Consider rv's X_1, X_2, ..., X_n iid but PMF/PDF is unknown but we know it has expectation mu and variance sigma-squared.

Let $T_n = X_1 + X_2 + \dots + X_n$

Let $\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + X_2 + \dots + X_n}{n}$

From 241, we know $E[\bar{X}_n] = \mu$, $Var[\bar{X}_n] = \frac{\sigma^2}{n}$

Let $Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \underbrace{\frac{\sqrt{n}}{\sigma}}_{a} \bar{X}_n + \underbrace{\frac{-\sqrt{n}}{\sigma}\mu}_{b}$ "$\bar{X}$ standardized"

$E[Z_n] = 0$

$Var[Z_n] = 1 = SD[Z_n]$

$\phi_{T_n}(t) \overset{P3}{=} \phi_{X_1}(t)\phi_{X_2}(t)\cdots\phi_{X_n}(t) \overset{P1}{=} \phi_X(t)^n$

$a = e^{\ln(a)}$

$\phi_{\bar{X}_n}(t) \overset{P7}{=} \phi_T\left(\frac{1}{n}t\right) = \phi_X\left(\frac{t}{n}\right)^n$

$\phi_{Z_n}(t) \overset{P2}{=} e^{itb}\phi_{\bar{X}_n}(at) = e^{-\frac{it\mu\sqrt{n}}{\sigma}\frac{\sqrt{n}}{\sqrt{n}}}\phi_X\left(\frac{\cancel{\sqrt{n}}}{\sigma}\frac{t}{\cancel{n}_{\sqrt{n}}}\right)^n$

$= e^{-\frac{it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n\right)} = e^{-\frac{it\mu\cancel{n}}{\sigma\sqrt{n}} + \cancel{n}\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)} \frac{\frac{1}{\cancel{1}}}{\frac{1}{n}}$

$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}} = e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}\right)} = \phi_{Z_n}(t)$

We want to investigate now $\lim_{n\to\infty} \phi_{Z_n}(t) = ?$

$= e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2 n}}} = e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$

let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n\to\infty \Rightarrow u \to 0$

L'Hopitals

$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}} = e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)^2}}$

$= e^{\frac{t^2}{2\sigma^2}\frac{\cancel{\phi_X(0)}\phi_X''(0) - \phi_X'(0)^2}{\cancel{\phi_X(0)}}} \overset{P0}{=} e^{\frac{t^2}{2\sigma^2}\left(\phi_X''(0) - \phi_X'(0)^2\right)}$

$\overset{P4}{=} e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X^2] - (iE[X])^2\right)} = e^{-\frac{t^2}{2\cancel{\sigma^2}}\underbrace{\left(E[X^2] - E[X]^2\right)}_{\cancel{\sigma^2}}} = e^{-\frac{t^2}{2}} = \phi_Z(t)$

P8 $\Rightarrow Z_n \xrightarrow{d} Z$ where Z has chf $\phi_Z(t) = e^{-\frac{t^2}{2}}$. $Z \sim f_Z(z) = ?$

Use P6 to find PDF of Z. Check $\phi_Z(t) \overset{?}{\in} L^1 \Rightarrow \underbrace{\int_{\mathbb{R}} e^{-t^2/2}dt}_{\text{Gaussian Integral}} = \sqrt{\pi} < \infty$ YES!

$f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\phi_Z(t)dt$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}e^{-\frac{t^2}{2}}dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)}dt$

$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 - \underbrace{\left(\frac{\sqrt{2}iz}{2}\right)^2}_{\frac{i^2z^2}{2}} = \frac{t^2}{2} + \cancel{2}\frac{\sqrt{2}iz}{\cancel{2}}\frac{t}{\sqrt{2}} + \cancel{\frac{i^2z^2}{2}} - \cancel{\frac{i^2z^2}{2}}$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2} e^{-\frac{z^2}{2}}dt = \frac{1}{2\pi}e^{-\frac{z^2}{2}}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2}dt$

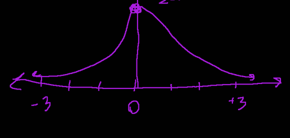
let $y = \frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2} \Rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{2}}$, $t\to\infty \Rightarrow y \to \infty$, $t \to -\infty \Rightarrow y \to -\infty$

Gaussian Integral

"Standard Normal rv"

$= \frac{1}{2\pi}e^{-z^2/2}\int_{\mathbb{R}} e^{-y^2}\sqrt{2}\,dy = \frac{1}{2\pi}e^{-z^2/2}\sqrt{2}\sqrt{\pi} = \frac{1}{\sqrt{2\pi}}e^{-z^2/2} = N(0,1)$

This completes the proof of the "central limit theorem" (CLT), the crown jewel of a basic probability class, one of the most useful results that probability has given to the world at large.



AKA Laplace's Second Error Distribution. It is the most famous and widely-used error distribution on Earth.

$$\boxed{\text{CLT: } X_1, \dots, X_n \text{ iid mean } \mu, \text{ Variance } \sigma^2, \quad \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)}$$

Let $\sigma > 0$

$Z \sim N(0,1)$, $X = \mu + \sigma Z \sim f_X(x) = ?$

$\phi_X(t) \overset{P2}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu - \sigma^2t^2/2}$

$f_X(x) = \frac{1}{\sigma}f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$

$E[Z] = \frac{\phi_Z'(0)}{i} = 0$, $Var[Z] = E[Z^2] - \cancel{E[Z]^2}^{0} = \frac{\phi_Z''(0)}{\underbrace{i^2}_{-1}} = 1$ ✓

$\phi_Z'(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -te^{-t^2/2}$, $\phi_Z''(t) = \frac{d}{dt}\left[\underbrace{-t}_{u}\underbrace{e^{-t^2/2}}_{v}\right] = -\left(-t^2e^{-t^2/2} + e^{-t^2/2}\right) = t^2e^{-t^2/2} - e^{-t^2/2}$

$E[X] = E[\mu + \sigma Z] = \mu$, $Var[X] = Var[\mu + \sigma Z] = \sigma^2$, $SD[X] = \sigma$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep. of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim f_T(t) = ?$

$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\phi_{X_2}(t) = e^{it\mu_1 - \sigma_1^2t^2/2}e^{it\mu_2 - \sigma_2^2t^2/2}$

$= e^{it(\mu_1 + \mu_2) - (\sigma_1^2 + \sigma_2^2)t^2/2} \overset{P1}{\Rightarrow} T \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim f_Y(y) = ?$ $g^{-1}(y) = \ln(y)$ $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|y|}$

$f_Y(y) = f_X(\ln(y))\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2}\frac{1}{|y|} = \frac{1}{\sqrt{2\pi\sigma^2y^2}}e^{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2}$

$= LogN(\mu, \sigma^2)$

Log-Normal distribution