CSPE32: Combinatrics and Graph Theory
Assignment-1

# Finding the $\gamma_r$ (Mixed Roman Domination Number) of $P_3$ O $P_n$

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CSE-B

The weak Roman domination number, denoted by $\gamma_r$(G), is the minimum weight of a WRDF in G

## Weak Roman Dominating Function requirements

We need to assign weights of 0, 1, or 2 to each vertex in such a way that:

- Every vertex with weight 0 has a neighboring vertex with weight $> 0$.
- If a vertex with weight 0 is changed to 1 then a weight of 1 is reduced from one of the adjacent vertices with weight$> 0$ and still point one satisfies while also not affecting the edges not involved

## Lexicographic Product

The **lexicographic product** of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, denoted $G[H]$, is a graph defined as follows:

- **Vertices**: The vertex set of $G[H]$ is $V(G) \times V(H)$, where each vertex is represented as a pair $(g, h)$ with $g \in V(G)$ and $h \in V(H)$.
- **Edges**: There is an edge between two vertices $(g_1, h_1)$ and $(g_2, h_2)$ in $G[H]$ if:
  1. $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$, or
  2. $\{g_1, g_2\} \in E(G)$.

Now consider $P_3$ and $P_n$, where $P_3 = \{u_1, u_2, u_3\}$ is the path graph on 3 vertices with edges $\{u_1u_2, u_2u_3\}$, and $P_n = \{v_1, v_2, \ldots, v_n\}$ is the path graph on $n$ vertices with edges $\{v_iv_{i+1} : 1 \leq i < n\}$.

- **Vertex Set**: The vertices of $P_3[P_n]$ are:

$$\{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n), (u_2, v_1), (u_2, v_2), \ldots, (u_2, v_n), (u_3, v_1), \ldots, (u_3, v_n)\}.$$

- **Edges**: Two vertices $(u_i, v_j)$ and $(u_k, v_l)$ are adjacent if:
  1. $u_i = u_k$ and $(v_j, v_l) \in E(P_n)$, or
  2. $(u_i, u_k) \in E(P_3)$ (i.e., $|i - k| = 1$).

Thus, the graph $P_3 \mathrm{O} P_n$ has the following structure:

- Three copies of $P_n$, corresponding to $u_1, u_2$, and $u_3$.
- **Cross connections**:
  - Every vertex in the first copy (corresponding to $u_1$) is connected to every vertex in the second copy (corresponding to $u_2$).
  - Every vertex in the second copy (corresponding to $u_2$) is connected to every vertex in the third copy (corresponding to $u_3$).

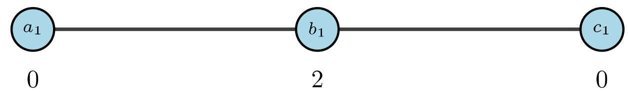
## Strategy for Assigning Weights

- We need to assign weight of 0 to each of the vertex.
- Then we start by assigning a weight of 1 to each vertex with max degree, if not satisfied we bump it to 2.
- If still not satisfied we go through the $b_n$ vertices as they are only the ones affected in terms of adjacent weight and try to assign a weight of 1 to any of them and see if it satisfies.
- If not satisfied then we just remove the extra weight given and just assign one of the vertices of $a_n$ or $c_n$ (either is fine) a weight of 2 as it will cover for all the $b_n$.

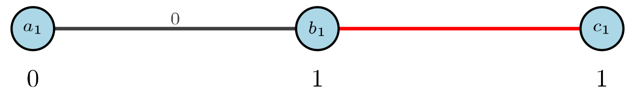
# Proof

## 1 $P_3$ O $P_1$

Starting with $P_3 = \{a, b, c\}$ and $P_1 = \{1\}$, the lexicographic product of the two gives two vertices $V = \{a_1, b_1\}$ and edges $E = \{(a_1, b_1)\}$.



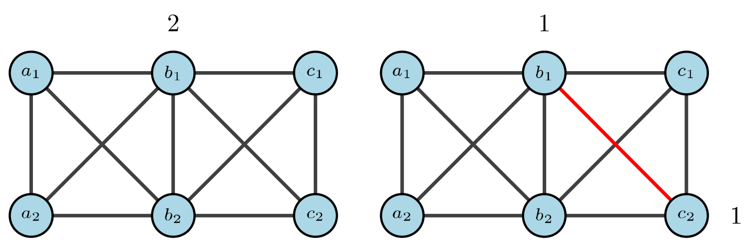

it satisfies the first function which is that $b_1$ with weight 2 is adjacent to $c_1$ , $a_1$ with weight 0 (i.e 1)




$\boxed{\textit{The } \gamma_r \textit{ for } \mathrm{P}_3 \text{ O } \mathrm{P}_1\text{=2}}$

## 2 $P_3$ O $P_2$

$P_3 = \{a, b, c\}$ and $P_2 = \{1, 2\}$, the lexicographic product of the two gives a complete graph with 4 vertices $V = \{a_1, b_1, a_2, b_2\}$ and edges $E = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)(a_1, a_2)(b_1, b_2)\}$

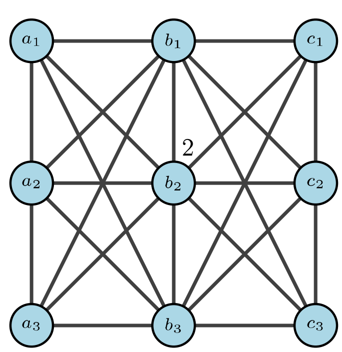



it satisfies as $b_1$ is connected to all vertices and when moved to another position it still has a weight of 1 to cover for the other vertices.

$\boxed{\textit{The } \gamma_r \textit{ for } \mathrm{P}_3 \text{ O } \mathrm{P}_2\text{=2}}$

## 3 $P_3$ O $P_3$

$P_2 = \{a, b\}$ and $P_3 = \{1, 2, 3\}$, the lexicographic product of the two gives as follows

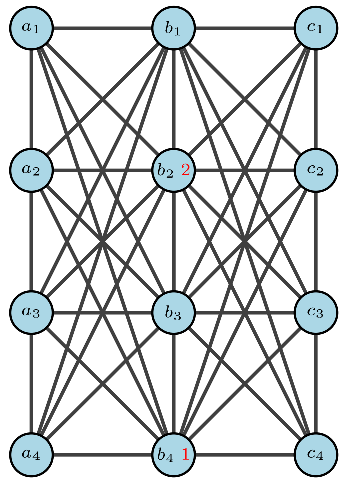



All the vertices are covered by $b_2$.
We can observe that $b_2$ covers for all vertices with its weight of 2. If we put 1 instead of 2 then when moved from $b_2$ to $a_1$ , $c_3$ remains undefended.

$\boxed{\textit{The } \gamma_r \textit{ for } \mathrm{P}_3 \text{ O } \mathrm{P}_3\text{=2}}$

# 4 $\mathbf{P_3}$ O $\mathbf{P_4}$

$P_3 = \{a, b, c\}$ and $P_4 = \{1, 2, 3, 4\}$, the lexicographic product of the two gives as follows

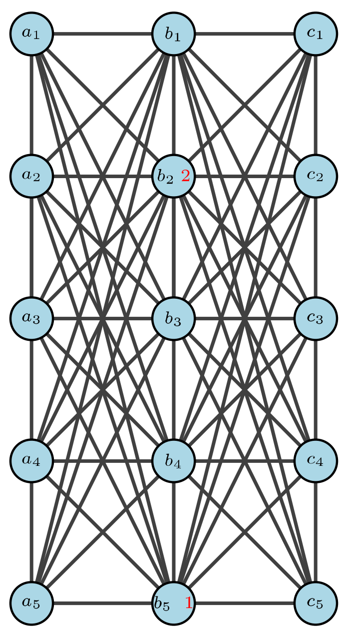



Here $b_4$ remains undefended so we just assign it a weight of 1 so it satisfies the first and second function.

$\boxed{\textit{The } \gamma_r \textit{ for } \mathrm{P_3 \ O \ P_4{=}3}}$

# 5 $\mathbf{P_3}$ O $\mathbf{P_5}$

$P_3 = \{a, b, c\}$ and $P_5 = \{1, 2, 3, 4, 5\}$, the lexicographic product of the two gives as follows




We assign $b_5$ the weight of 1 to cover for both the undefended $b_4$ and $b_5$. This also ensures coverage when weight moves from $b_5$ to any of the possible ways.

After this for $P_3$ O $P_n$ where n>5 we have $b_n$ vertices upto 6 where if we try to assign 1 to $b_5$. If moved to $b_4$ then $b_6$ remains undefended and when moved to $b_6$ , $b_4$ remains undefended so we instead of giving assigning 2 to $b_5$ we instead give one of the $a_n$ with weight of 2 which will be repeated.

$\boxed{\textit{The } \gamma_r\textit{; for } \mathrm{P_3 \ O \ P_5{=}3}}$

## Summary

The $\gamma_r$ for $P_3 \; O \; P_n$ is given by:

$$
\begin{aligned}
\text{The } \gamma_r \text{ for } P_3 \text{ O } P_1 &= 2,\\
\text{The } \gamma_r \text{ for } P_3 \text{ O } P_2 &= 2,\\
\text{The } \gamma_r \text{ for } P_3 \text{ O } P_3 &= 2,\\
\text{The } \gamma_r \text{ for } P_3 \text{ O } P_4 &= 3,\\
\text{The } \gamma_r \text{ for } P_3 \text{ O } P_5 &= 3,\\
\text{The } \gamma_r \text{ for the rest} &= 4,
\end{aligned}
$$

**Bonus next page**

## Bonus

Finding the $\gamma_R$ (Mixed Roman Domination Number) of $P_3 \odot P_n$:

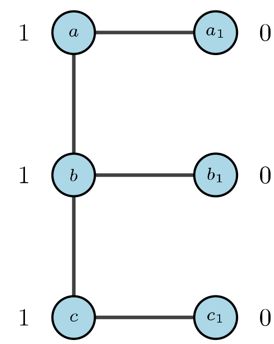
## Corona Product

One copy of P3 and 3 copies of Pn is taken. Every vertex in P3 is connected to all the vertices of the corresponding copy of Pn.

## Strategy for Assigning Weights

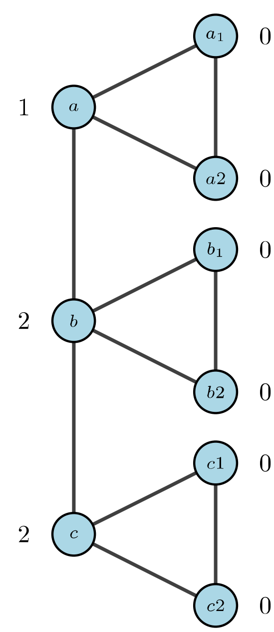
- Assign 2 to the vertices a,b,c.
- Assignment to the edges of each copy of Pn follows this pattern.
- 0 , 1 , 0,2, 0,2,0 , 0,2,0,1, 0,2,0,2 ,0,2,0,2,0 , 0,2,0,2,0,1 etc.

# 1 $P_3 \odot P_1$



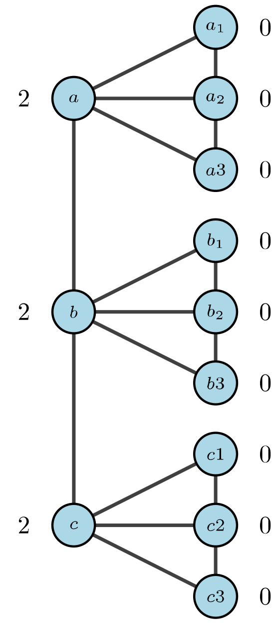

$\boxed{The\ \gamma_r\ for \mathrm{P}_3 \odot P_1{=}3}$

# 2 $P_3 \odot P_2$




$\boxed{The\ \gamma_r\ for \mathrm{P}_3 \odot P_2{=}1}$

## 3 $P_3 \odot P_3$




$\boxed{The\ \gamma_r\ for \mathrm{P}_3 \odot P_3 = 6}$

## Summary

The $\gamma_r$ for $P_3 \odot P_n$ is given by:

$$\text{The } \gamma_r \text{ for } P_3 \odot P_1 = 3,$$
$$\text{The } \gamma_r \text{ for } P_3 \odot P_2 = 3,$$
$$\text{The } \gamma_r \text{ for the rest} = 6.$$

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**Thank You**