10/05

$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$

$$f_{T_3}(t) = \int_{supp[T_2]} f_{X_1}^{old}(x) f_{X_3}^{old}(t-x) \mathbb{1}_{t-x \in supp[X_3]} dx$$

$$= \int_0^\infty x\lambda^2 e^{-\lambda x} \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{\underbrace{t-x\in(0,\infty)}_{x \leq t}} dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty x \mathbb{1}_{x \leq t} dx = \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t\in[0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t\in[0,\infty)} = \text{Erlang}(3,\lambda)$$

$$f_{T_4}(t) = \int_{supp[T_3]} f_{T_3}^{old}(x) f_{X_4}^{old}(t-x) \mathbb{1}_{t-x\in[0,\infty)} dx$$

$$= \int_0^\infty \frac{x^2}{2} \lambda^3 e^{-\lambda x} \lambda \overbrace{e^{-\lambda(t-x)}}^{e^{-\lambda t} e^{\lambda x}} \mathbb{1}_{t-x\in(0,\infty)} dx$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t x^2 dx\, \mathbb{1}_{t\in[0,\infty)} = \frac{t^3}{3\cdot 2} \lambda^4 e^{-\lambda t} \mathbb{1}_{t\in[0,\infty)} = \text{Erlang}(4,\lambda)$$

$$\sum_{i=1}^{k} X_i = T_k \sim \text{Eralng}(k,\lambda) := \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t\in[0,\infty)}$$

$supp[T_k] = [0,\infty)$

param. space $\lambda \in (0,\infty), k \in \mathbb{N}$

$$\text{Exp}(\lambda) = \text{Erlang}(1,\lambda) \qquad \sum_{i=1}^{k} \text{Exp}(\lambda) = \text{Erlang}(k,\lambda)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

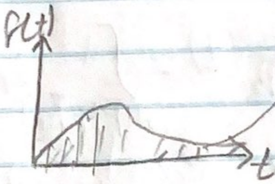
$$\text{Geom}(p) = \text{NegBin}(1,p) \qquad \sum_{i=1}^{k} \text{Geom}(p) = \text{NegBin}(k,p)$$

We will just do some pure math definitions. We will introduce the gamma family of functions. The "gamma function" is:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dt \qquad \text{e.g. } \Gamma(3) = \int_0^\infty \overbrace{t^2 e^{-t}}^{f(t)}\,dt = 2$$

We're only going to care about x being positive in this class

$$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t}\,dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\,dt}_{\Gamma(x,a)}$$



$a \in [0,\infty)$

$$\varphi(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1)$$

Lower regularized incomplete gemma function

$$P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1] \text{ proportion of the gamma function above } a$$

Is $\varphi(x,a) + P(x,a) = 1$

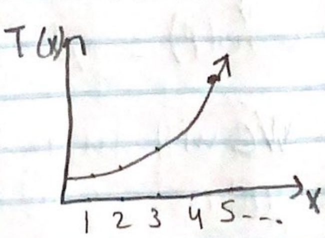
$\Gamma(1) := \int_0^\infty e^{-t}\,dt = 1$ This is the integral of the PDF for EXP(1) over its support

$\Gamma(x+1) = x\Gamma(x)$ Proved on the HW via integration by Parts.

$\Rightarrow \Gamma(2) = 1\Gamma(1) = 1\cdot 1,\ \Gamma(3) = 2\Gamma(2) = 2\cdot 1 = 2,\ \Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1$

for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

$$\Gamma(4.5) = 3.5\Gamma(3.5) = 3.5\cdot 2.5\,\Gamma(2.5) = 3.5\cdot 2.5\cdot 1.5\,\Gamma(1.5)$$
$$= 3.5\cdot 2.5\cdot 1.5\cdot 0.5\,\Gamma(0.5)$$



$$X \sim \text{Erlang}(\lambda) := \frac{x^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!} 1_{x\in[0,\infty)}$$

$$F_X(x) := P(X \le x) = \int_0^x \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!}\,dt = \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}} \overbrace{\int_0^x t^{k-1} e^{-\lambda t}\,dt}^{\frac{\gamma(k,\lambda x)}{\lambda^k}} = \frac{\lambda^k}{\Gamma(k)} \frac{\gamma(k,\lambda x)}{\lambda^k} = P(k,\lambda x)$$

Let's do some more boring calculus... for $c>0$,

$$\int_0^\infty t^{x-1} e^{-ct}\,dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\,du = \frac{1}{c^x}\int u^{x-1}e^{-u}\,du = \frac{\Gamma(x)}{c^x}$$

Let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c}du$, $t=0 \Rightarrow u=0$, $t\to\infty \Rightarrow u\to\infty$, $t=a \Rightarrow u=ac$,

$$\int_0^a t^{x-1} e^{-ct}\,dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u}\frac{1}{c}\,du = \frac{1}{c^x}\int_0^{ac} u^{x-1}e^{-u}\,du = \frac{\gamma(x,ac)}{c^x}$$

$$\int_a^\infty t^{x-1}e^{-ct}\,dt = \int_0^\infty \cdots\,dt - \int_0^a \cdots\,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x)-\gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$$

If $n \in \mathbb{N}$... $du = (n-1)t^{n-2}dt$, $v = -e^{-t}$

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}\,dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du = [-t^{n-1}e^{-t}]_a^\infty - \underbrace{\int_a^\infty -e^{-t}(n-1)t^{n-2}\,dt}_{\text{Iterate}}$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}\,dt = a^{n-1}e^{-a} + (n-1)\,\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\,\Gamma(n-2,a)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \cdots + \cdots (n-1)!\,\Gamma(1,a)\right)$$

$$e^{-a} = [-e^{-t}]_a^\infty = \int_a^\infty e^{-t}\,dt = \Gamma(1,a)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \cdots + \frac{a^0}{0!}\right) = e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

$$X \sim \text{Poisson}(\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} 1_{x\in\mathbb{N}_0}$$

$$F_X(x) = P(X\le x) = \sum_{t=0}^{x} \frac{e^{-\lambda}\lambda^t}{t!} = e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^t}{t!} = \frac{1}{\underbrace{x!}_{\Gamma(x+1)}} \overbrace{e^{-\lambda} x! \sum \frac{\lambda^t}{t!}}^{\Gamma(x+1,\lambda)}$$

$$= \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1,\lambda) \Rightarrow F_{T_1}(t) = P(1,\lambda t)$

$P(T_1 > 1) = 1 - F_{T_1}(1) = 1 - P(1,\lambda) = Q(1,\lambda)$

$N \sim \text{Poisson}(\lambda)$, $P(N=0) = F_N(0) = Q(1,\lambda)$

The first example of "Poisson Process, the link between waiting times in the Erlang and probability of events in a Poisson.

$f_{T_1}(t) = \text{Exp}(\lambda) = \text{Erlang}(1,\lambda)$

t (sec)

# of events in time between 0, 1 seconds is Poisson (lambda) distributed.