n,p)$

Homework

- find PMF of Binom(n,p) via induction