Lecture 7

$$p_T(t) = \sum_{x \in Supp[X]} p^{old}_X(x)\; p^{old}_Y(d-x)\; \mathbb{1}_{d-x \in Supp[Y]}$$

$x \in \{d, d+1, \dots\}$

$x-d \in \{0, 1, \dots\}$

$$= \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \; \frac{\lambda^{\overbrace{-(d-x)}^{x-d}} e^{-\lambda}}{\underbrace{(-(d-x))!}_{x-d}} \; \mathbb{1}_{d-x \in \{\dots,-1,0\}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} \; \mathbb{1}_{x \in \{d, d+1, \dots\}}$$

let $d' = -d = |d'|$

$$= e^{-2\lambda} \begin{cases} \text{if } d \le 0 & \displaystyle\sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!\,(x+d')!} \rightarrow \\ \text{if } d > 0 & \displaystyle\sum_{x \in \{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!\,(x-d)!} = \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!\,x'!} \rightarrow \end{cases}$$

let $x' = x - d$

$$\rightarrow = \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!\,(x+|d|)!} = \sum_{x \in \{0,1,\dots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!} := I_{|d|}(2\lambda)$$

$\rightarrow \underset{d=|d|}{=}$

(Modified Bessel Function)

$$= e^{-2\lambda}\, I_{|d|}(2\lambda)\, \mathbb{1}_{d \in \mathbb{Z}}$$

$$= \text{Skellam}(\lambda, \lambda)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$. From previous class, $X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$Supp[X] = \{0, 1, \dots\}$ $\quad Supp[Y] = \{\dots, -2, -1, 0, \dots\}$

$D = X_1 - X_2 \sim ?$ (difference)

$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$

$$P_Y(y) = \frac{e^{-\lambda} \lambda^{-y}}{(-y)!}$$

$Supp[X+Y] = Supp[X] + Supp[Y] = \mathbb{Z}$ (all integers)

convolution formula for independent discrete rv's.

$$p_T(t) = \sum_{x \in Supp[X]} p^{old}_X(x)\; p^{old}_Y(d-x)\; \mathbb{1}_{d-x \in Supp[Y]} \rightarrow$$

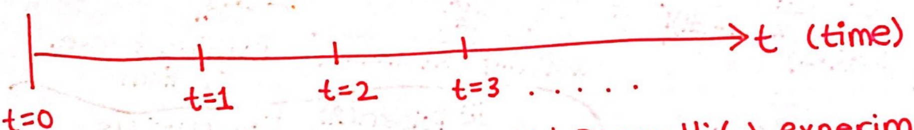
$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$$p_{X_1|T}(x,t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x,t-x)}{p_T(t)} = \frac{p_{X_1}(x)\,p_{X_2}(t-x)}{p_T(t)}$$

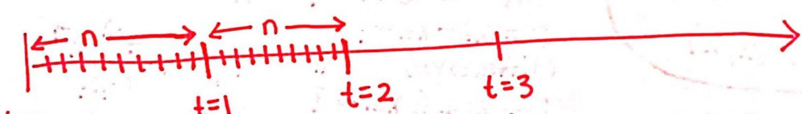
$$= \frac{\dfrac{\cancel{e^{-\lambda}}\,\cancel{\lambda^{x}}}{x!}\,\dfrac{\cancel{e^{-\lambda}}\,\cancel{\lambda^{t-x}}}{(t-x)!}}{\dfrac{\cancel{e^{-2\lambda}}\,(2\cancel{\lambda})^{t}}{t!}} = \frac{t!}{x!\,(t-x)!}\left(\frac{1}{2}\right)^{t} = \text{Binom.}(t, 1/2)$$

$$X_1 \sim \text{Geom.}(p) := \underbrace{(1-p)^x\, p\, \mathbb{1}_{x\in\{0,1,\dots\}}}_{p(x)}, \quad \text{Supp}[X_1] = \{0,1,\dots\}$$

$$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = \left(1 - (1-p)^x\right) \cancel{\mathbb{1}_{x\in\{0,1,\dots\}}}$$



In every "second", let's do n iid Bernoulli(p) experiments.



Let's call the resulting geometric random variables X_n, and it's unit realization is t.

$$p_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\dots,1,1+\frac{1}{n},\dots,2,\dots\}}$$

$F_{X_n}(x) = (1-(1-p)^{nx})\ \cancel{\mathbb{1}}$

where $\lambda \in (0,\infty)$

Let $n\to\infty,\ p\to 0,\ \lambda = np \to p = \frac{\lambda}{n}$ same as poisson

$$P_{X_\infty}(x) := \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}\frac{\lambda}{n}\,\mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}$$

Not a valid PMF!

$$\sum_{x\in Supp[X_\infty]} P_{X_\infty}(x) = 0$$

$\uparrow$

$$= \underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{x}}_{e^{-\lambda x}}\ \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0}\ \underbrace{\lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\mathbb{1}_{x\in[0,\infty)}} = 0\ \forall x!$$

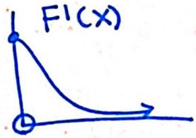
$\downarrow$

$Supp[X_\infty] = [0,\infty)$

$$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{x} = \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty)}$$

The PMF wasn't valid. Is the CDF valid? If so, I need to check 3 properties:

1. It's 0 as I go to negative infinity
2. It's 1 as I go to positive infinity and
3. It's an increasing function.



(1) $\lim_{x\to-\infty} (1-e^{-\lambda x})\,\mathbb{1}_{x\in[0,\infty)} = 0$ ✓

(2) $\lim_{x\to\infty} (1-e^{-\lambda x})\,\mathbb{1}_{x\in[0,\infty)} = 1 - \lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 1$ ✓

(3) $\frac{d}{dx}\left[(1-e^{-\lambda x})\,\mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x}\,\mathbb{1}_{x\in[0,\infty)}$

$\Rightarrow$ Valid CDF!

We now have a cts. r.v. Cts r.v.'s have the following properities:

$|Supp[X]| = |\mathbb{R}|$ uncountable infinity (the size of the continum)

- They do not have PMF's (because the probability of the rv being at the specific number is zero), but they do have CDF's.
- The derv. of the CDF is very useful function, it is called the probability density function (PDF) denoted $f(x)$. Note: discrete rv do not have PDF's.

$$f(x) := F'(x), \quad P(X \in [a,b]) = \underbrace{P(X \le b)}_{F(b)} - \underbrace{P(X \le a)}_{F(a)} = \int_a^b f(x)dx$$

By fundamental theorem of calculus.

$$\int_{\mathbb{R}} f(x) = 1 = \text{"} F(\cancel{\infty}) - F(\cancel{-\infty}) \text{"}$$

with $F(\infty) = 1$, $F(-\infty) = 0$.

$f(x) \ge 0$ since CDF's are increasing functions.

$\Rightarrow Supp[X] = \{x : f(x) > 0\}$

$X_1, \dots, X_n$, independent $\Rightarrow f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$ " JDF "

$$X \sim Exp(\lambda) := \underbrace{\underbrace{\lambda e^{-\lambda x}}_{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}, \quad F(x) = (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)}$$

Exponential rv

$\lambda \in (0, \infty)$ it's parameter space.

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) \qquad \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f_{\vec{X}}(\vec{x}) \, dx_1 \dots dx_K = 1$$

$$P(A) = \iint_A f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$