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$$P_T(t) = \sum P_X^{old}(x)\, P_{\underbrace{Y}_{x-d}}^{old}(d-x)\, \mathbb{1}_{\underbrace{d-x\in supp[Y]}_{x\in\{d,d+1,\dots\}}}$$

$$= \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{-(d-x)} e^{-\lambda}}{(\underbrace{-(d-x)}_{x-d})!} \mathbb{1}_{\overbrace{d-x\in\{\dots,-1,0\}}^{x-d\in\{0,1,\dots\}}}$$

$$= e^{-2\lambda} \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x\in\{d,d+1,\dots\}}$$

let $d' = -d = |d|$

$$= e^{-2\lambda} \begin{cases} \text{if } d\leq 0 & \displaystyle\sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} \\ \text{if } d>0 & \displaystyle\sum_{x\in\{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'\in\{0,1,\dots\}} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!(x')!} \end{cases}$$

Let $x' = x-d \iff x = x'+d$

$$\xrightarrow{d=|d|} \quad = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$$

$$I_{|d|}(2\lambda) := \sum_{x\in\{0,1,\dots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$

(Modified Bassel function of the first kind (it's a solution to a famous differential equation))

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d\in\mathbb{Z}}$$

$$= \text{skellam}(\lambda, \lambda)$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda),\ T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$$P_{X_1|T}(x,t) = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x,t-x)}{P_T(t)}$$

$$= \frac{P_{X_1}(x)\,P_{X_2}(t-x)}{P_T(t)}$$

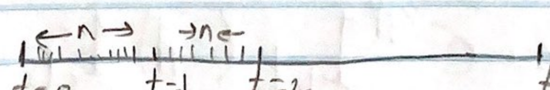
$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\,\frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!}\,\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

$$= \text{Bin}\left(t, \frac{1}{2}\right)$$

$X_1 \sim \text{Gemo}(p) := \overbrace{(1-p)^x p}^{P_{X_1}(x)}\ 1_{x\in\{0,1,\dots\}}$, $\text{supp}[X_1] = \{0,1,\dots\}$

$F_{X_1}(x) = P(X_1 \le x) = 1 - P(X_1 > x) = 1-(1-p)^x$

In every "second, Let's do n iid Bernoulli(p) experiments.



Let's call the resulting geometric rv X_n and its unit of realization is t.

$P_{X_n}(x) = (1-p)^{nx} p\ \ 1_{x\in\{0,\frac{1}{n},\frac{2}{n},\dots 1, 1+\frac{1}{n}, 1+\frac{2}{n},\dots 2,\dots\}}$

$F_{X_n}(x) = 1-(1-p)^{nx}$ where $\lambda \in [0,\infty)$

let $n \to \infty$, $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ same as Poisson

$$p_{X_\infty}(x) := \lim_{n\to\infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^{nx}}_{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x = e^{-\lambda x}} \frac{\lambda}{n} \mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}$$

$$\underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda x}} \underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} \underbrace{\lim \mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\mathbb{1}_{x\in[0,\infty)}}$$

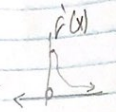
Not a vaild PMF! $\sum_{x\in \text{supp}[X_\infty]} p_{X_\infty}(x) = 0$ $= 0\ \forall x$!

$\text{supp}[X_\infty] = [0,\infty)$

$$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx} = \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty]}$$

The PMF wasn't vaild. Is the CDF vaild? If so I need to check three properties (1) it's 0 as I go to negative infinity, (2) it's 1 as I go to positive infinity and (3) it's an increasing function.

(1) $\lim_{x\to-\infty} \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty]} = 0$ ✓



(2) $\lim_{x\to\infty} \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty]} = 1 - \lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 1$ ✓

(3) $\frac{d}{dx}\left[\left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty)}\right] = \lambda e^{-\lambda x}\mathbb{1}_{x\in[0,\infty)} \geq 0$ ✓

$\Rightarrow$ vaild CDF!

we now have a continuous rv. continuous rv's have the following properties:

$|\text{supp}[X]| = |\mathbb{R}|$ uncountable infinity (the size of the continuum)

$f(x) = F'(x)$, $P(X\in[a,b]) = \overbrace{P(X\leq b)}^{F(b)} - \overbrace{F(X\leq a)}^{F(a)} = \int_a^b f(x)\,dx$ by fundamental thm calculus

$\int_{\mathbb{R}} f(x) = 1 = \text{"}F(\infty) - F(-\infty)\text{"}$ (with $F(\infty)=1$, $F(-\infty)=0$)

$f(x) \geq 0$ since CDF's are increasing functions

$\Rightarrow \text{supp}[X] = \{x : f(x) > 0\}$

$X_1 \dots X_n$ indp $\Rightarrow f_{X_1\dots X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$ joint density function

$$X \sim \text{Exp}(\lambda) := \underbrace{\underbrace{\lambda e^{-\lambda x}}_{f^{\text{old}}(x)} \mathbb{1}_{x\in[0,\infty]}}_{f(x)},\quad F(x) = \left(1-e^{-\lambda x}\right)\mathbb{1}_{x\in[0,\infty)}$$

Exponential rv

$\lambda \in (0,\infty)$

its parameter space

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$ $\quad \underbrace{\int_{\mathbb{R}}\dots\int_{\mathbb{R}}}_{k} f_{\vec{X}}(\vec{x})\,dx_1\dots dx_k = 1$

$k = 2$

$$P(A) = \iint_A f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$