M13

$$f_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j}\left(j f(x) F(x)^{j-1} (1-F(x))^{n-j} - (n-j) f(x) F(x)^{j} (1-F(x))^{n-j-1}\right)$$

$$= \sum_{j=k}^{n} \frac{n!}{\underset{(j-1)!}{j!}\,(n-j)!}\, j f(x) F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n} \frac{n!}{j!\,\underset{(n-j-1)!}{(n-j)!}}\,(n-j) f(x) F(x)^{j} (1-F(x))^{\overset{n-(j+1)}{n-j-1}}$$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x)F(x)^{j-1}(1-F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!\,\underbrace{(n-j-1)!}_{(n-(j+1))!}} f(x)F(x)^{j}(1-F(x))^{n-j-1}$$

reindexing trick for. Let $\ell = j+1 \to j = \ell - 1$ $\left[\begin{array}{l} j = k \to \ell = k+1 \\ j = n-1 \to \ell = n \end{array}\right.$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!\,(n-j)!} f(x) F(x)^{j-1} (1-F(x))^{n-j} - \sum_{\ell=k+1}^{n} \frac{n!}{(\ell-1)!\,(n-\ell)!} f(x) F(x)^{\ell-1} (1-F(x))^{n-\ell}$$

Note that both sum expressions are exactly the same, so when we subtract we're left with just the expression when $j = k$

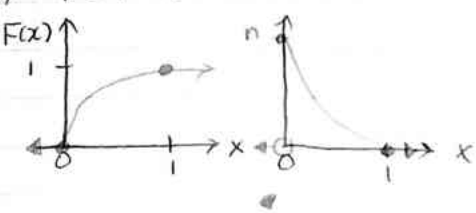
$$= \frac{n!}{(k-1)!\,(n-k)!} f(x) F(x)^{k-1} (1-F(x))^{n-k} = f_{X_{(k)}}(x) \quad \boxed{\checkmark}$$

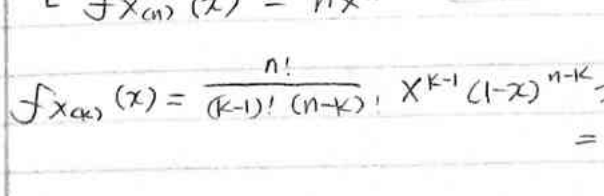
Let's make sure we can uncover the min/max formulas:

$$\left[\begin{array}{l} f_{X_{(1)}}(x) = \frac{n!}{(1-1)!\,(n-1)!} f(x) F(x)^{1-1} (1-F(x))^{n-1} = n f(x) (1-F(x))^{n-1} \\ f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f(x) F(x)^{n-1} (1-F(x))^{n-n} = n f(x) F(x)^{n-1} \end{array}\right.$$

$X_1, \cdots, X_n \overset{iid}{\sim} U[0,1] = 1\, \mathbb{1}_{x \in [0,1]}, \quad F(x) = x$

$$\left[\begin{array}{l} F_{X_{(1)}}(x) = 1 - (1-F(x))^n = 1 - (1-x)^n \\ F_{X_{(n)}}(x) = F(x)^n = x^n \end{array}\right.$$





$$\left[\begin{array}{l} f_{X_{(1)}}(x) = n(1-x)^{n-1} \\ f_{X_{(n)}}(x) = n x^{n-1} \end{array}\right.$$

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x \in [0,1]} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k+1-1} \mathbb{1}_{x \in [0,1]}$$

$$= \mathrm{Beta}(k, n-k+1)$$

$$\underbrace{X \sim \text{Gamma}(\alpha, \beta)}_{\text{Erlang}(\alpha_1, \beta)} \text{ independent of } \underbrace{Y \sim \text{Gamma}(\alpha_2, \beta)}_{\text{Erlang}(\alpha_2, \beta)},$$

$$T = X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

reasonable...

To prove this, we develop a new tool that makes it easier for us. That's "kernels", $K(x)$. For any PMF/PDF, we can decompose it into a normalization constant c and a kernel $K(x)$

$$P(x) = cK(x) \quad \text{and} \quad f(x) = cK(x)$$

$$\rightarrow p(x) \propto K(x), \quad f(x) \propto K(x)$$

$$1 = \sum P(x) = \sum cK(x) \Rightarrow \frac{1}{c} = \sum K(x) \Rightarrow c = \left(\sum K(x)\right)^{-1}$$

$$1 = \int_{\text{Supp}}^{\text{Supp}} f(x)\,dx = \int cK(x)\,dx \Rightarrow \frac{1}{c} = \int K(x)\,dx \Rightarrow c = \left(\int K(x)\,dx\right)^{-1}$$

This means that $K(x)$ is 1:1 with the PMF/PDF.

If you know $K(x)$. You know the distribution of the r.v.

Let's see some examples.

1. $X \sim \text{Bin}(n, p) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\dots,n\}}$

$$= \underbrace{n!(1-p)^n}_{c} \underbrace{\frac{1}{x!(n-x)!} \left(\frac{p}{1-p}\right)^x \mathbb{1} \dots}_{K(x)}$$

2. $X \sim \text{Weibull}(k, \lambda) = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}$

$$= \overbrace{k\lambda^k}^{c} \overbrace{y^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}}^{K(x)}$$

3. $X \sim \text{Gamma}(\alpha, \beta) = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}}_{K(x)}$

$$\propto x^a e^{-bx} \mathbb{1}_{x \geq 0}$$

$X \sim \text{Gamma}(\alpha_1, \beta)$ ind of $Y \sim \text{Gamma}(\alpha_2, \beta)$, $T = X + Y \sim f_T(t) = ?$

$$f_T(t) = \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\beta x} \frac{\beta^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2 - 1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t} e^{\beta x}} \underbrace{\mathbb{1}_{t-x \in [0,\infty)}}_{x-t \in (-\infty, 0]} dx$$

$x \in (-\infty, t]$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} dx \, \mathbb{1}_{t \geq 0}$$

$$\propto e^{-\beta t} \int_0^t x^{\alpha_1 - 1} (t-x)^{\alpha_2 - 1} dx \, \mathbb{1}_{t \geq 0}$$

$$= e^{-\beta t} t^{\alpha_1 - 1} t^{\alpha_2 - 1} \int_0^t \left(\frac{x}{t}\right)^{\alpha_1 - 1} \left(1 - \frac{x}{t}\right)^{\alpha_2 - 1} dx \, \mathbb{1}_{t \geq 0}$$

$$\left[\begin{array}{l} \text{let } u = \frac{x}{t} \Rightarrow \frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du, \quad x=0 \Rightarrow u=0, \\ \quad x = t \Rightarrow u = 1 \end{array}\right]$$

$$= e^{-\beta t} t^{\alpha_1 - 1} t^{\alpha_2 - 1} \int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} t\,du \, \mathbb{1}_{t \geq 0}$$

$$= e^{-\beta t} t^{\alpha_1 + \alpha_2 - 1} \underbrace{\int_0^1 u^{\alpha_1 - 1} (1-u)^{\alpha_2 - 1} du \, \mathbb{1}_{t \geq 0}}_{C(\alpha_1, \alpha_2)}$$

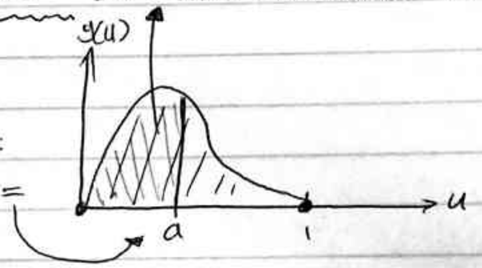
$$\propto e^{-\beta t} t^{\alpha_1 + \alpha_2 - 1} \mathbb{1}_{t \geq 0} \propto \text{Gamma}(\alpha_1 + \alpha_2, \beta) \quad \boxed{V}$$

* Beta function (a famous ubiquitous function)

$$B(\alpha, \beta) := \int_0^1 \underbrace{u^{\alpha-1} (1-u)^{\beta-1}}_{g(u)} du = \text{not available in closed form}$$

The "incomplete beta function" is:

$$B(a, \alpha, \beta) = \int_0^a u^{\alpha-1} (1-u)^{\beta-1} du =$$



The regularized incomplete beta function is:

$$I_a(\alpha, \beta) := \frac{B(a, \alpha, \beta)}{B(\alpha, \beta)}$$

Not a Bessel function!

Let's derive a beta function — gamma function identity

$$\text{Gamma}(\alpha_1 + \alpha_2, \beta) = \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2)} t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t} \mathbb{1}_{t \geq 0}$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1, \alpha_2) t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t} \mathbb{1}_{t \geq 0}$$

$$\Rightarrow B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \quad \text{Cool identity!}$$

$$X \sim \text{Beta}(\alpha, \beta) := \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]}, \quad \alpha, \beta > 0$$

$$1 = \int_{\text{supp}(x)} f(x) dx = \int_0^1 \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha, \beta)} \underbrace{\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx}_{B(\alpha, \beta)} = 1$$

$$F(x) = \int_0^x \frac{1}{B(\alpha, \beta)} y^{\alpha-1} (1-y)^{\beta-1} dy = \frac{1}{B(\alpha, \beta)} \int_0^x y^{\alpha-1} (1-y)^{\beta-1} dy = \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta)$$