

Lecture 07

09/29/2020.

$X_1, X_2 \sim \text{Poisson}(\lambda)$ from previous class

① $D = X_1 - X_2 \sim ?$

$$P_Y(y) = \frac{e^{-\lambda} \lambda^{-y}}{(-y)!}$$

$X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$\text{Supp}[X] = [0, 1, 2, \dots]$

② $D = X_1 + (-X_2) \sim ?$ $\text{Supp}[X+Y] = \text{Supp}[X] + \text{Supp}[Y] = 2\text{Supp}[Y] = \dots, -2, -1, 0]$

$$P_T(t) = \sum_{x \in \text{Supp}[X]} P_X(x) P_Y(d-x) \mathbb{1}_{d-x \in \text{Supp}[Y]}$$

$$= \sum_{x \in [0, 1, \dots, 3]} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{-(d-x)} e^{-\lambda}}{(-(d-x))!} \mathbb{1}_{d-x \in [0, 1, \dots, 3]}$$

$$= e^{-2\lambda} \sum_{x \in [0, 1, \dots, 3]} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \in [d, d+1, \dots, 3]}$$

$$= e^{-2\lambda} \begin{cases} \text{if } d \leq 0 & \sum_{x \in [0, 1, \dots, 3]} \frac{\lambda^{2x-d}}{x!(x-d)!} \\ \text{if } d > 0 & \sum_{x \in [d, d+1, \dots, 3]} \frac{\lambda^{2x-d}}{x!(x-d)!} \end{cases}$$

$$\text{let } x' = x - d \Leftrightarrow x = x' + d$$

$$\textcircled{1} = \sum_{x \in [0, 1, \dots, 3]} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$$

$$\textcircled{2} \stackrel{d=|d|}{=} \sum_{x \in [0, 1, \dots, 3]} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$$

$$= (2\lambda)^{|d|} \sum_{x \in [0, 1, \dots, 3]} \frac{\binom{2\lambda}{x}}{x!(x+|d|)!}$$

Modified Bessel Function of the First kind (It's a solution to a famous differential equation)

Skellam ($\lambda, -\lambda$)

this is used to model point spreads in sport games, photon noise,

$x_1, x_2 \sim \text{Poisson}(\lambda)$, $T = x_1 + x_2 \sim \text{Poisson}(2\lambda)$

$$P_{x_1, T}(x, t) = \frac{P_{x_1, T}(x, t)}{P_T(t)} = \frac{P_{x_1, x_2}(x, t-x)}{P_T(t)}$$

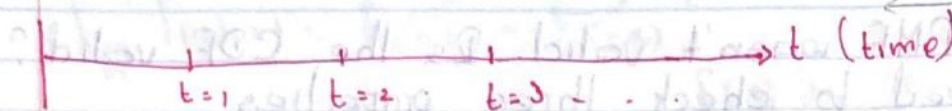
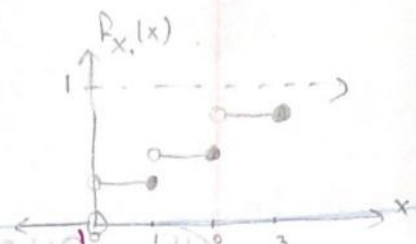
$$= \frac{P_{x_1}(x) P_{x_2}(t-x)}{P_T(t)} = \frac{\frac{e^{-\lambda} \lambda^x}{x!} \frac{e^{-\lambda} \lambda^{t-x}}{(t-x)!}}{\frac{e^{-2\lambda} (2\lambda)^t}{t!}}$$

$$= \frac{t!}{x!(t-x)!} \frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x} \left(\frac{1}{2}\right)^t = \text{Bin}(t, 1/2)$$

$X_1 \sim \text{Geom}(p) := (1-p)^x p \mathbb{1}_{x \in \{0, 1, \dots\}}$, $\text{Supp}[X_1] = \{0, 1, \dots\}$

$F_{X_1}(x) := P(X_1 \leq x) = 1 - P(X_1 > x) = (1 - (1-p)^x) \mathbb{1}_{x \in \{0, 1, \dots\}}$

a solution to a famous differential equation (it's a modified Bessel function of the first kind)



In every "second", let's do n iid Bernoulli(p) experiments.



Let's call the resulting geometric rv X_n and it's unit of realization is t .

$$P_{X_n}(x) = (1-p)^{nx} p \mathbb{1}_{x \in \{0, 1/n, 2/n, \dots\}, 1 + 1/n, 1 + 2/n, \dots, 2, \dots\}$$

$$F_{X_n}(x) = 1 - (1-p)^{nx}$$

let $n \rightarrow \infty$, $p \rightarrow 0$ but $\lambda = np$ where $\lambda \in (0, \infty)$
 $\Rightarrow P = \lambda/n$ same as Poisson

$$P_{X_n}(x) = \lim_{n \rightarrow \infty} (1 - \lambda/n)^{nx} \lambda/n \mathbb{1}_{x \in \{0, 1/n, \dots\}}$$

$$= \underbrace{\left(\lim_{n \rightarrow \infty} (1 - \lambda/n)^n \right)^x}_{e^{-\lambda x}} \cdot \underbrace{\lim_{n \rightarrow \infty} \lambda/n}_{0} \cdot \underbrace{\lim_{n \rightarrow \infty} \mathbb{1}_{x \in \{0, 1/n, \dots\}}}_{\mathbb{1}_{x \in [0, \infty)}} = 0 \quad \forall x!$$

$$F_{X_n}(x) = \lim_{n \rightarrow \infty} F_{X_n}(x) \quad \text{Supp}[X_n] = [0, \infty)$$

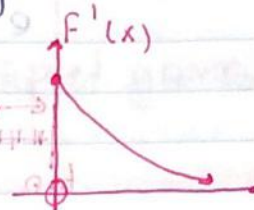
$$= \lim_{n \rightarrow \infty} 1 - (1 - \lambda/n)^{nx} = 1 - e^{-\lambda x} \mathbb{1}_{x \in [0, \infty)}$$

Not a valid PMF!

The PMF wasn't valid. Is the CDF valid? If so we need to check three properties:

- (1) It's 0 as x goes to negative infinity;
- (2) It's 1 as x goes to positive infinity and
- (3) It's an increasing function.

$$(1) \lim_{x \rightarrow -\infty} (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0, \infty)} = 0$$



$$(2) \lim_{x \rightarrow \infty} (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0, \infty)} = 1 - \lim_{x \rightarrow \infty} e^{-\lambda x} = 1$$

$$(3) d/dx [(1 - e^{-\lambda x}) \mathbb{1}_{x \in [0, \infty)}] = \lambda e^{-\lambda x} \mathbb{1}_{x \in [0, \infty)} \geq 0$$

\Rightarrow Valid CDF!

We now have a continuous rv X . Continuous rv's have the following properties:

$|\text{Supp}[X]| = |\mathbb{R}|$ uncountable infinity
(the size of the continuum)

They do not have PMF's (because the probability of the rv being at any specific number is zero) but they do have CDF's.

The derivative of the CDF is a very useful function, it is called the probability density function (PDF) denoted $f(x)$. (Note: discrete rv's do not have PDF's)

$$f(x) = F'(x), \quad P(X \in [a, b]) = P(X \leq b) - P(X \leq a)$$

by fundamental thm. calculus $\Rightarrow \int_a^b f(x) dx = 1$

$$\int_{\mathbb{R}} f(x) = 1 = \cancel{F(x)}^1 - \cancel{f(-\infty)}^0$$

$f(x) \geq 0$ since CDF's are increasing functions.

$$\Rightarrow \text{Supp}[X] = \{x : f(x) > 0\}$$

$$X_1, \dots, X_n \stackrel{\text{indep}}{\Rightarrow} f_{X_1, \dots, X_n}(x_1, \dots, x_n) \stackrel{\text{Joint density func.}}{=} \prod_{i=1}^n f_{X_i}(x_i)$$

$$X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{\substack{f_{\text{old}}(x) \\ f(x)}} \mathbb{1}_{x \in [0, \infty)}, F(x) = (1 - e^{-\lambda x}) \mathbb{1}_{x \in [0, \infty)}$$

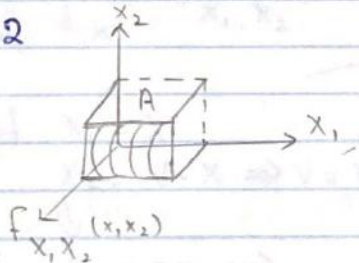
Exponential rv

$$\lambda \in (0, \infty)$$

its parameter space

$$\bar{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\bar{X}}(\bar{x}) \quad \underbrace{\int_{\mathbb{R}} \dots \int_{\mathbb{R}}}_{k} f_{\bar{X}}(\bar{x}) dx_1 \dots dx_k = 1$$

$k=2$



$$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2) dx_1 dx_2$$