Math 368 / 621 Fall 2020
Midterm Examination One

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Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 75 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. No food is allowed, only drinks.

Problem 1 [7 min] These are questions about indicator functions.

- [11 pt / 11 pts] Record the letter(s) of all the following that are **true**.

(a) $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{17\}} = 17$

(b) $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\{17\}} = 1$

(c) $\prod_{x\in\mathbb{R}} \mathbb{1}_{x\in\{17\}} = 17$

(d) $\prod_{x\in\mathbb{R}} \mathbb{1}_{x\in\{17\}} = 1$

(e) $\sum_{x\in\mathbb{R}} h(x)\mathbb{1}_{x\in\mathbb{N}} = \sum_{x\in\mathbb{N}} h(x)$ where h is a function.

Let X be a discrete rv with PMF $p(x)$, old-style PMF $p^{old}(x)$ and support $\mathbb{S}\mathrm{upp}[X]$. For any X,

(f) $\sum_{x\in\mathbb{R}} \mathbb{1}_{x\in\mathbb{S}\mathrm{upp}[X]} = 1$

(g) $\sum_{x\in\mathbb{R}} p^{old}(x) = 1$

(h) $\sum_{x\in\mathbb{S}\mathrm{upp}[X]} p^{old}(x) = 1$

(i) $\sum_{x\in\mathbb{R}} p^{old}(x)\mathbb{1}_{x\in\mathbb{S}\mathrm{upp}[X]} = 1$

(j) $\sum_{x\in\mathbb{R}} p(x) = 1$

(k) $\sum_{x\in\mathbb{R}} p(x)\mathbb{1}_{x\in\mathbb{S}\mathrm{upp}[X]} = 1$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 2 [8 min] Let

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{\boldsymbol{X}}(\boldsymbol{x}), \quad T := X_1 + X_2 \sim p_T(t), \quad X_1 \sim p_{X_1}(x) := \begin{cases} 5 & \text{w.p. } 0.2 \\ 10 & \text{w.p. } 0.8 \end{cases} \quad \text{independent of } \; X_2 \sim p_{X_2}(x) := \begin{cases} -5 & \text{w.p. } 0.1 \\ -10 & \text{w.p. } 0.9 \end{cases}$$

- [8 pt / 19 pts] Record the letter(s) of all the following that are **true**.

  (a) X_1, X_2 are identically distributed

  (b) $\mathbb{V}\text{ar}\left[\boldsymbol{X}\right] = \mathbb{V}\text{ar}\left[T\right]$

  (c) $T = \boldsymbol{a}\boldsymbol{X}$ where $\boldsymbol{a} = [1 \ 1]$

  (d) $p_T(t) = p_{X_1}(x) \star p_{X_2}(x)$

  (e) $p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{\boldsymbol{X}}(x_1, x_2)$

  (f) $p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{\boldsymbol{X}}(x_1, x_2) \mathbb{1}_{t = x_1 + x_2}$

  (g) $p_T(t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1}(x_1) p_{X_2}(x_2) \mathbb{1}_{t = x_1 + x_2}$

  (h) $p_T(t) = \sum_{x \in \mathbb{R}} p_{X_1}(x) p_{X_2}(t - x)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 3 [10 min] Consider the same setup as the previous problem:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim p_{\boldsymbol{X}}(\boldsymbol{x}), \quad T := X_1 + X_2 \sim p_T(t), \quad X_1 \sim p_{X_1}(x) := \begin{cases} 5 & \text{w.p. } 0.2 \\ 10 & \text{w.p. } 0.8 \end{cases} \quad \text{independent of } \; X_2 \sim p_{X_2}(x) := \begin{cases} -5 & \text{w.p. } 0.1 \\ -10 & \text{w.p. } 0.9 \end{cases}$$

- [10 pt / 29 pts] Record the letter(s) of all the following that are **true**.

  (a) $T \sim \text{Deg}(0)$

  (b) $T \sim \text{Binomial}\,(2,\, p)$ where p can be computed from $p_{X_1}(x)$ and $p_{X_2}(x)$

  (c) $\mathbb{S}\text{upp}\,[T] = \{-10, -5, 5, 10\}$

  (d) $p_{X_1}(x) = 0.2\mathbb{1}_{x=5} + 0.8\mathbb{1}_{x=10}$

  (e) $p_{X_1}(x) = 5\mathbb{1}_{x=0.2} + 10\mathbb{1}_{x=0.8}$

  (f) $p_T(t) = 0.2\mathbb{1}_{t=5} + 0.8\mathbb{1}_{t=10} + 0.1\mathbb{1}_{t=-5} + 0.9\mathbb{1}_{t=-10}$

  (g) $p_T(0) = p_{\boldsymbol{X}}(0, 0)$

  (h) $p_T(0) = p_{\boldsymbol{X}}(5, -5) + p_{\boldsymbol{X}}(10, -10)$

  (i) $p_T(0) = p_{X_1}(5) + p_{X_2}(-5) + p_{X_1}(10) + p_{X_2}(-10)$

  (j) $p_T(0) = 0.74$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 4 [8 min] These are questions about rv's we studied in class. Consider $X_1, X_2, X_3, \ldots \overset{iid}{\sim} \text{Bernoulli}(p)$.

- [9 pt / 38 pts] Record the letter(s) of all the following that are **true**.

(a) $X_1 + X_{17} \sim \text{Binomial}(17,\, p)$

(b) $X_1 + X_{17} \sim \text{Binomial}(2,\, p)$

(c) $X_1 + X_2 + X_3 + \ldots$ is a geometric rv

(d) $X_1 + X_2 + X_3 + \ldots$ is a negative binomial rv

(e) $[X_1\ X_2\ X_3]^\top$ is a multinomial rv

Let $T_n := \sum_{i=1}^n X_i$ where $n \in \mathbb{N}$

(f) $T_n \sim \text{Binomial}(n,\, p)$

(g) T_n will be approximately distributed as a Poisson(np) rv if n is large and p is small.

Let Y be the rv that counts the number of X_t's that are realized to be zero before the first X_t that is realized to be one i.e. $Y = \min\{t : X_t = 1\} - 1$.

(h) Y is a geometric rv

(i) Given that $Y = 4$, then X_3 is degenerate.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 5 [7 min] Consider $X_1, X_2, X_3, \dots \overset{iid}{\sim} \text{Geometric}(p)$. Let $T_n := \sum_{i=1}^{n} X_i$ and $T_m := \sum_{i=n+1}^{n+1+m} X_i$ where $n, m \in \mathbb{N}$.

- [7 pt / 45 pts] Record the letter(s) of all the following that are **true**.

(a) $\mathbb{S}\text{upp}[X_1] = \mathbb{S}\text{upp}[X_1 + X_2]$

(b) $T_n \sim p_{T_n}(t) = p^2 \sum_{x=0}^{\infty} (1-p)^x (1-p)^{t-x} \mathbb{1}_{t-x \in \{0,1,2,\dots\}}$

(c) $T_n \sim p_{T_n}(t) = p^2 \sum_{x=1}^{\infty} (1-p)^x (1-p)^{t-x} \mathbb{1}_{t-x \in \{1,2,\dots\}}$

(d) $T_n + T_n \sim \text{NegBin}(2n, p)$

(e) $T_m \sim \text{NegBin}(m, p)$

(f) $T_m \sim \text{NegBin}(n+m, p)$

(g) $T_n + T_m \sim \text{NegBin}(n+m, p)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 6 [6 min] Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T_n := \sum_{i=1}^n X_i$ and $\boldsymbol{X} = [X_1\ X_2\ \ldots\ X_n]^\top \sim p_{\boldsymbol{X}}$.

- [9 pt / 54 pts] Record the letter(s) of all the following that are **true**.

(a) $p_{X_1}(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

(b) $p_{X_1}(x) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{1,2,\ldots\}}$

(c) $p_{X_1}(x) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \{0,1,2,\ldots\}}$

(d) $p_{\boldsymbol{X}}(\boldsymbol{x}) = \displaystyle\prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \mathbb{1}_{x_i \in \{0,1,2,\ldots\}}$

(e) $T_n \sim \text{Poisson}(n\lambda)$

(f) $T_n \sim \text{Poisson}(\lambda/n)$

(g) $T_n \sim \text{Poisson}(\lambda)$

(h) As $n \to \infty$, T_n becomes more and more degenerate

(i) As $n \to \infty$, T_n becomes more and more like a $\text{Binomial}(n, \lambda/n)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 7 [10 min] Consider a bag of marbles with 5 red marbles, 4 green marbles, 6 blue marbles and 3 purple marbles. You sample (pick) 19 marbles from the bag by picking one at a time, recording its color and then putting that marble bag into the bag. Let X_1 count the number of red marbles in your sample, let X_2 sample the number of green marbles in your sample, let X_3 sample the number of blue marbles in your sample and let X_4 count the number of purple marbles in your sample. Let $\boldsymbol{X} = [X_1\ X_2\ X_3\ X_4]^\top \sim p_{\boldsymbol{X}}$.

- [11 pt / 65 pts] Record the letter(s) of all the following that are **true**.

(a) $p_{\boldsymbol{X}}(\boldsymbol{x}) = p_{X_1}(x_1)p_{X_2}(x_2)p_{X_3}(x_3)p_{X_4}(x_4)$

(b) X_1 is a binomial rv with $n = 19$

(c) $X_1 + X_2 + X_3 + X_4$ is degenerate

(d) $\boldsymbol{X} \sim \text{Multin}_4\left(18, \frac{1}{19}[5\ 4\ 6\ 3]^\top\right)$

(e) $\boldsymbol{X} \sim \text{Multin}_4\left(19, \frac{1}{18}[5\ 4\ 6\ 3]^\top\right)$

(f) $\boldsymbol{X} \sim \text{Multin}_{18}\left(19, \frac{1}{4}[5\ 4\ 6\ 3]^\top\right)$

(g) $p_{\boldsymbol{X}}(9, 2, 2, 6) = \frac{1}{18^4}\binom{19}{9,2,2,5}5^6 4^2 6^2 3^2$

(h) $p_{\boldsymbol{X}}(9, 2, 2, 0) = \frac{1}{18^4}\binom{19}{9,2,2}5^6 4^2 6^2$

(i) $p_{\boldsymbol{X}}(9, 2, 0, 0) = \frac{1}{18^4}\frac{19!}{2!}5^6 4^2$

(j) $p_{\boldsymbol{X}}(19, 0, 0, 0) = \frac{1}{18^4}\frac{19!}{19!}5^{19}$

(k) Given $X_1 = 3$ and $X_2 = 1$, $[X_3\ X_4]^\top$ is a multinomial rv with $K = 2$.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 8 [8 min] Consider the same situation as the previous problem: a bag of marbles with 5 red marbles, 4 green marbles, 6 blue marbles and 3 purple marbles. You sample (pick) 19 marbles from the bag by picking one at a time, recording its color and then putting that marble bag into the bag. Let X_1 count the number of red marbles in your sample, let X_2 sample the number of green marbles in your sample, let X_3 sample the number of blue marbles in your sample and let X_4 count the number of purple marbles in your sample. Thus, $\boldsymbol{X} = [X_1\ X_2\ X_3\ X_4]^\top \sim p_{\boldsymbol{X}} = \text{Multin}_4\left(19, \frac{1}{18}[5\ 4\ 6\ 3]^\top\right)$.

- [8 pt / 73 pts] Record the letter(s) of all the following that are **true**.

  (a) $\mathbb{E}[X_1] = 5/19$

  (b) $\mathbb{E}[X_1] = 19 \times 5/18$

  (c) $\mathbb{E}[X_2 + X_3] = 19 \times 5/18 + 19 \times 4/18$

  (d) $\mathbb{E}[X_2 + X_3] = 19 \times 6/18 + 19 \times 4/18$

  Let $\boldsymbol{c} = [1\ 2\ 3\ 4]^\top$ i.e. a 4-dimensional column vector of constants.

  (e) $\mathbb{E}[\boldsymbol{c} + \boldsymbol{X}] = \frac{19}{18}[1 \times 5 \quad 2 \times 4 \quad 3 \times 6 \quad 4 \times 3]^\top$

  (f) $\mathbb{E}[\boldsymbol{c} + \boldsymbol{X}] = \frac{19}{18}(1 \times 5 + 2 \times 4 + 3 \times 6 + 4 \times 3)$

  (g) $\mathbb{E}\left[\boldsymbol{c}^\top \boldsymbol{X}\right] = \frac{19}{18}[1 \times 5 \quad 2 \times 4 \quad 3 \times 6 \quad 4 \times 3]^\top$

  (h) $\mathbb{E}\left[\boldsymbol{c}^\top \boldsymbol{X}\right] = \frac{19}{18}(1 \times 5 + 2 \times 4 + 3 \times 6 + 4 \times 3)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 9 [11 min] Consider the same situation as the previous two problems: a bag of marbles with 5 red marbles, 4 green marbles, 6 blue marbles and 3 purple marbles. You sample (pick) 19 marbles from the bag by picking one at a time, recording its color and then putting that marble bag into the bag. Let X_1 count the number of red marbles in your sample, let X_2 sample the number of green marbles in your sample, let X_3 sample the number of blue marbles in your sample and let X_4 count the number of purple marbles in your sample. Thus, $\boldsymbol{X} = [X_1\ X_2\ X_3\ X_4]^\top \sim p_{\boldsymbol{X}} = \text{Multin}_4\left(19, \frac{1}{18}[5\ 4\ 6\ 3]^\top\right)$.

- [11 pt / 84 pts] Record the letter(s) of all the following that are **true**.

(a) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is a symmetric and diagonal matrix

(b) $\mathbb{C}\text{ov}[X_1, X_2 + X_3] = 2\mathbb{C}\text{ov}[X_1, X_2]$

(c) $\mathbb{C}\text{ov}[X_1, X_2 + X_3] = -19(50)/18^2$

(d) $\mathbb{V}\text{ar}[[1\ 1\ 1\ 1]\boldsymbol{X}] = 0$

(e) $\mathbb{V}\text{ar}[[1\ 1\ 1\ 1]\boldsymbol{X}] = [1\ 1\ 1\ 1]\mathbb{V}\text{ar}[\boldsymbol{X}][1\ 1\ 1\ 1]^\top$

(f) $\mathbb{V}\text{ar}[\boldsymbol{X}] = \frac{19}{18^2}\begin{bmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{bmatrix}$ where $a, b, c, d, e, f, g, h, i, j$ are integers.

(g) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is the matrix in the previous question and e, f, g, h, i, j are negative integers.

(h) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is the matrix in the previous question and $g = -15$.

(i) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is the matrix in the previous question and $b = -60$.

(j) The number of red marbles minus the number of purple marbles has variance $\frac{19}{18^2}(5 \times 14 + 2(5 \times 3) + 3 \times 16)$

(k) The number of red marbles minus the number of purple marbles has variance $\frac{19}{18^2}(5 \times 14 - 2(5 \times 3) + 3 \times 16)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.