Math 650.2 Problem Set 15

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Problem 1

Let X be a metric space and let E and Y be subsets of X such that $E \subset Y \subset X$. Is it possible for E to be open relative to Y but E is not open relative to X?

It suffices to show such a case when this is possible. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $E = (a, b)$ where a and b are real numbers and $a \neq b$. So E is open relative to Y since (a, b) is a neighborhood of $\mathbb{R}$ and is therefore open by Theorem 2.19.

However, E is not an open subset of X meaning that the line segment (a, b) is not open relative to $\mathbb{R}^2$. Before we prove this, we shall forewarn two abuses of notation. It is clear that in the paragraph above we had $(a, b) \subset \mathbb{R}$. So if $p \in (a, b)$ then $p =$ some real number. Now we are considering E as a subset of the overall metric space $\mathbb{R}^2$. So when we say that $p \in (a, b)$ we really mean that p is an **ordered pair and not a real number**. That is, $p = (c, 0)$ for $a < c < b$ since the interval (a, b) lays on the $x-$axis. It follows that our interval $E = (a, b)$ in $\mathbb{R}$ can now be described as $E = \{(x, 0) \mid a < x < b\}$. Having said this we continue the proof.

To show that E is not open in $\mathbb{R}^2$ we will use contradiction. Namely, assume that E is open relative to $\mathbb{R}^2$. Let $p \in E$ so $p = (c, 0)$ for $a < c < b$. The definition of open relative, states $\exists\ r > 0$ such that whenever $d(p, q) < r$, and $q \in X$ then $q \in E$. So we must try to contradict the statement that $\exists\ r > 0$ such that whenever $d(p, q) < r$, and $q \in \mathbb{R}^2$ then $q \in E$.

Consider the point $q = (c, r-\epsilon)$ where $0 < \epsilon < r$. Clearly $q \in \mathbb{R}^2$. Is $d(p, q) < r$? Well under the usual metric d on $\mathbb{R}^2$, we have $d(p, q) = d\big((c, 0), (c, r-\epsilon)\big) = \sqrt{((r-\epsilon) - 0)^2 + (c - c)^2} = \sqrt{(r-\epsilon)^2}$. Observe that $\sqrt{(r-\epsilon)^2} < r$ is a true statement since

$$\sqrt{(r-\epsilon)^2} < r \tag{1}$$
$$(r-\epsilon)^2 < r^2 \tag{2}$$

where the last line is obviously true. Going from Eq. 2 to Eq. 1 is valid and so we see that $d(p, q) < r$. However, $q = (c, r-\epsilon) \notin E$ since the second coordinate is not 0. So what have we shown?

We have shown that if $p \in E$ then regardless of what we choose $r > 0$ to be, there exists a point $q \in \mathbb{R}^2$ such that whenever $d(p, q) < r$ then $q \notin E$. Thus E is not open relative to

X. This problem motivates the idea of "open relative." In class, when we use to say that "E is open," we really meant "E is open in X." At that point of the course, we need not specify what E is open in because the only containment we had was $E \subset X$. However, since it is possible for E to be open in a subset of X without being open in X, we must develop this notion of "open relative."

Problem 2

Let X be a metric space and let $E \subset X$. Explain the difference of E being open in X verses E being open relative to X.

By definition, E is open in X if every point of E is an interior point of E. This leads us the the following chain of equivalent definitions.

$$\forall\, p \in E\big(\exists\, r > 0 \ s.t \ N_r(p) \subset E\big) \tag{3}$$

$$\forall\, p \in E\big(\exists\, r > 0 \ s.t \ \{q \mid d(p,q) < r\} \subset E\big) \tag{4}$$

$$\forall\, p \in E\Bigg(\exists\, r > 0 \ s.t \ \forall q \in E\big(d(p,q) < r \Rightarrow q \in E\big)\Bigg) \tag{5}$$

Compare this with the definition for E is open relative to X. By translating Rudin's definition on page 35 we have

$$\forall\, p \in E\Bigg(\exists\, r > 0 \ s.t \ \forall q \in X\big(d(p,q) < r \Rightarrow q \in E\big)\Bigg) \tag{6}$$

but since $q \in E$, and $E \subset X$, then we have

$$\forall\, p \in E\Bigg(\exists\, r > 0 \ s.t \ \forall q \in E\big(d(p,q) < r \Rightarrow q \in E\big)\Bigg) \tag{7}$$

so to say that E is open in X is the same as saying that E is open relative to X.

Problem 3

Prove or disprove the following statement: Let E be an open set and $E \subset Y$. Then E is open relative to Y.

Well first of all, there is a problem in the formulation of the statement. By Problem 2, we see that being open in a set and being open relative to a set mean the same thing. Moreover, prior to Rudin introducing the notion of open relative we would would always say "E is open." However this is implicitly assuming that $E \subset X$ where X is the larger metric space. Now with this notion of open relative, we must specify what E is open relative to. Is E open relative to a subset of X which is Y or is E open relative to Y? With this said, the question can have two meanings:

(a) Let E be an open subset relative to Y and $E \subset Y$. Then E is open relative to Y.

(b) Let E be an open subset relative to X and $E \subset Y$. Then E is open relative to Y.

If the statement means statement (a) then the statement is clearly true as argued by question 2 since E being open in Y has the same meaning as being open relative to Y.

Now if the statement means case (b) then we run into some trouble as again there can be more than one interpretation. We may have $E \subset X \subset Y$ or $E \subset Y \subset X$. In the former statement, where $E \subset X \subset Y$, the statement is false since the counterexample is precisely problem 1. So the only case left to settle is if $E \subset Y \subset X$ and E is an open subset of X, then does this implies E is open in Y? The answer is yes and here is why: Since E is open in X, then to each $p \in E$, there exists an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in X$. Since this holds for all $q \in X$ and since $Y \subset X$, then this holds for all $q \in Y$. Thus the statement "to each $p \in E$, there exists an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$" is true and so E is open in Y.

To summarize we have:

(a) Let E be an open subset relative to Y and $E \subset Y$. Then E is open relative to Y. This statement is true.

(b) Let $E \subset X \subset Y$ and let E be an open subset relative to X and $E \subset Y$. Then E is open relative to Y. This statement can be false.

(c) Let $E \subset Y \subset X$ and let E be an open subset relative to X and $E \subset Y$. Then E is open relative to Y. This statement is true.