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let X, Y iid Geom(p)

$P(X>Y) = ?$

$P(X>Y) = P(Y>X)$

why b/c it's iid

$$\underbrace{P(X>Y)+P(Y>X)}_{2P(X>Y)} + P(X=Y)$$

$$= 1-P(X=Y)$$

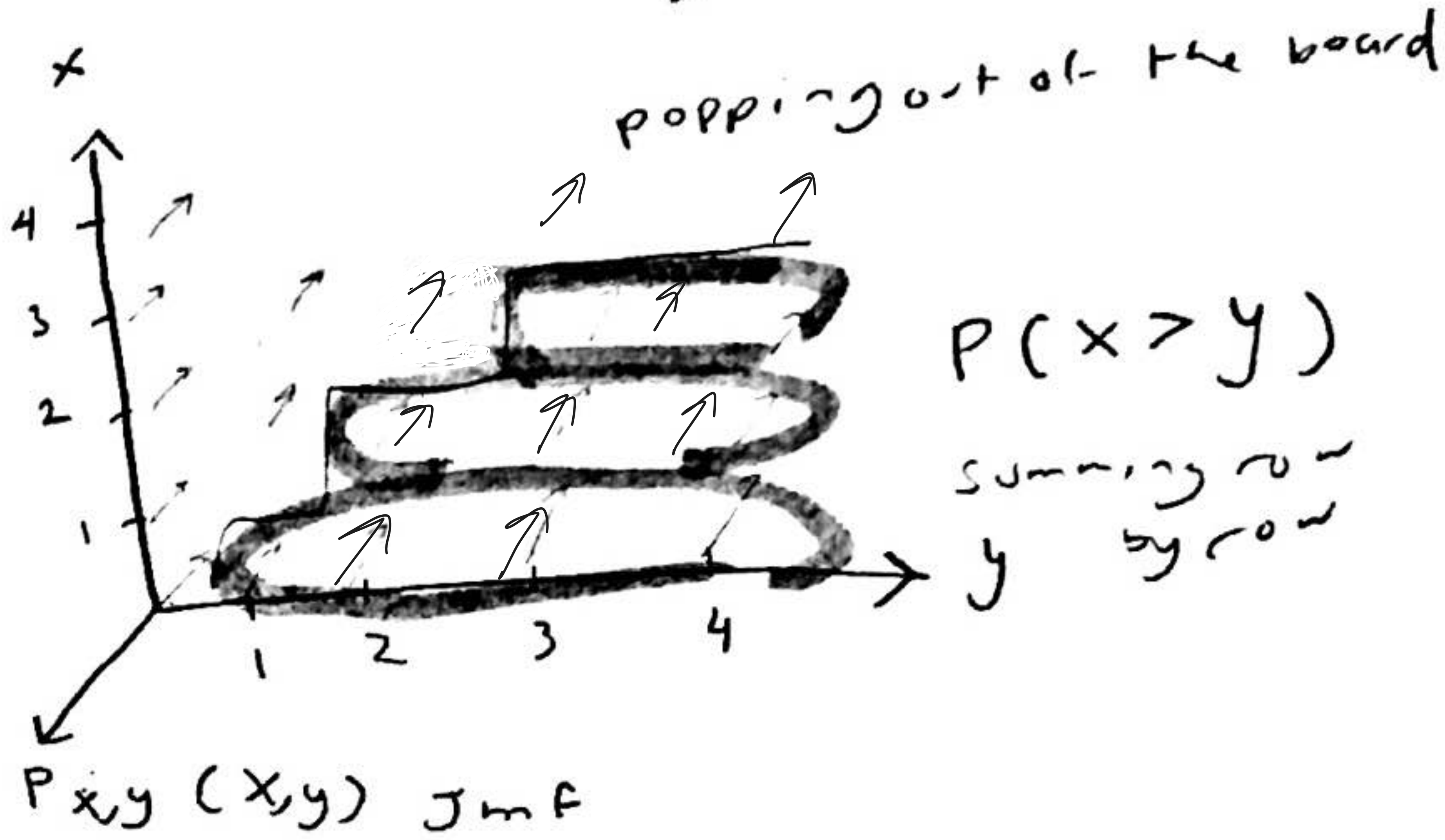
$$\Rightarrow P(X>Y) = \frac{1-P(X=Y)}{2} < \frac{1}{2}$$

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} P_{X,Y}(x,y)\, \mathbb{1}_{x>y}$$

the event (0,1) does not qualify

the event (1,1) " "

the event (2,1) qualifies



X, Y indep

$$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} P_X(x)\, P_Y(y)\, \mathbb{1}_{x>y}$$

$$= \sum_{x\in\{0,1\dots\}} \sum_{y\in\{0,1\dots\}} (1-p)^x p (1-p)^y p\, \mathbb{1}_{x>y}$$

$x \geq y+1 \Rightarrow x\in\{y+1, y+2 \dots$

summing over 2D doesn't matter which is held constant first?

we will flip it to make it easier

$$= p^2 \sum_{y\in\{0,1\dots\}} \sum_{x\in\{0,1\dots\}} (1-p)^x (1-p)^y\, \mathbb{1}_{x\in \underbrace{\{y+1, y+2, \dots, y+n\}}_{\text{defining the sum}}}$$

Plug into the scope of the sum

$$= p^2 \sum_{y\in\{0,1\dots\}} \sum_{x\in\{y+1,\dots\}} (1-p)^x (1-p)^y \rightarrow \text{is a constant!}$$

"rows" "cols" stepping up each time

$$= p^2 \sum_{y\in\{0,1\dots\}} (1-p)^y \sum_{x\in\{y+1, y+2\dots\}} (1-p)^x$$

like a usub

"Reindexing Trick"

$x' = x-(y+1) \in \{0,1,2\dots\}$

$\Rightarrow x = x'+y+1$

$$= p^2 \sum_{y\in\{0,1\dots\}} (1-p)^y \sum_{x'\{0,1\dots\}} (1-p)^{x'} (1-p)^y (1-p)$$

$$= p^2 (1-p) \sum_{y\in\{0,1\}} (1-p)^{2y} \sum_{x'\{0,1\dots\}} (1-p)^{x'}$$

? → *Geometric series formula

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \quad \text{for } a\in(-1,1)\backslash\{0\}$$

$$"\frac{1}{1-(1-p)} = \frac{1}{p}$$

$= p(1-p) \sum_{y \in \{0,1\dots\}} (1-p)^{2y}$

precalc: $(1-p)^{2y} = \left((1-p)^2\right)^y$

$= \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2} = \frac{1}{p(2-p)}$

$= \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p} < \frac{1}{2}$

Consider a bag of fruit that has apples and bananas, you now draw with <u>replacement</u> n samples from this bag and you count how many are apples and how many are bananas. let X_1 be the rv that counts the numbers of bananas. let P_1 be the probability of picking an apple and P_2 the prob of picking a banana

how is X_1 distributed
bernoulli n times binomial
n is the sample
prob apple is 1 ?

$X_1 \sim Bin(n, P_1)$

$X_2 \sim Bin(n, P_2)$

Sample $= X_1 + X_2$

* Are X_1 and X_2 independent? <u>NO</u> => dependent

$P(X_1 = x_1 \mid X_2 = x_2) \stackrel{?}{=} P(X_1 = x_1)$

$n=10 \quad P(X_1 = 6 \mid X_2 = 4) = 1 \neq \binom{10}{6} P_1^6 (1-P_1)$

$P(X_1 = 3 \mid X_2 = 4) = 0$

can't happen b/c $3+4 \neq 10$

$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ r.v vector

$\vec{X} \sim P_{\vec{X}}(\hat{x}_1, \hat{x}_2) = P_1 \cdot P_1 \cdots P$ x times

→ JMF

$= \frac{n!}{x_1!\,x_2!} P_1^{x_1} \cdot P_2^{x_2}$

many ways to get x_1 and x_2 apple back to 241
count the ways to get x_1 apples (not distinct)

$= \frac{n!}{x_1!\,x_2!} P_1^{x_1} \cdot P_2^{x_2} \, \mathbb{1}_{x_1+x_2=n} \, \mathbb{1}_{x_1 \{0,1\dots\}} \, \mathbb{1}_{x_2 \{0,1\dots\}}$

$\binom{n}{x_1, x_2}$

* multi choose notation

$\vec{X} \sim multi(n, \vec{P}) \quad \vec{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

$= \binom{n}{x_1, x_2} P_1^{x_1} \cdot P_2^{x_2}$

= Multinomial rv of dim = 2

Lets add cantaloupes to the bag let X_3 count the numbers of Cantaloupes and p_3 be the probability of drawing a Cantaloupe

makes it legal and compact

$$\vec{X} \sim \text{Multi}(n, \vec{p}) \quad \vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

jmf $p_{\vec{X}}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} \cdot p_2^{x_2} \cdot p_3^{x_3}$

$$= \left(\frac{n!}{x_1! x_2! x_3!}\right) p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1 + x_2 + x_3 = n}$$

- The general multinomial rv of dim $= K$ has PMF to

$$\vec{X} \sim \text{mult}(n, \vec{p}) := \binom{n}{x_1, x_2, \ldots, x_K} \prod_{k=1}^{K} p_k^{x_k}$$

$$\vec{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

prameter Space

$n \in \mathbb{N}$

makes sure it's not degenerate

$$\vec{p} \in \{ \vec{v} : \vec{v}_1 \cdot \vec{1} = 1, v_1 \in (0,1) \ldots v_K \in (0,1) \}$$

- need to add up to one
- can not be negative

Support Supp[X]

$$= \{ \vec{x} : \vec{x} \cdot \vec{1} = n, x_1 \in \{0, 1 \ldots n\}, \ldots, x_K \in \{0, 1, \ldots, n\} \}$$

I want to derive the Conditional pmf and the marginal pmf's in the case $K = 2$ (apples and bananas).

$$p_{x_1 | x_2}(x_1, x_2) = \frac{p_{x_1, x_2}(x_1, x_2)}{p_{x_2}(x_2)} = \deg(n - x_2)$$

$\leftarrow$ jmf

$\leftarrow$ marginal pmf

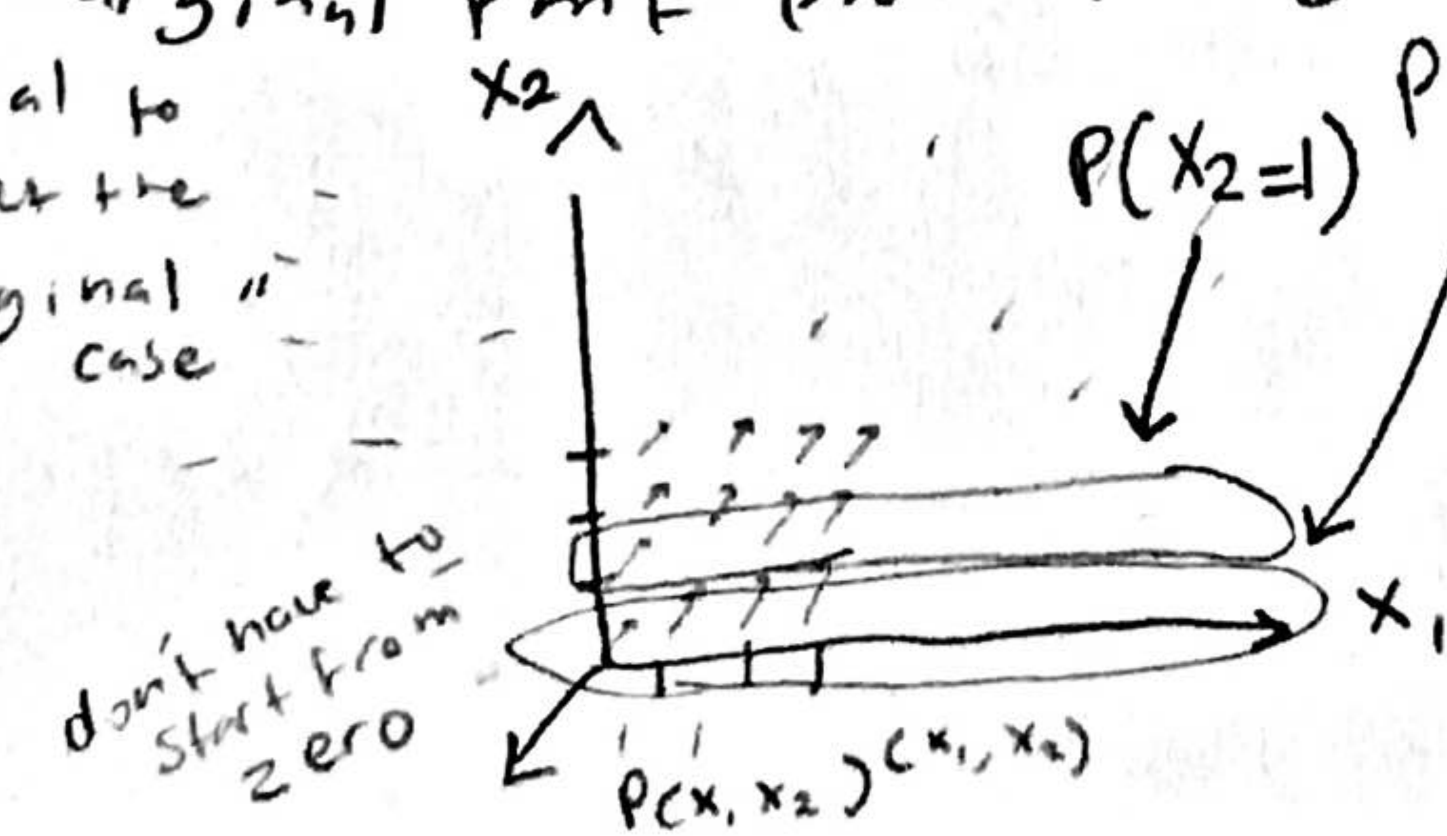
"you know what it has to be if you know x_2"

*How to prove the marginal pmf is binomial?

How do we compute the marginal pmf from the jmf?

$$p_{x_2}(x_2) = \sum_{x_1 \in \mathbb{R}} p_{x_1, x_2}(x_1, x_2)$$ "general to get the marginal case"

$$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$



$$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1\, x_2} p_1^{x_1} p_2^{x_2}$$

$$= p_2^{x_2} \sum_{x_1 \in \mathbb{R}} \frac{n!}{x_1!\, x_2!} p_1^{x_1} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1\dots\}} \mathbb{1}_{x_2 \in \{0,1\dots\}}$$

$$= p_2^{x_2} \frac{n!}{x_2!} \mathbb{1}_{x_2 \in \{0,1\dots\}} \sum_{x \in \mathbb{R}} \frac{p_1^{x_1}}{x!} \underbrace{\mathbb{1}_{x_1 = n - x_2}}_{x_1 \text{ is only one value } ! n - x_2} \mathbb{1}_{x_1 \in \{0,1\dots\}}$$

↳ also a constant over x_1

$$= p_2^{x_2} \frac{n!}{x_2!} \mathbb{1}_{x_2 \in \{0,1\dots\}} \frac{p_1^{n-x_2}}{(n-x_2)!} = \binom{n}{x_2} p_2^{x_2} p_1^{n-x_2}$$

$\binom{n}{x_2}$

becomes $Bin(n, p_2)$

Since $p_1 + p_2 = 1 \Rightarrow p_1 = 1 - p_2$

$$\binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2} ?$$

$$= \binom{n}{x_2} p_2^{x_2} p_1^{n-x_2} = Bin(n, p_2)$$