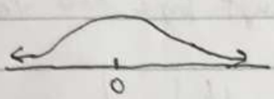
Wednesday October 14 2020

Lecture 11



$X \sim \text{logistic}(0,1) = \frac{e^x}{(1+e^x)^2} \simeq N(0,1)$ both with thicker tails

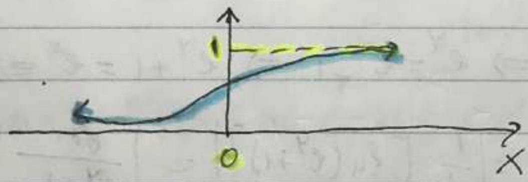
$E[X]=0,\ SD[X]=\frac{\pi}{\sqrt{3}} \simeq 1.8 > 1$

consider the shift and scale sigma > 0

$$Y = \mu + \sigma X \sim f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{|\sigma|} = \frac{e^{\frac{y-\mu}{\sigma}}}{\sigma(1+e^{\frac{y-\mu}{\sigma}})^2} = \text{Logistic}(\mu, \sigma)$$

Why is this called the "logistic distribution"? there's a famous function called the "logistic function". It has three parameters: L (maximum value), k (steepness), ~~mu~~ mu (center) and it is:

$$\ell(x) := \frac{L}{1+e^{-k(x-\mu)}} \overset{\text{if } L=1,\ k=1,\ \mu=0}{=} \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{e^x+1}$$ (standard logistic function).



$X \sim \text{logistic}(0,1)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{e^t}{(1+e^x)^2}\,dt = \int_1^{1+e^x} \frac{1-u}{u^2}\frac{du}{1-u} = [-u^{-1}]_1^{1+e^x} = 1 - \frac{1}{1+e^x} = \frac{e^x}{1+e^x}$$

Let $u = 1+e^t \Rightarrow e^t = 1-u \Rightarrow \frac{du}{dt} = e^t \Rightarrow dt = e^{-t}du = \frac{1}{1-u}du \Rightarrow t=-\infty \Rightarrow u=1,\ t=x \Rightarrow u = 1+e^x$

The "quantile" q or "percentile" $100q$ for r.v X is defined as the minimum x s.t $q \le P(X \le x) = F(x) \Leftrightarrow F(x) \ge q$. It is denoted $Q[X,q]$ where Q is the "quantile operator" (Not the upper incomplete regularized gamma function).

When $q = 0.5$, the quantile has a special name, the "median", $\text{Med}[X] := Q[X, q]$. Here's an example:

$$X \sim U(\{2, 4, 6, \ldots, 20\}) = \frac{1}{10} \mathbb{1}_{x \in \ldots}$$

x	P(x)	F(x)
2	0.1	0.1
4	0.1	0.2
6		0.3
8		0.4
10		0.5
12		0.6
14		0.7
16		0.8
18		0.9 ←
20	↓	1 ←

$$Q[X, 30\%] = 6 \qquad \text{med}[X] = 10$$

$$Q[X, 80\%] = 16$$

$$Q[X, 85\%] = 18 = Q[X, 0.9]$$

However, if X is a continuous r.v with "contiguous support" e.g $[0, 10]$, $[0, \text{infinity})$, all real numbers, etc and not something like $[0, 1]$ union $[2, 3]$. In the latter case, $F(x)$ is flat between $[1, 2]$ which means it's not invertible. In the former case, $F(x)$ is invertible.

$Q[X, q] = F_X^{-1}(q)$, and the inverse CDF is called appropriately, the "quantile function".

$$X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0} \Rightarrow F_X(x) = 1 - e^{-\lambda x} = q \Rightarrow 1 - q = e^{-\lambda x}$$

$$\Rightarrow \ln(1-q) = -\lambda x \Rightarrow x = -\frac{1}{\lambda} \ln(1-q) = \frac{1}{\lambda} \ln\left(\frac{1}{1-q}\right) = F_X^{-1}(q)$$

$$\text{Med}[X] = \frac{\ln(2)}{\lambda} = F_X^{-1}(0.5)$$

Quantile functions are not usually available in closed form since CDF's aren't even usually availble in closed form e.g.

$$X \sim \text{Erlang}(k, \lambda) \Rightarrow F_X(x) = P(k, \lambda x)$$

$\text{Med}[X] = x$ s.t $P(k, \lambda x) = 0.5$. Need a computer solver.

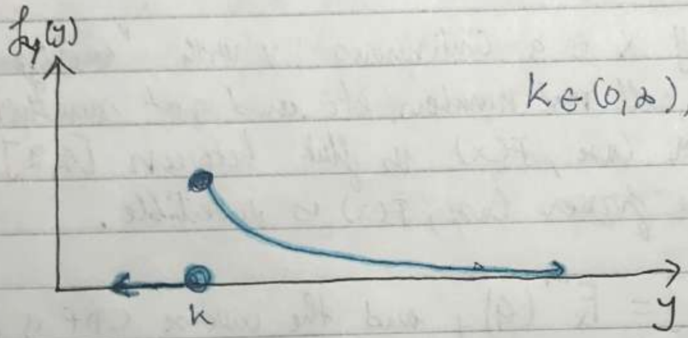
Let $X \sim \exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$, $Y = g(X) = ke^{X} \sim f_Y(y) = ?$

$$y = ke^{x} \Rightarrow \frac{y}{k} = e^{x} \Rightarrow x = \ln\left(\frac{y}{k}\right) = \ln(y) - \ln(k) = g^{-1}(y)$$

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \left|\frac{1}{y}\right| = \frac{1}{|y|}$$

$$f_Y(y) = f_X\left(\ln\left(\tfrac{y}{k}\right)\right)\frac{1}{|y|} = \frac{\lambda}{|y|} \underbrace{e^{-\lambda \ln\left(\frac{y}{k}\right)}}_{\ln\left(\left(\frac{y}{k}\right)^{-\lambda}\right)} \mathbb{1}_{\underbrace{\ln(y) - \ln(k) \in [0,\infty)}_{\underbrace{\ln(y) \in [\ln(k), \infty)}_{y \in [k, \infty).}}}$$

$$= \frac{\lambda}{y}\left(\frac{y}{k}\right)^{-\lambda} \mathbb{1}_{y \in [k,\infty)} = \text{Pareto I}(k, \lambda)$$



$k \in (0,\infty)$, $\lambda \in (0,\infty)$

$$F_Y(y) = \int_k^y \frac{\lambda}{k^{-\lambda}} \frac{1}{t^{\lambda+1}}\,dt = \frac{\lambda}{k^{-\lambda}}\left[-\frac{1}{\lambda t^{\lambda}}\right]_k^y = k^{\lambda}\left(\frac{1}{k^{\lambda}} - \frac{1}{y^{\lambda}}\right) = 1 - \left(\frac{k}{y}\right)^{\lambda}$$

$$\Rightarrow F_Y^{-1}(q) = k(1-q)^{-\frac{1}{\lambda}}$$ "typically test question"

This distribution was discovered by Vilfredo Pareto, an Italian economist in 1896 when he observed that 20% of the richest Italians owned 80% of the land (ie the wealth). This is known as the "Pareto Principle" and it corresponds to the pareto (1, 1.61) distribution.

Futher, the Pareto distribution is a waiting time / survival time model. It's used for [See wikipedia if you're interested]. Wealth, music talent, number of patents, ...

$$Z \sim \frac{1}{|-1|} f_X\left(\frac{z}{-1}\right) = e^{z}\mathbb{1}_{z\in(-\infty,0]}$$

$$X, Y \overset{iid}{\sim} \text{Exp}(1) = e^{-x}\mathbb{1}_{x\in[0,\infty)}, \quad D = X - Y = X + \underset{\sim}{(-Y)} \sim f_D(d) = ?$$

$$f_D(d) = \int_{Supp[X]} f_X^{old}(x)\, f_Z^{old}(d-x)\, \mathbb{1}_{d-x\in Supp(Z)}\, dx \overset{dx}{=} \int_0^\infty e^{-x} e^{d-x}\, \mathbb{1}_{d-x\in(-\infty,0]}\, dx$$

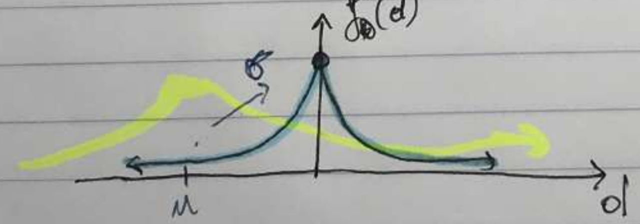
$$x-d\in[0,\infty)$$
$$x\in[d,\infty)$$

$$= e^d \int_0^\infty e^{-2x}\, \mathbb{1}_x[d,\infty)\, dx = e^d \begin{cases} \int_d^\infty e^{-2x}\,dx & \text{if } d\geq 0 \\ \int_0^\infty e^{-2x}\,dx & \text{if } d<0 \end{cases}$$

$$= e^d \begin{cases} \left[-\frac{1}{2}e^{-2x}\right]_d^\infty & \text{if } d\geq 0 \\ \left[-\frac{1}{2}e^{-2x}\right]_0^\infty & \text{if } d<0 \end{cases} = \frac{1}{2}e^d \begin{cases} \left[-e^{-2x}\right]_d^\infty & \text{if } d\geq 0 \\ \left[-e^{-2x}\right]_0^\infty & \text{if } d<0 \end{cases}$$

$$= \frac{1}{2}e^d \begin{cases} e^{-2d} & \text{if } d\geq 0 \\ 1 & \text{if } d<0 \end{cases} = \frac{1}{2}\begin{cases} e^{-d} & \text{if } d\geq 0 \\ e^{d} & \text{if } d<0 \end{cases} = \frac{1}{2}e^{-|d|}$$

↑ Laplace(0,1)
Std Laplace distribution
AKA "double exponential"



$$X = \mu + \sigma D \sim \text{Laplace}(\mu,\sigma) := \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$
$\sigma > 0$