Lec 18 11/16/2020

$Z \sim N(0,1)$, $Y = Z^2 \sim f_Y(y) = ?$ Not 1:1

$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P(Z \in [-\sqrt{y}, \sqrt{y}]) = 2P(Z \in [0, \sqrt{y}])$

$= 2(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{1/2}) = 2F_Z(\sqrt{y}) - 1$

$f_Y(y) = \frac{d}{dy}[2F_Z(\sqrt{y}) - 1] = 2(\frac{1}{2}y^{-1/2}) f_Z(\sqrt{y}) = y^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \ge 0}}$

$\propto y^{\overbrace{-1/2}^{1/2 - 1}} e^{-1/2 y} \mathbb{1}_{y \ge 0} \propto \text{Gamma}(1/2, 1/2)$

$Z_1, Z_2, \ldots, Z_k \overset{iid}{\sim} N(0,1)$, $Y = \underbrace{Z_1^2}_{\sim G(1/2, 1/2)} + \underbrace{Z_2^2}_{\sim G(1/2, 1/2)} + \ldots + \underbrace{Z_k^2}_{\sim G(1/2, 1/2)} \sim \text{Gamma}(k/2, 1/2)$

Note The beta is always 1/2 and the α is always $k/2$ so k is the only parameter. And because this is a common situation, we give it a special name:

$\text{Gamma}(k/2, 1/2) = \chi^2_k$ The "chi squared distri. with k "degrees of freedom" $k \in \mathbb{N}$

$E[Y] = k E[Z^2] = k$

$k=1, \Gamma(1/2) = \sqrt{\pi}$

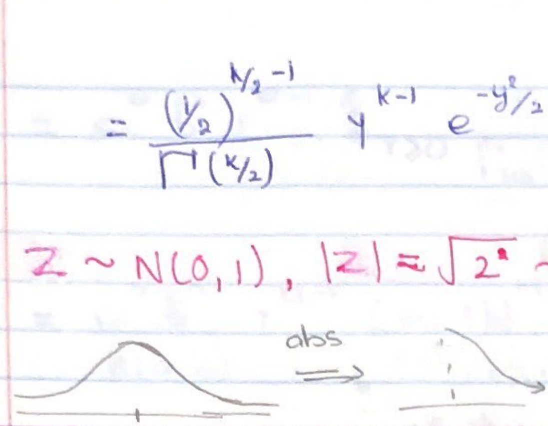
$\chi^2_k = \frac{(1/2)^{k/2}}{\Gamma(k/2)} y^{k/2 - 1} e^{-y/2} \mathbb{1}_{y \ge 0} \overset{\downarrow}{=} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} \mathbb{1}_{y \ge 0} = \chi^2_1$

$X \sim \chi^2_k$, $Y = \sqrt{X} \Rightarrow x = y^2 = g^{-1}(y)$, $|\frac{d}{dy}[g^{-1}(y)]| = |2y| = 2y$

$f_Y(y) = f_X(y^2) 2y = \frac{(1/2)^{k/2}}{\Gamma(k/2)} y^{k-2} e^{-y^2/2} (2y) \mathbb{1}_{\underbrace{y^2 \ge 0}_{y \ge 0}}$

$$= \frac{(1/2)^{k/2-1}}{\Gamma(k/2)} y^{k-1} e^{-y^2/2} \mathbb{1}_{y\ge 0} = \chi_k$$ the chi distribution with k degrees of freedom

$Z \sim N(0,1)$, $|Z| = \sqrt{Z^2} \sim \chi_1 = \sqrt{2/\pi}\, e^{-y^2/2}\, \mathbb{1}_{y\ge 0}$



$$= 2\underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right)}_{f_Z} \underline{\mathbb{1}_{y\ge 0}}$$

$X \sim \text{Gamma}(\alpha, \beta)$, $Y = cX$ where $c > 0$

$$f_Y(y) = \frac{1}{c} f_X(y/c) = \frac{1}{c}\, \beta^\alpha/\Gamma(\alpha)\, \underbrace{(y/c)^{\alpha-1}}_{y^{\alpha-1}/c^{\alpha-1}}\, e^{-\overbrace{\beta y/c}^{(\beta/c)y}}\, \mathbb{1}_{\overbrace{y/c>0}^{y>0}}$$

c^α

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-(\beta/c)y} \mathbb{1}_{y\ge 0} = \text{Gamma}(\alpha, \beta/c)$$

$X \sim \chi^2_k$, $Y = X/k \sim \text{Gamma}\left(k/2, \frac{1/2}{1/k}\right) = \text{Gamma}(k/2, k/2)$

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let $X_1 \sim \chi^2_{k_1}$ indep. of $X_2 \sim \chi^2_{k_2}$

let $U = X_1/k_1 \sim \text{Gamma}\left(\overset{a}{\widetilde{k_1/2}}, \overset{a}{\widetilde{k_1/2}}\right)$ indep of $V = X_2/k_2 \sim \text{Gamma}\left(\overset{b}{\widetilde{k_2/2}}, \overset{b}{\widetilde{k_2/2}}\right)$

$$R = U/V \sim f_R(r) = \int_{Supp[V]} f_U(rt)\, \mathbb{1}_{rt \in Supp[U]}\, f_V(t)\, |t|\, dt$$

$$= \int_0^\infty \frac{a^a}{\Gamma(a)} (rt)^{a-1} e^{-art}\, \mathbb{1}_{\frac{r\in(0,\infty)}{rt\in[0,\infty)}}\, \frac{b^b}{\Gamma(b)} t^{b-1} e^{-bt}\, t\, dt$$

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r\ge 0} \int_0^\infty t^{a+b-1} e^{-(ar+b)t}\, dt$$

$$= a^a b^b r^{a-1} \mathbb{1}_{r\geq 0} \underbrace{\frac{1}{\Gamma(a)\Gamma(b)} \Gamma(a+b)}_{\frac{1}{B(a,b)}} \cdot \frac{1}{(ar+b)^{a+b}}$$

$$= \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \underbrace{(ar+b)^{-(a+b)}}_{\underbrace{b^{-(a+b)}(1+a/b\, r)^{-(a+b)}}_{b^{-a}\cancel{b^{-b}}}} \mathbb{1}_{r\geq 0}$$

$$= \frac{(a/b)^a}{B(a,b)} r^{a-1} (1 + a/b\, r)^{-(a+b)} \mathbb{1}_{r>0}$$

$$= \frac{(k_1/k_2)^{k_1/2}}{B(k_1/2, k_2/2)} r^{k_1/2 - 1} (1 + k_1/k_2\, r)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r\geq 0}$$

$= F_{k_1,k_2}$ this is the "F distribution" or "Fisher-Snecedor Distribution" with k_1 numerator degrees of freedom and k_2 denominator degrees of freedom. $k_1, k_2 \in \mathbb{N}$

Let $Z \sim N(0,1)$, $X \sim \chi^2_k$, $W = Z/\sqrt{X/k} \sim f_W(w) = ?$

symmetric around 0

Consider $W^2 = Z^2/1 / X/k \sim F_{1,k}$

$F_{W^2}(w^2) = P(W^2 \leq w^2) = P(W \in [-w, w]) = F_W(w) - F_W(-w)$

Take derivatives

$d/dw [F_{W^2}(w^2)] = d/dw [F_W(w)] - d/dw [F_W(-w)]$

$\cancel{2} w f_{W^2}(w^2) = f_W(w) - {}^- f_W(-w) = \cancel{2} f_W(w)$

$$f_W(w) = w \underbrace{\frac{(1/k)^{1/2}}{B(1/2, k/2)}}_{\Gamma(\frac{k+1}{2}) / \Gamma(1/2)\Gamma(k/2)} \underbrace{(w^2)^{1/2 - 1}}_{w^{-1}} (1 + w^2/k)^{\frac{-1+k}{2}} \mathbb{1}_{\underset{w^2 \geq 0}{w \in \mathbb{R}}}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma(k/2)} \left(1 + w^2/k\right)^{-\frac{k+1}{2}} = T_k$$ Student's T distrs. with k degrees of freedom, discovered in 1908 by William Gosset while he was working at a beer factory.

If $k \to \infty$ $T_k \to Z$ Student's T distribution has the N(0,1) shape but just thicker tails.

$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$, $R = Z_1/Z_2 \sim \int_{\mathbb{R}} f(ru) f(u) |u|\, du$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-r^2u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} |u|\, du$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\frac{r^2+1}{2}u^2} |u|\, du + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} |u| du \right)$$

$$= \frac{1}{2\pi}\left(\underbrace{\int_{0}^{-\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du}_{\int_0^\infty \ldots du} + \int_{0}^{\infty} e^{-\frac{r^2+1}{2}u^2} |u|\, du \right)$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\frac{r^2+1}{2}u^2} u\, du$$

let $t = u^2 \Rightarrow dt/du = 2u \Rightarrow du = \frac{1}{2u} dt$, $u = 0 \Rightarrow t = 0$, $u \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\frac{r^2+1}{2}t} \cancel{u} \frac{1}{2\cancel{u}}\, dt = \frac{1}{2\pi} \frac{1}{\frac{r^2+1}{2}} \underbrace{\int_0^{\infty} \frac{r^2+1}{2} e^{-\frac{r^2+1}{2}t}\, dt}_{\int_0^1 \lambda e^{-\lambda x} dx = 1}$$

PDF exponent rv

$$= 1/\pi \; 1/(1+r^2) = \text{Cauchy}(0,1)$$

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let $X = c + \sigma R$, $R \sim \text{Cauchy}(0,1)$, $\sigma > 0$

$$X \sim \text{Cauchy}(c, \sigma) = \frac{1}{\sigma\pi} \frac{1}{1+\left(\frac{y-c}{\sigma}\right)^2}$$