HW1,1k

$$\sum_{x\in\mathbb{R}} \frac{1}{x!}\mathbb{1}_{x\in\mathbb{N}} = \sum_{x\in\mathbb{N}=\{1,2,\dots\}} \frac{1}{x!} = \sum_{x=1}^{\infty}\frac{1}{x!} = \Big(\underbrace{\sum_{x=0}^{\infty}\frac{1}{x!}}_{e}\Big) - \underbrace{\frac{1}{0!}}_{1} = e-1$$

$$\sum_{x=0}^{\infty}\frac{1}{x!} = e$$

2014,2e

$\vec{X}\sim \mathrm{Multin}_K(n,\vec{p})$, $\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_K\end{bmatrix}$ $\Rightarrow X_j \sim \mathrm{Bin}(n,p_j)$

$\vec{X}\,|\,X_j = x_j \sim \mathrm{Multin}_{K-1}(n-x_j, \vec{p}\,')$, $\vec{p}\,' = \begin{bmatrix} p'_1 \\ \vdots \\ p'_{K-1}\end{bmatrix}$, $p'_\ell := \frac{p_\ell}{1-p_j}$

$\vec{X}\sim \mathrm{Multin}_4\left(20, \begin{bmatrix} 5/14 \\ 6/14 \\ 2/14 \\ 1/14\end{bmatrix}\right)$, $\vec{C} = \vec{X}\,|\,X_1 = 5 \sim \mathrm{Multin}_3\left(15, \begin{bmatrix} 6/9 \\ 2/9 \\ 1/9\end{bmatrix}\right)$

$\vec{p}\,' = \frac{1}{1-\frac{5}{14}}\begin{bmatrix} 6/14 \\ 2/14 \\ 1/14\end{bmatrix}$

$$P(C_1 = 3) = \frac{15!}{3!\,12!}\left(\frac{6}{9}\right)^3\left(1-\frac{6}{9}\right)^{12}$$

$C_1 \sim \mathrm{Bin}(15, 6/9)$

$E[\vec{X}] = n\vec{p}$

$$\mathrm{Var}[\vec{X}] = \begin{cases} np_i(1-p_j) & \text{if } i=j \\ -np_ip_j & \text{if } i\neq j\end{cases}$$

$\in \mathbb{R}^{K\times K}$

2017,2f

$R = \vec{c}^{\,T}\vec{X}$, $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 2\end{bmatrix}$, $E[R] = E[\vec{c}^{\,T}\vec{X}] = \vec{c}^{\,T}\vec{\mu}\dots$

2017,2g

$$\mathrm{Var}[R] = \mathrm{Var}[\vec{c}^{\,T}\vec{X}] = \vec{c}^{\,T}\Sigma\vec{c}$$

HW2,2l

$\vec{X}\sim \mathrm{Multin}_K(n,\vec{p})$, $\vec{p} = \frac{1}{K}\vec{1}$

$$\sigma_{ij} = \mathrm{Cov}[X_i, X_j] = -np_ip_j = -n\frac{1}{K}\frac{1}{K} = -\frac{n}{K^2},\quad \lim_{K\to\infty}\sigma_{ij} = 0$$

If K is large then $X_i \sim \mathrm{Deg}(0)$, $X_j \sim \mathrm{Deg}(0)$

$$\mathrm{Cov}[X_i,X_j] = E[\underbrace{X_iX_j}_{0}] - \underbrace{\mu_i}_{0}\underbrace{\mu_j}_{0} = 0$$

$P(A)=1$, $P(B)=1$

$P(A\cap B) = 1$  $P(A\cap B) = P(A)P(B)$

HW2,2i

$\vec{X}\sim \mathrm{Multinom}_K(n,p)$  $\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$ where $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n \overset{iid}{\sim} \mathrm{Multin}_K(1,\vec{p})$

$\vec{X}_1, \dots, \vec{X}_r \overset{iid}{\sim} \mathrm{Multin}_K(n,p)$

$$\vec{X}_1 + \dots + \vec{X}_r = \underbrace{(\vec{X}_{11} + \dots + \vec{X}_{1n}) + \dots + (\vec{X}_{r1} + \dots + \vec{X}_{rn})}_{rn \text{ iid } \mathrm{Multin}_K(1,\vec{p})} = \mathrm{Multin}_K(nr, \vec{p})$$

$X_1, X_2 \overset{iid}{\sim}$ with $E[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$, $T = X_1 + X_2$

$\vec{Y} = \begin{bmatrix} X_1 \\ X_2 \\ T\end{bmatrix}$, $\Sigma_Y = \begin{bmatrix} \sigma^2 & 0 & \sigma^2 \\ 0 & \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 & 2\sigma^2\end{bmatrix}$

$\mathrm{Cov}[X_1, T]$
$= \mathrm{Cov}[X_1, X_1 + X_2]$
$= \mathrm{Cov}[X_1, X_1] + \mathrm{Cov}[X_1, X_2]$
$= \sigma^2$

2017,1f

$X_1, X_2, \dots, X_r, X_{r+1} \overset{iid}{\sim} \mathrm{Geom}(p) \Rightarrow X_1 + \dots + X_r \sim \mathrm{NegBin}(r,p) := \binom{x+r-1}{r-1}(1-p)^x p^r$

$X_1 + \dots + X_r + X_{r+1} \sim \mathrm{NegBin}(r+1, p) = \binom{x+r}{r}(1-p)^x p^{r+1}$

$\vec{X}\sim \mathrm{Multin}_4(1, \vec{p})$  $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0\end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0\end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0\end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1\end{bmatrix}$

$p_1$  $p_2$

$$= \binom{n}{x_1, x_2, x_3, x_4} \underbrace{p_1^{x_1}p_2^{x_2}p_3^{x_3}p_4^{x_4}}_{= p_1 \text{ or } p_2 \text{ or } p_3 \text{ or } p_4} =$$

$$\binom{1}{0,0,1,0} = \frac{1!}{0!\,0!\,1!\,0!} = \frac{1}{1\cdot 1\cdot 1\cdot 1} = 1$$

HW1,1f

$$\prod_{x\in\mathbb{R}} \mathbb{1}_{x\in\{1,2,3\}} = \mathbb{1}_{1\in\{1,2,3\}}\cdot\mathbb{1}_{2\in\{1,2,3\}}\cdot\mathbb{1}_{3\in\{1,2,3\}}\cdots$$

$$\underbrace{\mathbb{1}_{-3,482,643,128.94\in\{1,2,3\}}}_{0} = 0$$

HW1,2d

$X_1 \sim \begin{cases} 3 & \text{wp } 0.3 \\ 6 & \text{wp } 0.7\end{cases} = 0.3\,\mathbb{1}_{x=3} + 0.7\,\mathbb{1}_{x=6} = p_1(x)$

$X_2 \sim \begin{cases} 4 & \text{wp } 0.4 \\ 8 & \text{wp } 0.6\end{cases} = 0.4\,\mathbb{1}_{x=4} + 0.6\,\mathbb{1}_{x=8} = p_2(x)$

$$T \sim p_T(t) = \sum_{x\in\mathrm{Supp}[X_1]} p_1(x)\,p_2(t-x) = \sum_{x\in\{3,6\}}(0.3\,\mathbb{1}_{x=3} + 0.7\,\mathbb{1}_{x=6})(0.4\,\mathbb{1}_{\underbrace{t-x=4}_{t=4+x}} + 0.6\,\mathbb{1}_{\underbrace{t-x=8}_{t=8+x}})$$

$$= \sum_{x\in\{3,6\}} 0.12\,\mathbb{1}_{x=3}\mathbb{1}_{t=4+x} + 0.18\,\mathbb{1}_{x=3}\mathbb{1}_{t=8+x} + 0.28\,\mathbb{1}_{x=6}\mathbb{1}_{t=4+x} + 0.42\,\mathbb{1}_{x=6}\mathbb{1}_{t=8+x}$$

$$= 0.12\,\mathbb{1}_{t=7} + 0.18\,\mathbb{1}_{t=11} + 0.28\,\mathbb{1}_{t=10} + 0.42\,\mathbb{1}_{t=11}$$

Geometric series: $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}$  $c\in(-1,1)\setminus\{0\}$

$\sum_{i=17}^{\infty} c^i$  let $j = i-17 \Rightarrow i = j+17$

$$= \sum_{j=0}^{\infty} c^j c^{17} = c^{17}\sum_{j=0}^{\infty} c^j = \frac{c^{17}}{1-c}$$

HW1,2g

$X_1 \sim \mathrm{Bin}(n_1, p) \perp X_2 \sim \mathrm{Bin}(n_2, p)$, $T = X_1 + X_2$ (indep of)

$$p_T(t) = \sum_{x\in\mathrm{Supp}[X_1]} p_1(x)\,p_2(t-x) = \sum_{x\in\{0,1,\dots,n_1\}} \binom{n_1}{x}p^x(1-p)^{n_1-x}\binom{n_2}{t-x}p^{t-x}(1-p)^{n_2-t+x}$$

$\mathbb{1}_{t-x\in\mathrm{Supp}[X_2]}$

$$= p^t(1-p)^{n_1+n_2-t}\underbrace{\sum_{x\in\{0,1,\dots,n_1\}}\binom{n_1}{x}\binom{n_2}{t-x}}_{\binom{n_1+n_2}{t}} = \mathrm{Bin}(n_1+n_2, p)$$

2017,3b

$X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Bin}(n_0, p)$

$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n\end{bmatrix}$ ? ~~$\sim$~~ Multin

$\vec{X}\sim \mathrm{Multin}_K(n,\vec{p})$
$\Rightarrow X_j \sim \mathrm{Bin}(n, p_j)$

2017 3c

(c) [7 pt / 85 pts] A person goes to the grocery store and buys $n$ fruits. For each of his $n$ selections, he picks rambutans with probability $p$ otherwise he picks dragonfruits. If rambutans cost $a$ and dragonfruits cost $b$, find his expected bill.

$\vec{X} = \begin{bmatrix} X_1 \\ X_2\end{bmatrix} \sim \mathrm{Multi}\left(n, \begin{bmatrix} p \\ 1-p\end{bmatrix}\right)$ where $X_1$ denotes the # of rambutans and $X_2$ denotes the # of dragonfruits

$E[\vec{X}] = n\vec{p} = \vec{\mu}$

let $\vec{c} = \begin{bmatrix} a \\ b\end{bmatrix}$ $E[B] = E[\vec{c}^{\,T}\vec{X}] = \vec{c}^{\,T}\vec{\mu} = (a\ b)\,n\begin{bmatrix} p \\ 1-p\end{bmatrix} = n(ap + b(1-p))$

$B = \vec{c}^{\,T}\vec{X}$ denotes the bill

$$SE[B] = \sqrt{\mathrm{Var}[B]} = \sqrt{\mathrm{Var}[\vec{c}^{\,T}\vec{X}]} = \sqrt{\vec{c}^{\,T}\Sigma\vec{c}} =$$

$$\Sigma = \begin{bmatrix} np(1-p) & -np(1-p) \\ -np(1-p) & n(1-p)p\end{bmatrix} = np(1-p)\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}$$

$$\vec{c}^{\,T}\Sigma\vec{c} = np(1-p)\,[a\ b]\underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix}}_{\begin{bmatrix} a-b \\ b-a\end{bmatrix}} = np(1-p)(a(a-b) + b(b-a))$$