$A \in \mathbb{R}^{L\times k}$ (constants) $\qquad a_i := i^{th}$ row vector of A

$$\underset{\substack{(L\times k)(k\times 1)\\ L\times 1}}{E[A\vec{X}]} = \begin{bmatrix} E[a_{11}X_1 + a_{12}X_2 + \dots + a_{1k}X_k] \\ E[a_{21}X_1 + a_{22}X_2 + \dots + a_{2k}X_k] \\ \vdots \\ E[a_{L1}X_1 + a_{L2}X_2 + \dots + a_{Lk}X_k] \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{X}_1] \\ E[\vec{a}_2 \cdot \vec{X}_2] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}_k] \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\mu} \\ \vdots \\ \vec{a}_L \cdot \vec{\mu} \end{bmatrix} = A\vec{\mu}$$

$a \in \mathbb{R}^k$

$$\underset{\substack{(1\times k)(k\times 1)\\ \text{scalar}}}{Var[\vec{a}^T\vec{X}]} = Var[\overbrace{a_1X_1}^{Y_1} + \dots + \overbrace{a_kX_k}^{Y_k}] = Var[Y_1 + \dots + Y_k] = \sum_{i=1}^{k}\sum_{j=1}^{k} Cov[Y_i, Y_j]$$

$$= \sum\sum Cov[a_iX_i, a_jX_j]$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_j\sigma_{ij}$$

$$= \vec{a}^T \underset{Var[\vec{X}]}{\Sigma} \vec{a} \rightarrow (1\times k)(k\times k)(k\times 1) = \text{Scalar}$$

Let $V \in \mathbb{R}^{k\times k}$, $\vec{a} \in \mathbb{R}^k$

$$\vec{a}^T V\vec{a} = \vec{a}\cdot(V\vec{a}) = \vec{a}\begin{bmatrix} a_1V_{11} + \dots + a_kV_{1k} \\ a_1V_{21} + \dots + a_kV_{2k} \\ \vdots \\ a_1V_{k1} + \dots + a_kV_{kk} \end{bmatrix} = \begin{matrix} a_ia_jV_{11} + \dots + a_1a_kV_{1k} + \\ a_2a_1V_{21} + \dots + a_2a_kV_{2k} + \\ \vdots \\ a_ka_1V_{k1} + \dots + a_ka_kV_{kk} \end{matrix}$$

quadratic forms with V being the "determining matrix"

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_jV_{ij}$$

Application in finance Let $X_1, X_2, \dots, X_k$ be financial assets (e.g. stocks)
So let $w_1, \dots, w_k$ be the proportion allocated to each of these assets.
Let $\vec{\mu} = E[\vec{X}]$, $\Sigma = Var[X]$
$F = \vec{w}^T\vec{X}$ a r.v. modeling your portfolio
$\mu_F = E[F] = \vec{w}^T\vec{\mu}$, $Var[F] = \vec{w}^T\Sigma\vec{w}$
It's possible to pick w-vector to optimize the portfolio by minimizing the variance of returns, $Var[F]$, conditional on μ_F. This is called Markowitz optimal portfolio theory.
$\min_{\vec{w}} Var[F]$ subject to μ_F being constant and $\vec{w}^T\vec{1} = 1$.

$\vec{X} \sim Multi_k(n, \vec{p})$, $X_j \sim Bin(n, p_j)$

$$E[\vec{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$$

$$\mathrm{Var}[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & \leftarrow \sigma_{ij} \rightarrow & \\ & np_2(1-p_2) & & \\ \sigma_{ij} & & \ddots & \\ & & & np_k(1-p_k) \end{bmatrix}$$

$i \neq j$

$$\mathrm{Cov}[X_i, X_j] = E[X_iX_j] - E[X_i]E[X_j] = \underbrace{\sum_{x_i \in \mathbb{R}} \sum_{x_j \in \mathbb{R}} x_i x_j \, p_{X_iX_j}(x_i, x_j)}_{\text{complicated}} - n^2 p_i p_j$$

$$\vec{X} = \begin{bmatrix} X_i \sim \mathrm{Bin}(n, p_i) \\ \vdots \\ X_j \sim \mathrm{Bin}(n, p_j) \\ \vdots \end{bmatrix}$$

$X_i = \boxed{X_{1i}} + X_{2i} + \ldots + X_{ni}$ where $X_{1i}, \ldots, X_{ni} \overset{iid}{\sim} \mathrm{Bern}(p_i)$

$X_j = \boxed{X_{1j}} + X_{2j} + \ldots + X_{nj}$ where $X_{1j}, \ldots, X_{nj} \overset{iid}{\sim} \mathrm{Bern}(p_j)$

We've expressed the multinomial r.v. with $n \times k$ Bernoulli's.

$\rightarrow \vec{X} = \underline{\vec{X}_1} + \vec{X}_2 + \ldots + \vec{X}_n$ where $\vec{X}_1, \ldots, \vec{X}_n \overset{iid}{\sim} \mathrm{Multi}_k(1, \vec{p})$

$$\mathrm{Cov}[X_i, X_j] = \mathrm{Cov}[X_{1i} + \ldots + X_{ni}, X_{1j} + \ldots + X_{nj}]$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} \mathrm{Cov}[X_{li}, X_{mj}]$$

A lot of these covariances are zero due to independence. Which ones?

If l is different than m, the covariance is zero.

$$\hookrightarrow = \sum_{l=1}^{n} \mathrm{Cov}[X_{li}, X_{lj}]$$

$$= \sum_{l=1}^{n} \Big(\underbrace{E[X_{li}X_{lj}]} - \overbrace{E[X_{li}]}^{p_i} \overbrace{E[X_{lj}]}^{p_j}\Big) = \boxed{-np_ip_j}$$

b/c you can't get an apple & banana on one grab

$$= \sum_{x_{li} \in \{0,1\}} \sum_{x_{lj} \in \{0,1\}} x_{li} x_{lj} \, p_{X_{li}X_{lj}}(x_{li}, x_{lj}) = p_{X_{li},X_{lj}}(1,1) = 0$$

the only term that is nonzero is .

Uniform Discrete: $\rightarrow$ Start of Midterm 2 material

$$X \sim U(\{0,1,2,3\}) = p(x) = \begin{cases} 0 & \text{w.p } \frac{1}{4} \\ 1 & \text{w.p } \frac{1}{4} \\ 2 & \text{w.p } \frac{1}{4} \\ 3 & \text{w.p } \frac{1}{4} \end{cases} \qquad \mathrm{Supp}[X] = \{0,1,2,3\}$$

Generally, $X \sim U(A)$. $\mathrm{Supp}[X] = A$, $A \subset \mathbb{R}$ s.t $|A| < \infty$ & $A \neq \emptyset$

Create a new r.v $Y = -X = g(X)$, a very simple function

$\mathrm{Supp}[Y] = \{-3, -2, -1, 0\}$

$$P(y) = \begin{cases} -3 & \text{w.p } \frac{1}{4} \\ -2 & \text{w.p } \frac{1}{4} \\ -1 & \text{w.p } \frac{1}{4} \\ 0 & \text{w.p } \frac{1}{4} \end{cases}$$

Generally, for discrete r.v. X, is there a pattern?

$$p_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) =: p_X(-y)$$

$$\begin{aligned}
Supp[Y] &= \{z \mid p_Y(z) > 0\} \\
&= \{z \mid p_X(-z) > 0\} \\
&= \{-z \mid p_X(z) > 0\} \\
&= -\{z \mid p_X(z) > 0\} \\
&=: -Supp[X]
\end{aligned}$$

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1 \dots\}}$

In class we showed $T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$

$\to$ let $\underset{\text{difference}}{D} = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{-X_2}_{Y}$

$$= X + Y \quad , \quad Y \sim p_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\dots,-1,0\}}$$

$$p_D(d) = \sum_{x \in Supp[X]} p_X^{old}(x) p_Y^{old}(d-x) \mathbb{1}_{d-x \in Supp[Y]}$$

$$\begin{aligned}
Supp[D] &= Supp[X] + Supp[Y] \\
&= \{\dots, -1, 0, 1, \dots\} \\
&= \mathbb{Z} \text{ (all ints.)}
\end{aligned}$$