Lecture 20 11/23/20

Math 621
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$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \cdot \frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}}$$

$$= \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z}}{\sqrt{\underbrace{\frac{\frac{n-1}{\sigma^2}\cdot S^2}{n-1}}_{\chi^2_{n-1}}}} \sim T_{n-1}$$

due to Cochran's thm,
we know $\bar{X}$ & σ^2 are independent.

Multivariate Normal Distribution (MVN)

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}$,

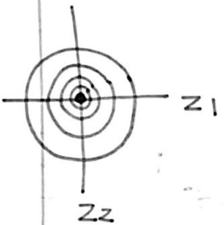
$Var[\vec{Z}] = I_n$.

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-1/2\,\vec{z}^T\vec{z}}$$

$= N_n(\vec{0}, I)$ Standard MVN (PDF $\leftrightarrow$)

$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$, $E[\vec{X}] = \vec{\mu}$, $Var[\vec{X}] = I_n \Rightarrow$

$\vec{X} \sim N_n(\vec{\mu}, I)$



$$\vec{X} = A\vec{Z} = \begin{bmatrix} z_1 \\ z_1 + z_2 \\ \vdots \\ z_1 + \dots + z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$$

but the components are dependent e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & 1 & 1 & 1 & \dots & 0 \\ \vdots & & & & & \\ \vdots & 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

$n \times n$.

$Cov[X_1, X_2] = Cov[Z_1, Z_1 + Z_2]$
$= \underbrace{Cov[Z_1, Z_1]}_{1} + \underbrace{Cov[Z_1, Z_2]}_{0}$
$= 1 \Rightarrow X_1, X_2$ dependent

Let's derive a general formula for the variance-covariance matrix of A (an $n \times n$ matrix of scalars) times a random vector X of dim n:

$$Var[A\vec{X}] := E[\overset{A\vec{X}\vec{X}^T A^T}{(A\vec{X})(A\vec{X})^T}] - E[A\vec{X}]E[A\vec{X}]^T$$
$$= A\,E[XX^T]\,A^T - A\,E[\vec{X}]\underbrace{(A\,E[\vec{X}])^T}_{E[X]^T A^T}$$

$$= A\underbrace{(E[XX^T] - E[\vec{X}]E[\vec{X}]^T)}_{\Sigma = Var[\vec{X}]}A^T = A\Sigma A^T$$

$\vec{X} = A\vec{Z}$, $Var[\vec{X}] = A I_n A^T = AA^T$.

Conjeume: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{X} = A\vec{Z} + \vec{A}$, $A \in \mathbb{R}^{n \times n}$, $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} \sim f_{\vec{X}}(\vec{x}) = ?$

$\underset{\shortparallel}{} g(\vec{z})$, $h(\vec{x}) = \vec{z}$ where hopefully g, h are inverses.

$\vec{z} = h(\vec{x}) = \underbrace{A^{-1}}_{B}(\vec{x} \cdot \vec{\mu}) \rightarrow$ in order for the inverse to exist.... A has to be invertible.

$= B\vec{x} - \beta\vec{\mu} = \begin{bmatrix} \vec{b_1} \cdot \vec{x} - \vec{b_1} \cdot \vec{\mu} \\ \vec{b_2} \cdot \vec{x} - \vec{b_2} \vec{\mu} \\ \vdots \\ \vec{b_n} \cdot \vec{x} - \vec{b_n} \cdot \vec{\mu} \end{bmatrix}$

* FACT:
$DD^{-1} = I$
$\rightarrow \det[DD^{-1}] = \det[I] = 1$
$\rightarrow \det[D] \det[D^{-1}] = 1$

$$J_n = \det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \det \overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B}$$

$= \det[A^{-1}]$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(x)) \, |J_n| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\underbrace{(A^{-1}(\vec{x} - \vec{\mu}))^T A^{-1}(\vec{x} - \vec{\mu})}_{(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})}}$$

$\cdot \frac{1}{|\det(A)|}$

$DD^{-1} = I \rightarrow (DD^{-1})^T = I^T = I$
$\rightarrow (D^{-1})^T D^T = I \rightarrow (D^{-1})^T = (D^T)^{-1}$

$$(CD)^{-1} = D^{-1} \cdot C^{-1}$$

$$= \frac{1}{\sqrt{(2\pi)^{a} \det[A]^{2}}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^{T} \underbrace{(A^{T})^{-1} A^{-1}}_{(AA^{T})^{-1}} (\vec{x}-\vec{\mu})}$$

let $\Sigma = AA^{T} = Var[\vec{X}]$

$\det[\Sigma] = \det[AA^{T}] = \det[A] \cdot \det[A^{T}] = \det[A]^{2}$

$$= \frac{1}{\sqrt{(2\pi)^{n} \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^{T} \Sigma^{-1} (\vec{x}-\vec{\mu})}$$

$= N_{n}(\vec{\mu}, \Sigma)$ * need Σ to be invertible *

A little bit of mult.variate characteristic fns:

$\phi_{x}(\vec{t}) = E[e^{i\vec{t}^{T}\vec{x}}] = E[e^{i(t_1 x_1 + \dots + t_n x_n)}] =$

$E[e^{it_1 x_1} \cdot \dots \cdot e^{it_n x_n}]$ $\overset{\text{if } x_1, \dots, x_n \text{ indep.}}{=}$

$\rightarrow E[e^{it_1 x_1}] \cdot \dots \cdot E[e^{it_n x_n}] = \phi_{x_1}(t_1)\, \phi_{x_2}(t_2) \cdot \dots \cdot \phi_{x_n}(t_n)$

(P_0) $\phi_{\vec{x}}(\vec{0}) = E[e^{i\vec{0}^{T}\vec{x}}] = 1$

(P_1) If 2 chf's are equal $\rightarrow$ the 2 rv's are equal in distribution.

(P_2) $\vec{Y} = A\vec{X} + \vec{b}$, $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^{m}$,
$\vec{x}$ in dim n $\Rightarrow$ $\vec{Y}$ in dim m.

$$\phi_{\vec{y}}(\vec{t}) : E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}] = E[e^{\underbrace{i\vec{t}^T A\vec{X}}_{(A^T\vec{t})^T}} e^{i\vec{t}^T b}]$$

$$= e^{i\vec{t}^T\vec{b}} \; E[e^{i(A^T\vec{t})^T\vec{X}}] = e^{i\vec{t}^T\vec{b}} \; \phi_{\vec{X}}(A^T\vec{t})$$

Let's derive the chf of the standard MVN

$$\phi_{\vec{Z}}(\vec{t}) = \prod_{i=1}^{n} \phi_Z(t_i) = \prod_{i=1}^{n} e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2}$$

$$= e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let's derive the chf of the general MVN.

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N(\vec{\mu}, \underbrace{AA^T}_{\Sigma})$$

$$\phi_{\vec{X}}(\vec{t}) \overset{P2}{=} e^{i\vec{t}^T\vec{\mu}} \; \phi_Z(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}} \; e^{\underbrace{-\frac{1}{2}(A^T\vec{t})^T A^T\vec{t}}_{\underbrace{\vec{t}^T AA^T\vec{t}}_{\Sigma}}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

$\vec{y} = B\vec{X} + \vec{c} \sim ?$ $\quad B \in R^{m \times n}, \; \vec{c} \in R^m$

$$\phi_{\vec{y}}(\vec{t}) \overset{P2}{=} e^{i\vec{t}^T\vec{c}} \; \phi_{\vec{X}}(B^T\vec{t}) = e^{i\vec{t}^T\vec{c}} \; e^{i\underbrace{(B^T\vec{t})^T}_{t^T B}\vec{\mu} - \frac{1}{2}\underbrace{(B^T\vec{t})^T}_{t^T B}\Sigma(B^T\vec{t})}$$

$$= e^{i\vec{t}^T(\vec{c}+B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T\vec{t}} \overset{P1}{\Rightarrow} \vec{y} \sim N_m(B\vec{\mu}+\vec{c}, B\Sigma B^T)$$ * if $B\Sigma B^T$ is invertible *

Let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$. Consider $(\vec{X} - \vec{\mu})^T$ =
$\Sigma^{-1}(\vec{X} - \vec{\mu}) \sim ?$

<u>Recall</u>: $\vec{Z} = A^{-1}(\vec{X} - \vec{\mu})$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1} A^{-1} = (A^{-1})^T A^{-1}$$

<u>Mahalanobis Distance</u>

$$= (\vec{X} - \vec{\mu})^T (A^{-1})^T A^{-1}(\vec{X} - \vec{\mu})$$
$$= (A^{-1}(\vec{X} - \vec{\mu}))^T A^{-1}(\vec{X} - \vec{\mu})$$
$$= \vec{Z}^T \vec{Z} \sim \chi^2_n.$$

This is kind of like distance in $\mathbb{R}^n$ adjusted for all the dependencies among the dimensions like a multivariate "z-score".