Lecture 20

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \cdot \frac{\sigma^2}{n-1}\cdot\frac{n-1}{\sigma^2}}} = \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{\chi^2_{n-1}}}{n-1}}} \sim T_{n-1}$$

we don't know σ so we use S instead

Due to Cochran's Thm, we know $\bar{X}$ and S^2 are independent.

Multivariate Normal Distribution MVN

$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}_n$,

$Var[\vec{Z}] = I_n$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$= N_n(\vec{0}, I)$ STD. MVN

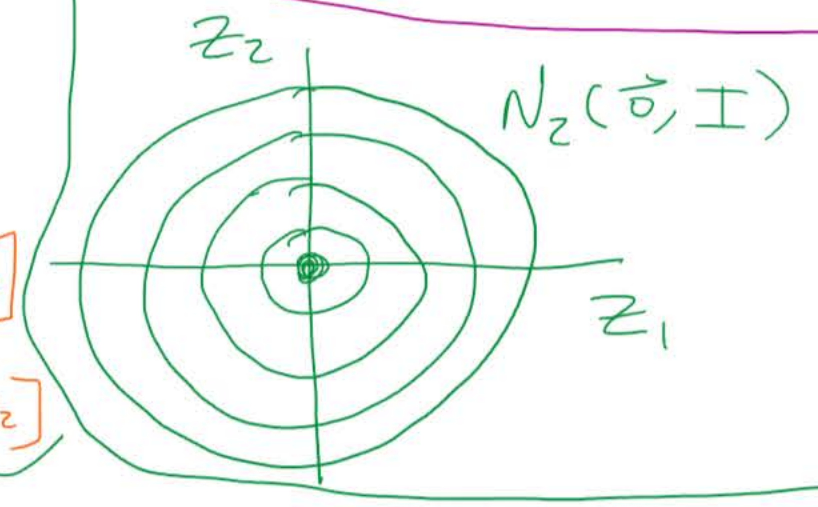
$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$, $E[\vec{X}] = \vec{\mu}$, $Var[\vec{X}] = I_n \Rightarrow \vec{X} \sim N_n(\vec{\mu}, I)$

$$\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1 + Z_2 \\ \vdots \\ Z_1 + \dots + Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$$

but the components are dependent e.g.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & & & \vdots \\ \vdots & & \ddots & & 0 \\ 1 & & & & 1 \end{bmatrix}}_{n \times n}$$

$$Cov[X_1, X_2] = Cov[Z_1, Z_1 + Z_2] = \underbrace{Cov[Z_1, Z_1]}_{1} + \underbrace{Cov[Z_1, Z_2]}_{0} = 1 \Rightarrow X_1, X_2 \text{ dep.}$$



Let's derive a general formula for the variance-covariance matrix of A (a $n \times n$ matrix of scalars) times a random vector X of dim n:

$$Var[A\vec{X}] := E[(A\vec{X})(A\vec{X})^T] - E[A\vec{X}]E[A\vec{X}]^T$$

$$= A E[\vec{X}\vec{X}^T]A^T - AE[\vec{X}](AE[\vec{X}])^T$$

$$= A(E[XX^T] - E[\vec{X}]E[\vec{X}]^T)A^T = A\Sigma A^T$$

→ HW: nothing

$$= A\left(\underbrace{E[XX^T] - E[\vec{X}]E[\vec{X}]^T}_{\Sigma = Var[\vec{X}]}\right)A^T = A\Sigma A^T$$ $\rightarrow$ HW: nothing changes for $m \times n$ matrix.

$\vec{X} = A\vec{Z}$, $Var[\vec{X}] = A I_n A^T = AA^T$ Conjecture: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{X} = A\vec{Z} + \vec{\mu}$, $A \in \mathbb{R}^{n\times n}$, $\vec{\mu} \in \mathbb{R}^n$, $\vec{X} \sim f_{\vec{X}}(\vec{x}) = ?$

$= g(\vec{z})$, $h(\vec{x}) = \vec{z}$ where g, h are (hopefully) inverses

$\vec{z} = h(\vec{z}) = \underbrace{A^{-1}}_{B}(\vec{x} - \vec{\mu}) \Rightarrow$ in order for the inverses to exist... A has to be invertible.

$$= B\vec{x} - B\vec{\mu} = \begin{bmatrix} \vec{b}_1 \vec{x} - \vec{b}_1 \vec{\mu} \\ \vdots \\ \vec{b}_n \vec{x} - \vec{b}_n \vec{\mu} \end{bmatrix} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

$$J_n = \det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \det \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \cdots & & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix}}_{B} = \det[A^{-1}] = \frac{1}{\det(A)}$$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_n| = \frac{1}{(2\pi)^{1/2}} e^{-1/2 \underbrace{A^{-1}(\vec{x}-\vec{\mu})^T A^{-1}(\vec{x}-\vec{\mu})}_{(\vec{x}-\vec{\mu})^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})}} \frac{1}{|\det[A]|}$$

$$DD^{-1} = I \Rightarrow (DD^{-1})^T = I^T = I \Rightarrow (D^{-1})^T D^T = I \Rightarrow (D^{-1})^T = (D^T)^{-1}$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(A)^2}} e^{-1/2(\vec{x}-\vec{\mu})^T \underbrace{(A^T)^{-1}A^{-1}}_{(AA^T)^{-1}}(\vec{x}-\vec{\mu})}$$

let $\Sigma = AA^T = Var[\vec{X}]$

$\det[\Sigma] = \det[AA^T]$

$= \det[A]\det[A^T]$

$= \det[A]^2$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-1/2(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})} = N_n(\vec{\mu}, \Sigma)$$

$\hookrightarrow$ need Σ to be invertible

A little bit of multivariate characteristic functions:

A little bit of multivariate characteristic functions:

$$\phi_{\vec X}(\vec t) := E\left[e^{i\vec t^T \vec X}\right] = E\left[e^{i(t_1X_1 + \dots + t_nX_n)}\right] = E\left[e^{it_1X_1} \cdot \dots \cdot e^{it_nX_n}\right]$$

if $X_1, \dots, X_n$ indep.

$$= E\left[e^{it_1X_1}\right] \cdot \dots \cdot E\left[e^{it_nX_n}\right] = \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) \cdot \dots \cdot \phi_{X_n}(t_n)$$

P0) $\phi_{\vec X}(\vec 0) = E\left[e^{i\vec 0^T \vec X}\right] = 1$

P1) if two chfs are equal $\Rightarrow$ the two rv's are equal in distribution

P2) $\vec Y = A\vec X + \vec b$, $A \in \mathbb{R}^{m\times n}$, $\vec b \in \mathbb{R}^m$, $\vec X$ is dim n $\Rightarrow \vec Y$ is dim m

$$\phi_{\vec Y}(t) := E\left[e^{i\vec t^T(A\vec X + \vec b)}\right] = E\left[e^{i\vec t^T A\vec X}\, e^{i\vec t^T \vec b}\right] = e^{i\vec t^T \vec b}\, E\left[e^{i(A^T\vec t)^T \vec X}\right]$$

$$= e^{i\vec t^T \vec b}\, \phi_{\vec X}(A^T\vec t)$$

Let's derive the chf of the std. MVN:

$$\phi_{\vec Z}(\vec t) = \prod_{i=1}^{n} \phi_Z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2} = e^{-\frac{1}{2}\vec t^T \vec t}$$

Let's derive the chf of the general MVN: $\vec X = A\vec Z + \vec\mu \sim N(\vec\mu, \overbrace{AA^T}^{\Sigma})$

$$\phi_{\vec X}(\vec t) \overset{P2}{=} e^{i\vec t^T\vec\mu}\, \phi_{\vec Z}(A^T\vec t) = e^{i\vec t^T\vec\mu}\, e^{-\frac{1}{2}\underbrace{(A^T\vec t)^T A^T\vec t}_{\vec t^T \underbrace{AA^T}_{\Sigma} \vec t}}$$

$$= e^{i\vec t^T\vec\mu - \frac{1}{2}\vec t^T\Sigma\vec t}$$

$\vec Y = B\vec X + \vec\mu \sim ?$ $B \in \mathbb{R}^{m\times n}$, $\vec c \in \mathbb{R}^m$

$$\phi_{\vec Y}(\vec t) \overset{P2}{=} e^{i\vec t^T\vec c}\, \phi_{\vec X}(B^T\vec t) = e^{i\vec t^T\vec c}\, e^{i(B^T\vec t)^T\vec\mu - \frac{1}{2}(B^T\vec t)^T\Sigma(B^T\vec t)}$$

$$= e^{i\vec t^T(\vec c + B\vec\mu) - \frac{1}{2}\vec t^T B\Sigma B^T\vec t} \quad \text{P1} \Rightarrow$$

$$= e^{i\vec{t}^T(\vec{c} + B\vec{\mu}) - \frac{1}{2}\vec{t}^T B\Sigma B^T \vec{t}} \overset{P1}{\Longrightarrow} \vec{Y} \sim N_{\mu}(B\vec{\mu} + \vec{c},\ B\Sigma B^T)$$

(if $B\Sigma B^T$ is invertible)

Let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$. Consider $(\vec{X}-\vec{\mu})^T \Sigma^{-1} (\vec{X}-\vec{\mu}) \sim ?$

$$= (\vec{X}-\vec{\mu})^T (A^{-1})^T A^{-1} (\vec{X}-\vec{\mu})$$

Recall: $\vec{Z} = A^{-1}(\vec{X}-\vec{\mu})$

$$= (A^{-1}(\vec{X}-\vec{\mu}))^T A^{-1}(\vec{X}-\vec{\mu})$$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1} A^{-1} = (A^{-1})^T A^{-1}$$

$$= \vec{Z}^T\vec{Z} \sim \chi^2_n$$

Mahalanobis Distance

This is kind of like distance in $\mathbb{R}^n$ adjusted for all the dependencies among the dimensions like a multivariate "z-score"

In one dimension $(x-\mu)(\sigma^2)^{-1}(x-\mu) = \dfrac{(x-\mu)^2}{\sigma^2} = \left(\dfrac{x-\mu}{\sigma}\right)^2 = z^2$