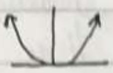
Monday 9 November 2020

Lecture 16

Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)| dx < \infty \}$ all the functions in this set are called "L1 integrable" or "absolutely integrable"

Are PDF's in the set L1?  $f(x) = x^2 \notin L^1$

If $f \in L^1 \Rightarrow \exists \hat{f}$, the "Fourier transform" of f:

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-i2\pi wx} f(x)\, dx = \mathcal{F}[f]$$

This is called "forward fourier Transform" or "fourier analysis". x is called the "time domain" and omega is called the frequency domain". One of Fourier's ideas is that functions in L1 can be decomposed into a sum of sines and cosines ~~and~~ with different frequencies, and amplitudes, and phases $|\hat{f}(\text{omega})|$, and phase shifts, Arg $[\hat{f}(\text{omega})]$.

Further, if $\hat{f} \in L^1$, then we can do a "reverse/inverse fourier transform" to restore our original function f:

$$f(x) = \int_{\mathbb{R}} e^{i2\pi xw} \hat{f}(w)\, dw = \mathcal{F}^{-1}[\hat{f}]$$

This is called the "inverse fourier transform" or Fourier Synthesis".

Fourier Inversion thm: If f and that are in L1, then f and that are 1:1

We define the Characteristic function (chf) for rv X as:

$$\phi_X(t) := E[e^{itX}] = \int_{\mathbb{R}} e^{itx} f_X(x)\,dx$$
$$= \sum_{x \in \mathbb{R}} e^{itx} P_X(x)$$

This is the Fourier transform with a different frequency unit $t = -2\pi\omega$.

The reason why we bother to take this crazy-looking transformation is that there are really powerful properties of the chf that will enable us to solve problems. Here are the main properties:

P_0 $\quad \phi_X(0) = E[e^{i(0)X}] = E[1] = 1 \quad \forall X, \forall t$

P_1 $\quad \phi_X(t) = \phi_Y(t) \iff X \stackrel{d}{=} Y$ "Uniqueness"

P_2 $\quad$ If $Y = aX + b$ where $a, b \in \mathbb{R}$ $\quad \phi_Y(t) = E[e^{it(aX+b)}] = E[e^{i\overbrace{at}^{t'}X} e^{itb}]$

$$= e^{itb} E[e^{it'X}] = e^{itb} \phi_X(t') = e^{itb} \phi_X(at)$$

P_3 $\quad X_1, X_2$ are independent. $T = X_1 + X_2$

$$\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}] = E[e^{itX_1}]E[e^{itX_2}] = \phi_{X_1}(t)\phi_{X_2}(t)$$

P_4 $\quad$ "Moment Generation"

We are able to interchange differentiation and integration here

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] = E\left[\frac{d}{dt}[e^{itX}]\right] = E[iX e^{itX}]$$

$$\phi_X'(0) = E[iX] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$

$$\phi_X''(t) = \frac{d}{dt}\left[E[iXe^{itX}]\right] = E\left[iX \frac{d}{dt}[e^{itX}]\right] = E[i^2 X^2 e^{itX}]$$

$$\phi_X''(0) = E[i^2 X^2] \Rightarrow E[X^2] = \frac{\phi_X''(0)}{i^2} \quad \ldots\ldots$$

$$E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}$$

(P5) $\phi_X(t) \in [-1, 1]$ ie the ChF exist $\forall x, \forall t$

$\Downarrow$

$|\phi_X(t)| \leq 1$ proof: $|E[e^{itX}]| = \left| \int_{\mathbb{R}} e^{itx} f_X(x)\, dx \right| \leq \int_{\mathbb{R}} |e^{itx} f_X(x)|\, dx$

$\leq \int_{\mathbb{R}} |e^{itx}|\, |f_X(x)|\, dx = \int_{\mathbb{R}} |\underset{a}{\cos(tx)} + i \underset{b}{\sin(tx)}|\, f_X(x)\, dx$

$= \int_{\mathbb{R}} \sqrt{\cos^2(tx) + \sin^2(tx)}\, f_X(x)\, dx = \int_{\mathbb{R}} f_X(x)\, dx = 1$ ✓

(P6) Inversion: If $\phi_X(t) \in L^1$ then $f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t)\, dt$

(P7) Levy's CDF formula

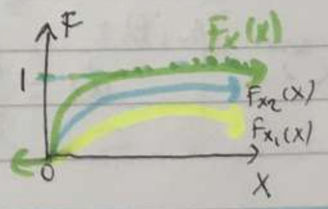
Probability

$$P(X \in [a, b]) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{itb}}{it} \phi_X(t)\, dt$$

(P8) Levy's Continuity Thm

Consider a sequence of rv's $X_1, X_2, \dots, X_n$. We define "X_n converges in distribution to X" (denoted $X_n \xrightarrow{d} X$) as:

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \quad \forall x \text{ "pointwise convergence"}$$



$$\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X.$$

If n large $\phi_{X_n}(t) \simeq \phi_X(t) \Rightarrow X_n \overset{d}{\simeq} X$

Define $M_X(t) = E[e^{tX}]$, the moment generating function (mgf)

Properties:

P_0 $M_X(0) = 1 \quad \forall X$

P_1 $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

P_2 $Y = aX + b$, $M_Y(t) = e^{tb} M_X(at)$

P_3 X_1, X_2 indep $T = X_1 + X_2$, $M_T(t) = M_{X_1}(t) M_{X_2}(t)$

P_4 $E[X^n] = M_X^{(n)}(0)$

It does not have P_5. Thus, sometime mgf's don't exist at all and sometimes it doesn't exist for certain values of t.

Chf's can do everything mgf's can do and more!! Thus, you don't need mgf's!

Ex 1 $X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0} \, dx$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x} \, dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - it)} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$

Ex 2 $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep of $X_2 \sim \text{Gamma}(\alpha, \beta)$, $T = X_1 + X_2$

$$\phi_T(t) \overset{P_2}{=} \phi_{X_1}(t) \phi_{X_2}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2}$$

$\overset{P_1}{\Rightarrow}$ $T \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$