$\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right]$

Consider a vector rv X with dimension n. Consider the following operation:

$$\phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) := E\left[e^{i[t\ 0\ \cdots\ 0]\vec{X}}\right] = E\left[e^{itX_1}\right] = \phi_{X_1}(t) \overset{P8}{\overset{P1}{\Rightarrow}} X_1 \sim f_{X_1}(x)$$

$$f_{X_1}(x) = \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{n-1} f_{X_1, X_2, \ldots, X_n}(x, u_1 \ldots u_{n-1})\, du_1 \cdots du_{n-1}$$

The bottom line is we can use multivariate chf's to immediately get marginal distributions.

$$\vec{X} \sim N(\vec{\mu}, \Sigma) \Rightarrow \phi_{\vec{X}}\left(\begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = e^{i[t\ 0 \cdots 0]\vec{\mu} - \frac{1}{2}[t\ 0 \cdots 0]\Sigma \begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

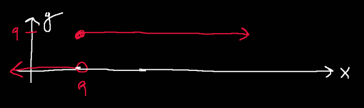
$$= e^{it\mu_1 - \frac{t}{2}[t\ 0 \cdots 0]\begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \vdots \\ \sigma_{n1} \end{bmatrix}}$$

$$= e^{it\mu_1 - t^2\sigma_1^2/2} \overset{P1}{\Rightarrow} X_1 \sim N(\mu_1, \sigma_1^2)$$

We now begin the unit on the "pure math" part of probability beginning with famous inequalities.

Let X be a rv with non-negative support ie. Supp[X] >= 0. Let a be a constant > 0. Consider the function:

$g(X) = a\mathbb{1}_{X \geq a}$



Is $a\mathbb{1}_{x \geq a} \leq x\ \ \forall x$? Consider two cases:

- $x < a \Rightarrow a\mathbb{1}_{x \geq a} = 0 \leq x$ because $Supp[X] \geq 0$ ✓
- $x \geq a \Rightarrow a\mathbb{1}_{x \geq a} = a \leq x$ because we assume $x \geq a$ ✓

$\Rightarrow a\mathbb{1}_{X \geq a} \leq X$

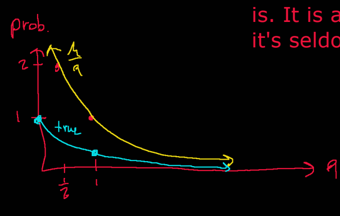
this rv has Supp $\{0,1\}$ $\Rightarrow$ it's Bernoulli(p) its expectation is p.

Now let's take the expectation of both sides:

$$E\left[a\mathbb{1}_{X \geq a}\right] \leq E[X] \Rightarrow a E\left[\mathbb{1}_{X \geq a}\right] \leq \mu \Rightarrow a P(X \geq a) \leq \mu$$

$\Rightarrow P(X \geq a) \leq \frac{\mu}{a}$

this is called Markov's inequality. It's a "tail bound" because it gives you an upper bound on what the probability of the "tail" is. It is a very "crude" bound which means it's seldom so useful and useless if a<mu.



let $\mu = 1 = E[X]$

e.g. $X \sim Exp(1)$

$P(X \geq a) = 1 - F_X(a) = e^{-a}$

$a = 1 \Rightarrow P(X \geq 1) = \frac{1}{e} \approx 0.37$

a	$P(X \geq a)$	Markov Bound	Chebyshev Bound	Chernoff Bound
2	0.1353	0.5	1	0.73576
5	0.0067	0.2	0.0635	0.09158
10	0.00004	0.1	0.012	0.00123

The Markov inequality has tons of corollaries:

- let $b = a\mu \Rightarrow P(X \geq b) \leq \frac{\mu}{b} \Rightarrow P(X \geq a\mu) \leq \frac{1}{a}$
- let $h(x)$ be a monotonically increasing function (so it's 1:1).

  $P(h(X) \geq h(a)) \leq \frac{E[h(X)]}{h(a)} \Rightarrow P(X \geq a) \leq \frac{E[h(X)]}{h(a)}$
- let $a = Quantile[X, p] \overset{\text{if } X \text{ continuous}}{=} F_X^{-1}(p)$

  $P(X \geq a) \leq \frac{\mu}{a} \Rightarrow 1 - F_X(a) \leq \frac{\mu}{a} \Rightarrow 1 - \cancel{F_X(F_X^{-1}}(p)) \leq \frac{\mu}{F_X^{-1}(p)}$

  $\Rightarrow 1 - p \leq \frac{\mu}{F_X^{-1}(p)} \Rightarrow F_X^{-1}(p) \leq \frac{\mu}{1-p}$ e.g. $Med[X] \leq 2\mu$
- Let X be any r.v. $\Rightarrow |X|$ is non-negative $\Rightarrow P(|X| \geq a) \leq \frac{E[|X|]}{a}$
- Let X be any rv with finite σ^2 (variance). Let $Y = (X - \mu)^2 \Rightarrow Y$ is non-neg.

  $P(Y \geq a^2) \leq \frac{E[Y]}{a^2} \Rightarrow P((X-\mu)^2 \geq a^2) \leq \frac{E[(X-\mu)^2]}{a^2}$ ← definition of variance

  $\Rightarrow P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$ Chebyshev's Inequality

So let's assume X is nonnegative and let's get this bound in a more user-friendly form.

$$P(|X-\mu| \geq a) = P(X - \mu \geq a \cup -(X-\mu) \geq a) = P(X - \mu \geq a) + P(-(X-\mu) \geq a)$$

$$= P(X - \mu \geq a) + P(X \leq \mu - a) \overset{\text{if } a \geq \mu}{\Longrightarrow}$$ second term is zero since X is assued non-negative

$$\Rightarrow P(|X-\mu| \geq a) = P(X - \mu \geq a) = P(X \geq a + \mu) \leq \frac{\sigma^2}{a^2} \overset{\text{let } b = \mu + a}{\Rightarrow} P(X \geq b) \leq \frac{\sigma^2}{(b-\mu)^2}$$

- Let X be any rv and $Y = e^{tX} \Rightarrow Y$ is non-neg $\forall t$.

  $\Rightarrow P(Y \geq c) \leq \frac{E[Y]}{c} \Rightarrow P(e^{tX} \geq c) \leq \frac{E[e^{tX}]}{c}$ ← mgf

  let $c = e^{ta}$

  $\Rightarrow P(e^{tX} \geq e^{ta}) \leq e^{-ta} M_X(t)$

  $\Rightarrow P(tX \geq ta) \leq e^{-ta} M_X(t)$

  $\overset{\text{if } t>0}{\Longrightarrow} P(X \geq a) \leq e^{-ta} M_X(t)\ \ \forall t > 0 \overset{t<0}{\Rightarrow} P(X \leq a) \leq e^{-ta} M_X(t)$

Since this works for all t and we are looking for the "best" i.e. the lowest upper bound, then just optimize over t:

$$\Rightarrow P(X \geq a) \leq \min_{t>0}\left\{e^{-ta} M_X(t)\right\} \text{ AND } P(X \leq a) \leq \min_{t<0}\left\{e^{-ta} M_X(t)\right\}$$

let $X \sim Exp(\lambda) \Rightarrow M_X(t) := E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$

$$= \lambda \frac{1}{t - \lambda}\left[e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{t-\lambda}\begin{cases} \infty - 1 & \text{if } t > \lambda \\ 0 - 1 & \text{if } t < \lambda \end{cases} = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda$$

otherwise the mgf doesn't exist!!!!

For X ~ Exp(1), the Chernoff bound is...

$$P(X \geq a) \leq \min_{t>0}\left\{e^{-ta} M_X(t)\right\} = \min_{t>0}\left\{e^{-ta}\frac{1}{1-t}\right\} \text{ if } t < 1$$

$$= \min_{t \in (0,1)} \overset{h(t)}{\left\{\frac{e^{-ta}}{1-t}\right\}} = \frac{e^{-(1-\frac{1}{a})a}}{1 - (1 - \frac{1}{a})} = \frac{e^{-a+1}}{\frac{1}{a}} = \frac{ae}{e^a}$$

$$h'(t) = \frac{(1-t)(-a)e^{-ta} - (e^{-ta})(-1)}{(1-t)^2} = \frac{a(t-1)e^{-ta} + e^{-ta}}{(1-t)^2} = \frac{e^{-ta}(a(t-1)+1)}{(1-t)^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow at - a + 1 = 0 \Rightarrow t_* = \frac{a-1}{a} = 1 - \frac{1}{a} \in (0,1)$$

true if $a > 1$

Let me tell you why the Chernoff bound is seldom useful. It requires the MGF. The MGF means you have the f(x) / p(x) and if you know these you may have F(x) which means you can calculate tail probabilities explicitly using numerical integration.