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Wednesday September 16th 2020

Lecture 6

$A \in \mathbb{R}^{L\times k}$ All constant

$$E[A\vec{x}] = \begin{bmatrix} E[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}X_k] \\ E[a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}X_k] \\ \vdots \\ E[a_{L1}x_1 + a_{L2}X_2 + \cdots + a_{Lk}X_k] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \vec{x}] \\ E[\vec{a}_2, \vec{x}] \\ \vdots \\ E[\vec{a}_L, \vec{x}] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \vec{\mu} \\ \vec{a}_2 \vec{\mu} \\ \vdots \\ \vec{a}_L, \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$(L\times k)(k\times 1) = L\times 1$

$\vec{a} \in \mathbb{R}^k$

$$Var[\vec{a}^T\vec{x}] = Var[\underbrace{a_1x_1}_{Y_1} + \cdots + \underbrace{a_kX_k}_{Y_k}] = Var[Y_1 + \cdots + Y_k] = \sum_{i=1}^{k}\sum_{j=1}^{k} Cov[Y_i, Y_j]$$

$(1\times k)(k\times 1)$ = Scalar

$$= \sum\sum Cov[a_i x_i, a_j X_j] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j \sigma_{ij} = \vec{a}^T \Sigma \vec{a} = Var[\vec{x}]$$

Let $V \in \mathbb{R}^{k\times k}$, $\vec{a} \in \mathbb{R}^k$

$$\vec{a}^T V \vec{a} = \vec{a}\cdot(V\vec{a}) = \vec{a}\cdot \begin{bmatrix} a_1 v_{11} + \cdots + a_k v_{1k} \\ a_1 v_{21} + \cdots + a_k v_{2k} \\ \vdots \\ a_1 v_{k1} + \cdots + a_k v_{kk} \end{bmatrix} = \begin{matrix} a_1 a_1 v_{11} + \cdots + a_1 a_k v_{1k} + \\ a_2 a_1 v_{21} + \cdots + a_2 a_k v_{2k} + \\ \vdots \\ a_k a_1 v_{k1} + \cdots + a_k a_k v_{kk} \end{matrix}$$

Quadratic Forms with V being the "determining matrix"

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j v_{kj}$$

Application in Finance. Let $X_1, X_2, \ldots, X_k$ be the return x financial assets (e.g. stocks). So let $w_1, w_2, \ldots, w_k$ be the portion allocated to each of these assets. Let $\vec{\mu} = E[X]$, $\Sigma = Var[\vec{x}]$

$F = \vec{w}^T\vec{X}$ a r.v modeling your portofolio

$\mu_F = E[F] = \vec{w}^T\vec{\mu}$, $Var[F] = \vec{w}^T\Sigma\vec{w}$

It's possible to pick w_vector to optimize the portofolio by minimizing the variance of returns, Var[F], conditional on mu_F. This is called "Markowitz optimal portofolio theory".

min $Var[F]$ subject to μ_F being constant and $\vec{w}^T\vec{1}=1$.

$\vec{X} \sim multin_k(n,\vec{p})$. $E[\vec{X}] = \begin{bmatrix}E[X_1]\\ E[X_2]\\ \vdots \\ E[X_k]\end{bmatrix} = \begin{bmatrix}np_1\\ np_2\\ \vdots\\ np_k\end{bmatrix} = n\vec{p}$

$X_j \sim bin(n,p_j)$

$$Var[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & & \sigma_{ij} \\ & np_2(1-p_2) & & \\ \sigma_{2j} & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

$i \neq j$

$\sigma_{ij} < 0$

$$Cov[X_i,X_j] = E[X_iX_j] - E[X_i]E[X_j] = \sum_{x_i \in \mathbb{R}}\sum_{x_j \in \mathbb{R}} x_i x_j P_{X_i,X_j}(x_i,x_j) - n^2 p_i p_j$$

$\vec{X} = \begin{bmatrix} \vdots \\ X_i \sim Bin(n,p_i) \\ \vdots \\ X_j \sim Bin(n,p_j) \\ \vdots \end{bmatrix}$

$i =$ Apple, $j =$ Banana

$X_i = X_{1i} + X_{2i} + \dots + X_{ni}$ where $X_{1i}, \dots, X_{ni} \overset{iid}{\sim} Bern(p_i)$

$X_j = X_{1j} + X_{2j} + \dots + X_{nj}$ where $X_{1j}, \dots, X_{nj} \overset{iid}{\sim} Bern(p_j)$

We've expressed the multinomial rv with $n \times k$ Bernoulli's

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$ where $\vec{X}_1, \dots, \vec{X}_n \overset{iid}{\sim} multin_k(1,\vec{p})$

Now we can express the covariance

$$Cov[X_i,X_j] = Cov[X_{1i}+\dots+X_{ni}, X_{1j}+\dots+X_{nj}] = \sum_{l=1}^{n}\sum_{m=1}^{n} Cov[X_{li}, X_{mj}]$$

A lot of these covariance are zero due to independence. Which ones?
So, if l is different than m, the covariance is zero

$$= \sum_{l=1}^{n} Cov[X_{li}, X_{lj}]$$

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$$= \sum_{\ell=1}^{n} \left(E[X_{\ell i}, X_{\ell j}] - \underbrace{E[X_{\ell i}]}_{p_i} \underbrace{E[X_{\ell j}]}_{p_j} \right) = \boxed{-n p_i p_j}$$

$$\stackrel{*}{=} \sum_{x_{\ell i} \in \{0,1\}} \sum_{x_{\ell j} \in \{0,1\}} x_{\ell i} x_{\ell j} P_{X_{\ell i}, X_{\ell j}}(x_{\ell i}, x_{\ell j}) \stackrel{*}{=} P_{X_{\ell i}, X_{\ell j}}(1,1) = 0$$

the only term that's nonzero is...

you can't get an apple and a banana on one grab.

End (Midterm I material)

(Midterm II) $X \sim U(\{0,1,2,3\}) = \begin{cases} 0 & \text{w.p } 1/4 \\ 1 & \text{w.p } 1/4 \\ 2 & \text{w.p } 1/4 \\ 3 & \text{w.p } 1/4 \end{cases}$

(Midterm 2)

Generally, $X \sim U(A)$

$Supp[X] = A$, $A \subset \mathbb{R}$ set $|A| < \infty$ and $A \neq \emptyset$

Create a new r.v $Y = -X = g(X)$, a very simple function.

$Supp[Y] = \{-3, -2, -1, 0\}$ $\qquad P(Y) = \begin{cases} -3 & \text{w.p } 1/4 \\ -2 & \text{w.p } 1/4 \\ -1 & \text{w.p } 1/4 \\ 0 & \text{w.p } 1/4 \end{cases}$

Generally, for discrete r.v X, is there a pattern?

~~Supp [Y]~~ $\neq$ $\quad P_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) = P_X(-y)$

$$Supp[Y] := \{z : P_Y(z) > 0\} = \{z : P_X(-z) > 0\} = \{-z : P_X(z) > 0\}$$

$$= -\{z : P_X(z) > 0\} =: Supp[X]$$

$X_1, X_2 \stackrel{iid}{\sim} \text{Poisson}(\lambda) := \dfrac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1,\ldots\}}$. In class we show: $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

difference of two r.v.!

Let $D = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} = X + Y,\ ,\ Y \sim P_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\ldots,-2,-1,0\}}$

$$P_D^{(d)} = \sum_{x \in Supp[X]} P_X^{old}(x)\, P_Y^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$

$Supp[D] = Supp[X] + Supp[Y]$
$= \{\ldots, -1, 0, 1, \ldots\}$
$= \mathbb{Z}$ (all intenger).

If $X_1, \ldots, X_k$ are independent, what is the vector matrix?

$$\Sigma = \text{diag}\{\sigma_1^2, \ldots, \sigma_k^2\} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \sigma_k^2 \end{bmatrix}$$

Rules about vector r.v expectations

$$E[ax, \vec{c}] = \begin{bmatrix} a\mu_1 + c_1 \\ a\mu_2 + c_2 \\ \vdots \\ a\mu_k + c_k \end{bmatrix} = a\vec{\mu} + c$$

$$E[\vec{a}^T x] = E[a_1 x_1 + \ldots a_k x_k] = a_1\mu_1 + \ldots + a_k\mu_k = \vec{a}^T\vec{\mu}$$