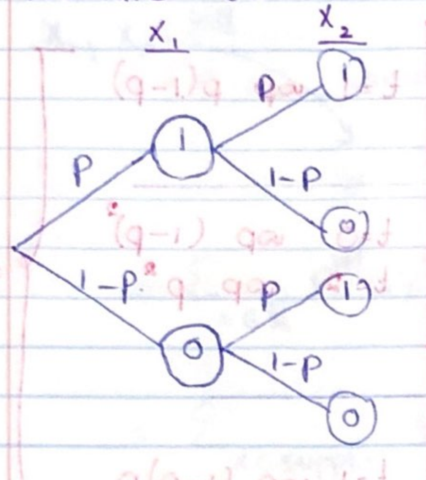
Lecture - 02 08/31/2020



$P_{\vec{X}}(x_1, x_2)$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0

Eg:-
$P(T=1) = P_{X_1,X_1}(1,0) + P_{X_1,X_2}(0,1)$

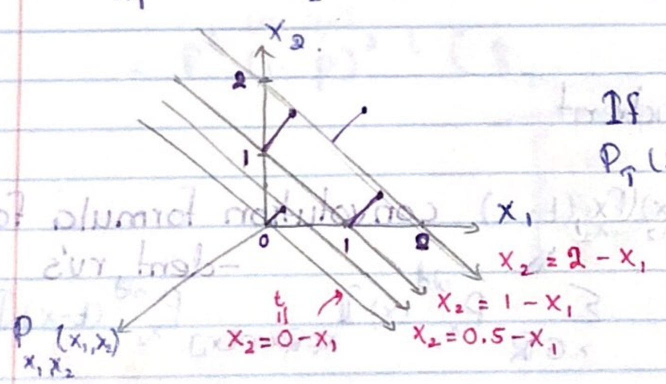
$$T \sim \begin{cases} 2 & \text{with probability } p^2 \\ 1 & \text{wp} \quad 2p(1-p) \\ 0 & \text{wp.} \quad (1-p)^2 \end{cases}$$

$$P(T) = P(T=t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} P_{X_1,X_2}(x_1, x_2) \mathbb{I}_{x_1 + x_2 = t}$$

$\Downarrow$

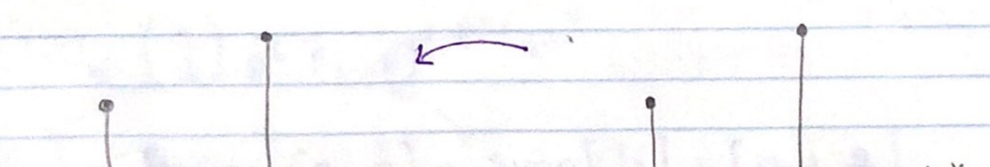
$x_2 = t - x_1$

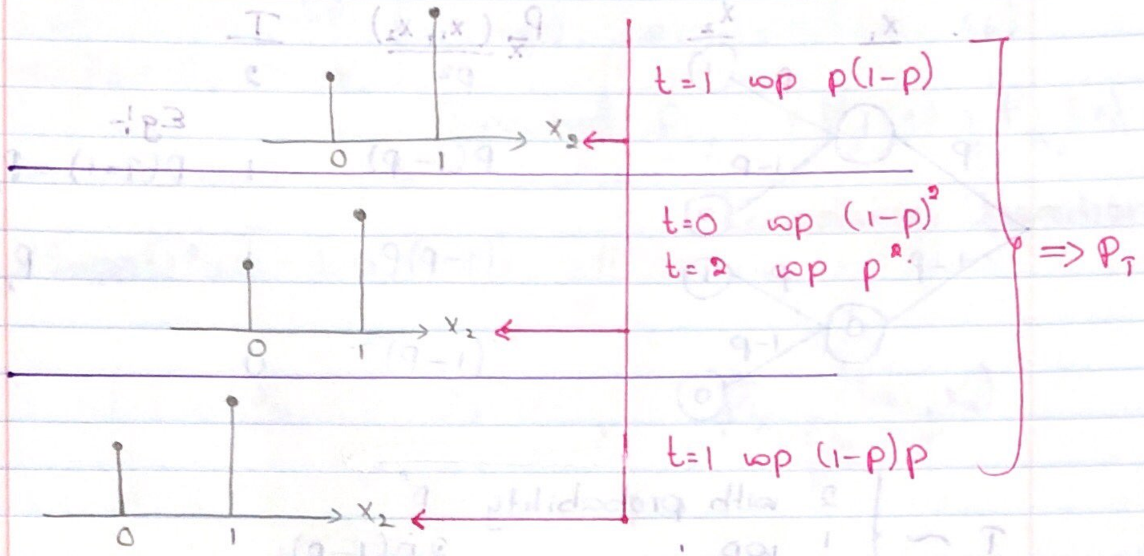
$\| P_{X_1}(x) * P_{X_2}(x)$



If $t = 2$.
$P_T(2) = \sum_{x_1 \in \mathbb{R}} P_{X_1 X_2}(x_1, 2 - x_1)$

"Convolve" means to "roll or coil together/entwine"





$$P_T(t) = \sum_{x_1\in\mathbb{R}} \sum_{x_2\in\mathbb{R}} P_{X_1,X_2}(x_1,x_2)\,\mathbb{1}_{x_2=t-x_1}$$

$$= \sum_{x\in\mathbb{R}} P_{X_1,X_2}(x,t-x)$$ general convolution formula

If X_1, X_2 independent

$$= \sum_{x\in\mathbb{R}} P_{X_1}(x)\,P_{X_2}(t-x)$$ convolution formula for independent rv's

$$= \sum_{x\in\mathbb{R}} P^{old}_{X_1}(x)\,\mathbb{1}_{x\in supp[X_1]}\,P^{old}_{X_2}(t-x)\,\mathbb{1}_{t-x\in supp[X_2]}$$

$$= \sum_{x\in Supp[X_1]} P^{old}_{X_1}(x)\,P^{old}_{X_2}(t-x)\,\mathbb{1}_{t-x\in Supp[X_2]}$$

$X_1, X_2 \overset{iid}{\sim}$ Convolution formula for iid rvs

$$= \sum_{x \in \mathbb{R}} P(x)\, P(t-x)$$

$$= \sum_{x \in \mathbb{R}} p^{old}(x)\, \mathbb{1}_{x \in Supp[X]}\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in Supp[X]} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}$$

$$T_2 \sim \sum_{x \in [0,1]} p^{x} (1-p)^{1-x}\, p^{t-x} (1-p)^{1-t+x}\, \mathbb{1}_{\underbrace{t-x \in [0,1]}_{t \in [x,\, x+1]}}$$

$$= p^{t} (1-p)^{2-t} \sum_{x \in [0,1]} \mathbb{1}_{t \in [x,\, x+1]}$$

$$= p^{t} (1-p)^{2-t} \underbrace{\left(\mathbb{1}_{t \in [0,1]} + \mathbb{1}_{t \in [1,2]} \right)}$$

$$\begin{cases} t = 0 \implies 1 \\ t = 1 \implies 2 \\ t = 2 \implies 1 \end{cases} = \binom{2}{t}$$

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$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}\, \mathbb{1}_{n \in \mathbb{N}}\, \mathbb{1}_{k \in [0,1,\dots,n]}$$

$$= \binom{2}{t} p^{t} (1-p)^{2-t}$$

$$= \text{Binom}(2, p), \quad Supp[T_2] = [0, 1, 2]$$

Generally $\text{Supp}[T] = \text{Supp}[X_1] + \text{supp}[X_2]$,
$A+B := \{a+b : a \in A,\ b \in B\}$

$$P_{T_2}(t) = \sum_{x \in \mathbb{R}} p(x)\, p(t-x)$$

$$= \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) = \underbrace{\binom{1}{x} p^x (1-p)^{1-x}}$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x}\binom{1}{t-x}$$

$$= p^t (1-p)^{2-t} \left(\binom{1}{0}\binom{1}{t} + \binom{1}{1}\binom{1}{t-1} \right)$$

$$= \binom{2}{t} p^t (1-p)^{2-t}$$

$$= \text{Binom}(2, p)$$

Recall Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$

$$T_3 = \overbrace{X_1 + X_2}^{T_2} + X_3 = X_3 + T_2 \sim P_{T_3}(t) = ?$$

$$P_{T_3}(t) = \sum_{x \in \text{supp}[X_3]} P^{old}_{X_3}(x)\, P_{T_2}(t-x)$$

$$= \sum_{x \in \{0,1\}} \left(p^x (1-p)^{1-x} \right) \left(\binom{2}{t-x} p^{t-x} (1-p)^{2-t+x} \right)$$

$$= p^t (1-p)^{3-t} \sum_{x \in [0,1]} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right)$$

$$= \binom{3}{t} p^t (1-p)^{3-t}$$

$$= \text{Binom}(3, p)$$

HW ....

$X_1, X_2 \overset{iid}{\sim} \text{Binom}(n,p) \rightarrow := \binom{n}{x} p^x (1-p)^{n-x}$

$T = X_1 + X_2 \sim ?$

$$P_T(t) = \sum_{x \in \mathbb{R}} p(x)\, p(t-x)$$

$$= \sum_{x \in R} \binom{n}{x} p^x (1-p)^{n-x} \binom{n}{t-x} p^{t-x} (1-p)^{n-t+x}$$

$$= p^t (1-p)^{2n-t} \underbrace{\sum_{x \in R} \binom{n}{x}\binom{n}{t-x}}$$

Vandermonde's identity gives us

$$= \binom{2n}{x} p^t (1-p)^{2n-t}$$

$$= \text{Binom}(2n, p)$$