1

Lecture #6

6/1/11

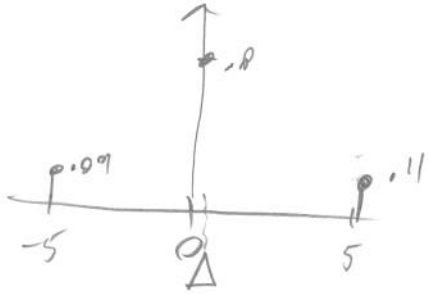
Admin

- HW due today
- Off Hrs today

Plan

- review r.v.'s, exp, var, SD
- Continuous r.v.'s
- 3 special r.v.'s
- IID
- Ch 9, 10.1, 10.3, 10.5
  Skip ch 10.2, 10.4

Book p219 X is a r.v. for \$100 of IBM stock



$\mu_X = E[X] = 5(.11) + 0(.80) + (-5)(.09) = \$0.10$

Where is expectation? $\sigma_X^2 = 4.97\ \$^2$, $\sigma_X = \$2.23$

Def: median: $Med(X)$ s.t. $P(X \geq Med(X)) = \frac{1}{2}$. No std. notation

What if I was day-trading \$1000 of IBM and the trade cost \$7?

What is my expectation and variance?

Elementary Transformation Theory

$Y = 10X - 7 \qquad E[Y] = 10E[X] - 7 = -6$ (not good)

$Var[Y] = 100\,Var(X) \approx 497\ \$^2$

$SD[Y] = 10\sigma_X \approx \$22.30$

General Rules:

$$E[a+bX] = a+b\,E[X]$$

$$Var[a+bX] = b^2\,Var[X]$$

$$SD[a+bX] = |b|\,SD[X]$$

ch 9.4: changing currencies

If X is in dollars, $\frac{X}{0.75}$ is in euros

Sharpe Ratio

Yearly / monthly return (%)

risk free return %

$$Sharpe(X) = \frac{E(X) - r_{free}}{\sigma_X}$$

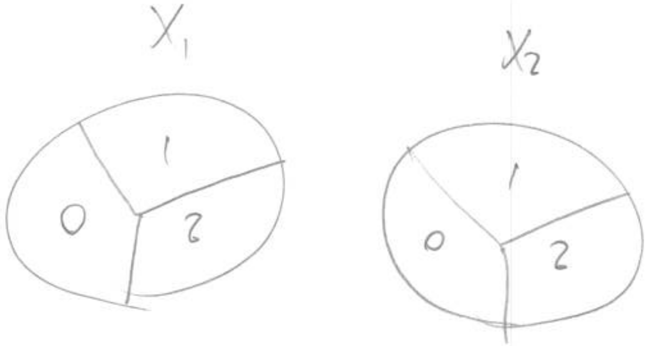
Std deviation

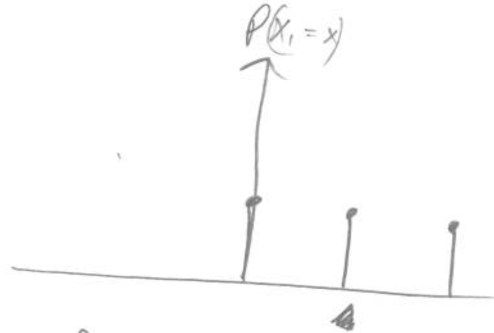
$\frac{\text{Signal}}{\text{Noise}}$ ratio

~~Verify ... formula:~~

~~$Var(X) = E[(X-\mu)^2]$~~

~~$= E[X^2 - 2\mu X + \mu^2]$~~





$$E[X_1] = E[X_2] = 1$$

$$Var[X_1] = Var[X_2] = E(X-1)^2 = \tfrac{1}{3}(-1)^2 + \tfrac{1}{3}0^2 + \tfrac{1}{3}1^2 = \frac{2}{3}$$

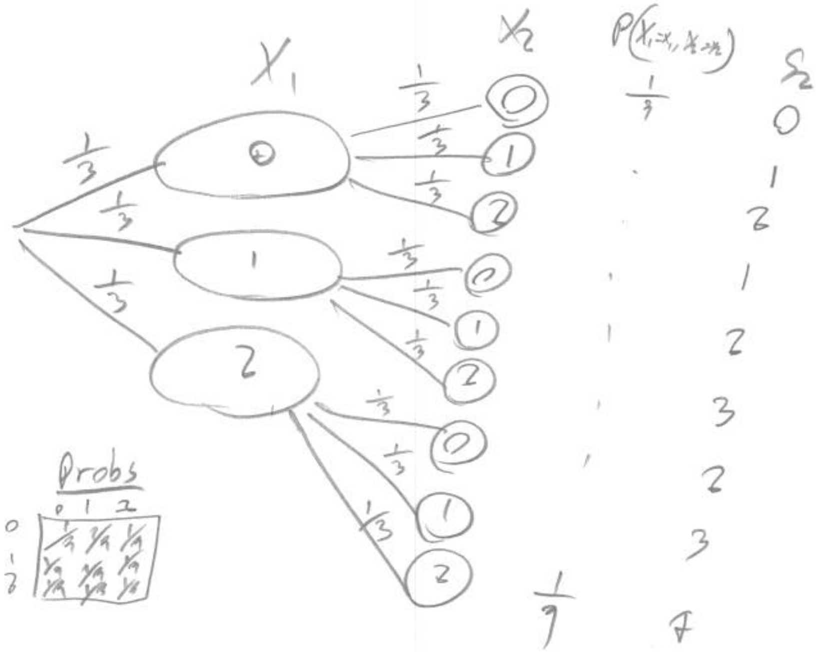
$$Support(X_1) = Support(X_2) = \{0, 1, 2\}$$

Def: Support is set of x s.t. $P(X=x) > 0$

Define a very special r.v. called the sum var: "S for sum"

$S_2 = X_1 + X_2$ book calls this "T" for "total"

What would this look like? Try drawing a tree:



Values of S_2

	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

Probs

	0	1	2
0	1/9	1/9	1/9
1	1/9	1/9	1/9
2	1/9	1/9	1/9

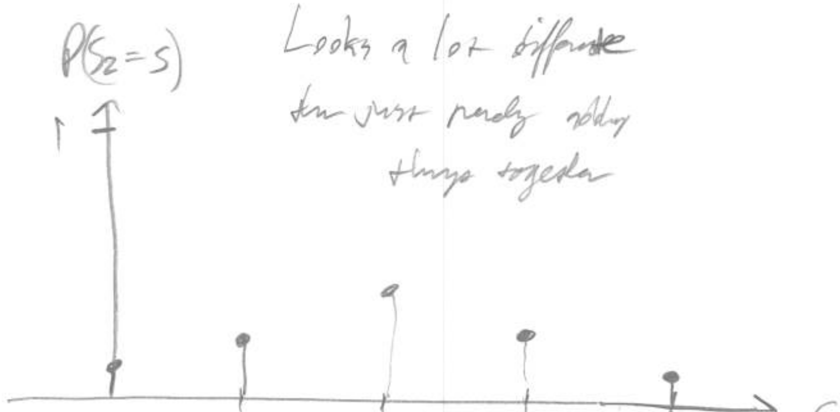
$P(S_2 = 0) = \frac{1}{9}$

$P(S_2 = 1) = \frac{2}{9}$

$P(S_2 = 2) = \frac{3}{9}$

$P(S_2 = 3) = \frac{2}{9}$

$P(S_2 = 4) = \frac{1}{9}$



Looks a lot different than just merely adding things together

Looks symmetric

What's expectation?

What's median?

symmetric ~~case~~ case: $E[X] = Med[X]$

Is it a valid r.v.? How do we know?

Support$(S_2) = \{0,1,2,3,4\}$ ⟶ SUPPORT IS ~~DIFFERENT~~ DIFFERENT!

Let's calc $E(S_2)$, $Var[S_2]$ ..

$$E[S_2] = 0\left(\frac{1}{9}\right) + 1\left(\frac{2}{9}\right) + 2\left(\frac{3}{9}\right) + 3\left(\frac{2}{9}\right) + 4\left(\frac{1}{9}\right) = 2$$

$$Var[S_2] = (-2)^2\left(\frac{1}{9}\right) + (-1)^2\left(\frac{2}{9}\right) + (0)^2\left(\frac{3}{9}\right) + (1)^2\left(\frac{2}{9}\right) + (2)^2\frac{1}{9}$$
$$= \frac{4}{3}$$

Do you see a pattern?

$$E[S_2] = E(X_1) + E(X_2) \qquad (1)$$

$$Var[S_2] = Var(X_1) + Var(X_2) \qquad (2)$$

It turns out these are rules. Formula (1) is <u>ALWAYS</u> true. Why? ... Can't prove this yet...

But: ... $\sum_{j=1}^{L}\sum_{i=1}^{K}(x_i + y_i)\, P(X=x_i, Y=y_i) = \sum_{i=1}^{K} x_i P(X=x_i) + \sum_{j=1}^{L} y_i P(Y=y_i)$

What about formula 2. Only true in the case where X_1, X_2 are independent. We ~~haven't~~ haven't defined this for r.v.'s yet, but you can sort of see why this is true: Knowing the spin of first spinner won't tell you anything about second spinner.

X_1 & X_2 are <u>more</u> than independent, they have the same distribution. Hence, they're ind. <u>and</u> identically distributed (iid). This is denoted as follows:

$$X_1, X_2 \overset{iid}{\sim} \begin{cases} 0 & \text{w.p } \frac{1}{3} \\ 1 & \text{w.p } \frac{1}{3} \\ 2 & \text{w.p } \frac{1}{3} \end{cases}$$

What about S_n? Sum of n iid spinners?

$$S_n = X_1 + X_2 + \dots + X_n$$

$$E[S_n] = E(X_1) + E(X_2) + \dots + E(X_n) = n\,E[X_1] = n\mu$$

all expectations same

$$Var[S_n] = Var[X_1] + Var[X_2] + \dots + Var[X_n]$$ independence

$$= n\,Var[X_1] = n\sigma^2$$

all have same variance

p230

$$SD[S_n] = \sqrt{Var[S_n]}$$ by def

$$= \sqrt{n\,Var[X_1]}$$ ~~by~~ by calc above

$$= \sqrt{n}\,\sigma$$

Let's define another r.v. also very special:

$$\overline{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{S_n}{n}$$

The "average" of r.v.'s is also itself a r.v.

$$E[\overline{X}_n] = E\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right]$$

$= \frac{1}{n} E[X_1 + \dots + X_n]$ What rule??

$= \frac{1}{n} n \, E(X_1)$ see above

$= E(X_1) = \mu$ makes sense?

a lot of sense...

$$Var(\overline{X}_n) = Var\left[\frac{X_1 + \dots + X_n}{n}\right]$$

$= \frac{1}{n^2} Var(X_1 + \dots + X_n)$ What rule???

$= \frac{1}{n^2} n \, Var(X_1)$ (see above)

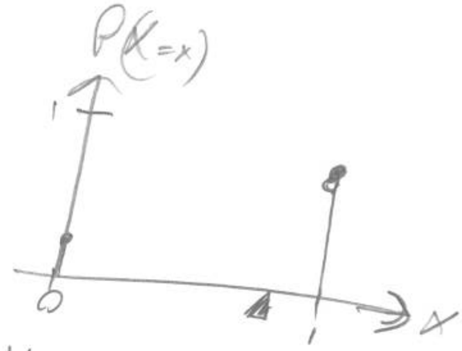
$= \frac{1}{n} Var X_1 = \frac{\sigma^2}{n}$

What does this tell you??

$$SD[\overline{X}_n] = \sqrt{Var(\overline{X}_n)} = \frac{\sigma}{\sqrt{n}}$$

WE WILL REVIEW THIS NEXT WEEK

More review from yesterday...

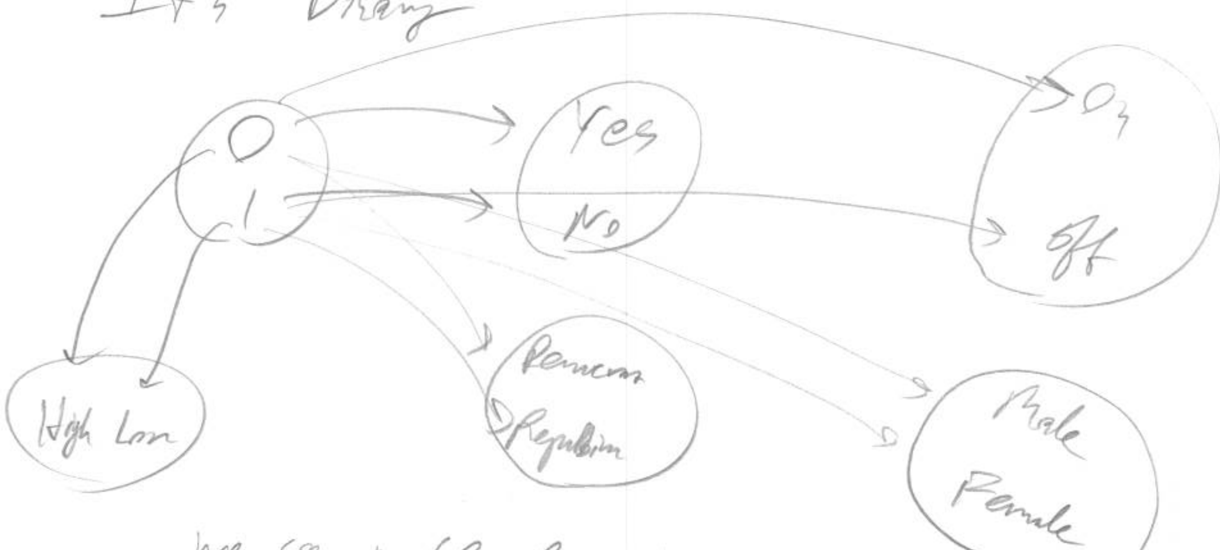


$$X \sim \begin{cases} 1 & \text{w.p. } 0.8 \\ 0 & \text{w.p. } 0.2 \end{cases}$$

$\mu = E[X] = 0.8 \quad \sigma^2 = Var[X] = (0.8)^2 0.2 + (0.2)^2 0.8 = .16$

This zero-one r.v. is super useful. Why?

It's Binary



$\cancel{p^2(1-p) + (1-p)^2 p}$
$\cancel{p^2 - p^3 + p - p^3}$
$\cancel{p^2 - p}$
$\cancel{p(1-p)}$

We can model lots of stuff

$E[S_{100}] = 100 \cdot 0.8 = 80$ We expect 80 yes's out of 100. Sense?

$Var[S_{100}] = 100 \cdot .16 = 16$

$SD[S_{100}] = 4$

We expect 80 ± 4 yes's out of 100

$E[\bar{X}_{100}] = 0.8$

$Var[\bar{X}_{100}] = \frac{0.16}{100} = 0.0016$ We expect $80\% \pm 4\%$ yes's as a percentage!

$SD[\bar{X}_{100}] = .04$

8

So general, it's called something. It's called "Bernoulli" family...

$X \sim \text{Bernoulli}(0.8) \triangleq \begin{cases} 1 & \text{w.p. } 0.8 \\ 0 & \text{w.p. } 0.2 \end{cases}$

type of r.v

"prob" of "success"

Let's do general case,

prob of success p could be anything $\in [0,1]$

$X \sim \text{Bernoulli}(p) \triangleq \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$E[X] = (1)(p) + (0)(1-p) = \boxed{p} = \mu$

$\text{Var}[X] = (1-p)^2 p + (0-p)^2(1-p)$

$= (1-2p+p^2)p + p^2(1-p)$

$= p - 2p^2 + \cancel{p^3} + p^2 - \cancel{p^3}$

$= p - p^2$

$= \boxed{p(1-p)} = \sigma^2$

p244/245

How do we count many successes? Sum of Bernoullis:

$$S_6 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

What's the probability of getting 4 successes?

$$\underbrace{P(X_1 = 1)\,P(X_2 = 1)\,P(X_3 = 1)\,P(X_4 = 1)}_{4 \text{ successes}}\,\underbrace{P(X_5 = 0)\,P(X_6 = 0)}_{2 \text{ failures}}$$

What is the probability of the above? By independence of the above

$$p^4 (1-p)^2$$

But... how many ways to order these?

$$S_1 S_2 S_3 S_4 F_1 F_2 = 6!$$

Now... don't care about order of successes $= 4!$, order of failures $= 2!$

$$\frac{6!}{4!\,2!} = \binom{6}{4}$$

$$P(4 \text{ successes in } 6 \text{ trials}) = \binom{6}{4} p^4 (1-p)^2$$

Very special indeed...

$$S_6 = X_1 + \dots X_6 \sim \text{Binomial}(\overset{\#\text{ trials}}{6}, \overset{\text{prob of success}}{p}) \triangleq \binom{6}{\underset{\#\text{ success}}{k}} p^k (1-p)^{6-k}$$

In general...

$$S_n = \underbrace{X_1 + \dots + X_n}_{\text{sum of } n \text{ Bernoullis}} \sim \text{Binomial}(n, p) \triangleq \binom{n}{k} p^k (1-p)^{n-k}$$

Let's calc Expec and Var...

$$E[S_n] = \sum_{k=1}^{n} k \overbrace{\binom{n}{k} p^k (1-p)^{n-k}}^{P(S_n = k)} \quad \text{or...}$$

$$= E[X_1 + \dots X_n] = n E(X_1) = \boxed{np}$$

$$\text{Var}(S_n) = \sum_{k=1}^{n} (k - np)^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \text{Var}[X_1 + \dots + X_n]$$

$$= n \text{Var}[X_1] = \boxed{np(1-p)}$$

p897/200

32 teach

Examples of same...