lec 14 Oct 26, 2020

$g: \mathbb{R}^n \to \mathbb{R}^n$ and 1 to 1 $\vec{X}, \vec{Y}$ are cont rv vectors of dim n s.t $\vec{Y} = g(\vec{X})$

Given $f_{\vec{X}}(\vec{x})$, find $f_{\vec{Y}}(\vec{y})$

recall what a multidimensional function is:

$Y_1 = g_1(X_1, \dots, X_n)$

$Y_2 = g_2(X_1, \dots, X_n)$

$\vdots$

$Y_n = g_n(X_1, \dots, X_n)$ $\vec{h} = \vec{g}^{-1}$

b/c it's 1 to 1 $X_1 = h_1(Y_1, \dots Y_n)$ $" \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$

$X_2 = h_2(Y_1, \dots Y_n)$

$\vdots$

$X_n = h_n(Y_1, \dots Y_n)$

using multivariable calc, you show that

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\underbrace{\vec{h}(\vec{y})}_{\text{gives you back the x}}) \left| J_n(\vec{y}) \right|$$ there is streaching and squishing

$\to$ the Jacobian determinant

$$= \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \dots & \frac{\partial h_1}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial h_n}{y_1} & \dots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

lets verify the convolution formula via this general change of variable formula.

$T = X_1 + X_2 \sim f_T(t)$

Recipe:

(1) find a "clever" $g$ so that

(2) we can find an $h$

(3) Compute $J_h$

(4) Compute the multidimensional change of variables formula

(5) Integrate out the "nuisance dimension" "clever" $g$

① $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$ let $Y_2 = X_2 = g_2(X_1, X_2)$

not $h$ yet

easy $h$
many ways of doing it

② $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2) \rightarrow$

$\rightarrow X_2 = Y_2 = h_2(\underline{\underline{Y_1, Y_2}})$ you need only the inputs

③ $J_h = \det \begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{bmatrix}$

$= \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - -1 \cdot 0 = 1$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h| = f_{\vec{X}}(y_1 - y_2, y_2)\,|1|$

$= f_{\vec{X}}(y_1 - y_2, y_2)$

⑤ $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\, dy_2$

$= \int_{\mathbb{R}} f_{X_1, X_2}(t - u, u)\, du$ ← exactly the general convolution formula

$\Rightarrow$ if ~~iid~~ indep $\int_{\mathbb{R}} f_{X_1}(t-u)\, f_{X_2}(u)\, du$

$= \text{if iid} \int_{\mathbb{R}} f(t-u)\, f(u)\, du = \int_{supp[X]} f(t-u)\, \mathbb{1}_{t-u \in supp[X]}\, f(t)\, du$

$= \int f_{X_1}^{old}(t-u)\, \mathbb{1}_{t-u \in supp[X_1]}\, f_{X_2}^{old}(t)\, du$

$R = \frac{X_1}{X_2} \sim f_R(r) = ?$

① $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2) \qquad Y_2 = X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2) \qquad X_2 = Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

④ $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\, |y_2|$

⑤ $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\, |y_2|\, dy_2$

$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u)\, |u|\, du$

$\int_{supp[X]} f^{old}(ru)\, \mathbb{1}_{ru \in supp[X]}\, f^{old}(u)\, |u|\, du$

if $X_1, X_2$ indep

$= \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\, |u|\, du \overset{\text{if iid}}{=} \int_{\mathbb{R}} f(ru)\, f(u)\, |u|\, du$

$\parallel$

$\int_{supp[X_2]} f_{X_1}^{old}(ru)\, \mathbb{1}_{ru \in supp[X_1]}\, f_{X_2}^{old}(u)\, |u|\, du$

$$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$$

① $Y_1 = \dfrac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$ $\quad Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

② $X_1 = Y_1(X_1 + X_2) = \cancel{Y_1(Y_2 + X_2)}$

$Y_1 Y_2 = h_1(Y_1, Y_2)$

$X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

③ $J_h = \det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{bmatrix} = y_2(1 - y_1) - -y_1 y_2$

$\cancel{\geq 0\, y_2}$

$= y_2 - \cancel{y_2 y_1 + y_1 y_2} = y_2$

④ $f_{\vec{y}}(\vec{y}) = f_{\vec{x}}(y_1 y_2, y_2 - y_1 y_2) \, |y_2|$

⑤ $f_R(r) = f_{y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{y}}(y_1, y_2)\, dy$

$= \int f_{\vec{x}}(y_1 y_2, y_2 - y_1 y_2) \, |y_2| \, dy_2$

$= \int_{\mathbb{R}} f_{\vec{x}}(ru, u - ru) \, |u| \, du$

$x_1, x_2$ indep

$\downarrow$

$= \int_{\mathbb{R}} f_{x_1}(ru) \, f_{x_2}(u - ru) \, |u| \, du$

$\underset{=}{\text{iid}} \int_{R} f(ru) \, f(u - ru) \, |u| \, du = \int_{\text{Sup}[X]} [\text{H.W}]$

$$\int_{\mathbb{R}} = \int_{x=-\infty}^{x=+\infty} \qquad \int_{\mathbb{R}} f(x)\,\mathbb{1}_{x\in S}\,dx = \int_S f(x)\,dx$$

Sum over the $u$

$$\int_{\mathbb{R}} f_X^{old}(ru)\,\cancel{\mathbb{1}_{ru\in supp[X_1]}}\, f_{X_2}^{old}(u-ru)\,\mathbb{1}_{u-ru\in supp[X_2]}\,|u|\,du$$

$$\frac{supp[X_1]}{r}$$

we don't have over $u$ so if we divide out the $r$ we get a set over just $u$

$X_1 \sim Gamma(\alpha_1, \beta)$ ind of $X_2 \sim Gamma(\alpha_2, \beta)$

$$R = \frac{X_1}{X_1+X_2} \sim f_R(r) = \int_{\frac{[0,\infty)}{r}} \cancel{\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1-1} e^{-\beta ru}}$$

$\to$ doesn't matter what $r$ is

$$= \int_{\frac{[0,\infty)}{r}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}(ru)^{\alpha_1-1}\cancel{e^{-\beta ru}}\frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(u-ru)^{\alpha_2-1}e^{-\beta(u-\cancel{ru})}\cdot \mathbb{1}_{u-ru\in[0,\infty)}\,\overset{u}{|u|}\,du$$

$$\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty (ru)^{\alpha_1-1}(u-ru)^{\alpha_2-1}e^{-\beta u}\,\mathbb{1}_{u-ru\in[0,\infty)}\,u\,du$$

$(ru)^{\alpha_1-1} \to r^{\alpha_1-1}u^{\alpha_1-1}$, $(u(1-r))^{\alpha_2-1} \to u^{\alpha_2-1}(1-r)^{\alpha_2-1}$

$u-ru\in[0,\infty) \;\; u(1-r)\in[0,\infty) \;\; u\in[0,\infty)$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1}(1-r)^{\alpha_2-1}\underbrace{\int_0^\infty u^{\alpha_1+\alpha_2-1}e^{-\beta u}\,du}_{j(\alpha_1,\alpha_2)}$$

$$\leftarrow \propto r^{\alpha_1-1}(1-r)^{\alpha_2-1} \quad \propto Beta(\alpha_1,\alpha_2)$$

$X_1 \sim \text{Gamm}(\alpha_1, \beta)$ indep of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

$\rightarrow r > 0$

$$R = \frac{X_1}{X_2} \sim f_R(r) = ?$$

$$f_R(r) = \int_{\text{supp}[X_2]} f_{X_1}^{\text{old}}(ru) \mathbb{1}_{ru \in \text{Supp}[X_1]} f_{X_2}^{\text{old}}(u) |u| \, du$$

$$= \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u} \underbrace{\mathbb{1}_{ru \in [0, \infty)}}_{u \in [0, \infty]} \rightarrow 1$$

$$\cdot \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} u \, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \int_0^\infty \underbrace{u^{\alpha_1 + \alpha_2 - 1} e^{-\beta(r+1)u} \, du}_{\frac{\Gamma(\alpha_1 + \alpha_2)}{(\beta(r+1))^{\alpha_1 + \alpha_2}}}$$

$$\frac{1}{\beta(\alpha_1, \alpha_2)}$$

$$\beta^{\alpha_1 + \alpha_2} (r+1)^{\alpha_1 + \alpha_2}$$

$$= \frac{1}{\beta(\alpha_1, \alpha_2)} \left( \frac{r}{r+1} \right)^{\alpha_1 + \alpha_2} \mathbb{1}_{r > 0}$$

!! Betaprime$(\alpha_1, \alpha_2)$

0.hr

$$R \sim \text{Betaprime}(\alpha_1, \alpha_2) := \underbrace{\frac{1}{\beta(\alpha_1, \alpha_2)}}_{C} \underbrace{\left( \frac{r}{r+1} \right)^{\alpha_1 + \alpha_2} \mathbb{1}_{r > 0}}_{k(r)}$$

kernal