Recall: $X_1 \sim \text{Multi}_2(n, \vec{p})$

we know $P(X_1, X_2) := P(X_1 = x_1 \mid X_2 = x_1)$

$$= \frac{P(X_1, X_2)}{P(X_2)} = \text{Deg}(n - X_2)$$

Last time $P(X_2) = \text{Bin}(n, p_2) = \text{Bin}(n, 1-p_1)$

Let $J_n := \{0, 1, \ldots, n\}$

$$P(X_1, X_2) = \frac{\binom{n}{x_1\, x_2} p_1^{x_1} p_2^{x_2}}{\binom{n}{x_2} p_2^{x_2} (1-x_2)^{n-x_2}} = \frac{\frac{n!}{x_1!\, x_2!} \mathbb{1}_{x_1+x_2=n} \mathbb{1}_{x_1\in J_n} \mathbb{1}_{x_2 \in J_n} p_1^{x_1} p_2^{x_2}}{\frac{n!}{x_2!\,(n-x_2)!} \mathbb{1}_{x_2\in J_n} p_2^{x_2} p_1^{n-x_2}}$$

Let $\mathbb{1}_A^u = \frac{\mathbb{1}_A}{\mathbb{1}_A} = \begin{cases}1 & \text{if } A\\ 0 & \text{if } A^c\end{cases}$ $\quad = \frac{(n-x_2)!}{x_1!} \mathbb{1}_{x_1=n-x_2} \mathbb{1}_{x_1\in J_n} \mathbb{1}^u_{x_2\in J_n} = \text{Deg}(n-x_2)\, \mathbb{1}^u_{x_2 \in J_n}$

Let's generalize this conditional prob!

$\vec{X} \sim \text{Multi}_k(n, \vec{p})$

$$P_{\vec{x}_{-j}}(\vec{x}_{-j}, x_j) = \frac{P_{\vec{x}}(\vec{x})}{P_{x_j}(x_j)} = \text{Multi}_{k-1}(n-x_j, ?) = \frac{\text{Multi}_k(n,\vec{p})}{\text{Bin}(n,p)}$$

$$= \frac{\binom{n}{x_1,\ldots,x_j,\ldots,x_k} p_1^{x_1}\cdots \cancel{p_j^{x_j}}\cdots p_k^{x_k}}{\binom{n}{x_j} \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}} = \frac{\frac{\cancel{n!}}{x_1!\cdots \cancel{x_j!}\cdots x_k!} \mathbb{1}_{x_1+\cdots+x_k=n} \mathbb{1}_{x_1\in J_n}\cdots \mathbb{1}_{x_k\in J_n} p_1^{x_1}\cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}}\cdots p_k^{x_k}}{\frac{\cancel{n!}}{\cancel{x_j!}\,(n-x_j)^{n-x_j}} \mathbb{1}_{x_j\in J_n}(1-p_j)^{n-x_j}}$$

Note: $p_1+\cdots+p_k=1 \Rightarrow p_1+\cdots+p_{j-1}+p_{j+1}+\cdots+p_k = 1-p_j$

$$= \frac{p_1}{1-p_j} + \cdots \frac{p_{j-1}}{1-p_j} + \frac{p_{j+1}}{1-p_j} + \cdots + \frac{p_k}{1-p_j} = 1$$

Note: $n - x_j = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k$

Let $n' = n - x_j$

$$= \frac{\frac{n!}{x_1!\cdots x_{j-1}!\, x_{j+1}!\cdots x_k!} \mathbb{1}_{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_k=n'} \mathbb{1}_{x_1\in J_n}\cdots \mathbb{1}_{x_{j-1}\in J_n} \mathbb{1}_{x_{j+1}\in J_n}\cdots \mathbb{1}_{x_k\in J_n}\, p_1^{x_1}\cdots p_{j-1}^{x_{j-1}} p_{j+1}^{x_{j+1}}\cdots p_k^{x_k}\, \mathbb{1}^u_{x_j\in J_n}}{(1-p_j)^{x_1+\cdots+x_{j-1}+x_{j+1}+\cdots+x_k}}$$

$= \text{Mult}_{k-1}(n', \vec{p}\,') \prod_{x_j \in J_n}^{4}$

if $\vec{X} \sim \text{Multi}_k(n, \vec{p})$ what is $E[\vec{X}]$, $Var[\vec{X}]$?

$E[aX + c] = aE[X] + c$ $\qquad E[\prod x_i] = \prod E[x_i]$

$E[\sum x_i] = \sum E[x_i] \overset{iid}{=} n\mu$ $\qquad \sigma^2 := var[X] := E[(X-\mu)^2]$

$\sigma := SD[X] := \sqrt{var[X]}$

$Var[X_1 + X_2] = \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E[X_1X_2] - \mu_1\mu_2}_{Cov(X_1 X_2)})$ $\overset{\text{if indep.}}{=} \sigma_1^2 + \sigma_2^2$

Covariance Rules

$Cov[X_1 X_1] = \sigma^2$ $\qquad Cov[aX_1, aX_2] = a_1 a_2 \sigma_{12}$

$Cov[X_1, X_2] = Cov[X_2, X_1]$ $\qquad Var[X_1 + \ldots + X_n] = \sum\sum Cov[X_i, X_j]$

$Cov[X_1 + X_2, X_3] = Cov[X_1, X_3] + Cov(X_2 X_3]$

Additionally:

$E[\vec{X}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}$ let $m := \begin{bmatrix} x_{11} & \ldots & x_{1m} \\ x_{n1} & & x_{nm} \end{bmatrix}$

$E[m] = \begin{bmatrix} \mu_{11} & \ldots & \mu_{1m} \\ \mu_{m1} & \ldots & \mu_{mn} \end{bmatrix}$ $\qquad Var[\vec{X}] := E[\vec{X}\vec{X}^T] - \vec{\mu}\vec{\mu}^T = \begin{bmatrix} var[X_1] & cov[X_1 X_2] & \ldots & Cov(X_1 X_k] \\ \vdots & Var[X_2] & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov[X_k X_1] & \ldots & \ldots & Var[X_k] \end{bmatrix}$

$= \Sigma$

if $X_1 \ldots X_k$ are indep what is the varcov matrix?

$\Sigma = \text{dig}\{\sigma_1^2, \ldots, \sigma_k^2\} = \begin{bmatrix} \sigma_1^2 & & 0 \\ 0 & \ddots & \sigma_k^2 \end{bmatrix}$

$$E[a X + \vec{c}] = \begin{bmatrix} a\mu_1 + c_1 \\ \vdots \\ a\mu_k + c_k \end{bmatrix} = a\vec{\mu} + \vec{c}$$

$$E[\vec{a}^T X] = E[a_1 x_1 + \dots + a_2 x_k] = a_1\mu_1 + \dots + a_k\mu_k = \vec{a}^T\vec{\mu}$$