$Z \sim N(0,1)$, $Y = Z^2 = g(Z)$ not 1:1

$$F_Y(y) := P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = \int_{-\sqrt{y}}^{+\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

f_Z symmetric

$$= 2 \int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 2\left(F_Z(\sqrt{y}) - \cancel{F_Z(0)}^{\frac{1}{2}}\right) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = \cancel{2} f_Z(\sqrt{y}) \frac{1}{\cancel{2}} y^{-\frac{1}{2}} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \ge 0}}$$

$$\propto y^{\underbrace{-\frac{1}{2}}_{\frac{1}{2}-1}} e^{-\frac{1}{2}y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$E[Y] = k\, E[Z^2] = k$

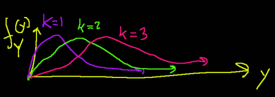
$Z_1, Z_2, \ldots, Z_k \overset{iid}{\sim} N(0,1)$ and $Y = Z_1^2 + Z_2^2 + \ldots + Z_k^2 \sim ?$

$$Y \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi^2_k = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{k/2-1} e^{-\frac{y}{2}} \mathbb{1}_{y \ge 0} \overset{k=1}{=} \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}} \mathbb{1}_{x \ge 0}$$

the only parameter here is k

chi-squared distribution with parameter k and this parameter is called "degrees of freedom"

$k \in \mathbb{N}$



$X \sim \chi^2_k$, $Y = \sqrt{X} \sim f_Y(y)$, $x = y^2 = g^{-1}(y)$, $\left|\frac{d}{dy} g^{-1}(y)\right| = 2y$

$$f_Y(y) = f_X(y^2)\, 2y = \frac{\left(\frac{1}{2}\right)^{k/2}}{\Gamma\left(\frac{k}{2}\right)} y^{k-2} e^{-\frac{y^2}{2}} 2y\, \mathbb{1}_{\underbrace{y^2 \ge 0}_{y \ge 0}}$$

$$= \frac{\left(\frac{1}{2}\right)^{k/2-1}}{\Gamma(k/2)} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y \ge 0} = \chi_k$$

this is the chi distribution with k degrees of freedom

$X \sim N(0,1)$, $|X| \sim ?$ $|X| = \sqrt{X^2} \sim \chi_1 = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \mathbb{1}_{y \ge 0} = 2\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

2 times PDF $N(0,1)$

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX \sim \frac{1}{c} \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left(\frac{x}{c}\right)^{\alpha-1}}_{x^{\alpha-1}/c^{\alpha-1}} e^{-\beta\frac{x}{c}}$ for $c > 0$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta/c)x} = \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

$X \sim \chi^2_k$, $Y = \frac{X}{k} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

$X_1 \sim \chi^2_{k_1}$ indep of $X_2 \sim \chi^2_{k_2}$, let $U = \frac{X_1}{k_1} \sim \text{Gamma}\left(\overset{a}{\frac{k_1}{2}}, \frac{k_1}{2}\right)$, $V = \frac{X_2}{k_2} \sim \text{Gamma}\left(\overset{b}{\frac{k_2}{2}}, \frac{k_2}{2}\right)$

$$R = \frac{X_1/k_1}{X_2/k_2} = \frac{U}{V} \sim \int_{Supp[U]} f_U(rt) f_V(t) |t| dt \quad \mathbb{1}_{rt \in [0,\infty]}$$

$$= \int_0^\infty t \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art} \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt} dt\, \mathbb{1}_{r \ge 0}$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r>0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t} dt$$

$$b^{-a} \cancel{b^{-b}} \qquad b^{-(a+b)} \left(1 + \frac{a}{b} r\right)^{-(a+b)}$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1} \mathbb{1}_{r>0} \frac{\Gamma(a+b)}{(ar+b)^{a+b}} = \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)} \mathbb{1}_{r>0}$$

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1} \left(1 + \frac{a}{b}\right)^{-(a+b)} \mathbb{1}_{r>0} = \frac{\left(\frac{k_1}{k_2}\right)^{k_1/2}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2}\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r>0}$$

$$= F_{k_1, k_2}$$

the "F distribution" or the "Fisher-Snecedor" distribution with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom, $k_1 \in \mathbb{N}$, $k_2 \in \mathbb{N}$

$Z \sim N(0,1)$ indep of $X \sim \chi^2_k$. Let $W = \frac{Z}{\sqrt{X/k}} \sim f_W(w)$

$$W^2 = \frac{Z^2}{X/k} \sim F_{1,k}$$

$$F_{W^2}(w^2) = P(W^2 \le w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$$

take d/dw of both sides

$$\frac{d}{dw}\left[F_{W^2}(w^2)\right] = \frac{d}{dw}\left[F_W(w)\right] - \frac{d}{dw}\left[F_W(-w)\right]$$

$$\Rightarrow 2w f_{W^2}(w^2) = f_W(w) - - f_W(w) \Rightarrow f_W(w) = w f_{W^2}(w^2)$$

$$\Rightarrow f_W(w) = \cancel{w} \frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} \underbrace{(w^2)^{\frac{1}{2}-1}}_{\cancel{w^{-1}}} \left(1 + \frac{w^2}{k}\right)^{-\frac{k+1}{2}} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\, \Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{w^2}{k}\right)^{-\frac{k+1}{2}} = T_k$$

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}} \Gamma\left(\frac{k}{2}\right)}$$

Student's T distribution with k degrees of freedom

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$ $R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f(u) |u| du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u| du$$

$$= \frac{1}{2\pi}\left(\underbrace{\int_{-\infty}^{0} e^{-\left(\frac{1+r^2}{2}\right)u^2} (-u) du}_{\int_0^{-\infty} \text{—}\, du \;=\; \int_0^{\infty} \text{—}\, du} + \int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\, du\right) = \frac{1}{\pi} \int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\, du$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2}\frac{1}{u} dt$, $u = 0 \Rightarrow t = 0$, $u \to \infty \Rightarrow t \to \infty$

PDF exponential

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{1+r^2}{2} t} \cancel{u} \frac{1}{2} \cancel{\frac{1}{u}} du = \frac{1}{2\pi} \frac{1}{\frac{1+r^2}{2}} = \frac{1}{\pi} \frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

$\sigma > 0$

$$X = \sigma R + c \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi} \frac{1}{1 + \left(\frac{r-c}{\sigma}\right)^2}$$

$$T_1 = \text{Cauchy}(0,1)$$