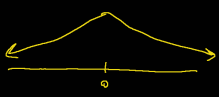
$X \sim \text{Cauchy}(0,1) = \frac{1}{\pi}\frac{1}{x^2+1}$

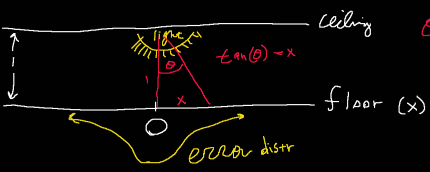
$E[X] = \int_{\mathbb{R}} x \frac{1}{\pi}\frac{1}{x^2+1}\,dx = \infty$ the expectation doesn't exist

$M_X(t) = \int_{\mathbb{R}} e^{tx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx$ does not exist



$\phi_X(t) = \int_{\mathbb{R}} e^{itx}\frac{1}{\pi}\frac{1}{x^2+1}\,dx = \dots = e^{-|t|}$, $\phi_X'(t) = -\frac{t}{|t|}e^{-|t|}$, $\phi_X'(0)$ dne

Let's derive the Cauchy distribution like the physicists found it.



$\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi}\mathbb{1}_{\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}$

$x = g(\theta)$

$\theta = g^{-1}(x) = \arctan(x)$

tangent is invertible between -pi/2 and pi/2.

$f_X(x) = f_\theta(g^{-1}(x))\left|\frac{d}{dx}[g^{-1}(x)]\right| = \frac{1}{\pi}\mathbb{1}_{\underbrace{\arctan(x)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}_{x\in\mathbb{R}}}\frac{1}{x^2+1} = \text{Cauchy}(0,1)$

Let $X_1,\dots,X_n \overset{iid}{\sim} N(\mu,\sigma^2) \Rightarrow \frac{X_i-\mu}{\sigma} = Z_i \sim N(0,1)$

$T_n \sim N(n\mu, n\sigma^2)$, $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, $S_n^2 = \frac{1}{n-1}\sum(X_i-\bar{X})^2 \sim f_{S_n^2}(s^2) = ?$

sample variance

$\Rightarrow Z_1,\dots,Z_n \overset{iid}{\sim} N(0,1)$

$\vec{Z}^T\vec{Z} = \sum_{i=1}^n Z_i^2 \sim \chi^2_n$

$= \sum_{i=1}^n\left(\frac{X_i-\mu}{\sigma}\right)^2 = \frac{\sum(X_i-\mu)^2}{\sigma^2}$

$\cancel{n\bar{X}^2}$ $\cancel{n\bar{X}\mu}$

$\bar{x}\sum X_i - \cancel{n\bar{X}^2} - \mu\sum X_i + \cancel{n\bar{X}\mu}$

$//$

$\sum X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu$

$\sum(X_i-\mu)^2 = \sum((X_i-\bar{X})+(\bar{X}-\mu))^2 = \sum(X_i-\bar{x})^2 + \underbrace{\sum 2(X_i-\bar{X})(\bar{X}-\mu)}_{0} + \sum(\bar{X}-\mu)^2$

$= \sum(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2$

$\frac{n(\bar{X}-\mu)^2}{\sigma^2} = \left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)^2 = \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$

$\Rightarrow \frac{\sum(X_i-\mu)^2}{\sigma^2} = \underbrace{\frac{n-1}{\sigma^2}S^2}_{\chi^2_{n-1}} + \underbrace{\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{Z^2\sim\chi^2_1} \sim \chi^2_n$

Maybe...

$U_1 \sim \chi^2_{K_1}$ indep. of $U_2 \sim \chi^2_{K_2}$

$\Rightarrow U_1 + U_2 \sim \chi^2_{K_1+K_2}$

In order for this "maybe" to be true, we need independence of those two terms i.e we need S^2 and Xbar to be independent. We need Cochran's Theorem to prove this.

$\vec{Z}^T\vec{Z} = \vec{Z}^T I\vec{Z} \sim \chi^2_n$ this scalar is called a "quadratic form"

Consider $\vec{Z}^T\underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0\end{bmatrix}}_{B_1}\vec{Z} = Z_1^2 \sim \chi^2_1$

Consider $\vec{Z}^T\underbrace{\begin{bmatrix}0 & \ddots & & 0\\ & & 1 & \\ 0 & & & \ddots \\ & & & 0\end{bmatrix}}_{B_i}\vec{Z} = Z_i^2 \sim \chi^2_1$

$\text{rank}[B_i] = 1$

$\sum\text{rank}[B_i] = n$

$\vec{Z}^T I\vec{Z} = \vec{Z}^T(B_1+B_2+\dots+B_n)\vec{Z} = \underbrace{\vec{Z}^T B_1\vec{Z}}_{\sim\chi^2_1} + \underbrace{\vec{Z}^T B_2\vec{Z}}_{\sim\chi^2_1} + \dots + \underbrace{\vec{Z}^T B_n\vec{Z}}_{\sim\chi^2_1} \sim \chi^2_n$

Conjecture: each of these quadratic forms is indpendent.

Cochran's Thm: If B_1 + B_2 + ... + B_k = I, k<=n and the sum of their ranks is n then you have two powerful results:

(a) $\vec{Z}^T B_j\vec{Z} \sim \chi^2_{\text{rank}[B_j]}$ and (b) $\vec{Z}^T B_{j_1}\vec{Z}$ is independent of $\vec{Z}B_{j_2}\vec{Z}$ $\forall_{j_1\neq j_2}$

Consider $\sum(Z_i-\bar{Z})^2 = \sum Z_i^2 - 2\overbrace{\sum Z_i}^{n\bar{Z}}\bar{Z} + \sum\bar{Z}^2 = \sum Z_i^2 - 2n\bar{Z}^2 + n\bar{Z}^2$

$= \sum Z_i^2 - n\bar{Z}^2$

let $\vec{1}_n =$ n-dim column vector of all ones $\bar{Z} = \frac{1}{n}\vec{1}^T\vec{Z} = \frac{1}{n}\vec{Z}^T\vec{1}$

$n\bar{Z}^2 = n\bar{Z}\bar{Z} = \cancel{n}\frac{1}{\cancel{n}}\vec{Z}^T\vec{1}\frac{1}{n}\vec{1}^T Z = \frac{1}{n}\vec{Z}^T\vec{1}\vec{1}^T\vec{Z} = \vec{Z}^T\underbrace{\left(\frac{1}{n}J_n\right)}_{B_2}\vec{Z}$

let $J_n = \vec{1}\vec{1}^T$, which is an n x n matrix of all ones

$\sum(Z_i-\bar{Z})^2 = \vec{Z}^T I\vec{Z} - \vec{Z}^T\left(\frac{1}{n}J_n\right)\vec{Z} = \vec{Z}^T\underbrace{\left(I-\frac{1}{n}J_n\right)}_{B_1}\vec{Z}$

$\vec{Z}^T\vec{Z} = \sum(Z_i-\bar{Z})^2 + n\bar{Z}^2 = \vec{Z}^T B_1\vec{Z} + \vec{Z}^T B_2\vec{Z}$

I want to use Cochran's thm on the above expression. So I need to make sure B_1 + B_2 = I and rank[B_1] + rank[B_2] = n.

$B_1 + B_2 = \left(I - \frac{1}{n}J_n\right) + \frac{1}{n}J_n = I$ ✓

$\text{rank}[B_2] = \text{rank}\left[\frac{1}{n}J_n\right] = \text{rank}[J] = 1$

$\text{rank}[B_1] = \text{rank}\left[I-\frac{1}{n}J\right] = n-1$

$\Rightarrow \sum\text{rank}[B_j] = 1 + n - 1 = n$ ✓

Thm from 231 class: if A is symmetric and idempotent (i.e. AA = A) then rank[A] = tr[A] = sum of A's diagonal entries.

$\left(I-\frac{1}{n}J\right)^T = I^T - \frac{1}{n}J^T = I - \frac{1}{n}J$ ✓

$J_3J_3 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix} = 3J$

$\left(I-\frac{1}{n}J\right)\left(I-\frac{1}{n}J\right) = II - \frac{1}{n}JI - \frac{1}{n}IJ + \frac{1}{n^2}JJ = I - \frac{2}{n}J + \cancel{\frac{1}{n^2}n}J = I - \frac{1}{n}J$ ✓

$\text{tr}\left[I-\frac{1}{n}J\right] = \left(1-\frac{1}{n}1\right) + \left(1-\frac{1}{n}1\right) + \dots + \left(1-\frac{1}{n}1\right) = n-1 = \text{rank}[B_1]$

Since the two conditions of Cochran's Thm are satisfied, we can apply it to get the two results:

$\Rightarrow \vec{Z}^T B_1\vec{Z} = \sum(Z_i-\bar{Z})^2 \sim \chi^2_{n-1}$ ind. of $\vec{Z}^T B_2\vec{Z} = n\bar{Z}^2 \sim \chi^2_1$

What does this have to do with our goal? Well, it's the same thing:

$\bar{Z} = \frac{Z_1+\dots+Z_n}{n} = \frac{\frac{X_1-\mu}{\sigma}+\dots+\frac{X_n-\mu}{\sigma}}{n} = \frac{\sum X_i - n\mu}{\sigma n} = \frac{\bar{X}-\mu}{\sigma}$

$\sum(Z_i-\bar{Z})^2 = \sum\left(\frac{X_i-\mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2 = \sum\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum(X_i-\bar{X})^2 = \frac{n-1}{\sigma^2}S^2$

$n\bar{Z}^2 = n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 = \left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right)^2 = \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$

$\underbrace{\frac{n-1}{\sigma^2}S^2}_{\sim\chi^2_{n-1}} + \underbrace{\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{\sim\chi^2_1} \sim \chi^2_n$

independent

Fisher proved this without Cochran's thm in 1925 and Geary proved in 1936 that this decomposition is exclusive to the iid normal rv model.

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, $\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim ?$ Not $N(0,1)$