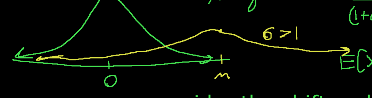


$X \sim \text{logistic}(0,1) = \frac{e^x}{(1+e^x)^2} \approx N(0,1)$ but with thicker tails

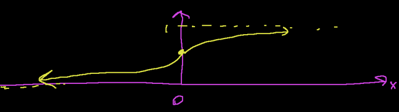
$E[X] = 0, \quad SD[X] = \frac{\pi}{\sqrt{3}} \approx 1.8 > 1.$

consider the shift and scale where sigma > 0.

$$Y = \mu + \sigma X \sim f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{|\sigma|} = \frac{e^{\frac{y-\mu}{\sigma}}}{\sigma\left(1+e^{\frac{y-\mu}{\sigma}}\right)} = \text{Logistic}(\mu, \sigma).$$

Why is this called the "logistic distribution"? There's a famous function called the "logistic function". It has three parameters: L (maximum value), k (steepness), mu (center) and it is:

$$\ell(x) := \frac{L}{1+e^{-k(x-\mu)}} \overset{\text{if } L=1,\, k=1,\, \mu=0}{=} \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{e^x+1}$$ (standard logistic function)



$X \sim \text{logistic}(0,1)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{e^t}{(1+e^t)^2}\,dt = \int_1^{1+e^x} \frac{\cancel{1-u}}{u^2}\frac{du}{\cancel{1-u}} = \left[-u^{-1}\right]_1^{1+e^x} = 1 - \frac{1}{1+e^x} = \frac{e^x}{1+e^x}$$

let $u = 1+e^t \Rightarrow e^t = 1-u \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du = \frac{1}{1-u}du \Rightarrow t=-\infty \Rightarrow u=1,\ t=x \Rightarrow u = 1+e^x$

The "quantile" q or "percentile" 100q for a rv X is defined as the minimum x s.t. $q \le P(X \le x) = F(x) \Leftrightarrow F(x) \ge q$. It is denoted Q[X, q] where Q is the "quantile operator" (not the upper incomplete regularized gamma function). When q = 0.5, the quantile has a special name, the "median", Med[X] := Q[X, q]. Here's an example:

$X \sim U(\{2,4,6,\ldots,20\}) = \frac{1}{10}\mathbb{1}_{x\in\ldots}$

x	p(x)	F(x)
2	0.1	0.1
4	0.1	0.2
6	↓	0.3
8		0.4
10		0.5
12		0.6
14		0.7
16		0.8
18		0.9 ←
20		1 ←

$Q[X, 30\%] = 6$

$Q[X, 80\%] = 16$

$Q[X, 85\%] = 18 = Q[X, 0.9]$

$\text{Med}[X] = 10$

However, if X is a continuous rv with "contiguous support" e.g. [0, 10], [0, infinity), all real numbers, etc and not something like [0,1] union [2,3]. In the latter case, F(x) is flat between [1,2] which means it's not invertible. In the former case, F(x) is invertible.

$Q[X, q] = F_X^{-1}(q)$, and the inverse CDF is called appropriately, the "quantile function".

$X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x}\mathbb{1}_{x\ge 0} \Rightarrow F_X(x) = 1 - e^{-\lambda x} = q \Rightarrow 1-q = e^{-\lambda x}$

$\Rightarrow \ln(1-q) = -\lambda x \Rightarrow x = -\frac{1}{\lambda}\ln(1-q) = \frac{1}{\lambda}\ln\left(\frac{1}{1-q}\right) = F_X^{-1}(q)$

$\text{Med}[X] = \frac{\ln(2)}{\lambda} = F_X^{-1}(0.5)$

Quantile functions are not usually available in closed form since CDF's aren't even usually available in closed form e.g.

$X \sim \text{Erlang}(k, \lambda) \Rightarrow F_X(x) = P(k, \lambda x)$

$\text{Med}[X] = x$ s.t. $P(k, \lambda x) = 0.5$. Need a computer solver.

$k > 0 \ldots$

$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x}\mathbb{1}_{x\ge 0}, \quad Y = g(X) = ke^X \sim f_Y(y) = ?$

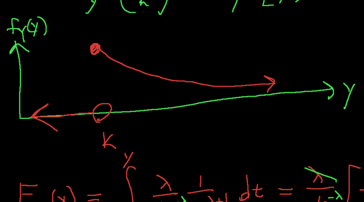
$y = ke^x \Rightarrow \frac{y}{k} = e^x \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k) = g^{-1}(y)$

$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{y}\right| = \frac{1}{|y|}$

$$f_Y(y) = \underbrace{f_X\left(\ln\left(\tfrac{y}{k}\right)\right)\frac{1}{|y|}}_{\frac{c}{y^{\lambda+1}}} = \frac{\lambda}{|y|}e^{\underbrace{-\lambda\ln\left(\frac{y}{k}\right)}_{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)}}\ \mathbb{1}_{\overbrace{\ln(y)-\ln(k)\in[0,\infty)}^{\overbrace{\ln(y)\in[\ln(k),\infty)}^{y\in[k,\infty)}}}$$

$$= \frac{\lambda}{y}\left(\frac{y}{k}\right)^{-\lambda}\mathbb{1}_{y\in[k,\infty)} = \text{Pareto I}(k, \lambda)$$

$k \in (0,\infty), \quad \lambda \in (0,\infty)$



$\frac{d}{dt}\left[-\frac{t^{-\lambda}}{\lambda}\right] = -\lambda\left(\frac{t^{-\lambda-1}}{\lambda}\right) = \frac{1}{t^{\lambda+1}}$

$$F_Y(y) = \int_k^y \frac{\lambda}{k^{-\lambda}}\frac{1}{t^{\lambda+1}}\,dt = \frac{\cancel{\lambda}}{k^{-\lambda}}\left[-\frac{1}{\cancel{\lambda}t^{\lambda}}\right]_k^y = k^{\lambda}\left(\frac{1}{k^{\lambda}} - \frac{1}{y^{\lambda}}\right) = 1 - \left(\frac{k}{y}\right)^{\lambda}$$

$$\Rightarrow F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}.$$

This distribution was discovered by Vilfredo Pareto, an Italian economist in 1896 when he observed that 20% of the richest Italians owned 80% of the land (i.e. the wealth). This is known as the "Pareto Principle" and it corresponds to the ParetoI(1, 1.161) distribution.

Further, the Pareto distribution is a waiting time / survival time model. It's used for [see wikipedia if you're interested]. Wealth, music talent, number of patents, ...

$Z \sim \frac{1}{|-1|}f_X\left(\frac{z}{-1}\right) = e^z\mathbb{1}_{z\in(-\infty,0]}$

$X, Y \overset{iid}{\sim} \text{Exp}(1) = e^{-x}\mathbb{1}_{x\in[0,\infty)}, \quad D = X - Y = X + \underbrace{(-Y)}_{Z} \sim f_D(d) = ?$

$$f_D(d) = \int_{\text{Supp}(X)} f_X^{old}(x)\, f_Z^{old}(d-x)\,\mathbb{1}_{d-x\in\text{Supp}(Z)}\,dx = \int_0^{\infty} e^{-x}e^{d-x}\,\mathbb{1}_{\overbrace{d-x\in(-\infty,0]}^{\overbrace{x-d\in[0,\infty)}^{x\in[d,\infty)}}}\,dx$$

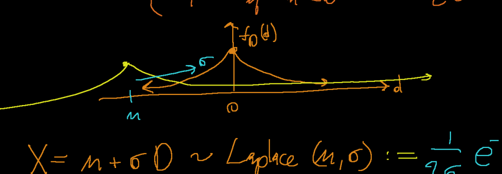
$$= e^d\int_0^{\infty} e^{-2x}\mathbb{1}_{x\in[d,\infty)}\,dx = e^d\begin{cases}\int_d^{\infty} e^{-2x}\,dx & \text{if } d \ge 0\\ \int_0^{\infty} e^{-2x}\,dx & \text{if } d < 0\end{cases}$$

$$= e^d\begin{cases}\left[-\frac{1}{2}e^{-2x}\right]_d^{\infty} & \text{if } d \ge 0\\ \left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} & \text{if } d < 0\end{cases} = \frac{1}{2}e^d\begin{cases}\left[-e^{-2x}\right]_d^{\infty} & \text{if } d\ge 0\\ \left[-e^{-2x}\right]_0^{\infty} & \text{if } d<0\end{cases}$$

$$= \frac{1}{2}e^d\begin{cases}e^{-2d} & \text{if } d \ge 0\\ 1 & \text{if } d < 0\end{cases} = \frac{1}{2}\begin{cases}e^{-d} & \text{if } d\ge 0\\ e^{d} & \text{if } d<0\end{cases} = \frac{1}{2}e^{-|d|}$$

$= \text{Laplace}(0,1)$

Std. laplace distr. AKA "double exponential"



$$X = \mu + \sigma D \sim \text{Laplace}(\mu, \sigma) := \frac{1}{2\sigma}e^{-\frac{|x-\mu|}{\sigma}}$$

$\sigma > 0$