Consider $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

$\uparrow$ possibly infinite sequence of iid Bernoullis

Let $X := \#$ of zeroes before the first one occurs $= \min\{t \mid B_t = 1\} - 1$

$p(0) = P(X=0) = P(\{1\}) = p$

$p(1) = P(X=1) = P(\{0,1\}) = (1-p)p$

$p(2) = P(X=2) = P(\{0,0,1\}) = (1-p)^2 p$

$\vdots$

$p(x) = P(X=x) = P(\{\underbrace{0,0,\ldots,0}_{x},1\}) = (1-p)^x p$

$\text{Supp}[X] = \{0,1,2,\ldots\}$

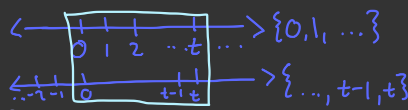
$X \sim \text{Geom}(p) := (1-p)^x p \mathbb{1}_{x \in \{0,1,\ldots\}}$

geometric r.v.

$X_1, X_2 \overset{iid}{\sim} \text{Geom}(p)$, $T_2 = X_1 + X_2 \sim p_{T_2}(t) = ?$

$p_{T_2}(t) = \sum_{x \in \text{Supp}[X]} p^{old}(x) p^{old}(t-x) \mathbb{1}_{t-x \in \text{Supp}[X]} = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p (1-p)^{t-x} p \mathbb{1}_{\underbrace{t-x \in \{0,1,\ldots\}}_{\underbrace{x-t \in \{\ldots,-1,0\}}_{x \in \{\ldots,t-1,t\}}}}$

$= (1-p)^t p^2 \sum_{x \in \{0,1,\ldots\}} \mathbb{1}_{x \in \{\ldots,t-1,t\}}$



$\hookrightarrow \{0,1,\ldots\} \cap \{\ldots,t-1,t\}$

$= (1-p)^t p^2 \sum_{x \in \{0,\ldots,t\}} 1 = (t+1)(1-p)^t p^2 = \text{NegBin}(2,p)$

the negative binomial r.v.

$\leftarrow T \rightarrow$

$\underbrace{0 \quad 0 \quad \ldots \quad 1 \quad 0 \quad 0 \quad 0 \quad \ldots \quad 0}_{t+1 \text{ realizations}} \quad \underset{t+2}{1}$ — $p \cdot p = p^2$

$\hookrightarrow$ thus $t+1$ possible locations for the first 1

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Geom}(p)$, $T_3 = X_1 + X_2 + X_3 \sim p_{T_3}(t) = ?$

$= X_3 + T_2$

$p_{T_3}(t) = \sum_{x \in \text{Supp}[X_3]} p^{old}_{X_3}(x) p^{old}_{T_2}(t-x) \mathbb{1}_{t-x \in \text{supp}[T_2]} = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p (t-x+1)(1-p)^{t-x} p^2 \mathbb{1}_{t-x \in \{0,1,\ldots\}}$

$= (1-p)^t p^3 \sum_{x \in \{0,1,\ldots\}} (t+1-x) \mathbb{1}_{x \in \{\ldots,t-1,t\}}$

$$\boxed{\sum_{x \in S} a + bx = \sum_{x \in S} a + \sum_{x \in S} bx = a \sum_{x \in S} 1 + b \sum_{x \in S} x}$$

$= (1-p)^t p^3 \sum_{x \in \{0,\ldots,t\}} (t+1) + -x = (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\ldots,t\}} 1 - \sum_{x \in \{0,\ldots,t\}} x \right)$

$= (1-p)^t t^3 \left(\underbrace{(t+1)(t+1) - \frac{t(t+1)}{2}}_{t^2 + 2t + 1 - \frac{t^2}{2} - \frac{t}{2}} \right) \rightarrow \frac{t^2+3t+2}{2} = \frac{(t+2)(t+1)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$

$= \binom{t+2}{2}(1-p)^t p^3 = \text{NegBin}(3,p)$

↳ we have $t+2$ realizations ; $t+2$ locations to put 2 ones in

$X_1, X_2, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p)$, $T_r := X_1 + X_2 + \ldots + X_r \sim \text{NegBinom}(r,p) = p_{T_r}(t)$

$p_{T_r}(t) = \binom{t+r-1}{r-1}(1-p)^t p^r \mathbb{1}_{t \in \{0,1,\ldots\}}$

$X \sim \text{Binom}(n,p)$

Let n be really large and p be really small, $n \to \infty$, $p \to 0$, but $\lambda = np$

Our goal is to get the PMF of X under this limit.

$\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$\lim\limits_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x\in\{0,\ldots,n\}} = \lim\limits_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \lim\limits_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x\in\{0,\ldots,n\}}$

$= \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!\,n^x} \lim\left(1-\frac{\lambda}{n}\right)^n \lim\left(1-\frac{\lambda}{n}\right)^{-x} = \frac{\lambda^x}{x!} \lim \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x \text{ terms}}}{\underbrace{n\cdot n \; n \cdots n}_{x \text{ terms}}} e^{-\lambda}(1)\mathbb{1}_{x\in\{0,1\ldots\}}$

$= \frac{\lambda^x e^{-\lambda}}{x!}\mathbb{1}_{x\in\{0,1,\ldots\}} \cancel{\lim \frac{n}{n}}^{1} \cancel{\lim \frac{n-1}{n}}^{1} \cdots \cancel{\lim \frac{n-x+1}{n}}^{1}$

$= \frac{\lambda^x e^{-\lambda}}{x!}\mathbb{1}_{x\in\{0,1,\ldots\}} = \text{Poisson}(\lambda)$, $\star\, \lambda \in (0,\infty)$ parameter space

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1 + X_2 \sim p_T(t) = ?$

$p_T(t) = \sum\limits_{x\in\{0,1\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \mathbb{1}_{\underbrace{t-x\in\{0,1,\ldots\}}_{x\in\{\ldots,t-1,t\}}}$

$= \lambda^t e^{-2\lambda} \sum\limits_{x\in\{0,\ldots,t\}} \frac{1}{x!(t-x)!} = \lambda^t e^{-2\lambda} \sum\limits_{x\in\{0,\ldots,t\}} \frac{t!}{t!}\frac{1}{x!(t-x)!} = \frac{\lambda^t e^{-2\lambda}}{t!} \underbrace{\sum\limits_{x\in\{0,\ldots,t\}} \binom{t}{x}}_{\binom{t}{0}+\ldots+\binom{t}{t} = 2^t}$

$= \frac{\lambda^t e^{-2\lambda}}{t!} 2^t = \frac{(2\lambda)^t e^{-2\lambda}}{t!} = \text{Poisson}(2\lambda)$