Math 650.2 Problem Set 14

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Problem 1

Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Prove that $y \in \bar{E}$.

Proof: Let us condition on y. Then we have two cases: either $y \in E$, or $y \notin E$.

Case One: If $y \in E$, then $y \in E \cup E'$ and so $y \in \bar{E}$.

Case Two: If $y \notin E$, and we want to show that $y \in \bar{E}$, then we better prove that $y \in E'$. That is, y is a limit point of E. Let us see if this is true. Since $y = \sup E$, then for every $r > 0$, there exists some point $x \in E$ such that $y - r < x < y$. This must be the case because if there is no point x such that x is between $y - r$ and y, then this implies that x is an upper bound of E but this would contradict the fact that y is the least upper bound of E. Thus the inequality $y - r < x < y$ shows that every neighborhood of y contains some point $x \neq y$ where $x \in E$. However, this is precisely the definition of a limit point and thus $y \in E'$ which implies that $y \in \bar{E}$.

It should be noted that we can make a stronger statement in the case where E is closed. This is because if E is closed, then every limit point of E is a point of E and thus $y \in E$ so it is trivial that $y \in \bar{E}$.

Problem 2

Suppose $Y \subset X$. Prove a subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof:

$\Rightarrow$ Let us assume that a subset E of Y is open relative to Y. Then by the definition of open relative, to each $p \in E$, there is an $r_p > 0$ such that $q \in E$ whenever $d(p, q) < r_p$ and $q \in Y$. Let us now construct the set $V_p = \{q \in X \mid d(p, q) < r_p\}$ for each $p \in E$. Observe that V_p represents a neighborhood of p and by Theorem 2.19, we know that V_p is an open set since every neighborhood is an open set. Now let $\{V_p\}$ be a collection of sets. Since each such V_p is open, it follows that $\cup_{p \in E} V_p$ is an open set by Theorem 2.24(a). For simplicity in notation, let us call $\cup_{p\in E} V_p = G$.

Now let us recall what we are trying to prove. We want to show that $E = Y \cap G$ where G is an open subset of X. To do so, we will show set equality. We first show that $E \subseteq G \cap Y$. Observe that for all $p \in E$, we have $p \in V_p$ because $d(p,p) < r_p$. Thus $p \in G$ since G is the union of all V_p's. Additionally, since $p \in E$ and $E \subseteq Y$ by assumption, then $p \in Y$. So we have shown that for every $p \in E$, $p \in G$ and $p \in Y$ and thus $E \subseteq G \cap Y$.

We now show that $G \cap Y \subseteq E$. Let $p \in G$ and let $p \in Y$. Well since $p \in G$, then $p \in V_p$ but this implies that $p \in E$ since E is open relative to Y. So for each $p \in G \cap Y$, we have shown that $p \in E$, and so $G \cap Y \subseteq E$. Thus $G \cap Y = E$.

$\Leftarrow$ We now assume that $E = Y \cap G$ for some open subset G of X. Since G is open in X, then to each point $p \in G$, there is a positive real number r_p such that the conditions $d(p,q) < r_p$ imply that $q \in G$. Simply put, for every $p \in G$, there is some neighborhood $N_{r_p}(p)$ such that $N_{r_p} \subset G$. Note that the neighborhood $N_{r_p}(p)$ contains all points $q \in X$ such that $d(p,q) < r_p$ but this is precisely the definition of V_p and so we have $N_{r_p}(p) = V_p \subset G$. Observe that $V_p \cap Y \subset E$ since $G \cap Y = E$ by assumption. However, $V_p \cap Y \subset E$ tells us to each $p \in E$, there is some $r > 0$ such that $q \in E$ whenever $d(p,q) < r$ and $q \in Y$ which is precisely what is meant when we say that E is open relative to Y.

Problem 3

Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact to relative to Y.

Proof:

$\Rightarrow$ Suppose that K is compact relative to X and $\{V_\alpha\}$ is a family of sets such that for each α, V_α is open relative to Y such that

$$K \subset \cup_\alpha V_\alpha$$

By Theorem 2.30, for each α, there exists a set G_α such that G_α is open relative to X and $V_\alpha = Y \cap G_\alpha$. Since $K \subset Y$ and

$$K \subset \cup_\alpha V_\alpha = \cup_\alpha (Y \cap G_\alpha) = Y \cap (\cup_\alpha G_\alpha)$$

we see that $K \subset Y \cap (\cup_\alpha G_\alpha)$ and therefore $K \subset \cup_\alpha G_\alpha$. Since K is compact relative to X, there exists a finite number of elements, $\alpha_1, \ldots \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$$

Using the assumption that $K \subset Y$ and $K \subset \cup_{j=1}^{n} G_{a_j}$ yields that

$$K \subset Y \cap (\cup_{j=1}^{n} G_{a_j}) = (Y \cap G_{\alpha_1}) \cup \ldots \cup (Y \cap G_{\alpha_n}) = V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$$

Since V_α was arbitrary, we have shown that every open cover of K has a finite subcover. Therefore, K is compact relative to Y.

$\Leftarrow$ Now suppose that $K \subset Y \subset X$ and K is compact to relative to Y. Let $\{G_\alpha\}$ be a collection of sets such that for each α, G_α is open relative to X and

$$K \subset \cup_\alpha G_\alpha$$

For each α, let $V_\alpha = Y \cap G_\alpha$. Now since $K \subset Y$, and $K \subset \cup_\alpha G_\alpha$, this implies that

$$K \subset Y \cap (\cup_\alpha G_\alpha)$$

Observe that V_α is an open cover for K. Since K is compact in Y, then we know there exists a finite number of elements in α, say $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $K \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$. Since

$$\cup_{j=1}^n V_{a_j} = \cup_{j=1}^n (Y \cap G_{a_j}) = Y \cap (\cup_{j=1}^n G_{\alpha_j})$$

and $K \subset Y$, it follows that $K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$. Since $\{V_\alpha\}$ was arbitrary, we conclude that every collection of sets that form an open cover of K has a finite subcover. Therefore, K is compact relative to X.