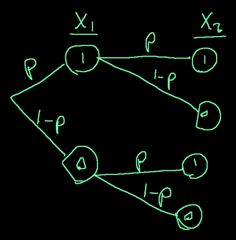


$p_{X_1,X_2}(x_1,x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0
1	

} mutually exclusive, collectively exhaustive events

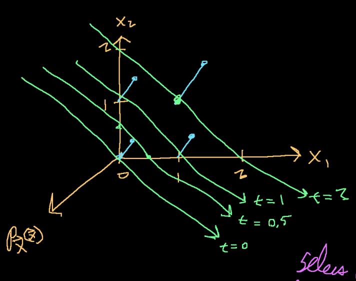
$$p_T(t) = P(T_2 = t) = \begin{cases} 2 & \text{wp } p^2 \\ 1 & \text{wp } 2p(1-p) \\ 0 & \text{wp } (1-p)^2 \end{cases}$$

---

slope = −1

$t = x_1 + x_2 \Rightarrow x_2 = t - x_1$

$P(T=0) \rightarrow x_2 = 0 - x_1 = -x_1$



General convolution formula

$$p_T(t) = \underbrace{\sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}}}_{\text{search through } \mathbb{R}^2} \underbrace{p_{X_1,X_2}(x_1,x_2)}_{\text{add up all probs}} \overbrace{\boxed{\mathbb{1}_{\substack{x_1 + x_2 = t \\ x_2 = t - x_1}}}}^{\text{selects events}} = \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$

if X_1, X_2 independent

$$= \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x) = \sum_{x \in \mathbb{R}} p^{old}_{X_1}(x)\, \boxed{\mathbb{1}_{x \in Supp[X_1]}}\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}$$

$$= \sum_{x \in Supp[X_1]} p^{old}_{X_1}(x)\, p^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x \in Supp[X_2]}$$

<- convolution formula for independent rvs / convolution formula for iid rvs ->

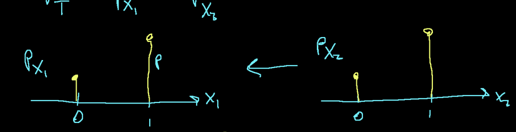
if $X_1, X_2 \overset{iid}{\sim}$

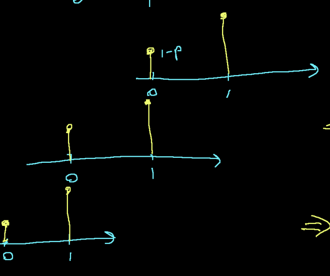
$$= \sum_{x \in \mathbb{R}} p(x) p(t-x) = \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in Supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]} = \sum_{x \in Supp[X]} p(x) p(t-x)\, \mathbb{1}_{t-x \in Supp X}$$

"Convolve" means to "roll, coil or entwine together"

$p_T = p_{X_1} * p_{X_2}$ convolution

Sum-product





$\Rightarrow$ $t = 1$ wp $p(1-p)$

$\Rightarrow$ $t = 0$ wp $(1-p)^2$,

$t = 2$ wp p^2

$\Rightarrow$ $t = 1$ wp $(1-p)p$

$$p_{T_2}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x}\, p^{t-x} (1-p)^{1-t+x}\, \mathbb{1}_{\underbrace{t-x \in \{0,1\}}_{t \in \{x, x+1\}}} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right)$$

$= \binom{2}{t}$

$$T_2 \sim \begin{cases} 0 & \text{wp } (1-p)^2 \\ 1 & \text{wp } 2p(1-p) \\ 2 & \text{wp } p^2 \end{cases}$$

$$= \binom{2}{t} p^t (1-p)^{2-t} = Binom(2, p)$$

$\binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{n \in \mathbb{N}} \mathbb{1}_{k \in \{0,1,\dots,n\}}$

$Supp[T] = Supp[X_1] + Supp[X_2]$

$A + B := \{a + b : a \in A, b \in B\}$

$\binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}$

$$X_1, X_2 \overset{iid}{\sim} Bern(p) := p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \binom{1}{x} p^x (1-p)^{1-x}$$

$$p_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x} = p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$

$\binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}$

$$= p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x} = p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right) = \binom{2}{t} p^t (1-p)^{2-t}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(p)$ $\quad T_3 = \overbrace{X_1 + X_2}^{T_2} + X_3 = X_3 + T_2 \sim p_{T_3}(t) = ?$

$$p_{T_3}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x} = p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right) = \binom{3}{t} p^t (1-p)^{3-t} = Binom(3, p)$$

HW: find PMF of $Binom(n, p)$ via induction

$X_1, X_2 \overset{iid}{\sim} Binomial(n, p)$, $T = X_1 + X_2 \sim ?$

$$p_T(t) = \sum_{x \in \mathbb{R}} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x} = p^t (1-p)^{2n-t} \sum_{x \in \mathbb{R}} \binom{n}{x}\binom{n}{t-x}$$

$$= \binom{2n}{t} p^t (1-p)^{2n-t} = Binom(2n, p)$$

Vandermonde's Identity