Math 650.2 Problem Set 13

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Problem 1

Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

We first take care of the easy part, namely is every point of every closed set $E \subset \mathbb{R}^2$ a limit point of E? Here is a simple counter example. Consider the set $E = \{(0,0)\}$. This set is closed because every limit point of E belongs to E since the set of limit points is just the empty set (Corollary to Theorem 2.20) and the empty set is a subset of every set. Therefore, E is closed. However, $(0,0)$ is not a limit point for E otherwise this would contradict the corollary to Theorem 2.20.

We now ask ourselves is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Let E be an open set in $\mathbb{R}^2$ and let $p \in E$. Then p is an interior point meaning that there exists at least one neighborhood $N_r(p) \subset E$. Additionally, we know that $N_r(p)$ contains infinitely many points of E (Theorem 2.20). We want to show that p is a limit point of E, that is every neighborhood of p contains a point $q \neq p$ such that $q \in E$. We have two cases, either the radius of the neighborhood of p is smaller than or greater than the radius of an arbitrary neighborhood.

Case One: Let $r \leq s$. Then $N_r(p) \subseteq N_s(p)$. Since $N_r(p)$ contains some distinct point $q \in E$, (because p is a limit point) then $q \in N_s(p)$ and so p is a limit point of E if $r \leq s$.

Case Two: We now assume that $r > s$. Well $N_s(p)$ contains infinitely many points by Theorem 2.20. However, we know something specific about these points. We have $N_s(p) \subset N_r(p) \subset E$, and so $N_s(p)$ contains infinitely many points of E and thus p is a limit point of E since we have shown that for all neighborhoods of p we can find at least one point $q \neq p$ such that $q \in E$.

Problem 2

Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

Let $p \in E^\circ$. We wish to show that p is an interior point of E°. Since $p \in E^\circ$, then p is an interior point of E. That means there exists some neighborhood, $N_r(p)$ such that $N_r(p) \subset E$. By Theorem 2.19, we know that $N_r(p)$ is open. That means if $q \in N_r(p)$, then there exists some $N_s(q)$ such that $N_s(q) \subset N_r(p) \subset E$. This shows that q is an interior point of E and so $q \in E^\circ$. Since q was arbitrary, we see that all points in $N_r(p)$ are interior points of E so $N_r(p) \subset E^\circ$. Thus we have shown that every point, $p \in E^\circ$ is an interior point of E° since we have shown for every point in E°, there exists a neighborhood $N_r(p) \subset E^\circ$ which proves E° is open.

(b) Prove that E is open if and only if $E^\circ = E$.

If $E^\circ = E$, then E is open by part (a). So assume that E is open. We wish to show that $E^\circ = E$. Since E is open, if $p \in E$, then there exists some $N_r(p) \subset E$ and so $p \in E^\circ$. This shows that $E \subseteq E^\circ$. Conversely, let $p \in E^\circ$. Then p is an interior point of E meaning there exists some neighborhood $N_r(p)$ such that $N_r(p) \subset E$. Since $p \in N_r(p)$, then $p \in E$ which shows that $E^\circ \subseteq E$. Thus we have shown both sets are subsets of each other which implies that $E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Let $p \in G$. Since G is open, p is an interior point of G meaning that we can find some neighborhood $N_r(p)$ such that $N_r(p) \subset G \subset E$. This tells us that every point in G is also an interior point of E since $N_r(p) \subset E$. So $p \in E^\circ$ which shows $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

We must show that $(E^\circ)^c = \bar{E}^c$. As usual, we will show set equality.

Let $p \in (E^\circ)^c$. Then $p \notin E^\circ$ so that means p is not an interior point of E. This means that for all neighborhoods, the intersection of $N_r(p)$ and E^c cannot be empty since all neighborhoods of p are not wholly contained in E. So we have two cases, either $p \in E^c$ or there exists some x where $p \neq x$ such that $x \in N_r(p) \cap E^c$.

Case One: If $p \in E^c$ then $p \in E^c \cup (E^c)' = \bar{E}^c$ and thus $(E^\circ)^c \subset \bar{E}^c$.

Case Two: Now assume that there exists some x where $p \neq x$ such that $x \in N_r(p) \cap E^c$. Then $p \in (E^c)'$ since we have shown p is a limit point of E^c. So $p \in E^c \cup (E^c)' = \bar{E}^c$ and thus $(E^\circ)^c \subset \bar{E}^c$.

By reversing the proof above we can show that $\bar{E}^c \subset (E^\circ)^c$. Let $p \in \bar{E}^c$. Then $p \in E^c$ or $p \in (E^c)'$.

Case One: If $p \in E^c$, then $p \notin E$ so p is not an interior point of E because all neighborhoods $N_r(p)$ are not wholly contained in E since $p \notin E$. Thus $p \notin E^\circ$ and so $p \in (E^\circ)^c$.

Case Two: If $p \in (E^c)'$, then p is a limit point of E^c so every neighborhood of p contains some point $q \in E^c$. Therefore, p cannot be an interior point of E so $p \notin E^\circ$ which implies that $p \in (E^\circ)^c$.

(e) Do E and $\bar{E}$ always have the same interiors?

E and $\bar{E}$ do not always have the same interiors. Let $E = \mathbb{Q}$ in $\mathbb{R}^1$. Then $\mathbb{Q}$ has no interior points because any neighborhood of a rational number contains irrational numbers since we proved set of irrationals numbers are dense in $\mathbb{R}$ by a previous homework exercise. Thus $\mathbb{Q}^\circ = \emptyset$.

Now consider $\bar{E} = \bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$ where $\mathbb{Q}'$ denotes the set of all limit points of $\mathbb{Q}$. We see that $\mathbb{Q}'$ is the set of irrationals since any neighborhood around any irrational number contains a rational. Thus, $\bar{\mathbb{Q}}$ equals to the union of the rationals and irrationals which tells us that $\bar{E} = \mathbb{R}$. Finally, $(\bar{E})^\circ = (\mathbb{R})^\circ = \mathbb{R}$ because $\mathbb{R}$ is open in itself. So we see that $\mathbb{Q}^\circ = \emptyset \neq (\bar{\mathbb{Q}})^\circ = \mathbb{R}$.

(f) Do E and E° always have the same closures?

Recall that the closure of A is the set $\bar{A} = A \cup A'$. Again letting $E = \mathbb{Q}$, we see that $(\bar{\mathbb{Q}}) = \mathbb{R}$ by the argument above. Additionally, the closure of E° is the closure of $\mathbb{Q}^\circ$ which is $\bar{\mathbb{Q}^\circ} = \bar{\emptyset} = \emptyset$ and so they do not always have the same closure.

Problem 3

Let X be an infinite set. For $p \in X$ and $q \in X$, defined

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is metric. Which subsets of the resulting metric space are open? Which are closed?

We first show that d is metric defined on X.

(a) It is clear that d is a real valued function.

(b) We must show that $d(p, q) > 0$ if $p \neq q$. However, this follows directly from our definition of d since $d(p, q) = 1$ if $p \neq q$.

(c) We must show that $d(p, p) = 0$ but this follows directly from our definition of d.

(d) We must show that $d(p, q) = d(q, p)$. We have two cases. If $p \neq q$, then we have $d(p, q) = 1 = d(q, p)$. Alternatively, if $p = q$, then we have $d(p, q) = 0 = d(q, p)$.

(e) We must show that $d(p, q) \leq d(r, p) + d(r, q)$ for any $r \in X$. Note that we have two cases. Either $p = q$ or $p \neq q$. If $p = q$ then $d(p, q) = 0$ but the right hand side is some nonnegative positive real number and thus $0 \leq d(r, p) + d(r, q)$ so the claim holds. Alternatively, assume that $p \neq q$. Note that in this case, if $p = r$ and $r = q$ then we would have violated the fact that $p \neq q$. So WLOG, assume that $p \neq r$. Then we have $d(p, q) = 1 \leq 1 + d(r, q) = d(r, p) + d(r, q)$ and since $d(r, q)$ is nonnegative, the claim holds. This proves that d is a metric.

We now wish to consider which sets are open. Recall that an open set, E, is defined to be any set in the metric space such that every point of E is an interior point of E. With the metric defined on X, it appears that any subset of X is open. To see this let E be an arbitrary subset of X. We have two cases.

In the case that $E = \emptyset$, then $E^{\circ} = \emptyset$ and so $E^{\circ} = E$ which shows that every limit point of E is a point in E and so E is open.

Now let E be any nonempty subset of X and let $p \in E$. Observe what happens when we consider the neighborhood $N_{1/2}(p)$. $N_{1/2}(p) = \{x \in X \mid d(x, p) < 1/2\}$. But by the definition of d, we have $N_{1/2}(p) = \{x \in X \mid d(x, p) < 1/2\} = \{p\}$. Observe that the limit points of $\{p\}$ is the empty set by the corollary to Theorem 2.20 and so the set of limit points belongs to $N_{1/2}(p)$. Note that $N_{1/2}(p) = \{p\} \subset E$. Therefore every point of E is an interior point of E since we have shown for every point in E, there exists some neighborhood $N_{1/2}(p)$ which is wholly contained in E. This is precisely the requirements to show that a set E is open.

We now consider which sets are closed. We know that every subset of X is open. By Theorem 2.23, we know that a set is open if and only if its complement is closed. Since all sets are open, it follows that all sets are closed. This is because if we take A to be any subset of X, we may set $A = E^c = X - E$. Thus, by conditioning on E, we can get any subset of X. Hence all subsets of X are closed.

Problem 4

Prove that the Cantor set is uncountable.

Let us first recap how the Cantor set is constructed. Let E_0 be the interval $[0, 1]$. Remove the segment $(1/3, 2/3)$ and let E_1 be the union of intervals $[0, 1/3]$ and $[2/3, 1]$. Remove the middle thirds of these intervals and let E_2 be the union of the intervals $[0, 1/9]$, $[2/3, 3/9]$, $[6/9, 7/9]$, and $[8/9, 1]$. Continuing this way we obtain a sequence of sets E_n such that $E_1 \supset E_2 \supset E_3 \supset \dots$. The set $P = \cap_{n=1}^{\infty} E_n$ is called the Cantor set. On the next page, we graphically show the first few iterations of the Cantor Set. We see that a point in the Cantor set is uniquely determined as an infinite sequence of $L's$ and $R's$. Alternatively, we can let $L = 0$ and $R = 1$, so any point in the Cantor set may be described as a unique infinite sequence of 0's and 1's. Let us assume that the Cantor set is countable. Then we can list the elements as follows:

$$\begin{aligned} a_1 &= a_{11}a_{12}a_{13}\dots\dots\dots \\ a_2 &= a_{21}a_{22}a_{23}\dots\dots\dots \\ a_3 &= a_{31}a_{32}a_{33}\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

where $a_{ij} = 0$ or 1. Moreover, since the Cantor set, P is countable, there exists a bijection $f : P \to \mathbb{N}$ where a_n maps to the natural number n. If we can show that there exists some number in the Cantor set which is missed by the general correspondence, then it shows that P is not countable because f is not a function on P. Let us attempt to construct this number which is missed by f. Call this number

$$b_k = \begin{cases} 0 & \text{if } a_{kk} = 1 \\ 1 & \text{if } a_{kk} = 0 \end{cases}$$

Let us look along the diagonal of our list of elements. Notice that b_k is no where in our list. To see this, assume that $b_k = a_n$ for some $n \in \mathbb{N}$. Then the a_{nn} term in a_n differs from the b_k term and since a binary representation is unique, these two numbers are different. Thus f is not a bijection, and so the Cantor set is uncountable. This reasoning is the same train of thought used to prove Theorem 2.14 which shows that the Cantor set is uncountable. Namely, let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable and the proof is exactly the proof given above.