lecture Oct, 26 (1)

Will be doing arbitary multivariable transformations of Variables. $\vec{g}: \mathbb{R}^n \to \mathbb{R}^n$ and invertible, let $\vec{X}, \vec{Y}$ be continious vector rve's both with dimension n and $\vec{Y} = \vec{g}(\vec{X})$

Given that,

Jdf of the vector X rve, find the Jdf of the vector Y rve, This generalizes what we did previously with Univariate change of Variables. Let's recall what this multivariate function looks like.

$$y_1 = g_1(x_1, \dots, x_n) \qquad \vec{g} = [g_1, \dots g_n]^T \qquad x_1 = h_1(y_1, \dots y_n)$$
$$y_2 = g_2(x_1, \dots, x_n) \qquad\qquad x_2 = h_2(y_1 \dots y_n)$$
$$\vdots \qquad \vec{g}^{-1} = \vec{h} = [h_1, \dots h_n]^T \qquad \vdots$$
$$y_n = g_n(x_1, \dots x_n) \qquad\qquad x_n = h_n(y_1 \dots y_n)$$

From multivariable Calculus you can show the multivariate change of Variables formula is:

$$f_{\vec{y}}(\vec{y}) = f_{\vec{x}}\left(\vec{h}(\vec{y})\right) \left| J_h(\vec{y}) \right|$$

where, $J_h := \det \begin{bmatrix} \partial h_1/\partial y_1 & \cdots & \partial h_1/\partial y_n \\ \vdots & & \\ \partial h_n/\partial y_1 & \cdots & \partial h_n/\partial y_n \end{bmatrix}$, This is called the "Jacobian determinant"

Let's use this formula to prove the convulution formula:

for, $T = X_1 + X_2 \sim f_T(t) = ?$

There are recipes for these types of Problems: (Set the first dimension y_1 = your target)

1. find a g so that...
2. you can find the h.
3. Compute the Jacobian deteminant J_h
4. Substitute 1-3 into multivariate change of Variables formula
5. Integrate the "nuisance dimensions(s)"

Step 1:

① $T = y_1 = X_1 + X_2 = g_1(X_1, X_2)$, let, $y_2 = X_2 = g_2(X_1, X_2)$ "Nuisence" (1)

② need h, $X_1 = y_1 - X_2$ again, $X_2 = y_2$

$$= y_1 - y_2$$
$$= h_1(y_1, y_2)$$
$$X_2 = y_2 = h_2(y_1, y_2)$$

from ①

③ $J_h = \det \begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = (1*1) - (-1*0) = 1$

④ $f_{\vec{y}}(\vec{y}) = f_{\vec{x}}(y_1 - y_2, y_2)\,|1|$ —— (11)

⑤ $f_T(t) = f_{y_1}(x_1) = \int_{\mathbb{R}} f_{\vec{y}}(y_1, y_2)\,dy_2$

$$= \int_{\mathbb{R}} f_{\vec{x}}(y_1 - y_2, y_2)\,dy_2 \quad [\text{from } 2]$$

$$= \int_{\mathbb{R}} f_{\vec{x}}(t - u, u)\,du$$

If X_1, X_2 - independent

$$\overset{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\,du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(t-u) f(u)\,du$$

$$\| $$

$$\int_{\text{Supp}[X_2]} f_{X_1}^{\text{old}}(t-u)\, \mathbb{I}_{t-u \in \text{Supp}[X_1]}\, f_{X_2}^{\text{old}}(u)$$

$$\int f^{\text{old}}(t-u)\, \mathbb{I}_{t-u \in \text{Supp}[X]}\, f^{\text{old}}(u)\,du$$

3

Ratio of 2 random variables:

$$R = \frac{X_1}{X_2} \sim f_R(y) = ?$$

1. $y_1 = \cancel{\text{target}}$

let,

$$R = y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2) \quad , \quad y_2 = X_2 = g_2(X_1, X_2)$$

2. $$X_1 = y_1 X_2 = y_1 y_2 = h_1(y_1, y_2) \quad , \quad X_2 = y_2 = h_2(y_1, y_2)$$

3. $$J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$$

4. $$f_{\vec{y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|$$

5. $$f_R(r) = f_{y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|\, dy_2$$

$$\boxed{= \int_{\mathbb{R}} f_{\vec{X}}(ru, u)\,|u|\, du}$$ → General formula

X_1, X_2 independent

$$\overset{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\,|u|\, du \overset{\text{if iid}}{\underset{\downarrow}{=}} \int_{\mathbb{R}} f(ru) f(u)\,|u|\, du$$

$$= \int_{\text{Supp}[X_2]} f_{X_1}^{\text{old}}(ru)\, \mathbb{1}_{ru \in \text{Supp}[X_1]}\, f_{X_2}^{\text{old}}(u)\,|u|\, du$$

$$\int_{\text{Supp}[X]} f^{\text{old}}(ru)\, \mathbb{1}_{ru \in \text{Supp}[X]}\, f^{\text{old}}(u)\,|u|\, du$$

Another Formula R

(4)

$$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = ? \quad , \; y_2 = X_1 + X_2$$

(1) $R = y_1 = \dfrac{X_1}{X_1+X_2} = g_1(X_1, X_2)$

(2) $X_1 = y_1(X_1+X_2) = y_1 y_2 = h_1(y_1, y_2)$, $X_2 = y_2 - X_1 = \boxed{y_2 - y_1 y_2} = h_2(y_1, y_2)$

(3)
$$J_h = \det\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix}$$
$$= y_2(1-y_1) - (-y_1 y_2)$$
$$= y_2 - \cancel{y_1 y_2} + \cancel{y_1 y_2}$$
$$= y_2$$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\, \overbrace{|y_2|}^{\det}$

(5)
$$f_R(r) = f_{y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2$$
$$= \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\, |y_2|\, dy_2$$
$$= \boxed{\int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)\, |u|\, du}$$

General formula

Special cases:

If $X_1 X_2 \sim$ Ind

$$= \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u - ru)\, |u|\, du \overset{\text{iid}}{=} \int_{\mathbb{R}} f(ru)\, f(u - ru)\, |u|\, du$$

$$\int_{\mathbb{R}} f^{old}(ru)\, \mathbb{1}_{ru \in Supp[X]}\, f^{old}(u-ru)\, \mathbb{1}_{u-ru \in Supp[X]}\, |u|\, du$$

$$\boxed{\int f^{old}_{X_1}(ru)\, \mathbb{1}_{ru \in Supp[X_1]}\, f^{old}_{X_2}(u-ru)\, \mathbb{1}_{u-ru \in Supp[X_2]}\, |u|\, du}$$

Formula (A)

Let's use the formula's:

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ Indep of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$,

we want $R = \dfrac{X_1}{X_1 + X_2} \sim f_R(y) = ?$

X_1, X_2 = waiting times always +ve

R = Proportion of the waiting time for the first gamma and thus $\text{Supp}[R] = [0, 1]$

Formula:

From formula (A)

$$f_R(r) = \int_{\mathbb{R}} \underbrace{\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \overbrace{(ru)^{\alpha_1 - 1}}^{r^{\alpha_1-1} \cdot u^{\alpha_1-1}} \cancel{e^{-\beta r u}} \, \mathbb{1}_{\overbrace{ru \in [0,\alpha)}^{u \in [0,\alpha)}}}_{f_{X_1}^{\text{old}}} \underbrace{\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (\underbrace{u - ru}_{u(1-r)})^{\alpha_2 - 1} \overbrace{e^{-\beta(u - ru)}}^{e^{-\beta u}} \, \mathbb{1}_{u - ru \in [0,\alpha)}}_{f_{X_2}^{\text{old}}} |u| \, du$$

$$= \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \underbrace{\int_0^{\alpha} u^{\alpha_1+\alpha_2-1} e^{-\beta u} \, du}_{\frac{\Gamma^{\alpha_1+\alpha_2}}{\cancel{\beta^{\alpha_1+\alpha_2}}}} \, \mathbb{1}_{r \in [0,1]} \quad \underbrace{u(1-r) \in [0,\alpha)}_{u \in [0,\alpha)}$$

$$\boxed{\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\frac{1}{\beta(\alpha_1,\alpha_2)}} = \frac{1}{B(\alpha_1,\alpha_2)}}$$

$$= \frac{1}{\beta(\alpha_1,\alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \mathbb{1}_{r \in [0,1]} = \text{Beta}(\alpha_1, \alpha_2)$$

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$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ Independent of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$,

Formula before

$R = \frac{X_1}{X_2} \sim f_R(r) = ?$

$$\boxed{f_R(r) = \int_{Supp[X_2]} f^{old}_{X_1}(rU)\, \mathbb{1}_{rU \in Supp[X_1]}\, f^{old}_{X_2}(U)\, |U|\, du}$$

$$= \int_0^{\infty} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(rU)^{\alpha_1-1}}_{r^{\alpha_1-1},\, U^{\alpha_1-1}} e^{-\beta r U} \underbrace{\mathbb{1}_{rU \in [0,\infty)}}_{\text{If } r = +ve} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} U^{\alpha_2-1} e^{-\beta U}\, U$$

$U \in [0, \infty]$

So, its always 1

$\mathbb{1}_{r>0}$

From 0 to ∞ its always +ve so $|U| = U$

$$= \frac{\beta^{\cancel{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \mathbb{1}_{r>0} \underbrace{\int_0^{\infty} U^{\alpha_1+\alpha_2-1} e^{-\beta(r+1)U}\, du}_{\text{from lec(9)}}$$

$$\frac{\Gamma(\alpha_1+\alpha_2)}{\underbrace{(\beta(r+1))^{\alpha_1+\alpha_2}}_{\beta^{\cancel{\alpha_1+\alpha_2}}(r+1)^{\alpha_1+\alpha_2}}}$$

$$= \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0} = \text{Beta Prime distribution}$$

$= \text{Beta Prime}(\alpha_1, \alpha_2)$

$$\boxed{= \frac{1}{B(\alpha_1,\alpha_2)} \frac{r^{\alpha_1-1}}{(r+1)^{\alpha_1+\alpha_2}} \mathbb{1}_{r>0} = \text{Beta Prime}(\alpha_1, \alpha_2)}$$