10/05/2020
Lecture 09
Math 621

$T_3 = X_1 + X_2 + X_3 = T_2 + X_3 \sim f_{T_3}(t) = ?$

$$f_{T_3}(t) = \int_{Supp[X_1]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx$$

$$= \int_0^\infty x \lambda^2 \cancel{e^{-\lambda x}} \lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t}\cancel{e^{\lambda x}}} \mathbb{1}_{\overbrace{t-x \in [0,\infty)}^{x \leq t}}\, dx$$

$$= \lambda^3 e^{-\lambda t} \int_0^\infty x\, \mathbb{1}_{x \leq t}\, dx = \lambda^3 e^{-\lambda t} \int_0^t x\,dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$$

$$f_{T_4}(t) = \int_{Supp[T_4]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \int_0^\infty \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}} \lambda \underbrace{e^{-\lambda(t-x)}}_{e^{-\lambda t}\cancel{e^{\lambda x}}} \mathbb{1}_{t-x \in [0,\infty)}\, dx$$

$$= \lambda^4 e^{-\lambda t} \frac{1}{2} \int_0^t x^2\,dx\, \mathbb{1}_{t \in (0,\infty)}$$

$$= \frac{t^3}{3 \cdot 2} \lambda^4 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$$

$$\sum_{i=1}^{k} X_i = T_k \sim \text{Erlang}(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in [0,\infty)}$$

$Supp[T_k] = [0, \infty)$

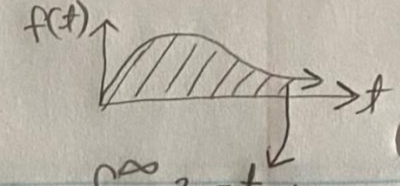
Param Space $\lambda \in (0,\infty)$, $k \in \mathbb{N}$

$\text{Exp}(\lambda) = \text{Erlang}(1, \lambda)$; $\sum_{i=1}^{k} \text{Exp}(\lambda) = \text{Erlang}(k, \lambda)$

$\Updownarrow$

$\text{Geom}(p) = \text{NegBin}(1, p)$ $\sum_{i=1}^{k} \text{Geom}(p) = \text{NegBin}(k, p)$

we will just do some pure math defination. introduce the gamma family of functions. "The gamma function" is:



$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$ e.g $\Gamma(3) = \int_0^\infty \underbrace{t^2 e^{-t}}_{f(t)}\, dt = 2$

we are going to care about x being positive in this class.

$\Gamma(x) = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)}$; $a \in [0, \infty)$

Lower incomplete gamma function | Upper incomplete gamma function

$Q(x,a) := \frac{\gamma(x,a)}{\Gamma(x)} \in [0,1)$ Proportion of the gamma function below a.

Lower regularized incomplete gamma function.

$P(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \in (0,1]$ Proportion of the gamna function above a.

Is $Q(x,a) + P(x,a) = 1$ (total 100% of the area)

$\Gamma(1) := \int_0^\infty e^{-t}\, dt = 1$ This is a intergal of the PDF for Exp(1) over it's support.

$\Gamma(x+1) = x\Gamma(x)$ Proved on the HW via integration by Par

$\Rightarrow \Gamma(2) = 1\,\Gamma(1) = 1 \cdot 1 = 1$

$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1 = 2$

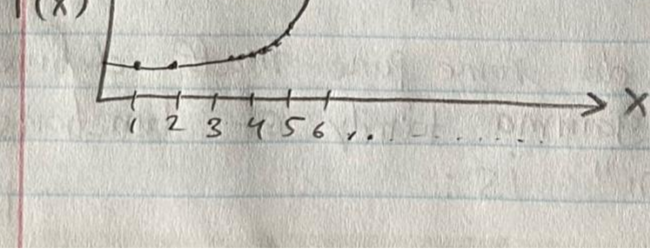
$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2$ ....

for $n \in \mathbb{N}$, $\boxed{\Gamma(n) = (n-1)!}$

$\Gamma(4.5) = 3.5\Gamma(3.5) = (3.5)(2.5)\Gamma(2.5) = 3.5 \cdot 2.5 \cdot 1.5\Gamma(1.5)$

$= 3.5.2.5.1.5.0.5.\Gamma(0.5).$

The gamma function is an "extension" if the factorial function valid for all positive numbers.



$$X \sim \text{Erlang}(\lambda) := \frac{x^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!}\mathbb{1}_{x\in[0,\infty)} \quad \frac{\gamma(k,\lambda x)}{\lambda^k}$$

$$F_X(x) := P(X = x) = \int_0^x \frac{t^{k-1}\lambda^k e^{-\lambda x}}{(k-1)!}dt = \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}}\int_0^{\infty} t^{k-1}e^{-\lambda t}\,dt =$$

Let's do some more Calc... for $c > 0$

$$\int_0^\infty t^{x-1}e^{-ct}\,dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}}e^{-u}\frac{1}{c}du = \frac{1}{c^x}\int_0^\infty u^{x-1}e^{-u}du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c}\,dt\,du$, $t=0 \Rightarrow u=0$, $t\to\infty \Rightarrow u\to\infty$

$$\int_0^a t^{x-1}e^{-ct}\,dt = \int_0^{ac}\frac{u^{x-1}}{c^{x-1}}e^{-u}\frac{1}{c}du = \frac{1}{c^x}\int_\infty^{ac}u^{x-1}e^{-u}du = \frac{\gamma(x,ac)}{c^x}$$

$$\int_a^\infty t^{x-1}e^{-ct}dt = \int_0^\infty '' \,dt - \int_0^a '' \,dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x,ac)}{c^x}$$

$$= \frac{\Gamma(x) - \gamma(x,ac)}{c^x} = \frac{\Gamma(x,ac)}{c^x}$$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)}\frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = P(k,\lambda x)$$

If $n \in \mathbb{N}$..

$du = (n-1)t^{n-2}dt$

$v = -e^{-t}$

$$\Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u}\underbrace{e^{-t}dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\,du =$$

$$= \left[-t^{n-1}e^{-t}\right]_a^\infty - \int_0^\infty -e^{-t}(n-1)t^{n-2}dt$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}dt = a^{n-1}e^{-a} + (n-1)\overbrace{\Gamma(n-1,a)}^{\text{iterate....}}$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\overbrace{\Gamma(n-2,a)}^{\text{iterate...}}\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + \dots + (n-1)!\,\Gamma(1,a)\right)$$

$$e^{-a} = \left[-e^{-t}\right]_a^\infty = \int_a^\infty e^{-t}dt =$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \dots + \frac{a^0}{0!}\right)$$

$$= e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!}$$

$$X \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^{x}}{x!}\,\mathbb{1}_{x\in\mathbb{N}_0}$$

$$F_X(x) := P(X \leqslant x) = \sum_{t=0}^{x} \frac{e^{-\lambda}\lambda^{t}}{t!} = e^{-\lambda}\sum_{t=0}^{x}\frac{\lambda^{t}}{t!}$$

$$= \underbrace{\frac{1}{x!}}_{\Gamma(x+1)}\underbrace{e^{-\lambda}\,x!\sum_{t=0}^{x}\frac{\lambda^{t}}{t!}}_{\Gamma(x+1,\lambda)} = \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

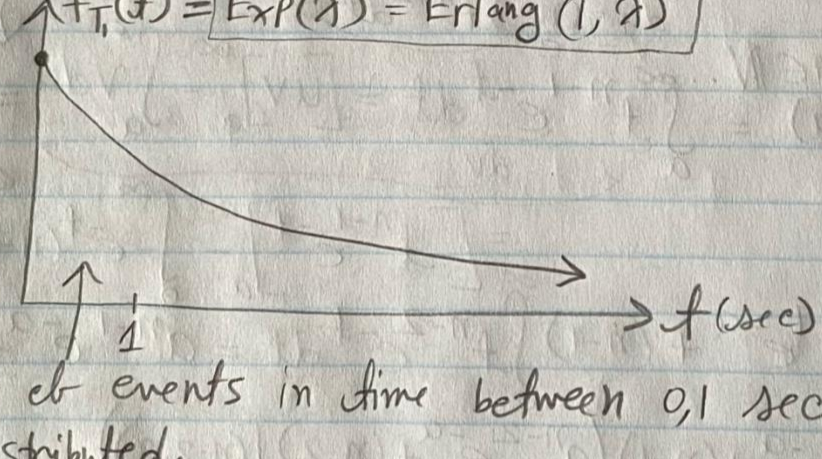
$$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1,\lambda) \Rightarrow F_{T_1}(t) = P(1,\lambda t)$$

$$P(T_1 > 1) = 1 - F_{T_1}(1) = 1 - P(1,\lambda) = Q(1,\lambda)$$

$$N \sim \text{Poisson}(\lambda),\ P(N=0) = F_N(0) = Q(1,\lambda)$$

The first example of the "Poisson Process", the link between waiting times in the Erlang and the probability of events in a Poisson.

$$f_{T_1}(t) = \boxed{\text{Exp}(\lambda) = \text{Erlang}(1,\lambda)}$$



# of events in time between 0,1 second is Poisson(λ) distributed.