Lecture 20 (1)

Math 621

11-23-2020

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{S^2 \cdot \frac{\sigma^2}{n-1} \cdot \frac{n-1}{\sigma^2}}}$$

$$= \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}} = \frac{\overbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}^{Z}}{\sqrt{\frac{\overbrace{\frac{n-1}{\sigma^2}S^2}^{\chi^2_{n-1}}}{n-1}}} \sim T_{n-1}$$

Due to Cochran's thm, we know $\bar{X}$ and s^2 are independent

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Multivariate Normal Distribution (MVN)

$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$, $E[\vec{Z}] = \vec{0}_n$

$Var[\vec{Z}] = \underbrace{I_n}_{\text{identity matrix}}$

$$\vec{Z} \sim f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f_Z(z_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum z_i^2}$$

$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N_n(\vec{0}, I)$$

$= $ Standard MVN.

$\vec{X} = \vec{Z} + \vec{\mu}$, $\vec{\mu} \in \mathbb{R}^n$, $E[\vec{X}] = \vec{\mu}$,

$Var[\vec{X}] = I_n$

$\Rightarrow \vec{X} \sim N_n(\vec{\mu}, I)$

(2) $\vec{X} = A\vec{Z} = \begin{bmatrix} Z_1 \\ Z_1 + Z_2 \\ \vdots \\ Z_1 + \cdots + Z_n \end{bmatrix} \begin{matrix} \sim N(0,1) \\ \sim N(0,2) \\ \vdots \\ \sim N(0,n) \end{matrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots \\ 1 & 1 & 1 & \cdots & & 1 \end{bmatrix}$

but the components are dependent.

e.g. $\text{cov}[X_1, X_2] = \text{cov}[Z_1, Z_1 + Z_2]$

$= \underbrace{\text{cov}[Z_1, Z_1]}_{1} + \underbrace{\text{cov}[Z_1, Z_2]}_{0}$

$= 1 \underset{\text{means}}{\Rightarrow} X_1, X_2$ dependent.

let's derive a general formula for the Variance-Covariance matrix of A (an $n \times n$ matrix of scalars) times a random vector X of dim n:

$Var[A\vec{X}] = E[\underbrace{(A\vec{X})(A\vec{X})^T}_{A\vec{X}\vec{X}^T A^T}] - E[A\vec{X}]E[A\vec{X}]^T$

$= A E[\vec{X}\vec{X}^T]A^T - A E[\vec{X}]\underbrace{(A E[\vec{X}])^T}_{E[\vec{X}]^T A^T}$

$= A \underbrace{(E[\vec{X}\vec{X}^T] - E[\vec{X}]E[\vec{X}]^T)}_{\Sigma = Var[\vec{X}]} A^T$

Σ sigma

$= A \Sigma A^T$

Σ Sigma

$\vec{X} = A\vec{Z}$, $Var[\vec{X}] = A I_n A^T = AA^T$

Conjecture: $\vec{X} \sim N(\vec{0}, AA^T)$

$\vec{X} = A\vec{Z} + \vec{\mu}$, $A \in \mathbb{R}^{n \times n}$, $\vec{\mu} \in \mathbb{R}^n$,

$\quad \parallel$

$g(\vec{Z})$, $\vec{X} \sim f_{\vec{X}}(\vec{x}) = ?$

$h(\vec{X}) = \vec{Z}$, where hopefully g, h are inverses

$h(\vec{x}) = A^{-1}(\vec{x} - \vec{\mu}) \Rightarrow$ in order for the inverse to exist... A has to be invertible

Let $A^{-1} = B$

So, $h(\vec{x}) = B(\vec{x} - \vec{\mu}) = B\vec{x} - B\vec{\mu}$

$$= \begin{bmatrix} \vec{b}_1 \vec{x} - \vec{b}_1 \vec{\mu} \\ \vec{b}_2 \vec{x} - \vec{b}_2 \vec{\mu} \\ \vdots \\ \vec{b}_n \vec{x} - \vec{b}_n \vec{\mu} \end{bmatrix} \begin{matrix} = h_1 \\ = h_2 \\ \\ = h_n \end{matrix}$$

(Jacobian)

$$J_n = \det \begin{bmatrix} \partial h_1/\partial x_1 & \cdots & \partial h_1/\partial x_n \\ \vdots & & \\ \partial h_n/\partial x_1 & \cdots & \partial h_n/\partial x_n \end{bmatrix}$$

$$= \det \overbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}^{B} = \det[A^{-1}] = \frac{1}{\det[A]}$$

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$DD^{-1} = I$

$\Rightarrow \det[DD^{-1}] = \det[I] = 1$

$\Rightarrow \det[D] \det[D^{-1}] = 1$

$$f_{\vec{X}}(\vec{x}) = f_{\vec{Z}}(h(\vec{x}))\,|J_n|$$

$$= \frac{1}{|\det[A]|} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}(A^{-1}(\vec{x}-\mu))^T \cdot}$$

$A^{-1}(\vec{x}-\vec{\mu})$

$(\vec{x}-\mu)^T (A^{-1})^T A^{-1}(\vec{x}-\vec{\mu})$

Math 231

$DD^{-1} = I \Rightarrow (DD^{-1})^T = I^T = I$

$\Rightarrow (D^{-1})^T D^T = I \Rightarrow (D^{-1})^T = (D^T)^{-1}$

$(CD)^{-1} = D^{-1} C^{-1}$

$$= \frac{1}{\sqrt{(2\pi)^n \det[A]^2}} e^{-\frac{1}{2}(\vec{x}-\mu)^T \underbrace{(A^T)^{-1} A^{-1}}_{(AA^T)^{-1}} (\vec{x}-\vec{\mu})}$$

Let $\Sigma = AA^T = \mathrm{Var}[\vec{x}]$

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$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T]$
$= \det[A]\det[A] = \det[A]^2$

So, $\dfrac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\mu)^T \Sigma^{-1}(\vec{x}-\vec{\mu})}$

$= N_n(\vec{\mu}, \Sigma)$, you need Σ sigma to be invertible

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A little bit of multivariate characteristic functions:

$\phi_{\vec{X}}(\vec{t}) = E[e^{i\vec{t}^T\vec{X}}] = E[e^{i(t_1X_1 + \cdots + t_nX_n)}]$

$= E[e^{it_1X_1} \cdot \cdots \cdot e^{it_nX_n}]$

if $X_1, \ldots, X_n$ independent

$= E[e^{it_1X_1}] \cdot \cdots \cdot E[e^{it_nX_n}]$

$= \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) \cdot \cdots \cdot \phi_{X_n}(t_n)$

(P_0) $\phi_{\vec{X}}(\vec{0}) = E[e^{i\vec{0}^T\vec{X}}] = 1$

(P_1) If two chf's are equal then the two rv's are equal in distribution

(P_2) $\vec{Y} = A\vec{X} + \vec{b}$, $A \in \mathbb{R}^{m\times n}$, $\vec{b} \in \mathbb{R}^m$,

$\vec{X}$ in dimension n

$\Rightarrow \vec{Y}$ in dim m.

$\phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}^T(A\vec{X}+\vec{b})}]$

$= E[e^{i\underbrace{\vec{t}^TA}_{(A^T\vec{t})^T}\vec{X}}\, e^{i\vec{t}^T\vec{b}}]$

$= e^{i\vec{t}^T\vec{b}} \, E\left[e^{i(A^T\vec{t})^T\vec{x}}\right]$

$= e^{i\vec{t}^T\vec{b}} \, \phi_{\vec{x}}(A^T\vec{t})$

Let's derive the ChF of the standard MVN:

$$\phi_{\vec{z}}(\vec{t}) = \prod_{i=1}^{n} \phi_z(t_i) = \prod_{i=1}^{n} e^{-t_i^2/2}$$

$$= e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

Let's derive the chf of the general MVN:

$$\vec{X} = A\vec{z} + \vec{\mu} \sim N(\vec{\mu}, \underbrace{AA^T}_{\Sigma})$$

$$\phi_{\vec{X}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{\mu}} \, \phi_{\vec{z}}(A^T\vec{t})$$

$$= e^{i\vec{t}^T\vec{\mu}} \, e^{-\frac{1}{2}\underbrace{(A^T\vec{t})^T A^T\vec{t}}_{\vec{t}^T \underbrace{AA^T}_{\Sigma} \vec{t}}}$$

$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

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$\vec{Y} = B\vec{X} + \vec{c} \sim ?$ $\quad B \in \mathbb{R}^{m \times n}$, $\vec{c} \in \mathbb{R}^m$

$$\phi_{\vec{Y}}(\vec{t}) \overset{(P2)}{=} e^{i\vec{t}^T\vec{c}} \, \phi_{\vec{X}}(B^T\vec{t})$$

$$= e^{i\vec{t}^T\vec{c}} \, e^{i\underbrace{(B^T\vec{t})^T}_{t^TB}\vec{\mu} - \frac{1}{2}\underbrace{(B^T\vec{t})^T}_{t^TB}\Sigma(B^T\vec{t})}$$

$$= e^{it^T(c + B\mu) - \frac{1}{2}t^TB\Sigma B^T t}$$

$$\overset{(P1)}{\Longrightarrow} \vec{Y} \sim N_m(B\vec{\mu} + \vec{c}, \; B\Sigma B^T)$$

(if $B\Sigma B^T$ is invertible)

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Let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$.

Consider $\underbrace{(\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu})}_{\text{called Mahalanobis Distance}} \sim ?$

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Recall: $\vec{z} = A^{-1}(\vec{x}-\vec{\mu})$

$$\Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$$

So,

$$= (x-\mu)^T (A^{-1})^T A^{-1}(x-\mu)$$

$$= (A^{-1}(x-\mu))^T A^{-1}(x-\mu)$$

$$= z^T z \sim \chi^2_n$$

PC Mahalanobis discovered this in 1936. He was India's founding father of statistics and founded the Indian Institute of Statistics.

This is kind of like distance in R^n adjusted for all the dependencies among the dimensions like a multivariate "z-score".

In one direction,

$$(x-\mu)(\sigma^2)^{-1}(x-\mu) = \frac{(x-\mu)^2}{\sigma^2} = \left(\frac{x-\mu}{\sigma}\right)^2 = z^2$$