lecture #2

5/23/11

Admin stuff

- webcafe coming soon
- need reminder for recording
- No class June 8/9 but class Fri Jun 3, Fri Jun 24
- Office Hrs 10:35-1PM M-W
- HW due Thurs?
- Working on 1st HW

Plan

- Applied set theory and more interesting examples
- Counting

Review: sets are unordered collections of things - whatever they may be. $a \in A$, $B \subseteq A$, $C \subset D$, E^c, $F \setminus G$, $H \cup I$ (OR), $J \cap K$ (and), 2^L, $\phi \subset M$, $|N|$

Remember our special set the universe of discourse, Ω which is called sample space. We are going to venture away from abstract sets such as

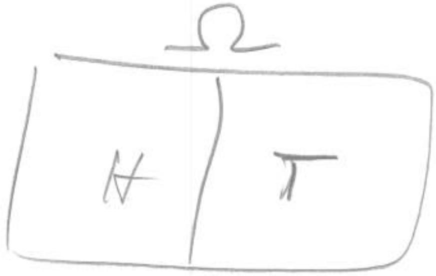
$$F = \{ \text{Jane}, \text{Susan}, \text{Mary} \}$$

and only talk about sets whose elements are actual <u>events</u>.

We care about probabilities of events.

Events actually need a chance of happening.

Consider a coin toss:



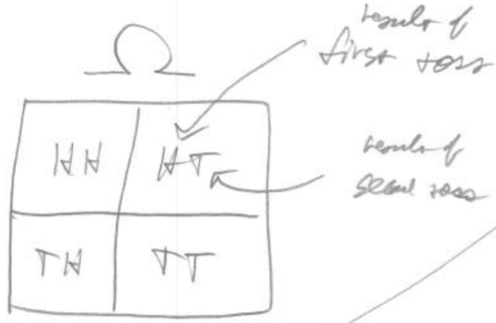
$|\Omega| = 2$

The standard mut. excl. colle. exh. ...

$\{H\} \cap \{T\} = \phi$, $\{H\} \cup \{T\} = \Omega$, $\{H\}^c = \Omega \setminus \{H\} = \{T\}$

Boring sample space!

Let's get more interesting. How about two coin tosses?



There are four events. Never mind that there are only two coins

What is $|\Omega| = 4$ Mut excl? Coll exh?

let $A = \{$ set with at least one H $\} = \{ HH, TH, HT \}$

are these different?

let $B = \{$ set with at least one T $\} = \{TT, TH, HT\}$

Are A, B mut excl ?

Are A, B coll exh ?

→ Test: $A \cap B \stackrel{?}{=} \emptyset$

→ Test $A \cup B \stackrel{?}{=} \Omega$

what about $A \setminus B$?

Still Boring! 3 coin tosses ?

Ω

HHH	HHT	HTT	TTT
TTH	THH	HTH	THT

$|\Omega| = 8$

$C = \{$ two tails $\} = \{HTT, TTH, THT\}$

$D = \{$ getting two tails and then not a tail $\} = \{TTH\}$

specific order

$D \subset C$

any order

How about 4 coin tosses ?

Ω

HHHH	HHHT	HHTT	HTTT
TTTT	TTTH	TTHH	THHH
HTHT	THTH	THHT	HTTH
HHTH	HTHH	THTT	TTHT

$|\Omega| = 16$

Our samples are getting bigger and bigger!

What about 5 tosses? $|\Omega| = 32$

What about 8 tosses? $|\Omega| = 256$

" $\quad$ n tosses? $|\Omega| = 2^n$

" $\quad$ 20 tosses? $|\Omega| \approx 1{,}000{,}000$

30 tosses? $|\Omega| \approx 1{,}000{,}000{,}000$

not an unreasonable # of tosses $\Rightarrow$ very unreasonable sample space

We're going to learn how to count. What was our reasoning above?

$$\underset{\text{first toss}}{\overset{\text{How many choices?}}{\underline{\;2\;}}} \cdot \underset{\text{second toss}}{\overset{\text{How many choices}}{\underline{\;2\;}}} \cdot \;\dots\; \cdot \underset{n^{th}\text{ toss}}{\overset{\text{How many choices?}}{\underline{\;2\;}}} = 2^n$$

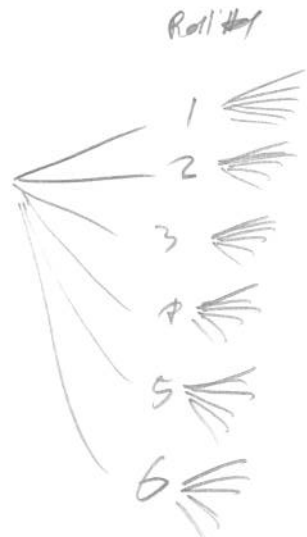
Imagine 3 die rolls. How big is sample space

$$\underset{\text{first roll}}{\overset{\#\text{choices?}}{\underline{\;6\;}}} \cdot \underset{\text{second roll}}{\overset{\#\text{choices?}}{\underline{\;6\;}}} \cdot \underset{\text{third roll}}{\overset{\#\text{choices?}}{\underline{\;6\;}}} = 6^3 = 216$$

4 dice rolls $|\Omega| = 6^4$

General method: go element-by-element and count the # choices and then multiply. Why multiply?

Examine 2 dice rolls:



a tree visualization serves as "proof"

An ℓ-sided coin tossed 4 times? $|\Omega| = \ell^4$

Let's do a slightly different type of problem.

Imagine 3 people Joe (J), Mary (M), Susan (S) sitting in front of you. How many ways to order them? Brute force it:

JMS, JSM, MJS, MSJ, SMJ, SJM $\Rightarrow |\Omega| = 6 \neq 3^3$

Why? Can't duplicate people

Stare at the above let's examine the first "elem", the second "elem", third "elem"

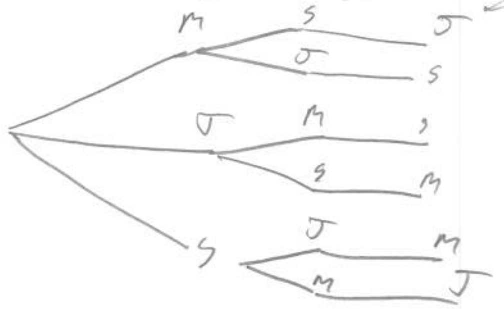
Use general method

why is this??

$$\underset{\text{1st Chair}}{\overset{\#\text{ choices?}}{\underline{3}}} \cdot \underset{\text{2nd chair}}{\underline{2}} \cdot \underset{\text{3rd chair}}{\underline{1}} = 3! = 6$$

Tree representation!

1st chr 2nd chair 3rd chr



$1\,ly = 10^{19} m$

$1 \cdot 10^{11}$

Now 5 people ...

$$\underset{\text{1st chr}}{\underline{5}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5! = 120$$

10 people

$$\underline{10} \cdot \underline{9} \cdot \ldots \cdot \underline{1} = 10! \approx 3.6 \text{ Million}$$

30 people

$30! \approx 2.65 \times 10^{32}$ diameter of the obs. universe in feet

truly an astronomical #

n people: $\boxed{n!}$

Permutations: how many different ways to order a fixed collection

What if there are less chairs than people?

Imagine 5 people, two chairs?

$$\underset{1^{st}\text{ chair}}{\underline{5}} \cdot \underset{2^{nd}\text{ chair}}{\underline{4}} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{5!}{3!}$$

10 people, 4 chairs?

$$\underline{10}\ \underline{9}\ \underline{8}\ \underline{7} = \frac{10!}{6!}$$

n people, k chairs

$$\underset{1}{\underline{n}}\ \underset{2}{\underline{n-1}} \cdots \underset{k}{\underline{n-k+1}} = \frac{n!}{(n-k)!} \quad \text{Notation } "{}_nP_k"$$

How about more chairs than people?

3 people, 10 chairs?

Invert the thinking ... assign chairs to people

$$\underset{\text{person 1}}{\overset{\#\text{chairs}}{\underline{10}}} \cdot \underset{\text{person 2}}{\underline{9}} \cdot \underset{\text{person 3}}{\underline{8}} = \frac{10!}{7!} \quad \text{also } {}_nP_k$$

Go back to 5 people. What if there are 5 chairs in a circle

and we don't care about the rotation. The configuration is the same to observers no matter where they stand.

We still have 5 choices, so...

$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5!$

But now we "<u>don't care</u>" which chair is "first"!

So pretend to look at this from all 5 views:

Hence, we need to divide out the 5 views

$$\Longrightarrow \frac{5!}{5} = 4! = \boxed{24}$$

This method of "dividing out" things we "don't care" about is really, really powerful

5 people, 3 chairs, but don't care about order in 3 chairs!

We know ${}_5P_3 = \frac{5!}{2!}$

But now we have 3 people sitting, they have many orders... how many?

$\underline{3} \cdot \underline{2} \cdot \underline{1} = 3!$

We don't care about the order

So let's divide $\frac{\frac{5!}{2!}}{3!} = \frac{5!}{3!\,2!} = \boxed{10}$

n people, k chairs, don't care about order

$\frac{n!}{(n-k)!\,k!}$ The number of "Combinations"

${}_nC_k$ or $\binom{n}{k}$

How many 5-card poker hands $\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$

How many royal flushes? 4 ! Prob $\Rightarrow$ 1.5 in a million