Math 650.2 Problem Set 10

Elliot Gangaram
elliot.gangaram@gmail.com

Problem 1

Prove Theorem 2.12 in Rudin.

Theorem 2.12: Let E_n, $n = 1, 2, 3 \ldots$, be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Proof: First let us take a moment and attempt to explain what this theorem is telling us. This theorem is saying that the union of a countable collection of countable sets is countable. When we use the word "collection", we mean the set $\{E_1, E_2, \ldots E_n, \ldots\}$. To prove this theorem, we will show there exists a one to one map $f : \bigcup_{n=1}^{\infty} E_n \to \mathbb{N}$. From Homework 7, Problem 5, it will then follow that S is countable.

We will first show that we can find disjoint countable sets B_n such that $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} B_n$. We can construct these sets as follows. Let

$B_1 = E_1$
$B_2 = E_2 - E_1$
$B_3 = E_3 - (E_1 \cup E_2)$
$\ldots$
$\ldots$
$\ldots$
$B_n = E_n - (E_1 \cup \ldots \cup E_{n-1})$
$\ldots$
$\ldots$
$\ldots$

It is clear that any two distinct B_n's are disjoint. To see this, consider two distinct arbitrary sets, B_j and B_k. WLOG, assume that $j < k$. If $s \in B_j$ then $s \in E_j - (E_1 \cup \ldots \cup E_{j-1})$

since that is how we defined B_j. This implies that $s \in E_j$. However $s \notin B_k$ because $B_k = E_k - (E_1 \cup \ldots \cup E_j \ldots \cup E_{k-1})$. Thus, $B_j \cap B_k = \varnothing$.

Now that we have shown that the intersection of the B_n's are empty, we must verify our claim that $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} B_n$. To do this, we will show set equality.

First observe by the way we have defined B_n, we have $B_n \subseteq E_n$. Thus this implies that $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} E_n$.

Now, let $x \in \bigcup_{n=1}^{\infty} E_n$. The only piece of information this tells us that x is in at least one of the E_n's. Since $n = 1, 2, 3, \ldots$, pick the smallest n_0 such that $x \in E_{n_0}$. We know the smallest n_0 exists by a previous homework exercise. Since $x \in E_{n_0}$, then $x \in B_{n_0}$ since $x \notin E_1 \cup E_2 \cup \ldots \cup E_{n_0-1}$ by the way we have constructed B_{n_0}. Therefore, $x \in \bigcup_{n=1}^{\infty} B_n$ and so we have established set equality. Using $\bigcup_{n=1}^{\infty} B_n$ instead of $\bigcup_{n=1}^{\infty} E_n$, we shall show that $\bigcup_{n=1}^{\infty} B_n$ is countable which implies that $\bigcup_{n=1}^{\infty} E_n$ is countable since they are the same set.

First observe that we are given by assumption that each E_n is countable. We have proven that a subset of a countable set is also countable and since $B_n \subseteq E_n$ it follows that B_n is countable. This tells us there exists a bijection $f_n : B_n \to \mathbb{N}$ for each n.

Let us list the prime numbers as $p_1 = 2, p_2 = 3, p_3 = 5 \ldots, p_n, \ldots$.

Define $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ as follows: if $x \in B_n$ then $f(x) = p_n^{f_n(x)}$.

As an example, if $x \in B_2$ then $f_2 : B_2 \to \mathbb{N}$ and $f_2(x)$ is some natural number. More concretely, assume that under f_2 we have $f_2(x) = 8$. We know f_2 exists because we are given that B_2 is countable. Then $f(x) = p_2^{f_2(x)} = p_2^8 = 3^8$.

Recall that we proved the intersection between any two distinct sets B_j and B_k is empty. Note that the function $f(x)$ relies on f_n. Since an element, x belongs to one and only one of the B_n's, then there is no ambiguity in writing $f(x)$.

The only thing left to prove is that f is injective. That is, if $f(x) = f(y)$ then $x = y$. To show this, we will show the contrapositive. Namely, assume $x \neq y$. We wish to prove that this implies $f(x) \neq f(y)$. However, note that x and y can be in the same set, or they may be elements in different sets. So we have two cases:

Case One: Let $x \in B_n$ and $y \in B_m$ and assume that $m \neq n$. Then $f(x) = p_n^{f_n(x)} \neq p_m^{f_m(y)} = f(y)$. The reason is that since $m \neq n$, then $p_n \neq p_m$. By the Fundamental Theorem of Arithmetic, $p_n^{f_n(x)}$ cannot be factored into a product of p_m's and this completes case one.

Case Two: Now assume x and y are in the same set, call it B_n. Recall that f_n is a bijection and hence one to one. This tells us that $f_n(x) \neq f_n(y)$. So $f(x) = p_n^{f_n(x)} \neq p_n^{f_n(y)} = f(y)$ by the Fundamental Theorem of Arithmetic because $f(x)$ and $f(y)$ have a different number of p_n's in their prime factorizations.

Thus f is one to one and therefore $\bigcup_{n=1}^{\infty} B_n$ is countable.

Problem 2

Prove the corollary to Theorem 2.12

Corollary: Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put

$$T = \bigcup_{\alpha \in A} B_\alpha$$

Then T is at most countable.

Proof: Since A is at most countable, then there exists a bijection $f : A \to \mathbb{N}$ defined by $f(\alpha) = n$. So we may write $T = \bigcup_{\alpha \in A} B_\alpha = \bigcup_{f(\alpha) \in \mathbb{N}} B_{f(\alpha)}$. Note that there are two cases. Either A is finite or A is countable.

If A is countable, then $A \sim \mathbb{N}$ and thus it follows directly from Theorem 2.12 that T is countable since $T = \bigcup_{\alpha \in A} B_\alpha = \bigcup_{f(\alpha) \in \mathbb{N}} B_{f(\alpha)} = \bigcup_{n=1}^{\infty} B_n$ which is precisely what we proved in Problem 1.

Now if A is finite, then $A \sim \mathbb{N}^*$ where the * indicates that A is a subset of $\mathbb{N}$. Then we have $T = \bigcup_{\alpha \in A} B_\alpha = \bigcup_{f(\alpha) \in \mathbb{N}^*} B_{f(\alpha)} = \bigcup_{n=1}^{k} B_n$ for some integer k. However, $\bigcup_{n=1}^{k} B_n$ is a subset of $\bigcup_{n=1}^{\infty} B_n$ and we proved that any subset of a countable set is countable. Since we have previously shown that $\bigcup_{n=1}^{\infty} B_n$ is countable, it follows that $\bigcup_{n=1}^{k} B_n$ is countable and hence T is countable.

Problem 3

Prove theorem 2.13 in Rudin.

Theorem 2.13: Let A be a countable set and let B_n be the set of all n-tuples $(a_1, a_2, \ldots a_n)$ where $a_k \in A$ $(k = 1, 2, \ldots, n)$ and the elements $a_1, a_2, \ldots a_n$ need not be distinct. Then B_n is countable.

It appears that Rudin's proof, to some degree, is incorrect as the concept of ordered n-tuples as not been defined. In any case, the following proof attempts to prove theorem 2.13.

Proof: To prove this statement we will use strong induction.

Base Case. Let $n = 1$. Then Rudin claims that B_1 is countable since $B_1 = A$ and A itself is countable.

Before continuing the proof, the following point should be noted. Since the above statement holds for any such A, take $A = \{1, 2, 3\}$. Then B_1 by definition equals to $\{(1), (2), (3)\}$. To claim that $B_1 = A$ is incorrect as the elements of B_1 are 1-tuples and the elements of A are just a few integers. More rigorously, if $x \in B_1$ then $x = (1), (2)$ or (3), and it is clear that $x \notin A$ which shows these sets are not equal. Thus, when Rudin makes the claim that $B_1 = A$ it appears that he is assuming the ordered pair (a) is the same as a. With this noted, we continue the proof.

Strong Inductive Step: Assume that the claim holds true for all n up to B_{n-1}. We would like to show that the claim holds for B_n. What do the elements in B_n look like? The elements of B_n, by definition are n-tuples. Let us denote these elements as $(b_1, b_2, \ldots, b_{n-1}, b_a)$ where $b_1, b_2, \ldots b_{n-1}, b_a \in A$. Rudin then writes these elements as (b, a) where $b \in B_{n-1}$ and $a \in A$. Here, he is implicitly assuming that in general $((x_1, x_2, \ldots, x_k), x_{k+1}) = (x_1, x_2, \ldots, x_k, x_{k+1})$.

Continuing with the notion that (b, a) where $b \in B_{n-1}$ and $a \in A$, we are led to the conclusion that the set of ordered pairs (b, a) is equivalent to A, meaning there exists a bijection from A to B_n. This bijection is given by $f(a) = (b, a)$. To prove this is a bijection note that if $f(a) = f(x)$ then $(b, a) = (b, x)$ and so $a = x$. We also see this function is onto since given any ordered pair in B_n, call it (x, y), then this ordered pair is mapped from the element $y \in A$. This bijection shows that B_n is countable which completes the proof.

Problem 4

Find an explicit bijection $\phi : \mathbb{N} \to \mathbb{Q}$.

Motivation: In the previous homework, we showed there exists a bijection between $\mathbb{N}$ and $\mathbb{Q}^+$. We now attempt to show there exists an explicit bijection between $\mathbb{N}$ and $\mathbb{Q}$. Using the result from Homework 9, it appears easier to first create a bijection between $\mathbb{Z}$ and $\mathbb{Q}$. The reason for doing so is because we have already created a bijection from the positive integers to the positive rationals. So it only seems natural that by adding in the negative integers, we can map them to the negative rationals and thus obtain a bijection. We do this as follows:

$$g(z) = \begin{cases} \dfrac{a_z}{a_{z+1}}, & \text{if } z > 0 \\ \\ -\dfrac{a_{-z}}{a_{-(z-1)}}, & \text{if } z < 0 \\ \\ 0, & \text{if } z = 0 \end{cases}$$

where the a_i term refers to the i^{th} term in Stern's diatomic series as defined in Homework 9, Problem 3.

We already referenced a proof by Northshield showing that $g(z) = \dfrac{a_z}{a_{z+1}}$ if $z > 0$ is a bijection from $\mathbb{N} \to \mathbb{Q}^+$ in Problem 3 of the previous homework.[1] Equivalently, we may write this as g is a bijection from $\mathbb{Z}^+$ to $\mathbb{Q}^+$ for $z > 0$. Now, it follows by the symmetry of the problem that $g(z) = -\dfrac{a_{-z}}{a_{-(z-1)}}$ is a bijection from $\mathbb{Z}^-$ to $\mathbb{Q}^-$ if $z < 0$. [2] That is, g is a bijection between the negative integers and the negative rationals. So we have covered all the positive and negative rationals. The only element in the rationals that is not accounted for is the zero element. So we shall have the integer 0 mapping to the rational number 0. However, g is a bijection from the integers to the rationals. We wish to find a bijection from the natural numbers to the rationals. So we shall now define the well-known bijection from the natural numbers to the integers.

$$h(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

[1]See Theorem 5.1 here.

[2]As an example, suppose we are given that $\psi : A \to B$ where $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$. Moreover, we are given that $\psi(1) = 4, \psi(2) = 5, \psi(3) = 6$. It is clear that ψ is a bijection. However, now consider $B' = \{4, 5, 6, -4, -5, -6\}$. A natural way to create a bijection is to enlarge set A as follows: $A' = \{1, 2, 3, -1, -2, -3\}$ and then define $\psi(1) = 4, \psi(2) = 5, \psi(3) = 6, \psi(-1) = -4, \psi(-2) = -5, \psi(-3) = -6$. This is essentially what we are doing here. We are given g is bijection from one set of positive integers to a set of positive rationals. We can then "extend" g by mapping the negative integers to the negative rationals. However, this leaves one integer, 0, and one rational number, 0, not accounted for so we let $g(0) = 0$ thus establishing a bijection.

We proved in Homework 7, Problem 3 that this is a bijection. It follows that $g \circ h : \mathbb{N} \to \mathbb{Q}$ is a bijection since the composition of two bijections is a bijection. We showed that if two functions are injective, then their composition is also injective in Homework 7, Problem 4. So we need to only check that the composition is surjective. However, this is trivial since all elements in $\mathbb{Q}$ comes from some element in $\mathbb{Z}$ and every element in $\mathbb{Z}$ comes from an element in $\mathbb{N}$ so it follows that every element in $\mathbb{Q}$ comes from an element in $\mathbb{N}$. Thus, we have an explicit bijection from $\mathbb{N}$ to $\mathbb{Q}$.

However, given a rational number, can we find what this rational number maps to in the set of natural numbers? The answer is yes and is given by the following piece-wise defined function which is symmetric to the function defined in the previous homework. We first define $g^{-1} : \mathbb{Q} \to \mathbb{Z}$ as

$$g^{-1}(q) = \begin{cases} 2f^{-1}(q-1)+1, & \text{if } q > 1 \\ 1, & \text{if } q = 1 \\ 2f^{-1}\left(\dfrac{q}{1-q}\right), & \text{if } 0 < q < 1 \\ 0, & \text{if } q = 0 \\ -2\left(f^{-1}\left(\dfrac{-q}{1+q}\right)\right), & \text{if } -1 < q < 0 \\ -1, & \text{if } q = -1 \\ -2(f^{-1}(-q-1)+1), & \text{if } q < -1 \end{cases}$$

where f^{-1}, is given as follows:

$$f^{-1}(1) = 1$$
$$f^{-1}(q) = 2f^{-1}\left(\frac{q}{1-q}\right) \text{ if } q < 1$$
$$f^{-1}(q) = 2f^{-1}(q-1)+1 \text{ if } q > 1$$

We now define the function $h^{-1} : \mathbb{Z} \to \mathbb{N}$ as follows:

$$h^{-1}(z) = \begin{cases} 2z, & \text{if } z > 0 \\ 1, & \text{if } z = 0 \\ -2z+1, & \text{if } z < 0 \end{cases}$$

Although I do not prove this formally, then $h^{-1} \circ g^{-1} : \mathbb{Q} \to \mathbb{N}$ is the bijection we are looking for.