Wed November 11-20.

Consider $X_1, X_2, \ldots X_n$ iid rv's of unknown pmf/pdf but we know it has expectation mu and variance sigsq (both finite).

Let $T_n := X_1 + X_2 + \ldots + X_n$, $E[T_n] = n\mu$, $Var(T_n) = n\sigma^2$

Let $\bar{X}_n = \frac{X_1 + \ldots + X_n}{n} = \frac{T_n}{n}$, $E[\bar{X}_n] = \mu$, $Var(\bar{X}_n) = \frac{\sigma^2}{n}$

Let $Z_n :=$

$$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu, \quad E(Z_n) = 0, \ Var(Z_n) =$$

$$1 = SD(Z_n)$$

" $\bar{X}_n$ standard"

$$\phi_{T_n}(t) \overset{P_3}{=} \phi_{X_1}(t) \cdots \phi_{X_n}(t) \overset{\text{identical } \Phi\ P_1}{=} \phi_X(t)^n$$

$$\phi_{\bar{X}_n}(t) \overset{(P_2)}{=} \phi_{T_n}\left(\frac{t}{n}\right) = \phi_X\left(\frac{t}{n}\right)^n$$

$$\phi_{Z_n}(t) = e^{-it\mu\sqrt{n}} \phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right) = e^{-\frac{it\mu\sqrt{n}}{\sigma}} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)^n =$$

$$= e^{\frac{-itun}{\sigma\sqrt{n}} + n\ln\left(\phi_x\left(\frac{t}{\sigma}\right.\right.}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_x\left(\frac{t}{\sigma(\sqrt{n})}\right)\right) - \frac{itu}{\sigma\sqrt{n}}}{\frac{t^2}{\sigma^2}}\right)} = \phi_{Z_n}(t)$$

We want to examine $\lim \phi_{Z_n}(t)$ and if we find its limiting chf, $\phi_{Z(t)}$ we can use P8 to show that $Z_n \to Z_n \xrightarrow{d} Z \Rightarrow Z_n \xrightarrow{d} Z$.

$$\lim \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{itu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}} =$$

$$e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_x(u)) - iu\mu}{u^2}}$$

let $u = \frac{t}{\sqrt{n}\sigma}$ if $n \Rightarrow \infty \Rightarrow u \to 0$

$$\overset{\text{H. rule}}{=} e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_x'(u)}{\phi_x(u)} - iu}{u}} = e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\phi_x(u)\phi_x''(u) - \phi_x'(u)^2}{\phi_x(u)^2}}$$

$$= e^{\frac{t^2}{2\sigma^2}\frac{\phi_x(0)\phi_x''(0) - \phi_x'(0)^2}{\phi_x(0)^2}} = e^{\frac{t^2}{2\sigma^2}\left(\phi_x''(0) - \phi_x'(0)^2\right)}$$

$$\overset{P4}{=} e^{\frac{t^2}{2\sigma^2}\left(i^2 E[X]^2 - (iE[X])^2\right)} = e^{-\frac{t^2}{2\sigma^2}\left(E[X^2] - E[X]^2\right)}$$

$$= e^{-\frac{t^2}{2}} = \phi_Z(t)$$