$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ From previous class $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$D = X_1 - X_2 \sim f_D(d) = ?$ (difference)

$D = \underbrace{X_1}_{X} + \underbrace{(-X_2)}_{Y} \sim ?$

$p_Y(y) = \dfrac{e^{-\lambda}\lambda^{-y}}{(-y)!}$ $\quad Supp[X] = \{0, 1, 2, \ldots\}$ $\quad Supp[Y] = \{\ldots, -2, -1, 0\}$

$Supp[D] = Supp[X+Y] = Supp[X] + Supp[Y] = \overset{\text{all integers}}{\mathbb{Z}}$

convolution formula for independent discrete rv's

$$p_D(d) = \sum_{x \in Supp[X]} p_X^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{\underbrace{d-x \in Supp[Y]}_{\substack{x \in \{d, d+1, d+2, \ldots\} \\ x-d \in \{0, 1, \ldots\}}}}$$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{e^{-\lambda}\lambda^x}{x!} \frac{e^{-\lambda}\lambda^{\overbrace{-(d-x)}^{x-d}}}{\underbrace{(-(d-x))!}_{(x-d)!}} \mathbb{1}_{d-x \in \{\ldots, -1, 0\}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x}{x!} \frac{\lambda^{x-d}}{(x-d)!} \mathbb{1}_{x \in \{d, d+1, \ldots\}}$$

let $x' = x - d \Rightarrow x = x' + d$

$$= e^{-2\lambda} \begin{cases} d > 0 & \displaystyle\sum_{x \in \{d, d+1, \ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x' \in \{0,1,\ldots\}} \frac{\lambda^{2(x'+d)-d}}{(x'+d)!\,(x'+d-d)!} = \sum_{x' \in \{0,1,\ldots\}} \frac{\lambda^{2x'+d}}{(x'+d)!\,x'!} \\ d \le 0 & \displaystyle\sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} = \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x+|d|}}{x!\,(x+|d|)!} \end{cases}$$

$d = |d|$

let $d' = -d = |d|$

$$I_{|d|}(2\lambda) := \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!\,(x+|d|)!}$$

Modified Bessel Function of the First kind (comes up in diff eq's)

$$= e^{-2\lambda} I_{|d|}(2\lambda)\mathbb{1}_{d \in \mathbb{Z}} = \text{Skellam}(\lambda, \lambda)$$

discovered in 1946

It's used to model point spreads in sports games, photo noise, etc

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$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \Rightarrow T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$\text{Bern}(\frac{1}{2})$ $\quad$ 1 2 3 . . . t

$$p_{X_1|T}(x, t) = \frac{p_{X_1,T}(x,t)}{p_T(t)} = \frac{p_{X_1,X_2}(x, t-x)}{p_T(t)} = \frac{p_{X_1}(x)\,p_{X_2}(t-x)}{p_T(t)}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!} \frac{\cancel{e^{-\lambda}}\lambda^{t-x}}{(t-x)!}}{\frac{\cancel{e^{-2\lambda}}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t = \text{Bin}\left(t, \frac{1}{2}\right)$$

$\left(\frac{\lambda}{2\lambda}\right)^t$

$B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$ $\quad p^{old}(x)$

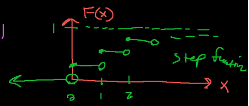
$$X_1 \sim \text{Geom}(p) := \underbrace{(1-p)^x p}_{p(x)} \mathbb{1}_{x \in \{0,1,\ldots\}}$$

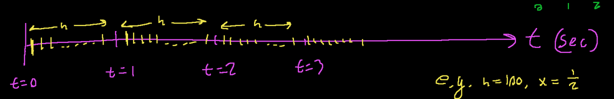
$b_1 \cdots b_7 = 0$

$\Updownarrow$

$x = 7$ $\quad X_1 > 7 \Rightarrow X_1 \in \{8, 9, 10, \ldots\}$

$$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = 1 - (1-p)^{x+1}$$





e.g. $n = 100$, $x = \frac{1}{2}$

Let there be n experiments in each second (time unit). x is in seconds...

$$p_{X_n}(x) = (1-p)^{nx} p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots\}}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx+1}$$

Let's put infinite experiments into every second (time unit), this is the limit as n goes to positive infinity. X_∞

And $p \to 0$ but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$ à la the Poisson, $\quad \lambda \in (0, \infty)$

$$p_{X_\infty}(x) = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n} \mathbb{1}_{x \in \{0, \frac{1}{n}, \ldots\}}$$

$Supp[X_\infty] = [0, \infty)$

$$= \Big(\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\Big)^x \lim_{n\to\infty}\frac{\lambda}{n} \lim_{n\to\infty} \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$$= e^{-\lambda x}(0)\mathbb{1}_{x \in [0,\infty)} = 0 \quad \forall x$$

This is not a PMF because $\displaystyle\sum_{x \in Supp[X_\infty]} p_{X_\infty}(x) = 0 \neq 1$

intuitive proof

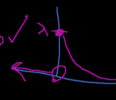
$$F_{X_\infty}(x) \overset{:= P(X_\infty \le x)}{=} \lim_{n \to \infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nx+1} = 1 - \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{e^{-\lambda x}} \cdot \cancel{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^1} = (1 - e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)}$$

Is this limiting CDF a legal CDF? If so, it must satisfy three conditions
(1) limit as x goes to negative infinity is zero
(2) limit as x goes to positive infinity is one
(3) increasing function i.e. its derivative is >= 0

(1) $\displaystyle\lim_{x \to -\infty} (1 - e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)} = 0$ ✓ $\quad$ since $\lambda > 0$

(2) $\displaystyle\lim_{x \to \infty} (1 - e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)} = 1 - \lim_{x\to\infty}\frac{1}{e^{\lambda x}} = 1$ ✓

(3) $\displaystyle\frac{d}{dx}\left[(1-e^{-\lambda x})\mathbb{1}_{x \in [0,\infty)}\right] = \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{old}(x)}\mathbb{1}_{x \in [0,\infty)}}_{f(x)} > 0$ ✓



$\Rightarrow F_{X_\infty}$ is a valid CDF... but of what rv?? A "continuous rv".

A continuous rv X has $Supp[X] \subseteq \mathbb{R}$ but $|Supp[X]| = |\mathbb{R}|$,

this size is known as "uncountable infinity" or the "size of the continuum". They also have no PMF, the P(X=x) is always zero for every x. But they have a CDF (continuous for the purposes of this class). And the derivative of the CDF is a very useful function, so it gets a special name which is the "probability density function" or just "density" (PDF) denoted f:

$$f(x) := F'(x), \quad P(X \in (a,b)) = \overbrace{P(X \le b)}^{F(b)} - \overbrace{P(X \le a)}^{F(a)} = \int_a^b f(x)\,dx$$

fundamental theorem of calculus

$f(x) \ge 0 \ \forall x$ (property of the CDF) $\Leftrightarrow Supp[X] = \{x : f(x) > 0\}$

$$\int_{\mathbb{R}} f(x)\,dx = 1 \qquad \int_{-\infty}^{\infty} f(x)\,dx = \overbrace{F(\infty)}^{1} - \overbrace{F(-\infty)}^{0} = 1 \text{ (properties of CDF)}$$

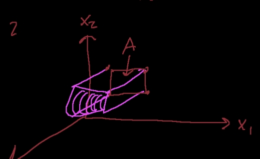
joint density function (JDF)

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_K}(x_K) = f(x_1) \cdot \ldots \cdot f(x_K)$$

all components continuous $\quad$ if $X_1, \ldots, X_K$ independent $\quad$ identically distr

$$\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{K} f_{\vec{X}}(\vec{x})\,dx_1 \cdots dx_K = 1$$

$K = 2$



$f_{X_1,X_2}(x_1,x_2)$

$$P(\vec{X} \in A) = \iint_A f_{\vec{X}}(\vec{x})\,dx_1\,dx_2$$

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exponential rv

$$X \sim \text{Exp}(\lambda) := \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f^{old}(x)} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}$$

support

$\lambda \in (0, \infty)$