Lecture - 14 10/26/2020

$\vec{g}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and 1:1, $\vec{X}, \vec{Y}$ continuous rv vectors of dim n s.t $\vec{Y} = g(\vec{X})$.

$\vec{g} = [g_1 \cdots g_n]^T$

Given $f_{\vec{X}}(\vec{x})$, find $f_{\vec{Y}}(\vec{y})$.

recall what a multidimensional function is: $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = \vec{h} = \vec{g}^{-1}$.

$y_1 = g_1(x_1, \ldots x_n)$, $\qquad x_1 = h_1(y_1, \ldots y_n)$,

$y_2 = g_2(x_1, \ldots, x_n)$, $\qquad \exists h_j$'s .. $x_2 = h_2(y_1, \ldots, y_n)$,

$\vdots \qquad\qquad \vdots$

$y_n = g_n(x_1, \ldots, x_n)$, $\qquad x_n = h_n(y_1, \ldots y_n)$

Using multivariable calculus, you can show that

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y})) \left| J_n(\vec{y}) \right|$$ the Jacobian determinant

$$J_n(\vec{y}) = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

Let's verify the convolution formula via this general change-of-variable formula.

$y_1 = T = X_1 + X_2 \sim f_T(t)$

Recipe:

(1) Find a "clever" g so that

(2) We can find an h.

(3) compute J_h

(4) Compute the multidimensional change of variables formula.

(5) Integrate out the "nuisance dimension"

a nuisance dimension

① $y_1 = X_1 + X_2 = g_1(x_1, x_2)$, $y_2 = X_2 = g_2(x_1, x_2)$

② $X_1 = y_1 - X_2 \stackrel{?}{=} y_1 - y_2 = h_1(y_1, y_2)$, $X_2 \stackrel{?}{=} y_2 = h_2(y_1, y_2)$

③ $$J_h = \det\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - -1 \cdot 0 = 1$$

④ $$f_{\vec{y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y})) \, |J_h| = f_{\vec{X}}(y_1 - y_2, y_2) \, |1|$$
$$= f_{\vec{X}}(y_1 - y_2, y_2)$$

⑤ $$f_T(t) = f_{y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\, dy_2 = \int_{\mathbb{R}} f_{X_1, X_2}(t-u, u)\, du$$

~~every~~ exactly the general convolution formula.

if X_1, X_2 indep if iid

$$= \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\, du \stackrel{}{=} \int_{\mathbb{R}} f(t-u) f(u)\, du$$

$$\|$$

$$\int_{Supp[X_2]} f_{X_1}^{old}(t-u)\, \mathbb{1}_{t-n \in Supp[X_1]}\, f_{X_n}^{old}(t)\, bu$$ equal

$$\int_{Supp[X]} f(t-u)\, \mathbb{1}_{t-n \in Supp[X]}\, f(t)\, du.$$

Q $R = \frac{X_1}{X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$ $\quad Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$ $\quad X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2) \left| y_2 \right|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2) \left| y_2 \right| dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u) \left| u \right| du.$

if X_1, X_2 indep

$\stackrel{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u) \left| u \right| du$ $\quad \stackrel{\text{if iid}}{=} \int_{\mathbb{R}} f(ru) f(u) \left| u \right| du.$

$\|$

$\int_{Supp[X_2]} f_{X_1}^{old}(ru) \mathbb{1}_{ru \in Supp[X_1]} f_{X_2}^{old}(u) \left| u \right| du.$ $\quad \|$

$\int_{Supp[X]} f^{old}(ru) \mathbb{1}_{ru \in Supp[X]} f^{old}(u) \left| u \right| du.$

Q $R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$, $\quad Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1(X_1 + X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$, $\quad X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2$

$= h_2(Y_1, Y_2)$

③ $J_h = \det\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix} = y_2(1-y_1) - -y_1 y_2 = y_2 - y_2 y_1 + y_1 y_2$

$= y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|\, dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)|u|\, du$

$\overset{x_1, x_2 \text{ indep}}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u - ru)|u|\, du \overset{iid}{=} \int_{\mathbb{R}} f(ru) f(u - ru)|u|\, du$

$\|$

$\int_{\mathbb{R}} f^{old}_{X_1}(ru)\, \mathbb{1}_{ru \in Supp[X_1]}\, f^{old}_{X_2}(u - ru)\, \mathbb{1}_{u - ru \in Supp[X_2]} |u|\, du$

$\frac{Supp[X_1]}{r}$

$\int_{x=-\infty}^{x=+\infty} f(x)\, \mathbb{1}_{x \in S}\, dx$

$\|$

$\int_S f(x)\, dx$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ ind. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

$$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = \int_{\frac{[0,\alpha)}{r} = [0,\alpha)} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} \cancel{e^{-\beta r u}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u - ru)^{\alpha_2 - 1} \overbrace{e^{-\beta(u - ru)}}^{e^{-\beta u} e^{+\beta r u}} \mathbb{1}_{u - ru \in [0,\alpha)} \overbrace{|u|}^{u} du.$$

$r \in [0,1]$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\alpha \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} \underbrace{(u - ru)^{\alpha_2 - 1}}_{\overbrace{(u(1-r))^{\alpha_2 - 1}}^{u^{\alpha_2 - 1}(1-r)^{\alpha_2 - 1}}} e^{-\beta u} \mathbb{1}_{\frac{u \in [0,\alpha)}{\frac{u(1-r) \in [0,\alpha)}{u - ru \in [0,\alpha)}}} u\, du.$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} (1-r)^{\alpha_2 - 1} \underbrace{\int_0^\alpha u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u} du}_{J(\alpha_1, \alpha_2)}$$

$$\propto r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \propto \text{Beta}(\alpha_1, \alpha_2)$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$,

$$R = \frac{X_1}{X_2} \sim f_R(r) = ? \qquad r > 0$$

$$f_R(r) = \int_{\text{Supp}[X_2]} f_{X_1}^{\text{old}}(ru) \mathbb{1}_{ru \in \text{Supp}[X_1]} f_{X_2}^{\text{old}}(u) |u| du.$$

$$= \int_0^\alpha \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} e^{-\beta r u} \mathbb{1}_{\frac{\cancel{u \in [0,\alpha)}}{ru \in [0,\alpha)}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} u\, du$$

$$= \underbrace{\frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}}_{1/B(\alpha_1, \alpha_2)} r^{\alpha_1 - 1} \underbrace{\int_0^\alpha u^{\alpha_1 + \alpha_2 - 1} e^{\overbrace{-\beta(r+1)}^{c} u} du.}_{\{\Gamma(\alpha_1 + \alpha_2) / (\beta(r+1))^{\alpha_1 + \alpha_2}\} \Longrightarrow}$$

$$\Longrightarrow \cancel{\beta^{\alpha_1+\alpha_2}} \quad (r+1)^{\alpha_1+\alpha_2}$$

$$= \frac{1}{\beta(\alpha_1, \alpha_2)} \left(\frac{r}{r+1}\right)^{\alpha_1+\alpha_2} \mathbb{1}_{r>0}$$

$$\|$$

$$\text{Beta Prime}(\alpha_1, \alpha_2)$$