Math 650.2 Problem Set 11

Elliot Gangaram
elliot.gangaram@gmail.com

Problem 1

Prove that all $k-$cells are convex when the metric space is $\mathbb{R}^k$ with the usual distance function.

Recall that if $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a k-cell. Intuitively, a $k-$cell in $\mathbb{R}^1$ is an interval, in $\mathbb{R}^2$ is a rectangle and $\mathbb{R}^3$ is a rectangular prism. Also recall that a set $E \subset \mathbb{R}^k$ is convex if

$$\lambda \mathbf{y} + (1-\lambda)\mathbf{z} \in E \tag{1}$$

whenever $\mathbf{y} \in E, \mathbf{z} \in E$, and $0 < \lambda < 1$. We would like to show that all $k-$cells are convex when the metric space is $\mathbb{R}^k$ under the usual metric.

Let E be some $k-$cell and let $\mathbf{y} \in E$ and $\mathbf{z} \in E$. Consider $\lambda \mathbf{y} + (1-\lambda)\mathbf{z}$. We would like to show this vector belongs to E. For simplicity call $(1-\lambda) = \beta$. So we have

$$\begin{aligned}
&\lambda \mathbf{y} + \beta \mathbf{z} \\
&= \lambda(y_1, y_2, \ldots, y_k) + \beta(z_1, z_2, \ldots, z_k) \\
&= (\lambda y_1, \lambda y_2, \ldots, \lambda y_k) + (\beta z_1, \beta z_2, \ldots, \beta z_k) \\
&= (\lambda y_1 + \beta z_1, \lambda y_2 + \beta z_2, \ldots, \lambda y_k + \beta z_k)
\end{aligned}$$

If we can show that $a_i \leq \lambda y_i + \beta z_i \leq b_i \; \forall i$, then we are done because this tells us $\lambda \mathbf{y} + (1-\lambda)\mathbf{z} \in E$. Let us use our assumptions that $\mathbf{y} \in E$ and $\mathbf{z} \in E$. This implies that

$$a_i \leq y_i \leq b_i \tag{2}$$

and

$$a_i \leq z_i \leq b_i \tag{3}$$

Multiplying Eq. 2 by λ yields

$$\lambda a_i \leq \lambda y_i \leq \lambda b_i \tag{4}$$

and multiplying Eq. 3 by β yields

$$\beta a_i \leq \beta z_i \leq \beta b_i \tag{5}$$

Adding Equations 4 and 5 and simplifying yields

$$\begin{aligned} \lambda a_i + \beta a_i &\leq \lambda y_i + \beta z_i \leq \lambda b_i + \beta b_i &(6)\\ = a_i(\lambda + \beta) &\leq \lambda y_i + \beta z_i \leq b_i(\lambda + \beta) &(7) \end{aligned}$$

But recall that $\lambda + \beta = \lambda + (1 - \lambda) = 1$ and so Equation 7 becomes

$$a_i \leq \lambda y_i + \beta z_i \leq b_i \tag{8}$$

which is precisely what we needed to show. Thus, $k-$cells are convex in $\mathbb{R}^k$.

Problem 2

Prove that all balls are convex when the metric space is $\mathbb{R}^k$ with the usual distance function.

Recall that the open (or closed) ball B with center $\mathbf{x}$ and radius r is defined to be the set of all $y \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$) where $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$. We would like to show that all balls are convex when the metric space is $\mathbb{R}^k$. Let $\mathbf{y}$ and $\mathbf{z}$ be two vectors inside the ball B centered at $\mathbf{x}$ of radius r. Observe that

$$\begin{aligned} &|\lambda\mathbf{y} + (1-\lambda)\mathbf{z} - \mathbf{x}| \\ &= |\lambda(\mathbf{y} - \mathbf{x}) + (1-\lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda|\mathbf{y} - \mathbf{x}| + (1-\lambda)|\mathbf{z} - \mathbf{y}| \\ &< \lambda r + (1-\lambda)r \\ &= r \end{aligned}$$

where going from the third line to the fourth line follows because $|\mathbf{y} - \mathbf{x}| < r$ and $|\mathbf{z} - \mathbf{x}| < r$ since we assumed that $\mathbf{y} \in B$ and $\mathbf{z} \in B$. Notice that this proof is for the open ball B. However, the same proof holds for the closed ball B since the $<$ in the fourth line becomes $\leq$ which still leads us to our desired result.

Problem 3

Find a metric space where closed balls are *not* convex.

In the previous question, we showed that $\mathbb{R}^k$ under the usual metric has the property that all open balls are convex. This leads us to a natural question - under what metric are balls not convex? Consider the metric space $\mathbb{R}^2$ along with the metric d where d is defined as follows: $d(\mathbf{x}, \mathbf{y}) = d\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}$. One must really check that d is a metric. It is clear that d is a real valued function and satisfies the first two properties of the definition of metric (see Rudin Definition 2.15). What is not so obvious is seeing that the function satisfies property (c). To see this, observe that

$$\begin{aligned}
&d(\mathbf{x},\mathbf{y}) \\
&= d\big((x_1,x_2),(y_1,y_2)\big) \\
&= \sqrt{|x_1-y_1|}+\sqrt{|x_2-y_2|} \\
&\le \sqrt{|x_1-z_1|}+\sqrt{|z_1-y_1|}+\sqrt{|x_2-z_2|}+\sqrt{|z_2-y_2|} \\
&= d(\mathbf{x},\mathbf{z})+d(\mathbf{z},\mathbf{y})
\end{aligned}$$

which proves that d is a metric on $\mathbb{R}^2$. Note that the inequality from line four follows from the fact[1] that $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ for $a,b\in\mathbb{R}^+$.

Now consider the ball B centered at $\mathbf{x}=\mathbf{0}=(0,0)$ whose radius is 1. To show that this set is not convex, we must show there exists two vectors, $\mathbf{u}$ and $\mathbf{v}$ in the ball B such that $\lambda\mathbf{u}+(1-\lambda)\mathbf{v}\notin B$. Let $\mathbf{u}=(0.75,0)$ and let $\mathbf{v}=(0,0.75)$. Note that these vectors are in B since $d(\mathbf{x},\mathbf{u})=d\big((0,0),(0.75,0)\big)=\sqrt{0.75}<1$ and similarly, $d(\mathbf{x},\mathbf{v})=\sqrt{0.75}<1$. Now consider $\lambda\mathbf{u}+(1-\lambda)\mathbf{v}$. Take $\lambda=\frac{1}{2}$. Is is true that $\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}$ is in B? If it is, then $d(\mathbf{x},\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v})<1$. Observe that

$$\begin{aligned}
&d\big(\mathbf{x},\tfrac{1}{2}\mathbf{u}+\tfrac{1}{2}\mathbf{v}\big) \\
&= d\big((0,0),\tfrac{1}{2}(0.75,0)+\tfrac{1}{2}(0,0.75)\big) \\
&= d\big((0,0),(0.375,0.375)\big) \\
&= \sqrt{|0-0.375|}+\sqrt{|0-0.375|} \\
&= \sqrt{0.375}+\sqrt{0.375} \\
&\approx 1.22>1
\end{aligned}$$

Thus the ball B of radius 1 centered at the origin is not convex.

Problem 4

Illustrate an intuitive meaning of the terms defined on page 32.

See attached.

[1]For proof of this fact, observe that $a+b\le a+b+2\sqrt{ab}$ and taking the square root of both sides yields $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$.