# THE ELECTRIC POTENTIAL

Gauss's Law is always true.

It may not always be useful!

If we can take advantage of the symmetries of a problem, Gauss's Law can be a very powerful tool.

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E} = -\nabla V$$

$$V(\boldsymbol{r}) \equiv \text{Electric Potential}$$

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0 \quad \Rightarrow \quad \boxed{\nabla^2 V = -\rho/\epsilon_0}$$ Poisson's Equation

In regions of no charge $\boxed{\nabla^2 V = 0}$ Laplace Equation

# THE ELECTRIC POTENTIAL

$$\nabla \times \boldsymbol{E} = 0 \quad \Leftrightarrow \quad \oint \boldsymbol{E} . d\boldsymbol{l} = 0$$

$$\boxed{V(\boldsymbol{r}) = -\int_{O}^{\boldsymbol{r}} \boldsymbol{E} . d\boldsymbol{l}}$$

O is some standard reference point

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{O}^{\boldsymbol{b}} \boldsymbol{E} . d\boldsymbol{l} + \int_{O}^{\boldsymbol{a}} \boldsymbol{E} . d\boldsymbol{l} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} . d\boldsymbol{l}$$

$$\boldsymbol{E} = -\nabla V \qquad \nabla^2 V = -\rho/\epsilon_0$$

We have reduced a vector problem to a scalar one. How?

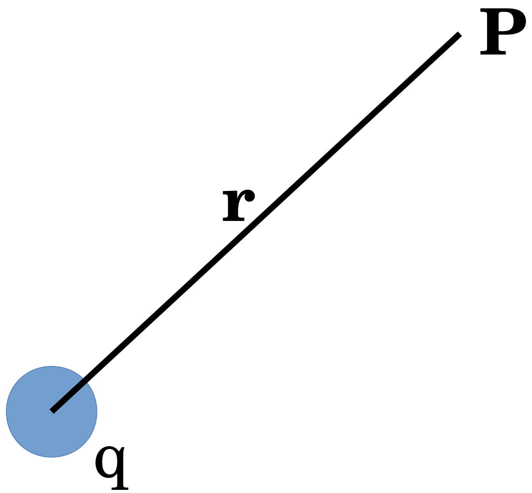
The choice of reference point is arbitrary. Changing the reference point does not change the potential difference or electric field between two point.

For most situations, $V(\infty) = 0$

Potential obeys the superposition principle.

Units: N-m/C, J/C, Volt

# POTENTIAL FOR LOCALIZED CHARGE DISTRIBUTIONS

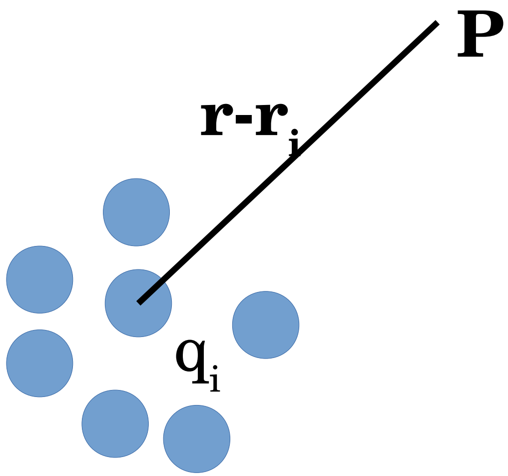


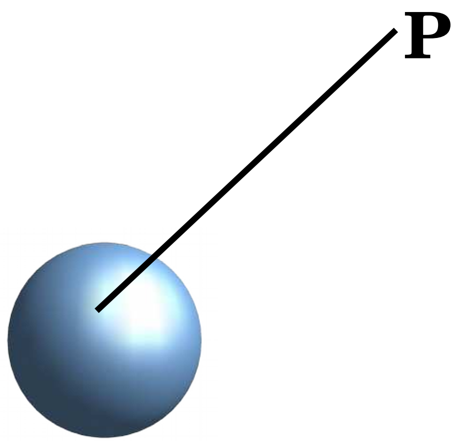

Point charge at origin

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r'}\right]_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{s}}|}$$



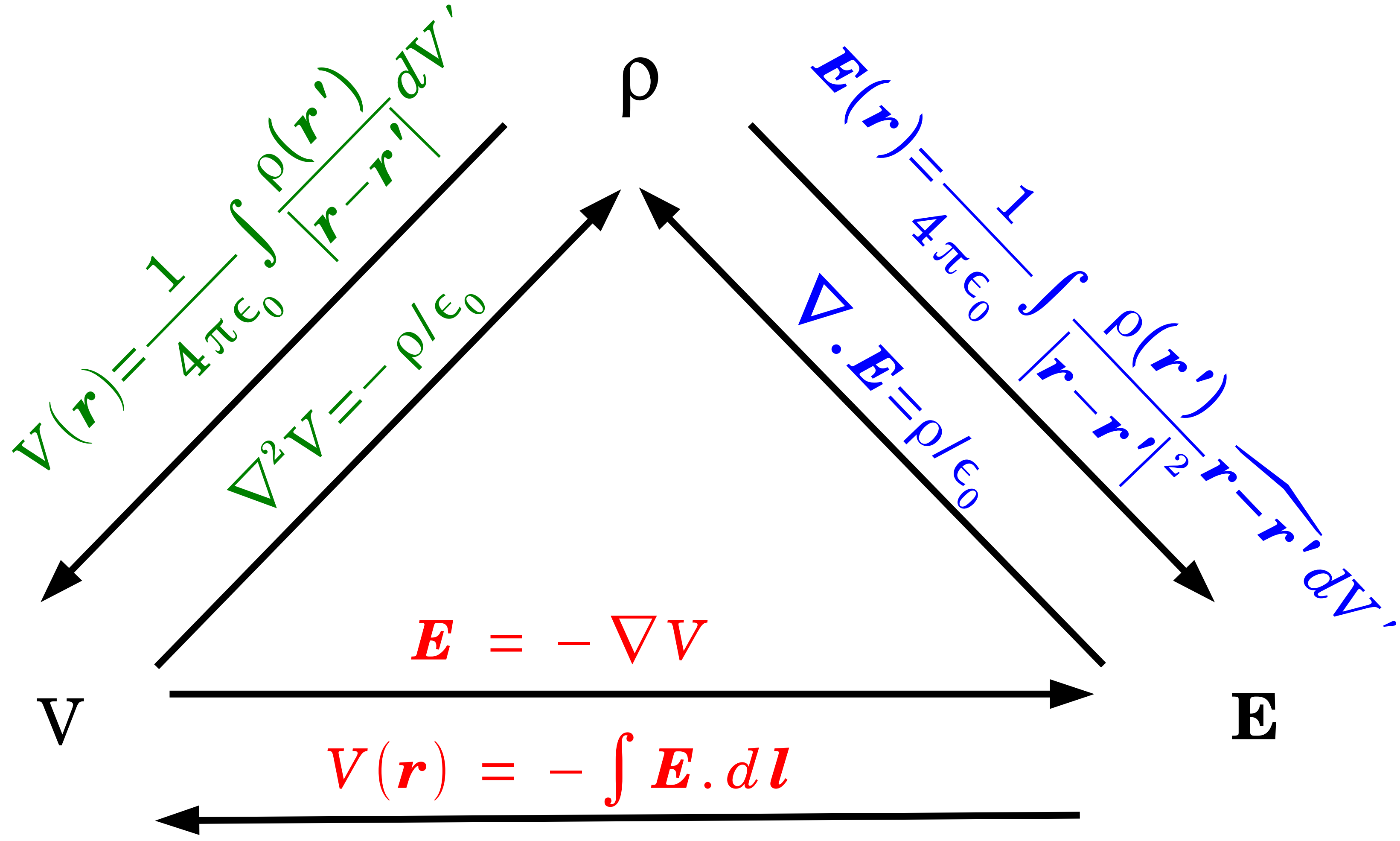




Superposition principle

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{i}}|}$$

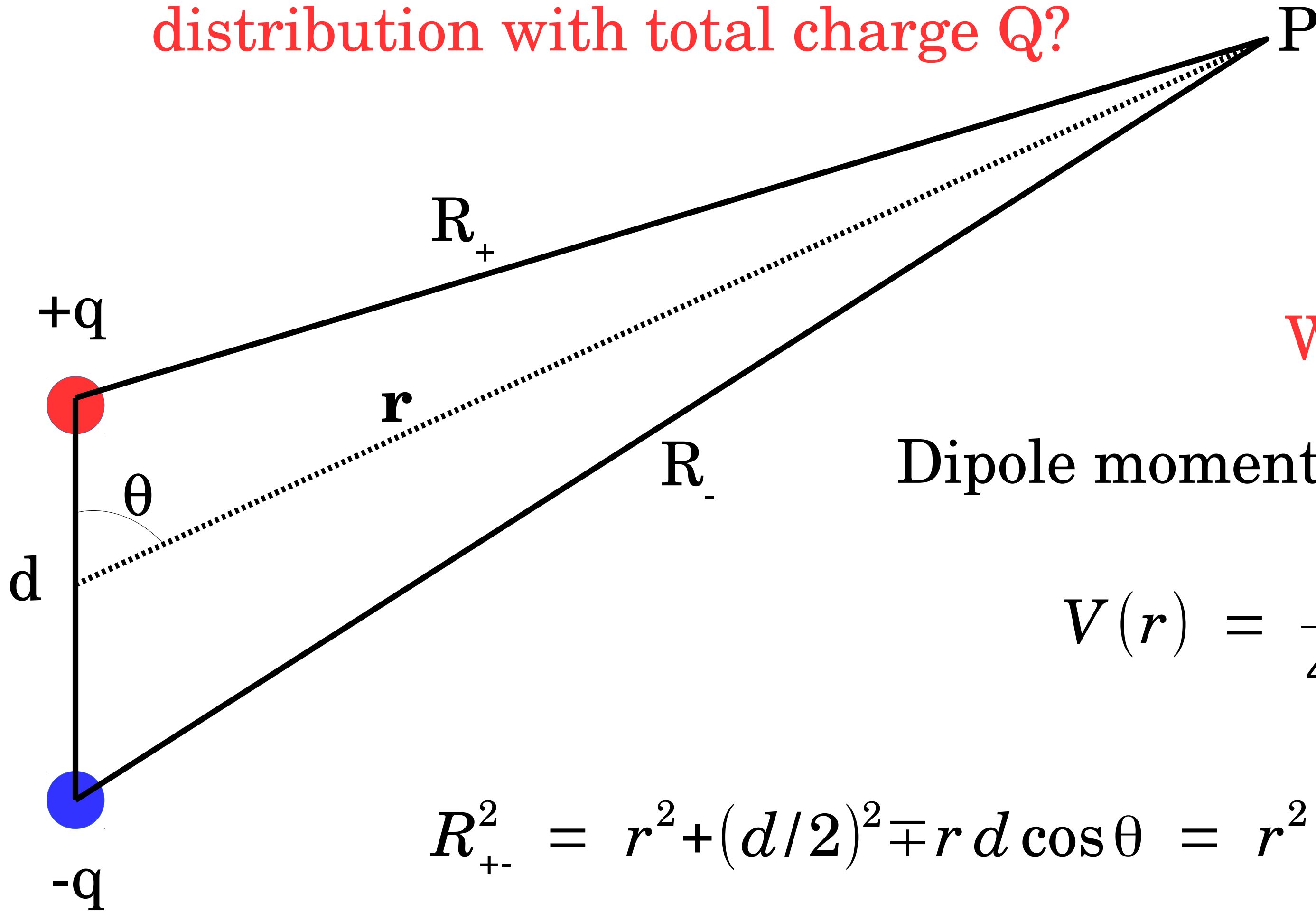
$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{q(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

# CHARGE, POTENTIAL AND ELECTRIC FIELD



# POTENTIAL FOR A DIPOLE

What is the potential far away from a localized charge distribution with total charge Q?




$$V \simeq \frac{Q}{4\pi\epsilon_0 r}$$

What if Q = 0?

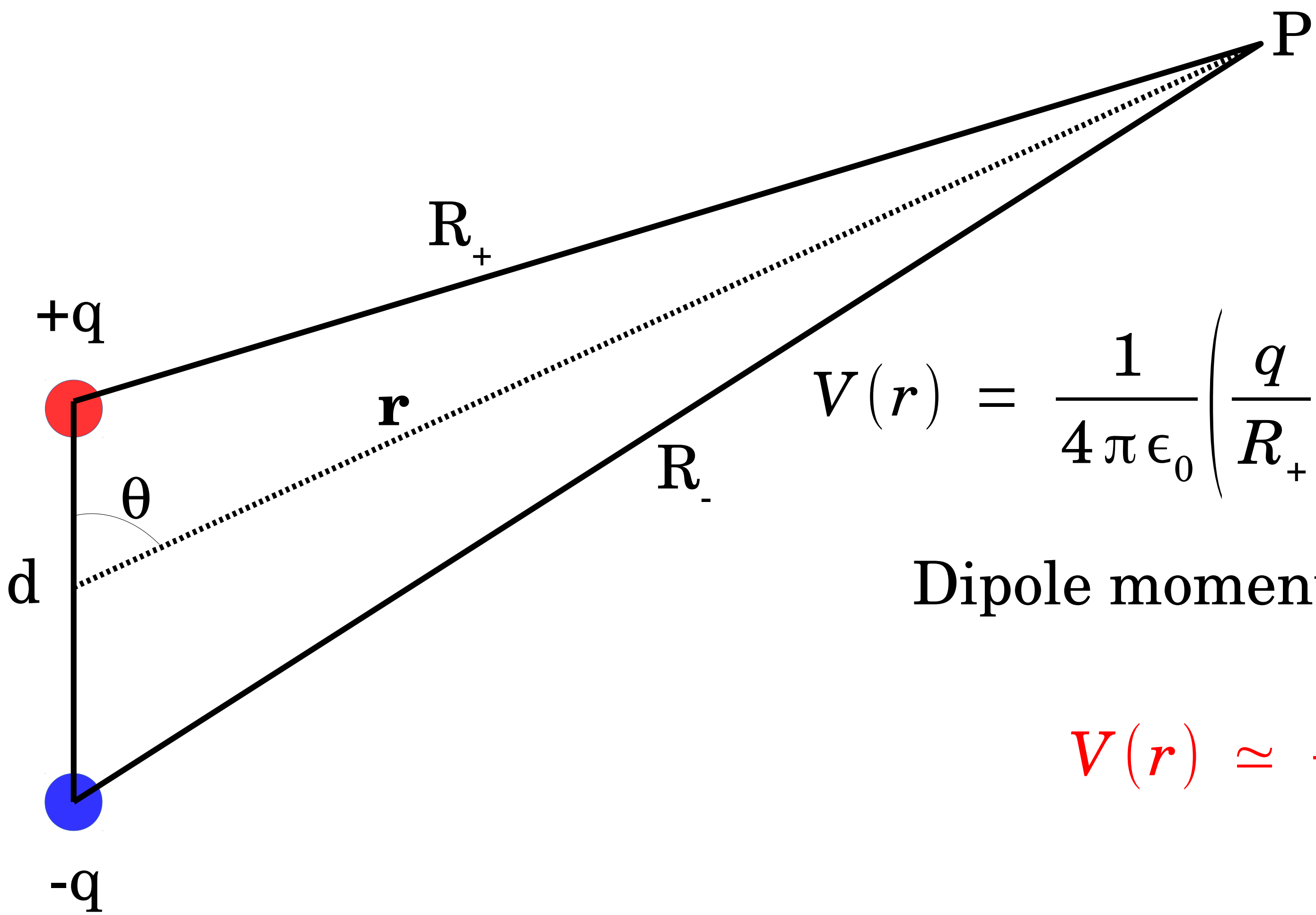
Dipole moment vector $\boldsymbol{p} = q\,d\,\hat{\boldsymbol{p}}$

$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_+} - \frac{q}{R_-}\right)$$

$$R_{+-}^2 = r^2 + (d/2)^2 \mp r\,d\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\text{If } r \gg d, \quad \frac{1}{R_{+-}} \simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

# POTENTIAL FOR A DIPOLE




$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_+} - \frac{q}{R_-}\right) \simeq \frac{1}{4\pi\epsilon_0}\frac{q\,d\cos\theta}{r^2}$$

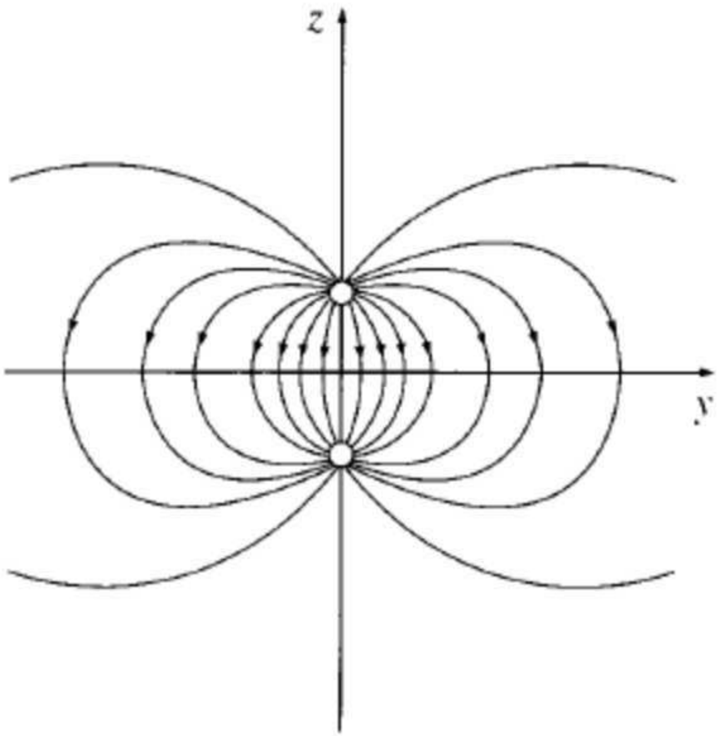
Dipole moment vector $\boldsymbol{p} = q\,d\,\hat{\boldsymbol{p}}$

$$V(r) \simeq \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}.\hat{\boldsymbol{r}}}{r^2}$$

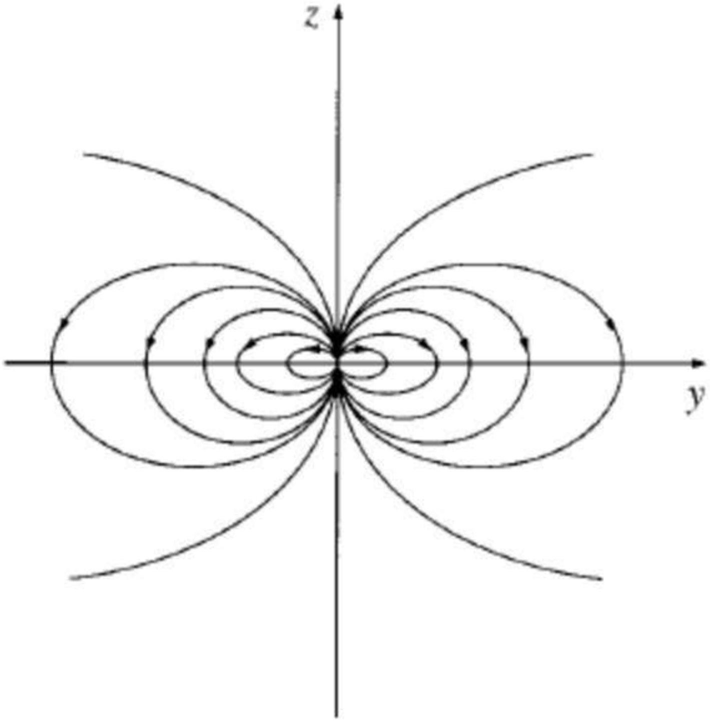
$$\boldsymbol{E} = -\nabla V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\boldsymbol{E} = \frac{p}{4\pi\epsilon_0}\left[\hat{\boldsymbol{r}}\frac{2\cos\theta}{r^3} + \hat{\boldsymbol{\theta}}\frac{\sin\theta}{r^3}\right] = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}.\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}$$

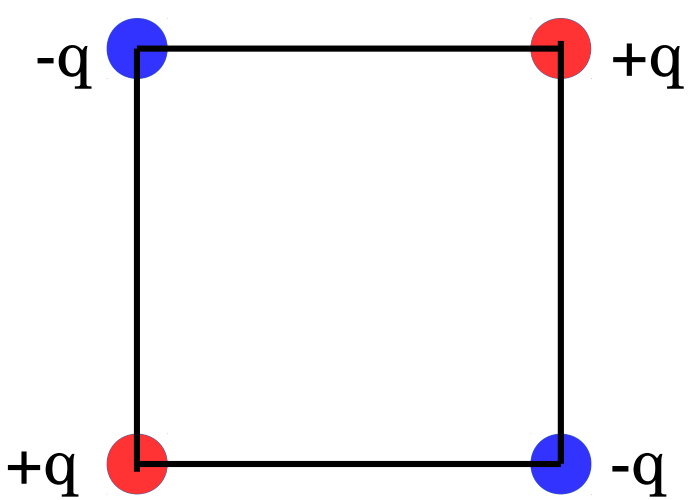
# DIPOLE, QUADRAPOLE AND SO ON...




Physical dipole



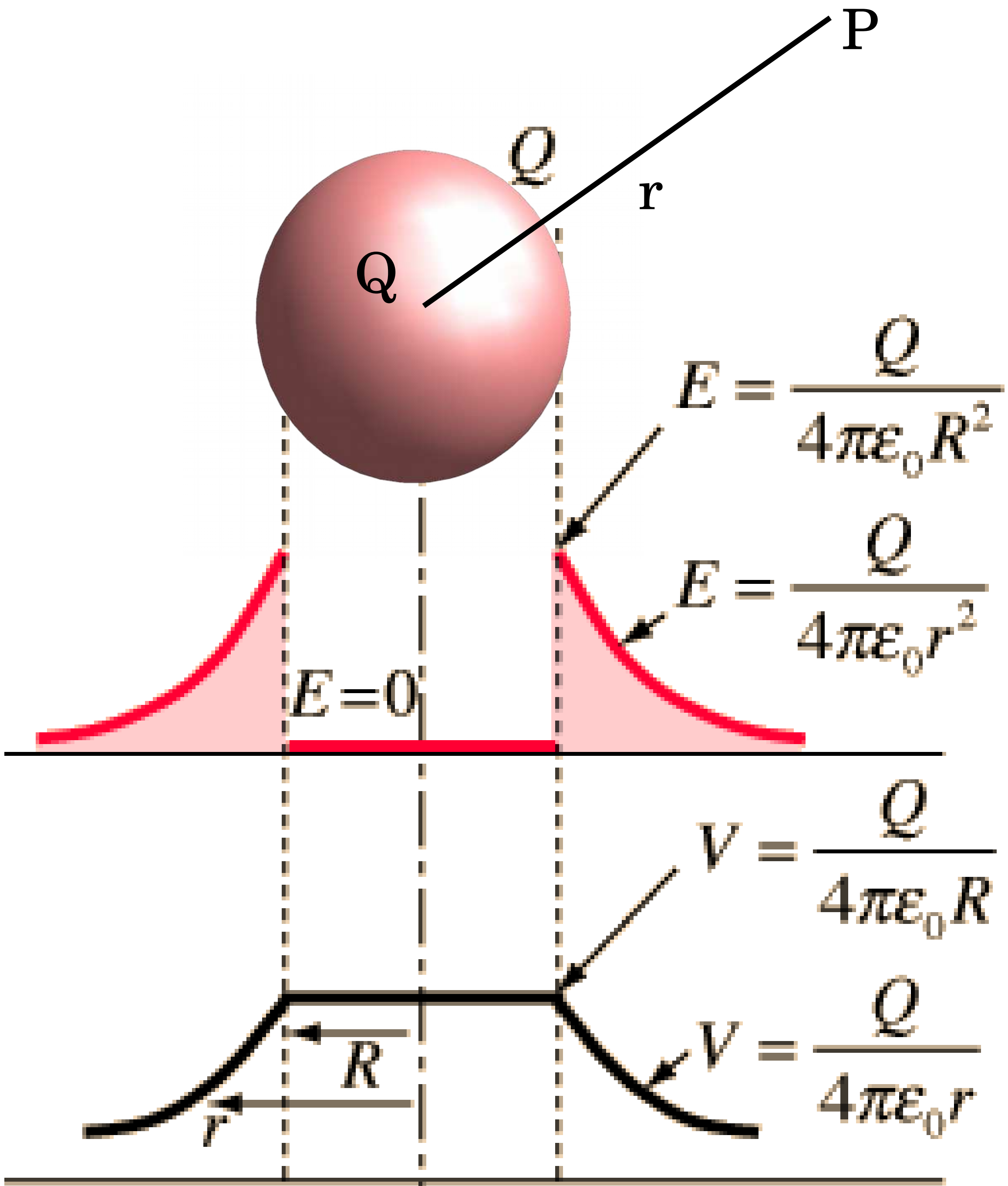

Pure dipole: Lim. $\left\{\begin{matrix} d \to 0 \\ q \to \infty \end{matrix}\right\}$ keeping $qd = p$ fixed




Quadrupole

$$V(r) \simeq \frac{1}{r^3}$$

# POTENTIAL OF A HOLLOW SPHERE

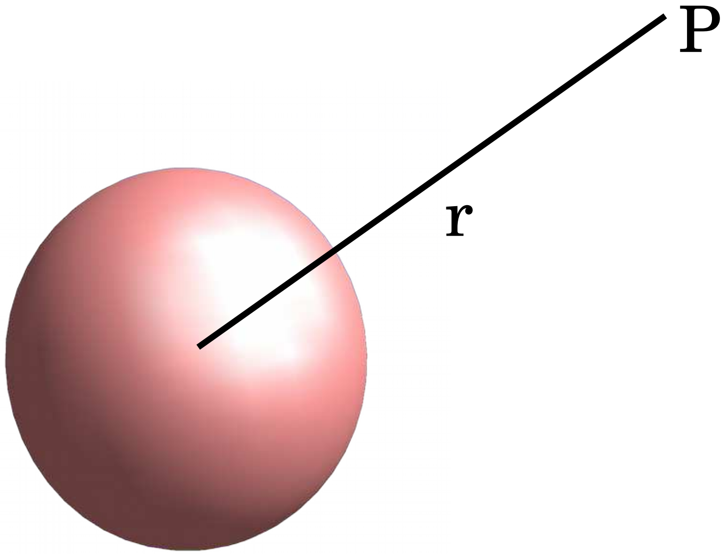



$$\boldsymbol{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r^2}\hat{\boldsymbol{r}} & r > R \\ 0 & r < R \end{cases}$$

$$V_{\text{outside}} = \frac{Q}{4\pi\epsilon_0 r}$$

$$V_{\text{inside}} = \frac{Q}{4\pi\epsilon_0 R}$$

# POTENTIAL OF A HOLLOW SPHERE




$$\boldsymbol{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r^2}\hat{\boldsymbol{r}} & r > R \\ 0 & r < R \end{cases}$$

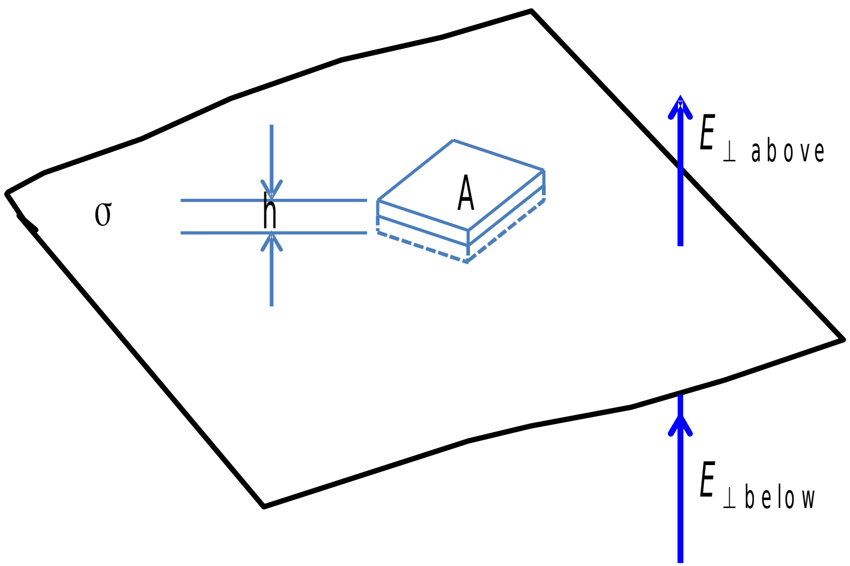
$$V_{\text{outside}} = \frac{Q}{4\pi\epsilon_0 r} \qquad V_{\text{inside}} = \frac{Q}{4\pi\epsilon_0 R}$$

$$\text{Charge density } \sigma = \frac{Q}{4\pi R^2}$$

$$\boldsymbol{E}_{\text{out}} = \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{\boldsymbol{r}} \quad \Rightarrow \quad \boldsymbol{E}_{\text{out}}^{\text{surf}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{r}}$$

$$\Delta \boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$$

# ELECTROSTATIC BOUNDARY CONDITIONS

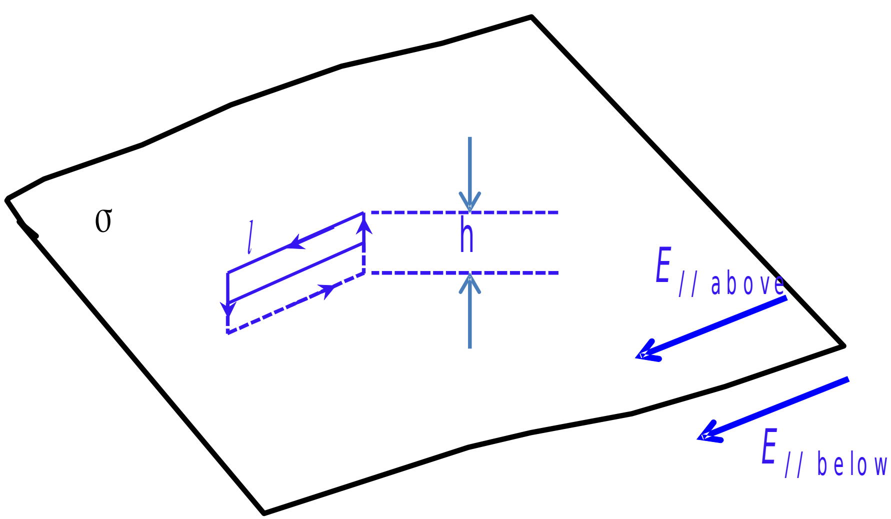



$$\oiint \boldsymbol{E}\,.\,\boldsymbol{dS} = \frac{Q_{enc}}{\epsilon_0}$$

$$\boldsymbol{E}_{\perp above} - \boldsymbol{E}_{\perp below} = \frac{\sigma}{\epsilon_0}$$

The normal component of the electric field may be discontinuous.

- Surfaces may be finite.
- The charge density may vary from place to place.
- Surface is not necessarily equipotential
- May be conducting or non-conducting




$$\oint \boldsymbol{E}\,.\,\boldsymbol{dl} = 0$$

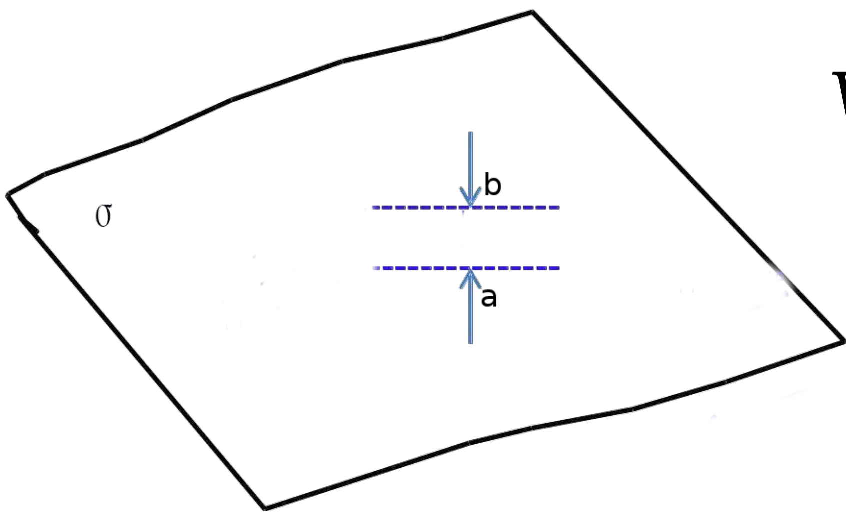
$$\boldsymbol{E}_{\parallel above} - \boldsymbol{E}_{\parallel below} = 0$$

The tangential component of the electric field is continuous.

# ELECTROSTATIC BOUNDARY CONDITIONS

$$\boldsymbol{E}_{above} - \boldsymbol{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}$$

$\hat{\boldsymbol{n}} \Rightarrow$ Unit vector perpendicular to surface, from below to above




$$V_{above} - V_{below} = -\int_a^b \boldsymbol{E}.d\boldsymbol{l}$$

As $d\boldsymbol{l} \rightarrow 0$ the line integral vanishes

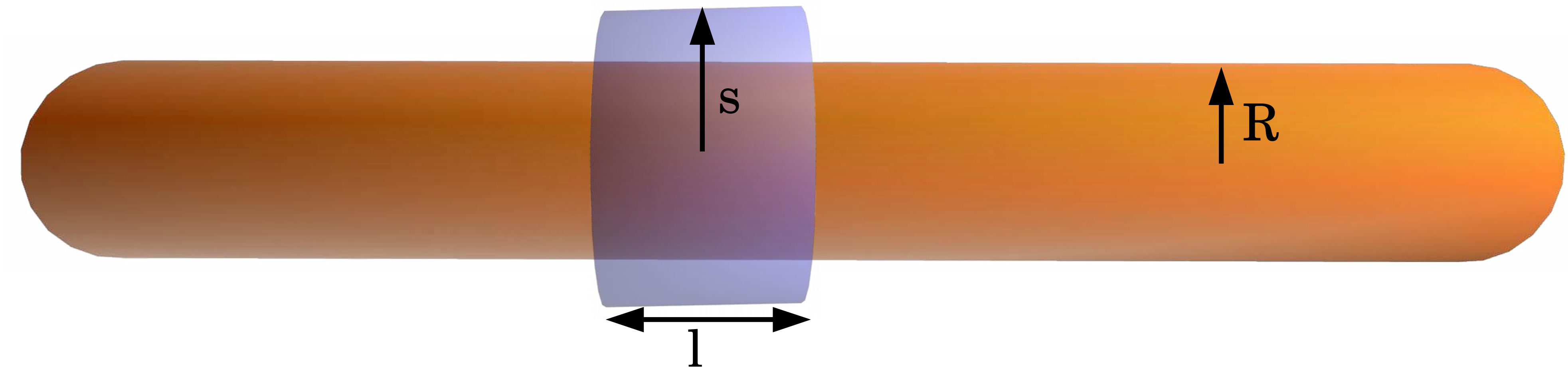
$$V_{above} = V_{below}$$

The discontinuity in **E** is finite

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

$$\frac{\partial V}{\partial n} \equiv \nabla V . \hat{\boldsymbol{n}}$$

# ELECTROSTATIC BOUNDARY CONDITIONS




Hollow cylindrical tube, with uniform surface charge density σ

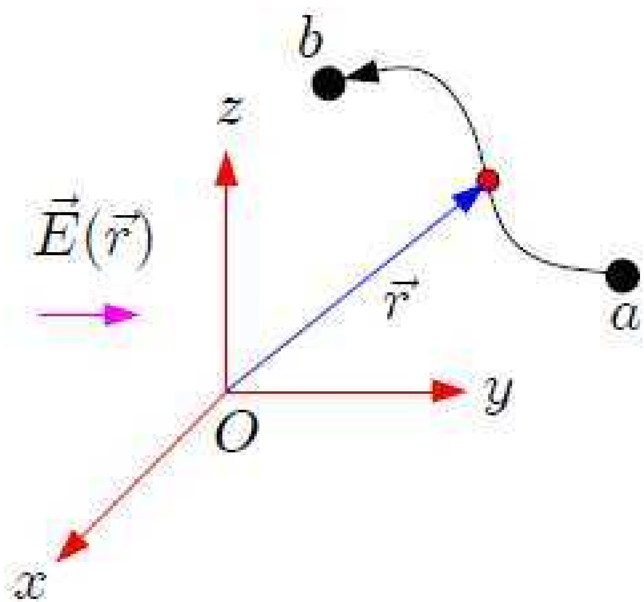
$$\oint \boldsymbol{E}_{\mathbf{out}} . d\,\boldsymbol{a} = E(2\pi s)l = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma}{\epsilon_0}(2\pi R)l$$

$$\boldsymbol{E}_{\mathbf{out}} = \frac{\sigma}{\epsilon_0}\frac{R}{s}\hat{\boldsymbol{s}} \qquad \Rightarrow \qquad \boldsymbol{E}^{\text{surf}}_{\text{out}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{s}}$$

$$\boldsymbol{E}_{\mathbf{in}} = 0$$

$$\Delta \boldsymbol{E} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{s}}$$

# WORK DONE TO MOVE A CHARGE




Work done on the charge

$$W = -q\int_a^b \boldsymbol{E}\,.\,d\boldsymbol{r} \quad \text{[always true]}$$

$$= q\left(V_b - V_a\right) \quad \text{[only if } \nabla\times\boldsymbol{E}=0\text{]}$$

The electrostatic force is conservative

The potential difference between two points **a** and **b** is the work per unit charge required to move a particle from **a** to **b**

How much work does it take to bring a charge from infinity to **r**?

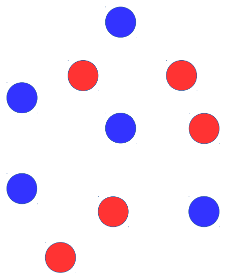
$$W = q[V(\boldsymbol{r}) - V(\infty)] = q\,V(\boldsymbol{r})$$

Units: Joules (or e-V)

Work needed to move 1 electron through 1 V.

# ENERGY OF A POINT CHARGE DISTRBUTION

**How much work does it take to assemble an entire collection of point charges?**



Work done to bring first charge, $W_1 = 0$

Work done to bring second charge from infinity to desired position,

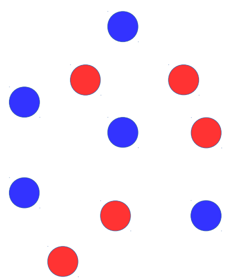
$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left( \frac{q_1}{|\boldsymbol{r_2} - \boldsymbol{r_1}|} \right)$$

Work done to bring third charge from infinity to desired position, against field due to first two charges,

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left( \frac{q_1}{|\boldsymbol{r_3} - \boldsymbol{r_1}|} + \frac{q_2}{|\boldsymbol{r_3} - \boldsymbol{r_2}|} \right)$$

# ENERGY OF A POINT CHARGE DISTRBUTION

**How much work does it take to assemble an entire collection of point charges?**



Total work done,

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\left(\frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}\right)$$

$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}\right)$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r}_i) \qquad \text{with,} \quad V(\boldsymbol{r}_i) = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}\right)$$

Represents the energy stored in the configuration

# ENERGY OF A CONTINUOUS CHARGE DISTRBUTION


$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r_i}) \quad \Rightarrow \quad W = \frac{1}{2}\int \rho V \, d\tau$$

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$$

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot E) V \, d\tau$$

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$$

$$W = \frac{\epsilon_0}{2}\left[-\int \boldsymbol{E} \cdot (\nabla V) \, d\tau + \oint V \boldsymbol{E} \cdot d\boldsymbol{a}\right]$$

$$= \frac{\epsilon_0}{2}\left[\int E^2 \, d\tau + \oint V \boldsymbol{E} \cdot d\boldsymbol{a}\right]$$

If we integrate over all space, surface term goes to zero!

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

# ENERGY OF A POINT CHARGE

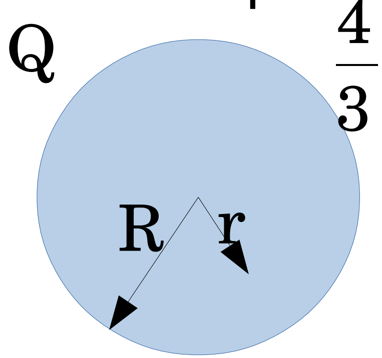
- $$W = \frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2 \int_{\text{all space}} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2} dr = \infty$$

The energy of a point charge is infinite!

No resolution within classical electromagnetism
Concept of a point charge has limitations.

Assume the point charge is an uniform sphere of radius R

$$\rho = \frac{Q}{\frac{4}{3}\pi r^3}$$




$$E_{\text{in}} = \frac{Q}{4\pi\epsilon_0 R^2}\frac{r}{R} \qquad (r<R)$$

$$E_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r>R)$$

$$W = \left(\frac{\epsilon_0}{2}\right)\left[\int_0^R E_{\text{in}}^2 \, d\tau + \int_R^\infty E_{\text{out}}^2 \, d\tau\right]$$ will converge

# WORK AND ENERGY

$$W_{\text{discrete}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}\right) \qquad W_{\text{continuous}} = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

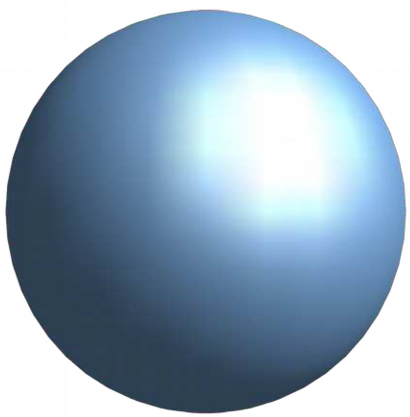
$W_{\text{discrete}}$ can be positive or negative, while $W_{\text{continuous}}$ is necessarily positive

Discrete $V(\boldsymbol{r}_i)$ is the potential due to all other charges but not $q_i$, while the continuous $V(\boldsymbol{r})$ is the full potential

The interaction energy of discrete charges assumes that the charges already existed, and no energy was spent in creating them. The self interaction of the electric field of a charge with itself was excluded.

For a continuous distribution, the exact charge at the point **r** is vanishingly small, so its contribution to the potential is zero.

# UNIFORMLY CHARGED SPHERE



Total charge Q    Radius of sphere R

Constant charge density $\rho = \frac{3Q}{4\pi R^3}$

What is the electric field? (Use Gauss's Law)

$$E_{\text{outside}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{r}} \qquad r > R$$

$$E_{\text{inside}} = \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\boldsymbol{r}} \qquad r < R$$

What is the electric potential?

$$V_{\text{outside}} = -\int_{\infty}^{r} \boldsymbol{E}_{\text{out}} \cdot d\boldsymbol{l}$$

$$= -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0}\frac{1}{r'^2}dr'$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$V_{\text{inside}} = -\int_{\infty}^{R} \boldsymbol{E}_{\text{out}} \cdot d\boldsymbol{l} + -\int_{R}^{r} \boldsymbol{E}_{\text{in}} \cdot d\boldsymbol{l}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \int_{R}^{r} \frac{Q}{4\pi\epsilon_0}\frac{r'}{R^3}dr'$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \frac{Q}{8\pi\epsilon_0}\frac{1}{R^3}\left(r^2 - R^2\right)$$

$$= \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

# UNIFORMLY CHARGED SPHERE – POTENTIAL ENERGY 1

$$
\begin{aligned}
W &= \frac{1}{2}\int \rho(\boldsymbol{r})V(\boldsymbol{r})\,d^3r \\
&= \frac{1}{2}\int_0^R \frac{3Q}{4\pi R^3}\cdot\frac{Q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right).\,4\pi r^2\,dr \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\int_0^R\left(3r^2-\frac{r^4}{R^2}\right)dr \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\left(R^3-\frac{R^5}{5R^2}\right) \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\left(\frac{4R^3}{5}\right)
\end{aligned}
$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE – POTENTIAL ENERGY 2

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

$$= \frac{\epsilon_0}{2} \left[ \int_0^R \left| E_{\text{in}}^2 \right| 4\pi r^2 dr + \int_R^\infty \left| E_{\text{out}}^2 \right| 4\pi r^2 dr \right]$$

$$= \frac{\epsilon_0}{2} \left[ \int_0^R \frac{Q^2}{16\pi^2 \epsilon_0^2} \frac{r^2}{R^6} . 4\pi r^2 dr + \int_R^\infty \frac{Q^2}{16\pi^2 \epsilon_0^2} \frac{1}{r^4} . 4\pi r^2 dr \right]$$

$$= \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{R^6} \int_0^R r^4 dr + \int_R^\infty \frac{1}{r^2} dr \right]$$

$$= \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{R^6} \frac{R^5}{5} + \frac{1}{R} \right] = \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{6}{5R} \right]$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE - POTENTIAL ENERGY 3

$$W = \frac{\epsilon_0}{2}\left[\int E^2 d\tau + \oint V \boldsymbol{E}.d\boldsymbol{a}\right]$$

Let us choose the volume of integration as the charged sphere itself

$$= \frac{\epsilon_0}{2}\left[\int_0^R \left|E_{\text{in}}^2\right| 4\pi r^2 dr + \oint V_{\text{surf}} \boldsymbol{E}_{\textbf{surf}}.d\boldsymbol{a}\right]$$

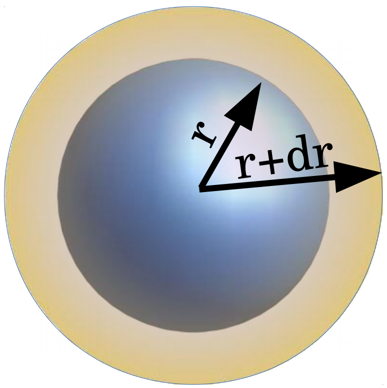
$$= \frac{\epsilon_0}{2}\left[\int_0^R \frac{Q^2}{16\pi^2\epsilon_0^2}\frac{r^2}{R^6}.4\pi r^2 dr + \oint \frac{Q}{4\pi\epsilon_0 R}\frac{Q}{4\pi\epsilon_0 R^2} R^2 \sin\theta\, d\theta\, d\phi\right]$$

$$= \frac{\epsilon_0}{2}\left[\left(\frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{5R}\right) + \left(\frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{R}\right)\right] = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{6}{5R}\right]$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE - POTENTIAL ENERGY 4

At an instant of time, we have a sphere of radius r and charge q




We bring in a charge dq, and smear it uniformly over the surface, increasing the radius to r+dr

Continue until final radius is R and final charge is Q

$$dq = 4\pi r^2 dr\,\rho = 4\pi r^2 \frac{3Q}{4\pi R^3} dr = \frac{3Q}{R^3} r^2 dr$$

$$q(r) = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi r^3 \frac{3Q}{4\pi R^3} = Q\frac{r^3}{R^3}$$

Work done in bringing charge dq and spreading over existing sphere is

$$dW = V(r)\,dq = \frac{1}{4\pi\epsilon_0}\frac{q(r)}{r}dq = \frac{1}{4\pi\epsilon_0} Q\frac{r^2}{R^3}\frac{3Q}{R^3}r^2 dr = \frac{3Q^2}{4\pi\epsilon_0 R^6} r^4 dr$$

$$W = \int_0^R dW = \frac{3Q^2}{4\pi\epsilon_0 R^6}\int_0^R r^4 dr = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# WHERE IS THE ENERGY STORED?

$$W = \frac{1}{2}\int \rho V \, d\tau \quad \Leftrightarrow \quad \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau$$

Integral over a charge distribution

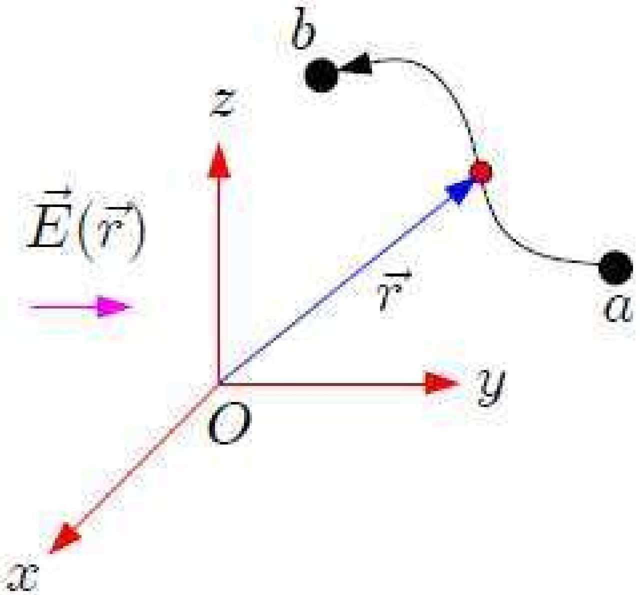
Integral over a field

Is the energy stored in the charge distribution or is it stored in the field?

Either! (as far as electrostatics is concerned)

$$\text{Energy density of the electrostatic field} = \frac{\epsilon_0}{2}E^2$$

# WORK DONE TO MOVE A CHARGE




Work done on the charge

$$W = -q\int_a^b \boldsymbol{E}\,.\,d\boldsymbol{r} \quad [\text{always true}]$$

$$= q\left(V_b - V_a\right) \quad [\text{only if } \nabla\times\boldsymbol{E}=0]$$

The electrostatic force is conservative

The potential difference between two points **a** and **b** is the work per unit charge required to move a particle from **a** to **b**

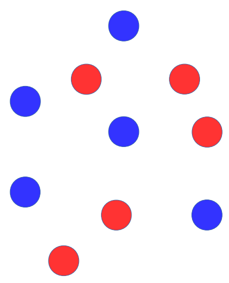
How much work does it take to bring a charge from infinity to **r**?

$$W = q[V(\boldsymbol{r}) - V(\infty)] = qV(\boldsymbol{r})$$

Units: Joules (or e-V) Work needed to move 1 electron through 1 V.

# ENERGY OF A POINT CHARGE DISTRBUTION

**How much work does it take to assemble an entire collection of point charges?**



Work done to bring first charge, $W_1 = 0$

Work done to bring second charge from infinity to desired position,

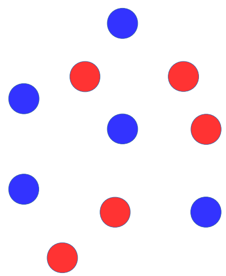
$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left( \frac{q_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} \right)$$

Work done to bring third charge from infinity to desired position, against field due to first two charges,

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left( \frac{q_1}{|\boldsymbol{r}_3 - \boldsymbol{r}_1|} + \frac{q_2}{|\boldsymbol{r}_3 - \boldsymbol{r}_2|} \right)$$

# ENERGY OF A POINT CHARGE DISTRBUTION

**How much work does it take to assemble an entire collection of point charges?**



Total work done,

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\left(\frac{q_i q_j}{\left|\boldsymbol{r_j}-\boldsymbol{r_i}\right|}\right)$$

$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_i q_j}{\left|\boldsymbol{r_j}-\boldsymbol{r_i}\right|}\right)$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r_i}) \quad \text{with,} \quad V(\boldsymbol{r_i}) = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_j}{\left|\boldsymbol{r_j}-\boldsymbol{r_i}\right|}\right)$$

Represents the energy stored in the configuration

# ENERGY OF A CONTINUOUS CHARGE DISTRBUTION


$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r_i}) \quad \Rightarrow \quad W = \frac{1}{2}\int \rho V \, d\tau$$

$$\nabla . \boldsymbol{E} = \rho / \epsilon_0$$

$$W = \frac{\epsilon_0}{2}\int (\nabla . E) V \, d\tau$$

$$\nabla . (f\vec{A}) = f(\nabla . \vec{A}) + \vec{A} . (\nabla f)$$

$$W = \frac{\epsilon_0}{2}\left[-\int \boldsymbol{E} . (\nabla V) d\tau + \oint V \boldsymbol{E} . d\boldsymbol{a}\right]$$

$$= \frac{\epsilon_0}{2}\left[\int E^2 d\tau + \oint V \boldsymbol{E} . d\boldsymbol{a}\right]$$

If we integrate over all space, surface term goes to zero!

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

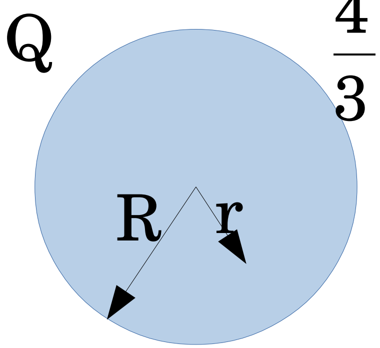
# ENERGY OF A POINT CHARGE

- $$W = \frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2 \int_{\text{all space}} \frac{q^2}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{q^2}{8\pi\epsilon_0}\int_0^{\infty} \frac{1}{r^2} dr = \infty$$

The energy of a point charge is infinite!

No resolution within classical electromagnetism
Concept of a point charge has limitations.

Assume the point charge is an uniform sphere of radius R




$$\rho = \frac{Q}{\frac{4}{3}\pi r^3}$$

$$E_{\text{in}} = \frac{Q}{4\pi\epsilon_0 R^2}\frac{r}{R} \qquad (r<R)$$

$$E_{\text{out}} = \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r>R)$$

$$W = \left(\frac{\epsilon_0}{2}\right)\left[\int_0^R E_{\text{in}}^2 \, d\tau + \int_R^{\infty} E_{\text{out}}^2 \, d\tau\right]$$ will converge

# WORK AND ENERGY

$$W_{\text{discrete}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{q_i q_j}{|\boldsymbol{r}_j - \boldsymbol{r}_i|}\right) \qquad W_{\text{continuous}} = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

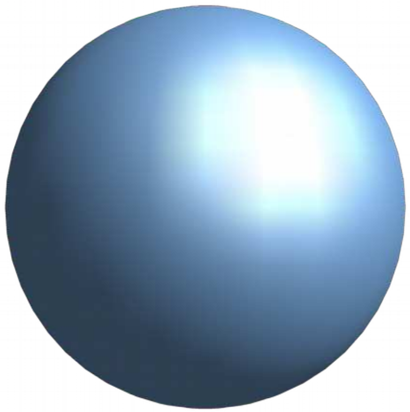
$W_{\text{discrete}}$ can be positive or negative, while $W_{\text{continuous}}$ is necessarily positive

Discrete $V(\boldsymbol{r}_i)$ is the potential due to all other charges but not $q_i$, while the continuous $V(\boldsymbol{r})$ is the full potential

The interaction energy of discrete charges assumes that the charges already existed, and no energy was spent in creating them. The self interaction of the electric field of a charge with itself was excluded.

For a continuous distribution, the exact charge at the point **r** is vanishingly small, so its contribution to the potential is zero.

# UNIFORMLY CHARGED SPHERE



Total charge Q Radius of sphere R

Constant charge density $\rho = \frac{3Q}{4\pi R^3}$

What is the electric field? (Use Gauss's Law)

$$\boldsymbol{E}_{\text{outside}} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{r}} \qquad r > R$$

$$\boldsymbol{E}_{\text{inside}} = \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\boldsymbol{r}} \qquad r < R$$

What is the electric potential?

$$V_{\text{outside}} = -\int_{\infty}^{r} \boldsymbol{E}_{\text{out}} . d\boldsymbol{l}$$

$$= -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0}\frac{1}{r'^2} dr'$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$V_{\text{inside}} = -\int_{\infty}^{R} \boldsymbol{E}_{\text{out}} . d\boldsymbol{l} + -\int_{R}^{r} \boldsymbol{E}_{\text{in}} . d\boldsymbol{l}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \int_{R}^{r} \frac{Q}{4\pi\epsilon_0}\frac{r'}{R^3} dr'$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{R} - \frac{Q}{8\pi\epsilon_0}\frac{1}{R^3}(r^2 - R^2)$$

$$= \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

# UNIFORMLY CHARGED SPHERE – POTENTIAL ENERGY 1

$$\begin{aligned}
W &= \frac{1}{2}\int \rho(\boldsymbol{r})V(\boldsymbol{r})\,d^3r \\
&= \frac{1}{2}\int_0^R \frac{3Q}{4\pi R^3}\cdot\frac{Q}{8\pi\epsilon_0 R}\left(3-\frac{r^2}{R^2}\right).\,4\pi r^2\,dr \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\int_0^R\left(3r^2-\frac{r^4}{R^2}\right)dr \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\left(R^3-\frac{R^5}{5R^2}\right) \\
&= \frac{3Q^2}{16\pi\epsilon_0 R^4}\left(\frac{4R^3}{5}\right)
\end{aligned}$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE - POTENTIAL ENERGY 2

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

$$= \frac{\epsilon_0}{2} \left[ \int_0^R \left| E_{\text{in}}^2 \right| 4\pi r^2 dr + \int_R^\infty \left| E_{\text{out}}^2 \right| 4\pi r^2 dr \right]$$

$$= \frac{\epsilon_0}{2} \left[ \int_0^R \frac{Q^2}{16\pi^2 \epsilon_0^2} \frac{r^2}{R^6} . 4\pi r^2 dr + \int_R^\infty \frac{Q^2}{16\pi^2 \epsilon_0^2} \frac{1}{r^4} . 4\pi r^2 dr \right]$$

$$= \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{R^6} \int_0^R r^4 dr + \int_R^\infty \frac{1}{r^2} dr \right]$$

$$= \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{R^6} \frac{R^5}{5} + \frac{1}{R} \right] = \frac{Q^2}{8\pi\epsilon_0} \left[ \frac{6}{5R} \right]$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE - POTENTIAL ENERGY 3

$$W = \frac{\epsilon_0}{2}\left[\int E^2 d\tau + \oint V \boldsymbol{E} . d\boldsymbol{a}\right]$$

Let us choose the volume of integration as the charged sphere itself

$$= \frac{\epsilon_0}{2}\left[\int_0^R \left|E_{\text{in}}^2\right| 4\pi r^2 dr + \oint V_{\text{surf}} \boldsymbol{E}_{\textbf{surf}} . d\boldsymbol{a}\right]$$

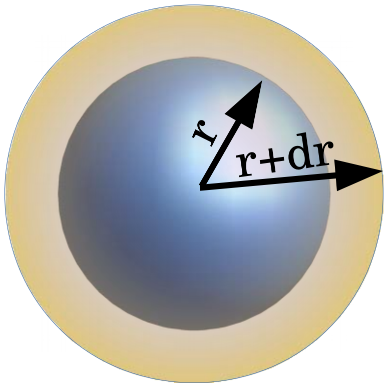
$$= \frac{\epsilon_0}{2}\left[\int_0^R \frac{Q^2}{16\pi^2\epsilon_0^2}\frac{r^2}{R^6} . 4\pi r^2 dr + \oint \frac{Q}{4\pi\epsilon_0 R}\frac{Q}{4\pi\epsilon_0 R^2} R^2 \sin\theta\, d\theta\, d\phi\right]$$

$$= \frac{\epsilon_0}{2}\left[\left(\frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{5R}\right) + \left(\frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{R}\right)\right] = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{6}{5R}\right]$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# UNIFORMLY CHARGED SPHERE – POTENTIAL ENERGY 4

At an instant of time, we have a sphere of radius r and charge q




We bring in a charge dq, and smear it uniformly over the surface, increasing the radius to r+dr

Continue until final radius is R and final charge is Q

$$dq = 4\pi r^2 dr\,\rho = 4\pi r^2 \frac{3Q}{4\pi R^3} dr = \frac{3Q}{R^3} r^2 dr$$

$$q(r) = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi r^3 \frac{3Q}{4\pi R^3} = Q\frac{r^3}{R^3}$$

Work done in bringing charge dq and spreading over existing sphere is

$$dW = V(r)\,dq = \frac{1}{4\pi\epsilon_0}\frac{q(r)}{r} dq = \frac{1}{4\pi\epsilon_0} Q\frac{r^2}{R^3}\frac{3Q}{R^3} r^2 dr = \frac{3Q^2}{4\pi\epsilon_0 R^6} r^4 dr$$

$$W = \int_0^R dW = \frac{3Q^2}{4\pi\epsilon_0 R^6}\int_0^R r^4 dr = \frac{3Q^2}{20\pi\epsilon_0 R}$$

# WHERE IS THE ENERGY STORED?

$$W = \frac{1}{2}\int \rho V \, d\tau \quad \Leftrightarrow \quad \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

Integral over a charge distribution

Integral over a field

Is the energy stored in the charge distribution or is it stored in the field?

Either! (as far as electrostatics is concerned)

$$\text{Energy density of the electrostatic field} = \frac{\epsilon_0}{2} E^2$$

# SUPERPOSITION PRINCIPLE FOR ENERGY?

If I have two electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_{2,}$
the total electric field at a point is $\boldsymbol{E}_{\text{tot}} = \boldsymbol{E}_1 + \boldsymbol{E}_2$
Is the total energy $W_{\text{tot}} = W_1 + W_2$?

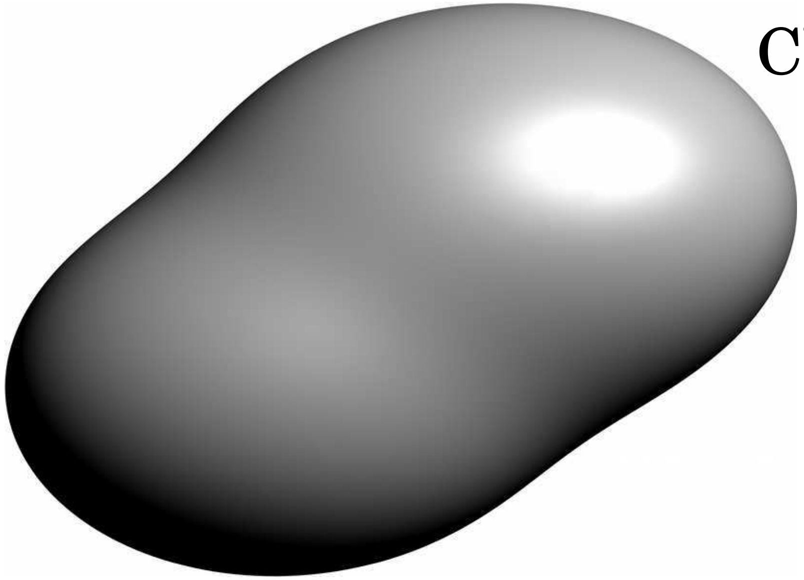
**NO!**

$$W_{\text{tot}} = \frac{\epsilon_0}{2} \int E_{\text{tot}}^2 \, d\tau = \frac{\epsilon_0}{2} \int (\boldsymbol{E}_1 + \boldsymbol{E}_2)^2 \, d\tau$$

$$= \frac{\epsilon_0}{2} \int \left( E_1^2 + E_2^2 + 2 \boldsymbol{E}_1 . \boldsymbol{E}_2 \right) d\tau$$

$$= W_1 + W_2 + \epsilon_0 \int \boldsymbol{E}_1 . \boldsymbol{E}_2 \, d\tau$$

# THE CONSERVATION OF CHARGE



Charge can neither be created nor destroyed

Total amount of charge is conserved.

$$I = -\oint_S \boldsymbol{J} \cdot d\boldsymbol{S}$$

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{S} = \frac{-\partial Q_V}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\int_V \rho \, dV\right) + \oint_S \boldsymbol{J} \cdot d\boldsymbol{S} = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J}\right) dV = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0}$$

Continuity Eqn.

Noether's Theorem: A continuous symmetry implies a conservation law.

# CONDUCTORS

- The Electric field is zero inside an conductor
- The charge density is zero inside a conductor
- Any net charge on a conductor resides on the surface

$$\nabla . \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\frac{1}{\sigma} \nabla . \boldsymbol{J} = \frac{\rho}{\epsilon_0}$$

$$-\frac{\partial \rho}{\partial t} = \frac{\sigma}{\epsilon_0} \rho$$

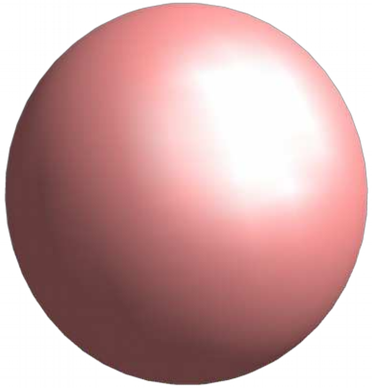
$$\rho = \rho(0) e^{-\frac{\sigma}{\epsilon_0} t}$$

If $\nabla . \boldsymbol{J} = 0$, excess charge is NOT in bulk

Charge MUST be conserved $\longrightarrow$ Excess charge goes from bulk to surface

For good metals $Cu, Ag, Au, \ldots \sigma/\epsilon_0 \approx 10^{20,}$ $\Rightarrow$ $\tau \approx 10^{-20} s$

# CONDUCTORS



$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\rho = \rho(0) e^{-\frac{\sigma}{\epsilon_0} t}$$

Since $\rho \to 0$ in the *long time* limit, $\nabla \cdot \boldsymbol{J} = 0$

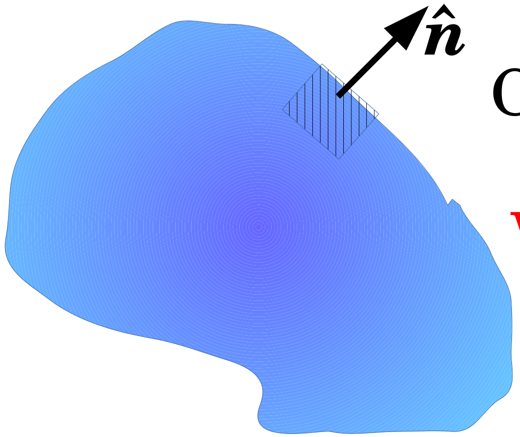
*If* $\boldsymbol{J} = 0, \;\Rightarrow\; \boldsymbol{E} = 0$ Pure electrostatics case

*If* $\boldsymbol{J}$ is a divergence-less vector we can have steady current flow

In either case, excess charge flows to boundary!

- If no current is flowing inside the conductor, then E = 0 inside
- The conductor is an equipotential if J = 0 (electrostatic case)
- The electric field just outside is perperndicular to the surface.

# ELECTROSTATIC PRESSURE ON CONDUCTORS




Consider a conductor with a surface charge density $\sigma$

What is the force experienced by a patch of area $\delta S$?

The force on the patch is due to the electric field created by all the other charges, $\boldsymbol{E}_{\text{other}}$

The total field is due to the patch, plus field due to everything else

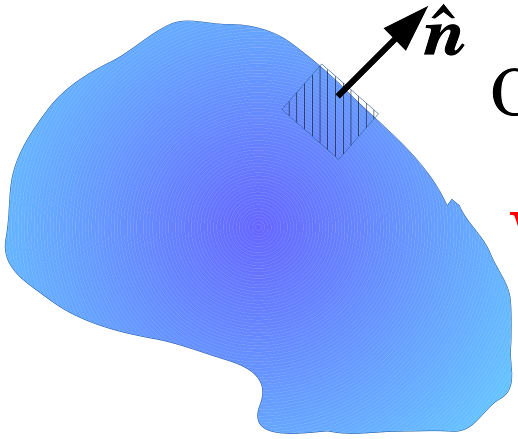
$$\boldsymbol{E}_{\text{total}} = \boldsymbol{E}_{\text{patch}} + \boldsymbol{E}_{\text{other}}$$

$$\boldsymbol{E}_{\text{total}} = \begin{cases} \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} & \text{Outside} \\ 0 & \text{Inside} \end{cases} \qquad \boldsymbol{E}_{\text{patch}} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} & \text{Outside} \\ \dfrac{-\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} & \text{Inside} \end{cases}$$

Discontinuous

$\boldsymbol{E}_{\text{other}} \rightarrow$ Continuous

# ELECTROSTATIC PRESSURE ON CONDUCTORS




Consider a conductor with a surface charge density $\sigma$

What is the force experienced by a patch of area $\delta S$?

$$\boldsymbol{E}_{\text{other}} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} = \frac{1}{2}\left(\boldsymbol{E}^{\text{out}}_{\text{total}} + \boldsymbol{E}^{\text{in}}_{\text{total}}\right)$$

Force on $\delta S$ = (Charge in $\delta S$) X (Field due to all charges not in $\delta S$)

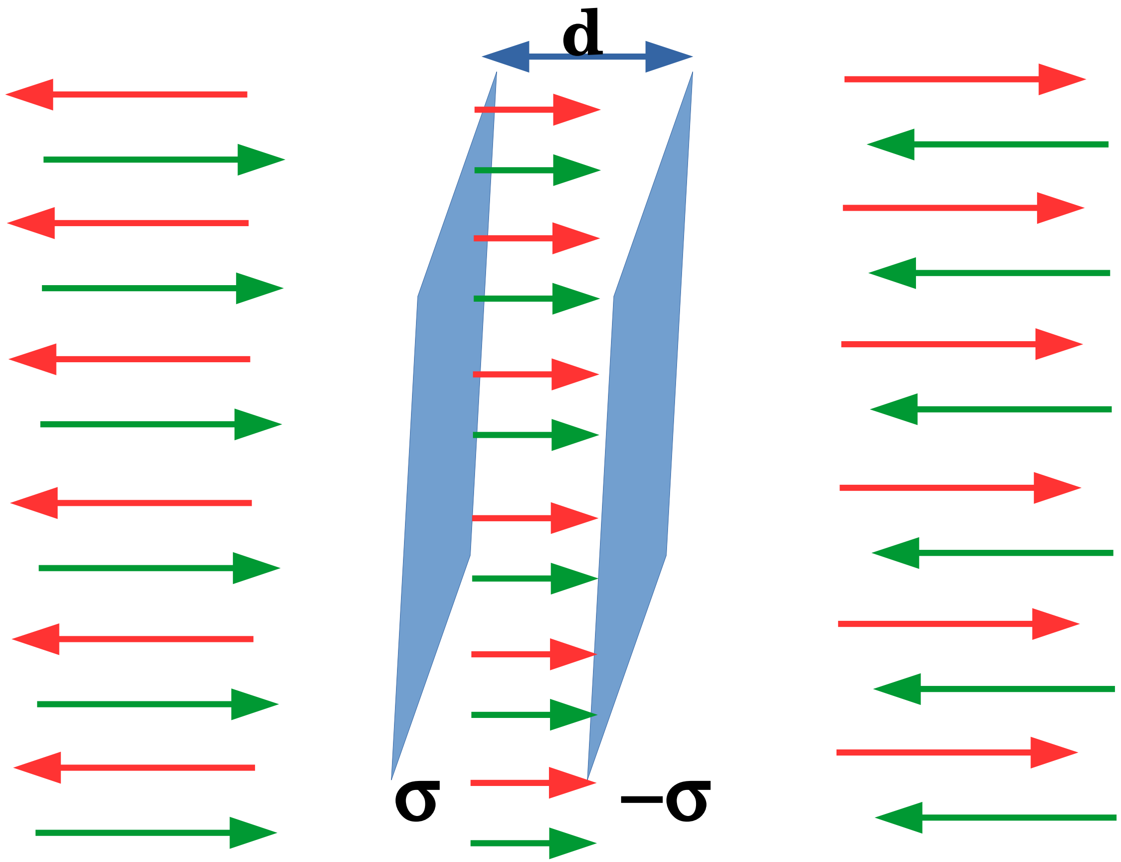
$$\delta \boldsymbol{F} = (\sigma\,\delta S).\left(\frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}\right) = \left(\frac{\sigma^2}{2\epsilon_0}\hat{\boldsymbol{n}}\right)\delta S$$

Electrostatic pressure,

in the outward direction

$$P = \frac{\delta \boldsymbol{F}}{\delta S} = \frac{\sigma^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2_{out}$$

# CAPACITANCE




$$\boldsymbol{E}_{\mathbf{in}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} = \frac{Q}{\epsilon_0 A}\hat{\boldsymbol{n}}$$

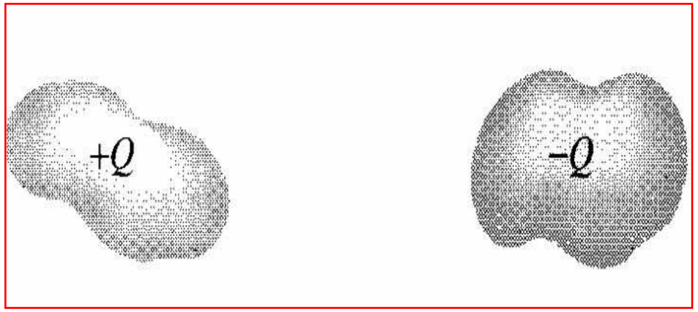
Potential difference between plates, $V = \left|E_{\text{in}}\right| d = \dfrac{Q\,d}{\epsilon_0 A}$

Charge $\propto$ Voltage difference

The constant of proportionality is the capacitance.

$$C = \frac{\epsilon_0 A}{d}$$

# CAPACITANCE




Two conductors with charge +Q and -Q

The conductors are an equipotential. They have a definite potential difference.

$$V = V_{+} - V_{-} = -\int_{(-)}^{(+)} \boldsymbol{E} \cdot d\boldsymbol{l}$$

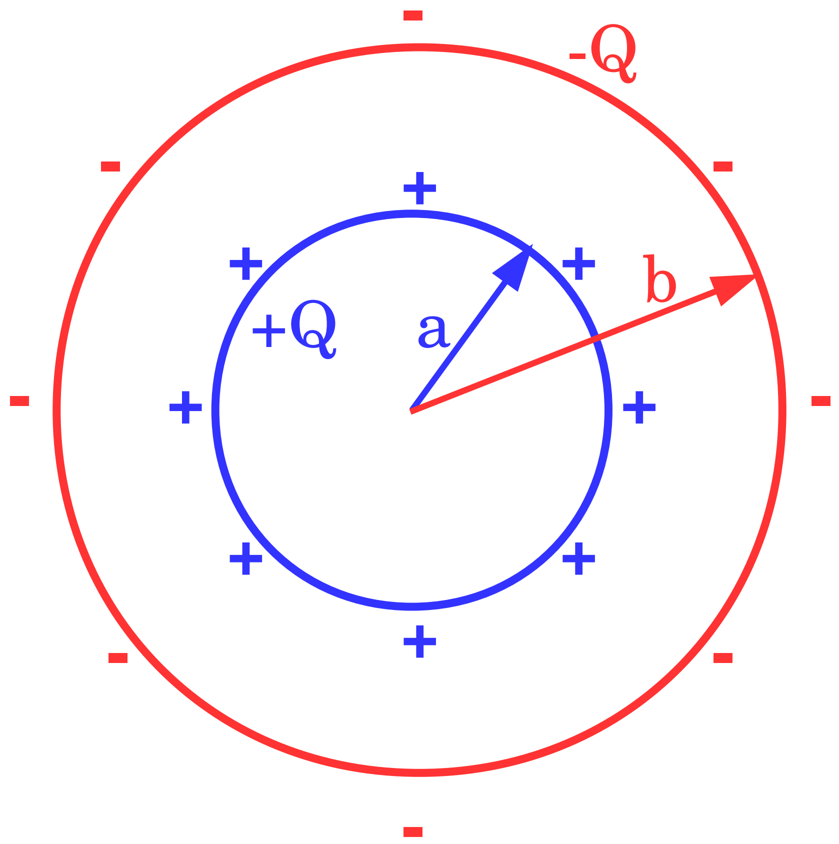
$V \propto$ total charge Q

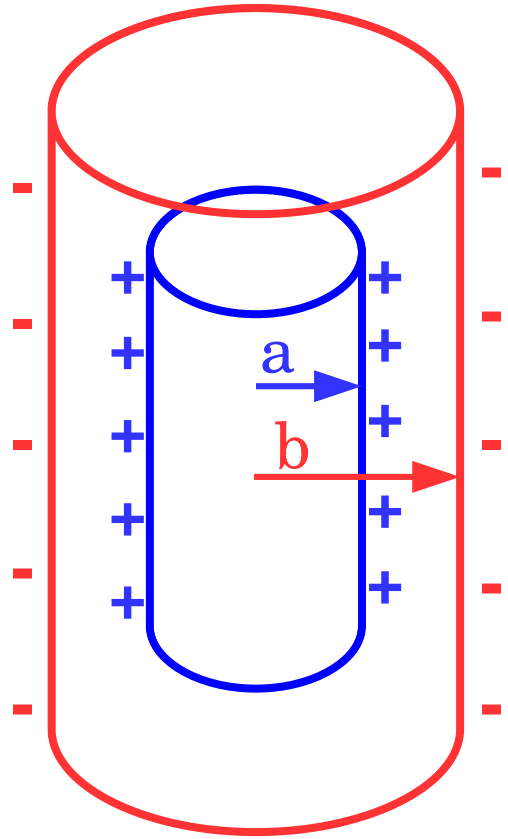
**CAPACITANCE** $\quad Q = CV$

Capacitance is a purely geometric quantity, depending on size, shape, and separation between the two conductors.

Units: Farads (Coulomb / Volt)

# CAPACITANCE



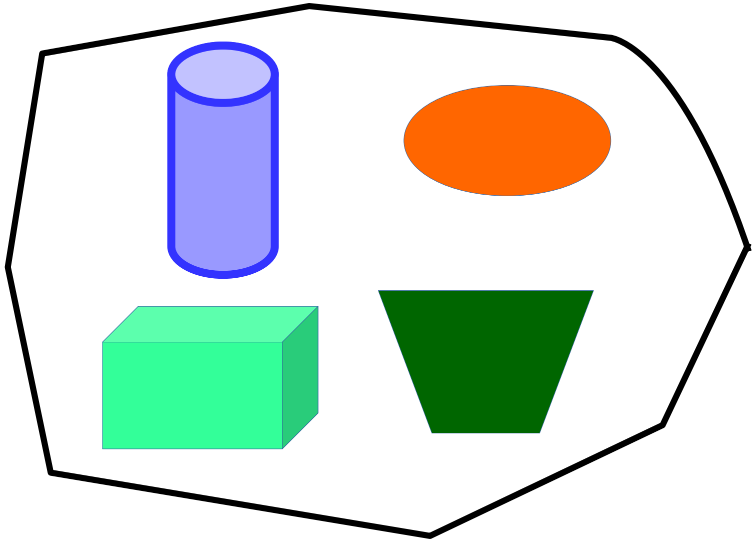

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

$$V = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$C = 2\pi\epsilon_0 L/\ln\left(\frac{b}{a}\right)$$

# CAPACITANCE



The charge $Q_1, Q_2, Q_3, \ldots, Q_n$ on each conductor is given

The conductors are at a potential $V_1, V_2, V_3, \ldots, V_n$

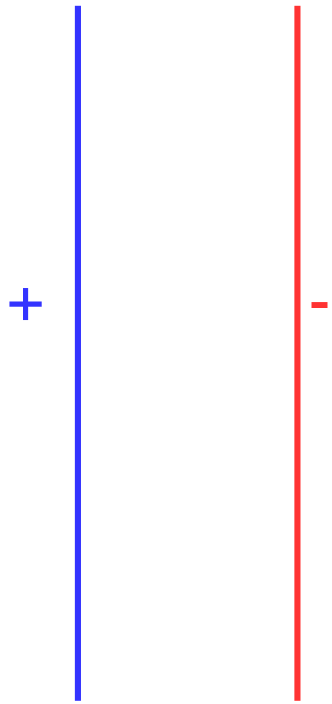
What is the most generic statement that one can make?

$$Q_i = \sum_j C_{ij} V_j \qquad \textit{with} \;\; C_{ij} = C_{ji}$$

The coefficients in this **linear** relation are the formal definition of **capacitance**.

For a single object, this reduces to, $Q = CV$

# WORK DONE IN ASSEMBLING A CAPACITOR




What is the work done to charge up the capacitor to the final amount Q?

$$V(q) = \frac{q}{C}$$

$$dW = V\,dq = \left(\frac{q}{C}\right)dq$$

$$W = \int_0^Q dW = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{1}{2}\frac{Q^2}{C}$$

$$W = \frac{1}{2}CV^2$$

# POISSON AND LAPLACE EQUATION

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\widehat{\boldsymbol{r}-\boldsymbol{r}'}\,dV'$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$$

$$\nabla^2 V = -\rho/\epsilon_0$$ **Poisson's equation**

$$\nabla^2 V = 0$$ **Laplace equation**

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

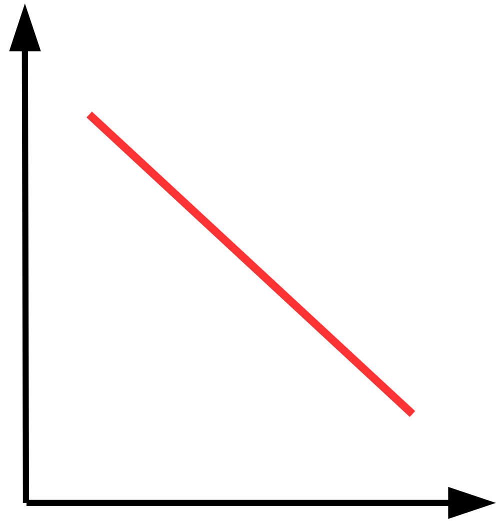
In a region where $\rho = 0$

# LAPLACE EQUATION IN ONE DIMENSION

If the potential V depends only on one variable

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$V(x) = m x + b$$



Once the boundary conditions are specified, the solution is determined.

Average value: $V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right] \quad \forall a$

Solutions to Laplace's equation can have no local maxima or minima

# LAPLACE EQUATION – AVERAGE VALUE THEOREM

A scalar function $V(\boldsymbol{r})$ satisfies $\nabla^2 V = 0$

Consider a sphere of radius $R$ : integrate $\nabla^2 V$ over the volume

$$\int_{vol} \nabla.(\nabla V)\, d\tau = \int_{surface} \nabla V . d\boldsymbol{S}$$

$$= \int [\boldsymbol{\epsilon}_r \frac{\partial V}{\partial r} + \boldsymbol{\epsilon}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \boldsymbol{\epsilon}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}] . d\boldsymbol{S}$$

$$= \int \frac{\partial V}{\partial r} R^2 \sin\theta\, d\theta\, d\phi$$

$$0 = R^2 \frac{\partial}{\partial r} \int_{surface} V(r,\theta,\phi) \sin\theta\, d\theta\, d\phi$$

The average value $\langle V(\theta,\phi)\rangle_r$ over a sphere is independent of $r$.

In the limit $r \to 0$, we must have $\langle V \rangle = V(0)$

$$V(\boldsymbol{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V\, da$$

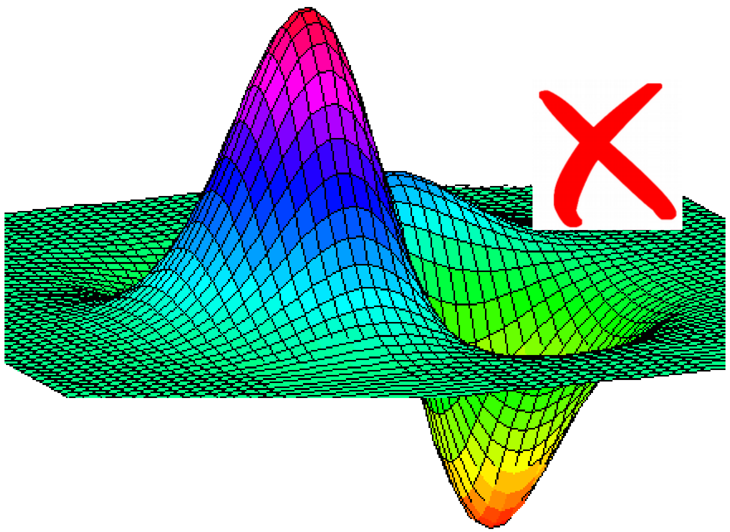
# LAPLACE EQUATION - AVERAGE VALUE THEOREM

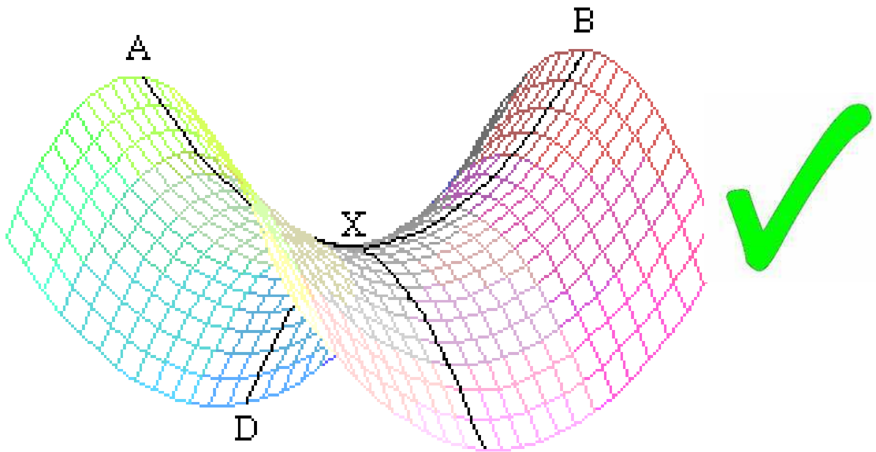
$$V(\boldsymbol{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V\, da$$

The value of V at a point **r** = Average value of V over a spherical surface of radius R centered at **r**

**V can have no local maxima or minima!**

All extreme values of V must occur at the boundaries



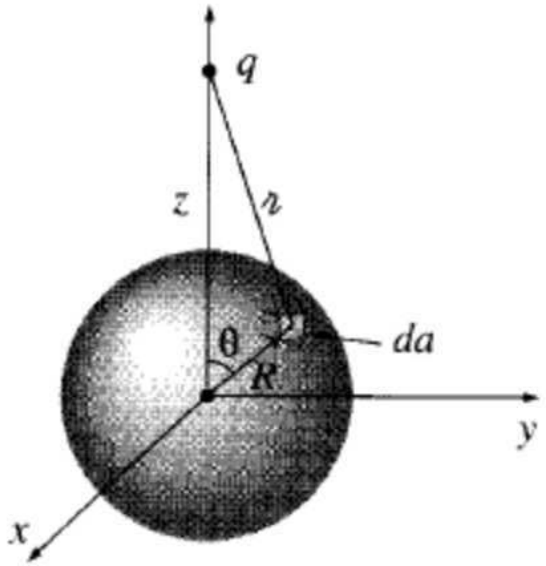



There can be saddle points

**e.g $V = x^2 - y^2$**

If the potential V is constant at **ALL** points on **ALL** boundaries, then necessarily V is constant everywhere.

# LAPLACE EQUATION – AVERAGE VALUE THEOREM




Consider a single point charge q

What is the average potential over a spherical surface of radius R if the charge is located outside the sphere?

$$V_{\text{avg}} = \frac{1}{4\pi R^2}\int \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}} R^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left[\sqrt{z^2+R^2-2zR\cos\theta}\right]_0^\pi$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\left[(z+R)-(z-R)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z} = \text{Potential at the center of the sphere}$$

# LAPLACE EQUATION - AVERAGE VALUE THEOREM

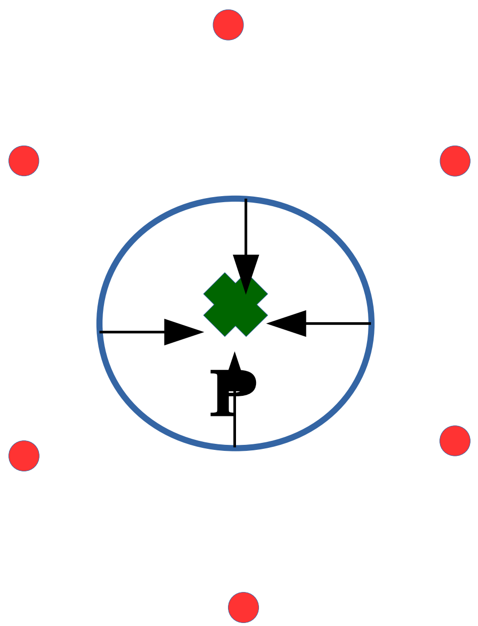
Can a charged particle be held in a stable equilibrium by electrostatic forces alone?

**NO!**

A stable equilibrium is a point of local minimum in the potential energy. However Laplace's equation allows for no minima in V.

**EARNSHAW'S THEOREM**

# EARNSHAW'S THEOREM




Consider a collection of charged particles which have somehow produced a stable equilibrium point at P.

If P is an equilibrium point, net force on an arbitrary test charge at P,

$$F_P = q\,\boldsymbol{E}(P) = 0 \quad \Rightarrow \quad \boldsymbol{E}(P) = 0$$

For a small displacement $\delta P$ about $P$, the force **must** point inwards,

$$\Rightarrow \quad \boldsymbol{E}(P + \delta P) \quad \text{must point inwards}$$

Construct a gaussian surface around P, $\oint_{\text{g.s.}} \boldsymbol{E}.d\boldsymbol{a} \; < \; 0$

Using the divergence theorem, this implies, $\int_{\text{vol.}} \nabla.\boldsymbol{E}.d\tau \; < \; 0$

In a charge free region, $\nabla.\boldsymbol{E} \; = \; 0$

**This contradicts our original assumption of equilibrium!**

# FIRST UNIQUENESS THEOREM

In order to solve Laplace's equation, suitable boundary conditions must be specified.

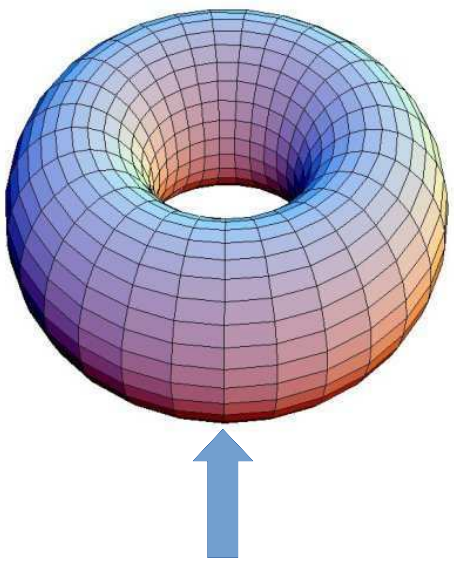
The solution to Laplace's equation in some volume is **uniquely** determined if the potential V is specified on the boundary surface.

Let there be two solutions to Laplace's equation, such that

$$\nabla^2 V_1 = 0 \qquad \text{and} \qquad \nabla^2 V_2 = 0$$

$$\text{Let, } V_3 = V_1 - V_2$$

$$\nabla^2 V_3 = 0$$



V specified on the surface

On the surface, $V_3 = 0$

$$\Rightarrow V_3 = 0 \text{ everywhere in the region}$$

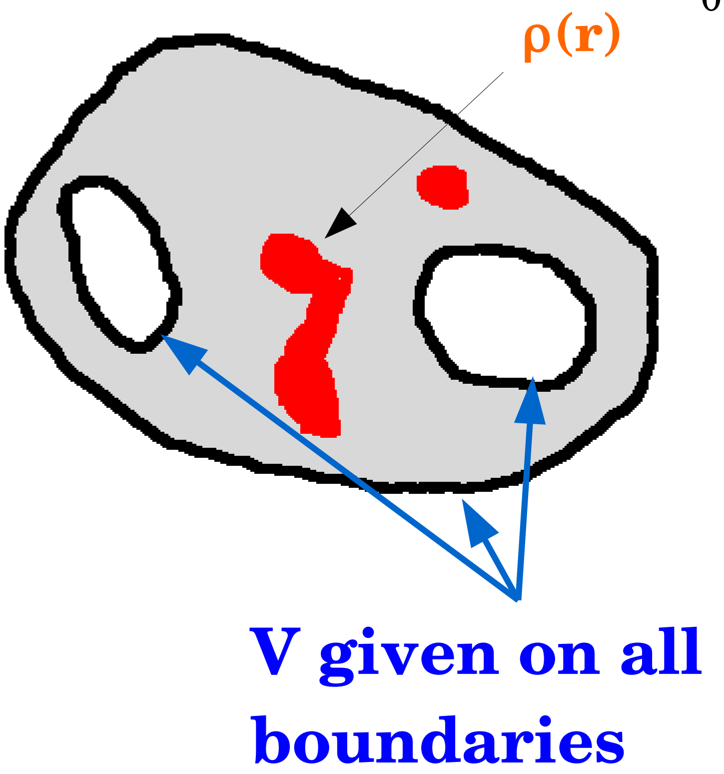
$$\Rightarrow V_1 = V_2$$

# FIRST UNIQUENESS THEOREM

The uniqueness theorem holds for Poisson's equation as well!

Let there be two solutions to Poisson's equation, such that

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$$




$$\text{Let, } V_3 = V_1 - V_2$$

$$\nabla^2 V_3 = 0$$

$$\text{On the surface, } V_3 = 0$$

$$\Rightarrow V_3 = 0 \ \text{ everywhere in the region}$$

$$\Rightarrow V_1 = V_2$$

If a "**guess**" satisfies the boundary conditions, then that **MUST** be the solution!

# WHY IS A METAL CAVITY A SHIELD

Arbitrary charges are outside the cavity (Q1...Qn)

Charges will be induced in the wall of the cavity.

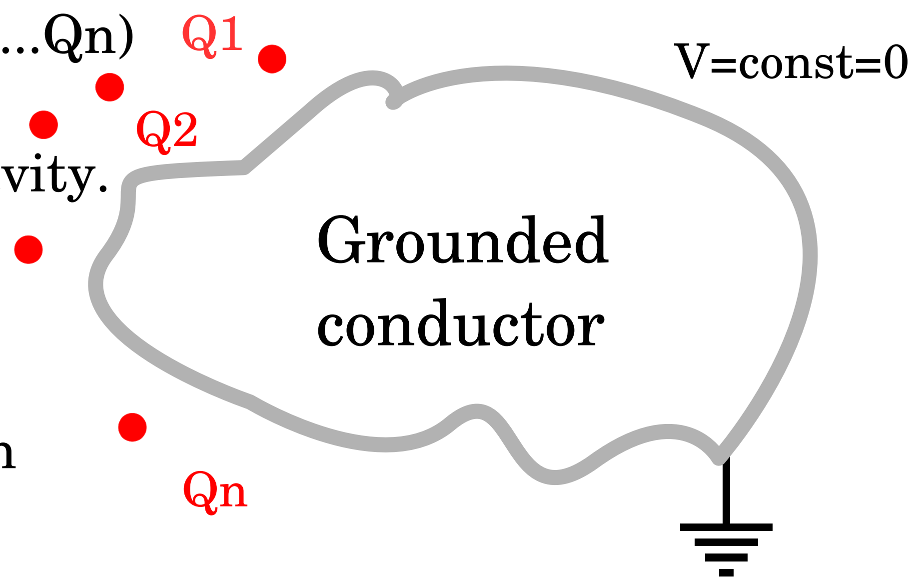
But the wall remains an equipotential.

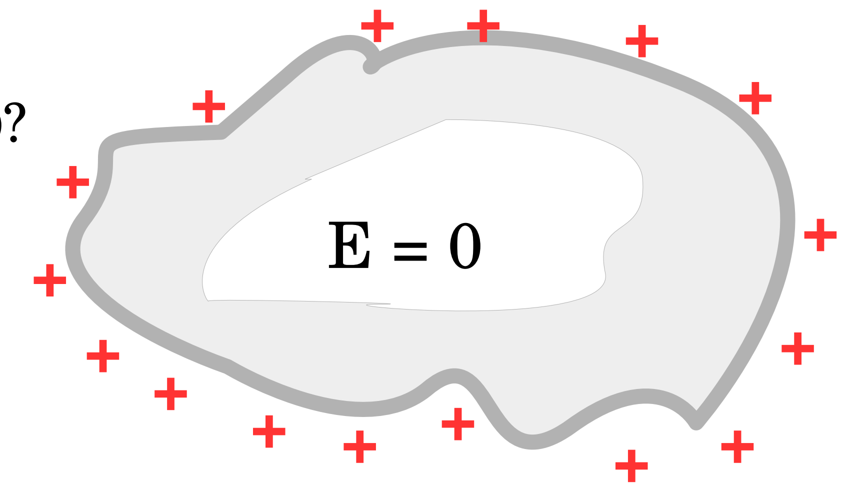
Inside the cavity V=0 is one possible solution satisfying the boundary conditions.

THAT IS THE UNIQUE SOLUTION.

What if the wall is not fixed at V=0 (i.e. floating)?

V=constant is still correct, but the constant will depend on the charge distribution outside.



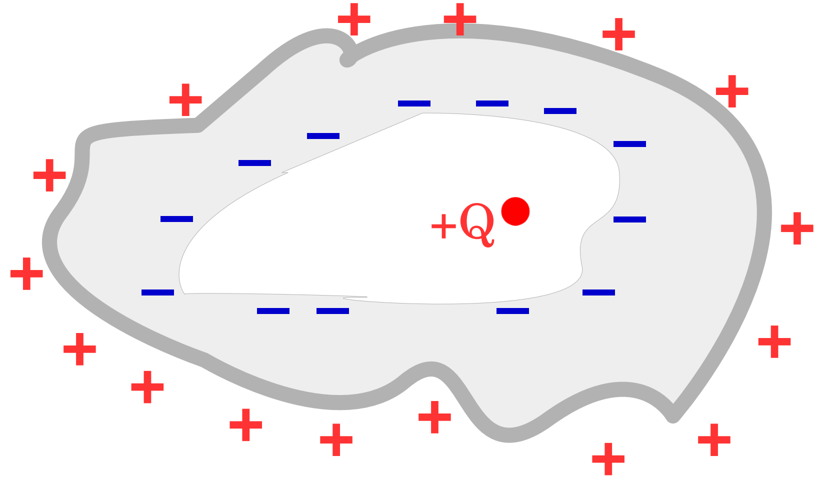

Floating conductor

# WHY IS A METAL CAVITY A SHIELD

The field inside the conductor is zero.

The field in the cavity is non-zero

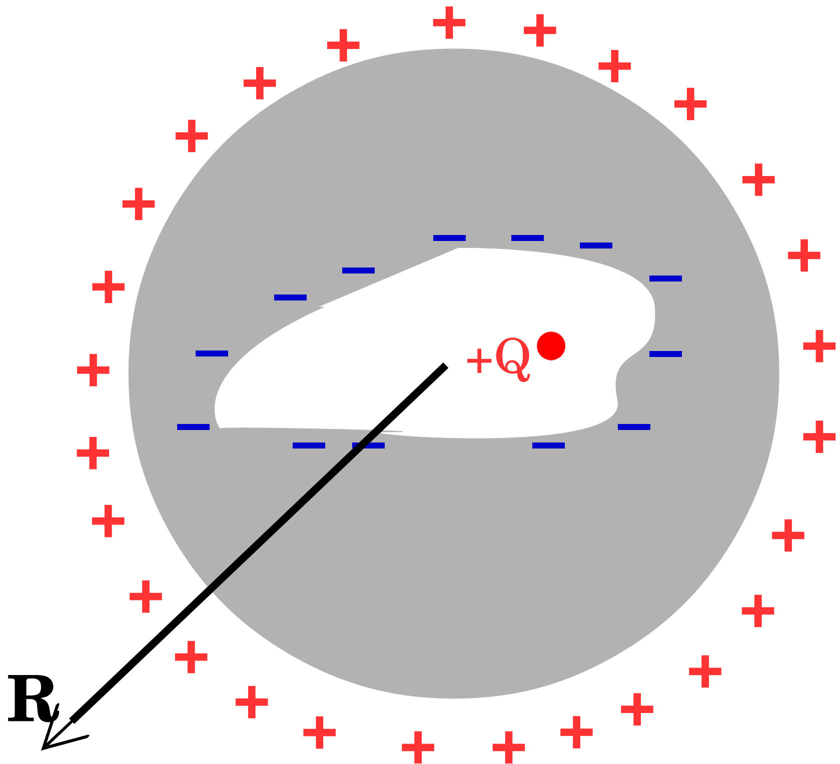
Equal amounts -Q and +Q on inner and outer surfaces.




Floating conductor

Surface is an equipotential
E = 0 inside
Charge density is uniform

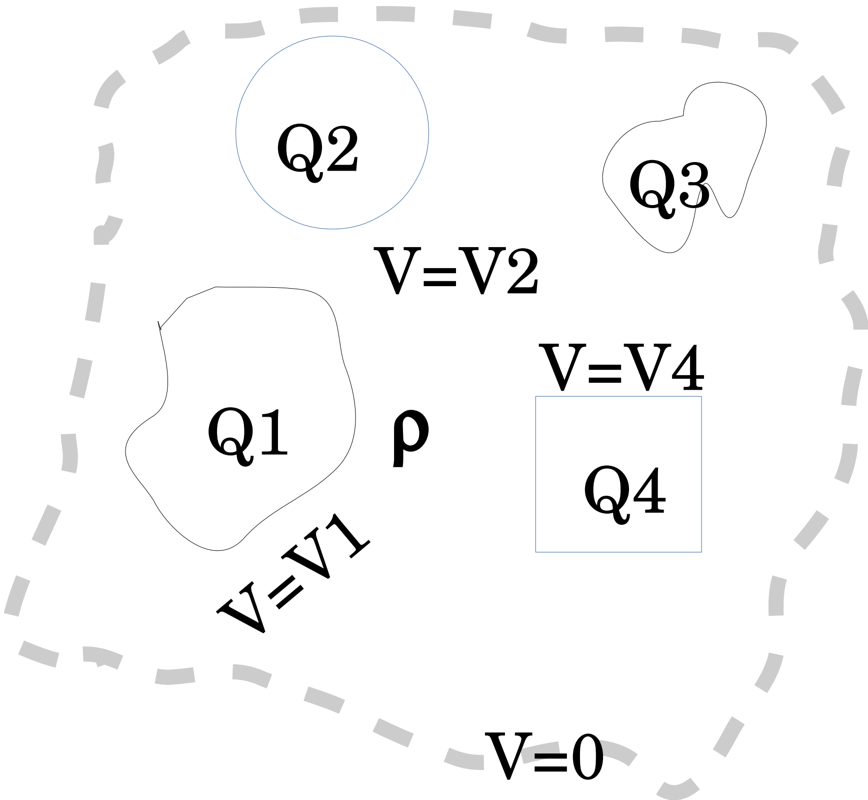
$$E(R)=\frac{Q}{4\pi\epsilon_0 R^2}$$




Irrespective of the location of Q inside the cavity!

# SECOND UNIQUENESS THEOREM

In a volume surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely specified if the total charge on each conductor is specified.




Let there be two fields

$\boldsymbol{E}_1$ and $\boldsymbol{E}_2$

satisfying the problem

In the space between the conductors,

$$\nabla \cdot \boldsymbol{E}_1 = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \cdot \boldsymbol{E}_2 = \frac{\rho}{\epsilon_0}$$

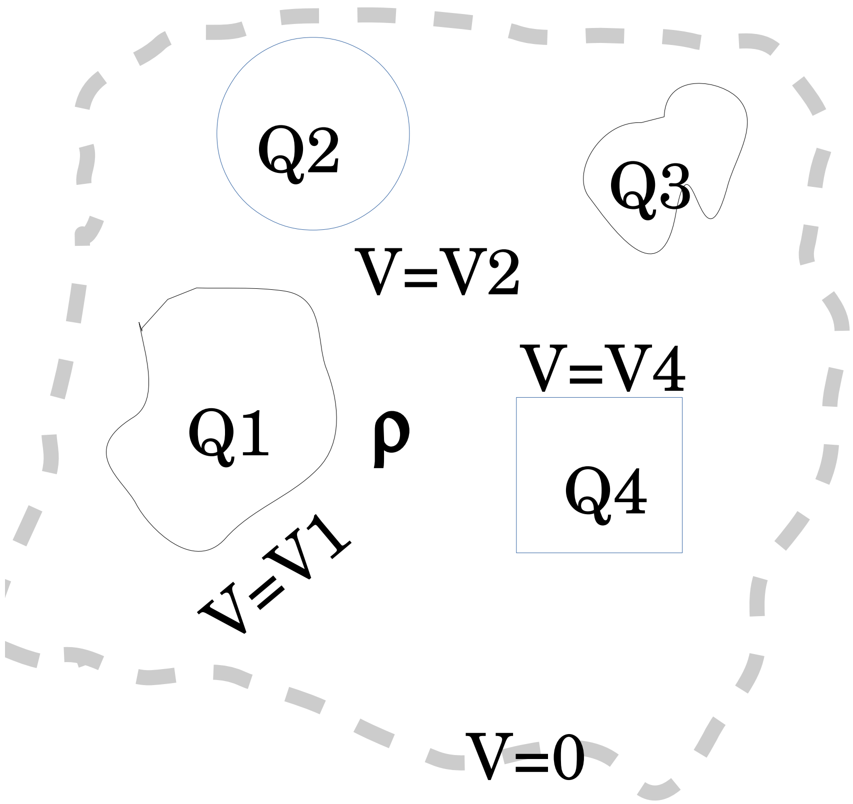
For the gaussian surface enclosing the $i^{th}$ conductor

$$\oint_{S_i} \boldsymbol{E}_1 \cdot d\boldsymbol{a} = Q_i/\epsilon_0 \quad \text{and} \quad \oint_{S_i} \boldsymbol{E}_2 \cdot d\boldsymbol{a} = Q_i/\epsilon_0$$

# SECOND UNIQUENESS THEOREM

For the outer boundary

$$\oint_{o.b.} \boldsymbol{E_1} . d\boldsymbol{a} = Q_{tot}/\epsilon_0 \quad \text{and} \quad \oint_{o.b.} \boldsymbol{E_2} . d\boldsymbol{a} = Q_{tot}/\epsilon_0$$




Let, $\boldsymbol{E_3} = \boldsymbol{E_1} - \boldsymbol{E_2}$

In the region between conductors

$$\nabla . \boldsymbol{E_3} = 0$$

Over each boundary surface

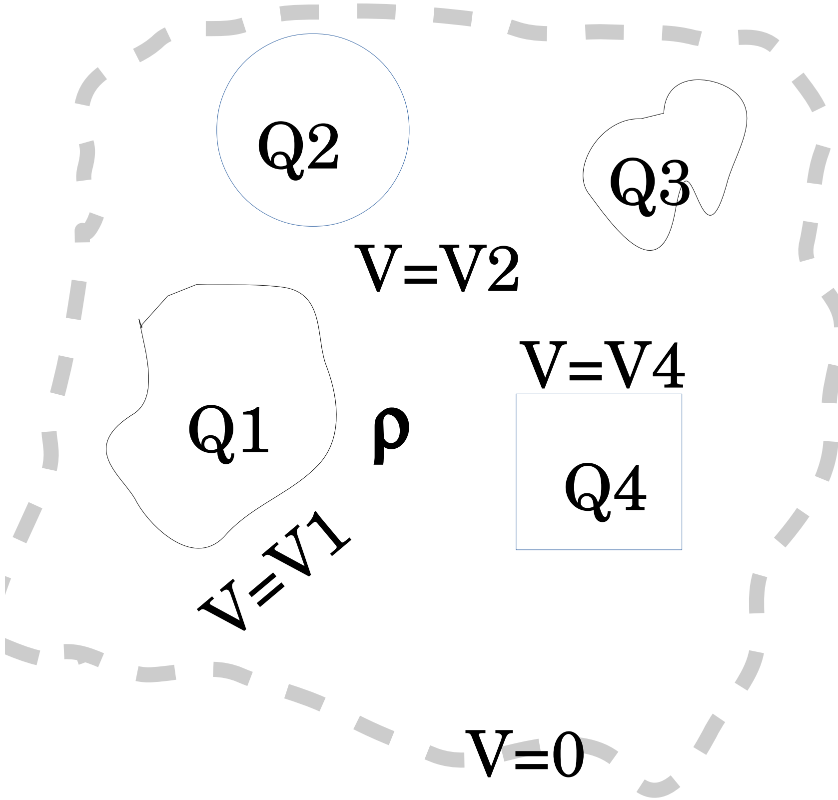
$$\oint \boldsymbol{E_3} . d\boldsymbol{a} = 0$$

Surface of a conductor is an equipotential

$\Rightarrow V_3 = constant$ over each surface

# SECOND UNIQUENESS THEOREM

$$\text{Now, } \nabla \cdot (V_3 \boldsymbol{E_3}) = V_3(\nabla \cdot \boldsymbol{E_3}) + \boldsymbol{E_3} \cdot (\nabla V_3) = -E_3^2$$




Integrating over the entire region

$$\int_{\text{whole region}} \nabla \cdot (V_3 \boldsymbol{E_3}) d\tau = \int_{\text{whole region}} -E_3^2 d\tau$$

Using divergence theorem

$$\int_{\text{whole region}} \nabla \cdot (V_3 \boldsymbol{E_3}) d\tau = \oint_{\text{all surfaces}} V_3 \boldsymbol{E_3} \cdot d\boldsymbol{a}$$

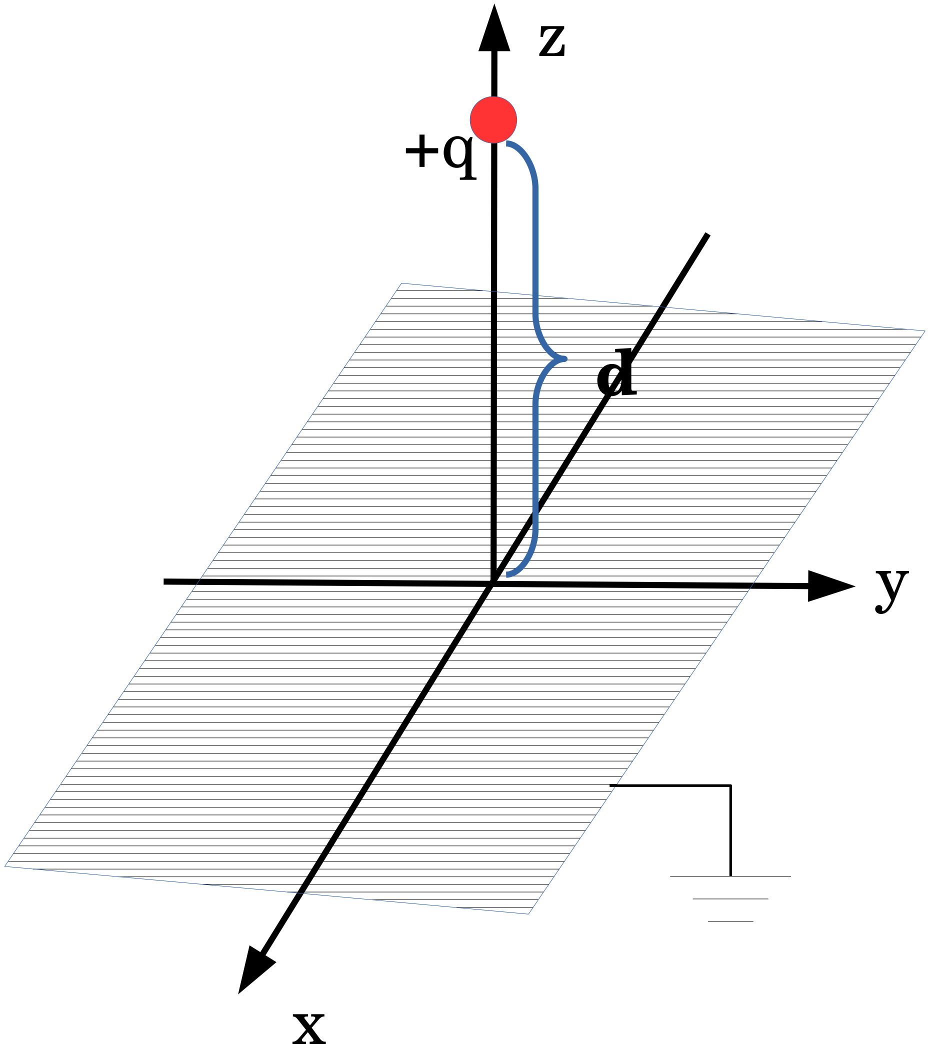
Since $V_3 = const.$ over all surfaces,

$$\oint_{\text{all surfaces}} V_3 \boldsymbol{E_3} \cdot d\boldsymbol{a} = 0$$

$$\text{Thus, } \int_{\text{whole region}} -E_3^2 d\tau = 0 \Rightarrow E_3 = 0 \Rightarrow \boldsymbol{E_1} = \boldsymbol{E_2}$$

# POINT CHARGE ABOVE A GROUNDED PLANE

A point charge q is at a distance d above an infinite grounded conducting plane located in the x-y plane. What is the potential in the region above the plane?




$$\nabla^2 V = -\frac{q}{\epsilon_0}\,\delta^3(x, y, z-d)$$

Boundary conditions:

$V = 0 \quad \text{at} \quad z = 0$

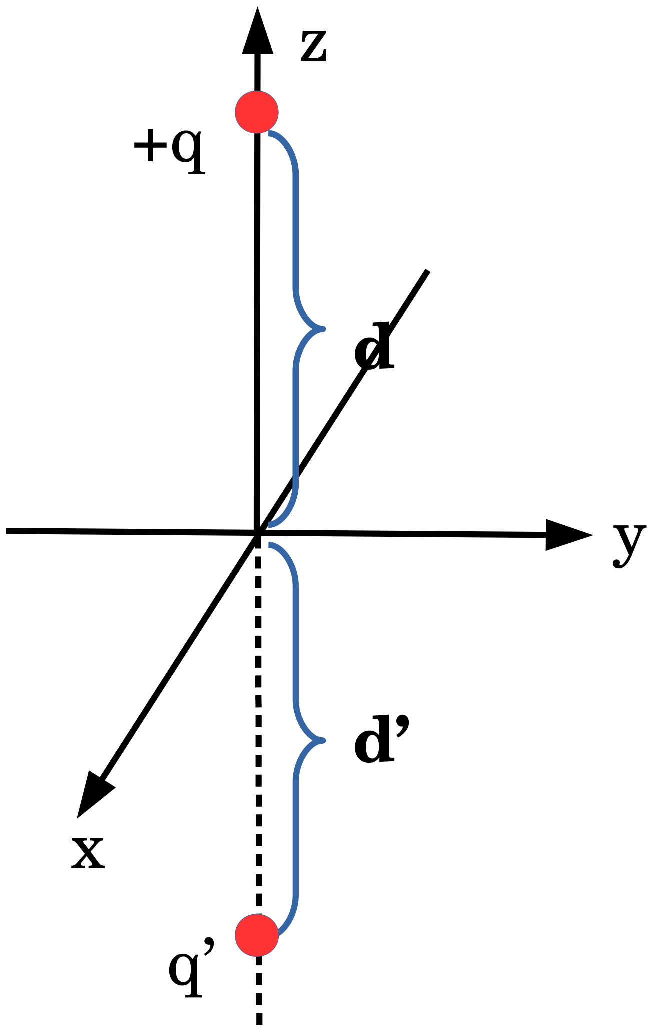
$V = 0 \quad \text{for} \quad x^2+y^2+z^2 \gg d^2$

Is $V(r) = \dfrac{q}{4\pi\epsilon_0 r}$? NO!

The charge q induces negative charge on the surface of the conductor. The total potential is due to the point charge and this induced surface charge.

# POINT CHARGE ABOVE A GROUNDED PLANE

Place a charge q' at a distance d' below the plane. Remove the conducting plane. Calculate the potential due to this ***new*** configuration of two point charges.




Potential at a point $P(x, y, z)$,

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d')^2}}\right]$$

Can we choose q' and d' such that the boundary conditions of the original problem are satified?

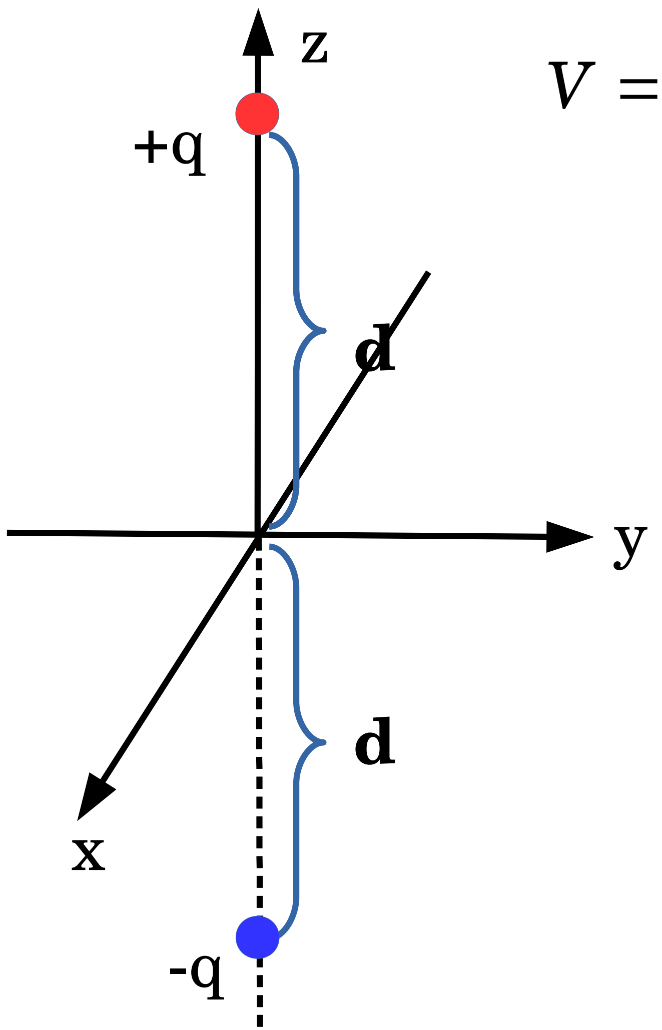
$$V = 0 \quad \text{at} \quad z = 0$$

$$\Rightarrow \frac{q}{\sqrt{x^2+y^2+d^2}} = -\frac{q'}{\sqrt{x^2+y^2+d'^2}}$$

$$\Rightarrow \boldsymbol{q' = -q}$$
$$\boldsymbol{d' = d}$$

$$V = 0 \quad \text{for} \quad x^2+y^2+z^2 \gg d^2$$

# POINT CHARGE ABOVE A GROUNDED PLANE




$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

What equation does this potential satisfy?

$$\nabla^2 V = -\frac{1}{\epsilon_0}\left[q\,\delta^3(x,y,z-d) - q\,\delta^3(x,y,z+d)\right]$$

If we focus on the upper half plane $z > 0$

$$\nabla^2 V = -\frac{q}{\epsilon_0}\,\delta^3(x,y,z-d)$$

with, $V = 0$ at $z = 0$

and, $V = 0$ for $x^2+y^2+z^2 \gg d^2$

This is precisely the definition of our original problem!

**Thus, by the uniqueness theorem, this potential must be the solution of our original problem!**

# POINT CHARGE ABOVE A GROUNDED PLANE

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad \text{for} \quad z > 0$$

The “virtual” charge is equal and opposite to the original source charge, and its distance is equal to the object distance.

## METHOD OF IMAGE CHARGES

Guess some configuration of image charges that satisfy Poisson’s equation in the region of interest and satisfies the boundary conditions. Uniqueness theorem guarantees that this MUST be the solution to the problem.

# POINT CHARGE ABOVE A GROUNDED PLANE

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad \text{for} \quad z > 0$$

What is the induced surface charge density on the conductor?

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$\sigma = -\frac{q}{4\pi}\frac{2d}{[x^2+y^2+d^2]^{3/2}}$$

Induced charge is negative

Maximum charge density at origin

$$Q_{\text{induced}} = \int \sigma\, da = -\frac{qd}{2\pi}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{\rho\, d\rho\, d\phi}{(\rho^2+d^2)^{3/2}} = \left.\frac{qd}{\sqrt{\rho^2+d^2}}\right|_0^{\infty} = -q$$

# POINT CHARGE ABOVE A GROUNDED PLANE

What is the force on the point charge due to the induced charges on the conductor?

$$\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}(-\hat{z})$$

What is the work done to bring in the point charge from infinity to d?

$$W = \int_{\infty}^{d} \boldsymbol{F}.d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{0}\frac{q^2}{4z^2}dz = -\frac{q^2}{16\pi\epsilon_0 d}$$

Note that this is half of the energy of a system with two charge +q and -q at a distance (2d) apart.

# METHOD OF IMAGE CHARGES

A charge distribution and some boundary conditions are given. The usual boundary conditions are fixed potentials over some surfaces. Solve for V(r) in a certain region.

1. Put some point charges (external charges) in the regions not part of the region where you need to solve for the potential.

2. Try to arrange these external charges such that the given charges plus the external charges produce the desired potential at the boundaries. Forget all else!

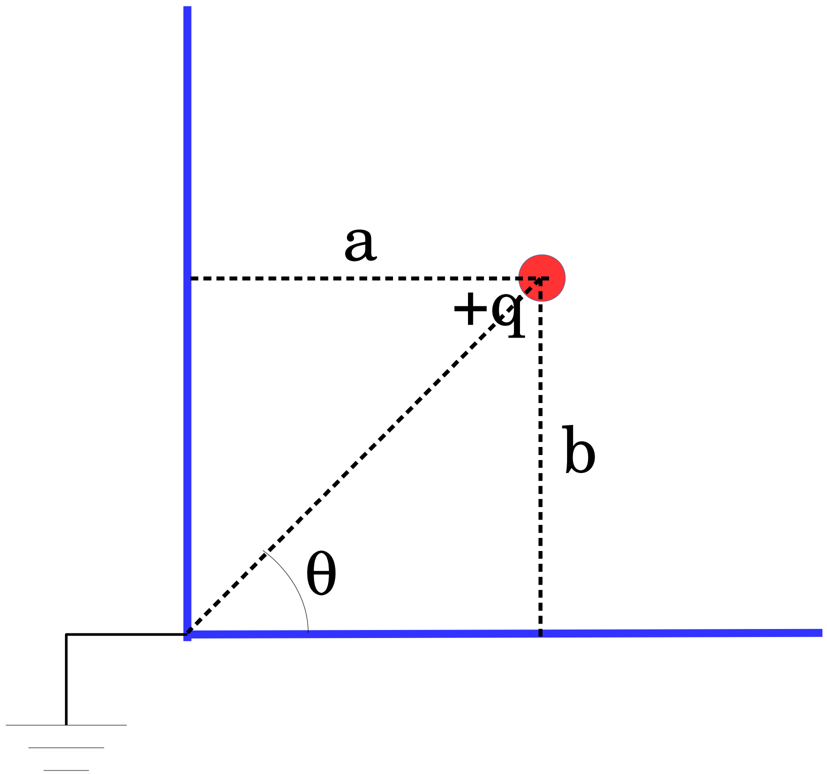
3. Calculate the total potential in the desired region using all the given charges PLUS the external charges.

4. The total field / potential produced by ALL the charges is the solution to the problem. **The extra charges are called IMAGE CHARGES**

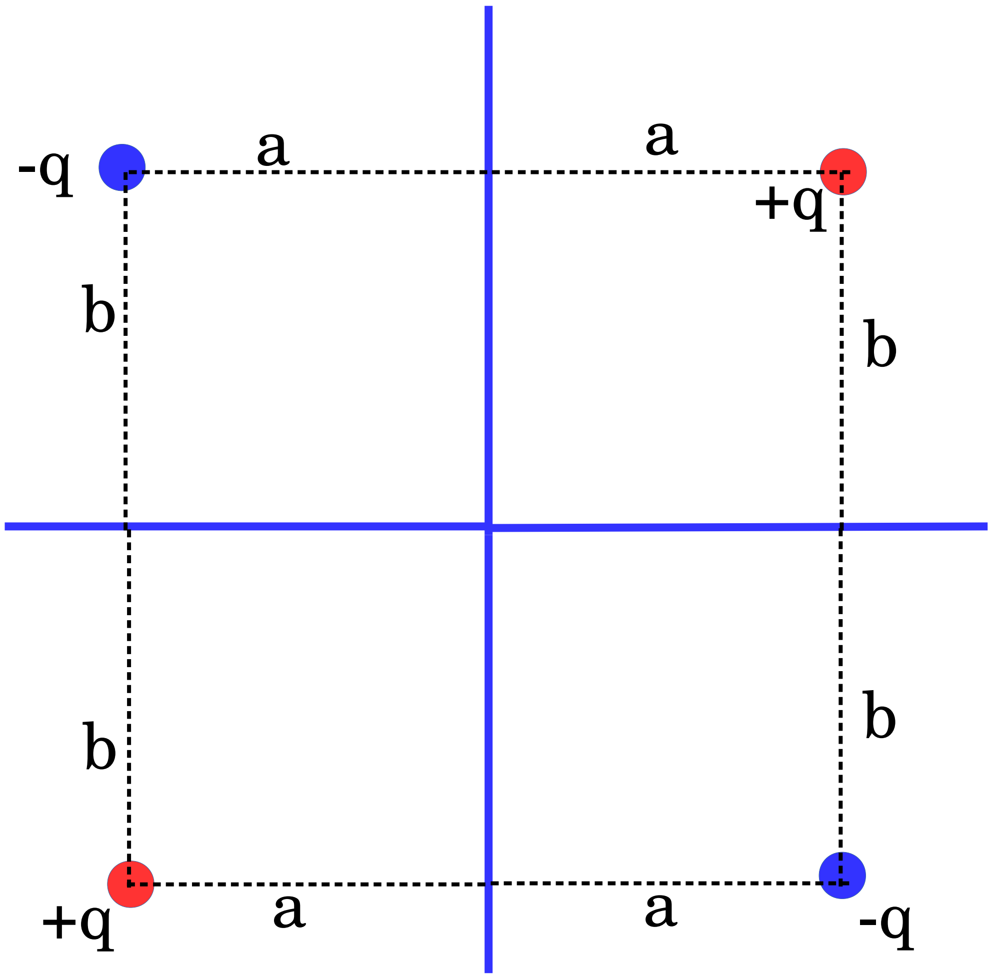
# POINT CHARGE + TWO SEMI-INFINITE PLANES

Consider two semi-infinite grounded, conducting planes intersecting along the z-axis, which is taken to be out of plane of the screen (x-y plane). A charge +q is located at (a,b).

- What is the potential in the region between the planes?
- What is the force on q due to the charges induced on the planes?



# POINT CHARGE + TWO SEMI-INFINITE PLANES




$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}+\frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right.$$
$$\left.-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]$$

# POINT CHARGE + TWO SEMI-INFINITE PLANES

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}}+\frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right.$$
$$\left.-\frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}}-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]$$

If, $x = 0,\ V = 0 \ \forall y$
If, $y = 0,\ V = 0 \ \forall x$

$$\nabla^2 V = -\frac{q}{\epsilon_0}\delta^3(x-a, y-b) \quad \text{in the} \{x>0, y>0\} \text{quadrant}$$

This potential satisfies the Poisson's equation and the Boundary conditions of the original problem. Thus Uniqueness Theorem guarantees this is the solution.

# POINT CHARGE + TWO SEMI-INFINITE PLANES

What is the force on the point charge due to the induced charges on the two conducting planes?

$$F_{-q\,at\,(a,-b)} = \frac{q^2}{4\pi\epsilon_0(2b)^2}(-\hat{y})$$

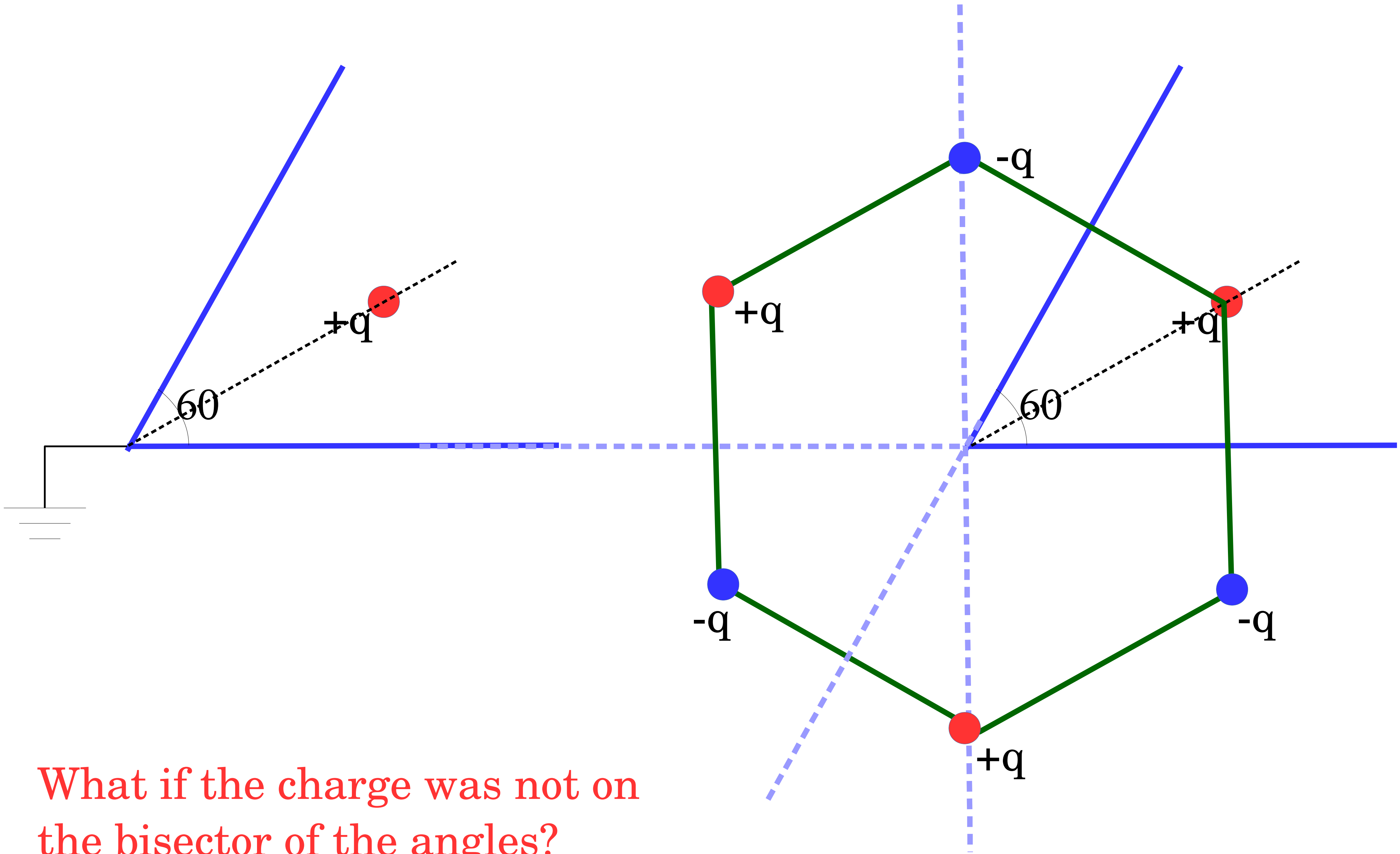
$$F_{-q\,at\,(-a,b)} = \frac{q^2}{4\pi\epsilon_0(2a)^2}(-\hat{x})$$

$$F_{+q\,at\,(-a,-b)} = \frac{q^2}{4\pi\epsilon_0(4(a^2+b^2))}\left[\cos\theta\,\hat{x}+\sin\theta\,\hat{y}\right]$$

$$F_x = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{4a^2}+\frac{a}{4(a^2+b^2)^{3/2}}\right]$$

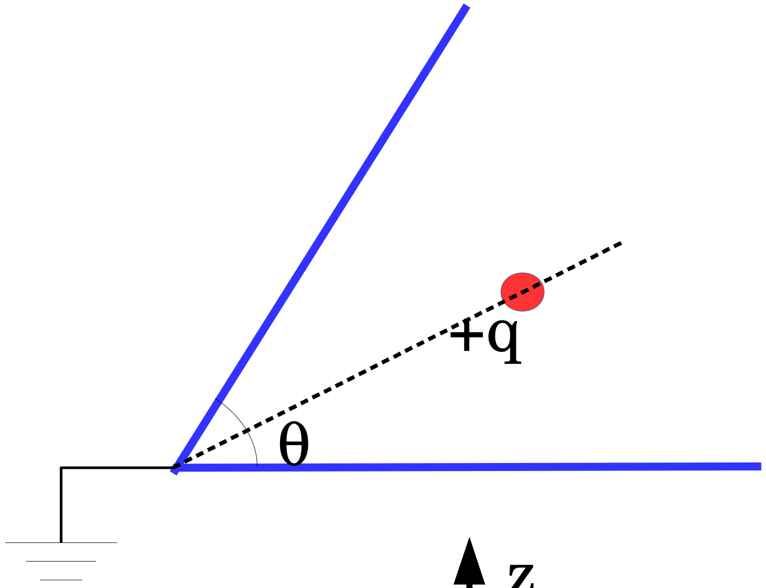
$$F_y = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{4b^2}+\frac{b}{4(a^2+b^2)^{3/2}}\right]$$

# POINT CHARGE + TWO PLANES AT AN ANGLE



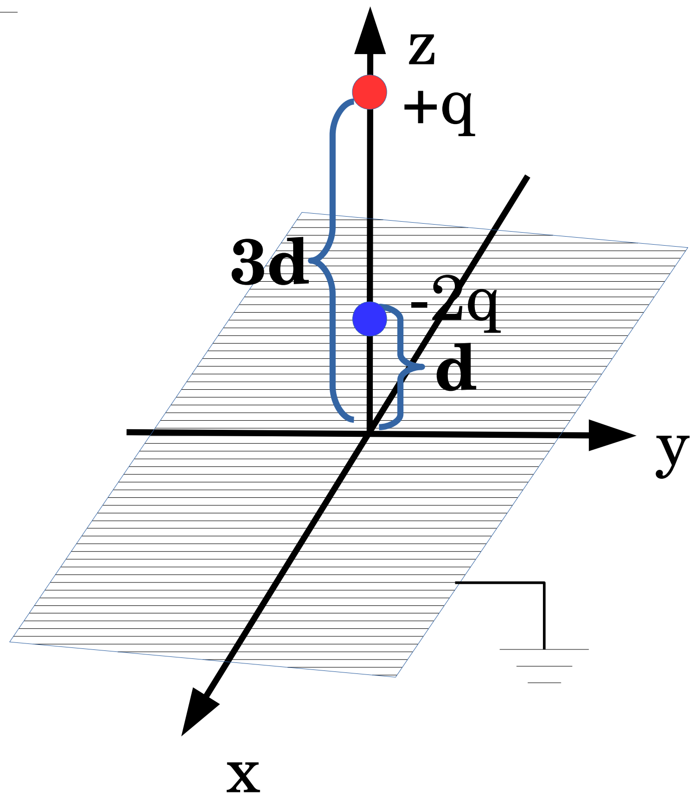

What if the charge was not on the bisector of the angles?

# POINT CHARGE + TWO PLANES AT AN ANGLE




This problem can be solved with a finite number of image charges if,

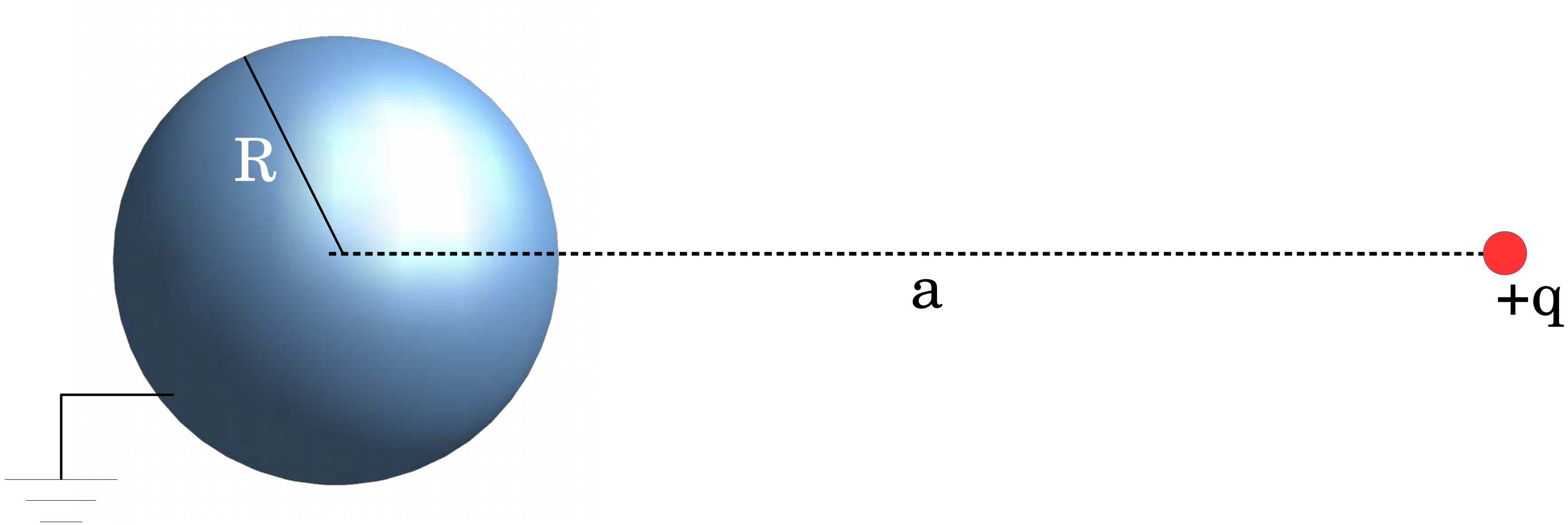
**(Integer) * θ = π**




Place image charge +2q at -d, and image charge -q at -3d

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

Consider a charge +q kept at a distance a from the centre of a grounded sphere of radius R. What is the potential at a point P outside the sphere?

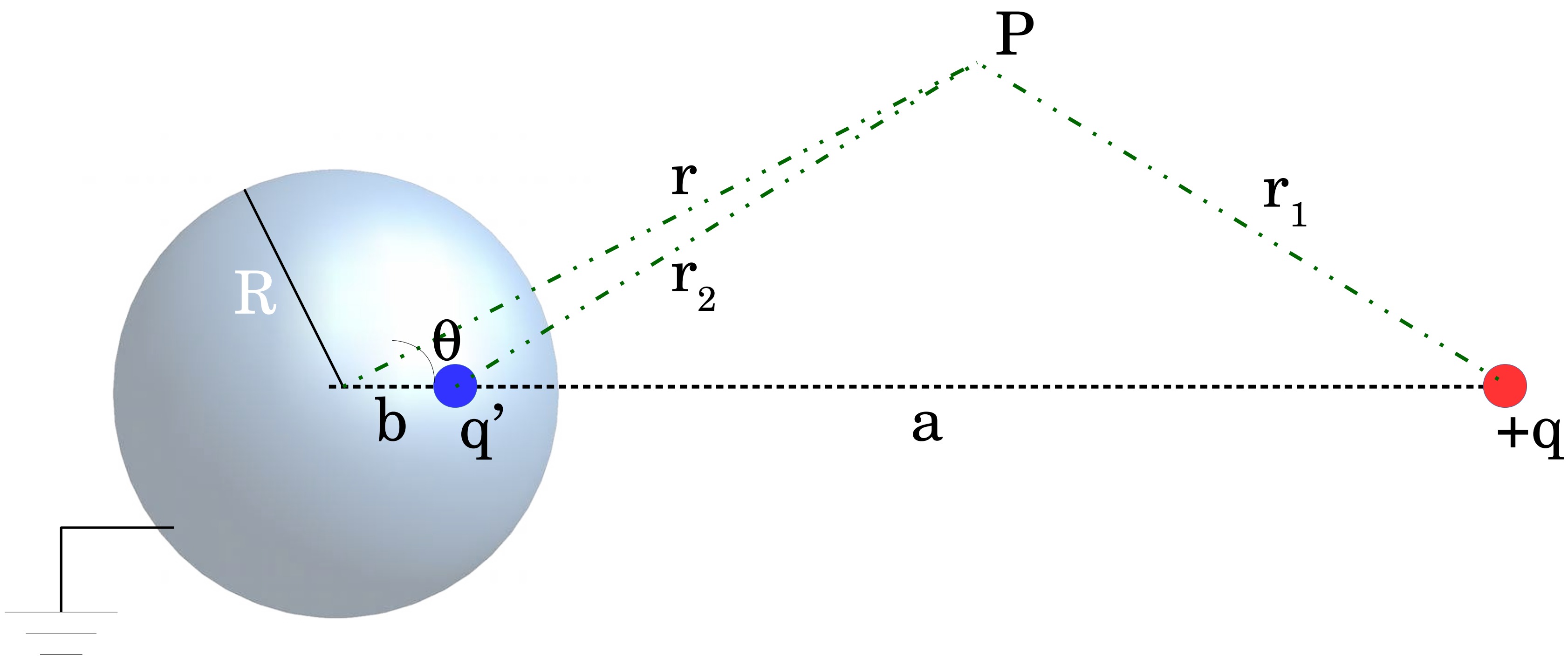



Boundary conditions:

$V = 0$ on the surface of the sphere

$V = 0$ for points far away from +q

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE



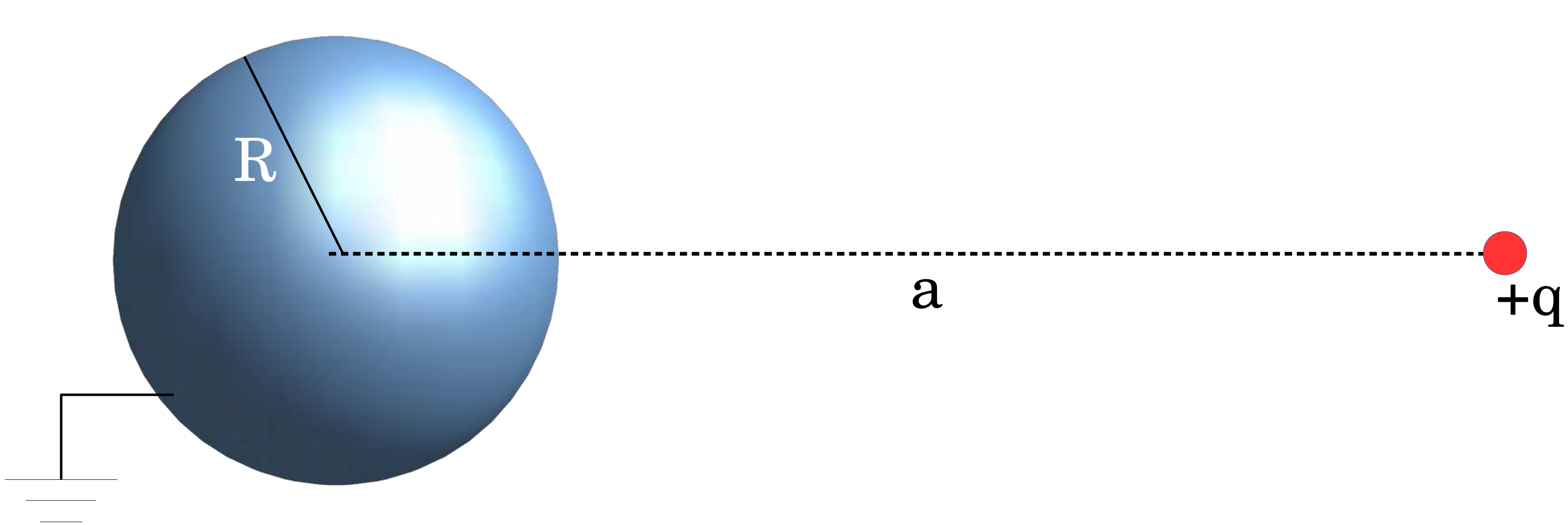

$$V_P = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}}+\frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}}\right]$$

Choose $q'$ and $b$ such that the potential is zero over the sphere

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

Consider a charge +q kept at a distance a from the centre of a grounded sphere of radius R. What is the potential at a point P outside the sphere?




Boundary conditions:

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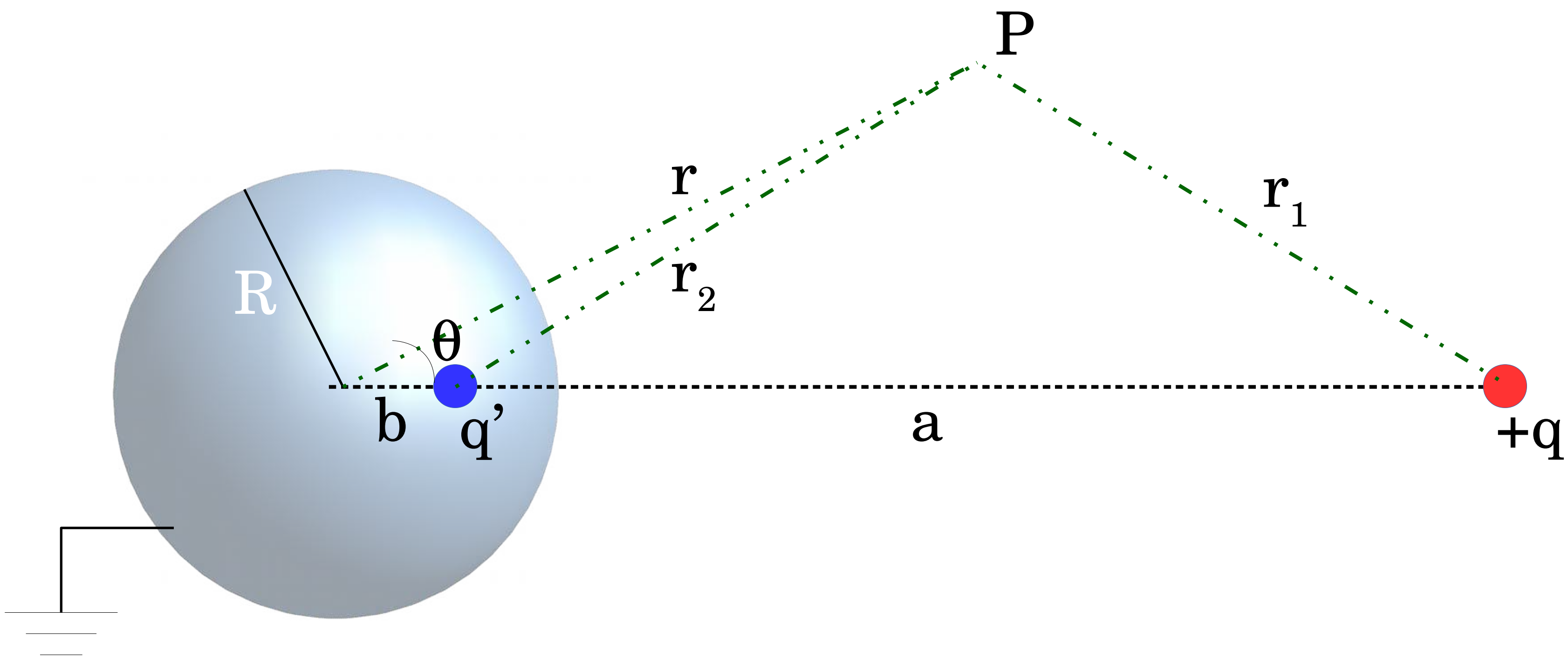
# POINT CHARGE OUTSIDE A CONDUCTING SPHERE




$$V_P = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br\cos\theta}}\right]$$

Choose $q'$ and $b$ such that the potential is zero over the sphere

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

$$V_P(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}}\right]$$

Boundary condition: $V(r=R,\theta) = 0 \;\; \forall \;\; \theta$

$$q^2(R^2+b^2-2bR\cos\theta) = q'^2(R^2+a^2-2aR\cos\theta)$$

Since this must hold true $\forall\ \theta$, we equate the coefficient of $\cos\theta$,

$$2bRq^2 = 2aRq'^2$$

$$\Rightarrow \;\; q' = -q\sqrt{\frac{b}{a}}$$

Substituting in the $\theta$ independent terms, we have

$$R^2+b^2 = \frac{b}{a}(R^2+a^2)$$

$$\Rightarrow \;\; ab = R^2$$

$$\Rightarrow \;\; b = R^2/a$$

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

$$V_P(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{R/a}{\sqrt{r^2+\left(\frac{R^4}{a^2}\right)-2\left(\frac{R^2}{a}\right)r\cos\theta}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{R}{\sqrt{r^2a^2+R^4-2R^2ar\cos\theta}}\right]$$

Surface charge density,

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n} = -\epsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=R}$$

$$\sigma = -\frac{q}{4\pi}\left[\frac{-R+a\cos\theta}{(R^2+a^2-2ar\cos\theta)^{3/2}} + \frac{R(a^2R-R^2a\cos\theta)}{(R^2a^2+R^4-2R^3a\cos\theta)^{3/2}}\right]$$

$$= \frac{q}{4\pi R}\frac{R^2-a^2}{(R^2+a^2-2ar\cos\theta)^{3/2}}$$

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

$$\sigma = \frac{q}{4\pi R}\frac{R^2-a^2}{\left(R^2+a^2-2ar\cos\theta\right)^{3/2}}$$

The surface charge density is negative, as expected.

$$Q_{\text{induced}} = \int \sigma\, da = \int_0^{\pi}\int_0^{2\pi} \sigma(R,\theta) R^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{q\left(R^2-a^2\right)}{4\pi R} 2\pi R^2 \int_0^{\pi} \frac{\sin\theta\, d\theta}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}$$

$$= \frac{qR\left(R^2-a^2\right)}{2}\int_{-1}^{+1}\frac{dx}{\left(R^2+a^2-2aR\,x\right)^{3/2}}$$

$$= \frac{qR\left(R^2-a^2\right)}{4aR}\left[\left(R^2+a^2-2aR\,x\right)^{-1/2}\right]_{+1}^{-1}$$

$$= -q\frac{R}{a}$$

# POINT CHARGE OUTSIDE A CONDUCTING SPHERE

What is the force on the charge due to the induced charges on the sphere?

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2 R a}{(a^2-R^2)^2}$$

What happens as $R \to \infty$?

Let, $d \to$ the distance of the charge from the surface of the sphere

$$a = d+R \quad \Rightarrow \quad b = \frac{R^2}{d+R}$$

Distance of the image charge from the surface

$$b' = R-b = R-\frac{R^2}{d+R} = \frac{Rd}{d+R} \to d \quad \text{as} \quad R\to\infty$$

What happens when the conductor is not grounded but held at some constant potential $V_0 \neq 0$?