Lec05

$X \sim \text{Multi}_2(n, \vec{p})$
$\Rightarrow k=2$

$$p_{X_1|X_2}(x_1, x_2) := P(X_1 = x_1 \mid X_2 = x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

Lec04. $p(x_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

Define $J_n := \{0, 1, \dots, n\}$

Define $\mathbb{1}^u_A := \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases} 1 & \text{if } A \\ \boxed{\frac{0}{0} \text{ not } A} \end{cases}$ — no Answer, Undefined if A^c

$$p_{X_1|X_2}(x_1, x_2) = \frac{\binom{n}{x_1\, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} (\underbrace{1-p_2}_{= p_1})^{n-x_2}}$$

$x_1 = 1 - x_2$

$$= \frac{\frac{\cancel{n!}}{x_1!\cancel{x_2!}} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} \cancel{p_2^{x_2}}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \mathbb{1}_{x_2 \in J_n} \cancel{p_2^{x_2}} p_1^{n-x_2}}$$

$$= \underbrace{\frac{(n-x_2)!}{x_1!}}_{= 1 \text{ IF } x_1 = n-x_2} \underbrace{\mathbb{1}_{x_1 = n-x_2}}_{\boxed{\begin{smallmatrix} x_1 = n-x_2 = 1 \\ \text{other } 0 \end{smallmatrix}}} \underbrace{\mathbb{1}_{x_1 \in J_n}}_{= 1 \text{ IF } x_1 = n-x_2} \underbrace{p_1^{x_1+x_2-n}}_{= 1 \text{ IF } x_1 = n-x_2} \mathbb{1}_{x_2 \in J_n}$$

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$$= \text{Deg}(n-x_2)\, \mathbb{1}^u_{x_2 \in J_n}$$

$P(A|B) = \frac{P(A,B)}{P(B)}$

IF $P(B) = 0 \Rightarrow P(A|B)$ undefined

Let's generalize this conditional probality a little Bit $\vec{X} = \text{Mult}_k(n, \vec{p})$

$$\underbrace{p_{\vec{X}_{-j} \mid X_j}}(\vec{x}_{-j}, x_j) = \frac{p_{\vec{X}}(\vec{x})}{p_{X_j}(x_j)} = \text{Multi}_{k-1}(n - x_j, ?)$$

this is vector without the j^{th} component

$$= \frac{\text{Multi}_k(n, \vec{p})}{\text{Bin}(n, p_j)} = \frac{\binom{n}{x_1 \dots x_k} p_1^{x_1} \dots \cancel{p_j^{x_j}} \dots p_k^{x_k}}{\binom{n}{x_j} \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}}$$

$x_1 \dots x_{j-1} + x_{j+1} \dots x_k = n - x_j$

$$= \frac{\frac{\cancel{n!}}{x_1! \dots \cancel{x_j!} \dots x_k!} \mathbb{1}_{x_1 + \dots + x_j + \dots + x_k = n} \mathbb{1}_{x_1 \in J_n} \dots \mathbb{1}_{x_j \in J_n} \dots \mathbb{1}_{x_k \in J_n}\ p_1^{x_1} \dots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}} \dots p_k^{x_k}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)!} \mathbb{1}_{x_j \in J_n} (1-p_j)^{n-x_j}}$$

* Note: $P_1 + \cdots P_k = 1 \Rightarrow P_1 + \cdots P_{j-1} + P_{j+1} + \cdots P_k = 1 - P_j$

$\Rightarrow$ Devide Both side By $1-P_j$

Let $n' := n - x_j$

$$= \underbrace{\frac{P_1}{1-P_j}}_{P_1'} + \cdots \underbrace{\frac{P_{j-1}}{1-P_j}}_{P_{j-1}'} + \underbrace{\frac{P_{j+1}}{1-P_j}}_{P_{j+1}'} + \cdots \underbrace{\frac{P_k}{1-P_j}}_{P_k'} = 1$$

* Note $n - x_j = x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k$ otherwise probability 0

$$= \underbrace{\frac{n'!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_k!}}_{= \binom{n'}{x_1 \cdots x_{j-1}, x_{j+1}, \cdots x_k}} \mathbb{1}_{x_1 + \cdots x_{j-1} + x_{j+1} + \cdots x_k = n'} \mathbb{1}_{x_1 \in J_n} \mathbb{1}_{x_{j-1} \in J_n} \mathbb{1}_{x_{j+1} \in J_n} \cdots \rightarrow$$

$$\rightarrow \underbrace{\frac{\mathbb{1}_{x_k \in J_n} P_1^{x_1} \cdots P_{j-1}^{x_{j-1}} \cdot P_{j+1}^{x_{j+1}} \cdots P_k^{x_k}}{(1-P_j)^{x_1 + \cdots x_{j-1} + x_{j+1} \cdots x_k}}}_{P_1'^{x_1} \cdots P_{j-1}'^{x_{j-1}} \cdot P_{j+1}'^{x_{j+1}} \cdots P_k'^{x_k}} \mathbb{1}^{u}_{x_j \in J_n}$$

$$= \text{Multir}_{k-1}(n', \vec{p}\,')\, \mathbb{1}^{u}_{x_j \in J_n}$$

$\vec{X} \sim \text{Multi}_k(n, \vec{p})$ what is $E[\vec{X}]$ $Var[\vec{X}]$

Review from Math 241 Let $X_1 \cdots X_n$ Be rv's and $a, c \in \mathbb{R}$

$$\begin{cases} E[aX + c] = aE[X] + c \\ E[\Sigma X_i] = \Sigma E[X_i] \overset{\text{id identicall distributed}}{=} n\mu \\ E[\Pi X_i] \overset{\text{IF independent}}{=} \Pi E[X_i] \end{cases}$$

$\sigma^2 := Var[X] = E[(X-\mu)^2]$ $\quad$ standard deviation $\sigma := SD[X] := \sqrt{Var[X]}$

$\quad \quad = E[X^2] - \mu^2$

$$Var[X_1+X_2] = E[((X_1+X_2) - (\mu_1+\mu_2))^2]$$

$$\overset{+2\mu_1\mu_2}{}$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2]$$

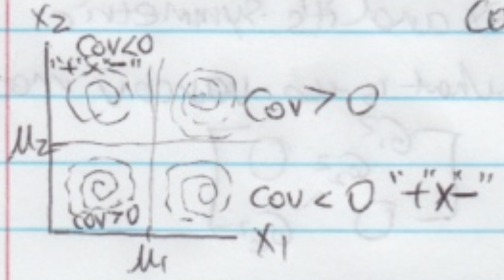
$$= E[X_1^2] + E[X_2^2] + \mu_1^2 + \mu_2^2 + 2E[X_1X_2] - 2\mu_1^2 - \cancel{2\mu_1\mu_2} - 2\mu_1\mu_2 - 2\mu_2^2 \cancel{+2\mu_1\mu_2}]$$

$$= \sigma_1^2 + \cancel{\mu_1^2} + \sigma_2^2 \cancel{+\mu_2^2} \cancel{+\mu_1^2} + \cancel{\mu_2^2} + 2E[X_1X_2] \overset{-2\mu_1^2}{} - 2\mu_1\mu_2 \cancel{-2\mu_2^2}$$

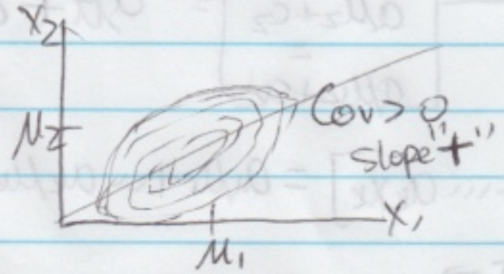
X_1, X_2 independent

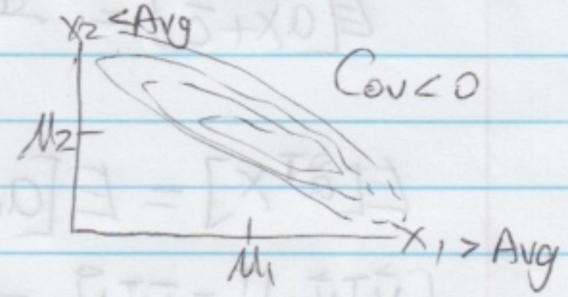
$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{\overset{=0}{E[X_1X_2] - \mu_1\mu_2}}_{Cov(X_1X_2)}) = \sigma_1^2 + \sigma_2^2 + 2\overbrace{Cov[X_1X_2]}^{\sigma_{12}} = \sigma_1^2 + \sigma_2^2 \quad \boxed{\text{If } X_1, X_2 \text{ independent}}$$

$Cov(X_1X_2)$ covariance of X_1 with X_2



HW: prove $Cov[X_1X_2] = E[\overset{+}{X_1 - \mu_1}][\overset{+}{X_2 - \mu_2}] \overset{=+}{}$





Covariance / Cov Rule

$Cov[X,X] = \sigma^2$

$Cov[X_1,X_2] = Cov[X_2,X_1]$

$Cov[X_1+X_2, X_3] = Cov[X_1,X_3] + Cov[X_2,X_3]$

$Cov[a_1X_1, a_2X_2] = a_1a_2\sigma_{1,2}$

$Var[X_1+\dots+X_n] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j]$ $\quad \sigma_1^2 + \sigma_2^2,$ $\underline{X_1 X_2 \text{ independent}}$

$$\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix} \quad \text{Let } M = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & & \\ X_{n1} & \cdots & X_{nm} \end{bmatrix}$$

$$E[M] = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1m} \\ \vdots & & \\ \mu_{n1} & \cdots & \mu_{nm} \end{bmatrix}$$

Capital letter sigma

$(k\times 1)(1\times k) = k\times k$ $\qquad k\times k$

$$\Sigma := Var[\vec{X}] := E[\underbrace{\vec{X}\vec{X}^T}_{\text{outer product}}] - \vec{\mu}\vec{\mu}^T = \begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \cdots & Cov[X_1,X_k] \\ Cov[X_2,X_1] & Var[X_2] & & Cov[X_2,X_k] \\ & & \ddots & Var[X_k] \\ Cov[X_k,X_1] & Cov[X_k,X_2] & \cdots & \end{bmatrix}$$

Variance-covariance (Varcov) matrix and it's symmetric

If $X_1 \cdots X_k$ are indepedent, what is the varcov matrix

$$\Sigma = \mathrm{diag}\{\sigma_1^2 \cdots \sigma_k^2\} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_k^2 \end{bmatrix}$$

Rules about vector w/ expectations

$$E[a\vec{X} + \vec{c}] = \cancel{a\mu_1 + c_1} \begin{bmatrix} a\mu_1 + c_1 \\ a\mu_2 + c_2 \\ \vdots \\ a\mu_k + c_k \end{bmatrix} = a\vec{\mu} + \vec{c}$$

$$E[\vec{a}^T X] = E[a_1X_1 + \cdots a_kX_k] = a_1\mu_1 + \cdots a_k\mu_k = \vec{a}^T\vec{\mu}$$

$$(\vec{v}_1^T\vec{v}_2)^T = \vec{v}_2^T\vec{v}_1 = \vec{v}_1^T\vec{v}_2$$