Lecture 09 MATH621 10/05/2020

$$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 \sim f_{T_3}(t) = ?$$

$$= \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx = \int_0^\infty x \lambda^2 e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)}\, \mathbb{1}^{[0,\infty)}_{t-x \in Supp[X_3]}$$

then $x \le t$

$$= e^{-\lambda t} \lambda^3 \int_0^\infty x\, \mathbb{1}_{x \le t}\, dx = \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = Erlang(3, \lambda)$$

$$T_4 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4 \sim f_{T_4}(t) = ?$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in Supp[X_4]}\, dx = \int_0^\infty \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}} \lambda e^{-\lambda t} \cancel{e^{\lambda x}}\, \mathbb{1}_{t-x \in [0,\infty)}$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^\infty x^2\, \mathbb{1}_{x \le t}\, dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = Erlang(4, \lambda)$$

$$T_k = \sum_{i=1}^{k} X_i \sim Erlang(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!}\, \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_k] = [0, \infty)$, $\lambda \in (0, \infty)$, $k \in \mathbb{N}$

$$\begin{array}{ccc} Exp(\lambda) & \Longrightarrow & Erlang(k, \lambda) \\ \Updownarrow & & \Updownarrow \\ Geom(p) & \Longrightarrow & NegBin(k, p) \end{array}$$

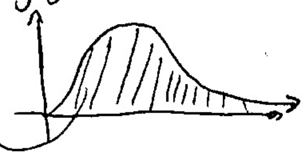
all discretes and continous waiting times.

I want to define the gamma family of functions. Beginning with the gamma function for x non-deg.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^a t^{x-1} e^{-t}\, dt + \int_a^\infty t^{x-1} e^{-t} dt = \gamma(x,a) + \Gamma(x,a)$$

↳ upper incomplete gamma
↳ Lower incomplete gamma

eg. $\Gamma(3) = \int_0^\infty t^2 e^{-t}\, dt$



$= 2$

$$\frac{\Gamma(x)}{\Gamma(x)} = 1 = \frac{\gamma(x,a) + \Gamma(x,a)}{\Gamma(x)} = \overbrace{\frac{\gamma(x,a)}{\Gamma(x)}}^{P(x,a)} + \overbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}^{Q(x,a)} = P(x,a) + Q(x,a) = 1$$

$P(x,a) \sim$ Lower Reg. Gamma Func.

$Q(x,a) \sim$ Upper Reg. Gamma Func.

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

Let $n \in \mathbb{N}$, $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \dots = (n-1)\dots(3)(2)(1)$
$= (n-1)!$

Let $x \in (0,\infty)$, $\Gamma(x) = (x-1)\Gamma(x-1) = \dots = (x-1)(x-2)\dots\Gamma(c)$ when c is $(0,1)$

The gamma func extends the factorial func to all positive #'s

$c > 0$

$$\int_0^\infty t^{x-1} e^{-ct}\, dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c} du = \frac{1}{c^x} \int u^{x-1} e^{-u} du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow \frac{du}{dt} = c \Rightarrow dt = \frac{1}{c} du$, $t = 0 \Rightarrow u$, $t \to \infty \Rightarrow u \to \infty$, $t = a$
$\Rightarrow u = ac$

Now lets integrate:

$$\int_0^a t^{x-1} e^{-ct} dt = \int \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c} du = \frac{1}{c^x} \int_0^{ac} u^{x-1} e^{-u} du = \frac{\gamma(x, ac)}{c^x}$$

$$\int_a^\infty t^{x-1} e^{-ct} dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$$

If $n \in \mathbb{N}$ lets derive!

$$\Gamma(n,a) = \int_a^\infty t^{n-1}\,dt\,e^{-t} = \int_a^\infty \underbrace{t^{n-1}}_{u}\,\underbrace{e^{-t}dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du =$$

$$v = \int dv = \int e^{-t}dt = -e^{-t}$$

$$\frac{du}{dt} = (n-1)t^{n-2} \Rightarrow (n-1)t^{n-2}dt$$

$$= \left[t^{n-1}(-e^{-t})\right]_a^\infty - \int -e^{-t}(n-1)t^{n-2}dt = e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}dt$$

$$a^{n-1}e^{-a} + (n-1)\Gamma(n-1,a) =$$

$$\cancel{a^{n-1}e^{-a} + (n-1)\left(e^{-a} + (n-2)\Gamma(n-2,a)\right)} = \cancel{e^{-a} + (n-1)\left(e^{-a} + (n-2)\left(e^{-a} + (n-3)\Gamma(n-3,a)\right)\right)}$$

$$= \cancel{e^{-a} + (n-1)\left(e^{-a} + (n-2)\left(e^{-a} + (n-3)\Gamma(n-3,a)\right)\right)}$$

$$= \cancel{e^{-a}\left(1 + (n-1)\left(1 + (n-2)\left(1 + (n-3)\Gamma(n-3,a)\right)\right)\right)}$$

$$a^{n-1}e^{-a} + (n-1)\Gamma(n-1,a) = e^{-a}\left(a^{n-1} + (n-1)\left(a^{n-2} + (n-2)\left(a^{n-3} + (n-3)\Gamma(n-3,a)\right)\right)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \frac{1}{(n-4)!}\Gamma(n-3,a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \dots + \frac{a^1}{1!} + \frac{a^0}{0!}\right) = e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

$$\Gamma(1,a) = \int_a^\infty e^{-t}dt = \left[-e^{-t}\right]_a^\infty = e^{-a}$$

Back to probability:

$$X \sim \text{Erlang}(k,\lambda) := \frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!}\mathbb{1}_{x \ge 0}$$

$$F_X(x) := P(X \le x) = \int_0^x \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!}dt = \frac{\lambda^k}{(k-1)!}\int_0^x t^{k-1}e^{-\lambda t}dt$$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)}\frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = \frac{\gamma(k,\lambda x)}{\Gamma(k)} = P(k,\lambda x)$$

$$1 - F_X(x) = 1 - P(k, \lambda x) = Q(k, \lambda x)$$

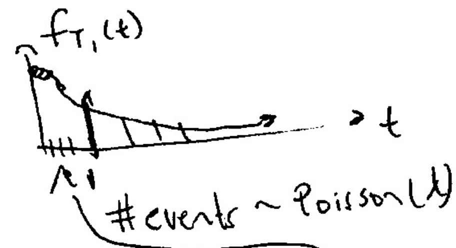
Lets say $X \sim \text{Poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$

CDF:

$$F_X(x) = P(X \le x) = \sum_{t=0}^{x} \frac{\lambda^t e^{-\lambda}}{t!} = e^{-\lambda} \sum_{t=0}^{x} \frac{\lambda^t}{t!} = e^{-\lambda} \frac{1}{x!} = \frac{\overbrace{1}^{\Gamma(x+1)}}{x!} \overbrace{e^{-\lambda} x! \sum_{t=0}^{x} \frac{\lambda^t}{t!}}^{\Gamma(x+1, \lambda)}$$

$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = Q(x+1, \lambda)$$

The relationship between the Erlang and the Poisson is known as the Poisson process.



$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1, \lambda)$

$P(T_1 > 1) = Q(1, \lambda)$

$N \sim \text{Poisson}(\lambda) \quad F_N(0) = P(N \le 0) = P(N = 0) = Q(1, \lambda)$