$$T_3 = \underbrace{X_1 + X_2}_{T_2} + X_3 \sim f_{T_3}(t) = ?$$

$$= \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f^{old}_{X_3}(t-x)\, \mathbb{1}_{t-x \in Supp[X_3]}\, dx$$

$$= \int_0^\infty x \lambda^2 \cancel{e^{-\lambda x}}\, \underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{\overbrace{t-x \in [0,\infty)}^{x \le t}}\, dx = \lambda^3 e^{-\lambda t} \int_0^\infty x \mathbb{1}_{x \le t}\, dx = \lambda^3 e^{-\lambda t} \int_0^t x\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{t^2}{2} \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(3, \lambda)$$

$$T_4 = X_1 + X_2 + X_3 + X_4 = T_3 + X_4 \sim f_{T_4}(t) = ?$$

$$f_{T_4}(t) = \int_{Supp[T_3]} f^{old}_{T_3}(x)\, f^{old}_{X_4}(t-x)\, \mathbb{1}_{t-x \in Supp[X_4]}\, dx$$

$$= \int_0^\infty \frac{x^2}{2} \lambda^3 \cancel{e^{-\lambda x}}\, \underbrace{\lambda e^{-\lambda(t-x)}}_{e^{-\lambda t} \cancel{e^{\lambda x}}}\, \mathbb{1}_{\overbrace{t-x \in [0,\infty)}^{x \le t}}\, dx$$

$$= \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^\infty x^2 \mathbb{1}_{x \le t}\, dx = \frac{1}{2} \lambda^4 e^{-\lambda t} \int_0^t x^2\, dx\, \mathbb{1}_{t \in [0,\infty)}$$

$$= \frac{1}{2 \cdot 3} t^3 \lambda^4 e^{-\lambda t} \mathbb{1}_{t \in [0,\infty)} = \text{Erlang}(4, \lambda)$$

$$T_k = \sum_{i=1}^{k} X_i \sim \text{Erlang}(k, \lambda) := \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in [0,\infty)}$$

$$Supp[T_k] = [0, \infty), \quad \lambda \in (0, \infty), \quad k \in \mathbb{N}$$

$$\text{Exp}(\lambda) \xRightarrow{\text{add}} \text{Erlang}(k, \lambda)$$

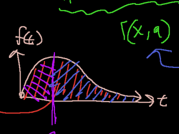
$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$ conceptually analogous

$$\text{Geom}(p) \xRightarrow{\text{add}} \text{NegBin}(k, p)$$

Let's do some pure math. I want to define the gamma family of functions. Beginning with the gamma function for x non-neg:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = \underbrace{\int_0^a t^{x-1} e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1} e^{-t}\, dt}_{\Gamma(x,a)} \qquad a > 0 \ldots$$

e.g. $\Gamma(3) = \int_0^\infty \overbrace{t^2 e^{-t}}^{f(t)}\, dt = 2$



Lower incomplete gamma function
Upper incomplete gamma function

$$1 = \frac{\Gamma(x)}{\Gamma(x)} = \frac{\gamma(x,a) + \Gamma(x,a)}{\Gamma(x)} = \overbrace{\frac{\gamma(x,a)}{\Gamma(x)}}^{P(x,a)} + \overbrace{\frac{\Gamma(x,a)}{\Gamma(x)}}^{Q(x,a)} = P(x,a) + Q(x,a) = 1$$

P(x,a) is called the "Lower regularized gamma function"
Q(x,a) is called the "Upper regularized gamma function"

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 \qquad X \sim \text{Exp}(1) = e^{-t} \mathbb{1}_{t \in [0,\infty)}$$

HW: $\Gamma(x+1) = x\Gamma(x)$

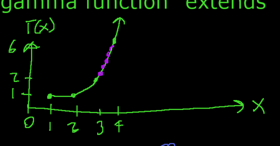
let $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \ldots = (n-1)(n-2) \cdot \ldots \cdot (3)(2)(1) = (n-1)!$$

let $x \in (0, \infty)$

$$\Gamma(x) = (x-1)\Gamma(x-1) = \ldots = (x-1)(x-2) \cdot \ldots \cdot \Gamma(c) \text{ where } c \in (0,1)$$

the gamma function "extends" the factorial function to all positive #'s



$c > 0$,

$$\int_0^\infty t^{x-1} e^{-ct}\, dt = \int_0^\infty \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\, du = \frac{1}{c^x} \int_0^\infty u^{x-1} e^{-u}\, du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow \frac{du}{dt} = c \Rightarrow dt = \frac{1}{c} du$, $t = 0 \Rightarrow u$, $t \to \infty \Rightarrow u \to \infty$, $t = a \Rightarrow u = ac$

$$\int_0^a t^{x-1} e^{-ct}\, dt = \int_0^{ac} \frac{u^{x-1}}{c^{x-1}} e^{-u} \frac{1}{c}\, du = \frac{1}{c^x} \int_0^{ac} u^{x-1} e^{-u}\, du = \frac{\gamma(x, ac)}{c^x}$$

$$\int_a^\infty t^{x-1} e^{-ct}\, dt = \frac{\Gamma(x)}{c^x} - \frac{\gamma(x, ac)}{c^x} = \frac{\Gamma(x, ac)}{c^x}$$

If $n \in \mathbb{N}$

$$\Gamma(n, a) := \int_a^\infty \underbrace{t^{n-1}}_{u} \underbrace{e^{-t} dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\, du = [t^{n-1}(-e^{-t})]_a^\infty - \int_a^\infty -e^{-t}(n-1)t^{n-2}\, dt$$

$$v = \int dv = \int e^{-t} dt = -e^{-t} \qquad \frac{du}{dt} = (n-1)t^{n-2} \Rightarrow du = (n-1)t^{n-2}\, dt$$

$$= a^{n-1} e^{-a} + (n-1) \int_a^\infty t^{n-2} e^{-t}\, dt = a^{n-1} e^{-a} + (n-1) \underbrace{\Gamma(n-1, a)}_{\text{repeat}} =$$

$$= a^{n-1} e^{-a} + (n-1)\left(a^{n-1} e^{-a} + (n-2)\Gamma(n-2, a)\right) = a^{n-1} e^{-a} + (n-1)\left(a^{n-2} e^{-a} + (n-2)\left(a^{n-3} e^{-a} + (n-3)\Gamma(n-3, a)\right)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)\left(a^{n-2} + (n-2)\left(a^{n-3} + (n-3)\Gamma(n-3, a)\right)\right)\right)$$

$$= e^{-a}\left(a^{n-1} + (n-1)a^{n-2} + (n-1)(n-2)a^{n-3} + (n-1)(n-2)(n-3)\Gamma(n-3, a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \frac{1}{(n-4)!}\Gamma(n-3, a)\right)$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \ldots + \frac{a^1}{1!} + \underbrace{\frac{a^0}{0!}}_{1}\right) = e^{-a}(n-1)! \sum_{i=0}^{n-1} \frac{a^i}{i!}$$

$$\Gamma(1, a) = \int_a^\infty e^{-t}\, dt = [-e^{-t}]_a^\infty = e^{-a}$$

Back to probability land...

$$X \sim \text{Erlang}(k, \lambda) := \frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!} \mathbb{1}_{x \ge 0}$$

CDF...

$$F_X(x) := P(X \le x) = \int_0^x \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!}\, dt\, \mathbb{1}_{x \ge 0} = \frac{\lambda^k}{\underbrace{(k-1)!}_{\Gamma(k)}} \overbrace{\int_0^x t^{k-1} e^{-\lambda t}\, dt}^{\frac{\gamma(k, \lambda x)}{\lambda^k}}$$

$$= \frac{\cancel{\lambda^k}}{\Gamma(k)} \frac{\gamma(k, \lambda x)}{\cancel{\lambda^k}} = P(k, \lambda x)$$

$$1 - F_X(x) = 1 - P(k, \lambda x) = Q(k, \lambda x)$$

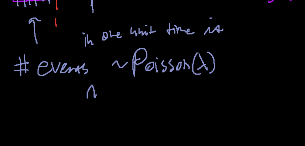
$$X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

CDF...

$$F_X(x) := P(X \le x) = \sum_{t=0}^{x} \frac{\lambda^t e^{-\lambda}}{t!} = e^{-\lambda} \sum_{t=0}^{x} \frac{\lambda^t}{t!} = \frac{1}{\underbrace{x!}_{\Gamma(x+1)}} \overbrace{e^{-\lambda} x! \sum_{t=0}^{x} \frac{\lambda^t}{t!}}^{\Gamma(x+1, \lambda)}$$

$$= \frac{\Gamma(x+1, \lambda)}{\Gamma(x+1)} = Q(x+1, \lambda)$$

the relationship between the Erlang and the Poisson is known as the "Poisson process"



$$T_1 \sim \text{Exp}(\lambda) = \text{Erlang}(1, \lambda)$$

$$P(T_1 > 1) = Q(1, \lambda) \quad \text{equal}$$

events in one unit time is $\sim$ Poisson(λ)

$$N \sim \text{Poisson}(\lambda)$$

$$F_N(0) = P(N \le 0) = P(N = 0) = Q(1, \lambda)$$