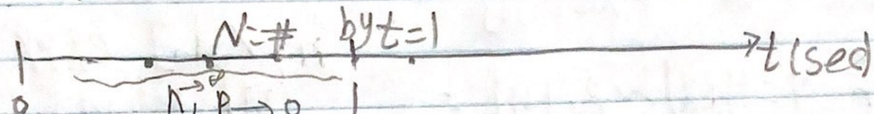
10/07

$T_k \sim \text{Erlang}(k,\lambda)$, $N \sim \text{Poisson}(\lambda)$ $X_1, X_2 \cdots \overset{iid}{\sim} \text{Exp}(\lambda)$

$P(T_k > 1)$

$= 1 - F_{T_k}(1) = \varphi(k,\lambda)$ $F_N(x) = \varphi(x+1,\lambda)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1)$ "Poisson process"



$k=5$ $\{T_5>1\} = \{X_1+X_2+X_3+X_4<1\} \cup \{X_1+X_2+X_3<1\} \cup$

$\{X_1+X_2<1\} \cup \{X_1<1\} \cup \{X_1>1\}$

$$\begin{array}{cc} 1-F_{T_5}(1) & F_N(4) \\ \| & \| \\ P(T_5>1) & = P(N\le 4) \end{array} = \{N=4\}\cup\{N=3\}\cup\{N=2\}\cup\{N=1\}\cup\{N=0\}$$

$$T \sim \text{Erlang}(k,\lambda) = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t\geq 0} = \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t\geq 0}$$ "Gamma"

$k \in \mathbb{N}, \lambda \in (0,\infty)$

$$T \sim \text{NegBin}(k,p) = \binom{k+t-1}{k-1}(1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$

$$= \frac{\Gamma(k+t)}{\Gamma(k)\,t!}(1-p)^t p^k \mathbb{1}_{t\in\mathbb{N}_0}$$

$k \in \mathbb{N}, p \in (0,1)$

~~when~~ if $k \in (0,\infty)$? Is the top PDF legal and the bottom PMF legal? Yes

$$\int_0^\infty \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)}\,dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\,t!}(1-p)^t p^k = 1$$

which means --- these are rv's

$X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} 1_{t \geq 0}$

$X \sim$ ~~EX~~ Negbin(k, p) - - - - - -

Transformations for Discrete rv's

$X \sim \text{Bern}(p)$, $Y = X + 3 \sim \begin{cases} 3 & \text{wp } 1-p \\ 4 & \text{wp } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} 1_{y-3 \in \{0,1\}}$

$\parallel$

$p^x (1-p)^{1-x} 1_{x \in \{0,1\}}$

how do I express the transformed pmf using the original pmf?

If $Y = g(X) \sim P_Y(y) = P_X(g^{-1}(y))$

$g^{-1}(y) \stackrel{\parallel}{=} X$

Is this formula general? No. This is only the formula for g invertible. If g non-invertible - - -

$X \sim U(\{1, 2, \ldots 10\}) = \frac{1}{10} 1_{x \in \{1, 2, \ldots, 10\}}$

$Y = \min\{X, 3\} \sim \begin{cases} 1 & \text{wp } \frac{1}{10} \\ 2 & \text{wp } \frac{1}{10} \\ 3 & \text{wp } P(X=3) + P(X=4) \ldots + P(X=10) = \frac{8}{10} \end{cases}$

If g invertible for supp of X

$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x) = \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$

$X \sim \text{Bin}(n, p)$, $Y = X^2 \sim P_Y(y) = P_X(g^{-1}(y)) = P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} 1_{\sqrt{y} \in \{0, 1, \ldots n\}}$

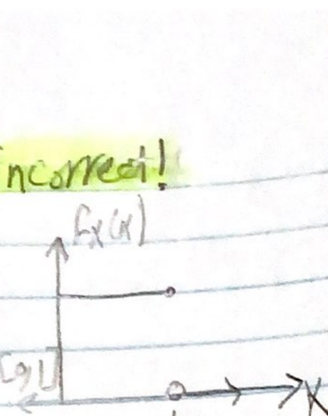
$\Downarrow$

$X = \sqrt{y} = g^{-1}(y)$

Transformations for continuous rv's $Y = g(X)$,

X is continuous.

for invertable g, $f_Y(y) \overset{?}{=} f_X(g^{-1}y)$ incorrect!



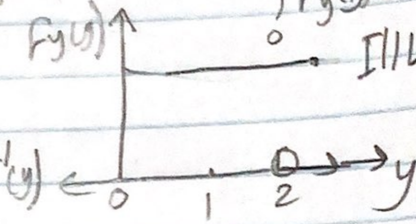
$X \sim U(0,1) = 1_{X\in[0,1]}$, $Y = 2X \sim f_Y(y)$

$= f_X(\frac{y}{2}) = 1_{\frac{y}{2}\in[0,1]}$

$= 1_{y\in[0,2]}$

$\int_0^2 f_Y(y) = 2-0=2$

Illegal



$F_Y(y) = P(Y\le y) = P(g(X)\le y) = P(X\le g^{-1}(y))$

$= F_X(g^{-1}(x))$

$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] = F_X'(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$

$= f_X(g^{-1}(x))\underbrace{\frac{d}{dy}[g^{-1}(y)]}_{\text{stretching}}$

if $g'<0$

$= P(X\ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$

$f_Y(y) = \frac{d}{dy}[\quad] = \frac{-d}{dy}[F_X(g^{-1}(x))] = f_X(g^{-1}(y))\left(\frac{-d}{dy}[g^{-1}(y)]\right)$

$\frac{d}{dy}[g^{-1}(y)] < 0$

$= f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$ for all g invertible

Let's derive some more rules! The most common invertible function is --- the straight line!

$Y = aX + C! \Rightarrow X = g^{-1}(y) = \frac{y-c}{a}$, $\left|\frac{d}{dy}[g^{-1}(y)]\right| = \frac{1}{|a|}$

s.t $a, c \in \mathbb{R}$

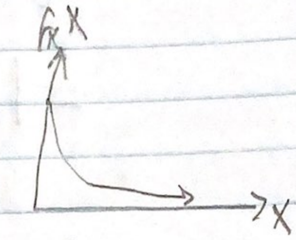
$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$ "shift and scale"

if $c=0$ just a scale --- $y = aX$

$$f_Y(y) = f_X\left(\frac{X}{a}\right)\frac{1}{|a|}$$

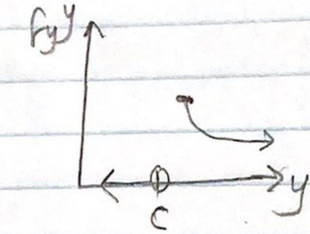
if $a=1$ just a shift $Y = X+c$

$$f_Y(y) = f_X(y-c)$$



$$X \sim Exp(\lambda) = \lambda e^{-\lambda x}\, 1_{x \geq 0}$$

$$Y = X + c = \lambda e^{-\lambda(y-c)}\, 1_{y \geq c}$$



$$X \sim Exp(1) = e^{-x}\, 1_{x \geq 0}$$

$$Y = g(x) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = y$$

$$\Rightarrow e^{y} = e^{x} - 1 \Longrightarrow e^{y}+1 = e^{x} \Rightarrow X = \ln(e^{y}+1) = g^{-1}(y)$$

$y \in \mathbb{R}$

$e^{y} \geq 0$

$e^{y}+1 \geq 1$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$$

$$f_Y(y) = f_X(\ln(e^{y}+1))\frac{e^{y}}{e^{y}+1} = \overbrace{e^{-\ln(e^{y}+1)}}^{e^{\ln\left(\frac{1}{e^{y}+1}\right)}}\, 1_{\ln(e^{y}+1) \geq 0}\frac{e^{y}}{e^{y}+1}$$

$$= \frac{1}{e^{y}+1}\,\frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2}\cdot\frac{e^{-2y}}{e^{-2y}} = \text{logistic } (0,1)$$

$$\parallel \quad \frac{e^{-y}}{(e^{-y}+1)^2}$$