Math 650.2 Homework 1

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Problem 1

If the number $q \in \mathbb{Q}$ is prime, prove that $\sqrt{q} \notin \mathbb{Q}$.

We will present three solutions to the above question.

Proof One: Let us assume that $\sqrt{q} \in \mathbb{Q}$ and try to reach some sort of contradiction. Since $\sqrt{q} \in \mathbb{Q}$, then there exists some integers x and y where x and y are relatively prime and $y \neq 0$ such that $\frac{x}{y} = \sqrt{q}$. Squaring both sides gives us $\left(\frac{x}{y}\right)^2 = q$ which implies $\frac{x^2}{y^2} = q$ so

$$x^2 = qy^2 \tag{1}$$

This shows that q divides into x^2 and since q is prime, then we know q divides into x. That is, there exists some integer k such that

$$x = qk \tag{2}$$

By Eq. 1 and Eq. 2 we have $qy^2 = x^2 = (qk)^2 = q^2k^2 = qqk^2$ so $qy^2 = qqk^2$ and thus

$$y^2 = qk^2 \tag{3}$$

This implies that q divides into y^2 and since q is prime, then we can say that q divides into y. However, since q divides into both x and y, then x and y are not relatively prime. Contradiction! Thus $\sqrt{q}$ is irrational.

Proof Two: Assume $\sqrt{p} \in Q$. Then there exists positive integers, m and n where m and n are relatively prime and $n \neq 0$ such that $\frac{m}{n} = \sqrt{p}$. This implies that $\frac{m^2}{n^2} = p$ which is the same as saying $m^2 = n^2p$.

Claim: m^2 has as an even amount of p's in it's prime factorization while n^2p has an odd amount of p's in its prime factorization.

By the Fundamental Theorem of Arithmetic we know the following: Let q be an integer such that $q > 1$. Then there exists a unique positive integer k, a unique set of k primes, $p_1 < p_2 < ... < p_k$ and a unique sequence of positive integers, $i_1, i_2, ..., i_k$ such that

$p_1^{i_1} p_2^{i_2} ... p_k^{i_k} = q$.

Using the Fundamental Theorem of Arithmetic, let us denote the prime p in the factorization of m by p_j. This implies that the exponent on p_j is i_j. Then in the prime factorization of m^2 the exponent of p_j will be $i_j + i_j$. Note that there exists only two possibilities. Either i_j is even or i_j is odd. If i_j is odd, then $i_j + i_j$ is even. If i_j is even, then $i_j + i_j$ is also even. Thus the exponent of p_j in the prime factorization of m^2 is even. We must now show that n^2p has an odd amount of p's in its prime factorization. By an argument similar to the one above, we know that n^2 must have an even amount of p's in its prime factorization. So n^2p will have an odd number of p's in its prime factorization.

However, by the Fundamental Theorem of Arithmetic, we know that for each positive integer, there exists a unique prime factorization. The fact that the left hand side of $m^2 = n^2p$ has an even amount of p's while the right hand side has an odd amount of p's tells us that these are two different integers. Contradiction! Thus $\sqrt{p} \notin Q$.

Proof Three: The following proof relies on a corollary which stems from the Rational Zeros Theorem.

The corollary states the following: Consider the polynomial equation $x^n + c_{n-1}x^{n-1} + ... + c_1x + c_0 = 0$ where the coefficients $c_0, c_1, ..., c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

So assume p is a prime number. We must show that $\sqrt{p}$ is not rational.

Let us consider the polynomial equation $x^2 - p = 0$. By the corollary we must find all integers which divides c_0. Since c_0 is prime, the only integers which divide into c_0 are $1, -1, p$, and $-p$. It is clear that x cannot be -1 or 1 because then we would end up with the equation $1 - p = 0$ but since p is a prime, p cannot be 1 to make the equation true. It is also clear that the solution cannot be p or $-p$ because substituting in p or $-p$ for x yields the following equation: $p^2 - p = 0$ which is true only in the case that $p = 1$ or $p = 0$ and this again violates our choice of p since p is suppose to be prime. This tells us that there is no rational solution to this equation. Note that since $\sqrt{p}$ is a solution, then $\sqrt{p}$ cannot be rational.

Problem 2

If the number $q \in \mathbb{Q}$ is prime, prove that $\sqrt[n]{q} \notin \mathbb{Q}$ where $n \in \mathbb{N}$ and $n > 1$.

Assume that the n^{th} root of q is rational. Then there exists integers x and y where x and y are relatively prime and $y \neq 0$ such that $\frac{x}{y} = q^{1/n}$. Raising both sides to the n^{th} power yields the following $\frac{x^n}{y^n} = q$ which shows

$$x^n = qy^n \tag{4}$$

This shows that q divides into x^n and since q is prime, then we can say that q divides into

x. That is, there exists some integer k such that

$$x = qk \tag{5}$$

By Eq. 4 and Eq. 5, $qy^n = x^n = (qk)^n = q^n k^n = q^2 q^{n-2} k^n$, so $qy^n = q^2 q^{n-2} k^n$. Dividing through by q leaves us with

$$y^n = qq^{n-2}k^n \tag{6}$$

which shows that q divides into y^n. But since q is prime, then we can say that q divides into y. However, since q divides into x and q divides into y, we have shown that x and y are not relatively prime. Contradiction! We are forced to conclude that the n^{th} root of q is irrational.

Problem 3

Let $A = \{q \in \mathbb{Q} : q^2 < 2\}$. Show that $\max\{A\}$ does not exist.

Assume $m \in A$ such that m is the largest number in A. We must show that there exists some element m' such that $m' > m$ and $m' \in A$.
To find a candidate for an m' first note that that $m^2 < 2$. Thus there exists some positive real number ε such that $m^2 + \varepsilon = 2$. This implies that $m^2 + \dfrac{\varepsilon}{2} < 2$, so we should take $m' = \sqrt{m^2 + \dfrac{\varepsilon}{2}}$

As noted earlier we must show that $m' > m$ and $m' \in A$.
To show that $m' > m$, we proceed as follows.

$$\begin{aligned} m' &> m \\ \sqrt{m^2 + \frac{\varepsilon}{2}} &> m \\ m^2 + \frac{\varepsilon}{2} &> m^2 \\ \frac{\varepsilon}{2} &> 0 \end{aligned}$$

Note that the last line is true based off our choice of ε. Since each of the steps above are reversible, it follows that $m' > m$. We must now show that $m' \in A$ by proving that $(m')^2 < 2$.

$$\begin{aligned} (m')^2 &< 2 \\ \left(\sqrt{m^2 + \frac{\varepsilon}{2}}\right)^2 &< 2 \\ m^2 + \frac{\varepsilon}{2} &< 2 \\ \frac{\varepsilon}{2} &< 2 - m^2 \end{aligned}$$

which is precisely how we choose our ε as seen in the discussion of ε on the previous page. Since each of the steps are reversible, it follows that $m' \in A$ and thus $\max\{A\}$ does not exist.