9/29/2020
Lecture 08
Math 621

$$P_T(d) = \sum_{x\in Supp[X]} p_X^{old}(x)\, p_Y^{old}(\underbrace{d-x}_{x-d})\, \mathbb{1}_{d-x\in Supp[Y]} \quad x\in\{d,d+1,d+2,\dots\}$$

$x-d\in\{0,1,\dots\}$

$$= \sum_{x\in\{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!}\, \frac{\lambda^{-(d-x)} e^{-\lambda}}{\underbrace{(-(d-x))}_{x-d}!}\, \mathbb{1}_{d-x\in\{\dots,-1,0\}}$$

$$= e^{-2\lambda} \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!}\, \mathbb{1}_{x\in\{d,d+1,\dots\}}$$

let $d'=-d$

let $d'=-d=|d'|$

$$= e^{-2\lambda}\begin{cases} \text{if } d\leqslant 0 \displaystyle\sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+d'}}{x!(x+d')!} = \sum_{x\in\{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} \\ \text{if } d>0 \displaystyle\sum_{x\in\{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} = \sum_{x'\in\{0,1,\dots\}} \frac{\lambda^{\overset{2x'+d}{2(x'+d)-d}}}{(x'+d)!\,x'!} \end{cases}$$

let $x'=x-d \iff x=x'+d$

$d=|d|$

$$I_{|d|}(2\lambda) := \sum_{x\in\{0,1,\dots\}} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!} \Leftarrow$$

Modified Bessel function of the first kind (it's a solution to a solution to a famous diffrantial eq.) Discovered in 1946

$$= e^{-2\lambda} I_{|d|}(2\lambda)\, \mathbb{1}_{d\in\mathbb{Z}}$$

$= \text{Skellem}(\lambda,\lambda) \Rightarrow$ This is used to model point spreads in sport games, photon noise

$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, $T = X_1+X_2 \sim \text{Poisson}(2\lambda)$

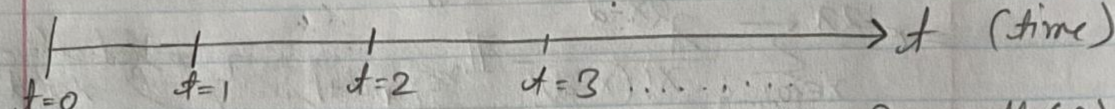
$$P_{X_1|T}(x,t) = \frac{P_{X_1,T}(x,t)}{P_T(t)} = \frac{P_{X_1,X_2}(x,t-x)}{P_T(t)} = \frac{P_{X_1}(x)\,P_{X_2}(t-x)}{P_T(t)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!}\,\frac{e^{-\lambda}\lambda^{t-x}}{(t-x)!}}{\frac{e^{-2\lambda}(2\lambda)^t}{t!}} = \frac{t!}{x!(t-x)!}\,\frac{\lambda^t}{(2\lambda)^t} = \binom{t}{x}\left(\frac{1}{2}\right)^t$$

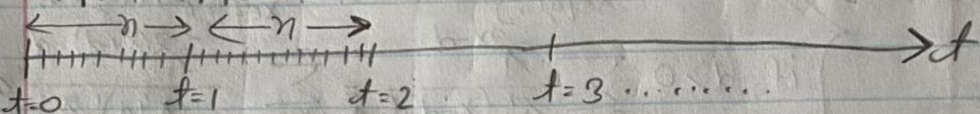
$$= \text{Bin}\left(t, \frac{1}{2}\right)$$

$$X_1 \sim \text{Geom}(p) := \overbrace{(1-p)^x p\, \mathbb{1}_{x\in\{0,1,\dots\}}}^{P(x)}, \quad \text{Supp}[X_1] = \{0,1,\dots\}$$

$$F_{X_1}(x) := P(X_1 \le x) = 1 - P(X_1 > x) = \left(1-(1-p)^x\right)$$



In every "second", let's do n iid Bernoulli(p) experiments.



Let's call the resulting geometric rv X_n.

$$P_{X_n}(x) = (1-p)^{nx} p\, \mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\dots,1,1+\frac{1}{n},1+\frac{2}{n},\dots,2,\dots\}}$$

$$F_{X_n}(x) = \left(1-(1-p)^{nx}\right)$$

Let $n \to \infty$, $p \to 0$, but $\lambda = np \Rightarrow p = \frac{\lambda}{n}$, same as Poisson. where $\lambda \in (0,\infty)$

$$P_{X_\infty}(x) = \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nx}}_{\underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^x}_{e^{-\lambda x}}} \underbrace{\frac{\lambda}{n}}_{\lim_{n\to\infty}\frac{\lambda}{n}} \underbrace{\mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\underbrace{\lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\dots\}}}_{\mathbb{1}_{x\in[0,\infty)}}}$$

$\longrightarrow \text{Supp}[X_\infty] = [0,\infty)$

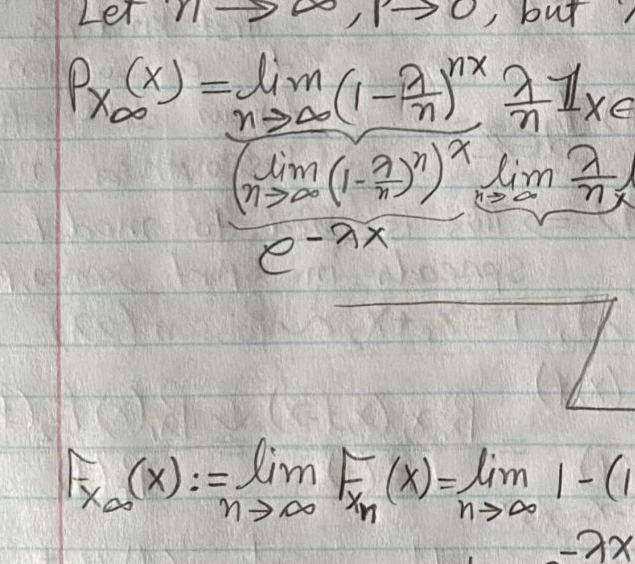
$$\sum_{x\in \text{Supp}[X_\infty]} P_{X_\infty}(x) = 0$$

$$\overset{n}{=} 0 \;\; \forall x!$$

Not a valid PMF!

$$F_{X_\infty}(x) := \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx}$$

$$= 1 - e^{-\lambda x}\, \mathbb{1}_{x\in[0,\infty)}$$



The PMF wasn't valid. Is the CDF valid? If so, I need to check three properties.

① It's 0 as I go to $-\infty$

② It's 1 as I go to $+\infty$

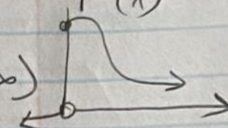
③ It's an increasing function.

check: ✓ (1) $\lim_{x \to -\infty} (1-e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)} = 0$

✓ (2) $\lim_{x \to \infty} (1-e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)} = 1 - \lim_{x \to \infty} \frac{1}{e^{\lambda x}} = 1$

✓ (3) $\frac{d}{dx}\left[(1-e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)}\right] = \lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}$

$F'(x)$



$\Rightarrow$ Valid CDF!

We now have a continuous rv. Continuous rv's have the following properties:

$|Supp[X]| = |\mathbb{R}|$ uncountable infinity (the size of the continuum)

They don't have PMF's, (b/c the probability of the rv being at any specific number is zero) but they <u>do have</u> CDF's.

The derivative of the CDF is a very useful function. It's called Probability density function (PDF) denoted $f(x)$. (Note: discrete rv's don't have PDF's)

$$f(x) := F'(x), \quad P(X \in [a,b]) = \underbrace{P(X \leq b)}_{F(b)} - \underbrace{P(X \leq a)}_{F(a)} = \int_a^b f(x)\,dx$$

By Fundamental Thm of Calc.

$$\int_{\mathbb{R}} f(x) = 1 = F(\infty)^{\nearrow 1} - F(-\infty)^{\nearrow 0}, \quad Supp[X] = \{x : f(x) > 0\}$$

$f(x) \geq 0$ since CDF's are increasing function.

$$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda x} \mathbb{1}_{x \in [0,\infty)}}_{f(x)}, \quad F(x) = (1-e^{-\lambda x}) \mathbb{1}_{x \in [0,\infty)}$$

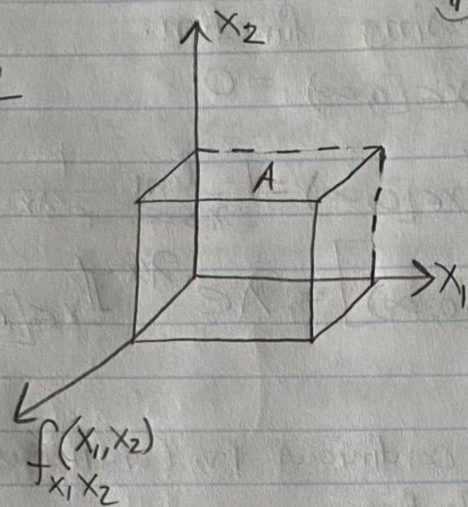
$\Rightarrow$ Exponential rv

$\lambda \in (0,\infty)$ it's parameter space.

$X_1, \dots X_n$ indepdt $\Rightarrow f_{X_1 \dots X_n}(x_1 \dots x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

If $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_k \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$, $\underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{k} f_{\vec{X}}(\vec{x})\, dx_1 \dots dx_k = 1$

$k = 2$



$$P(A) = \iint_A f_{X_1, X_2}(x_1, x_2)\, dx_1 dx_2$$