MIDSEM MODEL SOLUTION

1) The total energy is

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{\text{all pairs}\\(i,j)}} \frac{q_i q_j}{|r_i - r_j|}$$

Four sets of pairs

a) Charges $(-q,-q)$ along the sides of length L

No. of such pairs $= 4+4+2+2 = 12$ $\frac{1}{2}$

b) Charges $(-q,-q)$ along the face diagonals $(\sqrt{2}L)$

No. of such pairs $= \underbrace{2}_{\#/\text{face}} \times \underbrace{6}_{\#\text{ of faces}} = 12$ $\frac{1}{2}$

c) Charges $(-q,-q)$ along the body diagonals $(\sqrt{3}L)$

No. of such pairs $= 4$ $\frac{1}{2}$

d) Charges $(-q,+3q)$ from center to vertices $\left(\frac{\sqrt{3}L}{2}\right)$,

No. of such pairs $= 8$ $\frac{1}{2}$

The total electrostatic energy is

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{12q^2}{L} + \frac{12q^2}{\sqrt{2}L} + \frac{4q^2}{\sqrt{3}L} + \frac{8(-q)(3q)}{\sqrt{3}L/2}\right]$$

[No marks if negative sign omitted before Last term] 1/2 for writing the expression.

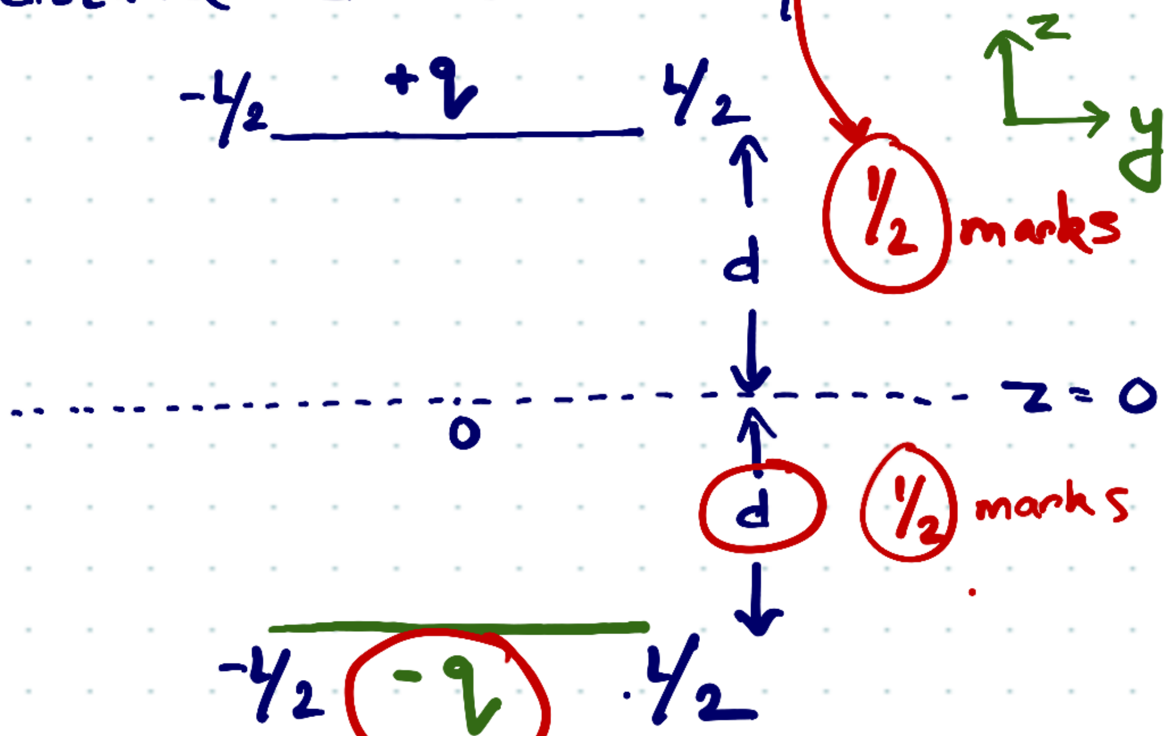
$$= \frac{1}{4\pi\varepsilon_0}\frac{12q^2}{L}\left[1 + \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{3}} - \frac{48}{\sqrt{3}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{12q^2}{L}\left[1 + \frac{1}{\sqrt{2}} - \frac{44}{\sqrt{3}}\right]$$

1/2 for final correct answer.

Any equivalent expression is acceptable.

2) a) The image is again a line charge at a distance 'd' below the plane at $z = 0$.



$\frac{1}{2}$ marks. No marks if sign is not mentioned.

b) The force between two elements of charge

$$dq_1 = \lambda\, dy_1 \quad \text{at} \quad (0, y_1, d)$$

and

$$dq_2 = \lambda\, dy_2 \quad \text{at} \quad (0, y_2, -d)$$

is

$$dF = \frac{1}{4\pi\varepsilon_0} \frac{\lambda^2\, dy_1\, dy_2}{(y_1 - y_2)^2 + 4d^2} \qquad \frac{1}{2}$$

From symmetry, after summing over all such elements of charge, only the z-components will survive,

$$\Rightarrow dF_z = dF\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda^2 dy_1 dy_2}{(y_1-y_2)^2+4d^2}\cdot\frac{2d}{\sqrt{(y_1-y_2)^2+4d^2}}$$

½ marks for the $\cos\theta$ component

The total force is then given by

$$\vec{F} = F(-\hat{z})$$

½ marks for the correct direction of force

with

$$F = \frac{\lambda^2}{4\pi\varepsilon_0}\int_{-l/2}^{l/2} dy_1 \int_{-l/2}^{l/2} dy_2 \frac{2d}{\left[(y_1-y_2)^2+4d^2\right]^{3/2}}$$

Let us first attempt the dy_1 integral

$$\mathcal{J} = \int_{-L/2}^{L/2} dy_1 \frac{2d}{\left[(y_1-y_2)^2+4d^2\right]^{3/2}}$$

Let, $y_1 - y_2 = u \Rightarrow dy_1 = du$

$$\Rightarrow \mathcal{J} = 2d \int_{-\left(\frac{L}{2}+y_2\right)}^{\frac{L}{2}-y_2} \frac{du}{\left(u^2+4d^2\right)^{3/2}} \quad \left(\text{Using the given integral}\right)$$

$$= \frac{1}{2d}\left[\frac{u}{\sqrt{u^2+4d^2}}\right]_{-\left(\frac{L}{2}+y_2\right)}^{\frac{L}{2}-y_2}$$

$$= \frac{1}{2d}\left[\frac{\frac{L}{2}-y_2}{\left[\left(\frac{L}{2}-y_2\right)^2+4d^2\right]^{1/2}} + \frac{\left(\frac{L}{2}+x_2\right)}{\left[\left(\frac{L}{2}+x_2\right)^2+4d^2\right]^{1/2}}\right]$$

1 mark for evaluating first integral correctly

Substituting back, we obtain,

$$F = \frac{\lambda^2}{4\pi\varepsilon_0}\frac{1}{2d}\int_{-L/2}^{L/2} \frac{\frac{L}{2}-y_2}{\left[\left(\frac{L}{2}-y_2\right)^2+4d^2\right]^{1/2}} + \frac{\frac{L}{2}+y_2}{\left[\left(\frac{L}{2}+y_2\right)^2+4d^2\right]^{1/2}}\, dy_2$$

Now, let
$$g' = \int_{-L/2}^{L/2} \frac{\frac{L}{2}\pm y_2}{\left[\left(\frac{L}{2}\pm y_2\right)^2+4d^2\right]^{1/2}}\, dy_2$$

Let, $u = \frac{L}{2} \pm y_2 \Rightarrow du = \pm\, dy_2$

$$\Rightarrow g' = \int_0^L \frac{u\,du}{\sqrt{u^2+4d^2}} = \left[\sqrt{u^2+4d^2}\right]_0^L$$

$$= \sqrt{L^2+4d^2} - 2d$$

1/2 marks for second integration

The total force is then given by

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda^2}{d}\left[\sqrt{L^2+4d^2} - 2d\right](-\hat{z})$$

1/2 marks for final correct answer. Any equivalent representation is acceptable.

c) In the limit $d \gg L$, the line charge effectively behaves as a point charge, in which case we can use the standard point charge result for the force,

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q}{4d^2} (-\hat{z})$$

1 mark

Either argument is acceptable.

This can also be obtained by taking the limit in the full force expression obtained in part b),

$$\vec{F} = \frac{\lambda^2}{2\pi\varepsilon_0} \left[\frac{\sqrt{L^2+4d^2}}{2d} - 1 \right] (-\hat{z})$$

$$= \frac{\lambda^2}{2\pi\varepsilon_0} \left[\left(1 + \left(\frac{L}{2d} \right)^2 \right)^{1/2} - 1 \right] (-\hat{z})$$

$$\overset{\text{Lt. } \frac{L}{2d} \to 0}{=} \frac{\lambda^2}{2\pi\varepsilon_0} \left[1 + \frac{1}{2} \frac{L^2}{4d^2} - 1 \right] (-\hat{z})$$

$$= \frac{(\lambda L)^2}{4\pi\varepsilon_0} \cdot \frac{1}{4d^2} (-\hat{z}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{4d^2} (-\hat{z})$$

3) $$V(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\left[\frac{A}{r} - Br\cos\theta + \frac{C}{r^2}\cos\theta\right]$$

a) $$\vec{E} = -\nabla V$$

$$= -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}\right)$$

$$\frac{\partial V}{\partial r} = \frac{1}{4\pi\varepsilon_0}\left[-\frac{A}{r^2} - B\cos\theta - \frac{2C}{r^3}\cos\theta\right]$$

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\varepsilon_0}\left[B\sin\theta - \frac{C}{r^3}\sin\theta\right]$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\left(\frac{A}{r^2} + \underline{B\cos\theta + \frac{2C}{r^3}\cos\theta}\right)\hat{r} + \left(\underline{-B\sin\theta + \frac{C}{r^3}\sin\theta}\right)\hat{\theta}\right]$$

$\frac{1}{2}$ mark for $\hat{r}$ component

$\frac{1}{2}$ mark for $\hat{\theta}$ component

b) As $r \to \infty$

$$\vec{E}_\infty = \frac{1}{4\pi\varepsilon_0}\left[B\cos\theta\,\hat{r} - B\sin\theta\,\hat{\theta} \right]$$

$\frac{1}{2}$ mark for correct field in (r, θ, φ) coordinates

Now, $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$$\Rightarrow \vec{E}_\infty = \frac{B}{4\pi\varepsilon_0}\hat{z}$$

$$\Rightarrow E^x_\infty = E^y_\infty = 0$$

$\frac{1}{2}$ marks

$$E^z_\infty = \frac{B}{4\pi\varepsilon_0}$$

c) Let us divide the electric field into three parts,

$$\vec{E}_1 = \frac{A}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

$$\vec{E}_2 = \frac{B}{4\pi\varepsilon_0}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$\vec{E}_3 = \frac{C}{4\pi\varepsilon_0}\left[\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

such that $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$

Now, we know, $\vec{\nabla}\cdot\left(\dfrac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

$\Rightarrow \vec{\nabla}\cdot\vec{E}_1 = \dfrac{A}{\varepsilon_0}\delta^3(\vec{r})$ (1) marks for correctly identifying delta-function.

$\vec{E}_2 = \vec{E}_0 = \dfrac{B}{4\pi\varepsilon_0}\hat{z}$

$\Rightarrow \vec{\nabla}\cdot\vec{E}_2 = \dfrac{\partial}{\partial z}\left(\dfrac{B}{4\pi\varepsilon_0}\right) = 0$ ($\frac{1}{2}$) marks

Finally,

$$\vec{\nabla}\cdot\vec{E}_3 = \frac{C}{4\pi\varepsilon_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2\cos\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^3}\right)\right]$$

($\frac{1}{2}$) marks for setting up

$$= \frac{C}{4\pi\varepsilon_0}\left[-\frac{2\cos\theta}{r^4} + \frac{2\sin\theta\cos\theta}{r^4\sin\theta}\right].$$

$$= 0$$

($\frac{1}{2}$) marks for this part

Thus, $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E}_1 + \vec{\nabla}\cdot\vec{E}_2 + \vec{\nabla}\cdot\vec{E}_3$

$$= \frac{A}{\varepsilon_0}\delta^3(\vec{r})$$

The change density is

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = A\delta^3(\vec{r})$$

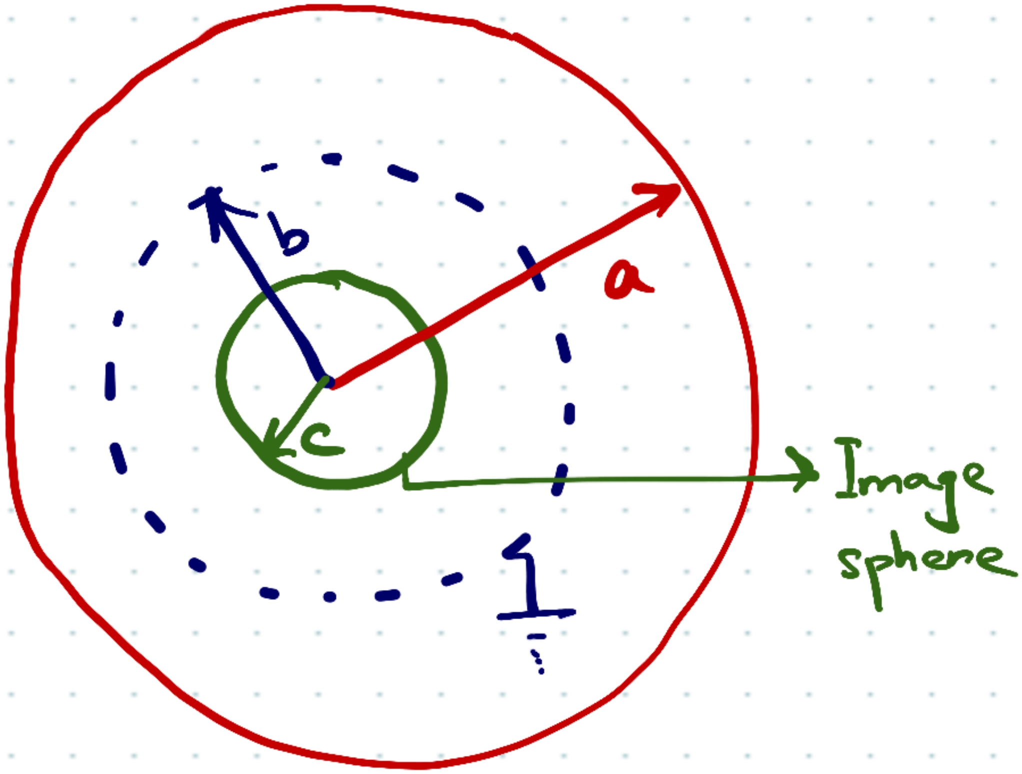
The total change is then,

$$Q = \int_{\text{all space}} \rho \, dV = A\int \delta^3(\vec{r})\, d^3\vec{r}$$

$$\Rightarrow \boxed{Q = A}$$

1/2 marks

4) a)



The image charge distribution is also a sphere, of radius $c = b^2/a$.

1/2 mark   1/2 mark

[Each element of charge dq on the original sphere will produce an image charge dq' on the surface of the image sphere.]

b) Since the surface of the sphere is uniformly charged, surface charge density is $\sigma = Q/4\pi a^2$

with, $Q = \int \sigma \, da = \int \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta \, d\theta \, d\varphi$

$$= \int dq$$

where, $dq = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\varphi$

This dq element of charge at $r = a$ produces an image charge dq' at $r = b^2/a$, with

$$dq' = -dq \frac{b}{a}$$

1 mark for identifying image charge element

$$= -\frac{Qb}{4\pi a} \sin\theta \, d\theta \, d\varphi$$

The total image charge is then,

$$Q_{image} = \int dq' = -\frac{Qb}{4\pi a} \int \sin\theta \, d\theta \, d\varphi$$

$$= -Qb/a$$

1/2 for correct answer.

c) Notice that the image setup is spherically symmetric, so we can use Gauss's Law to calculate electric fields, $(E_\theta = E_\phi = 0)$

for $b < r < a$

$$E_r \cdot 4\pi r^2 = \frac{Q_{encl.}^{\text{image sph.}}}{\varepsilon_0} = -\frac{Qb}{\varepsilon_0 a}$$

$$\Rightarrow \vec{E} = -\frac{Q}{4\pi\varepsilon_0}\frac{b}{a}\frac{\hat{r}}{r^2} \qquad b < r < a$$

1 mark

for $r > a$,

$$E_r \cdot 4\pi r^2 = \frac{Q_{encl.}^{\text{image sph}} + Q_{encl.}^{\text{orig. sphere}}}{\varepsilon_0}$$

$$\Rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0}\left(1 - \frac{b}{a}\right)\frac{\hat{r}}{r^2} \qquad r > a$$

1 mark

Note that since the sphere at $r = b$ is grounded, and there are no charges within this sphere, $\vec{E} = 0$ inside the sphere,

$$\Rightarrow \vec{E} = 0 \quad \text{for } r < b$$

$\frac{1}{2}$ mark

d) Now,

$$V(r) = -\int_{a} \vec{E}\cdot d\hat{r} = -\int_{\infty}^{a} E_r\, dr - \int_{a}^{r} E_r\, dr$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(1-\frac{b}{a}\right)\int_{\infty}^{a}\frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_0}\frac{b}{a}\int_{a}^{r}\frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(1-\frac{b}{a}\right)\frac{1}{a} + \frac{Q}{4\pi\varepsilon_0}\frac{b}{a}\left(\frac{1}{a}-\frac{1}{r}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{a}\left(1-\frac{b}{r}\right)$$ $\frac{1}{2}$ marks

$$\Rightarrow V\left(\frac{a+b}{2}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{a}\left(\frac{a}{a+b}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{1}{a+b}$$ $\frac{1}{2}$ marks