Math 650.2 Homework 3

Elliot Gangaram
elliot.gangaram@gmail.com

Problem 1

Prove propositions 1.18$c - e$ regarding ordered fields.

(a) Proposition 1.18c asserts the following: If $x < 0$ and $y < z$ then $xy > xz$.

Since $y < z$ it follows from Definition 1.17(i) that $y - y < z - y$ and hence $0 < z - y$. We are also given that $x < 0$ so it follows from Proposition 1.18(a) that $(-x) > 0$. By Definition 1.17(ii) we may write $(-x)(z-y) > 0$ and again by 1.18(a) this implies that $x(z-y) < 0$. As a result of 1.17(i), this implies that

$$\begin{aligned} x(z-y) + xy &< 0 + xy \\ xz - xy + xy &< xy \\ xz &< xy \\ xy &> xz \end{aligned}$$

(b) Proposition 1.18d asserts the following: If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.

Since we are in an ordered set, we have two cases. Either $x > 0$ or $x < 0$.

Case One: Assume that $x > 0$. By Definition 1.17(ii) $(x)(x) > 0$ and thus $x^2 > 0$.
Case Two: Assume that $x < 0$. It follows from Proposition 1.18(a) that $-x > 0$. Then by Definition 1.17(ii) we have $(-x)(-x) > 0$ but we know from Proposition 1.16(d) that $(-x)(-x) = (x)(x) = x^2$ and thus we have $x^2 > 0$.

Now to prove that $1 > 0$, note that $1 \neq 0$ so $1^2 > 0$ and thus $1 > 0$ since $1^2 = 1$ by the definition of 1 in field axiom ($M4$).

(c) Proposition 1.18e asserts the following: If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

We will first show that $\frac{1}{x} > 0$. We know that $x > 0$. Let $z \in F$ such that $z \leq 0$. Then $(-z)(x) \geq 0$ so $(z)(x) \leq 0$. By field axiom ($M5$) and Proposition 1.18(d), $(x)\left(\frac{1}{x}\right) = 1 > 0$ so it must be the case that $\frac{1}{x} > 0$ by 1.17(ii). By a similar argument, namely replacing x by y in the above, we can verify that $\frac{1}{y} > 0$. This tells us that

$$0 < \frac{1}{x} \tag{1}$$

and that

$$0 < \frac{1}{y} \tag{2}$$

Thus, $0 < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$. By Proposition 1.18($b$), since $x < y$ by assumption, then we have $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$ so $\left(\frac{1}{y}\right) < \left(\frac{1}{x}\right)$. Using this with Eq.(1) and Eq.(2) shows

$$0 < \frac{1}{y} < \frac{1}{x} \tag{3}$$

which is precisely what we wanted to prove.

Problem 2

Theorem 1.20

Prove the following theorem:
a) If $x \in R$, $y \in R$ and $x > 0$ then there is a positive integer n such that $nx > y$.
b) If $x \in R$, $y \in R$ and $x < y$ then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.

To prove statement (a) we will do so by contradiction. Assume $nx \leq y$, and let $S = \{nx \mid n = 1, 2, 3, 4...\}$. Since we have a nonempty subset of R which is bounded above by y, we may invoke the LUB property; that is, the supremum of S exists. Let sup $S = \alpha$. By the definition of the LUB, we have

$$nx \leq \alpha \tag{4}$$

Since this statement holds for all n in the set, it also holds for $n+1$. Thus we have

$$(n+1)x \leq \alpha \tag{5}$$

$$nx + x \leq \alpha$$

$$nx \leq \alpha - x \tag{6}$$

Recall that we assumed $x > 0$ so $\alpha - x < \alpha$. By Equation (6) we have $nx \leq \alpha - x$ and this contradicts α being the LUB because we have found an even smaller upper bound for S. This proves (a).

Armed with this tool, we will now use it to prove the second statement. Since $x < y$ this implies that $y - x > 0$ by Definition 1.17(i). We will now take $y - x$ to be our x in statement (a) and we will let 1 be our y in statement (a). Then there exists a positive integer n such that $n(y-x) > 1$. So we have

$$n(y-x) > 1$$
$$ny - nx > 1$$
$$ny > nx + 1 \tag{7}$$

Let us set Equation 7 aside and come back to it later. Let m be the smallest integer such that $m > nx$. How do we know such an m exists? Assume m does not exist. Then that means that there is no smallest integer greater than some real number, nx. Let $\lfloor nx \rfloor$ denote the integer part of nx. Then the successor of $\lfloor nx \rfloor$ is $\lfloor nx \rfloor + 1$. Clearly, $\lfloor nx \rfloor + 1 > nx$. So we have found an integer greater than nx but how do we know this is the smallest such integer? Assume that there is even a smaller integer than $\lfloor nx \rfloor + 1$. Then this integer can be represented as $\lfloor nx \rfloor + \epsilon$. However, since all distinct integers differ by at least 1, ϵ must be exactly 1 otherwise $\lfloor nx \rfloor + \epsilon$ would not be an integer for $0 < \epsilon < 1$. Thus, such an m exists and $m > nx$ and so

$$\frac{m}{n} > x \tag{8}$$

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Since m is the smallest integer such that $m > nx$, then $m - 1 \leq nx$. To see this, note if $m - 1 > nx$, then we would have $m > m - 1 > nx$ which contradicts our choice of m. So $m - 1 \leq nx$ implies $m \leq nx + 1$. So Equation 7 and $m \leq nx + 1$ implies

$$m \leq nx + 1 < ny \tag{9}$$

$$m < ny \tag{10}$$
$$\frac{m}{n} < y \tag{11}$$

Putting Eq.8 and Eq.11 tells us

$$x < \frac{m}{n} < y \tag{12}$$

To complete the proof, we must show $\frac{m}{n}$ is rational. Since m and n are defined to be integers and n cannot be zero since n is positive, then $\frac{m}{n} \in \mathbb{Q}$.

Problem 3

Chapter 1 Question 4

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Since α is a lower bound of E, we know that $\alpha \leq x$ for all $x \in E$. We are also given that β is an upper bound of E and so $x \leq \beta$ for all $x \in E$. By the transitivity of an ordered set, Definition 1.5(ii), we have $\alpha \leq x \leq \beta$ for all $x \in E$. From this we can conclude that $\alpha \leq \beta$.