Lec 16 11/09/2020

Define $L^1 := \{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \}$ "L1 integrable" or "absolutely integrable" functions.

Are all PDFs in the set L1? Yes

$$\int_{-\infty}^{\infty} |\lambda e^{-\lambda x} \mathbb{1}_{x\in[0,\infty)}|\,dx = \lambda \int_0^{\infty} e^{-\lambda x}\,dx = 1$$

If $f \in L^1 \Rightarrow \exists\, \hat{f}$, the "Fourier Transform" of f:

$$\hat{f}(\omega) := \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x)\,dx = F[f]$$

"forward fourier transform operator" AKA "Fourier analysis"

If $\hat{f} \in L^1 \Rightarrow$ then we can invert / reverse the Fourier transform via the "inverse / reverse fourier transform operator" to get the original f back AKA "Fourier synthesis":

$$f(x) = \int_{\mathbb{R}} e^{i2\pi x\omega} \hat{f}(\omega)\,d\omega$$

Fourier inversion thm: if f, fhat are in L1, then f and $\hat{f}$ are 1:1

f(x) is known as the "time domain" and $\hat{f}$(omega) is known as the "frequency domain". f(x) can be decomposed into a sum of sines and cosines with frequencies omega, amplitides given by $|\hat{f}(\text{omega})|$ and phase shifts given by $\text{Arg}[\hat{f}(\text{omega})]$.

Let X be a r.v. Define the characteristic function chf:

$$\phi_X(t) := E[e^{itX}] = \begin{cases} \int_{\mathbb{R}} e^{itx} f_X(x)\,dx & \text{if continuous.} \\ \sum_{x \in \mathbb{R}} e^{itx} P_X(x) & \text{if discrete} \end{cases}$$

The chf is the Fourier transform in a different unit $t = -2$ pi omega.

Properties of the chf:

(P0) $\phi_X(0) := E[e^{i(0)X}] = E[e^0] = 1$ for all rv's

(P1) $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$

(P2) $Y = aX + b$ for $a, b \in \mathbb{R}$

$$\phi_Y(t) = E[e^{it(aX+b)}] = E[e^{iatX} e^{itb}]$$
$$= e^{itb} E[e^{i\overset{t'}{at}X}] = e^{itb} \phi_X(t') = e^{itb} \phi_X(at)$$

(P3) $X_1, X_2 \overset{ind}{\sim}$ and $T = X_1 + X_2$

$$\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1} e^{itX_2}] \overset{\downarrow}{=} E[e^{itX_1}] E[e^{itX_2}]$$
$$= \phi_{X_1}(t)\, \phi_{X_2}(t)$$

(P4) "Moment generation"

conditions are satisfied to be able to interchange differen-tiation and integration

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] \overset{\Downarrow}{=} E[d/dt[e^{itX}]]$$
$$= E[iXe^{itX}]$$

$$\phi'_X(0) = E[iXe^{i(0)X}] = iE[X] \Rightarrow E[X] = \frac{\phi'_X(0)}{i}$$

$$\phi''_X(t) = \frac{d}{dt}\left[E[iXe^{itX}]\right] = E\left[iX \frac{d}{dt}[e^{itX}]\right]$$

$$= E[i^2X^2e^{itX}] \Rightarrow E[X^2] = \frac{\phi''_X(0)}{i^2}$$

$$\Rightarrow E[X^n] = \frac{\phi_X^{(n)}(0)}{i^n}$$ if the moment exists

(P5) $\phi_X(t) \in [-1,1]$ for all X, t hence it always exists.

$$\Updownarrow$$

$$|\phi_X(t)| \in [0,1]$$

$$\|$$

Proof $|E[e^{itX}]| \overset{\prime}{=} \left|\int_{\mathbb{R}} e^{itX} f(x)\,dx\right| \leq \int_{\mathbb{R}} |e^{itX} f(x)|\,dx$

$\|$ $\leq \int_{\mathbb{R}} |e^{itX}||f(x)|\,dx$

discrete same proof $= \int_{\mathbb{R}} \overbrace{|i\sin(tx) + \cos(tx)|}^{bi+a} f(x)\,dx$

$$= \int_{\mathbb{R}} \overbrace{\sqrt{\sin^2(tx) + \cos^2(tx)}}^{1}\, f(x)\,dx = 1$$

(P6) Inversion If $\phi_X(t) \in L^1$, then.

PDF: $f_X(x) = \int_{\mathbb{R}} e^{-itX} \phi_X(t)\,dt.$

(P7) Levy's CDF thm (works even if $\phi_X \notin L^1$)

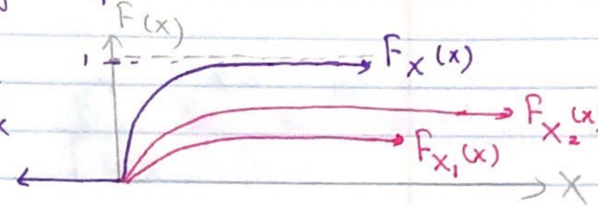
$$P(X \in [a,b]) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \phi_X(t)\, dt$$

(P8) Levy's continuity thm:

consider a sequence of rv's $X_1, X_2, \dots, X_n$. We define "X_n converges in distribution to X" and denote it $X_n \xrightarrow{d} X$

if the CDF of X_n converges pointwise to the CDF of X,

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \;\forall x$$



If $\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t) \;\forall t \implies X_n \xrightarrow{d} X$.

the distribution on the left (X_n) is becoming more and more like the distribution on the right (X).

(1) Define $M_X(t) := E[e^{tX}]$, the moment generating function (mgf)

(P0) $M_X(0) = E[e^{(0)X}] = 1$

(P1) $M_X(t) = M_Y(t) \implies X \stackrel{d}{=} Y$

(P2) $Y = aX + b \implies M_Y(t) = e^{tb} M_X(at)$

P3 X_1, X_2 ind, $T = X_1 + X_2$ then $M_T(t) = M_{X_1}(t)\, M_{X_2}(t)$

P4 $E[X^n] = M_X^{(n)}(0)$

but... mgf's sometimes don't exist!! And some times don't exist for all t.

I don't care about mgf's. Why? Because chf's can do everything they can do and much much more!

$X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x}\, dx$$

$$= \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ ind of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, $T = X_1 + X_2$

$$\phi_{X_1+X_2}(t) \overset{P3}{=} \phi_{X_1}(t)\, \phi_{X_2}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1} \left(\frac{\beta}{\beta - it}\right)^{\alpha_2}$$

$$= \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2} \overset{P1}{\Longrightarrow} \underbrace{X_1 + X_2}_{T} \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$