Math 650.2 Homework 4

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Problem 1

Denseness

(a) Prove that $\mathbb{Q}$ is dense in $\mathbb{Q}$.

Let $x \in \mathbb{Q}$ and let $y \in \mathbb{Q}$. WLOG, assume that $x < y$. We want to show that there exists a $q \in \mathbb{Q}$ such that $x < q < y$. I claim that q can be taken to be $\dfrac{x+y}{2}$. First note that by the closure property of the field $\mathbb{Q}$, we have $x + y \in \mathbb{Q}$ and $\dfrac{1}{2} \in \mathbb{Q}$, thus $\dfrac{x+y}{2} \in \mathbb{Q}$. We now must show that $x < \dfrac{x+y}{2} < y$. We will first show $x < \dfrac{x+y}{2}$.

$$\begin{aligned} x &< y \\ x + x &< x + y \\ 2x &< x + y \\ x &< \frac{x+y}{2} \end{aligned}$$

We must now show that $\dfrac{x+y}{2} < y$.

$$\begin{aligned} x &< y \\ x + y &< y + y \\ x + y &< 2y \\ \frac{x+y}{2} &< y \end{aligned}$$

Hence, $\mathbb{Q}$ is dense in $\mathbb{Q}$.

(b) Show that $\mathbb{R}$ is dense in $\mathbb{R}$.

Let $x \in \mathbb{R}$ and let $y \in \mathbb{R}$ such that $x < y$. We want to show that there exists a $q \in \mathbb{R}$ such that $x < q < y$. By the same argument above, let us take $q = \frac{x+y}{2}$. By the closure property of $\mathbb{R}$ it is clear that $\frac{x+y}{2} \in \mathbb{R}$. We will first show $x < \frac{x+y}{2}$.

$$\begin{aligned} x &< y \\ x + x &< x + y \\ 2x &< x + y \\ x &< \frac{x+y}{2} \end{aligned}$$

We must now show that $\frac{x+y}{2} < y$.

$$\begin{aligned} x &< y \\ x + y &< y + y \\ x + y &< 2y \\ \frac{x+y}{2} &< y \end{aligned}$$

(c) Prove that $\mathbb{R}$ is dense in $\mathbb{Q}$.

Let $x \in \mathbb{Q}$ and let $y \in \mathbb{Q}$ such that $x < y$. We want to show that there exists a $q \in \mathbb{R}$ such that $x < q < y$.
We will make use of the Archimedean Property. Since $x < y$ this implies that $y - x > 0$. Since 1 is a real number, then we may invoke the Archimedean property. That is,there exists a positive integer n such that $n(y - x) > 1$. So we have

$$\begin{aligned} n(y - x) &> 1 \\ ny - nx &> 1 \\ ny &> nx + 1 \qquad (1) \end{aligned}$$

Let us set Equation 1 aside and come back to it later. Let m be the smallest integer such that $m > nx$. This implies that

$$\frac{m}{n} > x \qquad (2)$$

.

Since m is the smallest integer such that $m > nx$, then $m - 1 \leq nx$. To see this, note if $m - 1 > nx$, then we would have $m > m - 1 > nx$ which contradicts our choice of m. So $m - 1 \leq nx$ implies $m \leq nx + 1$. So Equation 1 and $m \leq nx + 1$ implies

$$m \leq nx + 1 < ny \qquad (3)$$

$$m < ny \tag{4}$$

$$\frac{m}{n} < y \tag{5}$$

Putting Eq.2 and Eq.5 together tells us

$$x < \frac{m}{n} < y \tag{6}$$

To complete the proof, we must show $\frac{m}{n}$ is a real number. Since m and n are defined to be integers and n cannot be zero since n is positive, then $\frac{m}{n} \in \mathbb{Q}$ which is a subset of $\mathbb{R}$ and hence $\frac{m}{n} \in \mathbb{R}$.

Problem 2

(a) Prove Rudin's Theorem 1.20

Theorem 1.20 For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$

Proof: We first will show uniqueness. Assume that there exists two distinct y's, namely y_1 and y_2 such that $y_1^n = x$ and $y_2^n = x$. Since y_1 and y_2 are distinct, we have either $y_1 < y_2$ or $y_1 > y_2$. WLOG, assume $y_1 < y_2$. Then we have the following.

$$\begin{aligned} y_1 &< y_2 \\ y_1^n &< y_2^n \\ x &< x \end{aligned}$$

Contradiction! Thus, it must be the case that $y_1 = y_2$ and so y is unique.

Now we will prove the existence of such a y. To do so, let $E = \{t \mid t > 0, t \in \mathbb{R} \text{ and } t^n < x\}$. We will first show that E is not empty. To show this, it suffices to take $t = \frac{x}{1+x}$. Note that for sufficiently large x, $\frac{x}{1+x}$ is close to 1 but never equal to 1. Moreover, since $x > 0$, then the smallest $\frac{x}{1+x}$ can be is some number close to 0. Thus, $0 \leq t < 1$.

By Lemma 1, (see below) this implies $t^n \leq t < x$ and so there exists some $t \in E$.

Now that we know E is not empty, we will also show that E is bounded above. I claim that $1 + x$ is an upper bound of E. That is, for all $t \in E$, $t \leq x + 1$. To see this, we will do proof by contradiction. Assume that $t > 1 + x$, where $t \in E$. Then, by Lemma 2, we have $t^n \geq t > x$, but this implies that $t \notin E$ and we have reached our contradiction. This tells us that E is a nonempty subset of $\mathbb{R}$ which is bounded above. By the LUB property of $\mathbb{R}$, the LUB of E exists. Let $y = \sup E$.

We want to show that $y^n = x$. Well, since we are in an ordered set, if we can show that both $y^n < x$ and $y^n > x$ fails to hold, then it must mean that $y^n = x$. In order to see this, we will use the result from Lemma 4, namely

$$b^n - a^n < (b-a)nb^{n-1} \text{ where } 0 < a < b. \tag{7}$$

Case One: We will first show that $y^n < x$ leads to a contradiction. So assume $y^n < x$. Choose an h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}} \tag{8}$$

Note that the right hand side of the above equation is positive so it follows by the denseness of $\mathbb{R}$ in $\mathbb{R}$ that such an h exists. With regards to Equation 7, substitute in $a = y$ and $b = y + h$. It follows that

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n \tag{9}$$

Note the first inequality, $(y+h)^n - y^n < hn(y+h)^{n-1}$ is a result from Eq. 7 while $hn(y+h)^{n-1} < hn(y+1)^{n-1}$ follows since $h < 1$. The last inequality, $hn(y+1)^{n-1} < x - y^n$ comes directly from Eq. 8.

So, by Eq. 9, we have $(y+h)^n - y^n < x - y^n$ so $(y+h)^n < x$, which shows $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E.

Case Two: We now show that $y^n > x$ fails to hold. Assume $y^n > x$. Let

$$k = \frac{y^n - x}{ny^{n-1}} \tag{10}$$

Then we have $0 < k < y$. Why is $0 < k$? Well since $y^n > x$ the numerator of Equation 10 will always be bigger than 0. 0 divided by any nonzero real number, as seen in the denominator, will still give us 0. Hence $0 < k$. Moreover, we have $k < y$ since the ratio between the numerator and denominator of Equation 10 can be at most $y - \epsilon$ where ϵ is some small positive real number. To see this, let x be any positive real number as defined before and the smallest n can be is $n = 1$. Then as x approaches 0 from the right, $x = 0$ but can never actually be 0 based on the restriction of x. So this case tells us $k < y$. Now as x and n increases, the ratio between the numerator and denominator becomes smaller and so that ratio must be less than y.

Now if $y - k \leq t$, then this implies that

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x \tag{11}$$

Note that we get $y^n - t^n \leq y^n - (y-k)^n$ from using the assumption that $t \geq y-k$. From there, we use Equation 7, to show $y^n - (y-k)^n < kny^{n-1}$. Lastly, $kny^{n-1} = y^n - x$ stems from Equation 10. From the chain of inequalities, we see that $y^n - t^n < y^n - x$, which shows $t^n > x$ so $t \notin E$. This means our assumption was wrong, so for $t \in E$, we must have $y - k > t$. However this implies that $y - k$ is an upper bound of E. Since we assumed y is the least upper bound of E and since $y - k < y$, we have reached our contradiction.

Thus, it must be the case that $y^n = x$.

Lemmas

(a) **Lemma 1**: Let x be a real positive number, let n be a positive integer and let t have the following restrictions: $t = \dfrac{x}{1+x}$ and $t^n < x$. Prove that $t^n \leq t < x$.

Proof: We will first show $t^n \leq t$. Assume that $t^n > t$. Then, substituting in for t yields

$$\left(\frac{x}{1+x}\right)^n > \frac{x}{1+x}$$
$$\frac{x^n}{(1+x)^n} > \frac{x}{1+x}$$

$$x^n > x(1+x)^{n-1}$$

which is false since distributing and simplifying the right hand side will yield a x^n term accompanied by positive real numbers.

We now show $t < x$ by contradiction. Assume that $t \geq x$. Then we have

$$t \geq x$$
$$\frac{x}{1+x} \geq x$$

$$
\begin{aligned}
x &\geq x(1+x) \\
1 &\geq 1+x \\
0 &\geq x
\end{aligned}
$$

which is clearly false since x is defined to be positive.

(b) **Lemma 2**: Let x be a real positive number, let n be a positive integer and let t be such that $t > 1 + x$. Then $t^n \geq t > x$.

Proof: We will first show $t^n \geq t$ by contradiction. Assume that $t^n < t$. Then this implies that $t^{n-1} < 1$ which is false since $t > 1 + x$ and therefore $t > 1$ so this contradicts $t^{n-1} < 1$.

We now will show that $t > x$. Assume that $t \leq x$. Clearly t cannot equal x because this violates $t > 1 + x$. It is also clear that $t < x$ leads to a contradiction since this again violates $t > 1 + x$.

(c) **Lemma 3**: Let a and b be real numbers and let n be a positive integer. Then,

$$
b^k - a^k = (b-a)(b^{k-1} + b^{k-2}a + \ldots + a^{k-1}) \tag{12}
$$

Proof:

$$
(b-a)(b^{k-1} + b^{k-2}a + \ldots + a^{k-1})
$$

$$
= b^k + b^{k-1}a + b^{k-2}a^2 + b^{k-3}a^3 + \cdots + ba^{k-1} - ab^{k-1} - a^2b^{k-2} - \ldots - a^{k-1}b - a^k
$$

$$
\begin{aligned}
&= b^k + (b^{k-1}a - ab^{k-1}) + \ldots + (ba^{k-1} - a^{k-1}b) - a^k \\
&= b^k - a^k
\end{aligned}
$$

(d) **Lemma 4**: Let $0 < a < b$. Then $b^n - a^n < (b-a)nb^{n-1}$

Proof: Since $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-1})$, we must show that

$$
(b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-1}) < (b-a)nb^{n-1}
$$

Since $(b-a)$ is on both sides, the problem reduces to showing that

$$(b^{n-1} + b^{n-2}a + \ldots + a^{n-1}) < nb^{n-1}$$

Since $0 < a < b$, we have

$$(b^{n-1} + b^{n-2}a + \ldots + a^{n-1}) < (b^{n-1} + b^{n-2}b + \ldots + b^{n-1})$$

But notice that the right hand side is precisely nb^{n-1}, since $(b^{n-1}+b^{n-2}b+\ldots+b^{n-1}) = b^{n-1} + b^{n-1} + \ldots + b^{n-1}$ where there are a total of n b^{n-1}'s and thus we have proved what we needed to show:

$$(b^{n-1} + b^{n-2}a + \ldots + a^{n-1}) < nb^{n-1}$$

(e) **Lemma 5**: Suppose r is a rational number, x and y are real numbers, and $r < x + y$. Then there are rational numbers $s < x$ and $t < y$ with $r < s + t < x + y$.

Proof: Since $r < x + y$, then subtracting y and adding x to both sides yields $r < -y + x < 2x$). Dividing by 2 gives us $\frac{r-y+x}{2} < x$. By Theorem 1.20(b) in Rudin,there exists a rational number s such that

$$\frac{r-y+x}{2} < s < x.$$

Similarly, there exists a rational number t such that

$$\frac{r-x+y}{2} < t < y$$

Adding the above equations yields $r < s + t < x + y$.

Problem 3

Prove the following corollary of Theorem 1.20

(a) **Corollary**: If a and b are real numbers and n is a positive integer, then

$$(ab)^{1/n} = a^{1/n}b^{1/n}$$

Proof: Let $\alpha = a^{1/n}$ and let $\beta = b^{1/n}$. Then,

$$\alpha^n = (a^{1/n})^n = (a^{1/n})(a^{1/n})\ldots(a^{1/n}) = a^1$$

Similarly, $\beta^n = b^1$. Note that

$$ab = \alpha^n\beta^n = (\alpha\alpha\ldots\alpha)(\beta\beta\ldots\beta) = (\alpha\beta)(\alpha\beta)\ldots(\alpha\beta) = (\alpha\beta)^n$$

Since $ab = (\alpha\beta)^n$, then by Theorem 1.20, we have

$$\alpha\beta = (ab)^{1/n} \tag{13}$$

Also, based on our definition of α and β, we have

$$\alpha\beta = a^{1/n}b^{1/n} \tag{14}$$

By the uniqueness portion of Theorem 1.20, it therefore follows that

$$(ab)^{1/n} = a^{1/n}b^{1/n}$$

Problem 4

Rudin Chapter 1 Question 6

(a) Fix $b > 1$. If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$

First note that $\frac{m}{n} = \frac{p}{q}$ implies $mq = np$. We will use this fact in a few short moments. Let $y^n = b^m$. Using $y^n = b^m$, we have $y^{nq} = b^{mq} = b^{np}$, by our small fact mentioned above. As shown in the proof of the corollary of Theorem 1.20, we then may conclude that $(y^q)^n = (b^p)^n$. Since n^{th} roots are unique, we see that $y^q = b^p$. Taking the q^{th} roots of each side yields $y = (b^p)^{1/q}$. By a similar argument, we can show $y = (b^m)^{1/n}$. To see this, we have shown $y^{nq} = b^{mq}$ so $(y^n)^q = (b^m)^q$. Taking the q^{th} root yields $y^n = b^m$ and by taking the n^{th} root, we have $y = (b^m)^{1/n}$. Finally, since $y = (b^p)^{1/q}$ and $y = (b^m)^{1/n}$, by the uniqueness, we may conclude $(b^m)^{1/n} = (b^p)^{1/q}$. Since this value is well-defined, that is, the representation does not matter, it is sensible to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

To prove this, we first invoke the following lemma.
Lemma: If b is a real number greater than 1, $x, y \in \mathbb{Z}$, then $b^x b^y = b^{x+y}$.
Proof: $b^x b^y = (b_{x1} \times b_{x2} \ldots \times b_{xx})(b_{y1} \times b_{y2} \times \ldots \times b_{yy}) = b^{x+y}$ since b is multiplied a total of $x + y$ times.

We now continue with the original proof. Since r and s are rational, then there exists integers m, n, p, and q such that m, n, p, q are integers, $n \neq 0$, and $q \neq 0$, where $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Note that $r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$. By part (a) and by the corollary to

Theorem 1.20, we may write $b^{r+s} = (b^{mq+nq})^{1/nq} = (b^{mq}b^{np})^{1/nq} = (b^{mq})^{1/nq}(b^{np})^{1/nq} = (b^m)^{1/n}(b^p)^{1/q} = b^r b^s$.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(r)$$

We now must show that $b^r = \sup B(r)$. In order to prove this, we must show two things:

(a) We must show that b^r is an upper bound for $B(r)$, namely $b^r \geq x, \forall x \in B(r)$

(b) We must show that any element less than b^r fails to be an upper bound for $B(r)$. Namely if $b^r > \lambda$ then λ fails to be a upper bound for $B(r)$.

Let us now prove the first claim. Let us denote the element $x \in B(r)$ by b^s. By definition, s is rational and $s \leq r$ so $0 \leq r - s$ which implies that $1 \leq b^{r-s}$. (Note $b > 1$ by assumption). Multiplying by b^s gives us $b^s \leq b^r$, so b^r is an upper bound for $B(r)$. To prove the second claim, note that $r \leq r$ and so $b^r \in B(r)$. So if $\lambda < b^r$, then λ fails to be an upper bound for $B(r)$. Thus, $b^r = \sup B(r)$.

Since the numbers above are arbitrary, it holds for any such r so it makes sense to define $b^x = \sup B(x)$ provided that x is rational and $b > 1$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

If we can somehow show that $b^{x+y} \leq b^x b^y$ and also show that $b^x b^y \leq b^{x+y}$ then this would show $b^{x+y} = b^x b^y$. We will first attempt to show that $b^{x+y} \leq b^x b^y$. First note that $b^{x+y} = \sup B'(x+y)$ where $B'(z) = \sup \{b^r | r < z, r \in \mathbb{Q}\}$. For every $b^r \in B'(x+y)$, we have $r < x + y$ so by Lemma 5, there exists rational numbers $s < x$ and $t < y$ such that $r < s + t < x + y$. By part (b) and by the fact that $b > 1$, we have

$$b^r < b^{s+t} = b^{s+t} < b^{x+y}.$$

This suggests that $b^x b^y$ is an upper bound for $B'(x+y)$, and so $b^{x+y} \leq b^x b^y$.

We now show $b^x b^y \leq b^{x+y}$. If $r < x$ and $s < y$ then $r + s < x + y$. By part (b) and by using the fact that $b > 1$ suggests $b^x b^y = \sup \{b^r b^s | r < x, s < y, r, s \in \mathbb{Q}\}$. This implies that $b^x b^y \leq b^{x+y}$. Thus $b^x b^y = b^{x+y}$.

Problem 5

Rudin Question 7 Chapter 1. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, bu completing the following outline. (This x is called the logarithm of y to the base b).

(a) For any positive integer n, $b^n - 1 \geq n(b - 1)$.

We will prove this by induction. For our base case, $n = 1$, it is clear that $b - 1 \geq b - 1$ is a true statement. Assume this holds for $n = k$. We want to show that this holds for $k + 1$. That is,

$$\begin{aligned} b^{k+1} - 1 &\geq (k+1)(b-1) \\ b^k b^1 - 1 &\geq kb - b + b - 1 \\ b^k b^1 - b &\geq kb - b \\ b(b^k - 1) &\geq k(b-1) \end{aligned}$$

which is true since $b^k - 1 \geq k(b - 1)$ by hypothesis and b is a positive number greater than 1.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

Note this follows immediately by setting b in part (a) equal to $b^{1/n}$. This yields $b - 1 \geq n(b^{1/n} - 1)$ which is exactly what we needed to show. Note that the statement is still true because $b^{1/n}$ still satisfies the definition of b.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

Since $n > (b - 1)/(t - 1)$, then $(b - 1) < n(t - 1)$. By part (b) we may write $n(b^{1/n} - 1) \leq (b - 1) < n(t - 1)$ and so $n(b^{1/n} - 1) < n(t - 1)$ which shows $(b^{1/n} - 1) < t - 1$ and thus we may conclude $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

Let $t = \dfrac{y}{b^w}$. Then $t > 1$. We already showed $b^{1/n} < t$ so multiplying on both sides by b^w yields $b^w b^{1/n} < tb^w$ which shows $b^{w+(1/n)} < y$.

(e) If $b^w > y$ then $b^{w-(1/n)} > y$ for sufficiently large n.

Let $t = \dfrac{b^w}{y}$. Then $t > 1$. By part (c) we have $b^{1/n} < t$. Dividing throughout by $b^{1/n}$ yields $1 < \dfrac{t}{b^{1/n}}$. Substituting in for t gives us $1 < \dfrac{b^w}{yb^{1/n}}$ which gives us $y < b^{w-(1/n)}$.

(f) Let A be the set of all w such that $b^w < y$ and show that $x = \sup A$ satisfies $b^x = y$.

We will first show that $\sup A$ exists. Note that since $b > 1$, and $y > 0$ we may choose a sufficiently small w which will get us close to 0. This tells us that A is nonempty. Moreover, since $b^w < y$ for a sufficiently large w, A is bounded above. Since we have a nonempty subset of $\mathbb{R}$ which is bounded above, then the supremum exists. Let $\sup A = x$.
We must now show that $b^x = y$. To do this we will show that $b^x < y$ and $b^x > y$ lead to contradictions.

Assume $b^x < y$. Then by part (d), $b^{x+(1/n)} < y$. This tells us that $x + (1/n) \in A$. However, x is an upper bound of A and since, $x + (1/n) > x$ we have reached our contradiction.
Assume $b^x > y$. Then by part (e) $b^{x-(1/n)} > y$. However, $x - (1/n) < x$ and x is assumed to be the $\sup A$ so every element less than x fails to be an upper bound and thus we have reached our contradiction.
Therefore, it must be the case that $b^x = y$.

(g) Prove that this x is unique.

Assume that x is not unique. That is, there exists x and y such that $x = \sup A$ and $y = \sup A$, but $x \neq y$. Then, either $x > y$ or $x < y$. WLOG, assume that $x > y$. Then, be the definition of the supremum, every element less than x fails to be an upper bound for A. However, y is less than x and y is also a supremum and hence an upper bound as well. Contradiction!