geom, neg exp → erlang all related

lec10 368 10/7/20

$T_k \sim Erlang(k, \lambda)$, $N \sim Poisson(\lambda)$

$P(T_k > 1)$ $F_N(x) = Q(x+1, \lambda)$.

$= 1 - F_{T_k}(1) = Q(k, \lambda)$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1)$ "Poisson Process"

N: = # events by t=1

|—o—o—o—o—|—o——————| t (sec)

$k=5$ $\{T_5 > 1\}$ what does the event look like, what is T_5

it's waiting for 5 exponentials to happen

wait for all 5 If T_5 is >1 what is the 5th success

$X_1, X_2 \ldots \overset{iid}{\sim} Exp(\lambda)$

from the picture

$k=5$ $\{T_5 > 1\} = \{X_1 + X_2 + X_3 + X_4 < 1\} \cup \{X_1 + X_2 + X_3 < 1\} \cup$

all 5 could be after the 1 $\{X_1 + X_2 < 1\} \cup \{X_1 < 1\} \cup \{X_1 > 1\}$

break it into five parts

$= \{N=4\} \cup \{N=3\} \cup \{N=2\} \cup \{N=1\} \cup \{N=0\}$

$1 - F_{T_5}(1) = F_N(4)$

$P(T_5 >) = P(N \leq 4)$

Can you write in terms of the gamma function?

pdf

$T \sim Erlang(k, \lambda) = \dfrac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t \geq 0}$

time is cont

$= \dfrac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t \geq 0}$ $k \in \mathbb{N}$, $\lambda \in (0, \infty)$

$T \sim Neg\,bin(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

time is not cont

can you write this in term of the gamma function

$= \dfrac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

→ What if $k \in (0, \infty)$? is the top pdf legal
is the top pmf legal? yes!

$$\int_0^\infty \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} dt = 1 \quad \text{and}$$

two for free

$$\sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1 \quad \text{"extented negative bin"}$$

which means these are r.vs and inorder to have a r.v you need either a pdf or pmf

the average more flexible than the f

$X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} \mathbb{1}_{t>0}$

replace/substitute terms

extended
$X \sim \text{NegBin}(k, p) \ldots$
Ext NegBin

0, 1

then $x = y - 3$

Transformations of discrete rv's

$X \sim \text{Bern}(p)$, $Y = \underbrace{X+3}_{\text{how is that distributed}} \sim \; = \begin{cases} 3 & \text{w.p. } 1-p \\ 4 & \text{w.p. } p \end{cases} = p^{y-3}(1-p)^{1-(y-3)} \mathbb{1}_{y \in \{3,4\}}$

$\boxed{\begin{matrix} y-3 \in \{0,1\} \\ +3 \quad\quad +3 \end{matrix}}$

$\parallel$ 0, 1

inverse function

$p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}}$ how do I express the transformed pmf using the original pmf?

in general if $y = g(X) \sim p_y(y) = ?$ big hint

$p_X(g^{-1}(y))$ you can say $g^{-1}(x) = X$

Is this the general formula, what major assumtion did we make? No this is only for g invertible If g is non invertible ....

$X \sim U(\{1,2,\dots,10\}) = \frac{1}{10} \mathbb{1}_{x \in \{1,2,\dots,10\}}$

how is this distributed?

$Y = \min\{X, 3\} \sim \begin{cases} 1 \text{ wp } \frac{1}{10} \\ 2 \text{ wp } \frac{1}{10} \\ 3 \text{ wp } \frac{8}{10} \quad P(X=3) + \dots + P(X=10) \end{cases}$

hint to the general

$P_Y(y) = P_Y(x)$

$P_Y(y) = \sum_{\{x : g(x) = y\}} P_X(x)$ if it's not invertible gotta check everything

if invertible for supp[X] for every x just 1 y.

$= \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$

$X \sim Bin(n,p)$, $Y = X^2 \sim P_Y(y) =$

over the support of X it is invertible $x \neq -3$

$x = \sqrt{y} = g^{-1}(y)$ plug in

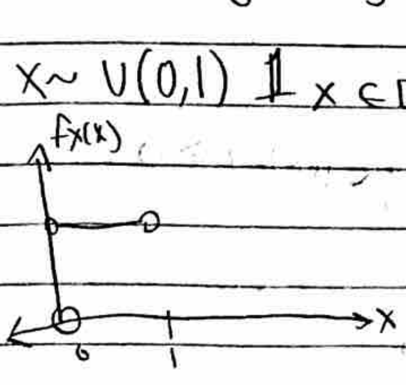
$P_Y(y) = P_X(g^{-1}(y)) = P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$

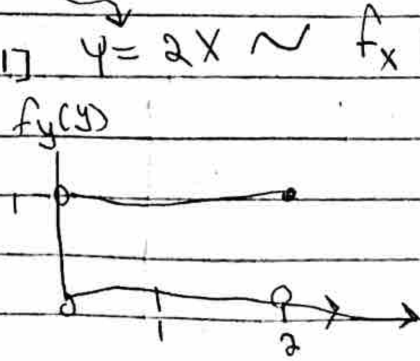
Transformations for Continous rv's $Y = g(x)$, X is cont

for invertible g, $f_Y(y) \stackrel{?}{=} f_X(g^{-1} y)$

$X \sim U(0,1)\ \mathbb{1}_{x \in [0,1]}$ $Y = 2X \sim f_X\left(\frac{y}{2}\right) = \mathbb{1}_{\frac{y}{2} \in [0,1]}$

$= \mathbb{1}_{y \in [0,2]}$





What do you think?

check if legal

$$\int_0^2 f_y(y) = 2 \neq 1$$ ILLEgal!

$f_y(y) = f_x(g^{-1}y)$ is incorrect!!

the above formula was derived using probabilities, Densities are not probabilities! But.... Cdf's are probabilities

Strategy! let's derive the cdf of y using the cdf of x. and then, like ~~as~~ when we did convolutions take the derivative to get the density of y

Convert cdf from y to x

g is invertible...

$$F_y(y) = P(y \le y) = P(g(x) \le y) =$$

if $g'>0$ cdf

$$P(X \le g^{-1}(y)) = F_x(g^{-1}(y))$$

density

$$f_y(y) = \frac{d}{dy}\left[F_y(y)\right] = \frac{d}{dy}\left[F_x(g^{-1}(y))\right] =$$

$$\cancel{(g^{-1})'(g) + (g)'(g^{-1})}$$

$$F_x'(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right]$$

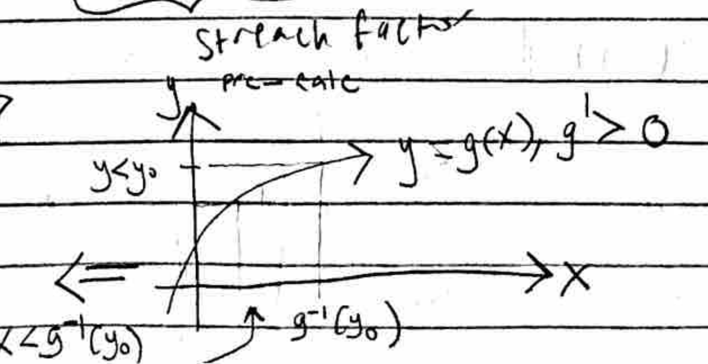
Part we were missing before!

not invertible care in HW

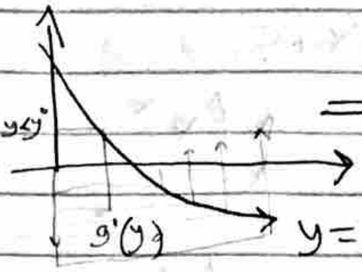
$$= f_x(g^{-1}(y))\underbrace{\frac{d}{dy}\left[g^{-1}(y)\right]}_{\text{streach factor}}$$

pre-calc

are we done?



Still invertible



$\Longrightarrow \quad x > g^{-1}(y_0)$

$y = g(x)\,,\; g' < 0$

you need two cases!

$= P(X \geq g'(y))$ if $g' < 0$

$= 1 - F_X(g^{-1}(y))$

$= f_y(y) = \frac{d}{dy}\left[\quad\right] = \frac{-d}{dy}\left[F_X(g'(y))\right]$

$= f_X(g^{-}(y))\left(-\frac{d}{dy}\left[g^{-1}(y)\right]\right)$

$\frac{d}{dy}\left[g^{-}(y)\right] < 0$ $\qquad = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$

we saw the same with the skalln

$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$ $*$

for all g invertible

lets derive some rules! The most common invertible function is... the straight line $y = ax + c$!

$X = g^{-1}(y) = \frac{y-c}{a}\,,\; \left|\left[\frac{d}{dy}\left[g^{-1}(y)\right]\right]\right| = \frac{1}{\|a\|}$

s.t $a, c \in \mathbb{R}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right)\frac{1}{|a|}$ "shift and scale"

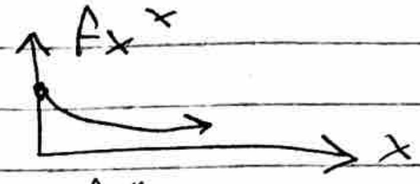
if $c=0$ just a scale ... $Y = aX$

$f_Y(y) = f_X\left(\frac{y}{a}\right)\frac{1}{|a|}$

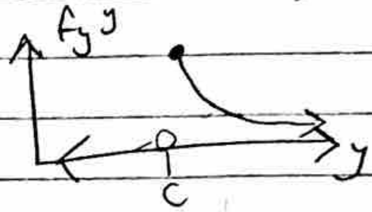
moving to the right is minus...

if $a = 1$ just a shift $y = x + c$

$f_y(y) = f_x(y - c)$



$X \sim Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$



$Y = X + c = \lambda e^{-\lambda(y-c)} \mathbb{1}_{y \geq 0}$

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \geq 0}$

$Y = g(X) = -\ln\left(\frac{e^{-x}}{1 - e^{-x}}\right) = \ln\left(\frac{1 - e^{-x}}{e^{-x}}\right)$

all g(x)

$= \ln(e^{x} - 1) = y$

$g^{-1}(y) \overset{?}{=} e^{y} = e^{x} - 1 = e^{y} + 1 = e^{x} \Rightarrow x = \ln(e^{y} + 1)$

$= g^{-1}(y)$

$\frac{d}{dy}\left[g^{-1}(y)\right] = \left|\frac{e^{y}}{e^{y}+1}\right| = \frac{e^{y}}{e^{y}+1}$

$\ln(0) = -\infty$

$f_y(y) = {}_{y \in \mathbb{R}} f_x(\ln(e^{y}+1)) \frac{e^{y}}{e^{y}+1}$

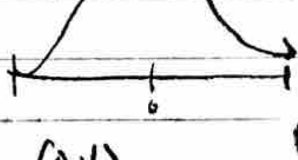
$e^{y} \geq 0$

$e^{y} + 1 \geq 1$

don't need indicator function support is everything

$= \underbrace{e^{-\ln(e^{y}+1)}}_{e^{\ln\left(\frac{1}{e^{y}+1}\right)}} \mathbb{1}_{\ln(e^{y}+1) \geq 0} \frac{e^{y}}{e^{y}+1}$

looks like a normal



$\frac{e^{-2y}}{e^{-2y}}$

$= \frac{1}{e^{y}+1} \frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2} = $ logistic (0,1)

$= \frac{e^{-y}}{(e^{-y}+1)^2}$