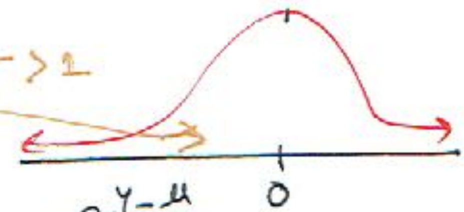
Logistic Distribution :-

$X \sim \text{logistic}(0,1) = \frac{e^x}{(1+e^x)^2} \approx N(0,1)$

$E[X] = 0, \ SD[X] = \frac{\pi}{\sqrt{3}} \approx 1.8 > 1$

$\sigma > 1$



$Y = \mu + \sigma X \sim f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{|\sigma|} = \frac{e^{\frac{y-\mu}{\sigma}}}{\sigma\left(1+e^{\frac{y-\mu}{\sigma}}\right)}$

consider the shift and scale where sigma > 0

$= \text{logistic}(m, \sigma)$

Q) Why it is called the logistic distribution ?

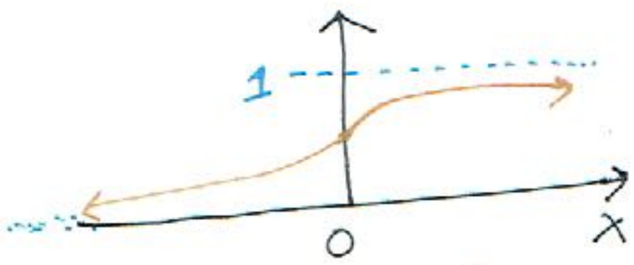
There is a famous function called "logistic function". It has three parameters: L (maximum value),

K (steepness)

mu (center) and it is:

$l(x) := \frac{L}{1+e^{-K(x-M)}} = \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{e^x+1}$ (Standard logistic function)

If $L=1, K=1, m=0$



$X \sim \text{logistic}(0,1)$

$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{e^t}{(1+e^t)^2}\,dt = \left[-u^{-1}\right]_1^{1+e^x}$

$= 1 - \frac{1}{1+e^x} = \frac{e^x}{1+e^x}$

let, $u = 1+e^t \Rightarrow e^t = 1-u \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du \Longrightarrow t = -\infty$

$\Downarrow$

$u = 1, \ t = x$

$\Downarrow$

$u = 1+e^x$

The "quantile" q, or Percentile 100q for rv X is defined as the minimum x s.t

$$q \leq (P X \leq x) = F(x) \Longleftrightarrow F(x) \geq q,$$ It is denoted $Q[X, q]$

when Q is the "quarantile operator" (not the upper incomplete regularited gamma function). When $q = 0.5$, the quantile has a special name, the "median", $\text{med}[X] := Q[X, \frac{1}{2}]$

Example:

$$X \sim U(\{2, 4, 6 \cdots 20\}) = \frac{1}{10} \mathbb{1}_{x \in} \cdots\cdots$$

x	px	F(x)
2	0.1	0.1
4	0.1	0.2
6	0.1	0.3
8	0.1	0.4
10	0.1	0.5
12		0.6
14	0.1	0.7
16	0.1	0.8
18	0.1	0.9
20	0.1	1

However, If x is a continious rv with "contigous support" e.g [0, 10], [0, Infinity), all real numbers, etc and not something like [0,1] union [2,3]. In the latter case, F(x) is flat between [1, 2]. which means its not inVertible, In the former cases, F(x) is inVertible

$Q[X, q] = F_X^{-1}(q)$, and the Inverse CDF is called appropriately, the "quarantile function"

what is $Q[X, 30\%]$ meaning, what is the 30th

$$Q[X, 30\%] = 6 \qquad \text{med}[X] = 10$$

$$Q[X, 80\%] = 16$$

$$Q[X, 85\%] = 18 = Q[X, 0.9]$$

Example of quantile function :-

$$X \sim \mathrm{Exp}(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

$$\Rightarrow F_X(x) = 1 - e^{-\lambda x} = q$$

$$\Rightarrow 1 - q = e^{-\lambda x}$$

$$\Rightarrow \ln(1-q) = -\lambda x$$

$$\Rightarrow x = -\frac{1}{\lambda} \ln(1-q) = \frac{1}{\lambda} \ln\left(\frac{1}{1-q}\right) = F_X^{-1}(q)$$

$$\underbrace{\mathrm{Med}[X]}_{F_X^{-1}(0.5)} = \frac{\ln(2)}{\lambda}$$

Quantile functions are not usually available in closed form since CDF's aren't even usually available in closed form

e.g. $X \sim \mathrm{Erlang}(k, \lambda) \Rightarrow F_X(x) = P(k, \lambda x)$

$\mathrm{Med}[X] = x$ s.t. $P(k, \lambda x) = 0.5$. Need a computer solver

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Let, $X \sim \mathrm{Exp}(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$, $k > 0$ ... $Y = g(x) = k e^{x} \sim f_Y(y) = ?$

$$y = k e^{x} \Rightarrow \frac{y}{k} = e^{x} \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k) = g^{-1}(y)$$

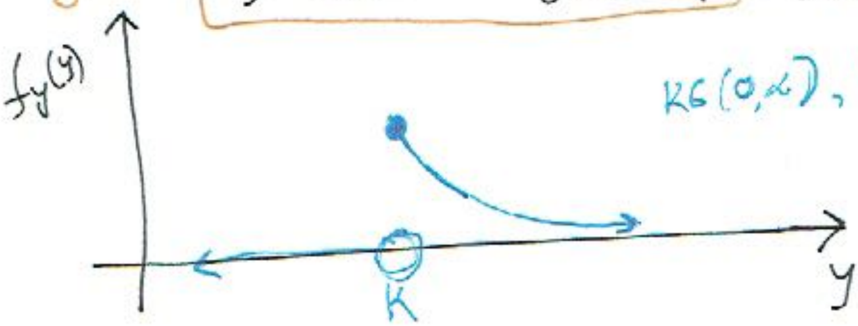
$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \left|\frac{1}{y}\right| = \frac{1}{|y|} =$$

y is always Positive

Plug :

$$f_Y(y) = f_X\left(\ln \frac{y}{k}\right) \frac{1}{|y|} = \frac{\lambda}{|y|} \overbrace{e^{-\lambda \ln \frac{y}{k}}}^{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)} \underbrace{\mathbb{1}_{\ln(y) - \ln(k) \in [0, \infty)}}_{\underbrace{\ln(y) \in [\ln(k), \infty)}_{y \in [k, \infty)}}$$

graph $= \boxed{\frac{\lambda}{y} \left(\frac{y}{k}\right)^{-\lambda} \mathbb{1}_{y \in [k, \infty)}} = \text{Pareto I}(k, \lambda)$



$k \in (0, \infty)$, $\lambda \in (0, \infty)$

$$F_Y(y) = \int^{y} \frac{\lambda}{k^{-\lambda}} \cdot \frac{1}{t^{\alpha+1}} \, dt$$

$$= \frac{\cancel{\lambda}}{k^{-\lambda}} \left[-\frac{1}{\cancel{\lambda} t^{\lambda}}\right]_k^y =$$

$$= k^{\alpha}\left(\frac{1}{k^{\alpha}} - \frac{1}{y^{\alpha}}\right) = 1 - \left(\frac{k}{y}\right)^{\alpha}$$

$$\Rightarrow \boxed{F_y^{-1}(q) = k(1-q)^{-1/\alpha}} \rightarrow \text{Imp For Exm}$$

*\# The distribution was discovered by Vilfredo Pareto, an Italian Economist in 1896 when he observed that 20% of the richest Italians owned 80% of the land (i.e the wealth). This is known as the "Pareto Principle" and it corresponds to the Pareto($1, 1.161$) distribution.

Further, the Pareto distribution is a waiting time/ Survival time model. Its used for. check wikipedia If you interested. weath, music talent, number of parents ...

$X, Y \overset{iid}{\sim} Exp(1) = e^{-x}\mathbb{1}_{x\in[0,\infty)}$, $D = X - Y = X + (-Y) \sim f_D(d) = ?$

$Z \sim \frac{1}{|-1|} f_X\left(\frac{z}{-1}\right) = e^{z}\mathbb{1}_{z\in(-\infty,0]}$

$$f_D(d) = \int_{Supp[X]} f^{old}(x) f_Z^{old}(d-x)\,\mathbb{1}_{d-x\in Supp(Z)}\,dx$$

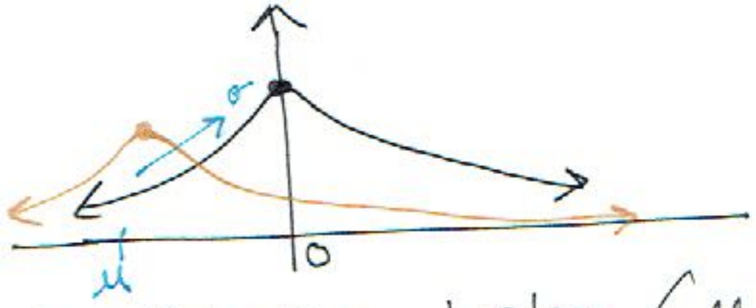
$$= \int_0^{\infty} e^{-x} e^{d-x}\,\underbrace{\mathbb{1}_{d-x\in\{-\infty\cdots -1,0\}}}_{\substack{x-d\in[0,\infty] \\ x\in[d,\infty)}}\,dx$$

$$= e^{d}\int_0^{\infty} e^{-2x}\,\mathbb{1}_{x\in[d,\infty)}\,dx = e^{d}\begin{cases}\int_d^{\infty} e^{-2x}dx & \text{If } d\geq 0\\ \int_0^{\infty} e^{-2x}dx & \text{If } d<0\end{cases}$$

$$= e^{d}\begin{cases}\left[-\frac{1}{2}e^{-2x}\right]_d^{\infty} & \text{If } d\geq 0\\ \left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} & \text{If } d<0\end{cases} = \frac{1}{2}e^{d}\begin{cases}e^{-2d} & \text{If } d\geq 0\\ 1 & \text{If } d<0\end{cases}$$

$$= \frac{1}{2}\begin{cases}e^{-d} & \text{If } d\geq 0\\ e^{d} & \text{If } d<0\end{cases} = \boxed{\frac{1}{2}e^{-|d|}}$$

Laplace($0, 1$), Std Laplace distr



AKA double exponential
Laplace (0, 1)

$X = \mu + \sigma D \sim \text{Laplace}(\mu, \sigma) := \boxed{\frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}}$

$\sigma > 0$