$X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X > Y) = ?$ $\frac{1}{2} \rightarrow$ good guess

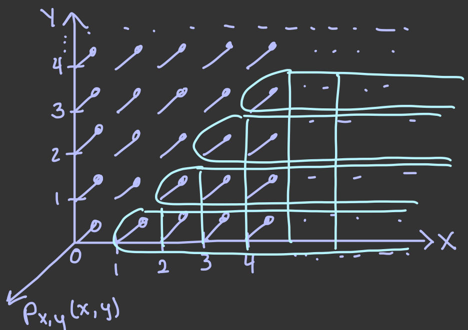
$P(X > Y) = P(Y > X)$

$P(X > Y) + P(Y > X) + P(\overset{\rightarrow 0}{X = Y}) = 1$

$2P(X > Y) = 1 - P(X = Y)$

$\Rightarrow P(X, Y) = \frac{1 - P(X = Y)}{2} < \frac{1}{2}$

$P(X, Y) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p_{X,Y}(x, y) \mathbb{1}_{x > y}$



$P(X > Y)$

$\rightarrow$ X, Y indep

$= \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p_X(x) p_Y(y) \mathbb{1}_{x > y}$

$= \sum_{x \in \{0,1..\}} \sum_{y \in \{0,1,...\}} (1-p)^x p (1-p)^y p \, \mathbb{1}_{x > y}$

$x \geq y + 1 \Rightarrow x \in \{y+1, y+2, ...\}$

$= p^2 \sum_{y \in \{0,1,..\}} \sum_{x \in \{0,1,..\}} (1-p)^x (1-p)^y \mathbb{1}_{x \in \{y+1, y+2, ...\}}$

$= p^2 \sum_{y \in \{0,1,..\}} \sum_{x \in \{y+1, y+2 ...\}} (1-p)^x (1-p)^y$

$= p^2 \sum_{y \in \{0,1,...\}} (1-p)^y \sum_{x \in \{y+1, y+2, ..\}} (1-p)^x$

Reindexing Trick - Kind of like U-sub.

$x' = x - (y+1) \in \{0, 1, 2, ...\}$

$\Rightarrow x = x' + y + 1$

$= p^2 \sum_{y \in \{0,1,...\}} \sum_{x' \in \{0,1,...\}} (1-p)^{x'} (1-p)^y (1-p)$

$= p^2 (1-p) \sum_{y \in \{0,1,..\}} (1-p)^{2y} \boxed{\sum_{x' \in \{0,1,..\}} (1-p)^{x'}} \rightarrow$ Geometric Series formula : $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$ for $a \in (-1, 1) \setminus \{0\}$

$= \frac{1}{1-(1-p)} = \frac{1}{p}$

$= p(1-p) \boxed{\sum_{y \in \{0,1,..\}} (1-p)^{2y}} = \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p - p^2} = \frac{1}{p(2-p)}$

$= \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p} < \frac{1}{2}$

Consider a bag of fruit that has apples and bananas. You now draw with replacement n samples from this bag and you count how many are apples and how many are bananas. Let X_1 be the r.v. that counts the number of apples and let X_2 be the r.v. that counts the number of bananas. Let p_1 be the prob. of picking an apple & p_2 be the prob. of picking a banana.

$\rightarrow X_1 \sim Bin(n, p_1)$, $X_2 \sim Bin(n, p_2)$

Are X_1 & X_2 independent? <u>NO</u>, they are dependent.

$P(X_1 = x_1 \mid X_2 = x_2) \stackrel{?}{=} P(X_1 = x_1)$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \vec{X} \sim p_{\vec{X}}(x_1, x_2) = \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{x_1 + x_2 = n} \mathbb{1}_{x_1 \in \{0,1,\dots,n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$$

$$\binom{n}{x_1, x_2}$$ multichoose notation

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$\vec{X} \sim Multi(n, \vec{p}) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$ → multinomial r.v. of dim = 2.

Let's add cantaloupes to the bag. Let X_3 count the number of cantaloupes & p_3 be the prob. of drawing a cantaloupe

$$\vec{X} \sim Multi(n, \vec{p}) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3} = \frac{n!}{x_1! \, x_2! \, x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \mathbb{1}_{x_1 + x_2 + x_3 = n}$$

$$Multi(n, \vec{p}) = p_{\vec{X}}(x_1, x_2, x_3)$$

The general multinomial r.v. of dim = K has PMF:

$$\vec{X} \sim Multi(n, \vec{p}) := \binom{n}{x_1, x_2, \dots, x_K} \prod_{k=1}^{K} p_k^{x_k}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix}$$

Parameter Space: $n \in \mathbb{N}$

$\vec{p} \in \{\vec{v} \mid \vec{v} \cdot \vec{1} = 1, v_1 \in (0,1), \dots, v_K \in (0,1)\}$

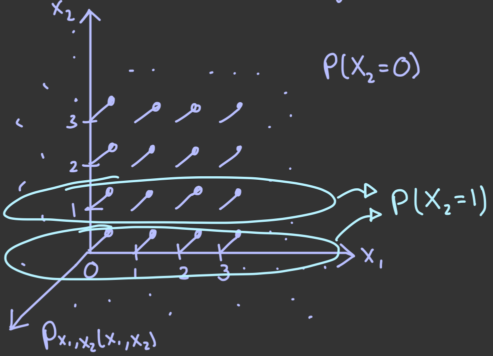
Support: $Supp[\vec{X}] = \{\vec{x} \mid \vec{x} \cdot \vec{1} = n, x_1 \in \{0,1,\dots,n\}, \dots, x_K \in \{0,1,\dots,n\}\}$

Want to derive the conditional PMF & the marginal PMF's in the case of $K=2$ (apples & bananas)

$$Deg(n - x_2) = p_{X_1 | X_2}(x_1, x_2) = \frac{p_{X_1, X_2}(x_1, x_2)}{p_{X_2}(x_2)} \begin{matrix} \rightarrow \text{JMF} \\ \rightarrow \text{marginal PMF} \end{matrix}$$

① How do we prove that the marginal PMF is Binomial?

② How do we compute the marginal PMF from the JMF?



$$p_{x_2}(x_2) = \sum_{x_1 \in \mathbb{R}} p_{x_1,x_2}(x_1,x_2)$$

$$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2}$$

$$= p_2^{x_2} \sum_{x_1 \in \mathbb{R}} \frac{n!}{x!\,x_2!} p_1^{x_1} \mathbb{1}_{\overbrace{x_1+x_2=n}^{x_1 = n-x_2}} \mathbb{1}_{x_1 \in \{0,1,\dots n\}} \mathbb{1}_{x_2 \in \{0,1,\dots,n\}}$$

$$= p_2^{x_2} \frac{n!}{x_2!} \mathbb{1}_{x_2 \in \{0,1,\dots n\}} \sum_{x_1 \in \mathbb{R}} \frac{p_1^{x_1}}{x_1!} \underbrace{\mathbb{1}_{x_1 = n-x_2}}_{x_1 \text{ is only one value. } n-x_2} \mathbb{1}_{x_1 \in \{0,1,\dots n\}}$$

$$= p_2^{x_2} \boxed{\frac{n!}{x_2!} \mathbb{1}_{x_2 \in \{0,\dots,n\}} \frac{p_1^{n-x_2}}{(n-x_2)!}}$$

$$= \binom{n}{x_2} p_2^{x_2} p_1^{n-x_2} \quad \text{, since } p_1 + p_2 = 1 \Rightarrow p_1 = 1 - p_2$$

$$= \text{Binom}(n, p_2)$$