Lecture #5
5/31/11

Admin Stuff

- HW due Wed (tomorrow)
- 2nd HW coming out Wed due following Wed
- Class on Friday
- No class next Wed / Thurs

Plan

Random Variables
Expectation
Variance
Std Dev
} almost all of ch. 9

Future:
Special r.v.'s ch 11
Association ch 10

Last time we talked about odds against winning (E)

$$\text{Odds(win)} = \frac{1 - P(\text{win})}{P(\text{win})}$$ ← ratio of prob not winning to prob of winning

If winning is rolling a 6 then $\text{Odds}(\{6\}) = \frac{5/6}{1/6} = 5:1$

We likened this to a fair game and drew a picture:

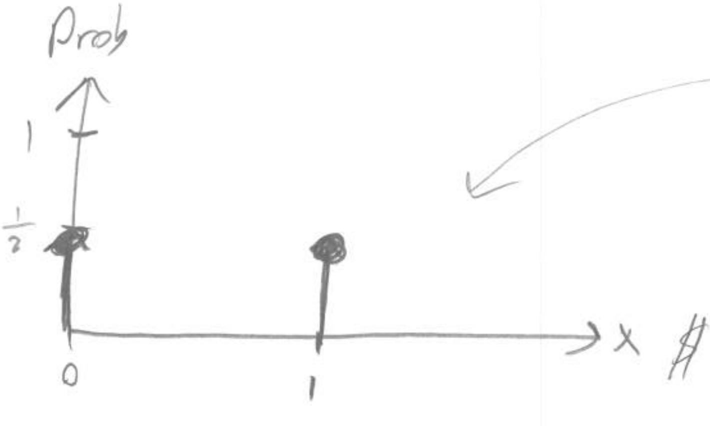
$$\text{expect} = (5)\tfrac{1}{6} + (-1)\tfrac{5}{6} = \tfrac{5}{6} - \tfrac{5}{6} = 0$$ (fair)

We intuitively used the notion of <u>expected value</u>. But, more so, we created a <u>statistical model</u> that has clear payoffs and clear prob's. (Ch 9) random variable

What is a (r.v.)? r.v. ≠ recreational vehicle

Before we define it... Imagine once again, flipping a coin...

What if we map a heads to winning \$1, tails to \$0.
We've mapped Ω's elements to a numerical value (\$ in this case). but we've kept our probabilities of the heads or tails.
Can we graph the outcomes vs. the prob?



note: ~~the book draws a (•) on top, I'll draw an arrow (easier to see on the board)~~. Feel free to draw it any way you wish

The book (p198) calls this a "prob. distr." but it is more accurately called a "prob mass function" (PMF) (also on p.198).

When we look at the PMF, we can see the prob's of all the outcomes.

A r.v.: "describes probs for an uncertain future event"

How do we notate this? X Why?

Historical anachronism... r.v.'s are actually functions!

$$X(H) = 1, \quad X(T) = 0$$

functions whose states have probabilities:

how can $X=1$? only if Heads happens

$$P(X=1) = P(\{H\}) = \frac{1}{2}$$

$$P(X=0) = P(\{T\}) = \frac{1}{2}$$

how can $X=0$? only if tails flips

Let's talk about another coin flip. This time, an unfair coin. Let's say it flips Heads w.p. 90% and we have the same mappings.

Sometimes, we'll denote this like so:

$$X \sim \begin{cases} 1 & \text{w.p. } .9 \\ 0 & \text{w.p. } .1 \end{cases}$$

when do the pieces happen? They have different probabilities

distributed as

piecewise function

What is PMF?

better notation for y-axis

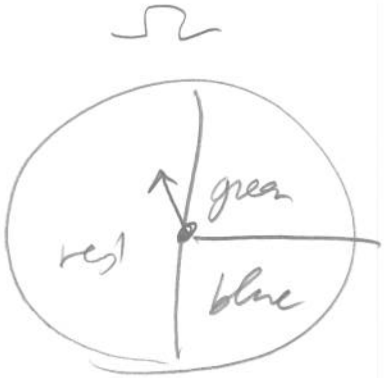
Once again

$P(X=1) = 0.9$

$P(X=0) = 0.1$

This r.v. is very special and has its own name... next lecture...

Let's do another example:



Consider X s.t.

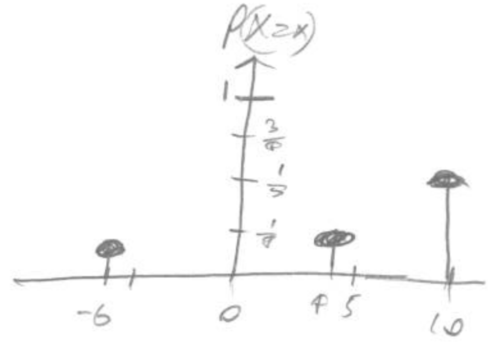
If red: win \$10

If green: win \$4

If blue: lose \$6

Who really cares what Ω is? X is just an abstraction... who knows where randomness comes from anyway?

$$X \sim \begin{cases} 10 & \text{w.p. } \frac{1}{2} \\ 4 & \text{w.p. } \frac{1}{4} \\ -6 & \text{w.p. } \frac{1}{4} \end{cases}$$



Side:
Quest: $P(X>0) = P(X=4) + P(X=10) = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$

What do we expect to happen in the long run?

We expect to get the weighted average:

$$\text{expect} = \left(\frac{1}{2}\right) 10 + \left(\frac{1}{4}\right) 4 + \left(\frac{1}{4}\right) -6 = 0$$

This "expected value" is also called "expectation" of the r.v.

In general: (p 201)

$$E[X] = p_1 x_1 + p_2 x_2 + \dots p_K x_K = \sum_{i=1}^{K} x_k \underbrace{P(X=x_k)}_{\text{prob mass}}$$

expectation operator

r.v. that we take expectation of

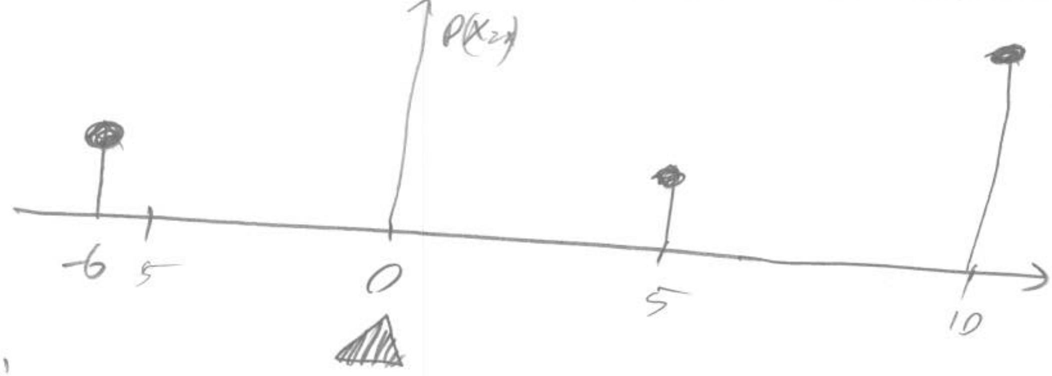
Let's take expectation of our variable above:

$E[X] = 0$

The weighted avg is just like the weighted avg in Physics

The fulcrum goes in the place where the plank would be balanced exactly.

Our r.v.'s numerical outcome is balanced at zero:



Boring

One more r.v.: Random Walk: ~~+1 w.p. 1/2~~

$$X \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

Why is this called a random walk?

$$E[X] = \frac{1}{2}(1) + \frac{1}{2}(-1) = \frac{1}{2} - \frac{1}{2} = 0$$

What are the expected values of some casino games?

Blackjack: many complicated strategies... basically (major approx) 0.5% house edge if you memorize some simple rules

$$X \sim \begin{cases} 1 & \text{w.p. } 0.495 \\ -1 & \text{w.p. } 0.505 \end{cases}$$

$$E[X] = 1 \cdot .495 + -1 \cdot .505 = -\$0.01$$

"You lose 1 cent on every deal on average"

Expected Value is so special, we give it a special symbol:

$$\mu \triangleq E[X] = \sum_{i=1}^{K} x_i P(X = x_i)$$

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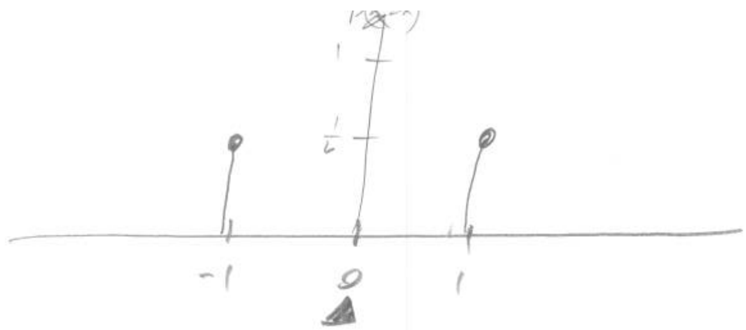
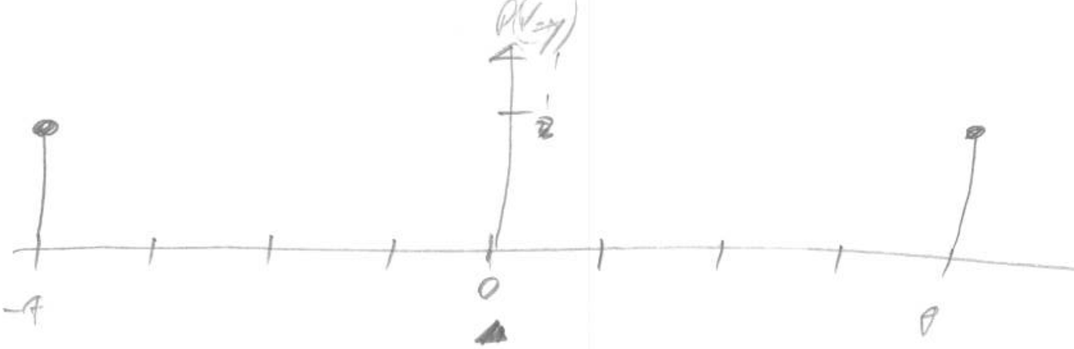
expected value is a good start in explaining our r.v. X. However, it is very limited... consider the following two r.v.'s, let's call them X and Y:



bet \$1 on a coin flip



bet \$4 on a coin flip

$E(X)=0$ and $E(Y)=0$

but Y's values are much ~~further~~ further from its expected value than X's values... We say Y has higher "variance" than X... because it "varies away from the mean more than X". We use expected squared deviation:

$$\sigma^2 \triangleq Var[X] = E\left[\underbrace{(X-E(X))^2}_{\text{how much sqd deviation}}\right] = \sum_{i=1}^{K} (x_i-\mu)^2 P(X=x_i)$$

special symbol

expected

$Var[X] = (1-0)^2\cdot\frac{1}{2} + (-1-0)^2\cdot\frac{1}{2} = \frac{1}{2}\$^2$

$Var[Y] = (4-0)^2\cdot\frac{1}{2} + (-4-0)^2\cdot\frac{1}{2} = 16\$^2$

big difference

Y is much more "diffuse" OR "dispersed" than X. Y is much more haphazard, much more chaotic...

Q: Can there be a r.v. with no variance?

Units on variance: initial units squared...

What if we want a measure of variance with the same units?

$$\sigma \triangleq SD[X] = SE[X] = \sqrt{Var[X]}$$

"standard deviation" "standard error"

$SD[X] = \sqrt{\frac{1}{2} \$^2} = \$0.25$, $SD[Y] = \sqrt{16 \$^2} = \$4$

Sometimes this is denoted 0 ± 0.25, or 0 ± 4

where $\pm$ indicates SD.

If time, do $E[X \pm c] = E(X) \pm c$, $Var[X \pm c] = Var(X)$

Tomorrow $E[cX] = cE(X)$, $Var[cX] = c^2 Var X \Rightarrow SD(cX) = |c| SD[X]$