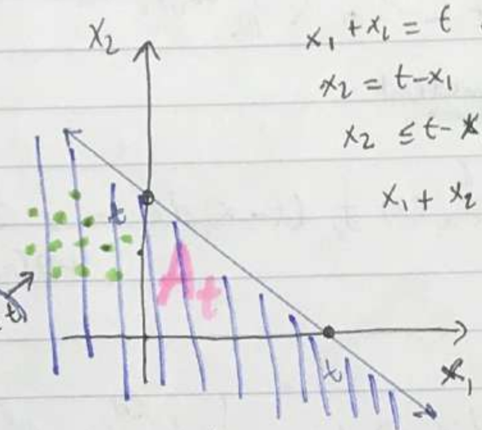
Wednesday September 30th 2020

Lecture 8

$T = X_1 + X_2 \sim f_T(t) = ?$ It also has CDF $f_T(t) = F_T'(t)$ ← CDF method: compute the convolution

$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{X_1 X_2}(x_1, x_2)\, dx_2\, dx_1$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{t-x_1} f_{X_1 X_2}(x_1, x_2)\, dx_2 \right) dx_1 \quad \text{when } T \le t$$



$x_1 + x_2 = t \iff$
$x_2 = t - x_1$
$x_2 \le t - x_1 \iff$
$x_1 + x_2 \le t$

$$= \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1, X_2}(x, v-x)\, dv\, dx = \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1, X_2}(x, v-x)\, dx\, dv$$

let $x_1 = x$, $x_2 = v - x \Rightarrow dx_2 = dv$, $x_2 = -\infty \Rightarrow v = -\infty$, $x_2 = t - x \Rightarrow v = t$

$$\Rightarrow f_T(t) = \left[\int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1, X_2}(x, v-x)\, dx\, dv \right]'$$

Leibnitz's Rule for derivatives of integral functions.

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\, dy\right] = g(x, b(x))\, b'(x) + g(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[g(x,y)\right] dy$$

If the derivative is with respect to third variable, t, then:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} \cdots\cdots \right] = \left(g(x, b(t))\right) b'(t) + g(x, a(t))\, a'(t) + \underbrace{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\left[g(x,y)\right] dy}_{0 \text{ since it's a constant}}$$

If one g the bounds is constant then....

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\,dy\right] = g(x,b(t))\,b'(t) + g(x,c)\underbrace{\frac{d}{dt}[c]}_{0}$$

$$f_T(t) = \frac{d}{dt}\left[\int_{-\infty}^{t}\left(\int_{\mathbb{R}} f_{X_1,X_2}(x,v-x)\,dx\right)\right] = \int_{\mathbb{R}} f_{X_1,X_2}(x,t-x)\,dx$$

general convolution formula.

X_1, X_2 independent

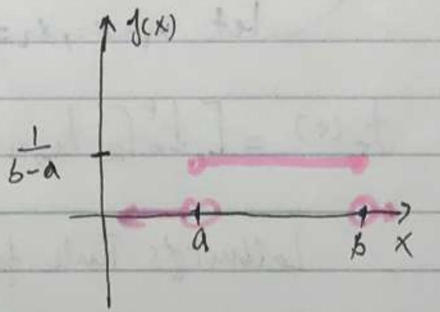
$$= \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\,dx = \int_{Supp[X_1]} f^{old}_{X_1}(x)\, f^{old}_{X_2}(t-x)\, \mathbb{1}_{t-x\in Supp[X_2]}\,dx$$

$X_1, X_2 \overset{iid}{\sim}$

$$= \int_{\mathbb{R}} f(x)\,f(t-x)\,dx = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x\in Supp[X_2]}\,dx$$

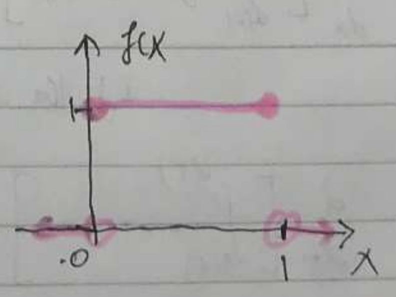
Continuous uniform random variable.

$$X \sim U(a,b) = \overbrace{\underbrace{\frac{1}{b-a}}_{f^{old}(x)}\ \mathbb{1}_{x\in[a,b]}}^{f(x)}$$



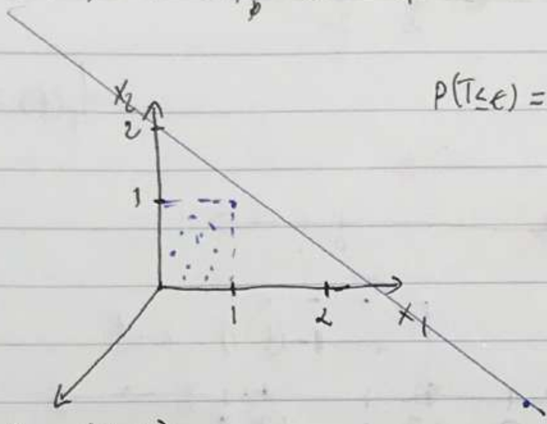
The "Standard uniform" r.v is when $a=0$, $b=1$

$$X \sim U(0,1) = \mathbb{1}_{x\in[0,1]}$$



③

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$ CDF method first

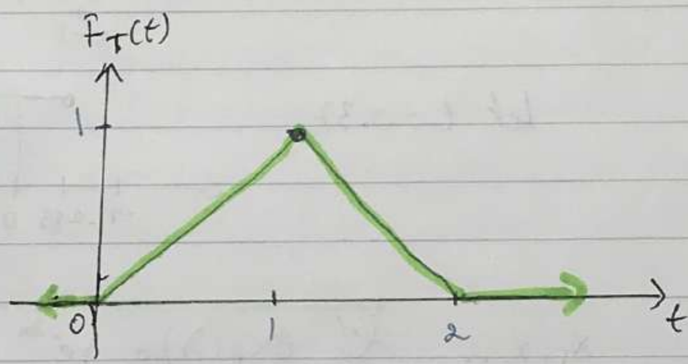


$P(T \le t) = F_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \frac{t^2}{2} & \text{if } t \in [0,1], \quad t \in (1,2) \\ 1 & \text{if } t \ge 2 \end{cases}$

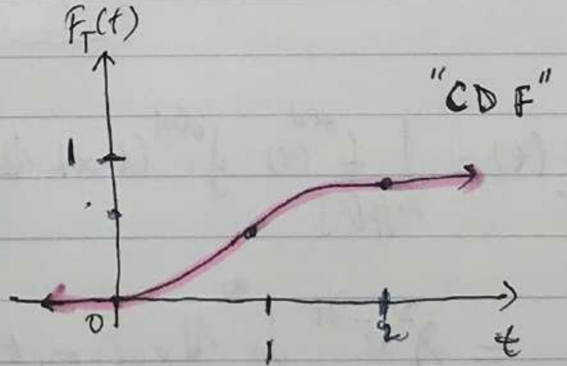
$\frac{t^2}{2} - \cancel{2}\left(\frac{(t-1)^2}{\cancel{2}}\right) = \frac{t^2}{2} - (t^2 - 2t + 1)$

$f_{X_1,X_2}(x_1,x_2)$

$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t & \text{if } t \in (1,2] \\ 0 & \text{if } t \ge 2 \end{cases}$



$Supp[T] = [0,2]$

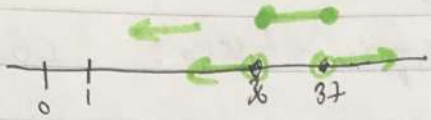


Let's try to derive the PDF of T using the convolution formula

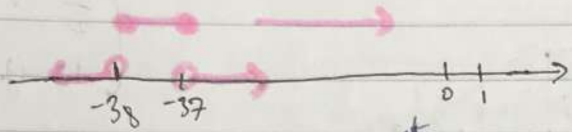
$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx = \int_0^1 (1)(1)\, \mathbb{1}_{\underbrace{t-x \in [0,1]}_{\substack{x-t \in [-1,0] \\ x \in [t-1,t]}}}\, dx$$

$$= \int_0^1 \underbrace{\mathbb{1}_{x \in [t-1,t]}}_{f(x)}\, dx$$

let $t = 37$

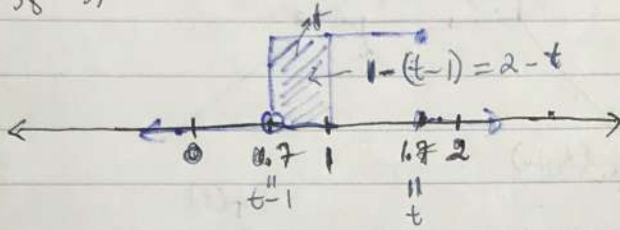


Let $t = -37$

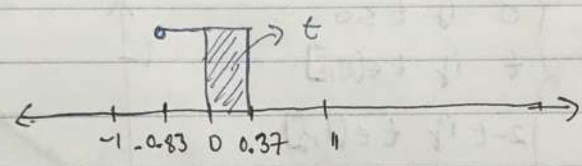


$$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } t \in (0,1] \\ 2-t = 1-(t-1) & \text{if } t \in [1,2] \\ 0 & \text{if } t > 2 \end{cases}$$

let $t = 1.7$



let $t = 0.37$



$X_1, X_2, \ldots \overset{iid}{\sim} \underbrace{\underbrace{Exp(\lambda)}_{f^{old}(x)} = \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}$, $T_2 = X_1 + X_2 \sim f_T(t) = ?$

$$f_{T_2}(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx = \int_0^\infty \lambda e^{-\lambda x} \lambda^{-\lambda(t-x)} \underbrace{\underbrace{\mathbb{1}_{t-x \in [0,\infty)}}_{x-t \in (-\infty,0]}}_{x \in (-\infty,t)} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^\infty \mathbb{1}_{x \in (-\infty,t]}\, dx = \lambda^2 e^{-\lambda t} \int_0^t dx$$

$$= t \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)}$$

$$\boxed{= Erlang(2,\lambda) = f_{T_2}(t)}$$