$\vec{g}: \mathbb{R}^n \to \mathbb{R}^n$ and $1:1$, continuous $\vec{X}, \vec{Y}$ rv vectors of dim n s.t. $\vec{Y} = g(\vec{X})$.

"$[g_1 \ldots g_n]^T$"

Given $f_{\vec{X}}(\vec{x})$, find $f_{\vec{Y}}(\vec{y})$.

$\vec{h} = \vec{g}^{-1} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$

recall what a multidimensional function is:

$$Y_1 = g_1(X_1, \ldots, X_n), \qquad X_1 = h_1(Y_1, \ldots, Y_n),$$
$$Y_2 = g_2(X_1, \ldots, X_n), \quad \exists h_j\text{'s} \ldots \quad X_2 = h_2(Y_1, \ldots, Y_n),$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$Y_n = g_n(X_1, \ldots, X_n) \qquad X_n = h_n(Y_1, \ldots, Y_n)$$

using multivariable calculus, you can show that

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))\,|J_h(\vec{y})|$$

the Jacobian determinant

$$J_h(\vec{y}) = \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$$

Let's verify the convolution formula via this general change-of-variables formula.

Y_1 := $T = X_1 + X_2 \sim f_T(t)$

Recipe:
(1) Find a "clever" g so that
(2) We can find an h
(3) Compute J_h
(4) Compute the multidimensional change of variables formula
(5) Integrate out the "nuisance dimension"

(1) $Y_1 = X_1 + X_2 = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$ — a nuisance dimension

(2) $X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1\cdot 1 - -1\cdot 0 = 1$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(\vec{h}(\vec{y}))|J_h| = f_{\vec{X}}(y_1 - y_2, y_2)|1| = f_{\vec{X}}(y_1 - y_2, y_2)$

(5) $f_T(t) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{X}}(y_1 - y_2, y_2)\, dy_2 = \int_{\mathbb{R}} f_{X_1,X_2}(t-u, u)\, du$ — exactly the general convolution formula

if X_1, X_2 indep.
$$\stackrel{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(t-u) f_{X_2}(u)\, du \stackrel{\text{if iid}}{=} \int_{\mathbb{R}} f(t-u) f(u)\, du$$
$$= \int_{Supp[X_2]} f^{dd}_{X_1}(t-u)\, \mathbb{1}_{t-u \in Supp[X_1]}\, f^{dd}_{X_2}(u)\, du$$
$$= \int_{Supp[X]} f(t-u)\, \mathbb{1}_{t-u \in Supp[X]}\, f(u)\, du$$

$R = \frac{X_1}{X_2} \sim f_R(r) = ?$

(1) $Y_1 = \frac{X_1}{X_2} = g_1(X_1, X_2)$, $Y_2 = X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1 X_2 = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} = y_2$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|$

(5) $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2)\,|y_2|\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(ru, u)|u|\, du$

if X_1, X_2 indep
$$\stackrel{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u)|u|\, du \stackrel{\text{if iid}}{=} \int_{\mathbb{R}} f(ru) f(u)|u|\, du$$
$$= \int_{Supp[X_2]} f^{dd}_{X_1}(ru)\, \mathbb{1}_{ru \in Supp[X_1]}\, f^{dd}_{X_2}(u)\,|u|\, du$$
$$= \int_{Supp[X]} f^{dd}(ru)\, \mathbb{1}_{ru \in Supp[X]}\, f^{dd}(u)|u|\, du$$

$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = ?$

(1) $Y_1 = \frac{X_1}{X_1 + X_2} = g_1(X_1, X_2)$, $Y_2 = X_1 + X_2 = g_2(X_1, X_2)$

(2) $X_1 = Y_1(X_1 + X_2) = Y_1 Y_2 = h_1(Y_1, Y_2)$, $X_2 = Y_2 - X_1 = Y_2 - Y_1 Y_2 = h_2(Y_1, Y_2)$

(3) $J_h = \det \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{bmatrix} = y_2(1-y_1) - -y_1 y_2 = y_2 - \cancel{y_2 y_1} + \cancel{y_1 y_2} = y_2$

(4) $f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|$

(5) $f_R(r) = f_{Y_1}(y_1) = \int_{\mathbb{R}} f_{\vec{Y}}(y_1, y_2)\, dy_2 = \int_{\mathbb{R}} f_{\vec{X}}(y_1 y_2, y_2 - y_1 y_2)\,|y_2|\, dy_2$

$$= \int_{\mathbb{R}} f_{\vec{X}}(ru, u - ru)|u|\, du$$

$$\int_{x=-\infty}^{x=+\infty} f(x)\, \mathbb{1}_{x \in S}\, dx = \int_S f(x)\, dx$$

X_1, X_2 indep
$$\stackrel{\downarrow}{=} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u - ru)|u|\, du \stackrel{\text{iid}}{=} \int_{\mathbb{R}} f(ru) f(u - ru)|u|\, du$$
$$= {*} \int_{Supp[X_1]} f^{dd}_{X_1}(ru)\, \cancel{\mathbb{1}_{ru \in Supp[X_1]}}\, f^{dd}_{X_2}(u - ru)\, \mathbb{1}_{u - ru \in Supp[X_2]}\,|u|\, du$$

$X_1 \sim Gamma(\alpha_1, \beta)$ ind. of $X_2 \sim Gamma(\alpha_2, \beta)$ $\quad e^{-\beta u} \cancel{e^{+\beta r u}}$

$R = \frac{X_1}{X_1 + X_2} \sim f_R(r) = \int \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} \cancel{e^{-\beta r u}} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u - ru)^{\alpha_2 - 1} e^{-\beta(u - ru)}$

$\Downarrow$ $r \in [0, 1]$ $\quad \frac{[0, \infty)}{r} = [0, \infty)$ $\quad u \in [0, \infty)$ $\quad u(1-r) \in [0, \infty)$

$$\mathbb{1}_{u - ru \in [0, \infty)} |u|\, du = \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} \underbrace{(u - ru)^{\alpha_2 - 1}}_{(u(1-r))^{\alpha_2 - 1} = u^{\alpha_2 - 1}(1-r)^{\alpha_2 - 1}} e^{-\beta u}\, \mathbb{1}_{u - ru \in [0, \infty)}\, u\, du$$

$$= \frac{\beta^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u}\, du}_{j(\alpha_1, \alpha_2)} \propto r^{\alpha_1 - 1}(1-r)^{\alpha_2 - 1} \propto Beta(\alpha_1, \alpha_2)$$

$X_1 \sim Gamma(\alpha_1, \beta)$ indep of $X_2 \sim Gamma(\alpha_2, \beta)$, $R = \frac{X_1}{X_2} \sim f_R(r) = ?$ $\quad r > 0$

$$f_R(r) = \int_{Supp[X_2]} f^{dd}_{X_1}(ru)\, \mathbb{1}_{ru \in Supp[X_1]}\, f^{dd}_{X_2}(u)\, |u|\, du$$

$$= \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \underbrace{(ru)^{\alpha_1 - 1}}_{r^{\alpha_1 - 1} u^{\alpha_1 - 1}} e^{-\beta r u}\, \cancel{\mathbb{1}_{ru \in [0, \infty)}}\, \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u}\, u\, du \qquad u \in [0, \infty)$$

$$= \frac{\cancel{\beta^{\alpha_1 + \alpha_2}}}{\underbrace{\Gamma(\alpha_1)\Gamma(\alpha_2)}} r^{\alpha_1 - 1} \underbrace{\int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\overbrace{\beta(r+1)}^{c} u}\, du} = \frac{1}{B(\alpha_1, \alpha_2)} \frac{r^{\alpha_1 - 1}}{(1+r)^{\alpha_1 + \alpha_2}} \mathbb{1}_{r > 0}$$

$$\frac{1}{B(\alpha_1, \alpha_2)} \qquad \frac{\Gamma(\alpha_1 + \alpha_2)}{\underbrace{(\beta(r+1))^{\alpha_1 + \alpha_2}}_{\cancel{\beta^{\alpha_1 + \alpha_2}}(r+1)^{\alpha_1 + \alpha_2}}}$$

$\stackrel{!!}{=}$ $BetaPrime(\alpha_1, \alpha_2)$