SUMMARY OF LECTURE-1

1. Del operator $\rightarrow$ $\nabla = \left\{ \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right\}$

operating on scalar (T) $\quad \vec{\nabla} T \rightarrow$ Acts as a vector.

2. operating on vectors $\rightarrow$ $\vec{\nabla}\cdot\vec{v}$ and $\vec{\nabla}\times\vec{v}$

Divergence and curl

Sum rules, product rules,

3. Second derivatives:

on $\vec{\nabla} T$,
$$\vec{\nabla}\cdot(\vec{\nabla}T) = \nabla^2 T$$
$$\cancel{\vec{\nabla}\times(\vec{\nabla}T)}$$

on $\vec{v}$
$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$
$$\cancel{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})}$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}),$$

$$\vec{\nabla}\cdot\vec{A} = 0$$
$$\vec{A} = (\vec{\nabla}\times\vec{B})$$

Mathematical preliminaries.

1. **Derivative**

$f \equiv f(x)$ function of x.

$\frac{df}{dx} \rightarrow$ How much f changes with an infinitesimal change in x.

Scalar field $T \equiv T(x, y, z)$.

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}$ partial derivatives

$$dT = \left(\hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z}\right) \cdot \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \vec{\nabla} T \cdot d\vec{l}$$

where $\vec{\nabla} T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z}$ is the gradient of a scalar field.

Gradient of a scalar field = vector field.

$dT = |\vec{\nabla} T| \, |d\vec{l}| \cos\theta$. for a fixed $|d\vec{l}|$

dT is maximum for $\theta = 0$.

$\vec{\nabla} T$ = direction of maximum increase in scalar field.

Gradient operator

$$\nabla = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)$$ Vector operator

$r = \sqrt{x^2 + y^2 + z^2}$

$\vec{\nabla} r = ?$

Three typs of operation

1) operate on a scalar fied $\rightarrow$ Gradient: $\vec{\nabla} T$

2) Scalar product with another vector $\rightarrow$ Divergeme. $\vec{\nabla} \cdot \vec{v}$

3) Cross product with another vector fied $\rightarrow$ Curl. $\vec{\nabla} \times \vec{v}$

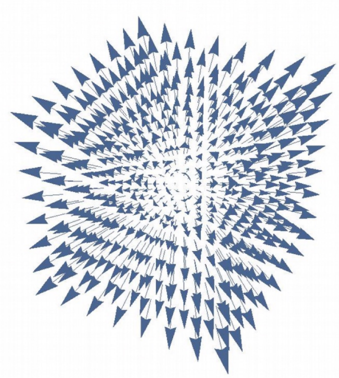
Divergeme: $\vec{v} = (v_x \hat{x} + v_y \hat{y} + v_z \hat{y})$

$$\vec{\nabla} \cdot \vec{v} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \right)$$

Scalar.

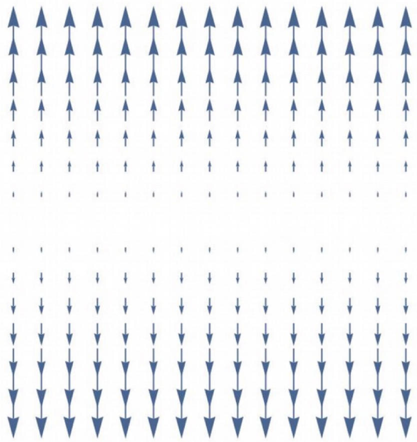
$\vec{v} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{\nabla} \cdot \vec{v} = \underline{3}$



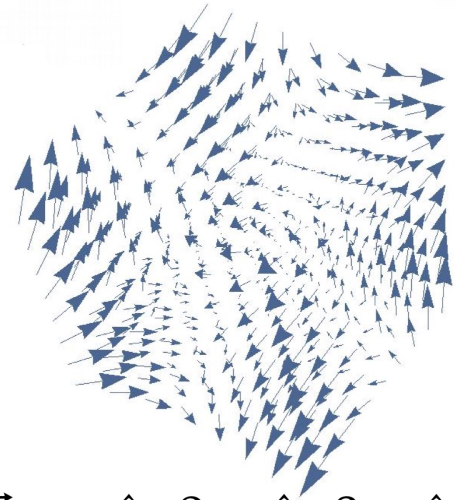
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla . \vec{v} = 3$



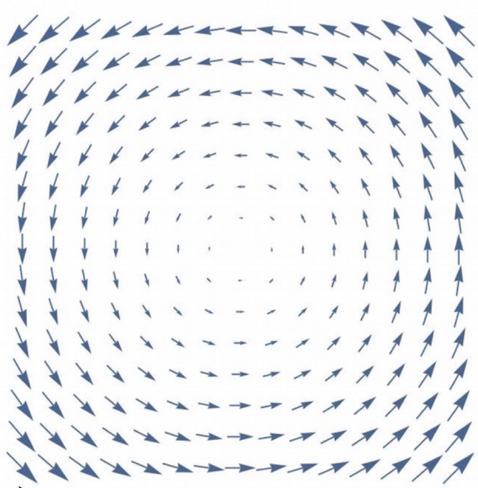
$\vec{v} = z\,\hat{z}$

$\nabla . \vec{v} = 1$



$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$

$\nabla . \vec{v} = y + 2\,z + 3\,x$



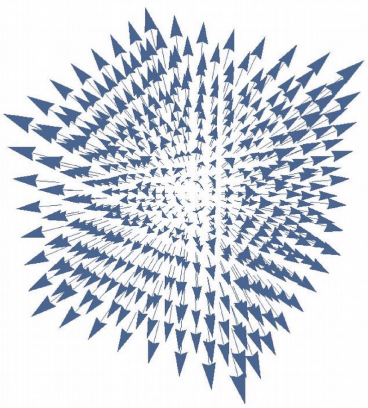
$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla . \vec{v} = 0$

The curl of a vector field.

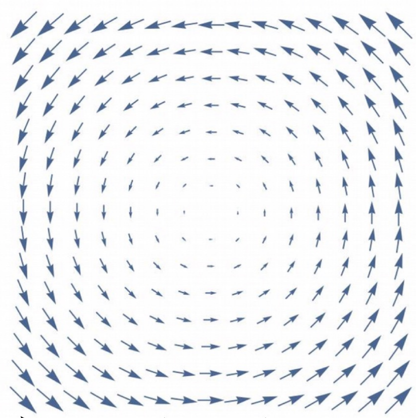
$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



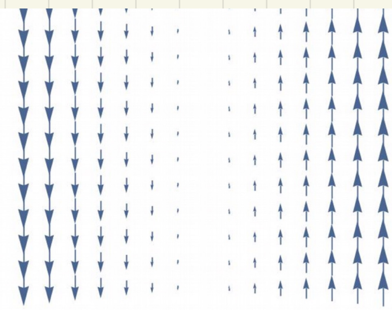
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla \times \vec{v} = 0$



$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla \times \vec{v} = 2\,\hat{z}$



$\vec{v} = x\,\hat{y}$

$\vec{\nabla} \times \vec{v} = \hat{z}$

How much a vector field curls around a point.

SUM RULES

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla.(\vec{A}+\vec{B})=(\nabla.\vec{A})+(\nabla.\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

MULTIPLICATION BY A CONSTANT

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$

$$\nabla.(k\vec{A})=k(\nabla.\vec{A})$$

$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

PRODUCT RULES

$$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\nabla.(f\vec{A})=f(\nabla.\vec{A})+\vec{A}.(\nabla f)$$

$$\nabla.(\vec{A}\times\vec{B})=\vec{B}.(\nabla\times\vec{A})-\vec{A}.(\nabla\times B)$$

$$\nabla\times(f\vec{A})=f(\nabla\times\vec{A})-\vec{A}\times(\nabla f)$$

$$\underline{\nabla\times(\vec{A}\times\vec{B})}=(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}+\vec{A}(\nabla.\vec{B})-\vec{B}(\nabla.\vec{A})$$

$$(\vec{B}\cdot\vec{\nabla})\vec{A}$$

$$\left(B_x\frac{\partial}{\partial x},\ B_y\frac{\partial}{\partial y},\ B_z\frac{\partial}{\partial z}\right)$$

$$(A_x, A_y, A_z)$$

$$\vec\nabla\cdot(f\vec A) = f(\vec\nabla\cdot\vec A) + \vec A\cdot(\vec\nabla f)$$

$$\left(\hat x\frac{\partial}{\partial x} + \hat y \frac{\partial}{\partial y} + \hat z\frac{\partial}{\partial z}\right)\cdot\left(\hat x f A_x + \hat y (f A_y) + \hat z (f A_z)\right)$$

$$\frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y)+\frac{\partial}{\partial z}(fA_z)$$

$$= \frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x} + \frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y} + A_z\frac{\partial f}{\partial z} + f\frac{\partial A_z}{\partial z}.$$

$$= \vec\nabla f\cdot\vec A + f(\vec\nabla\cdot\vec A).$$

$$\vec\nabla(\vec A\cdot\vec B) = \vec A\times(\vec\nabla\times\vec B)+\vec B\times(\vec\nabla\times\vec A) + (\vec A\cdot\vec\nabla)\vec B + (\vec B\cdot\vec\nabla)\vec A.$$

X- component.

$$[\vec\nabla(\vec A\cdot\vec B)]_x = \frac{\partial}{\partial x}(A_xB_x + A_yB_y + A_zB_z)$$

$$[\vec A\times(\vec\nabla\times\vec B)]_x = A_y(\nabla\times\vec B)_z - A_z(\vec\nabla\times\vec B)_y.$$

$$= A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)$$

$$[\vec B\times(\vec\nabla\times\vec A)]_x = B_y(\vec\nabla\times\vec A)_z - B_z(\vec\nabla\times A)_y$$

$$= B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial}{\partial y}A_x\right) - B_z\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right)$$

$$[(\vec A\cdot\vec\nabla)\vec B]_x = \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)B_x = A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}$$

$$\left[\vec{\nabla}(\vec{A}\cdot\vec{B})\right]_x = \frac{\partial}{\partial x}\left(A_x B_x + A_y B_y + A_z B_z\right)$$

$$\left[\vec{A}\times(\vec{\nabla}\times\vec{B})\right]_x = A_y\,(\nabla\times\vec{B})_z - A_z\,(\vec{\nabla}\times\vec{B})_y$$

$$= A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$

$$\left[\vec{B}\times(\vec{\nabla}\times\vec{A})\right]_x = B_y\,(\vec{\nabla}\times\vec{A})_z - B_z\,(\vec{\nabla}\times A)_y$$

$$= B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial}{\partial y}A_x\right) - B_z\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right)$$

$$\left[(\vec{A}\cdot\vec{\nabla})\vec{B}\right]_x = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x =$$

$$A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}$$

$$\left[(\vec{B}\cdot\vec{\nabla})\vec{A}\right]_x = B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}$$

$$\left.\begin{aligned}
&A_y\frac{\partial B_y}{\partial x} - \cancel{A_y\frac{\partial B_x}{\partial y}} - \cancel{A_z\frac{\partial B_x}{\partial z}} + A_z\frac{\partial B_z}{\partial x}\\
&+ B_y\frac{\partial A_y}{\partial x} - \cancel{B_y\frac{\partial A_x}{\partial y}} - \cancel{B_z\frac{\partial A_x}{\partial z}} + B_z\frac{\partial A_z}{\partial x}\\
&+ A_x\frac{\partial B_x}{\partial x} + \cancel{A_y\frac{\partial B_x}{\partial y}} + \cancel{A_z\frac{\partial B_x}{\partial z}}\\
&+ B_x\frac{\partial A_x}{\partial x} + \cancel{B_y\frac{\partial A_x}{\partial y}} + \cancel{B_z\frac{\partial A_x}{\partial z}}
\end{aligned}\right\}
\begin{aligned}
&\frac{\partial}{\partial x}(A_y B_y) + \frac{\partial}{\partial x}(A_z B_z)\\
&+ \frac{\partial}{\partial x}(A_x B_x)\\
&= \vec{\nabla}(\vec{A}\cdot\vec{B})_x
\end{aligned}$$

Second derivatives

$\vec{\nabla} T$ $\to$ Gradient : Vector

$\vec{\nabla} \cdot \vec{A}$ $\Rightarrow$ Divergence : Scalar

$\vec{\nabla} \times \vec{A}$ $\to$ Curl : vector.

$$\nabla \cdot \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] \to$$

$\vec{\nabla} \cdot (\vec{\nabla} T)$ $\to$ Divergence of gradient

$\vec{\nabla} \times (\nabla T)$ $\to$ Curl of a gradient.

$$\nabla \cdot \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}$$

$\nabla (\vec{\nabla} \cdot \vec{A})$ $\to$ Gradient of divergence.

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$ $\to$ Divergence of a 'curl'

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ $\to$ Curl of a curl.

Divergence of Gradient

$$\vec{\nabla}\cdot(\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplacian : $\nabla^2 T$ Important in E&M theory

Laplacian of $\vec{v}$ {vector}.

$$\nabla^2 \vec{v} = \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

$\underline{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}$?

Curl of a Gradient.

$$\vec{\nabla}\times\left(\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} = \hat{x}\left[\frac{\partial^2 T}{\partial y\,\partial z} - \frac{\partial^2 T}{\partial y\,\partial z}\right] + \hat{y}\left[\frac{\partial^2 T}{\partial x\,\partial z} - \frac{\partial^2 T}{\partial z\,\partial x}\right] + \dots = 0$$

$$\underline{\vec{\nabla}\times(\vec{\nabla}T) = 0}$$

3) Gradient of divergence.

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \neq \nabla^2 \vec{A}$$

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \hat{x}\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \hat{y}\frac{\partial}{\partial y}\left(\qquad\right) + \hat{z}\frac{\partial}{\partial z}\left(\qquad\right).$$

4) Divergence of curl.

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] = 0$$

5) $$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial x} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix}$$

$$(\vec{\nabla}\times\vec{\nabla}\times\vec{A})_x = \frac{\partial^2 A_y}{\partial x\,\partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x\,\partial z}$$

$$= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\,\partial y} + \frac{\partial^2 A_z}{\partial x\, A_z} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}$$

$$= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})\Big|_x = \nabla(\vec{\nabla} \cdot \vec{A})_x - \nabla^2 A\Big|_x$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

$\vec{\nabla} \cdot (\vec{\nabla} T) \rightarrow$ Divergence of gradient. **Laplacian.**

~~$\vec{\nabla} \times (\nabla T) \rightarrow$ Curl of a gradient.~~

$\nabla(\vec{\nabla} \cdot \vec{A}) \rightarrow$ Gradient of divergence.

~~$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \rightarrow$ Divergence of a curl~~

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \rightarrow$ Curl of a curl.