let $X, Y \overset{iid}{\sim} Geom(p)$

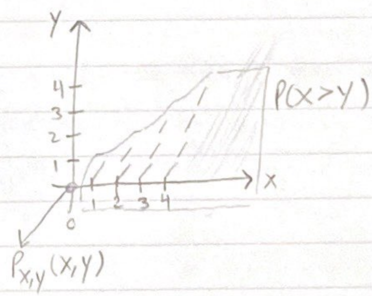
$P(X>Y) = ?$ $\quad P(X>Y) = P(Y>X)$

$P(X>Y) + P(Y>X) + P(X=Y) = 1$

$\Rightarrow 2P(X>Y) + P(X=Y) = 1$

$\Rightarrow P(X>Y) = \frac{1-P(X=Y)}{2} < 1/2$ since $P(X=Y) = 0$

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \overbrace{P_{X,Y}(x,y)}^{P_X(x)P_Y(y)} \mathbb{1}_{x>y}$$




$$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} P_X(x) P_Y(y) \mathbb{1}_{x>y}$$

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$$= \sum_{y\in\mathbb{R}} P_Y(y) \sum_{x\in\mathbb{R}} P_X(x) \mathbb{1}_{x>y}$$

$$= \sum_{y\in\{0,1,\dots\}} p(1-p)^y \sum_{x\in\{0,1,\dots\}} p(1-p)^x \mathbb{1}_{\substack{x>y \\ x\geq y+1}}$$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x\in\{y+1,y+2,\dots\}} (1-p)^x$$

let $x' = x-(y+1) \Rightarrow x' \in \{0,1,\dots\}$

$\Rightarrow x = x' + y + 1$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x'\in\{0,1,\dots\}} (1-p)^{x'} (1-p)^y (1-p)^1 = p^2(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y} \boxed{\sum_{x'\in\{0,1,\dots\}} (1-p)^{x'}}$$

$\frac{1}{1-(1-p)} = 1/p$

Geometric Series

$\sum_{i=0} a^i = \frac{1}{1-a}$

(for $a \in (-1,1)$ but not 0)

$$= p^2(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y} \left(\frac{1}{p}\right)$$

$= \frac{1}{1-(1-p)^2} = \frac{1}{p(2-p)}$

$$= p(1-p) \sum_{y\in\{0,1,\dots\}} ((1-p)^2)^y$$

$$= \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p} < 1/2$$

Bag of fruit of apples, bananas

Draw with replacement $n$ times 10

let $X_1 = \#$ apples, $p_1 = p(\text{apple})$

$\Rightarrow X_1 \sim Bin(n, p)$

Draw $n$ with replacement

$X_1 = \#$ apples, $X_2 = \#$ bananas

$X_1 \sim Bin(n, p_1)$ $X_2 \sim Bin(n, p_2)$

Are $X_1$ and $X_2$ independent?

Since $X_1 + X_2 = n \Rightarrow X_1, X_2$ dependent

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{X} \sim P_{\vec{X}}(\vec{x}) = P_{\vec{X}}(x_1, x_2) = \frac{n!}{x_1!\,x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$$

$\binom{n}{x_1, x_2}$ multichoose

$$\Rightarrow \vec{X} \sim multi(n, \vec{p}) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} \quad \text{multinomial r.v of dim} = 2$$

$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

Since $X_1, X_2$ are dependent, we cannot factor this jmf.

Bag of fruit now has Cantaloupes. $p_3$ and $x_3$

$$\vec{X} \sim multi(n, \vec{p}) = \underbrace{\binom{n}{x_1, x_2, x_3}} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \underbrace{\frac{n!}{x_1!\,x_2!\,x_3!}} p_1^{x_1} p_2^{x_2} p_3^{x_3} \underbrace{\mathbb{1}_{x_1+x_2+x_3=n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}} \mathbb{1}_{x_3 \in \{0,1,\dots,n\}}}$$

in general, if there are $K$ types of fruit (# categories) then the general multinomial r.v of dim $K$ is:

$$\vec{X} \sim multi(n, \vec{p}) = \binom{n}{x_1, x_2, \dots, x_K} \prod_{k=1}^{K} p_k^{x_k}$$

parameter space: $n \in \mathbb{N}$, $\vec{p} \in \{\vec{v} : \vec{v} \cdot \vec{1} = 1, v_1 \in (0,1), \dots, v_K \in (0,1)\}$

support: $Supp[\vec{X}] = \{\vec{x} : \vec{x} \cdot \vec{1} = n, x_1 \in \{0,1,\dots,n\}, \dots, x_K \in \{0,1,\dots,n\}\}$

$\vec{X} \sim \text{multi}(n, [\overset{\swarrow K=2}{p}, 1-p]) = \binom{n}{x_1, x_2} p^{x_1}(1-p)^{x_2}$

$P(X_1 = x_1 \mid X_2 = x_2) \overset{?}{=} \underset{\substack{\shortparallel \\ Bin(n, p_1)}}{P(X_1 = x_1)}$ Dependent? $\underline{no}$ cuz $x_1 + x_2 = n$

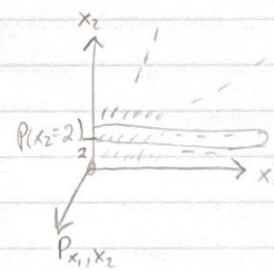
$\shortparallel$

$Deg(n - x_2) \Rightarrow$ Dependent!

Conditional pmf

$P_{X_1|X_2}(x_1, x_2) := \frac{\overset{\swarrow \text{jmf}}{P_{X_1,X_2}(x_1,x_2)}}{P_{X_2}(x_2)} \leftarrow$ marginal pmf of $X_2$

We have to show: $X_2 \sim Bin(n, p_2)$




$P_{X_2}(x_2) = P(X_2 = x_2) = \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1, x_2)$

"margining out $x_1$"

$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1, x_2} p^{x_1}(1-p)^{x_2}$

$= \sum_{x_1 \in \mathbb{R}} \frac{n!}{x_1! x_2!} p^{x_1}(1-p)^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$

$= \frac{n!}{x_2} (1-p)^{x_2} \mathbb{1}_{x_2 \in \{0,\dots,n\}} \underbrace{\sum_{x_1 \in \{0,\dots,n\}} \frac{p^{x_1}}{x_1!}}_{} \underbrace{\mathbb{1}_{x_1 = n - x_2}}_{}$

$= \frac{n!}{x_2!}(1-p)^{x_2} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}} \frac{p^{n-x_2}}{(n-x_2)!} = \binom{n}{x_2} p^{n-x_2}(1-p)^{x_2} = Bin(n, 1-p)$

margining a multinomial to yield one dimension is binomial.