M368 10/5

$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$ $\quad e^{-\lambda t} e^{\lambda x}$

$$f_{T_3}(t) = \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx = \int_0^\infty x\lambda^2 e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)}\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty x\, \mathbb{1}_{x \le t}\, dx = \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t \in [0,\infty)} = \frac{t^2}{2} \lambda^3 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = Erlang(3,\lambda)$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in [0,\infty)}\, dx = \int_0^\infty \frac{x^2}{2} \lambda^3 e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)}\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t x^2\, dx\, \mathbb{1}_{t \in [0,\infty)} = \frac{t^3}{3 \cdot 2} \lambda^4 e^{-\lambda t}\, \mathbb{1}_{t \in [0,\infty)} = Erlang(4,\lambda)$$

$$\sum_{i=1}^{K} X_i = T_K \sim Erlang(K,\lambda) := \frac{t^{K-1} \lambda^K e^{-\lambda t}}{(K-1)!}\, \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_K] = [0,\infty)$

param space $\lambda \in (0,\infty)$, $K \in \mathbb{N}$

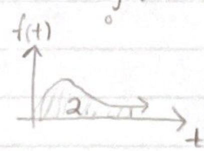
$Exp(\lambda) = Erlang(1,\lambda) \qquad \sum_{i=1}^{K} Exp(\lambda) = Erlang(K,\lambda)$

$\Updownarrow \qquad \Updownarrow$

$Geom(p) = NegBin(1,p) \qquad \sum_{i=1}^{K} Geom(p) = NegBin(K,p)$

Some pure math defs, introducing gamma family of functions.

"Gamma Function:" $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$, eg: $\Gamma(3) = \int_0^\infty t^2 e^{-t}\, dt = 2$



$$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)}$$ upper incomplete gamma func.

lower incomplete...

P $Q(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1)$ proportion of the gamma function below a.

Lower regularized incomplete gamma function.

Q $P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1]$ proportion of the gamma function above a.

$Q(x,a) + P(x,a) = 1$

$\Gamma(1) := \int_0^\infty e^{-t}\,dt = 1$ this is the integral of the pdf for Exp(1) over its supp.

$\Gamma(x+1) = x\Gamma(x)$ proved on HW via int. by parts

$\Rightarrow \Gamma(2) = 1\Gamma(1) = 1 \cdot 1,\quad \Gamma(3) = 2\Gamma(2) = 2\cdot 1 = 2,\ \Gamma(4) = 3\Gamma(3) = 3\cdot 2\cdot 1 = 6$

for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

$\Gamma(4.5) = 3.5\,\Gamma(3.5) = \ldots$

The gamma function is an "extension" of the factorial function valid for all positive numbers.

$X \sim Erlang(\lambda) = \frac{x^{k-1}\lambda^k e^{-\lambda k}}{(k-1)!}\mathbb{1}_{x\in[0,\infty)}$

$F_X(x) := P(X \le x) = \int_0^x \frac{t^{k-1}\lambda^k e^{-\lambda t}}{(k-1)!}\,dt = \underbrace{\frac{\lambda^k}{(k-1)!}}_{\Gamma(k)} \overbrace{\int_0^x t^{k-1}e^{-\lambda t}\,dt}^{\frac{\gamma(k,\lambda x)}{\lambda^k}} = \frac{\lambda^k}{\Gamma(k)}\frac{\gamma(k,\lambda x)}{\lambda^k} = P(k,\lambda x)$

More calculus: for $c>0$,

$\int_0^\infty t^{x-1}e^{-ct}\,dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u}\frac{1}{c}\,du = \frac{1}{c^x}\int_0^\infty u^{x-1}e^{-u}\,du = \frac{\Gamma(x)}{c^x}$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c}du$, $t=0 \Rightarrow u=0$, $t\to\infty \Rightarrow u\to\infty$, $t=a \Rightarrow u=ac$

$\int_0^a t^{x-1}e^{-ct}\,dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}}e^{-u}\frac{1}{c}\,du = \frac{1}{c^x}\int_0^{ac} u^{x-1}e^{-u}\,du = \frac{\gamma(x,ac)}{c^x}$

$\int_a^\infty t^{x-1}e^{-ct}\,dt = \int_0^\infty \cdots\,dt - \int_0^a \cdots\,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x} = \frac{\Gamma(x)-\gamma(x,ac)}{c^x} =$

More calculus: if $n \in \mathbb{N}$...

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t}\,dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du = \left[t^{n-1}\cdot(-e^{-t})\right]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}\,dt$$

$$du = (n-1)t^{n-2}\,dt \qquad v = -e^{-t}$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}\,dt = a^{n-1}e^{-a} + (n-1)\underbrace{\Gamma(n-1,a)}_{\text{iterate}}$$

$$= a^{n-1}e^{-a} + (n-1)\Big(a^{n-2}e^{-a} + (n-2)\overbrace{\Gamma(n-2,a)}^{\text{iterate}}\Big)$$

$$\int_a^\infty e^{-t}\,dt = [-e^{-t}]_a^\infty = e^{-a}$$

$$\| $$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \dots + (n-1)!\,\Gamma(1,a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \dots + \frac{a^0}{0!}\right) = e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

~

$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!}\mathbb{1}_{x\in\mathbb{N}_0}$$

$$F_X(x) := P(X \le x) = \sum_{t=0}^{x}\frac{e^{-\lambda}\lambda^t}{t!} = e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^t}{t!} = \frac{1}{\underbrace{x!}_{\Gamma(x+1)}}\overbrace{e^{-\lambda}x!\sum_{t=0}^{x}\frac{\lambda^t}{t!}}^{\Gamma(x+1,\lambda)}$$

$$= \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

~

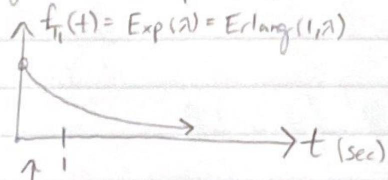
$$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1,\lambda) \Rightarrow F_{T_1}(t) = P(1,\lambda t)$$

$$P(T_1 > 1) = 1 - F_T(1) = 1 - P(1,\lambda) = Q(1,\lambda)$$

$$\|$$

$$N \sim \text{Poisson}(\lambda),\quad P(N=0) = F_N(0) = Q(1,\lambda)$$

the first example of the "poisson process", the link between waiting times in the Erlang and the probability of events in a Poisson.



λ of events in time btwn 0, 1 secs is Poisson(λ) distributed