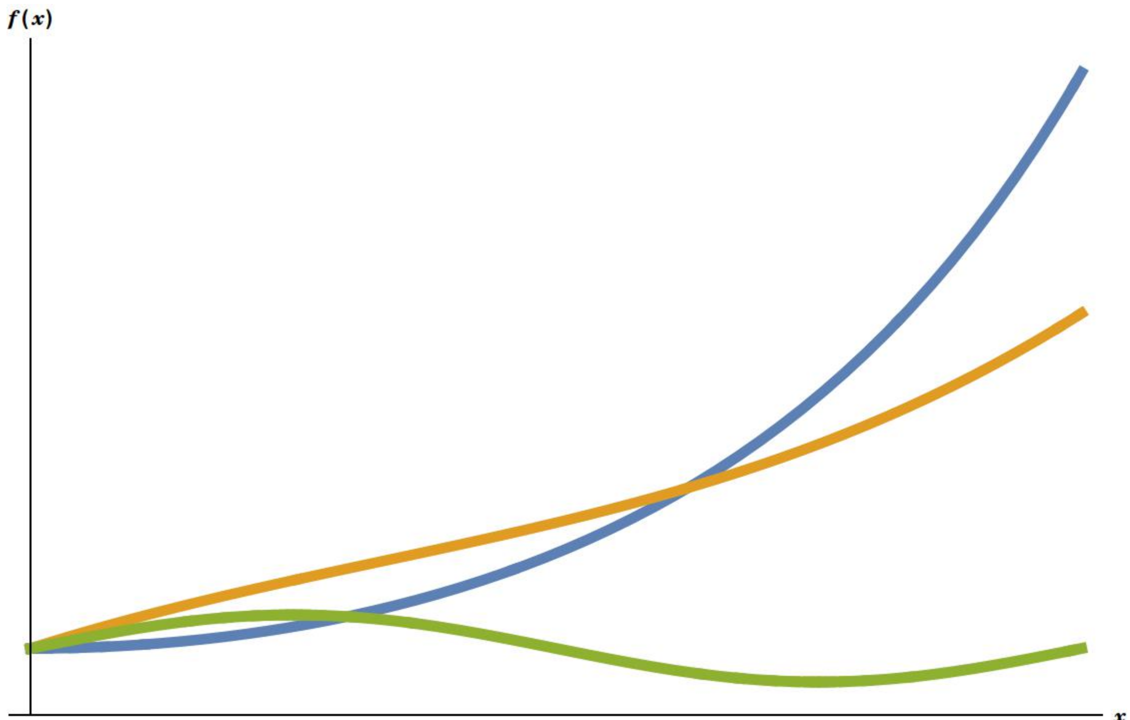
MATHEMATICAL PRELIMINARIES

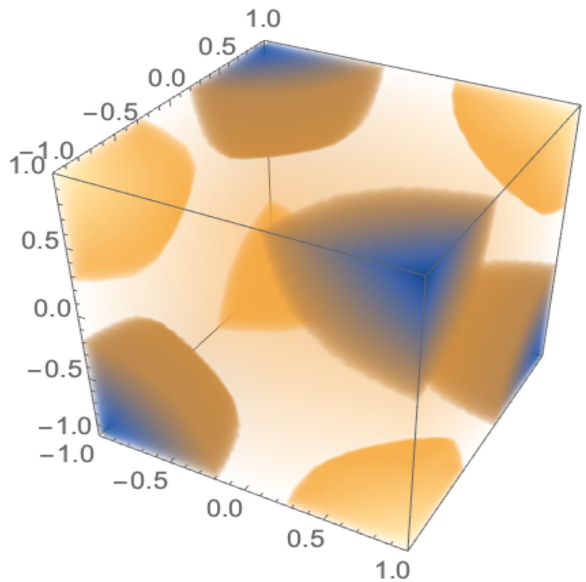
DERIVATIVE OF A FUNCTION



$$\frac{df(x)}{dx}$$

The derivative of a function measures how the function f(x) changes as we change x.

GRADIENT OF A SCALAR FIELD



Temperature T(x,y,z)

How fast does the temperature vary?

Depends on the direction we look at!

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = (\nabla T).(d\vec{l})$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}$$

Gradient of a scalar field is a vector.

In fact $\nabla T(x, y, z)$ is a vector field!

The gradient points in the direction of maximum increase of the field.

THE DEL OPERATOR

$$\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

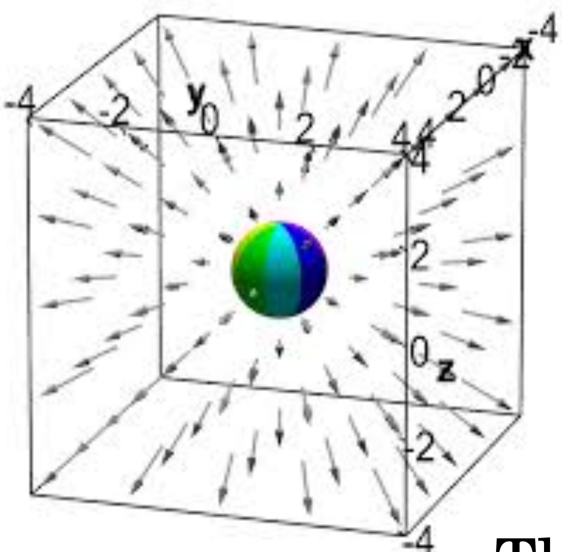
This is a ***vector* operator**. It needs to acts upon a quantity to have any meaning.

- Multiplication by a scalar ➔ Gradient of a scalar field ∇T
- Dot product with vector ➔ Divergence of a vector field $\nabla . \vec{v}$
- Cross product with vector ➔ Curl of a vector field $\nabla \times \vec{v}$

DIVERGENCE OF A VECTOR FIELD

$$\nabla . \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) . \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z}\right)$$

$$\nabla . \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

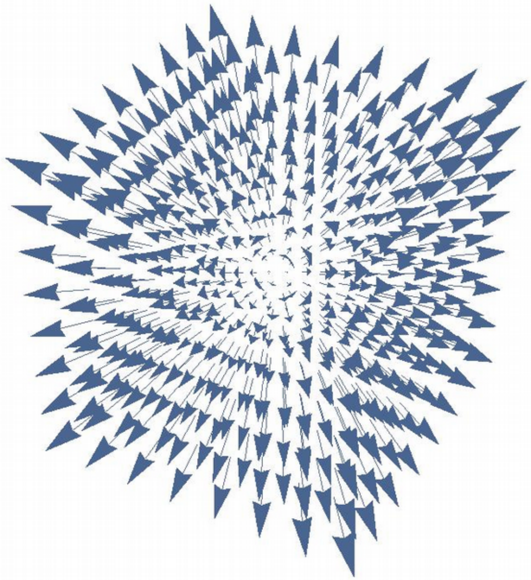


The divergence is a measure of how much the vector spreads out from the point in question.

The divergence of a vector is a scalar quantity.

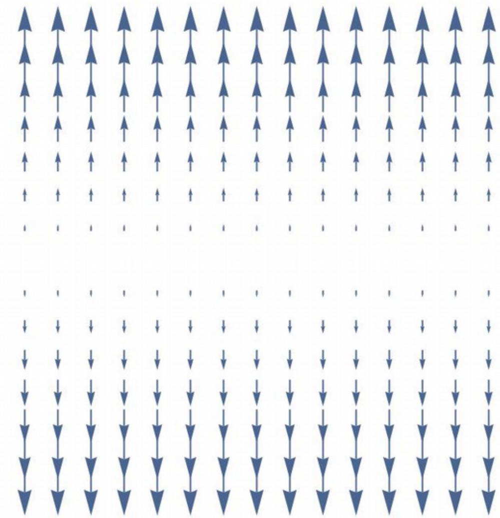
A point of positive divergence is a source, a point of negative divergence is a sink.

DIVERGENCE OF A VECTOR FIELD



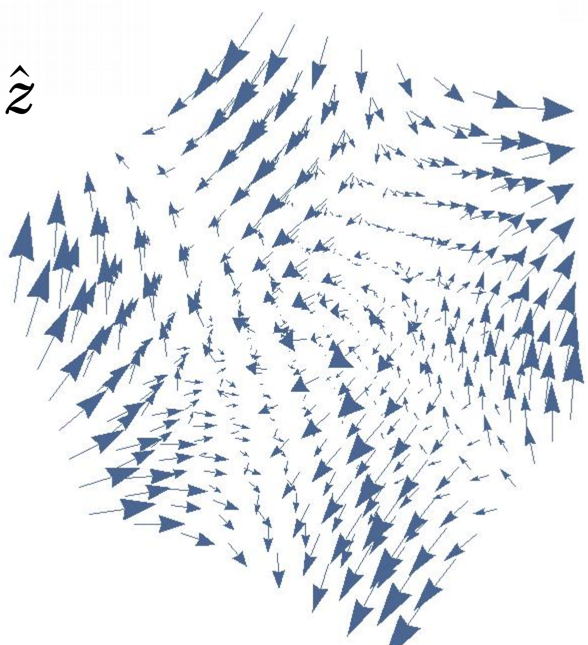
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla . \vec{v} = 3$



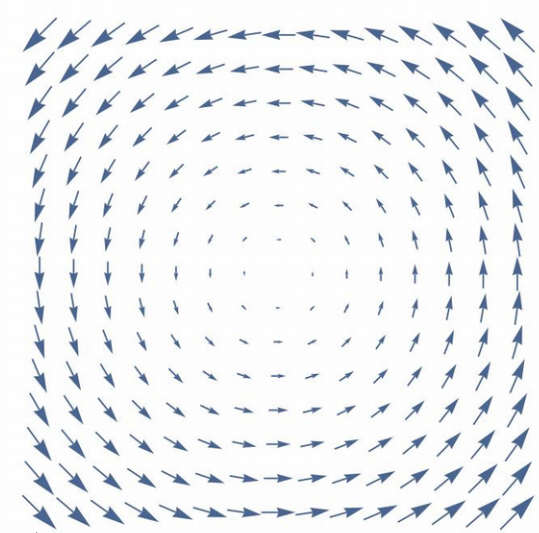
$\vec{v} = z\,\hat{z}$

$\nabla . \vec{v} = 1$



$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$

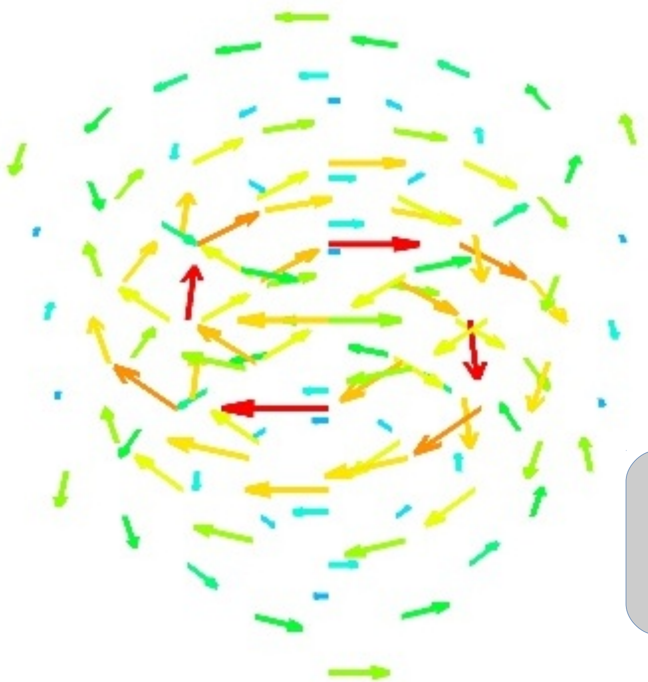
$\nabla . \vec{v} = y + 2\,z + 3\,x$



$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla . \vec{v} = 0$

CURL OF A VECTOR FIELD



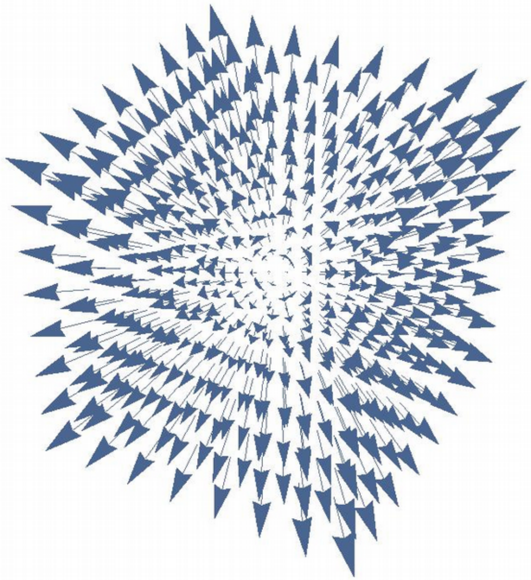
$$\nabla\times\vec{v}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

$$\nabla\times\vec{v}=\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

The curl of a vector is a measure of how much the vector curls around the point in question.

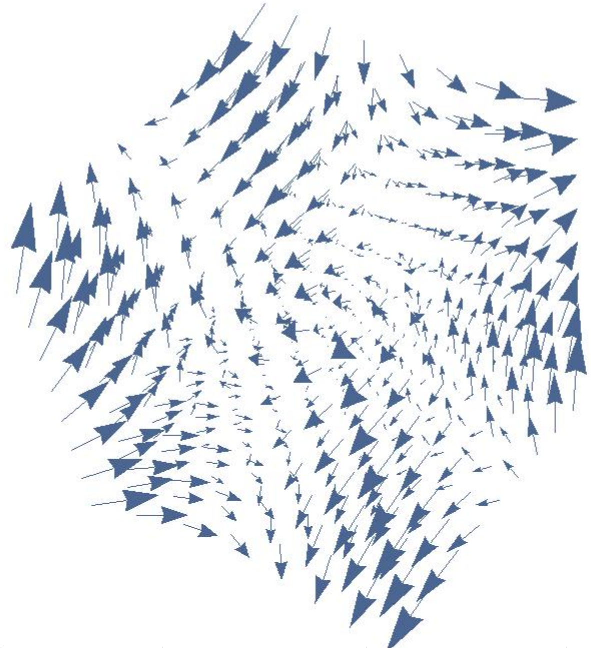
The curl of a vector is a vector itself.

CURL OF A VECTOR FIELD



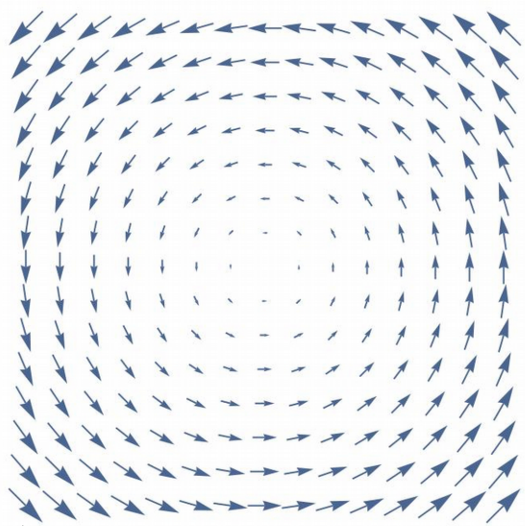
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla \times \vec{v} = 0$



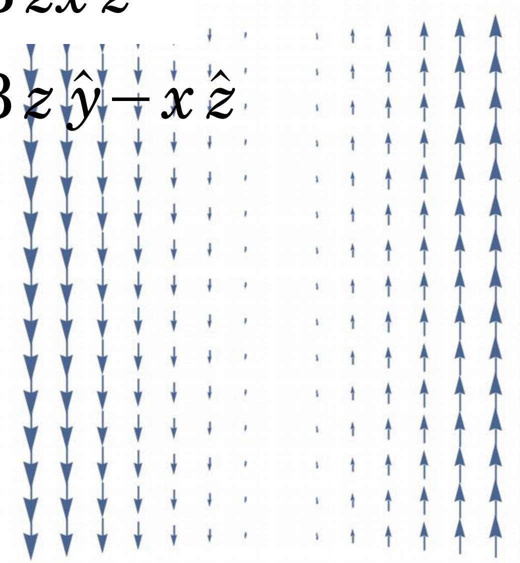
$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$

$\nabla \times \vec{v} = -2\,y\,\hat{x} - 3\,z\,\hat{y} - x\,\hat{z}$



$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla \times \vec{v} = 2\,\hat{z}$



$\vec{v} = x\,\hat{y}$

$\nabla \times \vec{v} = \hat{z}$

SUM RULES

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla.(\vec{A}+\vec{B})=(\nabla.\vec{A})+(\nabla.\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

MULTIPLICATION BY A CONSTANT

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$

$$\nabla.(k\vec{A})=k(\nabla.\vec{A})$$

$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

PRODUCT RULES

$$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$$
$$+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\nabla.(f\vec{A})=f(\nabla.\vec{A})+\vec{A}.(\nabla f)$$

$$\nabla.(\vec{A}\times\vec{B})=\vec{B}.(\nabla\times\vec{A})-\vec{A}.(\nabla\times B)$$

$$\nabla\times(f\vec{A})=f(\nabla\times\vec{A})-\vec{A}\times(\nabla f)$$

$$\nabla\times(\vec{A}\times\vec{B})=(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$$
$$+\vec{A}(\nabla.\vec{B})-\vec{B}(\nabla.\vec{A})$$

GRADIENT OF DOT PRODUCT OF TWO VECTORS - PROOF

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B})=\hat{x}\left\{A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$\vec{B}\times(\nabla\times\vec{A})=\hat{x}\left\{B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$(\vec{A}.\nabla)\vec{B}=\hat{x}\left\{\left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)B_x\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$(\vec{B}.\nabla)\vec{A}=\hat{x}\left\{\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)A_x\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$\Rightarrow \text{RHS}=\hat{x}\left\{A_y\frac{\partial B_y}{\partial x}-A_y\cancel{\frac{\partial B_x}{\partial y}}-A_z\cancel{\frac{\partial B_x}{\partial z}}+A_z\frac{\partial B_z}{\partial x}+B_y\frac{\partial A_y}{\partial x}-B_y\cancel{\frac{\partial A_x}{\partial y}}-B_z\cancel{\frac{\partial A_x}{\partial z}}+B_z\frac{\partial A_z}{\partial x}\right.$$

$$\left.+A_x\frac{\partial B_x}{\partial x}+A_y\cancel{\frac{\partial B_x}{\partial y}}+A_z\cancel{\frac{\partial B_x}{\partial z}}+B_x\frac{\partial A_x}{\partial x}+B_y\cancel{\frac{\partial A_x}{\partial y}}+B_z\cancel{\frac{\partial A_x}{\partial z}}\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$=\hat{x}\left\{\frac{\partial}{\partial x}(A_xB_x+A_yB_y+A_zB_z)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}=\nabla(\vec{A}.\vec{B})$$

SECOND DERIVATIVES

Divergence of a gradient $\nabla.(\nabla T)$

Curl of a gradient $\nabla\times(\nabla T)$

Gradient of a divergence $\nabla(\nabla.\vec{v})$

Divergence of curl $\nabla.(\nabla\times\vec{v})$

Curl of curl $\nabla\times(\nabla\times\vec{v})$

SECOND DERIVATIVES

Divergence of a gradient $\nabla.(\nabla T)$

$$\nabla.(\nabla T)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right).\left(\frac{\partial T}{\partial x}\hat{x}+\frac{\partial T}{\partial y}\hat{y}+\frac{\partial T}{\partial z}\hat{z}\right)$$

$$=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}$$

Laplacian: $\nabla^2 T\equiv\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}$

Laplacian of a vector $\nabla^2\vec{v}=(\nabla^2 v_x)\hat{x}+(\nabla^2 v_y)\hat{y}+(\nabla^2 v_z)\hat{z}$

SECOND DERIVATIVES

Gradient of a divergence $\nabla(\nabla.\vec{v})$

$$\nabla(\nabla.\vec{v})=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right).\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

$$=\hat{x}\left(\frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{y}\left(\frac{\partial}{\partial y}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial z}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$\nabla^2\vec{v}\neq\nabla(\nabla.\vec{v})$$

The Laplacian of a vector is not the same as gradient of the divergence.

SECOND DERIVATIVES

Curl of curl $\nabla\times(\nabla\times\vec{v})$

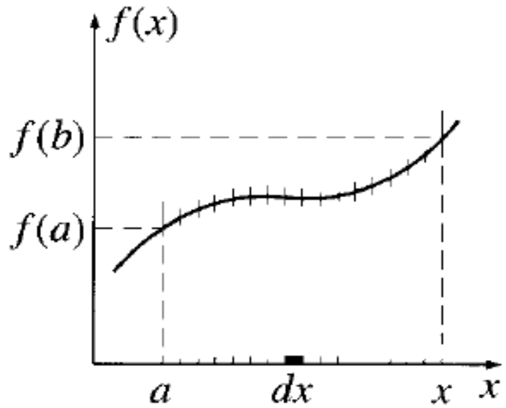
$$\nabla\times(\nabla\times\vec{v})=\nabla\times\left(\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right)$$

$$=\hat{x}\left\{\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$=\hat{x}\left\{\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)-\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\right\}+\ldots$$

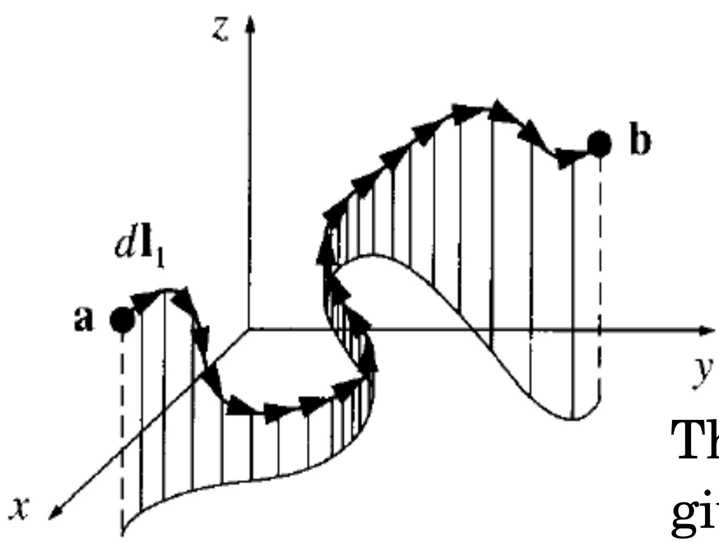
$$\nabla\times(\nabla\times\vec{v})=\nabla(\nabla.\vec{v})-\nabla^2\vec{v}$$

FUNDAMENTAL THEOREM OF GRADIENTS



$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

The integral of a derivative over an interval is the value of the function at the end points or boundaries.



Scalar field T(x,y,z) $\quad dT = (\nabla T).(d\vec{l})$

$$\int_{a\,\mathcal{P}}^{b} (\nabla T).(d\vec{l}) = T(b) - T(a)$$

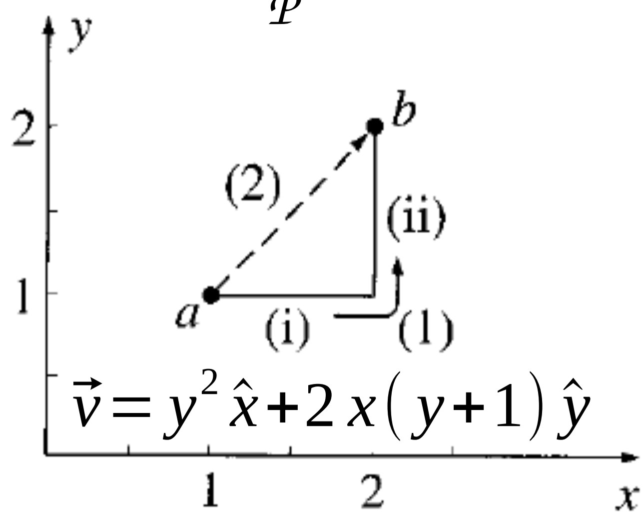
The line integral of a gradient of a scalar field is given by the value of the function at the boundaries.

Path Independence - Is this obvious?

LINE INTEGRALS

For any general vector field $\vec{v}$

$$\int_{a \, \mathcal{P}}^{b} \vec{v}.(d\vec{l})$$ depends on the path $\mathcal{P}$!



$$\vec{v} = y^2\,\hat{x} + 2x(y+1)\,\hat{y}$$

For a gradient field, $\oint (\nabla T).(d\vec{l}) = 0$

If $\vec{v} \equiv \nabla T$, then $\vec{v}$ is a conservative field

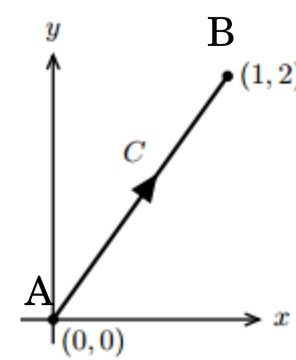
Electric field **E** is conservative.

$$\int_{a \; 1}^{b} \vec{v}.(d\vec{l}) = 11$$

$$\int_{a \; 2}^{b} \vec{v}.(d\vec{l}) = 10$$

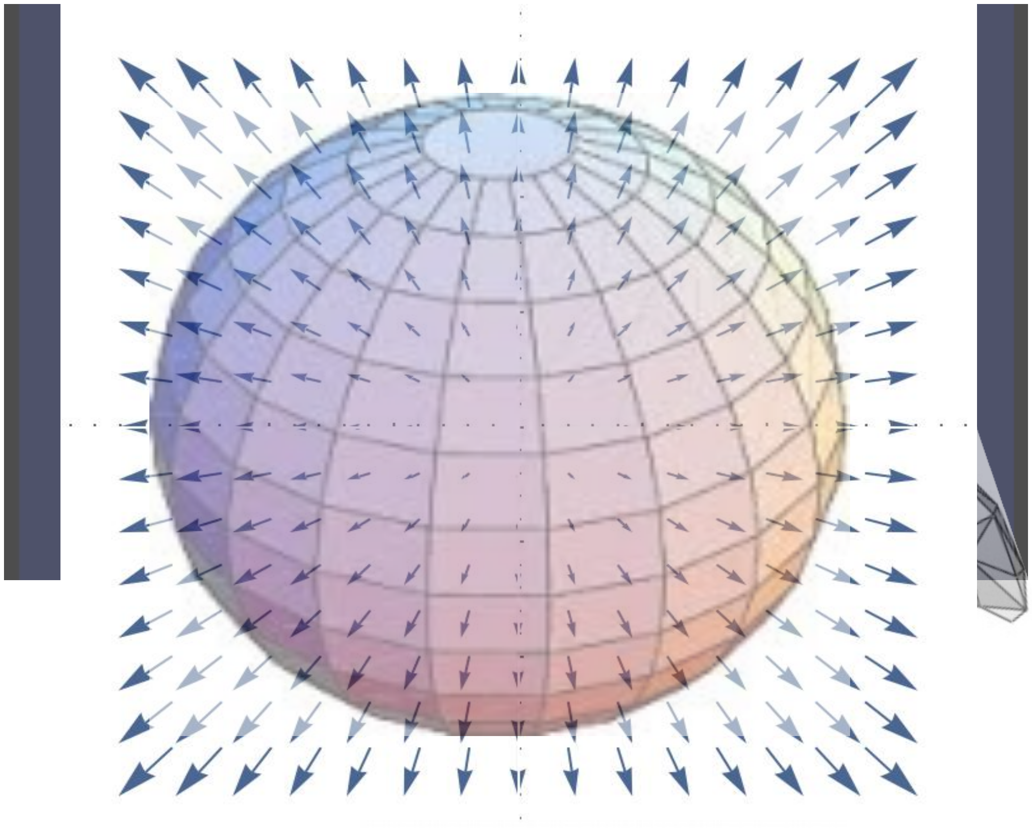
$$\Rightarrow \oint \vec{v}.(d\vec{l}) \neq 0$$

$$f(x,y) = xy^3 + xy^2$$



$$\int_{A}^{B} \nabla f.(d\vec{l}) = ?$$

FLUX OF A VECTOR FIELD



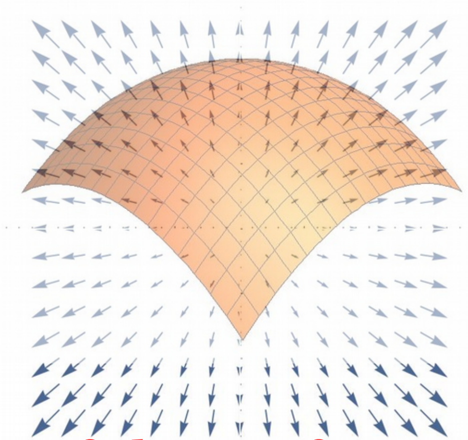
The volume of water flowing out through the surface per unit time,

Flux $\oiint \vec{v} \, . \, d\vec{S}$

$d\vec{S} \Rightarrow$ Infinitesimal area element, direction perpendicular to surface

$\oiint \Rightarrow$ Integral over a closed surface

Sign convention: Outward is positive



For the surface integral over an open surface

$$\iint \vec{v} \, . \, d\vec{S}$$

the sign is arbitrary

Does the value of the surface integral depend on the surface? Yes, except...

GAUSS'S THEOREM / DIVERGENCE THEOREM

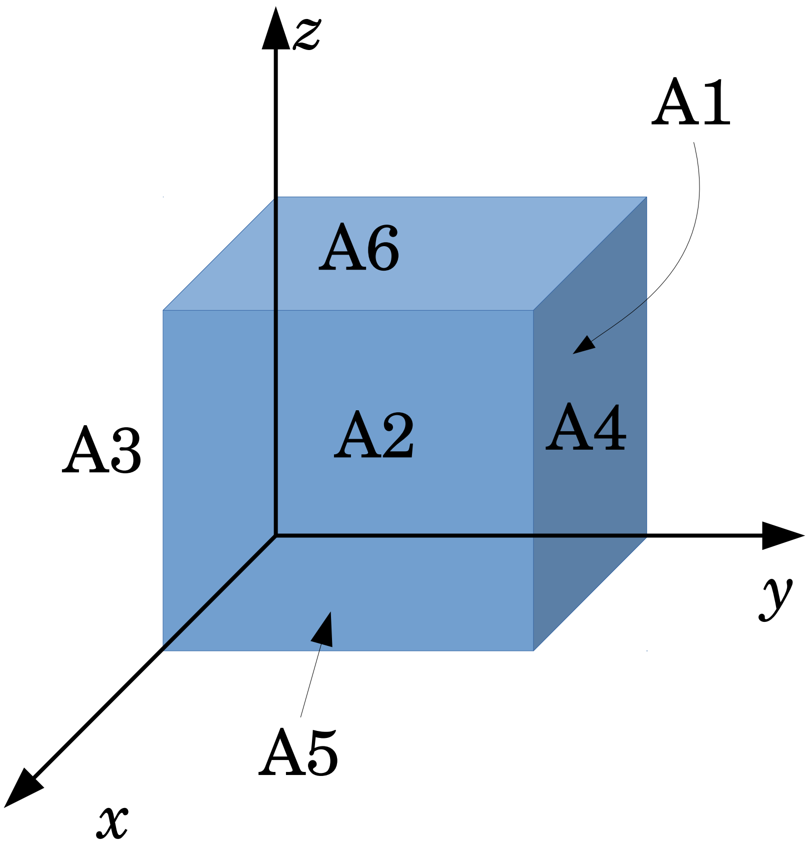
$$\oiiint_{V} (\nabla . \vec{v}) \; dV = \oiint_{S} \vec{v} . d\vec{S}$$

where, S is the surface bounding a volume V

The volume integral of the divergence of a function over a region is equal to the value of the function at the boundary (the surface enclosing the volume).

RHS measures the flux through the closed surface S.
LHS counts the sources and sinks within the region bounded by S

GAUSS'S THEOREM / DIVERGENCE THEOREM



$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad e \leqslant z \leqslant f$$

$$\iint_{A_1} \vec{v} . d\vec{S} + \iint_{A_2} \vec{v} . d\vec{S}$$

$$= -\int_e^f \int_c^d v_x(a, y, z)\, dy\, dz + \int_e^f \int_c^d v_x(b, y, z)\, dy\, dz$$

$$= \int_e^f \int_c^d \left(v_x(b, y, z) - v_x(a, y, z) \right) dy\, dz$$

$$= \int_e^f \int_c^d \int_a^b \left(\frac{\partial v_x}{\partial x} dx \right) dy\, dz = \oiiint_V \frac{\partial v_x}{\partial x} dV$$

Repeating over the other 4 sides and adding,

$$\oiint_S \vec{v} . d\vec{S} = \oiiint_V (\nabla . \vec{v})\ dV$$

STOKES THEOREM

$$\iint_S (\nabla \times \vec{v}) . d\vec{S} = \oint_{\wp} \vec{v} . d\vec{l}$$

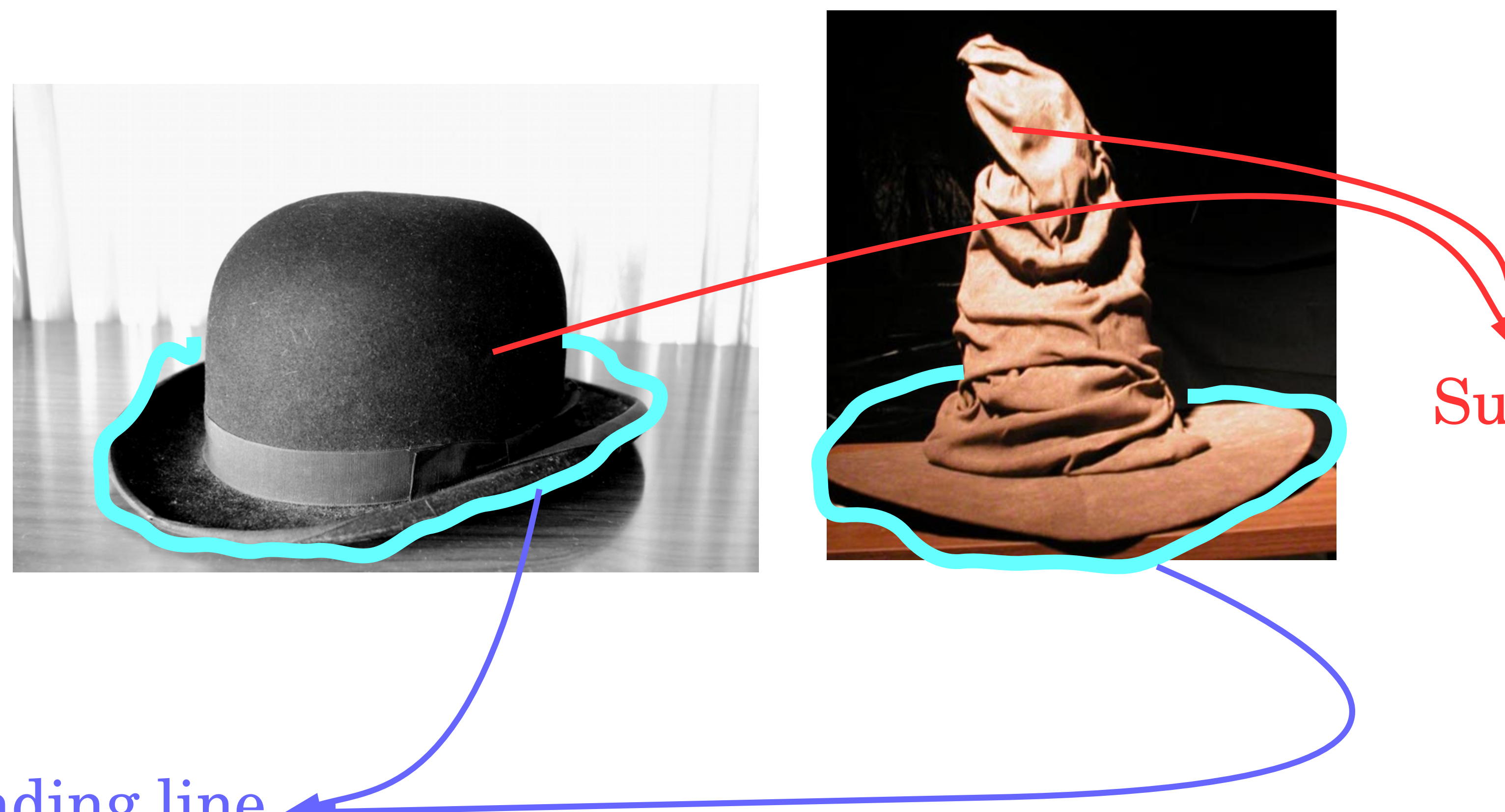
The flux of the curl of a vector through a surface S is equal to the closed line integral of the vector function over the bounding line of the surface.

What is the sign of the line integral?
What is the sign of the surface integral?

Right hand rule: If the fingers point in the direction of the line integral, the thumb fixes the direction of $d\vec{S}$

STOKES THEOREM

The same bounding line can enclose many surfaces!



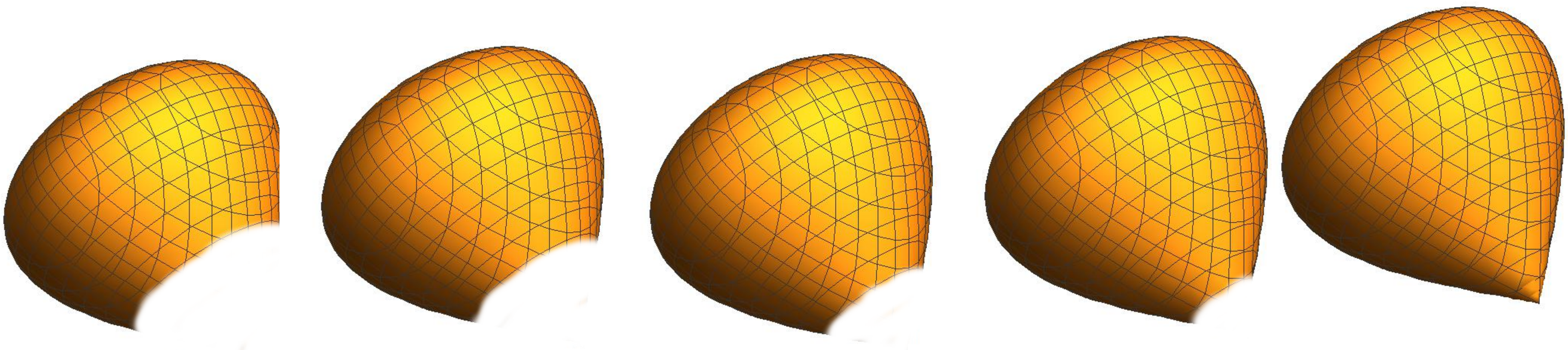
Surface

Bounding line

The surface integral can be over ***ANY*** surface that shares a common bounding line!

STOKES THEOREM

What happens if we shrink the boundary line?



$$\oiint_S (\nabla \times \vec{v}).d\vec{S} = 0$$

COROLLARIES

$\nabla\times\nabla\varphi=?$

$$\iint_S(\nabla\times\vec{v}).d\vec{S}=\oint\vec{v}.d\vec{l}$$ Stokes Theorem

If, $\vec{v}=\nabla\varphi$

$$\oint\nabla\varphi.d\vec{l}=0$$ Gradient Theorem

Combining, we get,

$$\iint_S(\nabla\times\nabla\varphi).d\vec{S}=0$$ for any surface S

$$\Rightarrow\nabla\times\nabla\varphi=0$$ for any scalar function φ

$\nabla.\nabla\times\vec{A}=?$

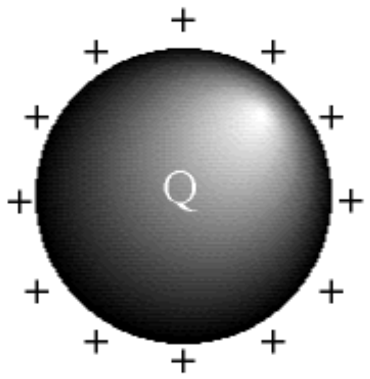
$$\oiint_S(\nabla\times\vec{A}).d\vec{S}=0$$ Stokes Theorem

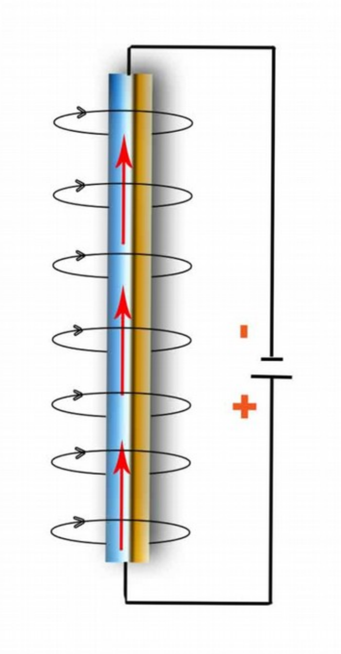
$$\oiiint\nabla.(\nabla\times\vec{A})dV=\oiint(\nabla\times\vec{A}).d\vec{S}=0$$ Gauss's Theorem

$$\Rightarrow\nabla.(\nabla\times\vec{A})=0$$ for any vector function $\vec{A}$

CURVILINEAR COORDINATE SYSTEMS

The symmetries of a problem can dictate the most efficient *choice* of the coordinate system.





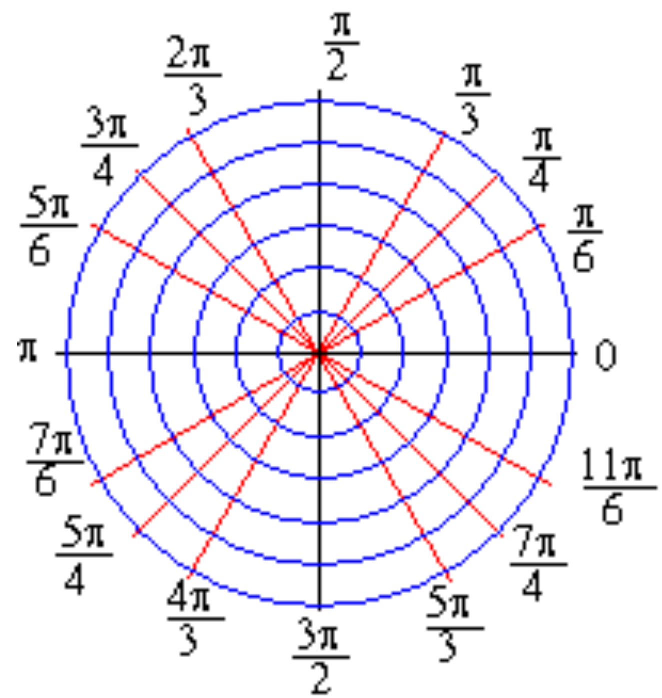
Any coordinate system will do for any problem, however, some coordinate systems will be easier!

PLANE POLAR COORDINATES

STEP 1: Write down the relation with (x,y) co-ordinates

$$x = r \cos\theta$$
$$y = r \sin\theta$$

STEP 2: Draw the coordinate grid



How do r=constant lines look?
How do θ= constant lines look?

STEP 3: What happens when the independent variables are changed infinitesimally?

$$\delta x = \cos\theta\, \delta r - r \sin\theta\, \delta\theta$$
$$\delta y = \sin\theta\, \delta r + r \cos\theta\, \delta\theta$$

PLANE POLAR COORDINATES

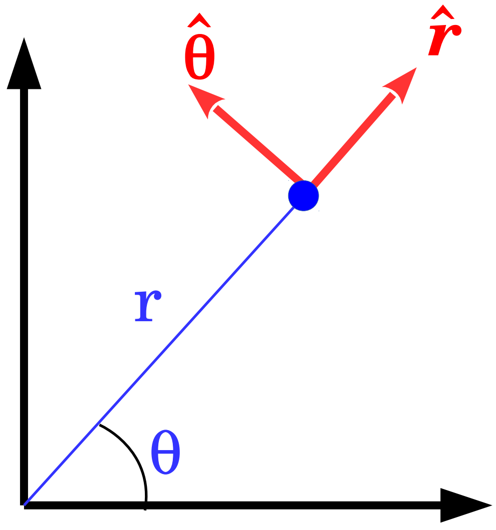
STEP 4: Which direction would we move, if only one variable was changed?

$\underline{\delta\theta=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y=(\hat{x}\cos\theta+\hat{y}\sin\theta)\,\delta r=\hat{\boldsymbol{r}}\,\delta r$$

$\underline{\delta r=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y=(-\hat{x}\sin\theta+\hat{y}\cos\theta)\,r\,\delta\theta=\hat{\boldsymbol{\theta}}\,r\,\delta\theta$$



$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\hat{r}\,.\,\hat{\theta}=(-\cos\theta\sin\theta)+(\sin\theta\cos\theta)=0$$

Curvilinear, but still orthogonal

PLANE POLAR COORDINATES

STEP 5: What happens to an element of distance, or arclength?

$$d\vec{l} = \delta r \, \hat{r} + r \, \delta\theta \, \hat{\theta}$$

Compare with the Cartesian case, $d\vec{l} = \delta x \, \hat{x} + \delta y \, \hat{y}$

Can it be $d\vec{l} = \delta r \, \hat{r} + \delta\theta \, \hat{\theta}$? Why not?

In general, for a curvilinear coordinate system, the scale factors are not unity!

Scale factor gives a measure of how much a change in the coordinate changes the position of a point.

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} = \sqrt{r^2 \sin^2\theta + r^2 \cos^2\theta} = r$$

$$ds^2 = d\vec{l} \, . \, d\vec{l} = \delta r^2 + r^2 \delta\theta^2$$

There are no cross terms in the arclength expression $O(\delta r \, \delta\theta)$ for orthogonal coordinate systems.

PLANE POLAR COORDINATES

STEP 6: What happens to the element of area?

We take a small step in the $\hat{r}$ direction, and a small step in the $\hat{\theta}$ direction. What is the infinitesimal area enclosed by these two perpendicular vectors?

$$d l_r = \delta r \qquad d l_\theta = r\,\delta\theta$$

$$dA = d l_r\, d l_\theta = r\,\delta\theta\,\delta r$$

STEP 7: What is the gradient?

$$\begin{aligned} dT &= \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta \\ &= [some \quad fn].\, d\vec{l} \\ &= [\nabla T].(\delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}) \end{aligned}$$

$$\boxed{\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta}}$$

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$$

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

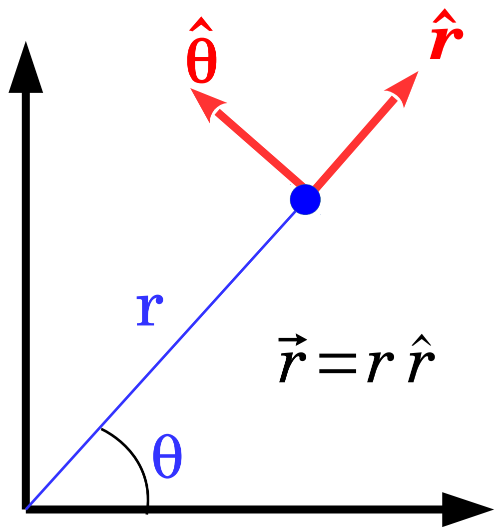
$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}$$

PLANE POLAR COORDINATES

STEP 8: What are the velocity components, when a particle's motion is described using polar coordinates?

$$\vec{v} = \frac{d}{dt}(r\,\hat{r}) = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$



The unit vectors here are not constant, unlike for cartesian coordinates, and must be themselves differentiated.

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\begin{pmatrix}\dot{\hat{r}}\\ \dot{\hat{\theta}}\end{pmatrix} = \dot{\theta}\begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}$$

$$\Rightarrow \vec{v} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$

$v_r = \dot{r} \rightarrow$ *radial* velocity component

$v_\theta = r\,\dot{\theta} \rightarrow$ *circumferential* velocity component

$$\vec{v} = \frac{\delta\vec{l}}{\delta t} = \frac{\delta}{\delta t}(\delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}) = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

PLANE POLAR COORDINATES

STEP 9: What are the acceleration components, when a particle's motion is described using polar coordinates?

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)$$

$$= \dot{\theta}\;\dot{r}\,\hat{\theta} + \ddot{r}\,\hat{r} - \dot{\theta}\,r\,\dot{\theta}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta}\text{¿}$$

$$= \left(\ddot{r} - \dot{\theta}^2 r\right)\hat{r} + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\theta}$$

Radial acceleration Circumferential acceleration

If the force on the particle is central, then which quantity is conserved?

What can you say about the matrix connecting the two sets and the inverse relation ?

CURVILINEAR COORDINATE SYSTEMS - PRESCRIPTION

Write down the relation with (x,y) co-ordinates

Draw the coordinate grid

What happens when the independent variables are changed infinitesimally?

Which direction would we move, if only one variable was changed?

What happens to an element of distance?

What happens to an element of area?

What is the gradient?

What are the velocity components?

What are the acceleration components?

You’re now set to solve problems in this coordinate system