$X \sim \text{Multi}_{\underset{\hookrightarrow k=2}{2}}(n, \vec{p})$

$^{*}p_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 | X_2 = x_2) = \frac{p(x_1, x_2)}{p(x_2)}$

$\hookrightarrow = \text{Deg}(n - x_2)$

Last time $p(x_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

$$\star = \frac{\binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2}}$$

Define $J_n := \{0, 1, \ldots, n\}$

$$= \frac{\frac{\cancel{n!}}{x_1! \cancel{x_2!}} \mathbb{1}_{\overbrace{x_1+x_2=n}^{x_1=n-x_2}} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} \cancel{p_2^{x_2}}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \mathbb{1}_{x_2 \in J_n} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

Define: $\mathbb{1}^{\upsilon}_A = \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 \text{ if } A \\ \text{undefined if } A^c \end{cases}$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{\substack{=1 \text{ if} \\ x_1 = n-x_2}} \mathbb{1}_{x_1 = n-x_2} \underbrace{\mathbb{1}_{x_1 \in J_n}}_{\substack{=1 \text{ if} \\ x_1 = n-x_2}} \underbrace{p_1^{x_1+x_2=n}}_{\substack{=1 \text{ if} \\ x_1 = n-x_2}} \mathbb{1}^{\upsilon}_{x_2 \in J_n} = \text{Deg}(n-x_2) \mathbb{1}^{\upsilon}_{x_2 \in J_n}$$

$p(A|B) = \frac{p(A,B)}{p(B)}$; if $p(B) = 0 \Rightarrow$ <u>$p(A|B)$ is undefined</u>

Let's generalize this conditional probability a little bit:

$\vec{X} \sim \text{Multi}_k(n, \vec{p})$

$p_{\vec{X}_{-j}|X_j}(\vec{x}_{-j}, x_j) = \frac{p_{\vec{X}}(\vec{x})}{p_{X_j}(x_j)} = \text{Multi}_{k-1}(n - x_j, ?)$

this is the vector component w/o the j^{th} component.

$$= \frac{\text{Multi}_k(n, \vec{p})}{\text{Bin}(n, p_j)} = \frac{\binom{n}{x_1, \ldots, x_j, \ldots, x_k} p_1^{x_1} \cdots p_j^{x_j} \cdots p_k^{x_k}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-x_j}}$$

$$= \frac{\frac{\cancel{n!}}{x_1! \cdots \cancel{x_j!} \cdots x_k!} \mathbb{1}_{\overbrace{x_1+\ldots+x_j+\ldots+x_k=n}^{x_1+\ldots+x_{j-1}+x_{j+1}+\ldots=n-x_j}} \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_j \in J_n} \cdots \mathbb{1}_{x_k \in J_n} p_1^{x_1} \cdots p_j^{x_j} \cdots p_k^{x_k}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)!} \mathbb{1}_{x_j \in J_n} (1-p_j)^{n-x_j}}$$

Note $p_1 + \ldots + p_k = 1 \Rightarrow p_1 + \ldots + p_{j-1} + p_{j+1} + \ldots + p_k = 1 - p_j$, divide both sides by $1-p_j$

$$\Rightarrow \underbrace{\frac{p_1}{1-p_j}}_{p_1'} + \ldots + \underbrace{\frac{p_{j-1}}{1-p_j}}_{p_{j-1}'} + \underbrace{\frac{p_{j+1}}{1-p_j}}_{p_{j+1}'} + \ldots + \underbrace{\frac{p_k}{1-p_j}}_{p_k'} = 1$$

Let $n' = n - x_j$

Note: $n - x_j = x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_k$ o/t prob. 0.

$$= \frac{n'!}{x_1! \cdots x_{j-1}! \cdot x_{j+1}! \cdots x_k!} \mathbb{1}_{x_1+\ldots+x_{j-1}+x_{j+1}+\ldots+x_k=n'} \mathbb{1}_{x_1 \in J_n} \cdots \mathbb{1}_{x_{j-1} \in J_n} \mathbb{1}_{x_{j+1} \in J_n} \cdots \mathbb{1}_{x_k \in J_n} \cdot \underbrace{\frac{p_1^{x_1} \cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \cdots p_k^{x_k}}{(1-p_j)^{x_1+\ldots+x_{j-1}+x_{j+1}+\ldots+x_k}}}_{p_1'^{x_1} \cdots p_{j-1}'^{x_{j-1}} p_{j+1}'^{x_{j+1}} \cdots p_k'^{x_k}} \mathbb{1}^{\upsilon}_{x_j \in J_n}$$

$$= \text{Multi}_{k-1}(n', \vec{p}') \mathbb{1}^{\upsilon}_{x_j \in J_n}$$

$\vec{X} \sim \text{Multi}_k(n, \vec{p})$, what is $E[\vec{X}]$? $Var[\vec{X}]$?

Review from 241. Let $X_1, \dots, X_n$ be r.v.'s & $a, c \in \mathbb{R}$

-> $E[aX+c] = aE[X] + c$

$E[\sum X_i] = \sum E[X_i] = n\mu$ -> identically distributed

$E[\prod X_i] = \prod E[X_i] =$ -> if independent.

$\sigma^2 = Var[X] := E[(X-\mu)^2]$, $\sigma := SD[X] := \sqrt{Var[X]}$

$= E[X^2] - \mu^2$ standard deviation

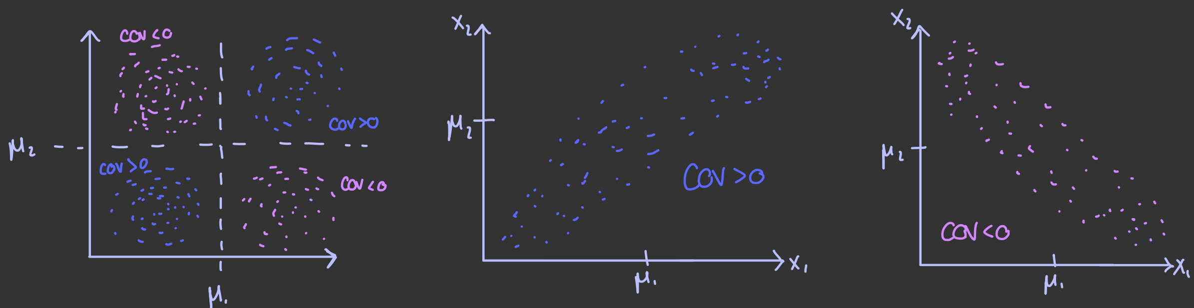
$$Var[X_1+X_2] = E[((X_1+X_2)-(\mu_1+\mu_2))^2]$$
$$= E[X_1^2+X_2^2+\mu_1^2+\mu_2^2+2X_1X_2-2X_1\mu_1-2X_1\mu_2-2X_2\mu_1-2X_2\mu_2+2\mu_1\mu_2]$$
$$= E[X_1^2]+E[X_2^2]+\mu_1^2+\mu_2^2+2E[X_1X_2]-2\mu_1^2-2\mu_1\mu_2-\cancel{2\mu_2\mu_1}-2\mu_2^2+\cancel{2\mu_1\mu_2}$$
$$= \sigma_1^2+\cancel{\mu_1^2}+\sigma_2^2+\cancel{\mu_2^2}+\cancel{\mu_1^2}+\cancel{\mu_2^2}+2E[X_1X_2]-\cancel{2\mu_1^2}-2\mu_1\mu_2-\cancel{2\mu_2^2}$$
$$= \sigma_1^2+\sigma_2^2+2(\underline{E[X_1X_2]-\mu_1\mu_2})$$
$$= \sigma_1^2+\sigma_2^2+2\underbrace{COV[X_1,X_2]}_{\sigma_{12}}$$

-> $COV[X_1,X_2]$: covariance of X_1 with X_2

$$= \sigma_1^2+\sigma_2^2$$

-> if X_1, X_2 independent

HW: $Cov[X_1,X_2] = E[(X_1-\mu_1)(X_2-\mu_2)]$



Covariance Rules:

$Cov[X, X] = \sigma^2$

$Cov[X_1, X_2] = Cov[X_2, X_1]$

$Cov[X_1+X_2+X_3] = Cov[X_1, X_3] + Cov[X_2+X_3]$

$Cov[a_1X_1, a_2X_2] = a_1a_2\sigma_{12}$

$Var[X_1+\dots+X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j]$

$\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}$, let $m = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$ -> $E[m] := \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \vdots \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$

$$\mathrm{Var}[\vec{X}] := E[\underbrace{\vec{X}\vec{X}^T}_{\text{Outer product}}] - \vec{\mu}\vec{\mu}^T = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_k] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2,X_k] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_k,X_1] & \mathrm{Cov}[X_k,X_2] & \cdots & \mathrm{Var}[X_k] \end{bmatrix}$$

$(k\times 1)(1\times k) = k\times k$

Variance - covariance (varcov) matrix and it is symmetric

If $X_1, \ldots, X_k$ are independent, what is the varcov matrix?

$\Sigma = \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_k^2\}$ → diagonal matrix

Rules about vector r.v. expectations

$$E[aX+\vec{c}] = \begin{bmatrix} a\mu_1+c_1 \\ a\mu_2+c_2 \\ \vdots \\ a\mu_k+c_k \end{bmatrix} = a\vec{\mu}+\vec{c}$$

$(\vec{V}_1^T\vec{V}_2)^T = \vec{V}_2^T\vec{V}_1 = \vec{V}_1^T\vec{V}_2$

$$E[\vec{a}^TX] = E[a_1X_1 + \ldots + a_kX_k] = a_1\mu_1 + \ldots + a_k\mu_k = \vec{a}^T\vec{\mu}$$