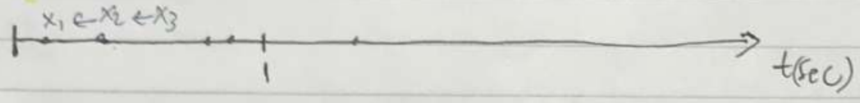
Wednesday, October 7 2020

# Lecture 10

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Exp}(\lambda)$

$T_k \sim \text{Erlang}(k, \lambda)$ $\qquad$ $N \sim \text{Poisson}(\lambda)$

$P(T_k > 1) = 1 - F_{T_k}(1) = Q(k, \lambda)$ $\qquad$ $F_N(x) = Q(x+1, \lambda)$




$k = 5$

$\{T_5 > 1\} = \{x_1 + x_2 + x_3 + x_4 < 1\} \cup \{x_1 + x_2 + x_3 < 1\} \cup \{x_1 + x_2 < 1\} \cup \{x_1 < 1\} \cup \{x_1 > 1\}$

let $N$ be # events before 1 sec

$= \{N = 4\} \cup \{N = 3\} \cup \{N = 2\} \cup \{N = 1\} \cup \{N = 0\}$

$$\Rightarrow \underset{1 - F_{T_5}(1)}{\underbrace{P(T_5 > 1)}} = P(N \le 4) = F_N(4)$$

$\Rightarrow 1 - F_{T_k}(1) = F_N(k-1) = Q(k, \lambda)$ "Poisson process"

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$T \sim \text{Erlang}(k, \lambda) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0} = \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t} \mathbb{1}_{t \ge 0}$ $\quad k \in \mathbb{N},\ \lambda \in (0, \infty)$

$T \sim \text{NegBin}(k, p) = \binom{k+t-1}{k-1} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} = \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0}$

$k \in \mathbb{N},\ p \in (0,1)$

For both, what if $k \in (0, \infty)$? Are both r.v.'s still "legal"?

Yes we can show that both:

$$\int_0^\infty \frac{\lambda^k}{\Gamma(k)} t^{k-1} e^{-\lambda t}\, dt = 1 \quad \text{and} \quad \sum_{t=0}^{\infty} \frac{\Gamma(k+t)}{\Gamma(k)\, t!} (1-p)^t p^k = 1$$

We just derived two new famous rv's:

- $X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta(x)} \ \forall x \geq 0$
- $X \sim \text{Ext Neg Bin}(k, p) = \frac{\Gamma(k,t)}{\Gamma(k)!} (1-p)^t p^k \ \forall t \in \mathbb{N}_0$ "the extended negative binomial"

Transformation of Discrete rv's

$$X \sim \text{Bern}(p) := p^x (1-p)^{1-x} \ \forall x \in \{0,1\} = \begin{cases} 1 & \text{w.p } p \\ 0 & \text{w.p } 1-p \end{cases}$$

$$Y = x+3 \sim \begin{cases} 4 & \text{w.p } p \\ 3 & \text{w.p } 1-p \end{cases} = p^{y-3} (1-p)^{1-(y-3)} \ \forall y-3 \in \{0,1\}$$

$x = y-3 = f^{-1}(y)$

I want to find the PMF of Y using the PMF of X:

$$Y = g(x) \sim P_Y(y) = P_X(g^{-1}(Y))$$

What assumption did I make when I "derived" this formula?
I assumed an inverse function exist ie g is invertible. If not...

$$X \sim U(\{1, 2, \ldots, 10\}) = \begin{cases} 1 & \text{w.p } \frac{1}{10} \\ 2 & \text{w.p } \frac{1}{10} \\ \vdots \\ 10 & \text{w.p } \frac{1}{10} \end{cases}, \quad Y = g(x) = \min(x, 3) \sim \begin{cases} 1 & \text{w.p } \frac{1}{10} \\ 2 & \text{w.p } \frac{1}{10} \\ 3 & \text{w.p } P(X=3) + P(X=4) + \ldots + P(X=10) = 8/10 \end{cases}$$

$$Y = g(x) \sim P_Y(y) = \sum_{\{x : y = g(x)\}} P_X(x) \underset{\text{if } g \text{ is invertible}}{=} \sum_{\{x : x = g^{-1}(y)\}} P_X(x) = P_X(g^{-1}(y))$$

one element

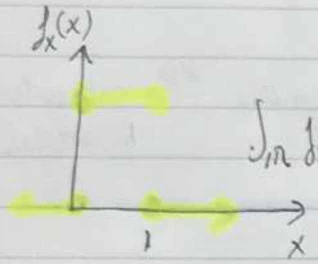
- $X \sim \text{bin}(n, p), \ Y = X^3 \sim P_X(\sqrt[3]{y}) = \binom{n}{\sqrt[3]{y}} p^{\sqrt[3]{y}} (1-p)^{n-\sqrt[3]{y}} \ \forall \sqrt[3]{y} \in \{0, 1, \ldots, n\}$

$g^{-1}(y) = \sqrt[3]{y}$

$$Y = X^2 \sim P_X(\sqrt{y}) = \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} \mathbb{1}_{\sqrt{y} \in \{0,1,\dots,n\}}$$

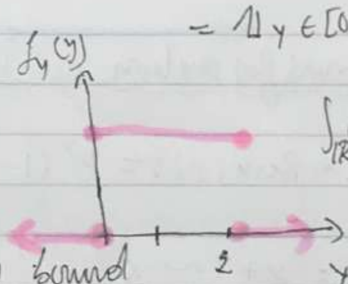
Transformation for continuous r.v.'s:

For $g$ invertible $f_Y(y) \overset{?}{=} f_X(g^{-1}(y))$ "Not working"




$\int_{\mathbb{R}} f_X(x)\,dx = 1$

let $X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$, $Y = 2X \overset{?}{\sim} f_X(g^{-1}(y))$
$= f_X(\frac{y}{2}) = \mathbb{1}_{\frac{y}{2} \in [0,1]}$
$= \mathbb{1}_{y \in [0,2]}$




$\int_{\mathbb{R}} f_Y(y)\,dy = 2$

Game over

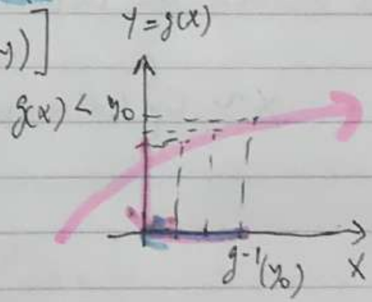
PDF's are not probabilities!! So this was bound to fail because we used them as probabilities. However, CDF's "are" probabilities.

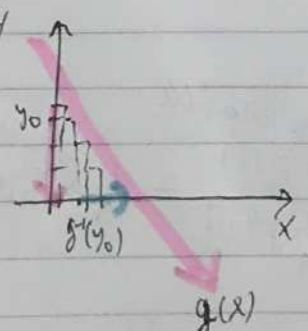
$g$ is invertible

$$F_Y(y) := P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

always positive

$$\Rightarrow F_X'(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)] = f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$




$g$ is invertible and $g' < 0$

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = f_X(g^{-1}(y)) \left(-\frac{d}{dy}[g^{-1}(y)]\right)$$

always negative

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} \left[ g^{-1}(y) \right] \right|$ general rule

We can derive a less general but very useful corollary rule:

$Y = \underline{aX + c} \sim f_Y(y) = ?$ Shift and scale (Shift by c and Scale by a)

$g(x)$ is invertible $\Rightarrow g^{-1}(y) = \frac{y-c}{a}$ $\left| \frac{d}{dy} [g^{-1}(y)] \right| = \left| \frac{1}{a} \right| = \frac{1}{|a|}$

$f_Y(y) = f_X\left(\frac{y-c}{a}\right) \frac{1}{|a|}$

$Y = aX \sim f_X\left(\frac{y}{a}\right) \frac{1}{|a|}$, $Y = X + c \sim f_X(y-c)$

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$X \sim Exp(1) = e^{-x} \mathbb{1}\, x \geq 0$

$Y = g(x) = -\ln\left(\frac{e^{-x}}{1-e^{-x}}\right) = \ln\left(\frac{1-e^{-x}}{e^{-x}}\right) = \ln(e^{x}-1) = g(x) \sim f_Y(y)$ invertible

$y = \ln(e^{x}-1) \Rightarrow e^{y} = e^{x} - 1 \Rightarrow e^{y} + 1 = e^{x} \Rightarrow x = \ln(e^{y}+1) = g^{-1}(y)$

$\left| \frac{d}{dy} [g^{-1}(y)] \right| = \left| \frac{d}{dy} [\ln(e^{y}+1)] \right| = \left| \frac{e^{y}}{e^{y}+1} \right| = \frac{e^{y}}{e^{y}+1}$ $e^{y} \geq 0 \Rightarrow y \in \mathbb{R}$

$f_Y(y) = f_X(\ln(e^{y}+1)) \cdot \frac{e^{y}}{e^{y}+1} = e^{-\ln(e^{y}+1)} \cdot \mathbb{1}\, \underset{e^{y}+1 \geq 1 \Rightarrow}{\ln(e^{y}+1) \geq 0} \cdot \frac{e^{y}}{e^{y}+1}$

$= \frac{1}{e^{y}+1} \frac{e^{y}}{e^{y}+1} = \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{e^{-2y}}{e^{-2y}} = \frac{e^{-y}}{(e^{-y}+1)^2} =$ Logistic $(0,1)$ standard logistic.