SUMMARY(!) Lecture-3

Fundamental theorem of differentiation : $\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$

Gradient operator: : $\vec{\nabla} T$, $\vec{\nabla} \cdot \vec{v}$, $\vec{\nabla} \times \vec{v}$.

$\oint \vec{\nabla} T \cdot d\vec{l} = 0$

Fundamental theorem of gradients : $\int_a^b \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$

Gauss's theorem : $\iiint \vec{\nabla} \cdot \vec{v} \, dv = \underbrace{\oint\!\!\oint \vec{v} \cdot d\vec{a}}_{\text{flux.}}$

Stoke's theorem : $\iint (\vec{\nabla} \times \vec{v}) \cdot da = \oint \vec{v} \cdot d\vec{l}$

SUMMARY - (2)

Curvilinear coordinates.

Why? Symmetry of the problem.

Plane polar coordinates:

(1) Relation with Cartesian coordinates. $\{r, \theta\} \rightarrow \{x, y\}$.

(2) Constant curves.

(3) Infinitesimal change.

(4) Unit vectors.

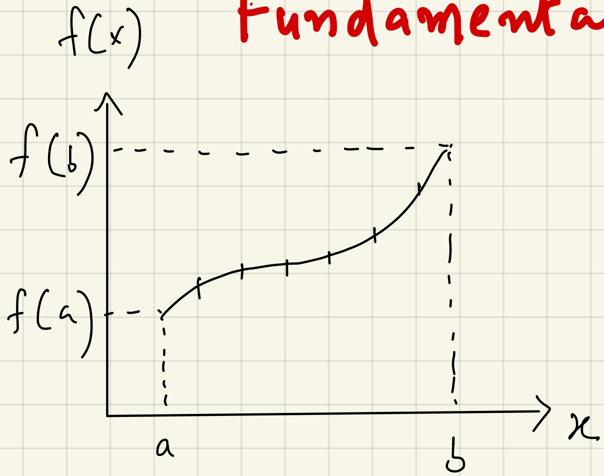
(5) Distance and (6) Area elements.

$\nabla^2 \vec{V} =$

(7) Gradient. $\vec{V} = \{v_1, v_2, v_3\}$.

(8) Time derivatives. $\{\nabla^2 v_1, \nabla^2 v_2, \nabla^2 v_3\}$

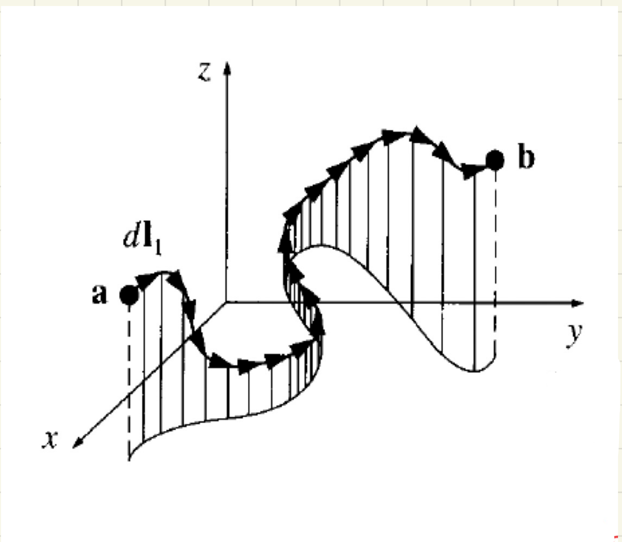
Fundamental theorem of gradients.



$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

Scalar field $T = T(x, y, z)$

$$dT = \vec{\nabla} T \cdot d\vec{l}$$

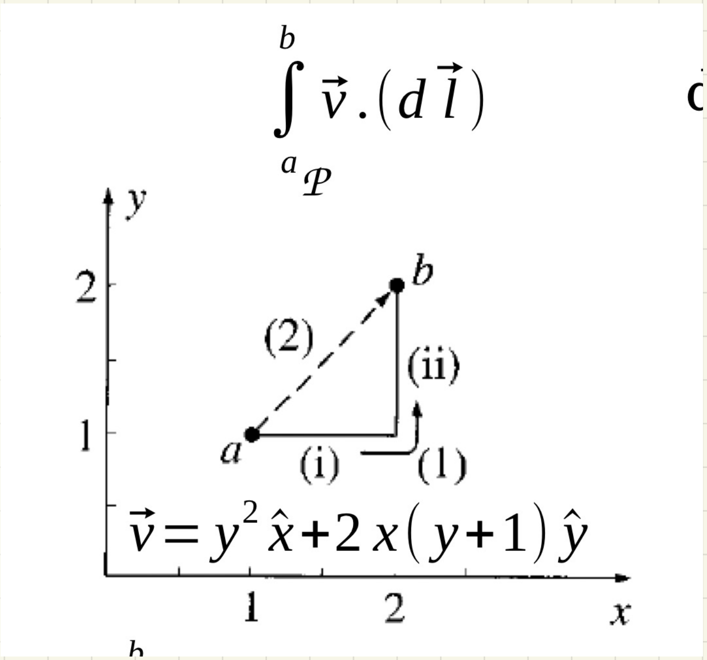


$$\int_{a(P)}^{b} (\vec{\nabla} T \cdot d\vec{l}) = T(b) - T(a)$$

The value of integral only depends on the end points.

Does not depend on path.

Path independence is not always true for any general vector integrals!



$$\vec{V} = y^2\,\hat{x} + 2x\,(y+1)\,\hat{y}$$

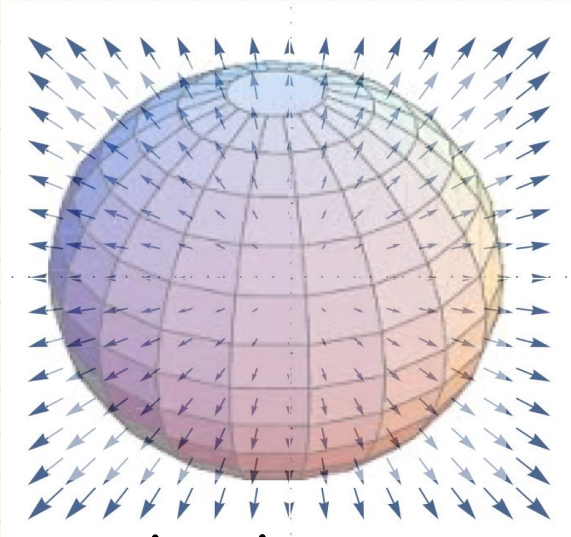
$\int \vec{V}.d\vec{l}$ depends on path.

$\oint \vec{V}.d\vec{l} \neq 0$ for such fields.

Path independence is a property of Conservative fields!

Conservative field : $\vec{V} = \nabla T$ (vector can be expressed as a gradient

Surface integral



closed surface

$d\vec{s}$ = Area element (vector)

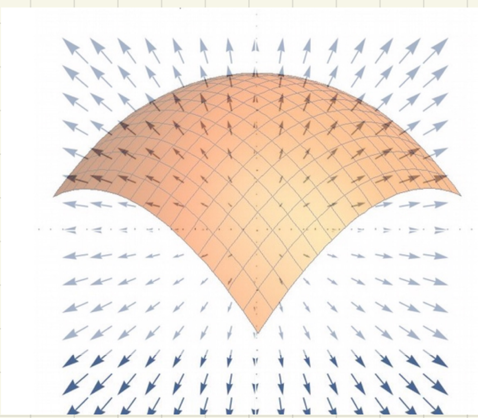
direction : outward normal (closed surface)

: arbitrary normal (open surface)

Flux: of $\vec{v}$ $\oint\!\!\oint \vec{v} \cdot d\vec{s}$

Integral over a closed surface.

$d\vec{s}$ = positive, if pointing outward



open surface.

$\iint \vec{v} \cdot d\vec{s}$

Integral over an open surface:

Sign of $d\vec{s}$ is arbitrary.

GAUSS' THEOREM / DIVERGENCE THEOREM

$$\iiint_V (\vec{\nabla}\cdot\vec{v})\,dV = \oiint_S \underline{\vec{v}\cdot d\vec{S}}$$

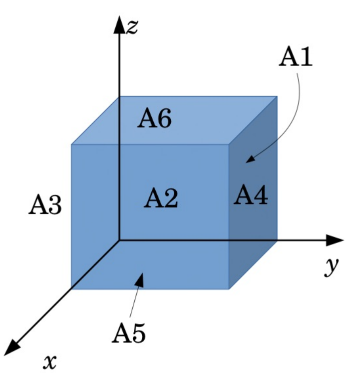
$\leftarrow$ } Flux over a closed surface.

$\downarrow$ volume integral | $\downarrow$ Surface integral (Surface that encloses Volume)

$\vec{\nabla}\cdot\vec{v}$ measures the sources and sinks

$\oiint \vec{v}\cdot d\vec{S}$ measures the total flux.

GAUSS'S THEOREM / DIVERGENCE THEOREM



$$\iint_{A_1} \vec{v}\cdot d\vec{S} + \iint_{A_2} \vec{v}\cdot d\vec{S}$$

$$= -\int_e^f\int_c^d v_x(a,y,z)\,dy\,dz + \int_e^f\int_c^d v_x(b,y,z)\,dy\,dz$$

$$= \int_e^f\int_c^d \left(v_x(b,y,z) - v_x(a,y,z)\right)dy\,dz$$

$$= \int_e^f\int_c^d\int_a^b \left(\frac{\partial v_x}{\partial x}dx\right)dy\,dz = \iiint_V \frac{\partial v_x}{\partial x}dV$$

$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad e \leqslant z \leqslant f$

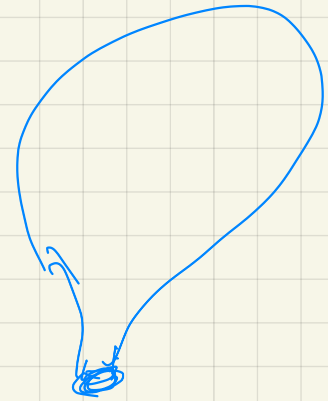
Repeating over the other 4 sides and adding,

$$\oiint_S \vec{v}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{v})\ dV$$

Gauss' theorem can be easily demonstrated on a cubic surface!

STOKES THEOREM

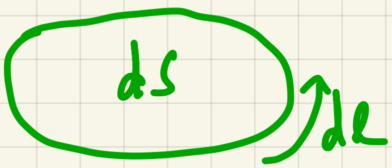
$$\iint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{s} = \oint_{?} \vec{v} \cdot d\vec{l}$$

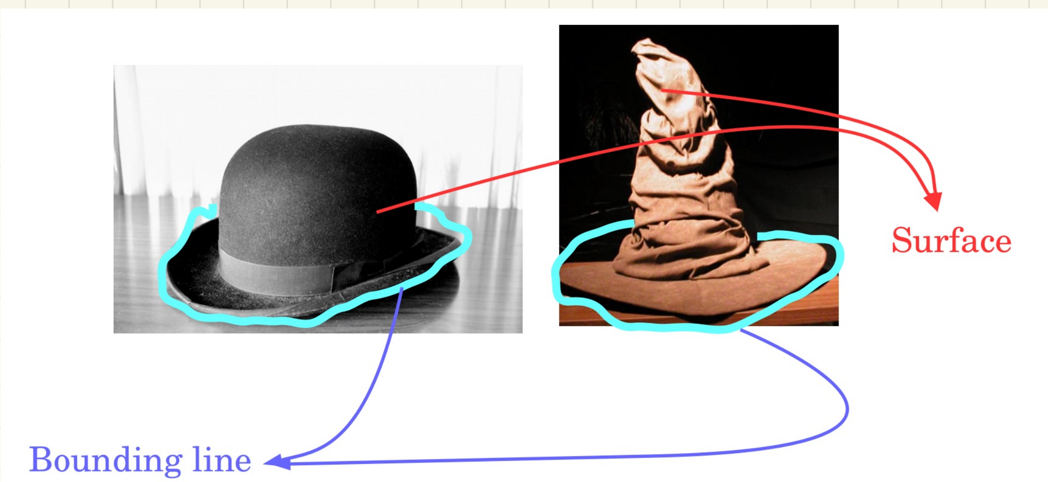


Integral over an arbitrary surface.

Integral over the bounding curve of the surface.

Sign: right hand rule.





The surface integral can be over ***ANY*** surface that shares a common bounding line!

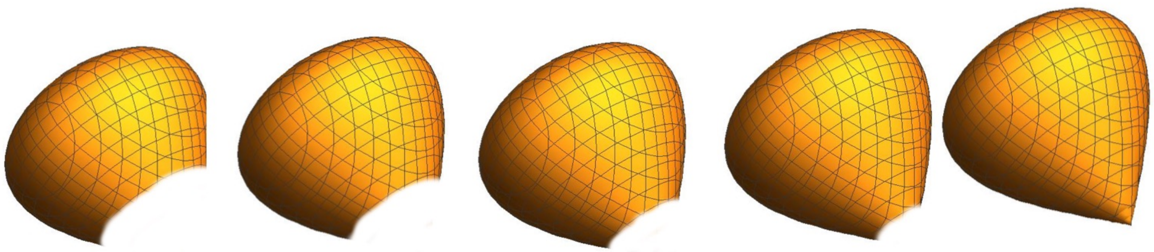
In general the flux depends on the surface.

$\iint \vec{B} \cdot ds \rightarrow$ flux of B

But $\vec{B} = \vec{\nabla} \times \vec{v}$, it only depends on the bounding line.

STOKES THEOREM

What happens if we shrink the boundary line?



$$\oiint_S (\nabla\times\vec{v}).d\vec{S}=0$$

Consistency check

Stokes' theorem.

$$\iint (\vec{\nabla}\times\vec{v})\, ds = \oint \vec{v}.d\vec{l}$$

if

$$\vec{v} = \nabla T$$

$$\oint \vec{\nabla} T . d\vec{l} = 0,$$

and $\vec{\nabla}\times(\nabla \vec{T}) = 0$

Gauss' theorem.

$$\iiint (\vec{\nabla}\cdot\vec{v})\, d\tau = \oiint \vec{v} . d\vec{s}$$

$$\vec{v} = \vec{\nabla}\times\vec{A}$$

$$\iiint \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\, d\tau = \oiint (\vec{\nabla}\times\vec{A}).ds = 0$$

$d\vec{l} = 0$ for a closed surface.

Curvilinear coordinate system.

Why?

Choice of coordinate system: Mathematical convenience!

Relation between cartesian and curvilinear systems.

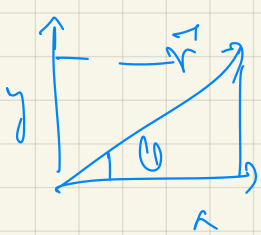
(1) Plane polar (2D)

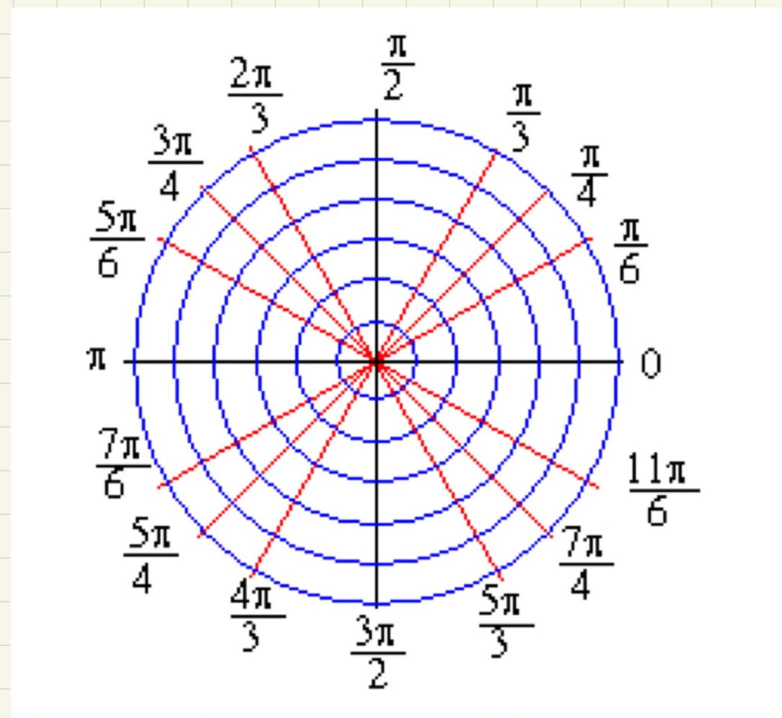
$$x = r\cos\theta$$
$$y = r\sin\theta$$

} step-1

r constant curves
θ constant curves.

} step-2





Infinitesimal changes

$$x = x(r, \theta) \qquad y = y(r, \theta)$$

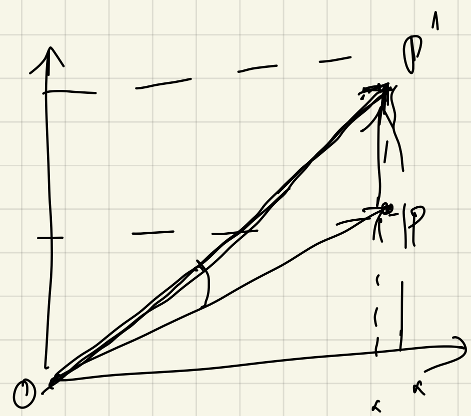
$$\delta x = \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta \qquad \delta y = \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta$$

$$\left.\begin{aligned} \delta x &= \cos\theta\, \delta r - r\sin\theta\, \delta\theta \\ \delta y &= \sin\theta\, \delta r + r\cos\theta\, \delta\theta \end{aligned}\right\} \text{Infinitesimal changes. (step-3)}$$

Step-4 what are $\hat{r}$ and $\hat{\theta}$?

$\hat{r} \rightarrow \delta\theta = 0$

$\hat{\theta} \rightarrow \delta r = 0$

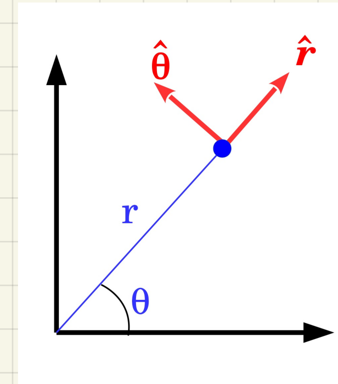


$\vec{P}' - \vec{P}$

$$\delta\theta = 0$$

$$\delta x\,\hat{x} + \delta y\,\hat{y} = (\hat{x}\cos\theta + \hat{y}\sin\theta)\,\delta r$$

$$= \underline{\hat{r}\,\delta r}$$



$$\delta r = 0$$

$$\hat{x}\,\delta x + \hat{y}\,\delta y = (-\hat{x}\sin\theta + \hat{y}\cos\theta)\,r\,\delta\theta = \underline{\hat{\theta}\,r\,\delta\theta}$$

$$\frac{d}{dt}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$\hat{r}$ and $\hat{\theta}$ are not constant vectors in space.

$$\underline{\hat{r}\cdot\hat{\theta} = 0}$$

Step-5: Distance element

$$d\vec{l} = \delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}$$

(Scale factors are not unity)

Scale factor:

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2} = 1 \qquad ds^2 = d\vec{l}\cdot d\vec{l} = \delta r^2 + r^2\,\delta\theta^2$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} = r$$

Step-6 Area element:

$$dl_r = \delta r \qquad dl_\theta = r\,\delta\theta$$

$$dA = dl_r\, dl_\theta = r\,\delta r\,\delta\theta$$

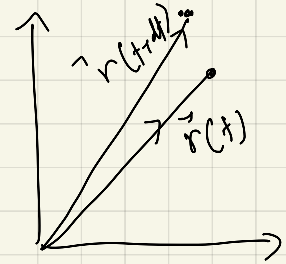
Step. 7

$$dT = \frac{\partial T}{\partial r}\,\delta r + \frac{\partial T}{\partial \theta}\,\delta\theta \qquad T = T(r,\theta).$$

$$= [\vec{\nabla} T]\cdot d\vec{l} = \vec{\nabla} T\cdot\left(\delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}\right).$$

$$\vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} \qquad \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$



$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta.$$

$$\frac{d\hat{r}}{dt} = -\hat{x}\sin\theta\,\dot{\theta} + \hat{y}\cos\theta\,\dot{\theta}$$

Step. 8: time derivatives

$$\vec{v} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$\begin{pmatrix}\dot{\hat{r}}\\ \dot{\hat{\theta}}\end{pmatrix} = \dot{\theta}\begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \dot{\theta}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

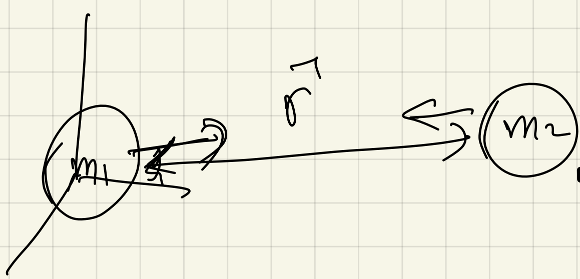
$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \qquad \vec{F} = f\hat{r}$$

$$= \dot{r}\dot{\theta}\hat{\theta} + \ddot{r}\hat{r} - \dot{\theta}r\dot{\theta}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta}$$

$$\vec{a} = \underbrace{\left(\ddot{r} - \dot{\theta}^2 r\right)}_{\text{radial acceleration}}\hat{r} + \underbrace{\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)}_{\text{circumferential acceleration}}\hat{\theta}$$

Central force, which quantity is conserved!



$$\vec{V} = \{v_i\}. \quad \vec{\nabla}F = \underbrace{\frac{\partial F}{\partial x}}_{\partial_i}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}.$$

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{V} = \partial_i v_i \\ \text{Outer product: } \partial_i v_j \end{array}\right\}.$$

Central force.

$F_\theta = 0.$ $\quad F_\theta = ma_\theta = m r\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0$

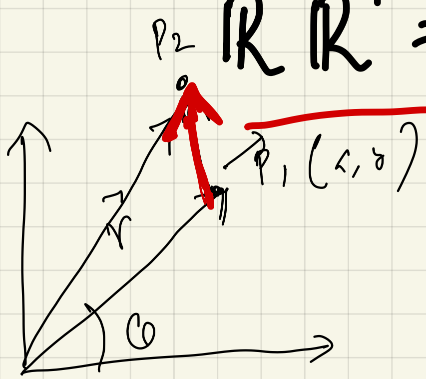
$$= \frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$mr^2\dot{\theta} =$ constant.

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

$R \to$ matrix.

$RR^T = \mathbb{I} \to$ Orthogonal matrix



$$\begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \end{pmatrix} = \dot{R} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \dot{R} R^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix}$$

$\dot{R} R^T =$ <u>anti symmetric</u>. Why?

$$\underline{R R^T = \mathbb{I}}$$

$$\underline{\frac{d}{dt}\left(R R^T\right)} = \dot{R} R^T + \underbrace{R\left(\dot{R}^T\right)}_{\left(\dot{R} R^T\right)^T} = 0 \quad \Big\} \Rightarrow \text{anti sym. property!}$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\hat{r}\cdot\hat{\theta} = 0$$

$$\vec{r} = r\,\hat{r}\,, \qquad d\vec{r} = d[r\,\hat{r}] = \underline{dr\,\hat{r} + \hat{r}\,d[\hat{r}]}$$

$$d\hat{r} = \left\{-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right\} d\theta$$

$$d\hat{r} = \hat{\theta}\,d\theta$$

$$\underline{d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}}$$

$$d\vec{A} = dr\,\hat{r} \times r\,d\theta\,\hat{\theta}$$

$$d\vec{A} = \underline{r\,dr\,d\theta}\,\hat{z}$$

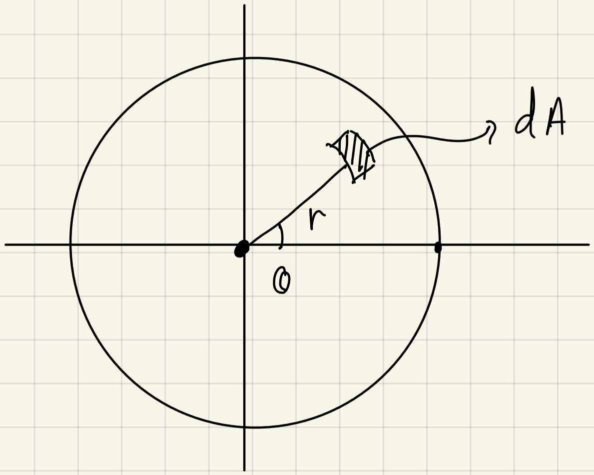
Circle with origin at the centre.

Range of r $\quad 0 \longrightarrow R$

Range of θ $\quad 0 \longrightarrow 2\pi$

for a finite plane.

Total area $\quad A = \int_0^R \int_0^{2\pi} r \, d\theta \, dr = \int_0^R \int_0^{2\pi} r \, dr \, d\theta .$



r and θ ranges are constants

Order of integration does not matter!

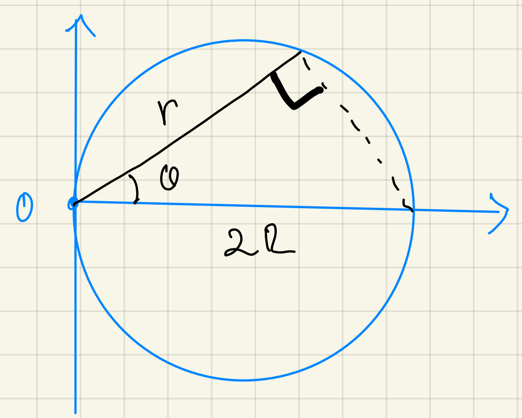
$A = \underline{\pi R^2}$

Same problem → different origin.

Origin on the rim of the circle

θ range $-\pi/2 \longrightarrow +\pi/2$

r range is not a constant!



$$r = 2R\cos\theta.$$

$$A = \int_{-\pi/2}^{\pi/2}\int_{0}^{2R\cos\theta} r\,dr\,d\theta$$

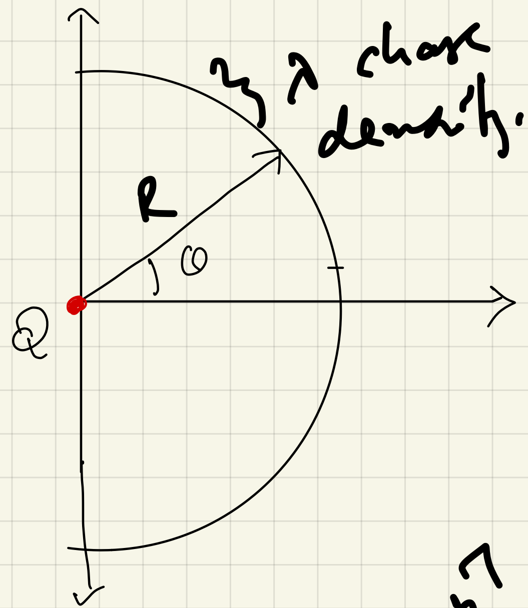
$$= 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = R^2 \int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\,d\theta$$

$$A = \pi R^2$$

When the range are not constants, **order does matter**

$\hat{r}$ and $\hat{\theta}$ are not constants!

Integration example.



$$d\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda Q\, dl}{R^2} \hat{r}$$

$dl = R\, d\theta$

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q\lambda}{R} \int_{-\pi/2}^{\pi/2} d\theta\, \hat{r}$$

$$\hat{r} = \cos\theta\, \hat{x} + \sin\theta\, \hat{y}$$

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Q\lambda}{R} \int_{-\pi/2}^{\pi/2} (\cos\theta\, \hat{x} + \sin\theta\, \hat{y})\, d\theta$$

$$\underline{\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{Q\lambda}{r} \sin\theta \Big|_{-\pi/2}^{\pi/2} \hat{x}}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

$$\vec{a} = a_r \hat{r} + a_\theta \hat{\theta}$$

<u>Kepler's Second law,</u> $a_\theta = 0$

Kepler's first law. $f(r) = -\frac{\text{constant}}{r^2}$

Equation of ellipse. polar form

$$r = \frac{a}{(1+e\cos\theta)}$$ also, $r^2\dot{\theta} = l$ constant..

$$\dot{r} = \frac{ae}{(1+e\cos\theta)^2}\sin\theta\,\dot{\theta} = \frac{le\sin\theta}{a}$$

$$\ddot{r} = \frac{le}{a}\cos\theta\,\dot{\theta} = \frac{l^2}{ar^2}\left(\frac{a}{r} - 1\right) = \underline{\frac{l^2}{r^3} - \frac{l^2}{ar^2}}$$

In general, for central force.

$$r^2\dot{\theta} = l$$

$$r\dot{\theta}^2 = \frac{l^2}{r^3}$$

$$\vec{F} = m\vec{a} = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$= m\left(\frac{l^2}{r^3} - \frac{l^2}{ar^2} - \frac{l^2}{r^3}\right)$$

$$\vec{F} = -\frac{ml^2}{a}\left(\frac{1}{r^2}\right)$$

Attraction inverse-square law.

Spherical polar coordinates

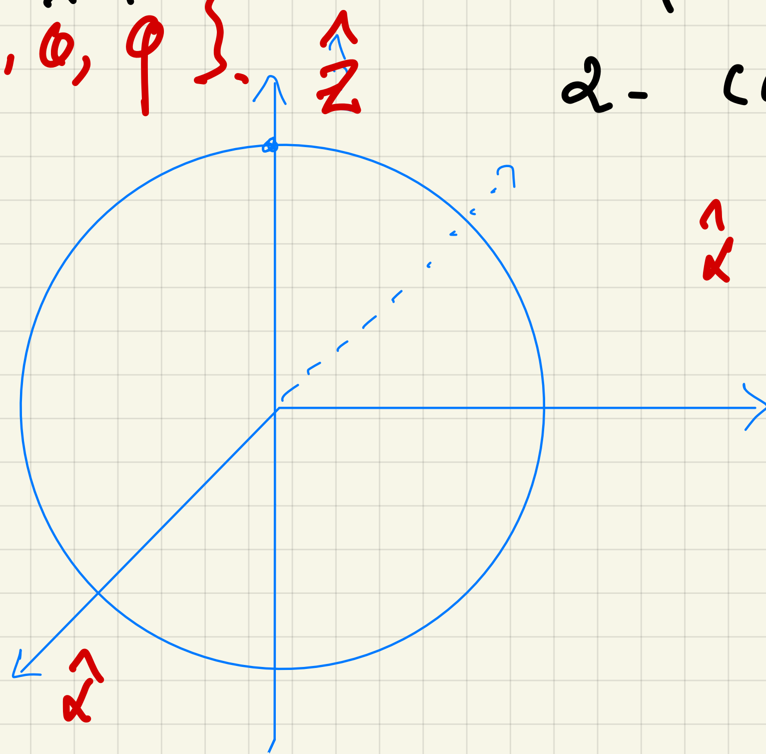
Plane polar $\Rightarrow$ 2D space.

3D - curvilinear system!

<u>Step - 1</u> Define 'constant' surfaces!

defined by 2- coordinates

Spherical polar $\{\hat{r}, \hat{\theta}, \hat{\varphi}\}$.



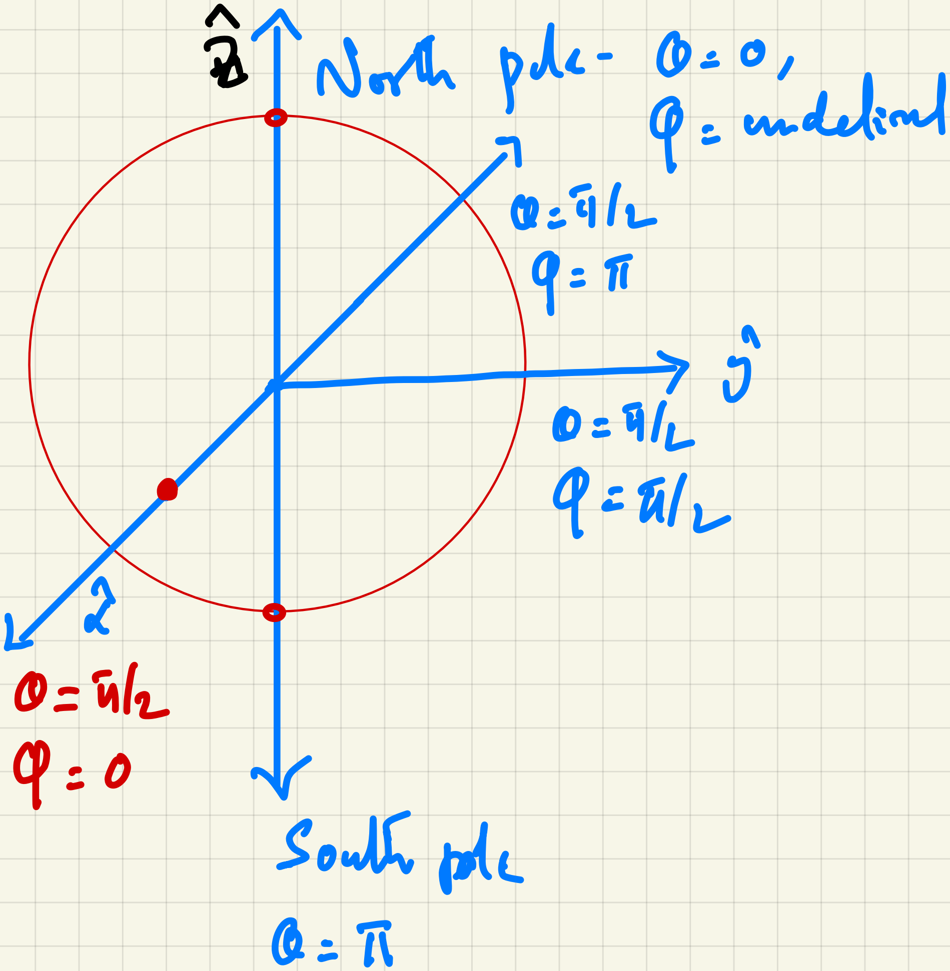
$r = r_0 \rightarrow$ Spherical surface.

$\hat{z}$ axis intersects with sphere.

$\underline{\theta = 0}$, $\varphi \rightarrow$ undefined

$\hat{x} \rightarrow$ axis intersects with sphere

$\varphi = 0,\ \theta = \pi/2$



θ range $\rightarrow$ $0 \rightarrow \pi$

ϕ range $-$ $0 \rightarrow 2\pi$

General displacement on the surface of a sphere R $\{\theta, \varphi\}$.

$$d\vec{r} = R\, d\theta\, \hat{\theta} + R \sin\theta\, d\varphi\, \hat{\varphi}$$

$$d\vec{A}_{\theta\varphi} = \underline{R^2 d\theta \sin\theta\, d\varphi\, \hat{r}}$$

Similarly,

$$d\vec{A}_{r\theta} = dr\,\hat{r} \times r\,d\theta\,\hat{\theta} = r\,dr\,d\varphi\,\hat{\varphi}$$

$$d\vec{A}_{\varphi r} = r\sin\theta\,dr\,d\varphi\,\hat{\theta}$$

Volume element:

$$dV = (dx\,\hat{x} \times dy\,\hat{y}) \cdot dz\,\hat{z}$$

Box-product (cartesian).

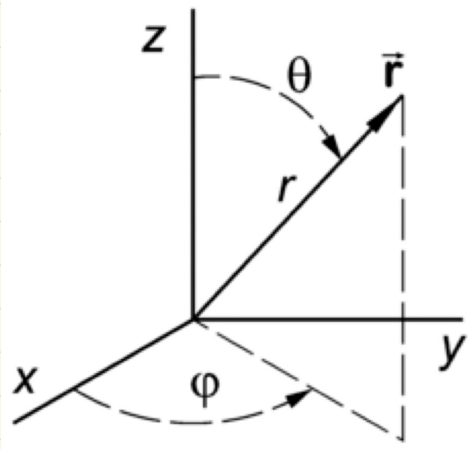
Spherical polar,

$$dV = (dr\,\hat{r} \times r\,d\theta\,\hat{\theta}) \times r\sin\theta\,d\varphi\,\hat{\varphi}$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$
\begin{aligned}
x &= r\sin\theta\cos\phi\\
y &= r\sin\theta\sin\phi\\
z &= r\cos\theta
\end{aligned}
$$

$$r^2 = x^2+y^2+z^2$$



$r\in[0,\infty]$
$\theta\in[0,\pi]$
$\phi\in[0,2\pi]$

$$\delta x = \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta + \frac{\partial x}{\partial \varphi}\delta\varphi$$ and so on...

$$\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{pmatrix}\begin{pmatrix}\delta r\\ \delta\theta\\ \delta\phi\end{pmatrix}$$

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}$$

$$\hat{r}.\hat{\theta}=\hat{r}.\hat{\phi}=\hat{\theta}.\hat{\phi}=0$$

$\hat{r} \rightarrow \delta\theta, \delta\varphi = 0$

$\hat{\theta} \rightarrow \delta r, \delta\varphi = 0$

$\hat{\varphi} \rightarrow \delta r, \delta\theta = 0$

Differentials distance $d\vec{r}$

$$\left.\begin{aligned} d\vec{r} &= dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \\ d\vec{r} &= \sum_{i=1}^{3} dx_i\,\hat{x}_i \end{aligned}\right\}$$ Cartesian.

$$ds^2 = \underset{\sim}{d\vec{r}} \cdot d\vec{r}.$$

General curvilinear coordinates. $d\vec{r} =$

$$d\vec{r} = \sum_{i=1}^{3} h_i\, du_i\, \hat{u}_i$$ $h_i =$ Scale factor.

$$d\vec{r} = h_1\, du_1\, \hat{u}_1 + h_2\, du_2\, \hat{u}_2 + h_3\, du_3\, \hat{u}_3$$

Spherical polar : $\hat{u}_1 = \hat{r}$, $\hat{u}_2 = \hat{\theta}$, $\hat{u}_3 = \hat{\varphi}$

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

$$d\vec{r} = r\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\hat{\varphi}$$

Gradient.

$$df = \vec{\nabla} f \cdot d\vec{r}.$$

$$\vec{\nabla} f = \hat{u}_1 \frac{\partial f}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial f}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial f}{h_3 \partial u_3}$$

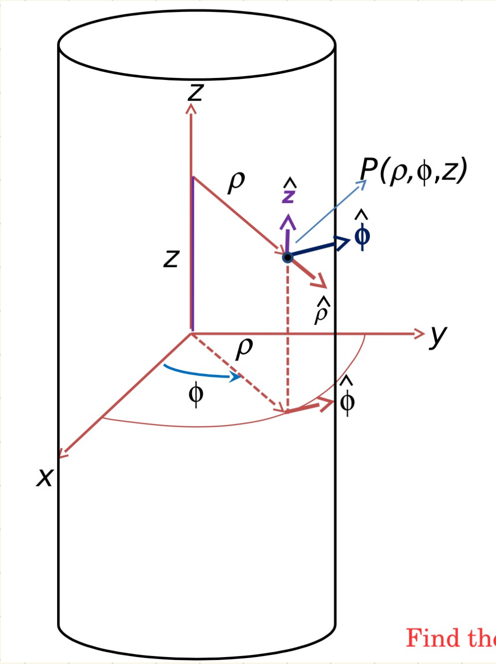
Spherical polar: $\hat{r} \dfrac{\partial f}{\partial r} + \hat{\theta} \dfrac{1}{r} \dfrac{\partial f}{\partial \theta} + \hat{\varphi} \dfrac{1}{r \sin \varphi} \dfrac{\partial f}{\partial \varphi}.$

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\varphi} \end{pmatrix} = M \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} , \quad \boldsymbol{M} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$M^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \boldsymbol{M}^T = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\varphi}} \end{pmatrix} = \dot{M} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \dot{M} M^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\varphi} \end{pmatrix}$$

$$\dot{M} M^T = ?$$



$\{\hat{\rho}, \hat{\varphi}, \hat{z}\}$. Coordinate system.

$$\begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$\delta \vec{l} = \delta\rho\,\hat{\rho} + \rho\,\delta\phi\,\hat{\phi} + \delta z\,\hat{z}$$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$dV = \rho\, d\rho\, d\phi\, dz$$

$$\vec{\nabla} T = \hat{\rho} \frac{\partial T}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial T}{\partial \varphi} + \hat{z} \frac{\partial T}{\partial z}$$

GENERAL ORTHOGONAL CURVILINEAR COORDINATES

System	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	r sin θ
Cylindrical	ρ	ϕ	z	1	ρ	1

Gradient of a scalar field

$$dT = \frac{\partial T}{\partial u_i}\delta u_i = \nabla T . d\boldsymbol{l} = [(\nabla T)_i \boldsymbol{\epsilon_i}](h_i \delta u_i \boldsymbol{\epsilon_i})$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\boldsymbol{\epsilon_1} + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\boldsymbol{\epsilon_2} + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\boldsymbol{\epsilon_3} = \frac{1}{h_i}\frac{\partial T}{\partial u_i}\boldsymbol{\epsilon_i}$$