Lecture 08
Math 621
09-30-2020

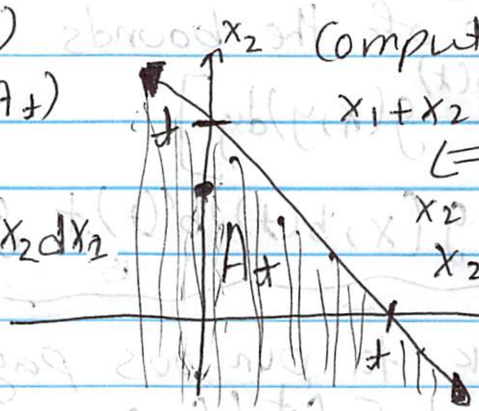
Tama C.

$T = X_1 + X_2 \sim f_T(t) = ?$

It also has a CDF $F_T(t) \Rightarrow f_T(t) = F_T'(t)$

CDF method to compute the convolution

$F_T(t) = P(T \le t)$
$= P(\vec{X} \in A_t)$



$x_1 + x_2 = t$
$\Leftrightarrow$
$x_2 = t - x_1$
$x_2 \le t - x_1$
$\Leftrightarrow$ $x_1 + x_2 \le t$

$= \iint_{A_t} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_2$

$= \int_{\mathbb{R}} \left(\int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2 \right) dx_1$

Let $x_1 = x$, $x_2 = v - x \Rightarrow dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$,
$x_1 = t - x \Rightarrow v = t$

$= \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\,dv\,dx$

$= \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx\,dv$

$\Rightarrow f_T(t) = \frac{d}{dt}[?]$

Calculus

Leibnitz's Rule for derivatives of integral functions:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\,dy\right] = g(x,b(x))\,b'(x) + g(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}[g(x,y)]\,dy$$

Partial derivative

If the derivative is with respect to a third variable, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} \dots\right] =$$

$$g(x, b(t))\,b'(t) + g(x, a(t))\,a'(t) + \int_{a(t)}^{b(t)} \underbrace{\frac{\partial}{\partial t}[g(x,y)]}_{0}\,dy \to 0$$

If one of the bounds in constant then

$$\frac{d}{dt}\left[\int_c^{b(t)} g(x,y)\,dy\right]$$

$$= g(x, b(t))\,b'(t) + g(x,c)\,\frac{d}{dt}[c] \to 0$$

Go back to previous page;

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\left(\int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\,dx\right)dv\right]$$

$$= \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\,dx$$

$\Longrightarrow$ General Convolution Formula

X_1, X_2 independent,

$$= \int_{\mathbb{R}} f_{X_1}(x)\,f_{X_2}(t-x)\,dx$$

Odd form

$$= \int_{Supp[X_1]} f_{X_1}^{odd}(x)\,f_{X_2}^{odd}(t-x)\,\mathbb{1}_{t-x \in Supp[X_2]}\,dx$$

$X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(x)\,f(t-x)\,dx$$

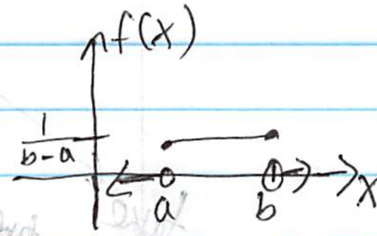
Old style

$$= \int_{Supp[X]} f^{odd}(x)\,f^{odd}(t-x)\,\mathbb{1}_{t-x \in Supp[X]}\,dx$$

——— o ———

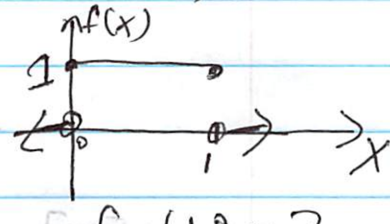
Continuous Uniform rv

$$X \sim U(a,b) = \underbrace{\overbrace{\frac{1}{b-a}\,\mathbb{1}_{x\in[a,b]}}^{f(x)}}_{f^{odd}(x)}$$



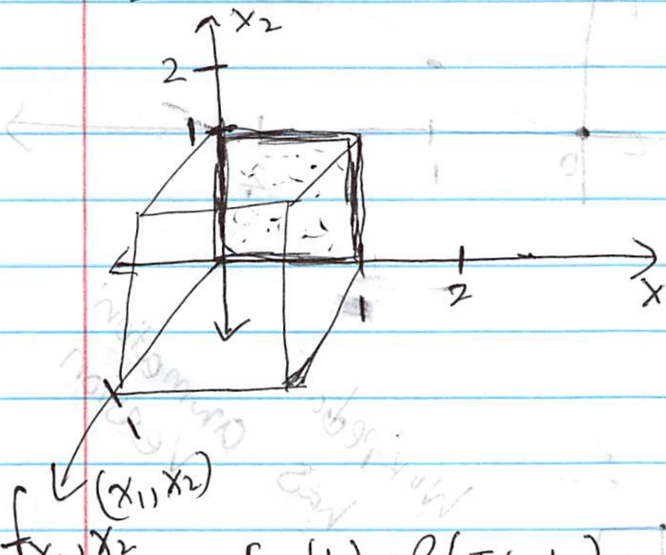
"Standard Uniform" rv is when $a=0, b=1$

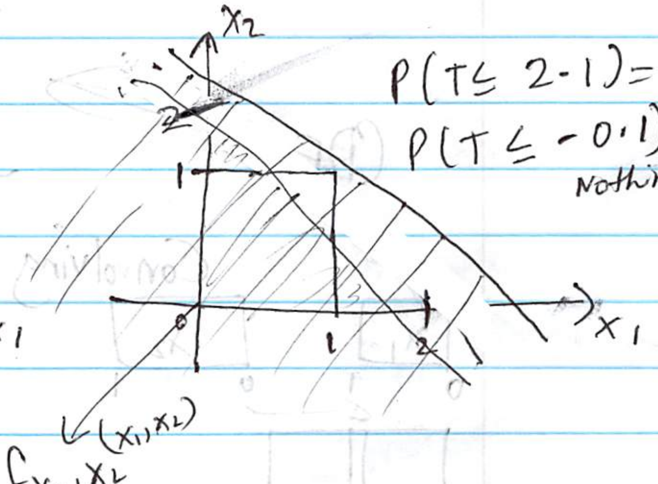
$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$$



$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$

CDF method first:





$P(T \le 2.1) = 1$

$P(T \le -0.1) = 0$

Nothing

$$F_T(t) = P(T \le t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t^2}{2} & \text{if } t \in (0,1] \\ -\frac{t^2}{2}+2t-1 & \text{if } t \in (1,2) \\ 1 & \text{if } t \ge 2 \end{cases}$$

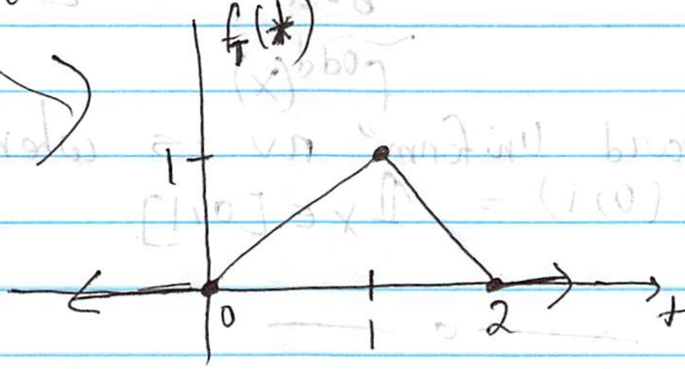
$$\frac{t^2}{2} - 2\left(\frac{(t-1)^2}{2}\right)$$

$$= \frac{t^2}{2} - (t^2 - 2t + 1)$$

$$= -\frac{t^2}{2} + 2t - 1$$

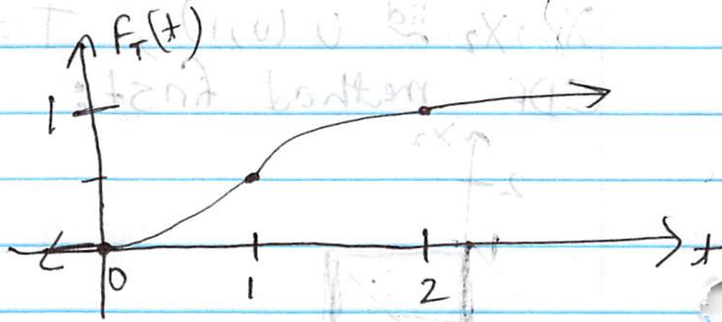
$$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in [0,1] \\ 2-t & \text{if } t \in (1,2) \\ 0 & \text{if } t \ge 2 \end{cases}$$

take derivatives

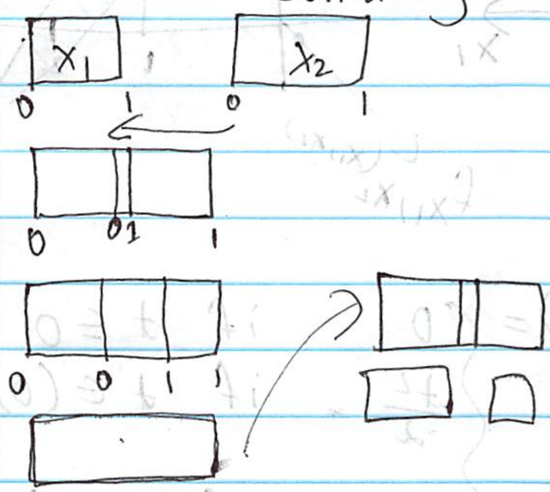


$\text{Supp}[T] = [0, 2]$

CDF



Convolving



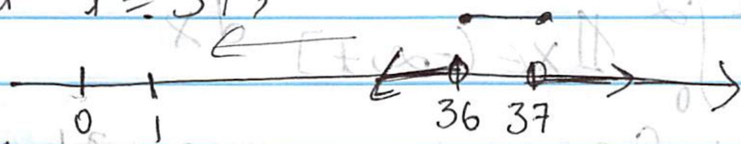
Wikipedia has animated version

Let's try to derive the PDF of T using the Convolution formula:

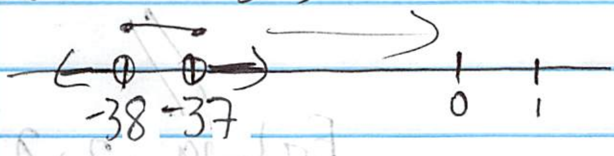
$$\underset{iid}{\Rightarrow} f_T(t) = \int_{\text{Supp}[X]} f^{odd}(x) f^{odd}(t-x) \mathbb{1}_{t-x \in \text{Supp}[X]} \, dx$$

$$= \int_0^1 (1)(1)\, \mathbb{1}_{\overbrace{\overbrace{t-x\in[0,1]}^{x-t\in[-1,0]}}^{x\in[t-1,t]}}\, dx = \int_0^1 \mathbb{1}_{x\in[t-1,t]}\, dx$$

Let $t = 37$,



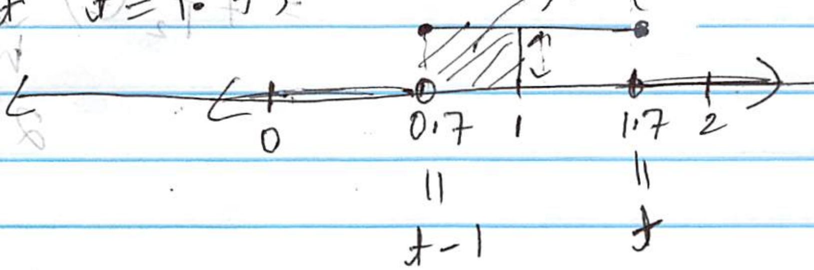
Let $t = -37$,



$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \in [0,1) \\ 0 & \text{if } t \geq 2 \\ 2-t & \text{if } t \in (1,2) \end{cases}$$

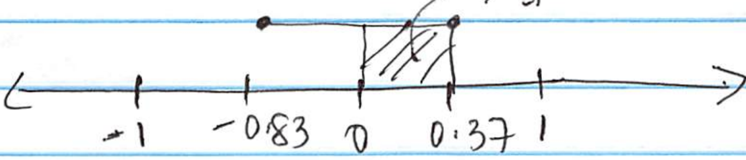
Let $t = 1.7$,

$1-(t-1) = 2-t$



Let $t = 0.37$,

t



$$X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda) \doteq \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{odd}(x)}\, \mathbb{1}_{x\in[0,\infty)}}_{f(x)},$$

$$T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$$

$$f_{T_2}(t) = \int_{Supp[X]} f^{odd}(x)\, f^{odd}(t-x)\, \mathbb{1}_{t-x\in Supp[X]}\, dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \underbrace{\lambda e^{-\lambda(t-x)}}_{\lambda e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{\underbrace{\underbrace{t-x \in [0,\infty)}_{x-t \in (-\infty,0]}}_{x \in (-\infty,t]}} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty, t]} \, dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx = t \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)}$$

$$\parallel$$

Erlang $(2, \lambda)$

$= f_{T_2}(t)$