Monday october 26th 2020

$g: \mathbb{R}^n \to \mathbb{R}^n$ and 1:1, $\vec{X}, \vec{Y}$ rv vector of dim n s.t

$\vec{Y} = g(\vec{X})$

Given $f_X(\vec{x})$, find $f_{\vec{Y}}(\vec{y})$

Recall what a multidimensional function is:

$Y_1 = g_1(X_1, \dots, X_n)$, $\quad X_1 = h_1(Y_1, \dots, Y_n)$

$Y_2 = g_2(X_1, \dots, X_n)$, $\exists h_j \quad X_2 = h_2(Y_1, \dots, Y_n)$

$Y_n = g_n(X_1, \dots, X_n)$ $\quad X_n = h_n(Y_1, \dots, Y_n)$

Using multivariable Calculus, you can show that

$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h^{-1}(\vec{y})) \, |J_h(\vec{y})|$ The Jacobian determinant

$$= \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \dots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & \dots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

- Let's verify the convolution formula via this general change of variables formula.

$T = X_1 + X_2 \sim f_T(t)$

Recipe:

(1) find a "clever" g so that

(2) we can find an h

(3) Compute J_h

(4) Compute the multidimensional change of variable formula

(5) Integrate out the "nuisance dimension".

① $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$ $X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} \partial h_1 / \partial y_1 & \partial h_1 / \partial y_2 \\ \partial h_2 / \partial y_1 & \partial h_2 / \partial y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - (-1 \cdot 0) = 1$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y})) |J_h| = f_{\vec{X}}(y_1 - y_2, y_2) |1| = f_{\vec{X}}(y_1 - y_2, y_2)$

⑤ $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\, dy_2 = \int_{\mathbb{R}} f_{X_1, X_2}(t - u, u)\, du$

general convolution formula.