Lecture 6

$A \in \mathbb{R}^{L \times K}$ $\qquad a_{i\cdot} :=$ i^{th} row vector of A

$$E[A\vec{X}] = \begin{bmatrix} E a_{11}X_1 + a_{12}X_2 + \cdots a_{1K}X_K \\ E a_{21}X_1 + a_{22}X_2 + \cdots a_{2K}X_K \\ \vdots \\ E a_{L1}X_1 + a_{L2}X_2 + \cdots a_{LK}X_K \end{bmatrix} = \begin{bmatrix} E[\vec{a}_1 \cdot \vec{x}_1] \\ E[\vec{a}_2 \vec{X}] \\ \vdots \\ E[\vec{a}_L \cdot \vec{X}] \end{bmatrix} = \begin{bmatrix} a_1 \cdot \vec{\mu} \\ a_2 \mu \\ \vdots \\ a_L \mu \end{bmatrix} = A\vec{\mu}$$

$(L \times K)(K \times \ell) = L \times \ell$

$\vec{a} \in \mathbb{R}^K$

$$\underset{\substack{(1\times K)(K\times 1) \\ \text{Scalar}}}{Var[\vec{a}^T \vec{X}]} = Var[\underbrace{a_1 X_1}_{Y_1} \cdots \underbrace{a_K X_K}_{Y_K}] = Var[Y_1 + \cdots Y_K] = \sum_{i=1}^{K}\sum_{j=1}^{K} cov(Y_i Y_j)$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} Cov(a_i X_i, a_j X_j) = \sum_{i=1}^{K}\sum_{j=1}^{K} a_{ij}\sigma_{ij} = \underset{(1\times K)(K\times K)(K\times 1)}{\vec{a}^T \underset{\substack{\| \\ Var[\vec{x}]}}{\Sigma} \vec{a}} \text{ Scalar}$$

matrix formed

Proof

let $\overset{\Sigma}{V} \in \mathbb{R}^{K \times K}$ $\vec{a} \in \mathbb{R}^K$

$$\vec{a}^T V \vec{a} = \vec{a} \cdot (V\vec{a}) = \vec{a} \cdot \begin{bmatrix} a_1 v_{11} + \cdots a_K v_{1K} \\ a_1 v_{21} + \cdots a_K v_{2K} \\ \vdots \\ a_1 v_{K1} + \cdots a_K v_{KK} \end{bmatrix} = \begin{matrix} a_1 a_1 v_{11} + \cdots a_1 a_K v_{1K} + \\ a_2 a_1 v_{21} \cdots a_2 a_K v_{2K} + \\ \vdots \\ a_K a_1 v_{K1} + \cdots a_K a_K v_{KK} \end{matrix}$$

Quadratic Form with V Being the "Determining matrix"

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} a_i a_j v_{ij}$$

Application in Finance. Let $X_1, X_2 \cdots X_K$ be Returns Financial assets (eg: Stocks). So, let $w_1, w_2 \cdots w_K$ be the proportion allocated to each of these assets. Let $F = \vec{W}^T \vec{X}$ a rv modeling your portfolio

$\mu = E[\vec{X}]$ $\quad \Sigma = Var[\vec{X}]$

$E[F] = \vec{w}^T \mu \quad Var[F] = \vec{w}^T \Sigma \vec{w}$

It's possible to pick w-vector to optimize the portfolio by minimizing the variance of returns $Var[F]$. Conditional on μ_F

[This is called "Markowitz optimal portfolio theory]

$\min_{\vec{w}} Var[F]$ subject to μ_F being constant and $\vec{W}^T \cdot \vec{1} = 1$

$\vec{X} \sim \text{Multin}_k(n, \vec{p})$. $E[\vec{X}] \overset{\text{defn}}{=} \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_k] \end{bmatrix} = \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_k \end{bmatrix} = n\vec{p}$

$X_j \sim \text{Bin}(n, p_j)$

$$\text{Var}[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & \sigma_{ij} \\ & np_2(1-p_2) & \\ \sigma_{ij} & & np_k(1-p_k) \end{bmatrix}$$

Complicated

$$\underset{i \neq j}{\text{Cov}}[X_i X_j] = E[X_i X_j] - E[X_i]E[X_j] = \sum_{x_i \in \mathbb{R}} \sum_{x_j \in \mathbb{R}} x_i x_j p_{X_i X_j}(x_i x_j)$$

FAIL

$- n^2 p_i p_j =$ i=Apple j=Banana

$\vec{X} =$

$$\begin{bmatrix} \vdots \\ X_i \sim \text{Bin}(n, p_i) \\ \vdots \\ X_j \sim \text{Bin}(n, p_j) \\ \vdots \end{bmatrix}$$

$\cdot X_i = X_{1i} + X_{2i} + \dots X_{ni}$ where $X_{1i} \dots X_{ni} \overset{iid}{\sim} \text{Bern}(p_i)$

$X_j = X_{1j} + X_{2j} + \dots X_{nj}$ wher $X_{2j} \dots X_{nj} \overset{iid}{\sim} \text{Bern}(p_j)$

We've expressed multinomial with $n \times k$ Bernoulli's

$\vec{X} = \vec{X}_1 + \vec{X}_2 + \dots \vec{X}_n$ when $\vec{X}_1 \dots \vec{X}_n \overset{iid}{\sim} \text{Multinormal}_k(1, \vec{p})$

$\text{Cov}[X_i, X_j] = \text{Cov}[X_{1i} + \dots X_{ni}, X_{1j} + \dots X_{nj}] =$

$$= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \text{Cov}[X_{\ell i}, X_{mj}]$$

A lot of these covariances are 0 zero due to indepedent.
IF ℓ ~~doesn't~~ is different than m, the covariance is zero 0

$$= \sum_{\ell=1}^{n} \text{Cov}[X_{\ell i}, X_{\ell j}] = \sum_{\ell=1}^{n} \Big(E[X_{\ell i} \cdot X_{\ell j}] - \overset{p_i}{E[X_{\ell i}]} \overset{p_j}{E[X_{\ell j}]} \Big) = -np_i p_j$$

more apple less Banana

$$= \sum_{x_{\ell i} \in \{0,1\}} \sum_{x_{\ell j} \in \{0,1\}} x_{\ell i} x_{\ell j} \, p_{X_{\ell i} X_{\ell j}}(x_{\ell i} x_{\ell j})$$

Bernoulli

only term that is not 0 is

$$= p_{X_{\ell i} X_{\ell j}}(1,1) = 0 \text{ ZERO}$$

you can't get an apple and Banana on one Grab

midterm 1

Midterm 2

uniform discrete

$X \sim U(\{0,1,2,3\}) = P(X) \begin{cases} 0 & wp\ \frac{1}{4} \\ 1 & wp\ \frac{1}{4} \\ 2 & wp\ \frac{1}{4} \\ 3 & wp\ \frac{1}{4} \end{cases}$

$Supp[X] = \{0,1,2,3\}$

Generall $X \sim U(A)$

$Supp[X] = A$, $A \subset \mathbb{R}$ such that $|A| < \infty$ and $|A| \geq 1$ (OR $A \neq \phi$)

Create a new rv. $Y = -X = g(X)$, a very simple Function

$Supp[Y] = \{-3,-2,-1,0\}$ $\quad P(Y) = \begin{cases} -3 & wp\ \frac{1}{A} \\ -2 & wp\ \frac{1}{A} \\ -1 & wp\ \frac{1}{A} \\ 0 & wp\ \frac{1}{A} \end{cases}$

Generally, For discrete rv X, is there a pattern?

~~$Supp[Y] = \{z : P_Y(z) > 0$~~

$P_Y(y) := P(Y=y) = P(-X=y) = P(X=-y) := P_X(-y)$

$Supp[Y] = \{z : P_Y(z) > 0\} = \{z : P_X(-z) > 0\} = \{-z, P_X(z) > 0\}$ Let $z' = -z$

$= -\{z : P_X(z) > 0\} = Supp[X]$

$X_1\ X_2 \overset{iid}{\sim} Poisson(\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \{0,1,\dots\}}$

In class we show $X_1 + X_2 \sim Poisson(2\lambda)$ $\quad Poisson(\lambda_1 + \lambda_2)$

Let $\overset{\text{difference}}{D} = X_1 - X_2 = \underbrace{X_1}_{X} + \underbrace{-X_2}_{Y} = X + Y$, ~~$Y \sim P_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!}$~~

$Y \sim P_Y(y) = \frac{e^{-\lambda}\lambda^{-y}}{(-y)!} \mathbb{1}_{y \in \{\dots,-1,0\}}$

$P_D(d) = \sum_{x \in Supp[X]} P_X^{old}(x)\, P_Y^{old}(d-x) \mathbb{1}_{d-x \in Supp[Y]}$

$Supp[D] = Supp[X] + Supp[Y]$
$= \{\dots,-1,0,1,\dots\}$
$= \mathbb{Z}$
all integers

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