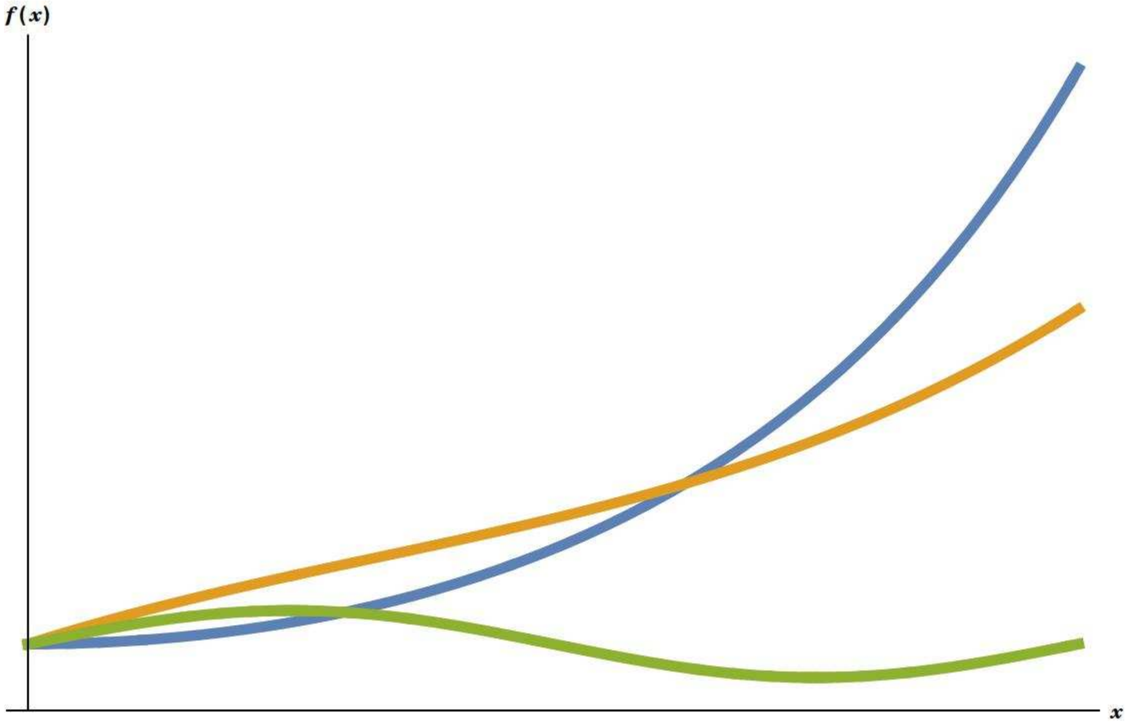
MATHEMATICAL PRELIMINARIES

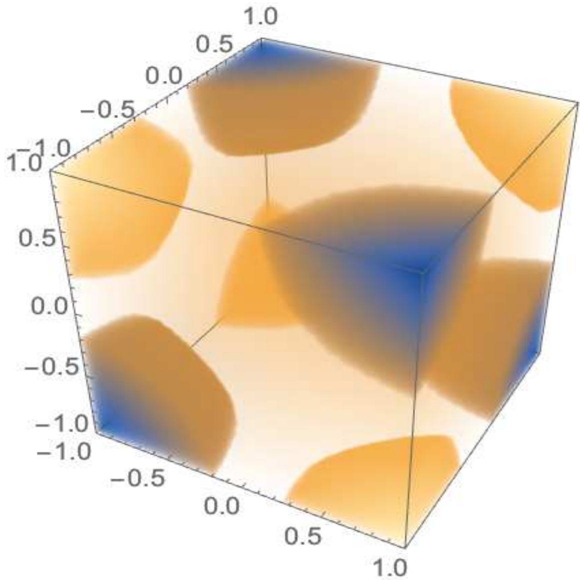
DERIVATIVE OF A FUNCTION



$$\frac{df(x)}{dx}$$

The derivative of a function measures how the function f(x) changes as we change x.

GRADIENT OF A SCALAR FIELD



Temperature T(x,y,z)

How fast does the temperature vary?

Depends on the direction we look at!

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = (\nabla T).(d\vec{l})$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}$$

Gradient of a scalar field is a vector.

In fact $\nabla T(x, y, z)$ is a vector field!

The gradient points in the direction of maximum increase of the field.

THE DEL OPERATOR

$$\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

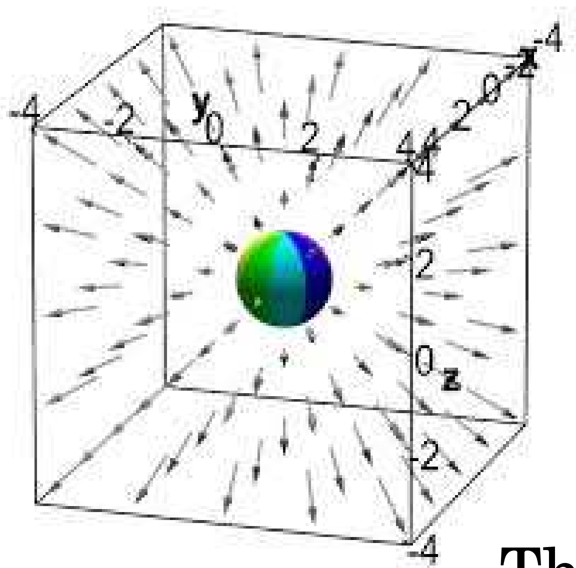
This is a ***vector* operator**. It needs to acts upon a quantity to have any meaning.

- Multiplication by a scalar → Gradient of a scalar field ∇T
- Dot product with vector → Divergence of a vector field $\nabla . \vec{v}$
- Cross product with vector → Curl of a vector field $\nabla \times \vec{v}$

DIVERGENCE OF A VECTOR FIELD

$$\nabla . \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right).\left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}\right)$$

$$\nabla . \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

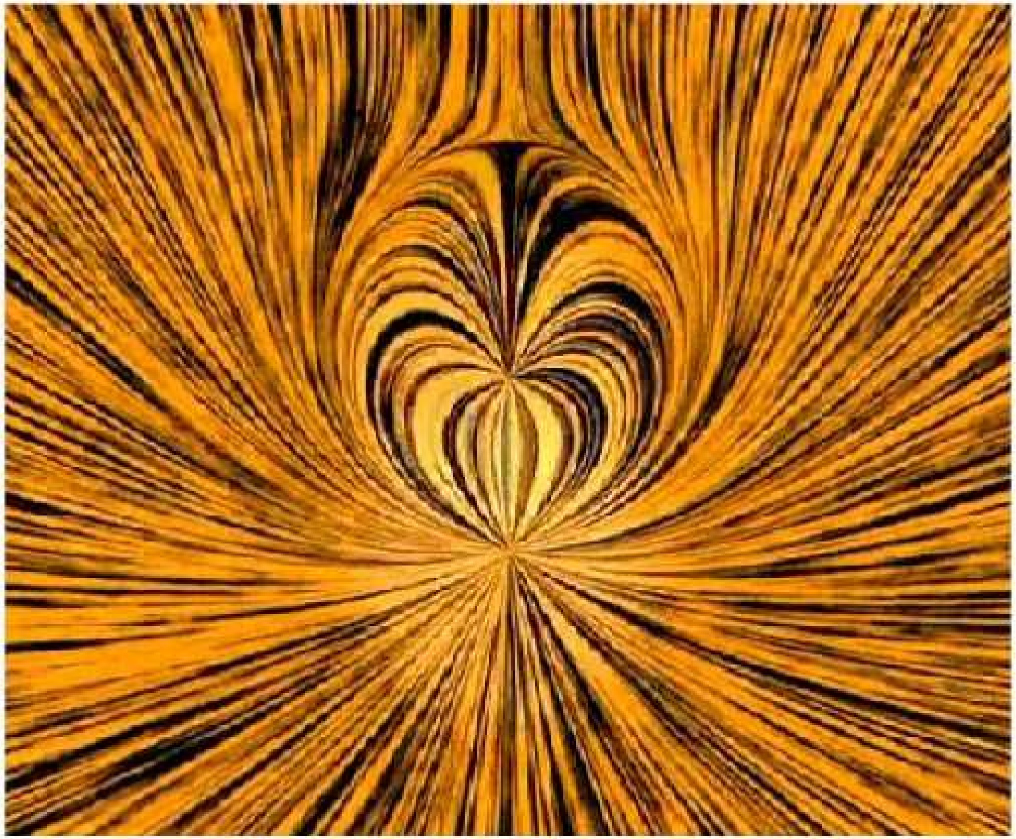


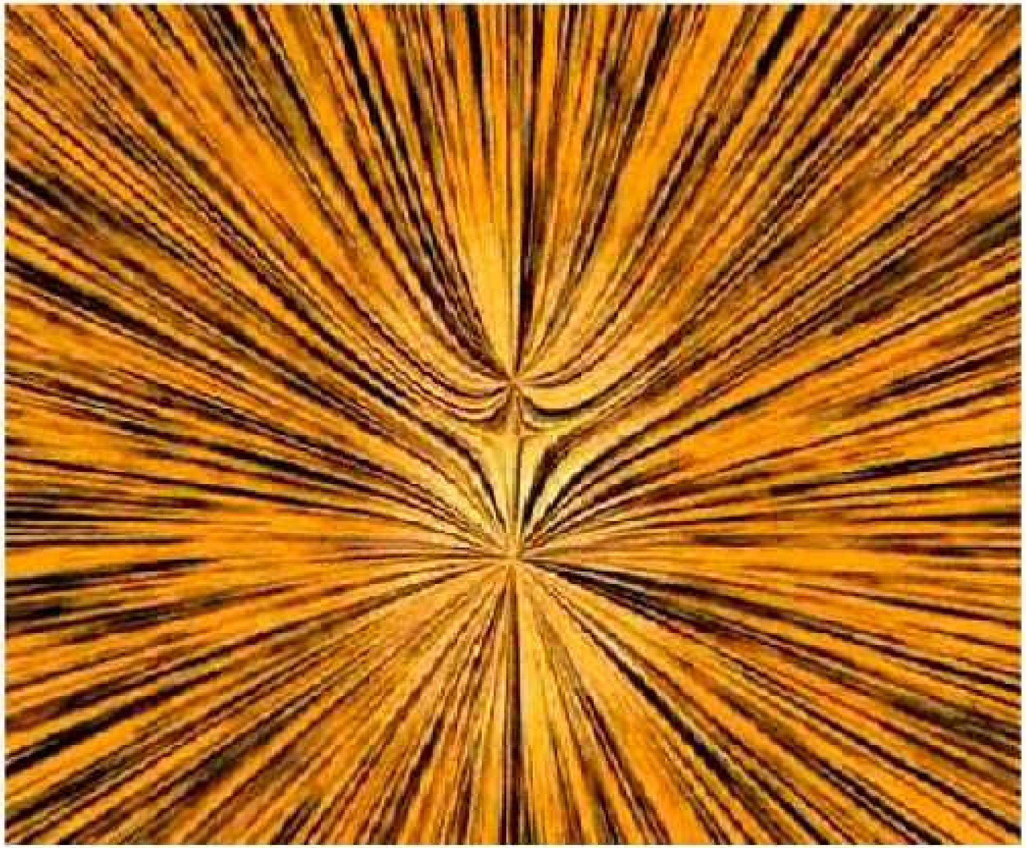
The divergence is a measure of how much the vector spreads out from the point in question.

The divergence of a vector is a scalar quantity.

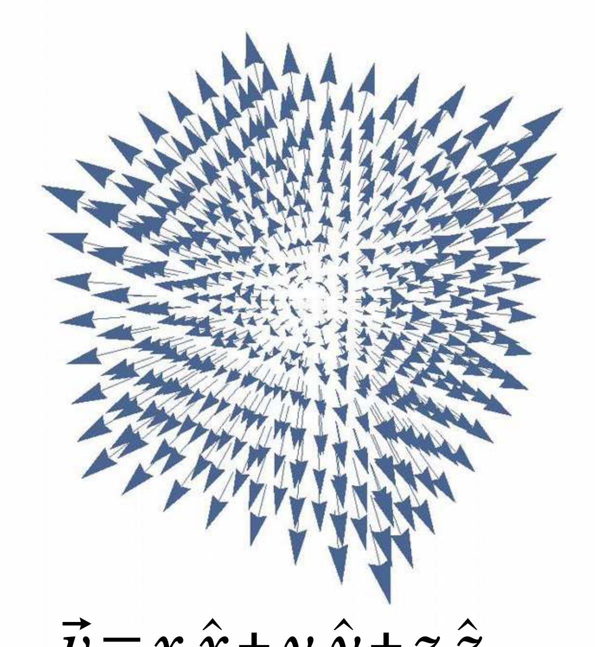
A point of positive divergence is a source, a point of negative divergence is a sink.

DIVERGENCE OF A VECTOR FIELD



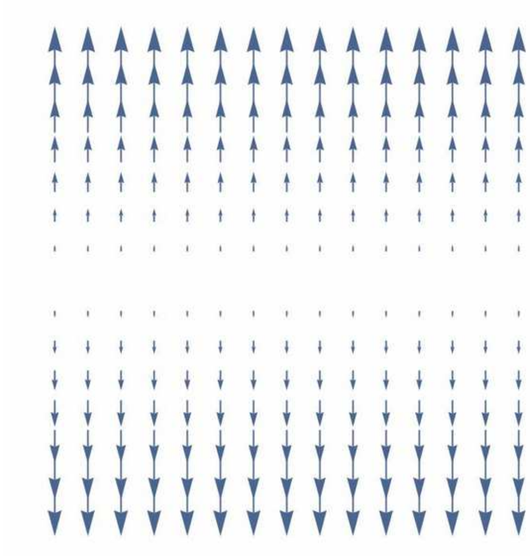


DIVERGENCE OF A VECTOR FIELD



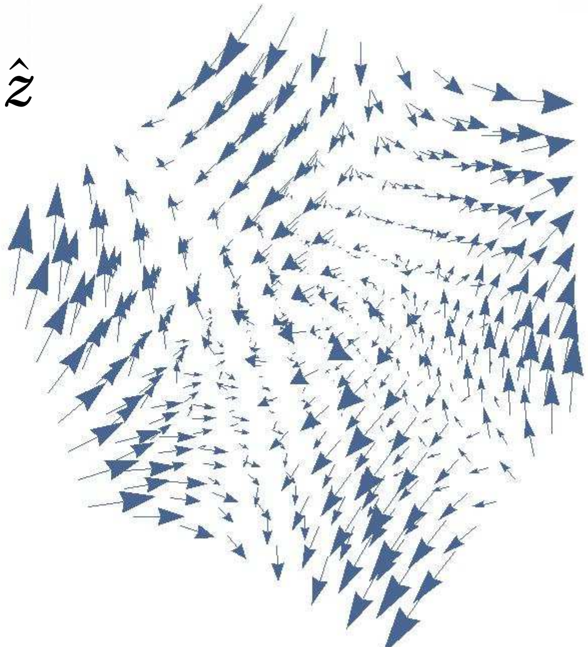
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla . \vec{v} = 3$



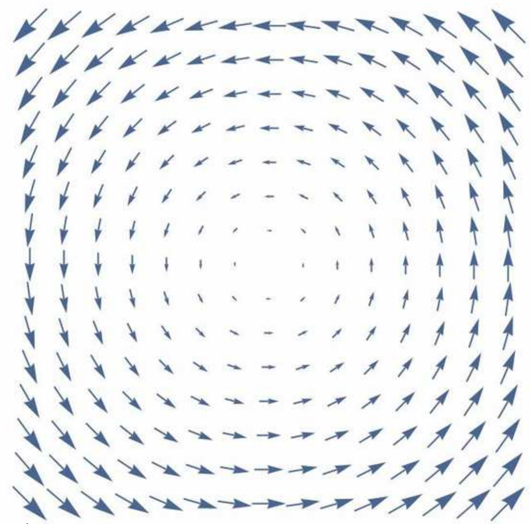
$\vec{v} = z\,\hat{z}$

$\nabla . \vec{v} = 1$



$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$

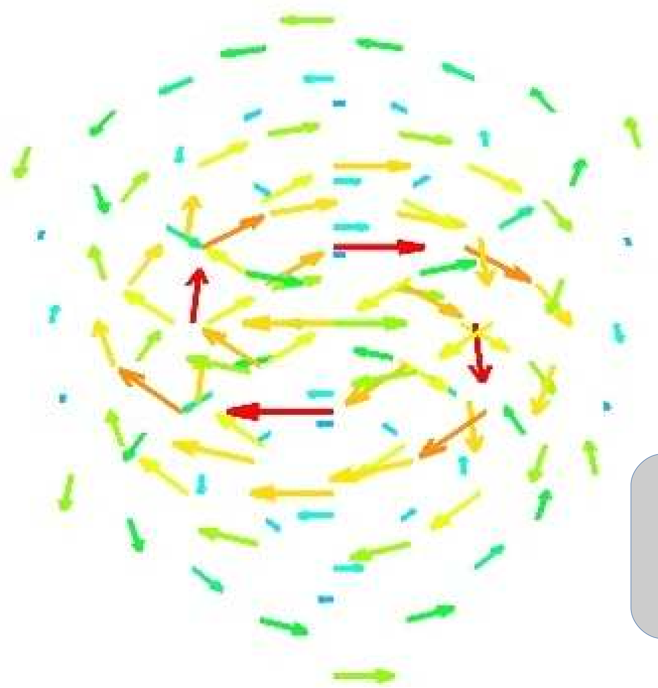
$\nabla . \vec{v} = y + 2\,z + 3\,x$



$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla . \vec{v} = 0$

CURL OF A VECTOR FIELD



$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

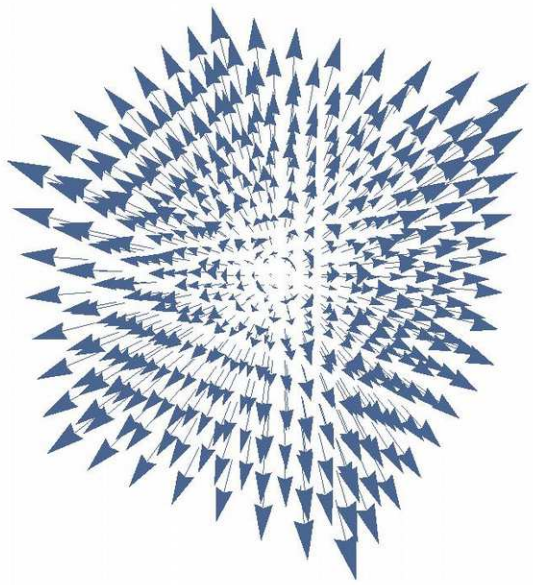
$$\nabla \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

The curl of a vector is a measure of how much the vector curls around the point in question.

The curl of a vector is a vector itself.

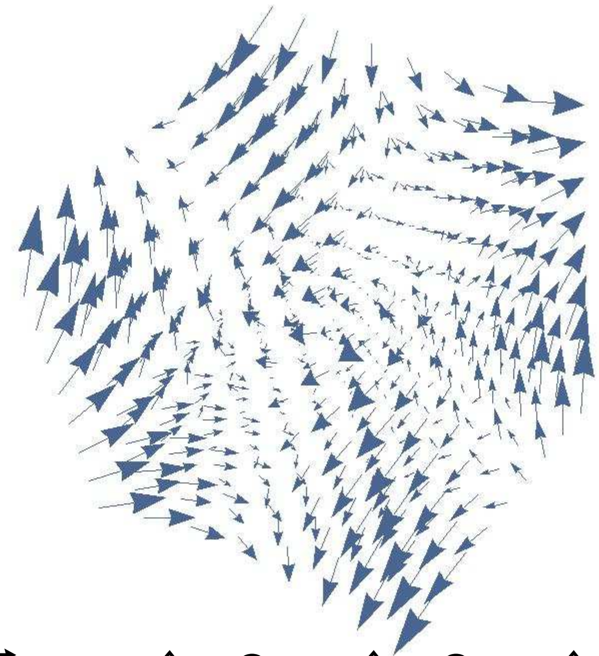
CURL OF A VECTOR FIELD

CURL OF A VECTOR FIELD



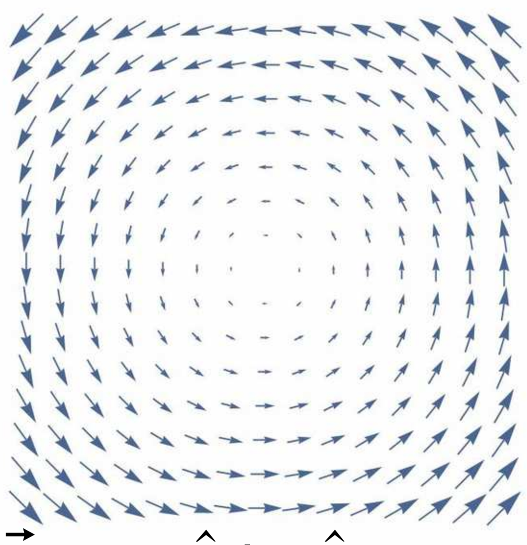
$$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\nabla \times \vec{v} = 0$$



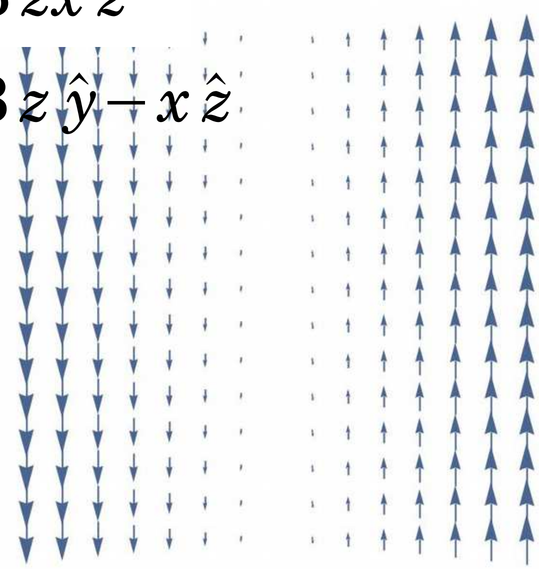
$$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$$

$$\nabla \times \vec{v} = -2\,y\,\hat{x} - 3\,z\,\hat{y} - x\,\hat{z}$$



$$\vec{v} = -y\,\hat{x} + x\,\hat{y}$$

$$\nabla \times \vec{v} = 2\,\hat{z}$$



$$\vec{v} = x\,\hat{y}$$

$$\nabla \times \vec{v} = \hat{z}$$

SUM RULES

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla.(\vec{A}+\vec{B})=(\nabla.\vec{A})+(\nabla.\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

MULTIPLICATION BY A CONSTANT

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$

$$\nabla.(k\vec{A})=k(\nabla.\vec{A})$$

$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

PRODUCT RULES

$$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$$
$$+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\nabla.(f\vec{A})=f(\nabla.\vec{A})+\vec{A}.(\nabla f)$$

$$\nabla.(\vec{A}\times\vec{B})=\vec{B}.(\nabla\times\vec{A})-\vec{A}.(\nabla\times B)$$

$$\nabla\times(f\vec{A})=f(\nabla\times\vec{A})-\vec{A}\times(\nabla f)$$

$$\nabla\times(\vec{A}\times\vec{B})=(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$$
$$+\vec{A}(\nabla.\vec{B})-\vec{B}(\nabla.\vec{A})$$

GRADIENT OF DOT PRODUCT OF TWO VECTORS - PROOF

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B})=\hat{x}\left\{A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$\vec{B}\times(\nabla\times\vec{A})=\hat{x}\left\{B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$(\vec{A}.\nabla)\vec{B}=\hat{x}\left\{\left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)B_x\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$(\vec{B}.\nabla)\vec{A}=\hat{x}\left\{\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)A_x\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$\Rightarrow \text{RHS} =\hat{x}\left\{A_y\frac{\partial B_y}{\partial x}-\cancel{A_y\frac{\partial B_x}{\partial y}}-\cancel{A_z\frac{\partial B_x}{\partial z}}+A_z\frac{\partial B_z}{\partial x}+B_y\frac{\partial A_y}{\partial x}-\cancel{B_y\frac{\partial A_x}{\partial y}}-\cancel{B_z\frac{\partial A_x}{\partial z}}+B_z\frac{\partial A_z}{\partial x}\right.$$

$$\left.+A_x\frac{\partial B_x}{\partial x}+\cancel{A_y\frac{\partial B_x}{\partial y}}+\cancel{A_z\frac{\partial B_x}{\partial z}}+B_x\frac{\partial A_x}{\partial x}+\cancel{B_y\frac{\partial A_x}{\partial y}}+\cancel{B_z\frac{\partial A_x}{\partial z}}\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$=\hat{x}\left\{\frac{\partial}{\partial x}\left(A_xB_x+A_yB_y+A_zB_z\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}=\nabla(\vec{A}.\vec{B})$$

SECOND DERIVATIVES

Divergence of a gradient $\nabla.(\nabla T)$

Curl of a gradient $\nabla\times(\nabla T)$

Gradient of a divergence $\nabla(\nabla.\vec{v})$

Divergence of curl $\nabla.(\nabla\times\vec{v})$

Curl of curl $\nabla\times(\nabla\times\vec{v})$

SECOND DERIVATIVES

Divergence of a gradient $\nabla . (\nabla T)$

$$\nabla . (\nabla T) = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) . \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplacian: $\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

Laplacian of a vector $\nabla^2 \vec{v} = (\nabla^2 v_x) \hat{x} + (\nabla^2 v_y) \hat{y} + (\nabla^2 v_z) \hat{z}$

SECOND DERIVATIVES

Gradient of a divergence $\nabla(\nabla.\vec{v})$

$$\nabla(\nabla.\vec{v})=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right).\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

$$=\hat{x}\left(\frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{y}\left(\frac{\partial}{\partial y}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial z}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$\nabla^2\vec{v}\neq\nabla(\nabla.\vec{v})$$

The Laplacian of a vector is not the same as gradient of the divergence.

SECOND DERIVATIVES

Curl of curl $\nabla \times (\nabla \times \vec{v})$

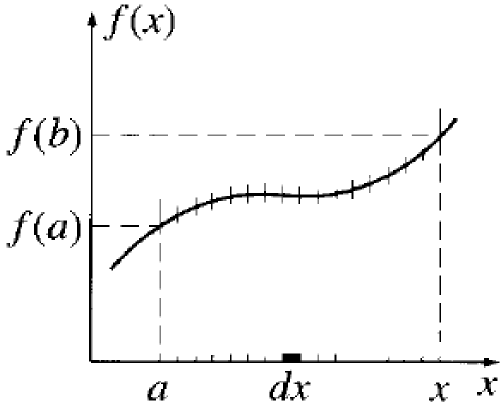
$$\nabla \times (\nabla \times \vec{v}) = \nabla \times \left(\hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right)$$

$$= \hat{x}\left\{ \frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$= \hat{x}\left\{ \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \right\} + \ldots$$

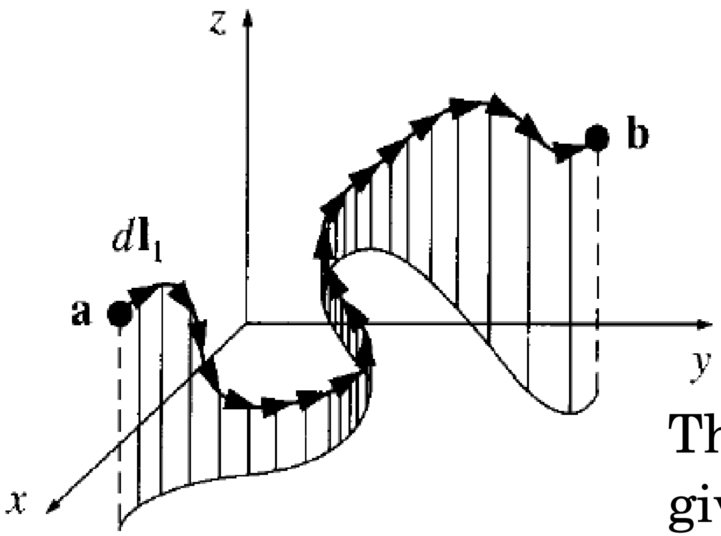
$$\nabla \times (\nabla \times \vec{v}) = \nabla (\nabla . \vec{v}) - \nabla^2 \vec{v}$$

FUNDAMENTAL THEOREM OF GRADIENTS



$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

The integral of a derivative over an interval is the value of the function at the end points or boundaries.



Scalar field T(x,y,z) $\quad dT = (\nabla T).(d\vec{l})$

$$\int_{a\,\mathcal{P}}^{b} (\nabla T).(d\vec{l}) = T(b) - T(a)$$

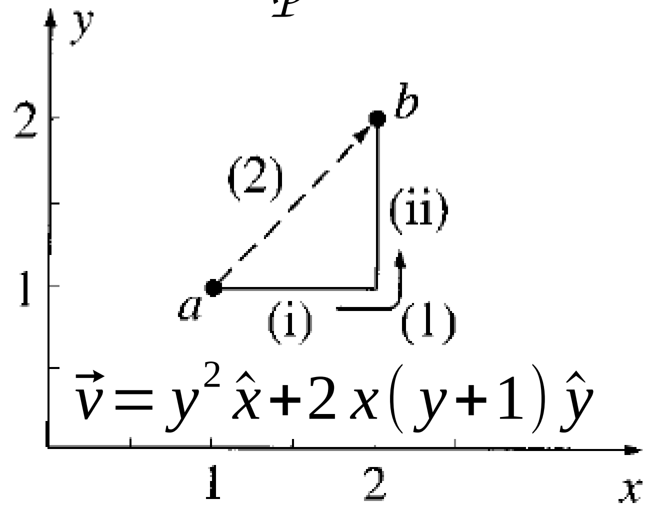
The line integral of a gradient of a scalar field is given by the value of the function at the boundaries.

Path Independence - Is this obvious?

LINE INTEGRALS

For any general vector field $\vec{v}$

$$\int_{a \, \mathcal{P}}^{b} \vec{v}.(d\vec{l})$$ depends on the path $\mathcal{P}$!



$$\vec{v} = y^2 \hat{x} + 2x(y+1)\hat{y}$$

For a gradient field, $\boxed{\oint (\nabla T).(d\vec{l}) = 0}$

If $\vec{v} \equiv \nabla T$, then $\vec{v}$ is a conservative field

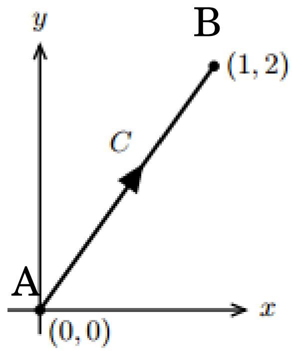
Electric field **E** is conservative.

$$\int_{a\,1}^{b} \vec{v}.(d\vec{l}) = 11$$

$$\int_{a\,2}^{b} \vec{v}.(d\vec{l}) = 10$$

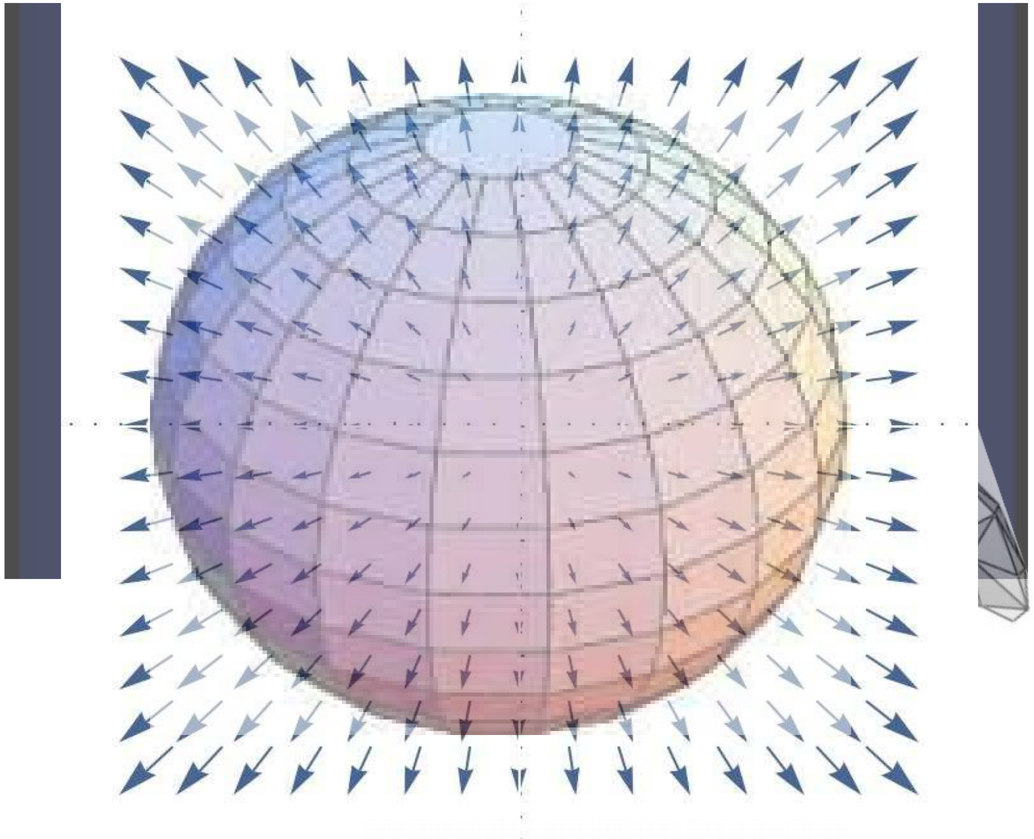
$$\Rightarrow \oint \vec{v}.(d\vec{l}) \neq 0$$

$$f(x,y) = xy^3 + xy^2$$



$$\int_{A}^{B} \nabla f.(d\vec{l}) = ?$$

FLUX OF A VECTOR FIELD



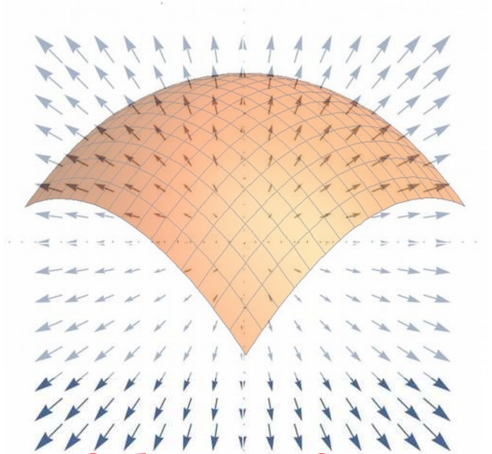
The volume of water flowing out through the surface per unit time,

Flux $\oiint \vec{v}\,.\,d\vec{S}$

$d\vec{S} \Rightarrow$ Infinitesimal area element, direction perpendicular to surface

$\oiint \Rightarrow$ Integral over a closed surface

Sign convention: Outward is positive



For the surface integral over an open surface

$$\iint \vec{v}\,.\,d\vec{S}$$

the sign is arbitrary

Does the value of the surface integral depend on the surface? Yes, except...

GAUSS'S THEOREM / DIVERGENCE THEOREM

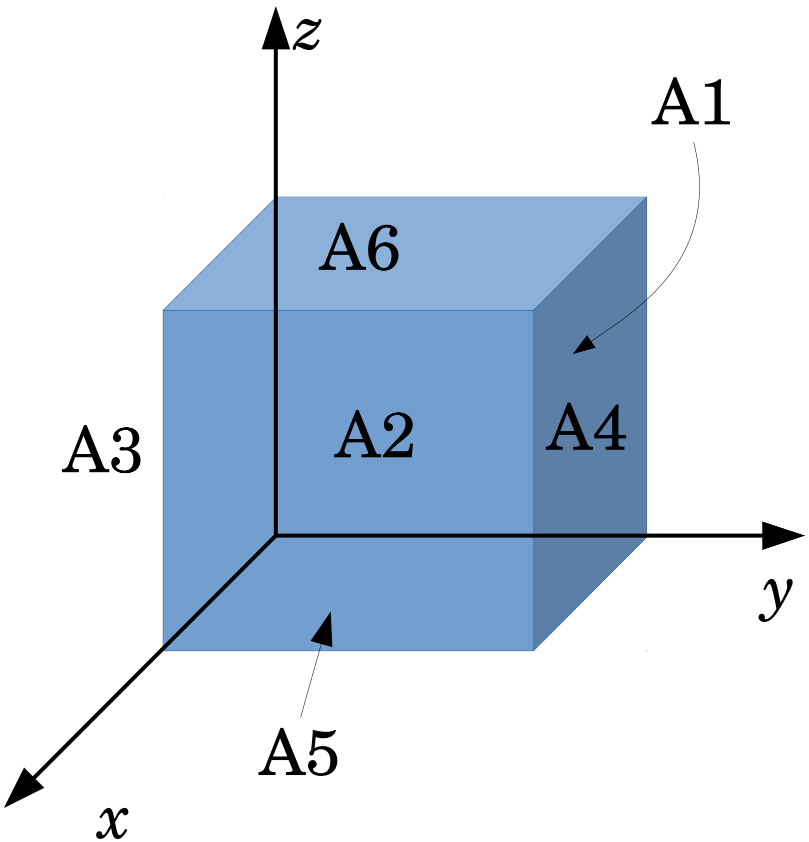
$$\oiiint_V (\nabla . \vec{v}) \; dV = \oiint_S \vec{v} . d\vec{S}$$

where, S is the surface bounding a volume V

The volume integral of the divergence of a function over a region is equal to the value of the function at the boundary (the surface enclosing the volume).

RHS measures the flux through the closed surface S.
LHS counts the sources and sinks within the region bounded by S

GAUSS'S THEOREM / DIVERGENCE THEOREM



$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad e \leqslant z \leqslant f$$

$$\iint_{A_1} \vec{v} \cdot d\vec{S} + \iint_{A_2} \vec{v} \cdot d\vec{S}$$

$$= -\int_e^f \int_c^d v_x(a, y, z)\, dy\, dz + \int_e^f \int_c^d v_x(b, y, z)\, dy\, dz$$

$$= \int_e^f \int_c^d \left(v_x(b, y, z) - v_x(a, y, z)\right) dy\, dz$$

$$= \int_e^f \int_c^d \int_a^b \left(\frac{\partial v_x}{\partial x} dx\right) dy\, dz = \oiiint_V \frac{\partial v_x}{\partial x} dV$$

Repeating over the other 4 sides and adding,

$$\oiint_S \vec{v} \cdot d\vec{S} = \oiiint_V (\nabla \cdot \vec{v})\; dV$$

STOKES THEOREM

$$\iint_S (\nabla \times \vec{v}) . d\vec{S} = \oint_{\wp} \vec{v} . d\vec{l}$$

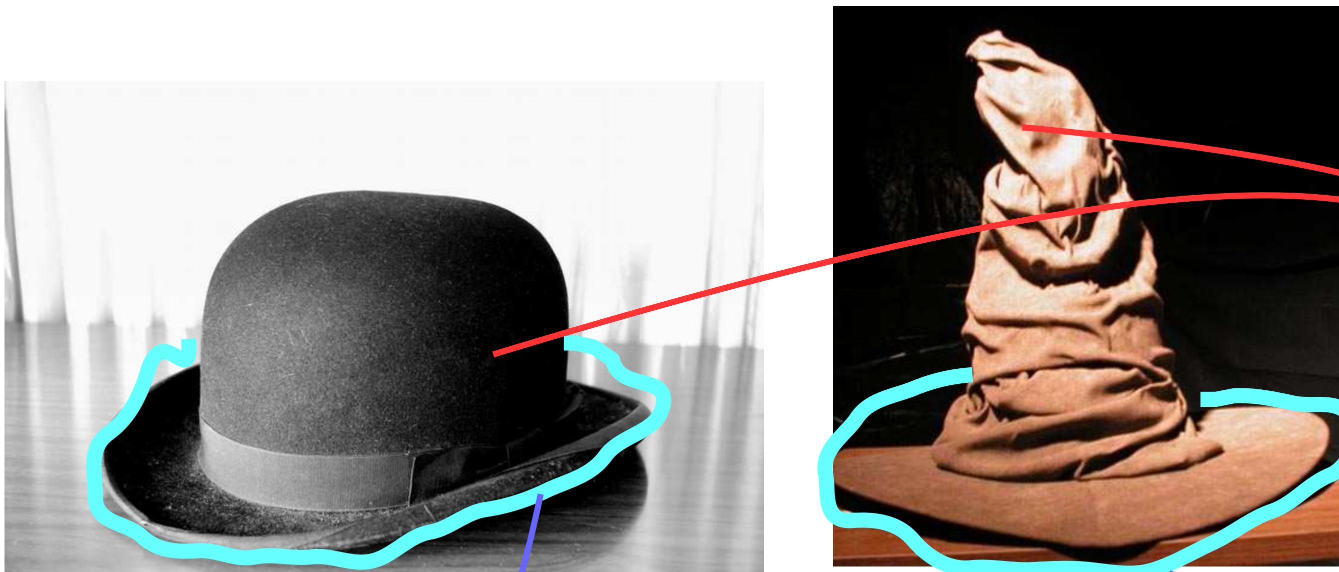
The flux of the curl of a vector through a surface S is equal to the closed line integral of the vector function over the bounding line of the surface.

What is the sign of the line integral?
What is the sign of the surface integral?

Right hand rule: If the fingers point in the direction of the line integral, the thumb fixes the direction of $d\vec{S}$

STOKES THEOREM

The same bounding line can enclose many surfaces!



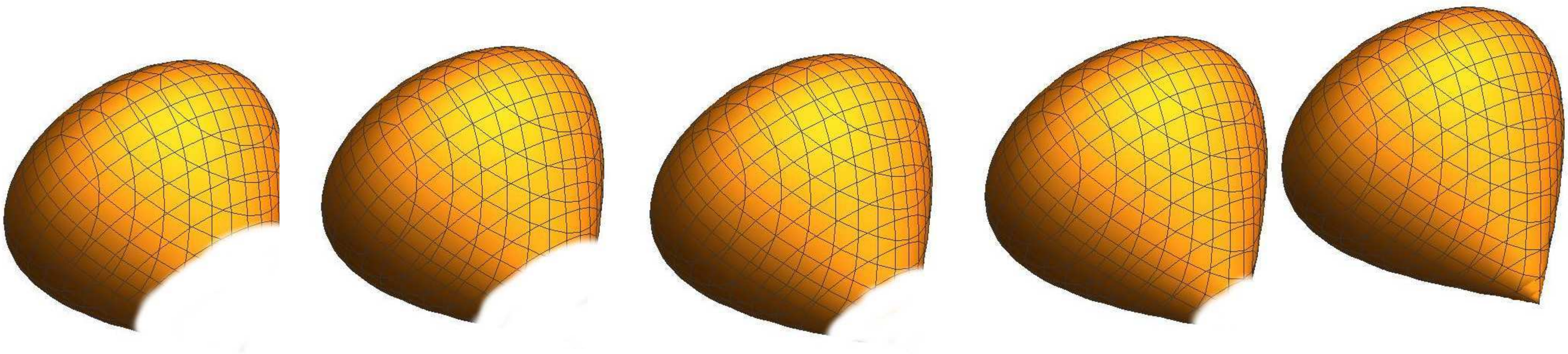
Surface

Bounding line

The surface integral can be over ***ANY*** surface that shares a common bounding line!

STOKES THEOREM

What happens if we shrink the boundary line?



$$\oiint_S (\nabla \times \vec{v}) . d\vec{S} = 0$$

COROLLARIES

$\nabla\times\nabla\varphi=?$

$$\iint_S (\nabla\times\vec{v}).d\vec{S}=\oint\vec{v}.d\vec{l}$$ Stokes Theorem

If, $\vec{v}=\nabla\varphi$

$$\oint\nabla\varphi.d\vec{l}=0$$ Gradient Theorem

Combining, we get,

$$\iint_S (\nabla\times\nabla\varphi).d\vec{S}=0$$ for any surface S

$$\Rightarrow\nabla\times\nabla\varphi=0$$ for any scalar function φ

$\nabla.\nabla\times\vec{A}=?$

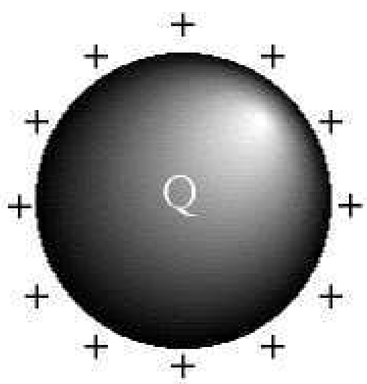
$$\oiint_S (\nabla\times\vec{A}).d\vec{S}=0$$ Stokes Theorem

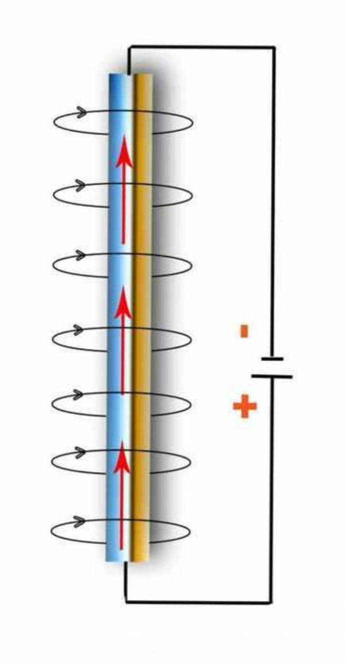
$$\oiiint\nabla.(\nabla\times\vec{A})dV=\oiint(\nabla\times\vec{A}).d\vec{S}=0$$ Gauss's Theorem

$$\Rightarrow\nabla.(\nabla\times\vec{A})=0$$ for any vector function $\vec{A}$

CURVILINEAR COORDINATE SYSTEMS

The symmetries of a problem can dictate the most efficient *choice* of the coordinate system.





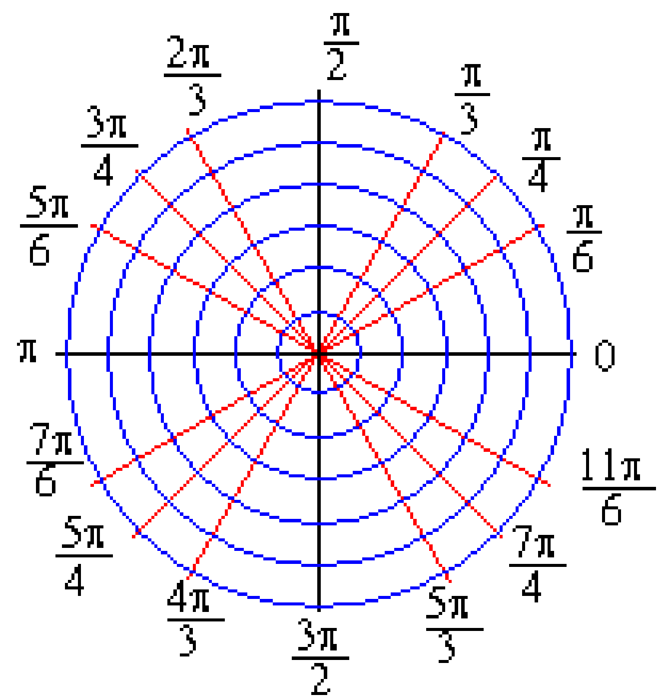
Any coordinate system will do for any problem, however, some coordinate systems will be easier!

PLANE POLAR COORDINATES

STEP 1: Write down the relation with (x,y) co-ordinates

$$x = r \cos\theta$$
$$y = r \sin\theta$$

STEP 2: Draw the coordinate grid



How do r=constant lines look?
How do θ= constant lines look?

STEP 3: What happens when the independent variables are changed infinitesimally?

$$\delta x = \cos\theta\,\delta r - r\sin\theta\,\delta\theta$$
$$\delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta$$

PLANE POLAR COORDINATES

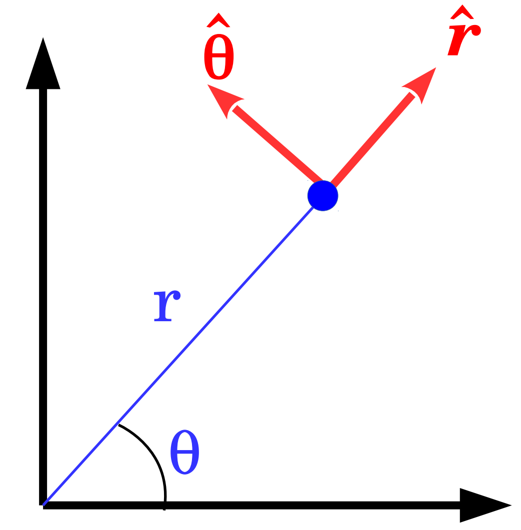
STEP 4: Which direction would we move, if only one variable was changed?

$\underline{\delta\theta=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y=\left(\hat{x}\cos\theta+\hat{y}\sin\theta\right)\delta r=\hat{\boldsymbol{r}}\,\delta r$$

$\underline{\delta r=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y=\left(-\hat{x}\sin\theta+\hat{y}\cos\theta\right)r\,\delta\theta=\hat{\boldsymbol{\theta}}\,r\,\delta\theta$$



$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\hat{r}\,.\,\hat{\theta}=\left(-\cos\theta\sin\theta\right)+\left(\sin\theta\cos\theta\right)=0$$

Curvilinear, but still orthogonal

PLANE POLAR COORDINATES

STEP 5: What happens to an element of distance, or arclength?

$$d\vec{l} = \delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}$$

Compare with the Cartesian case, $d\vec{l} = \delta x\,\hat{x} + \delta y\,\hat{y}$

Can it be $d\vec{l} = \delta r\,\hat{r} + \delta\theta\,\hat{\theta}$? Why not?

In general, for a curvilinear coordinate system, the scale factors are not unity!

Scale factor gives a measure of how much a change in the coordinate changes the position of a point.

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$$

$$ds^2 = d\vec{l}\,.\,d\vec{l} = \delta r^2 + r^2\,\delta\theta^2$$

There are no cross terms in the arclength expression $O(\delta r\,\delta\theta)$ for orthogonal coordinate systems.

PLANE POLAR COORDINATES

STEP 6: What happens to the element of area?

We take a small step in the $\hat{r}$ direction, and a small step in the $\hat{\theta}$ direction. What is the infinitesimal area enclosed by these two perpendicular vectors?

$$dl_r = \delta r \qquad dl_\theta = r\,\delta\theta$$

$$dA = dl_r\, dl_\theta = r\,\delta\theta\,\delta r$$

STEP 7: What is the gradient?

$$\begin{aligned} dT &= \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta \\ &= [some \quad fn].\, d\vec{l} \\ &= [\nabla T].(\delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}) \end{aligned}$$

$$\boxed{\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta}}$$

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$$

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

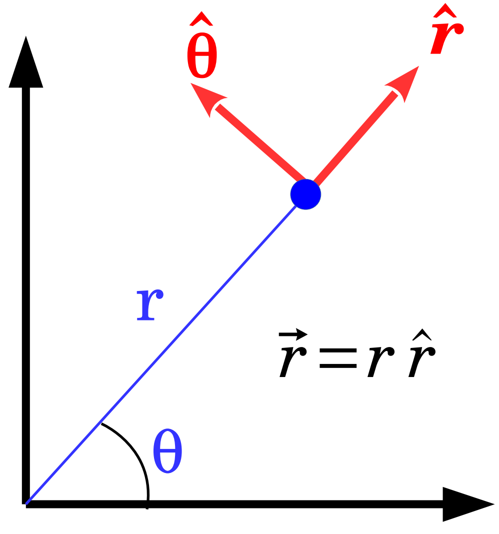
$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).$$
$$(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}$$

PLANE POLAR COORDINATES

STEP 8: What are the velocity components, when a particle's motion is described using polar coordinates?

$$\vec{v}=\frac{d}{dt}(r\,\hat{r})=\dot{r}\,\hat{r}+r\,\dot{\hat{r}}$$



The unit vectors here are not constant, unlike for cartesian coordinates, and must be themselves differentiated.

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\begin{pmatrix}\dot{\hat{r}}\\ \dot{\hat{\theta}}\end{pmatrix}=\dot{\theta}\begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\dot{\theta}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}$$

$$\Rightarrow\vec{v}=\dot{r}\,\hat{r}+r\,\dot{\hat{r}}=\dot{r}\,\hat{r}+r\,\dot{\theta}\,\hat{\theta}$$

$v_r=\dot{r}\rightarrow$ *radial* velocity component

$v_\theta=r\,\dot{\theta}\rightarrow$ *circumferential* velocity component

$$\vec{v}\;=\;\frac{\delta\,\vec{l}}{\delta\,t}=\;\frac{\delta}{\delta\,t}(\delta\,r\,\hat{r}+r\,\delta\,\theta\,\hat{\theta})=\;\frac{dr}{dt}\hat{r}+r\frac{d\,\theta}{dt}\hat{\theta}$$

PLANE POLAR COORDINATES

STEP 9: What are the acceleration components, when a particle's motion is described using polar coordinates?

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)$$

$$= \dot{\theta}\,\dot{r}\,\hat{\theta} + \ddot{r}\,\hat{r} - \dot{\theta}\,r\,\dot{\theta}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} \text{¿}$$

$$= \underbrace{\left(\ddot{r} - \dot{\theta}^2 r\right)}_{\text{Radial acceleration}}\hat{r} + \underbrace{\left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)}_{\text{Circumferential acceleration}}\hat{\theta}$$

If the force on the particle is central, then which quantity is conserved?

What can you say about the matrix connecting the two sets and the inverse relation ?

CURVILINEAR COORDINATE SYSTEMS - PRESCRIPTION

Write down the relation with (x,y) co-ordinates

Draw the coordinate grid

What happens when the independent variables are changed infinitesimally?

Which direction would we move, if only one variable was changed?

What happens to an element of distance?

What happens to an element of area?

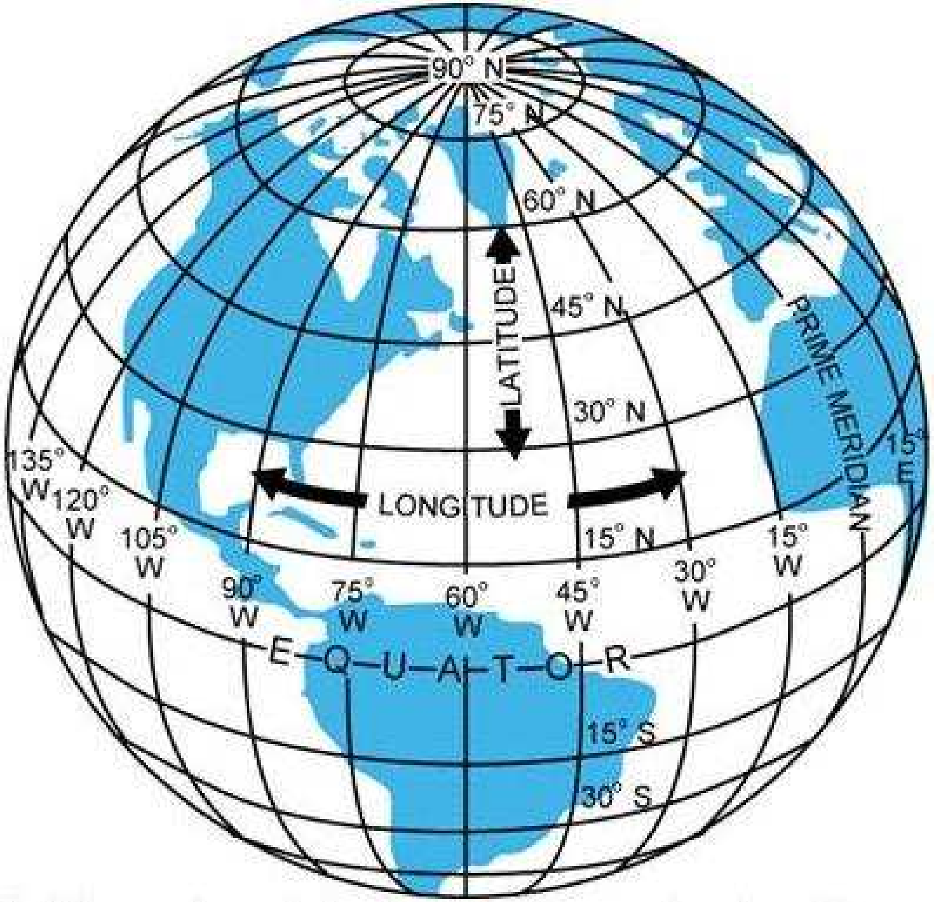
What is the gradient?

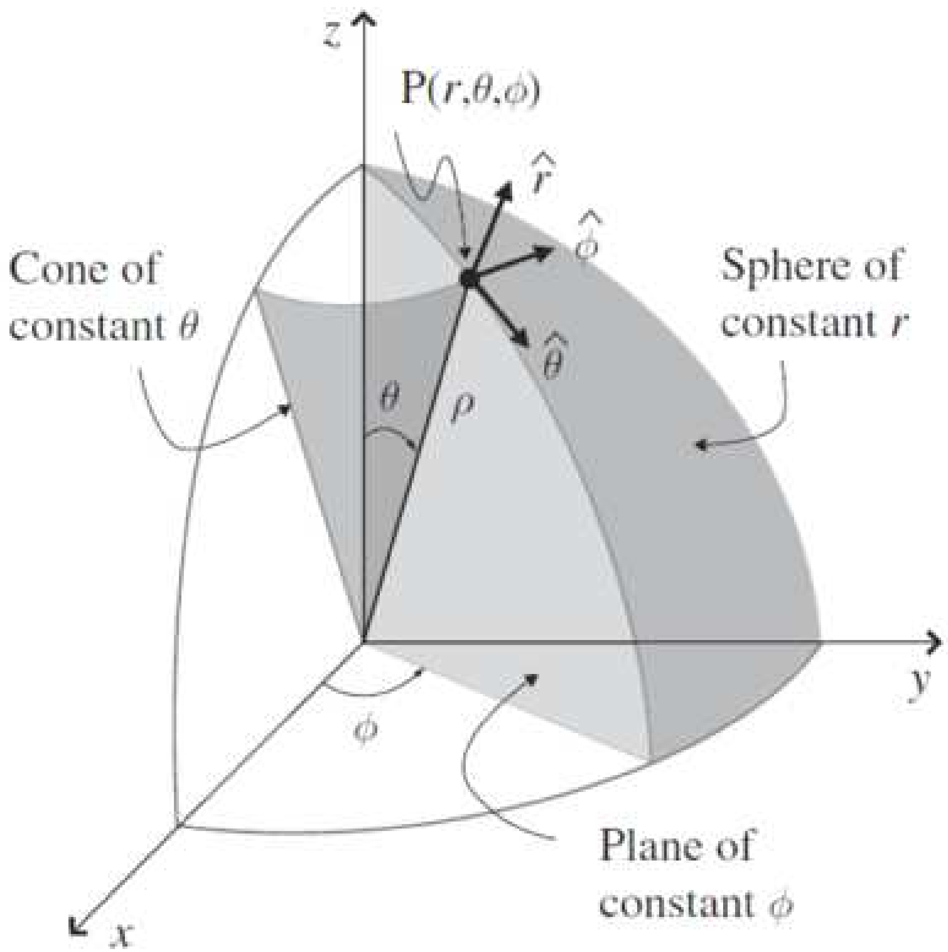
What are the velocity components?

What are the acceleration components?

You're now set to solve problems in this coordinate system

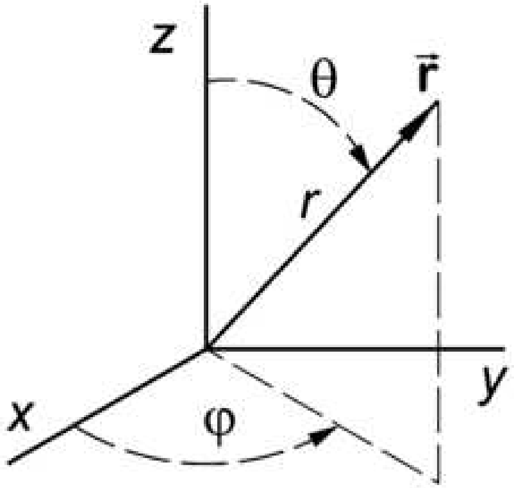
SPHERICAL POLAR COORDINATES





SPHERICAL POLAR COORDINATES

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$



$r\in[0,\infty]$
$\theta\in[0,\pi]$
$\phi\in[0,2\pi]$

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta\theta \\ \delta\phi \end{pmatrix}$$

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$\hat{r}.\hat{\theta}=\hat{r}.\hat{\phi}=\hat{\theta}.\hat{\phi}=0$$

SPHERICAL POLAR COORDINATES

$$dl_r = \delta r \qquad dl_\theta = r\,\delta\theta \qquad dl_\phi = r\sin\theta\,\delta\phi$$

Infinitesimal displacement $d\vec{l} = \delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta} + r\sin\theta\,\delta\phi\,\hat{\phi}$

Arclength $ds^2 = d\vec{l}\,.\,d\vec{l} = \delta r^2 + r^2\,\delta\theta^2 + r^2\sin^2\theta\,\delta\phi^2$

$$dV = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

Area element:

$$\textit{If} \quad r = \textit{constant} \quad (\textit{surface of a sphere}) \quad \delta r = 0$$

$$dA = dl_\theta\,dl_\phi = r^2\sin\theta\,d\theta\,d\phi$$

$$\textit{If} \quad \theta = \textit{constant}, \quad \delta\theta = 0$$

$$dA = dl_r\,dl_\phi = r\sin\theta\,dr\,d\phi$$

$$\textit{If} \quad \phi = \textit{constant}, \quad \delta\phi = 0$$

$$dA = dl_r\,dl_\theta = r\,dr\,d\theta$$

SPHERICAL POLAR COORDINATES

Gradient of a scalar field

If we have a function $T(r, \theta, \phi)$ then we want

$$dT = \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial \theta}\delta\theta + \frac{\partial T}{\partial \phi}\delta\phi$$

$$= \nabla T . \delta \vec{l}$$

Now, $\delta \vec{l} = \hat{r}\,\delta r + \hat{\theta}\, r\,\delta\theta + \hat{\phi}\, r \sin\theta\,\delta\phi$

The gradient in the spherical polar system can be identified as

$$\nabla T = \hat{r}\frac{\partial T}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial T}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

SPHERICAL POLAR COORDINATES

Velocity and acceleration

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \boldsymbol{M} = \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \boldsymbol{M}^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \qquad \begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\phi}} \end{pmatrix} = \dot{\boldsymbol{M}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \boldsymbol{M}^T = \begin{pmatrix} \sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi \\ \sin\theta\,\sin\phi & \cos\theta\,\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\dot{\boldsymbol{M}} = \begin{pmatrix} \cos\theta\,\cos\phi\,\dot{\theta} - \sin\theta\,\sin\phi\,\dot{\phi} & \cos\theta\,\sin\phi\,\dot{\theta} + \sin\theta\,\cos\phi\,\dot{\phi} & -\sin\theta\,\dot{\theta} \\ -\sin\theta\,\cos\phi\,\dot{\theta} - \cos\theta\,\sin\phi\,\dot{\phi} & -\sin\theta\,\sin\phi\,\dot{\theta} + \cos\theta\,\cos\phi\,\dot{\phi} & -\cos\theta\,\dot{\theta} \\ -\cos\phi\,\dot{\phi} & -\sin\phi\,\dot{\phi} & 0 \end{pmatrix}$$

SPHERICAL POLAR COORDINATES

Velocity and acceleration

$$\begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{\phi}} \end{pmatrix} = \dot{\boldsymbol{M}}\,\boldsymbol{M}^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dot{\theta} & \sin\theta\ \dot{\phi} \\ -\dot{\theta} & 0 & \cos\theta\ \dot{\phi} \\ -\sin\theta\ \dot{\phi} & -\cos\theta\ \dot{\phi} & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

Why are the diagonal terms zero? Can you see the physical implication?

Notice that the matrix connecting the two vectors is anti-symmetric.

This was also the case in the plane polar co-ordinates. But we didn't mention it there.

The problem for velocity and acceleration components can now be completed...

SPHERICAL POLAR COORDINATES

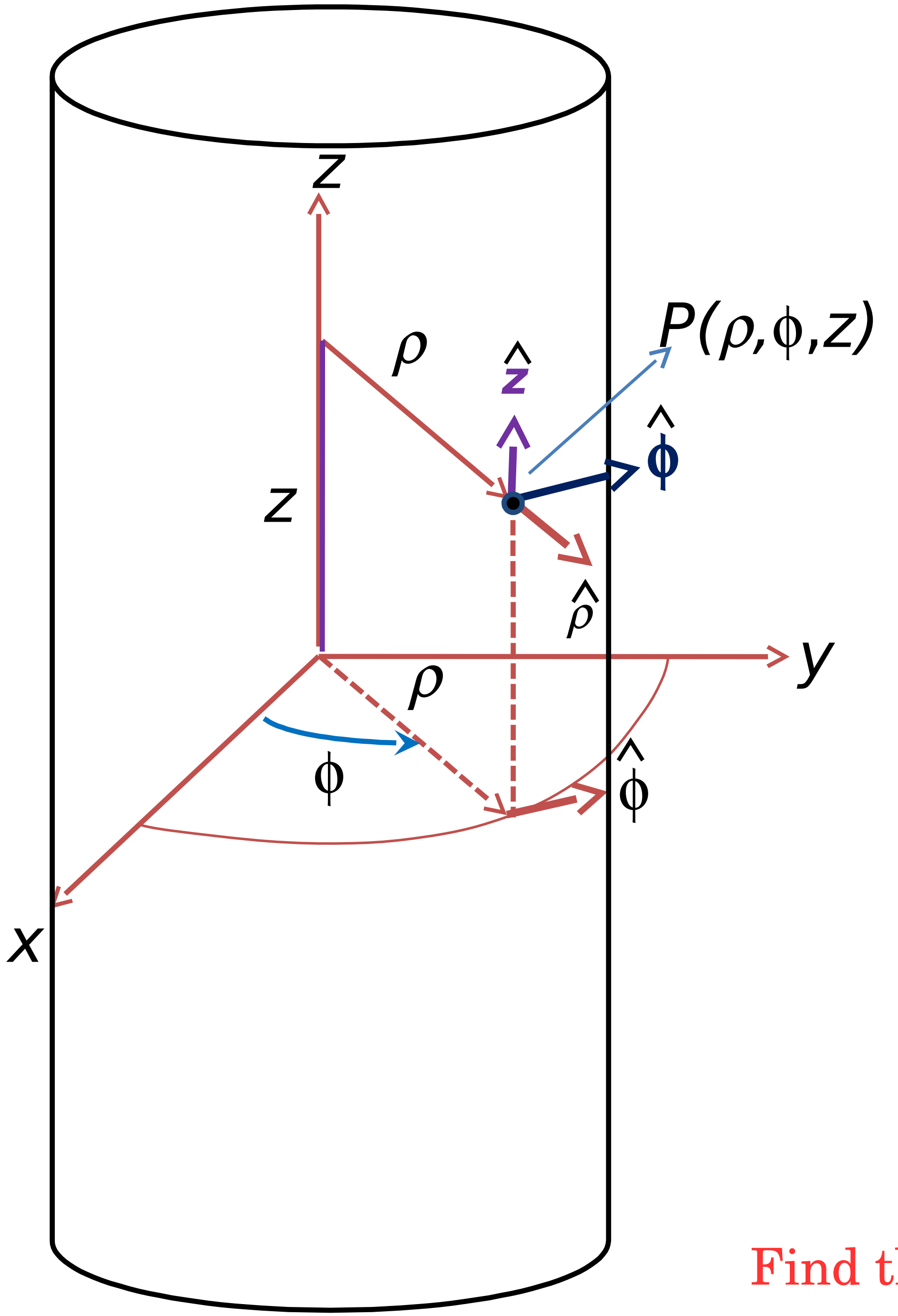
$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\vec{a} \;=\; \hat{r}\left(\ \ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\phi}^2\,\sin^2\theta\ \right) \;+\; \hat{\theta}\left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\ \text{-}\ r\,\dot{\phi}^2\sin\theta\cos\theta\right) \;+$$
$$\hat{\phi}\left(r\,\ddot{\phi}\sin\theta + 2\,\dot{r}\,\dot{\phi}\sin\theta + 2r\,\dot{\theta}\,\dot{\phi}\cos\theta\right)$$

We now have all ingredients to solve dynamical coordinates in this system.

CYLINDRICAL POLAR COORDINATES



$$x = \rho\cos\phi \qquad 0 \leqslant \rho \leqslant \infty$$
$$y = \rho\sin\phi \qquad 0 \leqslant \phi \leqslant 2\pi$$
$$z = z \qquad 0 \leqslant z \leqslant \infty$$

$$\begin{pmatrix} \hat{r} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

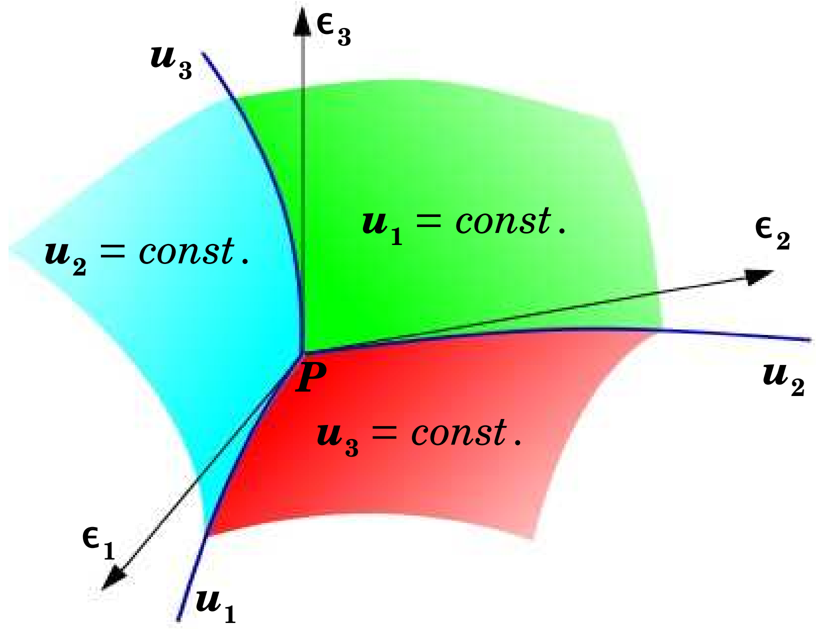
$$\delta\vec{l} = \delta\rho\,\hat{\rho} + \rho\,\delta\phi\,\hat{\phi} + \delta z\,\hat{z}$$
$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$$dV = \rho\, d\rho\, d\phi\, dz$$

$$\boxed{\nabla T = \hat{\rho}\frac{\partial T}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial T}{\partial \phi} + \hat{z}\frac{\partial T}{\partial z}}$$

Find the velocity and acceleration components

GENERAL ORTHOGONAL CURVILINEAR COORDINATES

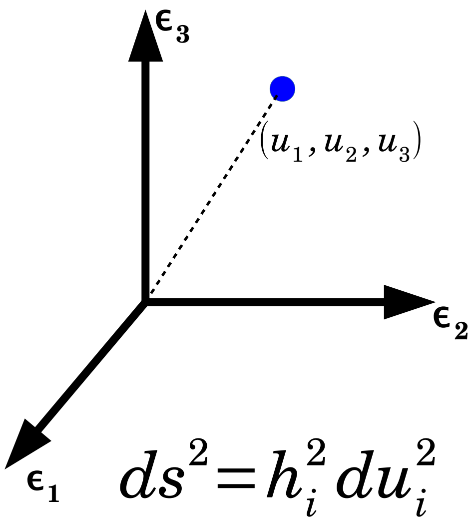


Identify a point in space by its three coordinates (u_1, u_2, u_3)

If the system is orthogonal,

$$\epsilon_i . \epsilon_j = 0 \quad \forall \quad i \neq j$$

$$\epsilon_i . \epsilon_j = \delta_{ij}$$



The unit vectors in general are functions of position, since their direction can vary from point to point.

The infinitesimal displacement vector is

$$d\boldsymbol{l} = \epsilon_1 h_1 du_1 + \epsilon_2 h_2 du_2 + \epsilon_3 h_3 du_3$$

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}$$

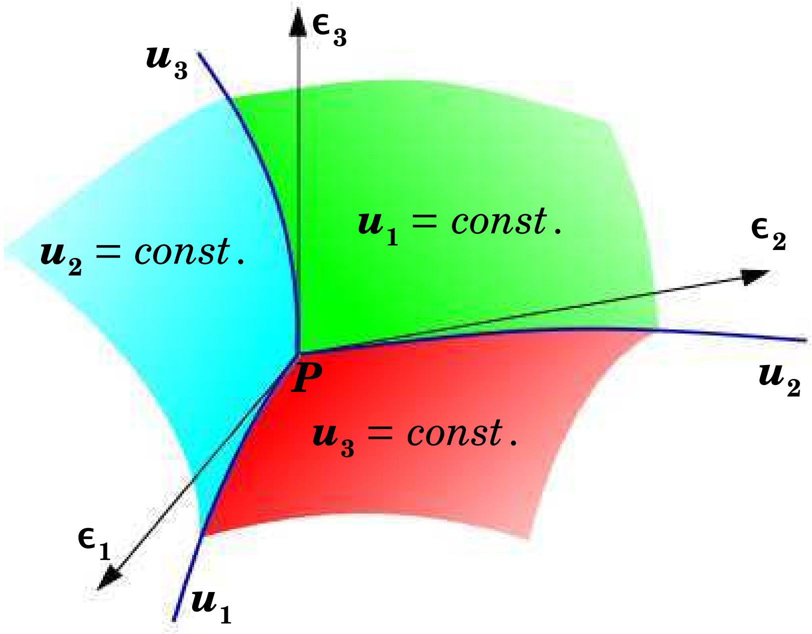
$$ds^2 = h_i^2 du_i^2$$

$$dV = \prod_i h_i du_i$$

$$= \sum_i \epsilon_i h_i du_i \; = \; \epsilon_i h_i du_i$$

REPEATED INDEX IMPLIES SUMMATION

GENERAL ORTHOGONAL CURVILINEAR COORDINATES



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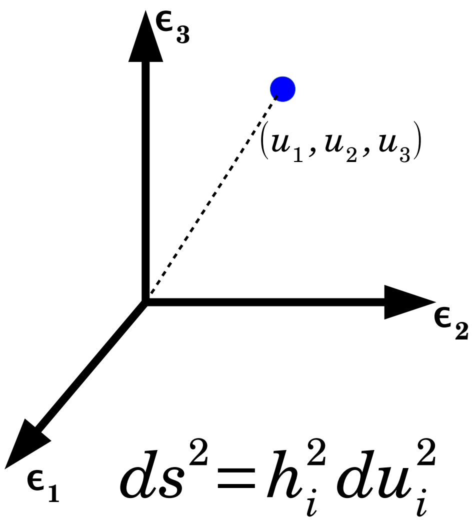
$$\boldsymbol{\epsilon}_i \cdot \boldsymbol{\epsilon}_j = \delta_{ij}$$



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$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}$$

$$= \sum_i \boldsymbol{\epsilon}_i h_i du_i = \boldsymbol{\epsilon}_i h_i du_i$$

REPEATED INDEX IMPLIES SUMMATION

$$ds^2 = h_i^2 du_i^2$$

$$dV = \prod_i h_i du_i$$

GENERAL ORTHOGONAL CURVILINEAR COORDINATES

System	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical	r	θ	φ	1	r	r sin θ
Cylindrical	ρ	φ	z	1	ρ	1

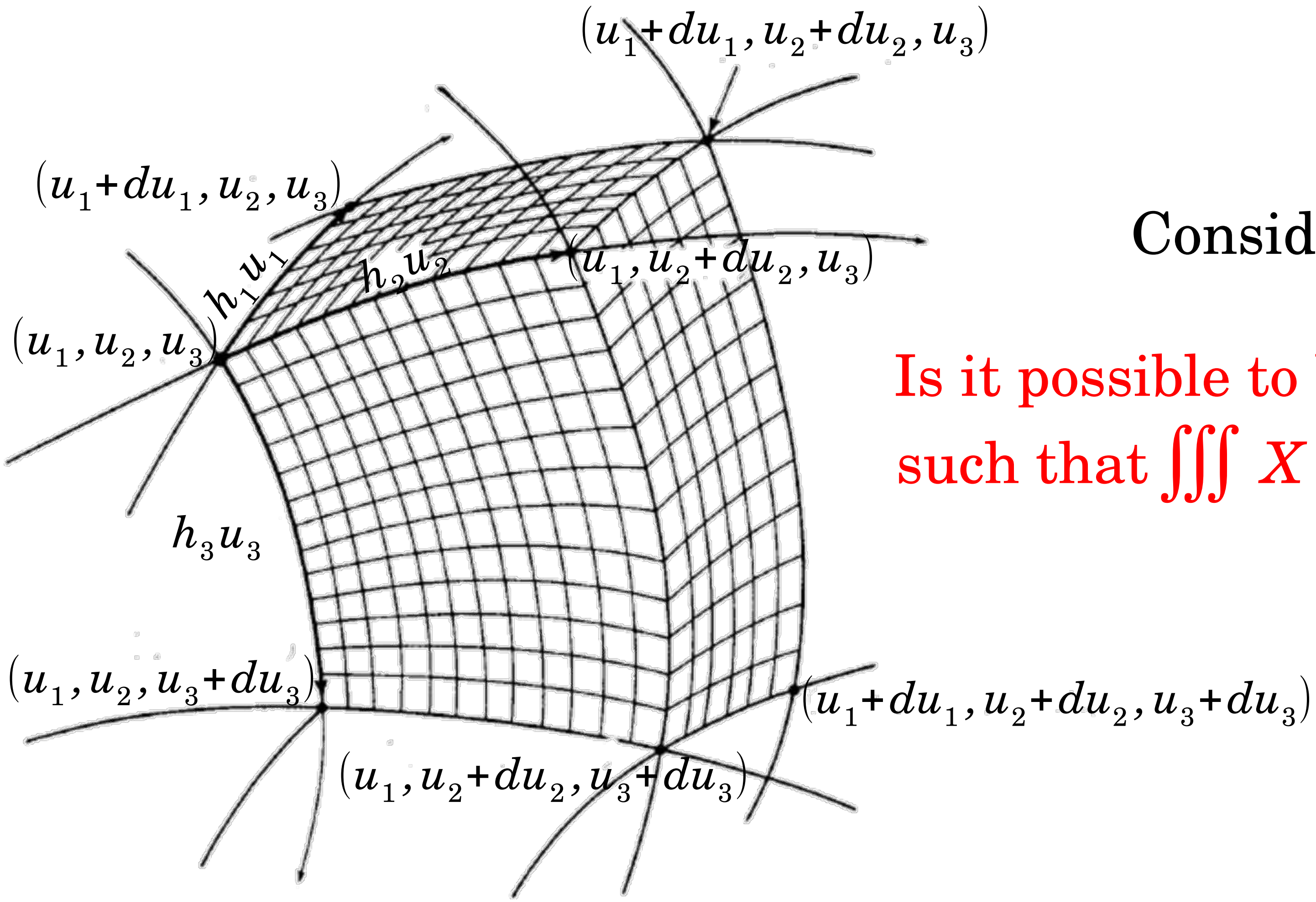
Gradient of a scalar field

$$dT = \frac{\partial T}{\partial u_i}\delta u_i = \nabla T . d\boldsymbol{l} = [(\nabla T)_i \boldsymbol{\epsilon_i}](h_i \delta u_i \boldsymbol{\epsilon_i})$$

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\boldsymbol{\epsilon_1} + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\boldsymbol{\epsilon_2} + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\boldsymbol{\epsilon_3} = \frac{1}{h_i}\frac{\partial T}{\partial u_i}\boldsymbol{\epsilon_i}$$

Does it match with the gradient for the three coordinate systems derived previously?

DIVERGENCE IN CURVILINEAR COORDINATES



Consider a vector $\vec{\boldsymbol{F}}$.

Is it possible to have a function $\boldsymbol{X}(\boldsymbol{F})$ such that $\iiint X(\boldsymbol{F})dV = \oiint \boldsymbol{F}.d\boldsymbol{S}$?

Consider the inifinitesimal volume generated by starting at the point (u_1, u_2, u_3) and increasing each of the coordinates in succession by an inifinitesimal amount.

DIVERGENCE IN CURVILINEAR COORDINATES

Volume element $dV = (h_1 h_2 h_3)\, du_1\, du_2\, du_3$

For the front surface $d\boldsymbol{S} = -(h_2 h_3)\, du_2\, du_3\, \boldsymbol{\epsilon_1}$

Flux through the front surface $[\boldsymbol{F}.d\boldsymbol{S}]_F = -(h_2 h_3 F_1)\, du_2\, du_3$

Flux through the back surface $[\boldsymbol{F}.d\boldsymbol{S}]_B = (h_2 h_3 F_1)\big|_{u_1+du_1} du_2\, du_3$

$$= (h_2 h_3 F_1)\, du_2\, du_3 + \frac{\partial}{\partial u_1}(h_2 h_3 F_1)\, du_1\, du_2\, du_3$$

Front and back together give a contribution

$$[\boldsymbol{F}.d\boldsymbol{S}]_F + [\boldsymbol{F}.d\boldsymbol{S}]_B = \frac{\partial}{\partial u_1}(h_2 h_3 F_1)\, du_1\, du_2\, du_3 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(h_2 h_3 F_1)\, dV$$

Similarly, the left and right sides together give a contribution

$$[\boldsymbol{F}.d\boldsymbol{S}]_L + [\boldsymbol{F}.d\boldsymbol{S}]_R = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}(h_1 h_3 F_2)\, dV$$

and $[\boldsymbol{F}.d\boldsymbol{S}]_T + [\boldsymbol{F}.d\boldsymbol{S}]_B = \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial u_3}(h_1 h_2 F_3)\, dV$

DIVERGENCE IN CURVILINEAR COORDINATES

Adding the flux from the individual sides, the total flux is

$$\boldsymbol{F}\,.\,d\boldsymbol{S}=\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1)+\frac{\partial}{\partial u_2}(h_1 h_3 F_2)+\frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]dV$$

The coefficient of dV defines the $\nabla\,.\,\boldsymbol{F}$ in curvilinear coordinates

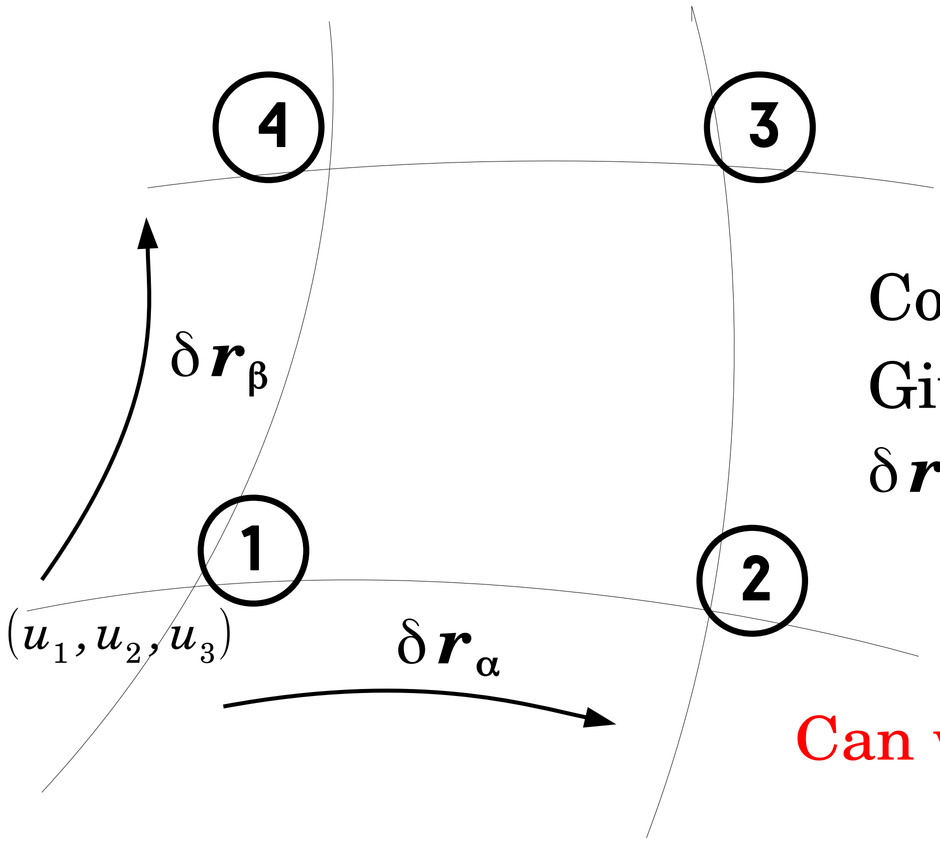
$$\boxed{\nabla\,.\,\boldsymbol{F}=\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1)+\frac{\partial}{\partial u_2}(h_1 h_3 F_2)+\frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]}$$

Cartesian: $\nabla\,.\,\boldsymbol{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$

Spherical: $\nabla\,.\,\boldsymbol{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi)$

Cylindrical: $\nabla\,.\,\boldsymbol{F}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}$

CURL OF VECTOR FIELDS



Consider a vector field $\boldsymbol{F}$.
Given two arbitrary displacements $\delta \boldsymbol{r}_\alpha$ and $\delta \boldsymbol{r}_\beta$ around the point (u_1, u_2, u_3),

Can we define a function $\boldsymbol{X}(\boldsymbol{F})$ such that

$$\iint \boldsymbol{X}(\boldsymbol{F}).d\boldsymbol{S} = \oint \boldsymbol{F}.d\boldsymbol{l}$$

$$\delta \boldsymbol{r}^\alpha = \boldsymbol{\epsilon}_1 h_1 \delta u_1^\alpha + \boldsymbol{\epsilon}_2 h_2 \delta u_2^\alpha + \boldsymbol{\epsilon}_3 h_3 \delta u_3^\alpha$$
$$\delta \boldsymbol{r}^\beta = \boldsymbol{\epsilon}_1 h_1 \delta u_1^\beta + \boldsymbol{\epsilon}_2 h_2 \delta u_2^\beta + \boldsymbol{\epsilon}_3 h_3 \delta u_3^\beta$$

CURL OF VECTOR FIELDS

What is the area element?

$$d\boldsymbol{S} = \delta \boldsymbol{r}^{\alpha} \times \delta \boldsymbol{r}^{\beta} = \begin{vmatrix} \boldsymbol{\epsilon}_1 & \boldsymbol{\epsilon}_2 & \boldsymbol{\epsilon}_3 \\ h_1 \delta u_1^{\alpha} & h_2 \delta u_2^{\alpha} & h_3 \delta u_3^{\alpha} \\ h_1 \delta u_1^{\beta} & h_2 \delta u_2^{\beta} & h_3 \delta u_3^{\beta} \end{vmatrix}$$

What is the flux of the vector ***X(F)*** through this area?

$$\begin{aligned} X(\boldsymbol{F}).d\boldsymbol{S} = \;& X_1 h_2 h_3 [\delta u_2^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_2^{\beta}] \\ & - X_2 h_1 h_3 [\delta u_1^{\alpha} \delta u_3^{\beta} - \delta u_3^{\alpha} \delta u_1^{\beta}] \\ & + X_3 h_1 h_2 [\delta u_1^{\alpha} \delta u_2^{\beta} - \delta u_2^{\alpha} \delta u_1^{\beta}] \end{aligned}$$

CURL OF VECTOR FIELDS

Consider the pair of paths $(1\rightarrow 2)$ and $(3\rightarrow 4)$

$$\boldsymbol{F}.\delta\boldsymbol{l}\big|_{1\rightarrow 2} = F_1 h_1 \delta u_1^{\alpha} + F_2 h_2 \delta u_2^{\alpha} + F_3 h_3 \delta u_3^{\alpha}$$

$$\boldsymbol{F}.\delta\boldsymbol{l}\big|_{3\rightarrow 4} = \left[F_i h_i + (\nabla F_i h_i).\delta\boldsymbol{r}^{\beta}\right](-\delta u_i^{\alpha}) \qquad (i=1,2,3)$$

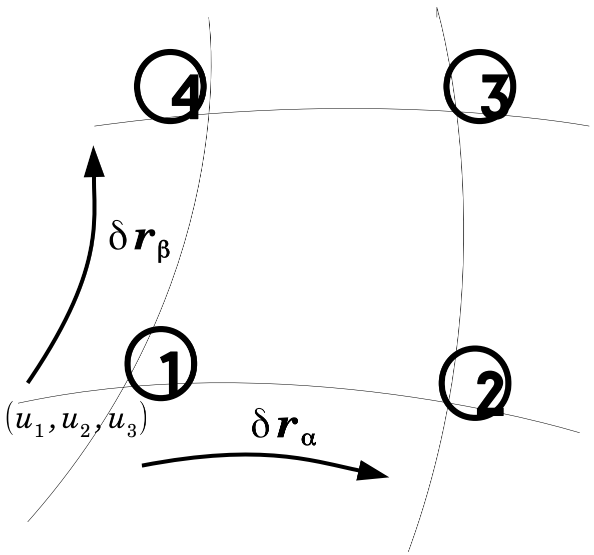
Write contributions from $\boldsymbol{F}.\delta\boldsymbol{l}\big|_{2\rightarrow 3}$ & $\boldsymbol{F}.\delta\boldsymbol{l}\big|_{4\rightarrow 1}$ similarly.

Full path gives:

$$(\nabla\boldsymbol{F}.\delta r^{\beta}).\delta r^{\alpha} - (\nabla\boldsymbol{F}.\delta r^{\alpha}).\delta r^{\beta}$$

$$= \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\beta}\right] h_k \delta u_k^{\alpha} - \sum_{k,i}\left[\frac{1}{h_k}\frac{\partial F_i h_i}{\partial u_k}\delta u_i^{\alpha}\right] h_k \delta u_k^{\beta}$$

$$= \sum_{k,i}\left[\frac{\partial F_i h_i}{\partial u_k} - \frac{\partial F_k h_k}{\partial u_i}\right]\delta u_i^{\beta}\delta u_k^{\alpha}$$



CURL OF VECTOR FIELDS

Comparing coefficients,

$$X_1 h_2 h_3 = \left[\frac{\partial F_3 h_3}{\partial u_2} - \frac{\partial F_2 h_2}{\partial u_3} \right]$$

The function X(F) is then,

So $$\boldsymbol{X}(\boldsymbol{F}) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \boldsymbol{\epsilon_1} & h_2 \boldsymbol{\epsilon_2} & h_3 \boldsymbol{\epsilon_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \equiv \begin{cases} \nabla \times \boldsymbol{F} \\ curl\, \boldsymbol{F} \\ rot\, \boldsymbol{F} \end{cases}$$

We have $\iint \nabla \times \boldsymbol{F} . d\boldsymbol{S} = \oint \boldsymbol{F} . d\boldsymbol{l}$ $\quad (Stoke's\ theorem)$

GRADIENT, DIVERGENCE, CURL – GENERALIZED COORDINATES

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\boldsymbol{\epsilon_1} + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\boldsymbol{\epsilon_2} + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\boldsymbol{\epsilon_3} = \frac{1}{h_i}\frac{\partial T}{\partial u_i}\boldsymbol{\epsilon_i}$$

$$\nabla . \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]$$

$$\begin{aligned}\nabla \times \boldsymbol{F} = \frac{1}{h_1 h_2 h_3}\Bigg[& h_1 \boldsymbol{\epsilon_1}\left\{\frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2)\right\} \\ & + h_2 \boldsymbol{\epsilon_2}\left\{\frac{\partial}{\partial u_3}(h_1 F_1) - \frac{\partial}{\partial u_1}(h_3 F_3)\right\} \\ & + h_3 \boldsymbol{\epsilon_3}\left\{\frac{\partial}{\partial u_1}(h_2 F_2) - \frac{\partial}{\partial u_2}(h_1 F_1)\right\}\Bigg]\end{aligned}$$

IS EVERYTHING CONSISTENT? - GRADIENT

$$f=\rho^3\sin\phi$$

$$\nabla f= 3\rho^2\sin\phi\,\hat{\rho}+\rho^2\cos\phi\,\hat{\phi}$$

$$f=x^2y+y^3$$

$$\nabla f=2xy\,\hat{x}+(3y^2+x^2)\,\hat{y}$$

$$\begin{aligned}\nabla f&=3\rho^2\sin\phi(\cos\phi\,\hat{x}+\sin\phi\,\hat{y})+\rho^2\cos\phi(-\sin\phi\,\hat{x}+\cos\phi\,\hat{y})\\&=[2\rho^2\sin\phi\cos\phi]\,\hat{x}+[3\rho^2\sin^2\phi+\rho^2\cos^2\phi]\,\hat{y}\\&=2xy\,\hat{x}+(3y^2+x^2)\,\hat{y}\end{aligned}$$

$$f=\frac{1}{r}$$

$$\nabla f= -\frac{1}{r^2}\hat{r}$$

$$f=\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f=-\frac{1}{r^2}(\sin\theta\cos\phi\,\hat{x}+\sin\theta\sin\phi\,\hat{y}+\cos\theta\,\hat{z})=-\frac{1}{r^2}\left(\frac{x}{r}\hat{x}+\frac{y}{r}\hat{y}+\frac{z}{r}\hat{z}\right)$$

$$=-\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{x}+\frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{y}+\frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{z}\right)$$

IS EVERYTHING CONSISTENT? - DIVERGENCE

$$\boldsymbol{F}=\rho^3\hat{\rho}+\rho z\hat{\phi}+\rho z\sin\phi\,\hat{z}$$

$$\nabla.\boldsymbol{F}=\frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{\partial}{\partial\phi}(F_\phi)+\frac{\partial}{\partial z}(\rho F_z)\right\}=4\rho^2+\rho\sin\phi$$

$$\boldsymbol{F}=(x^3+xy^2-yz)\hat{x}+(x^2y+y^3+xz)\hat{y}+yz\hat{z}$$

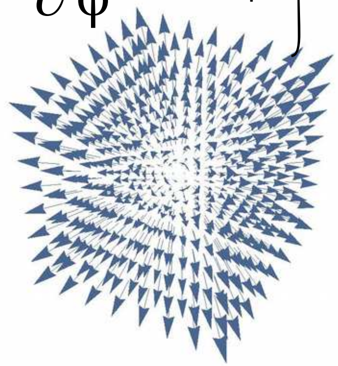
$$\nabla.\boldsymbol{F}=4(x^2+y^2)+y$$

$$\boldsymbol{F}=r\hat{\boldsymbol{r}}$$

$$\nabla.\boldsymbol{F}=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta F_r)+\frac{\partial}{\partial\theta}(r\sin\theta F_\theta)+\frac{\partial}{\partial\phi}(rF_\phi)\right\}=3$$

$$\boldsymbol{F}=x\hat{x}+y\hat{y}+z\hat{z}$$

$$\nabla.\boldsymbol{F}=3$$



IS EVERYTHING CONSISTENT? - CURL

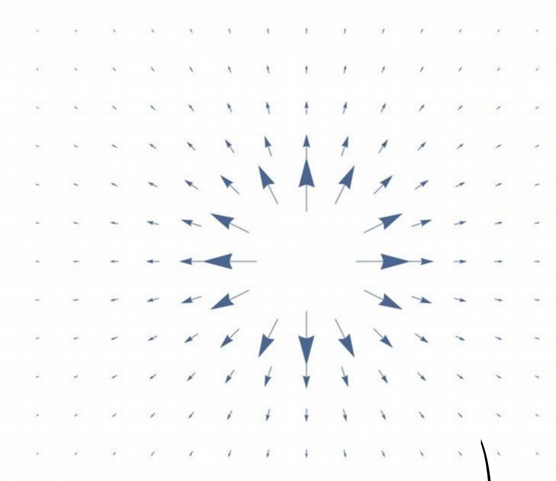
$$\boldsymbol{F} = \rho^3\,\hat{\rho} + \rho\, z\,\hat{\phi} + \rho\, z \sin\phi\,\hat{z}$$

$$\nabla \times \boldsymbol{F} \;=\; \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\,\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \rho^3 & \rho^2 z & \rho\, z \sin\phi \end{vmatrix} = \left(z\cos\phi - \rho\right)\hat{\rho} + z\sin\phi\,\hat{\phi} + 2\,z\,\hat{z}$$

$$\boldsymbol{F} = \left(x^3 + x\,y^2 - y\,z\right)\hat{x} + \left(x^2\,y + y^3 + xz\right)\hat{y} + y\,z\,\hat{z}$$

$$\boldsymbol{F} = r^k\,\hat{\boldsymbol{r}}$$

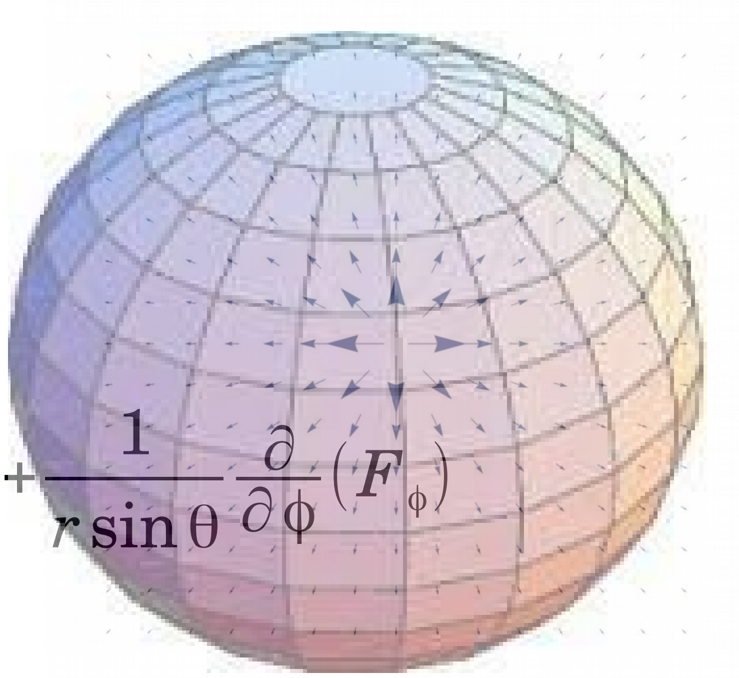
$$\nabla \times \boldsymbol{F} \;=\; \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ r^k & r\,.\,0 & r\sin\theta\,.\,0 \end{vmatrix} = 0$$



$$\boldsymbol{F} = \left(\frac{x}{\left(x^2+y^2+z^2\right)^{(k+1)/2}}\,\hat{x} + \frac{y}{\left(x^2+y^2+z^2\right)^{(k+1)/2}}\,\hat{y} + \frac{z}{\left(x^2+y^2+z^2\right)^{(k+1)/2}}\,\hat{z}\right)$$

DIVERGENCE OF 1/R²

Consider the vector function $\boldsymbol{v}=\dfrac{\hat{\boldsymbol{r}}}{r^2}$



What is $\nabla.\boldsymbol{v}$?

$$\nabla.\boldsymbol{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi)$$

$$\nabla.\boldsymbol{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)=0$$

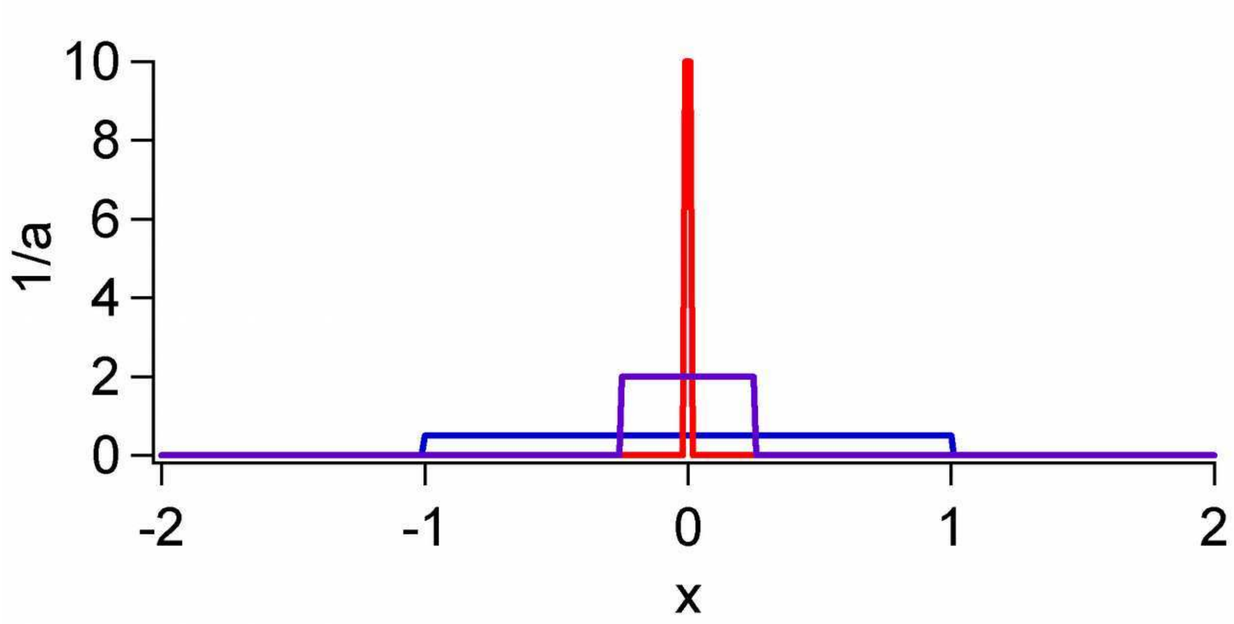
$$\iiint\nabla.\boldsymbol{v}\,dV=0$$

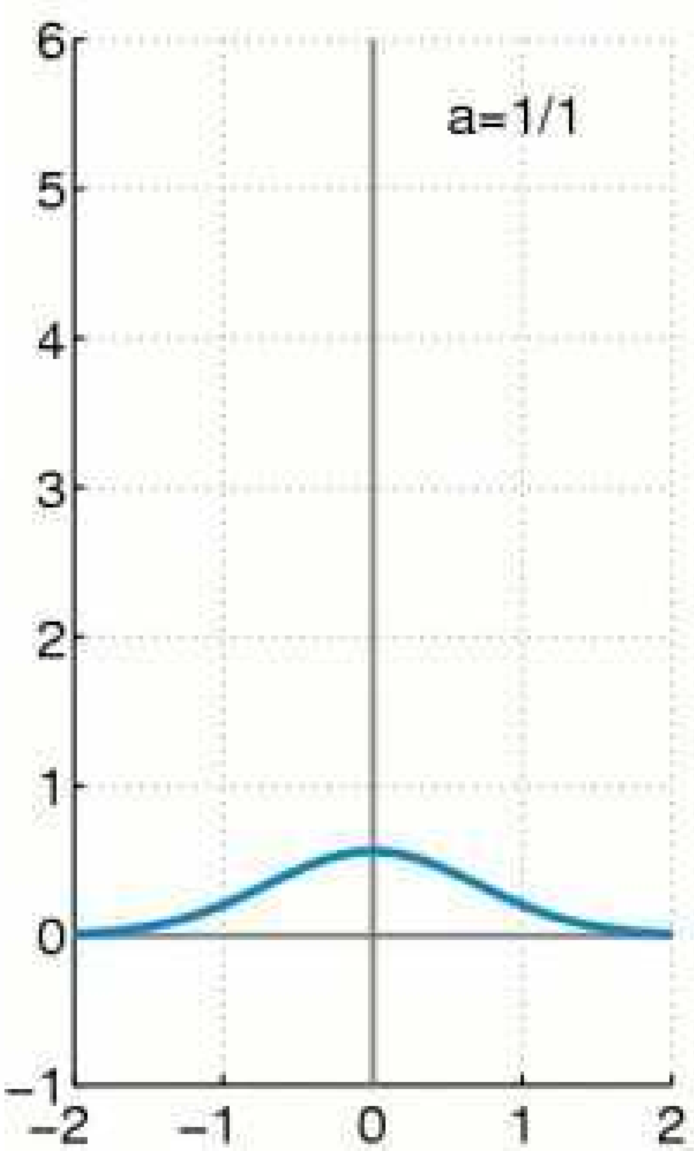
Something is wrong!

$$\oint\boldsymbol{v}.d\boldsymbol{a}=\iint\left(\frac{1}{R^2}\hat{\boldsymbol{r}}\right).(R^2\sin\theta\,d\theta\,d\phi\,\hat{\boldsymbol{r}})=\left(\int_0^\pi\sin\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right)=4\pi$$

THE DIRAC DELTA FUNCTION

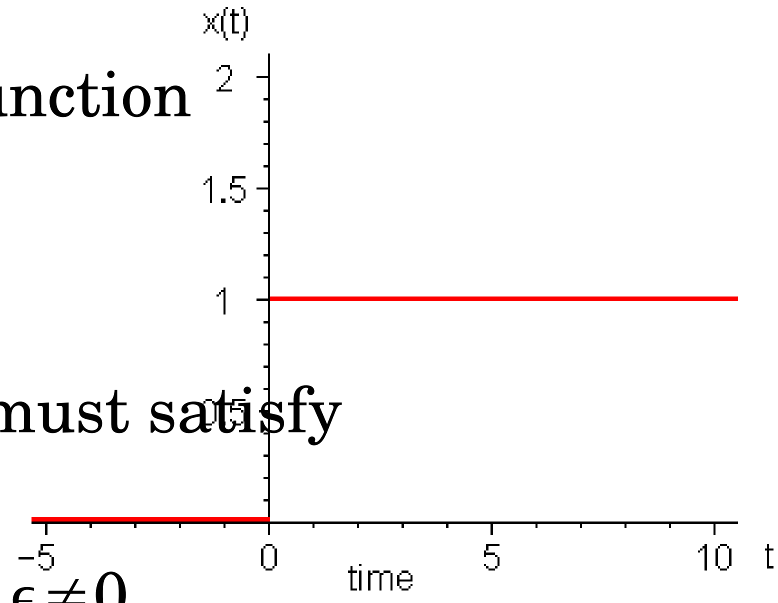
$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$$





THE DIRAC DELTA FUNCTION

Consider a simpler example: the step function



$$x(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$\dfrac{dx}{dt} = ?$ looks like zero everywhere but must satisfy

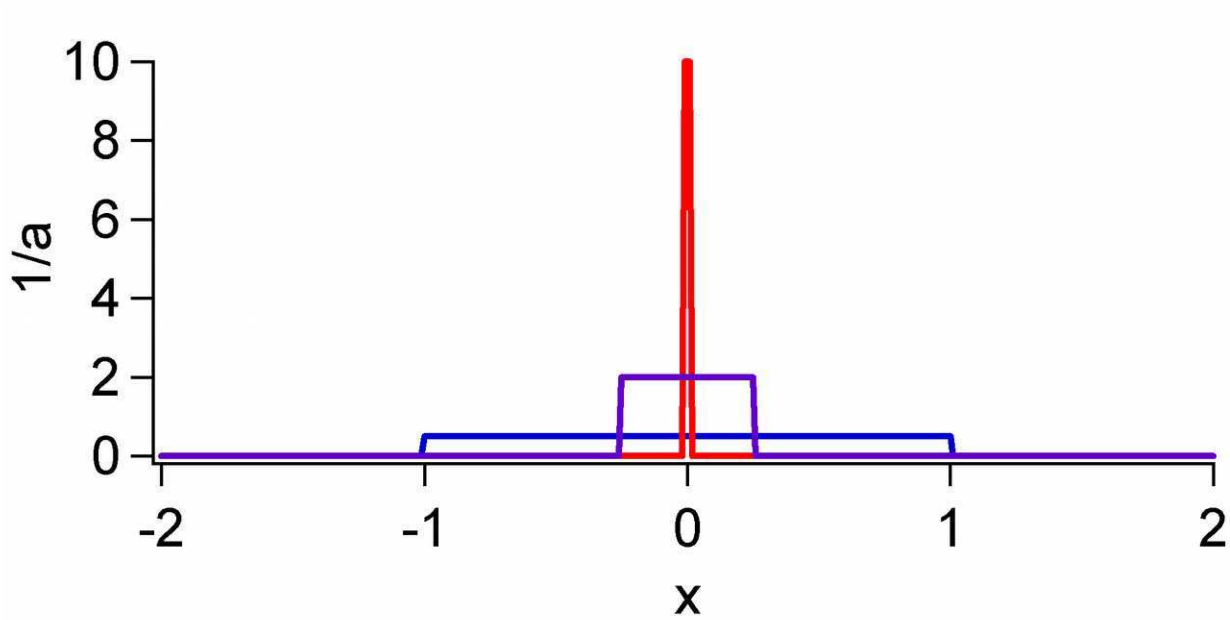
$$\int_{0-|\epsilon|}^{0+|\epsilon|} \left(\frac{dx}{dt}\right) dt \;=\; x(|\epsilon|) \;-\; x(-|\epsilon|) \;=\; 1 \quad \forall \quad \epsilon \neq 0$$

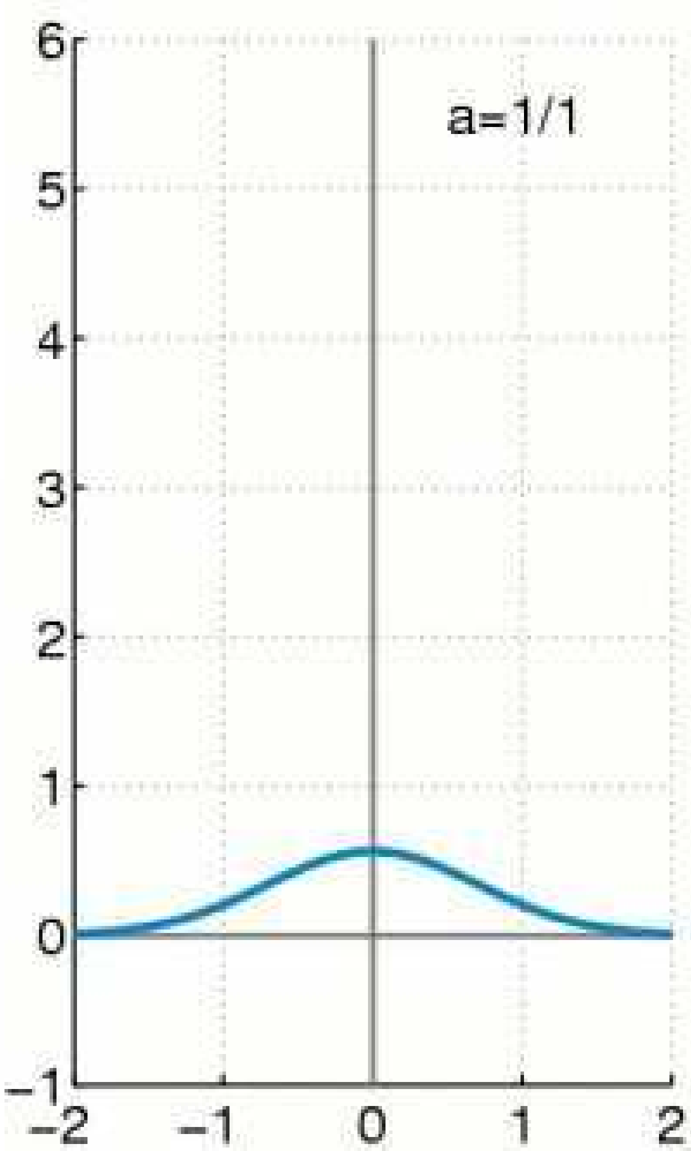
Such integrable singularities are treated by ***defining*** the δ function

$$\int_a^b \delta(x-x_0) f(x)\, dx \;=\; \begin{cases} f(x_0), & if\ x_0\, is\, within\, the\, limits \\ 0 & otherwise \end{cases}$$

The Delta function is not a proper function, but rather the limit of a sequence of functions. It’s called a generalized function or distribution, and is properly defined only within an integral.

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$$





THE 3D DIRAC DELTA FUNCTION

$$\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$$

$$\int_{\text{all space}} \delta(\boldsymbol{r})\, dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\, dx\, dy\, dz = 1$$

$$\int_{\text{all space}} f(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{a})\, dV = f(\boldsymbol{a})$$

$$\iiint \nabla \cdot \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\, dV = 4\pi$$

$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2} = 4\pi\delta^3(\boldsymbol{r})$$

Prove:

$$\delta(-x) = \delta(x)$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$x\frac{d}{dx}(\delta(x)) = -\delta(x)$$

UNIQUE SPECIFICATION OF A VECTOR FIELD

If we specify the curl of a function $\nabla \times \boldsymbol{F} = \boldsymbol{C}$,
can we determine the function $\boldsymbol{F}$?

NO. $\boldsymbol{F}' = \boldsymbol{F} + \nabla \phi$ will give same result!

If we specify the divergence of a function $\nabla . \boldsymbol{F} = D$,
can we determine the function $\boldsymbol{F}$?

NO. $\boldsymbol{F}' = \boldsymbol{F} + \nabla \times \boldsymbol{A}$ will give same result!

If we specify the divergence **AND** the curl of a function

$$\nabla \times \boldsymbol{F} = \boldsymbol{C} \quad \textbf{AND} \quad \nabla . \boldsymbol{F} = D,$$

can we determine the function $\boldsymbol{F}$?

YES! (Provided the boundary conditions are specified)

HELMHOLTZ'S THEOREM

HELMHOLTZ'S THEOREM

If $\quad \nabla \times \boldsymbol{F} = \boldsymbol{C}$ **AND** $\nabla . \boldsymbol{F} = D,$

Note that, necessarily $\nabla . \boldsymbol{C} = 0$

We claim $\quad \boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$

where $\quad U(\boldsymbol{r}) = \dfrac{1}{4\pi}\displaystyle\int \frac{D(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$

and $\quad \boldsymbol{W}(\boldsymbol{r}) = \dfrac{1}{4\pi}\displaystyle\int \frac{\boldsymbol{C}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,dV'$

$$\nabla . \boldsymbol{F} = -\nabla^2 U = -\frac{1}{4\pi}\int D(\boldsymbol{r}')\nabla^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)dV' = \int D(\boldsymbol{r}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')\,dV'$$

$$= D(\boldsymbol{r})$$

HELMHOLTZ'S THEOREM

$$\nabla\times\boldsymbol{F}=\nabla\times(\nabla\times\boldsymbol{W})=-\nabla^2\boldsymbol{W}+\nabla(\nabla\,.\,\boldsymbol{W})$$

$$-\nabla^2\boldsymbol{W}=-\frac{1}{4\pi}\int\boldsymbol{C}(\boldsymbol{r}')\nabla^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)dV'=\int\boldsymbol{C}(\boldsymbol{r}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')\,dV'=\boldsymbol{C}(\boldsymbol{r})$$

$$4\pi\nabla\,.\,\boldsymbol{W}=\int\boldsymbol{C}(\boldsymbol{r}').\nabla\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)dV'=-\int\boldsymbol{C}(\boldsymbol{r}').\nabla'\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)dV'$$

$$=\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'.\boldsymbol{C}(\boldsymbol{r}')\,dV-\oint\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{C}\,.\,d\boldsymbol{a}$$

$$=0\qquad(\text{if C goes to zero sufficiently rapidly})$$

$$\nabla.(f\vec{A})=f(\nabla.\vec{A})+\vec{A}.(\nabla f)$$

How rapidly? **C** and D must go to zero more rapidly than $1/r^2$

HELMHOLTZ'S THEOREM

Is this solution unique?

YES, as long as the vector field $\boldsymbol{F}(\boldsymbol{r})$ itself goes to zero at infinity

If the divergence D(**r**) and the curl **C(r)** of a vector function **F(r)** are specified, and if they both go to zero faster than $1/r^2$ as r goes to infinity, and if **F(r)** itself goes to zero as r goes to infinity, then **F(r)** is uniquely given by

$$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$$

MAXWELL'S EQUATIONS

$$\nabla . \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations specify the divergence and curl of the electric and magnetic fields. Using Helmholtz's theorem, we can then determine the electric and magnetic fields from Maxwell's equations.

MAXWELL'S EQUATIONS - ELECTROSTATICS

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0$$

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\left(\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathfrak{R}}\, dV'\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\mathfrak{R}^2}\,\hat{\mathfrak{R}}\, dV'$$

$$\mathfrak{R} = \boldsymbol{r} - \boldsymbol{r}'$$

Coulomb's Law!

For a continuous surface charge, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\boldsymbol{r}')}{\mathfrak{R}^2}\,\hat{\mathfrak{R}}\, da'$

For a continuous line charge, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\boldsymbol{r}')}{\mathfrak{R}^2}\,\hat{\mathfrak{R}}\, dl'$

For a collection of discrete charges, $\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathfrak{R}_i^2}\,\hat{\mathfrak{R}}_{\boldsymbol{i}}$

ELECTROSTATICS

ELECTROSTATICS

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0$$
$$\nabla \times \boldsymbol{E} = 0$$

Maxwell's Equations for Electrostatics

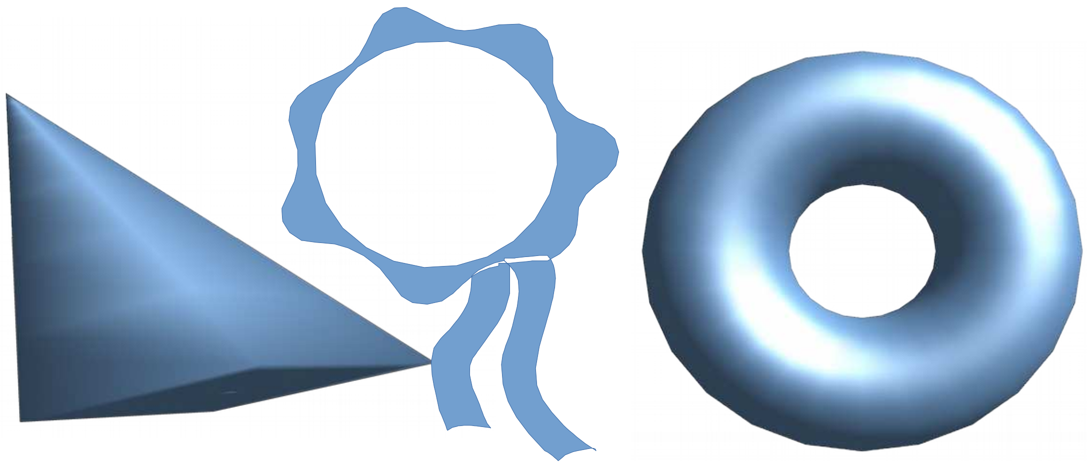
$$\xrightarrow{\text{Helmholtz Theorem}}$$

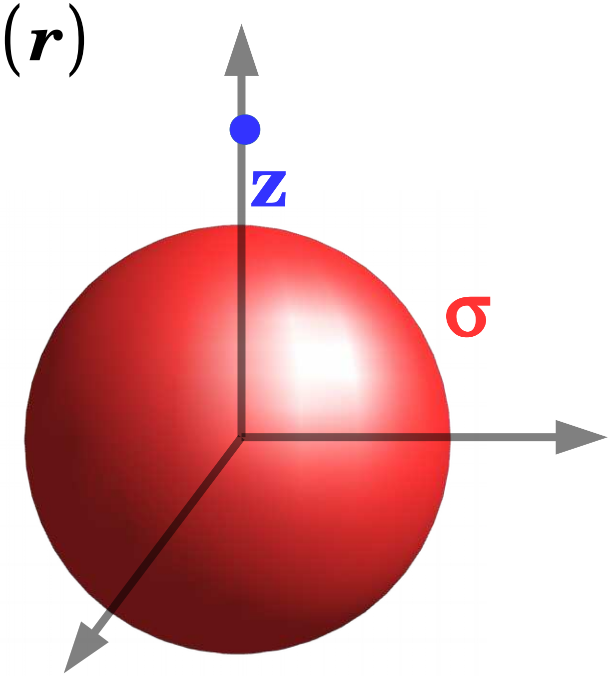
$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}')}{\Re^2} \hat{\boldsymbol{\Re}}\, dV'$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\Re_i^2} \hat{\boldsymbol{\Re}}_{\boldsymbol{i}}$$

Coulomb's Law

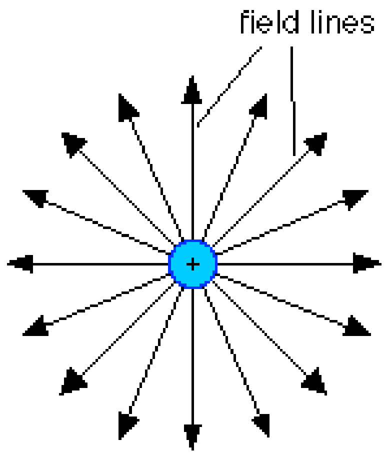
Force on a test charge Q, $\boldsymbol{F}(\boldsymbol{r}) = Q\,\boldsymbol{E}(\boldsymbol{r})$



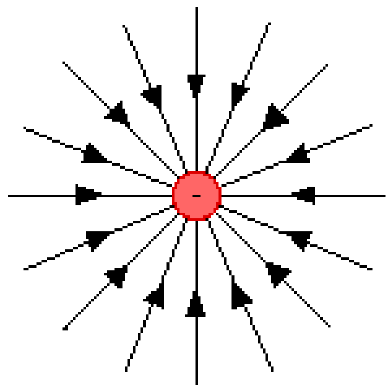


In principle, we're done with Electrostatics!

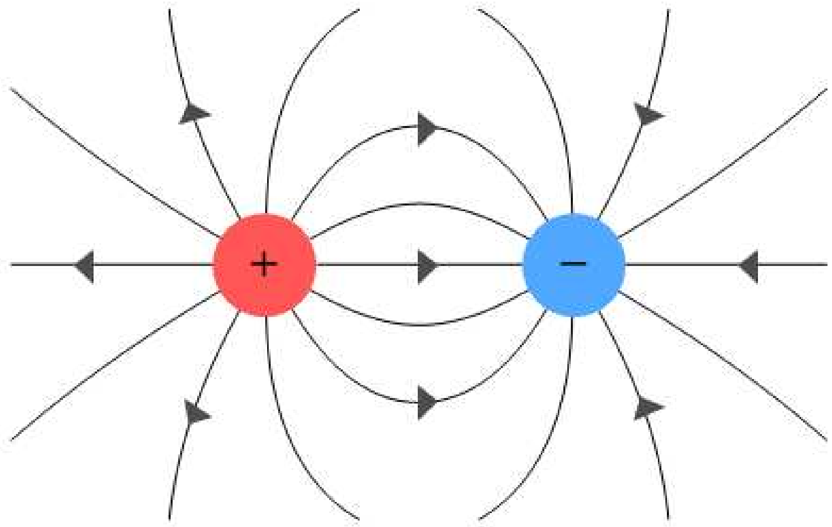
ELECTRIC FIELD LINES



The electric field from an isolated positive charge

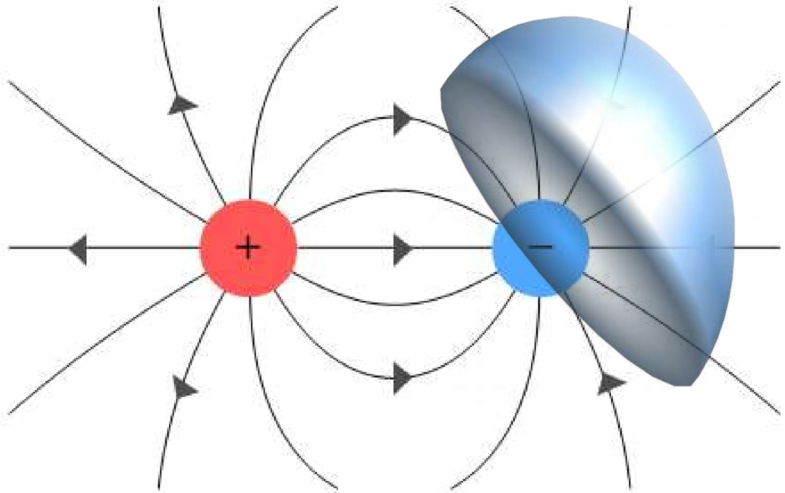


The electric field from an isolated negative charge



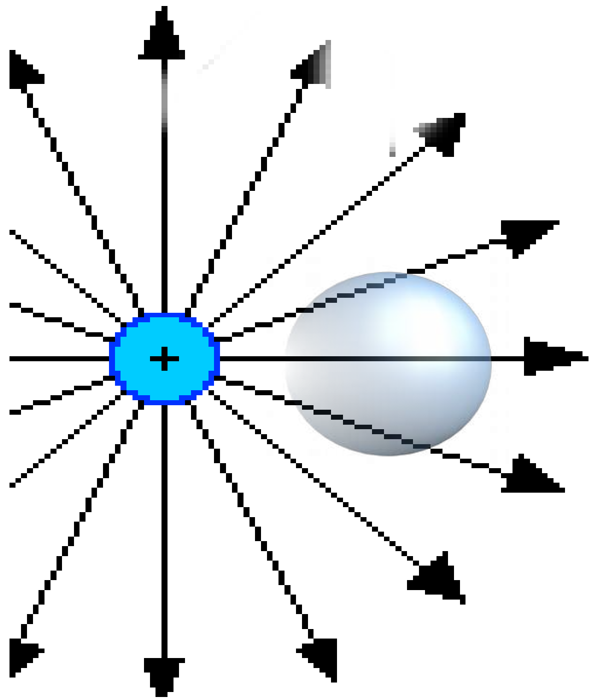
- Field lines begin on positive charges
- Field lines end on negative charges, or they extend upto infinity
- The strength of the field is indicated by the density of the field lines
- Field lines can never cross

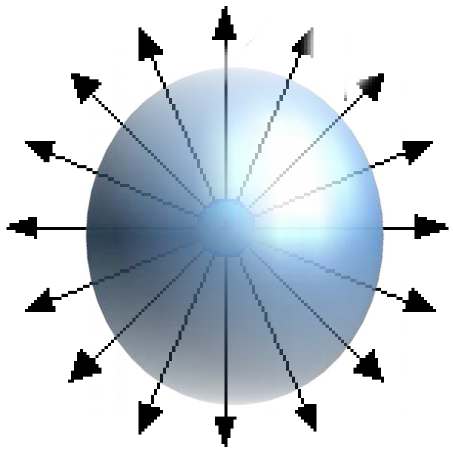
ELECTRIC FIELD LINES – CLOSED SURFACE

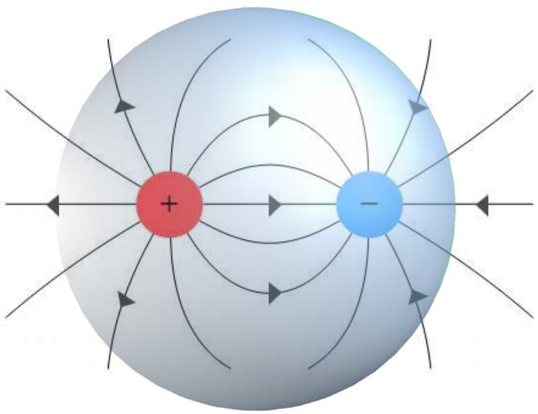


Flux $\Phi_E = \oint_S \boldsymbol{E} . d\boldsymbol{S}$

Flux $\propto$ Number of field lines

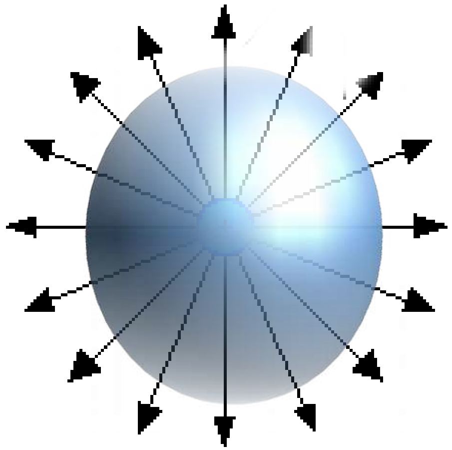






Flux through a closed surface is a measure of the total charge inside the surface – Gauss's Law

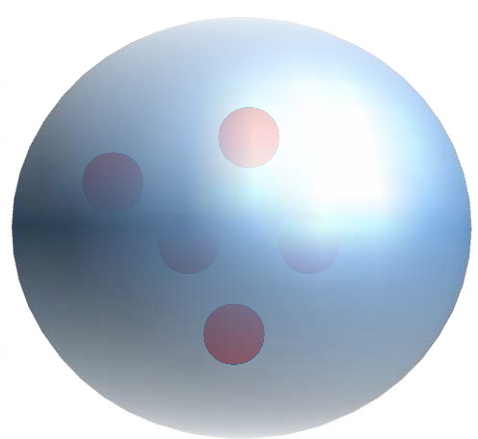
GAUSS'S LAW



Point charge q at origin

$$\oint \boldsymbol{E}.d\boldsymbol{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\boldsymbol{r}}\right)(r^2 \sin\theta\, d\theta\, d\phi\, \hat{\boldsymbol{r}}) = \frac{q}{\epsilon_0}$$

Principle of superposition $\quad \boldsymbol{E} = \sum_{i=1}^{N} \boldsymbol{E}_i$

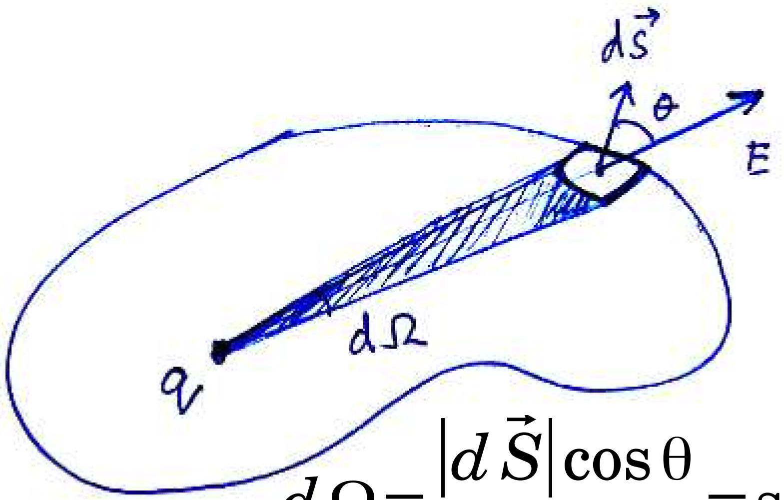


Collection of point charges q_i

$$\oint \boldsymbol{E}.d\boldsymbol{a} = \sum_{i=1}^{N}\left(\oint \boldsymbol{E}.d\boldsymbol{a}\right) = \sum_{i=1}^{N}\left(\frac{q_i}{\epsilon_0}\right)$$

$$\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{1}{\epsilon_0}\boldsymbol{Q}_{enc}$$

GAUSS'S LAW



$$\int_{surface} \boldsymbol{E}.d\boldsymbol{S} = \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{\hat{\boldsymbol{r}}.d\boldsymbol{S}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\int_{surface}\frac{|d\boldsymbol{S}|\cos\theta}{r^2}$$

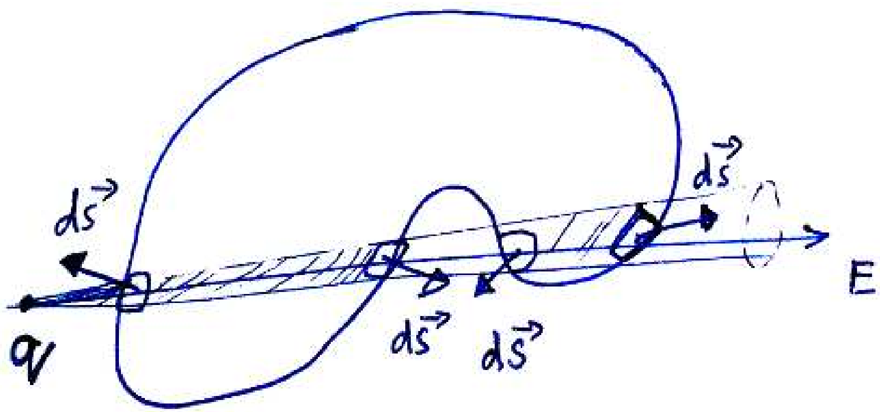
$$= \frac{q}{4\pi\epsilon_0}\int_{surface} d\Omega$$

$$= \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

$$d\Omega = \frac{|d\vec{S}|\cos\theta}{r^2} = \sin\theta\, d\theta\, d\phi$$

The solid angle subtended by a surface S is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere.

A measure of how large an object appears to an observer looking from that point



If the point is located outside then the contributions exactly cancel

Use superposition principle ---> Add contribution from each charge

GAUSS'S LAW - DIFFERENTIAL FORM

$$\int_{surface} \boldsymbol{E} . d\boldsymbol{S} = \int_{vol} \nabla . \boldsymbol{E} \, dV$$

$$\frac{1}{\epsilon_0} Q_{enc} = \int_{vol} \frac{\rho(\boldsymbol{r})}{\epsilon_0} dV$$

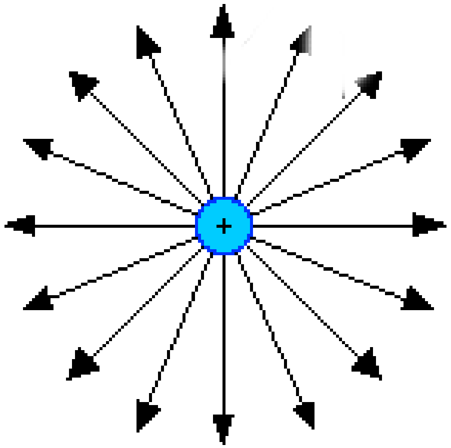
$$\oint_S \boldsymbol{E} . d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{enc} \Rightarrow \nabla . \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

$$\begin{tikzcd} \nabla . \boldsymbol{E} = \dfrac{\rho(\boldsymbol{r})}{\epsilon_0} \arrow[r, bend left, "\text{Helmholtz Theorem}"] & \boldsymbol{E} = \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{r^2} \hat{\boldsymbol{r}} \arrow[l, bend left, "\text{Gauss's Law}"] \end{tikzcd}$$

Valid for moving charges!

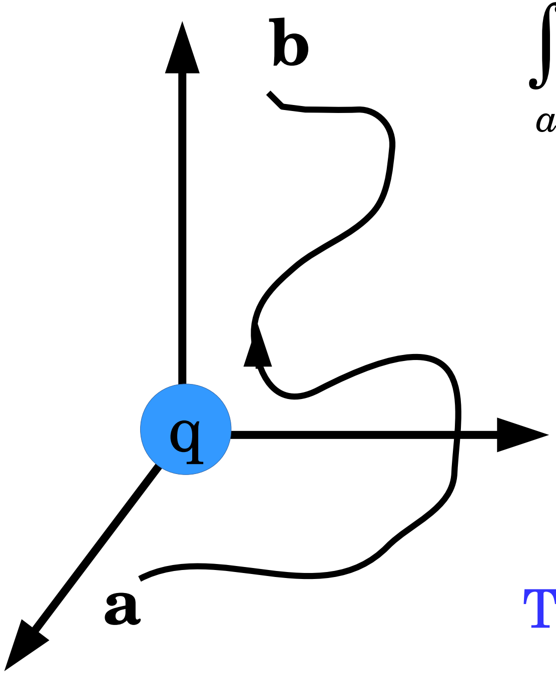
Only for static charges

CURL OF THE ELECTRIC FIELD



$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

$$\nabla\times\boldsymbol{E} = 0$$



$$\int_a^b \boldsymbol{E}.d\boldsymbol{l} = \int_a^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}\right).(dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}})$$

$$\int_a^b \boldsymbol{E}.d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

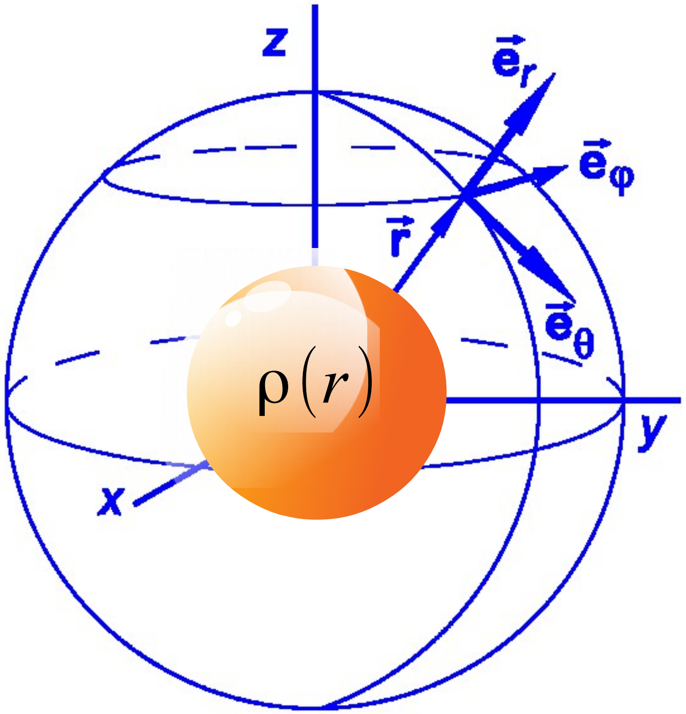
$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \overset{\text{Stokes Theorem}}{\Rightarrow} \quad \nabla\times\boldsymbol{E} = 0$$

True for any charge configuration due to superposition!

Valid only for static charges

GAUSS'S LAW + SYMMETRY - SPHERE

Consider a spherically symmetric charge distribution $\rho(r)$



$E_\phi = 0$ Why?

Rotate about the z-axis

$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\phi = 0$$

$E_\theta = 0$ Why?

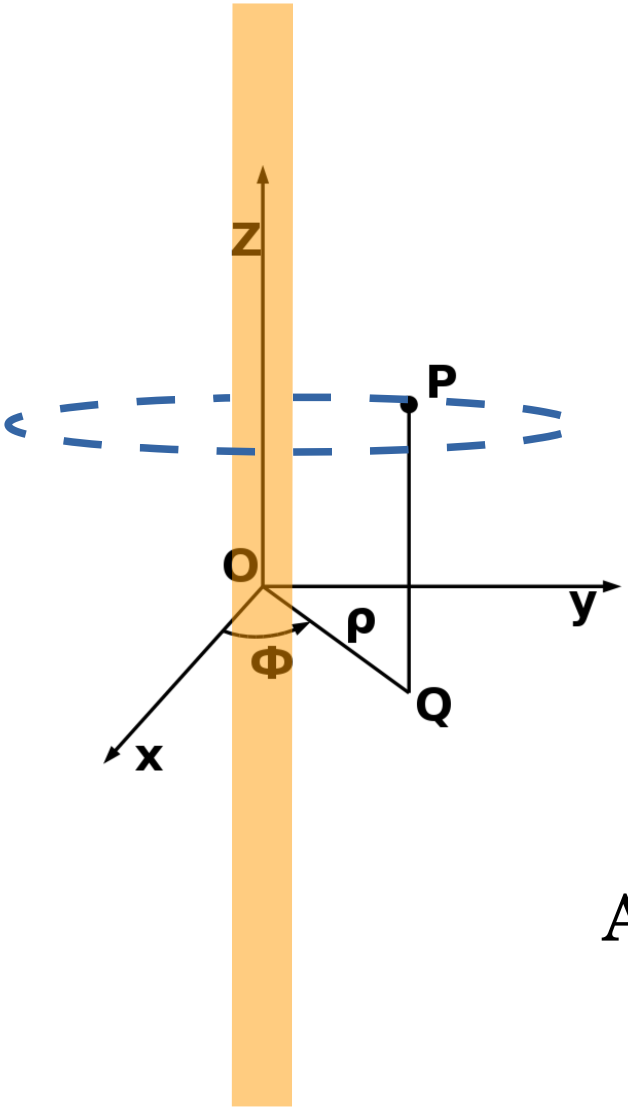
Rotate about the x-axis

$$\oint \boldsymbol{E}.d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\theta = 0$$

Apply Gauss's Law: $$E_r.4\pi R^2 = \frac{1}{\epsilon_0}\int_0^R \rho(r) 4\pi r^2.dr$$

GAUSS'S LAW + SYMMETRY - CYLINDER

Consider a long, narrow wire with a charge per unit length λ



$E_\phi = 0$ Why?

Rotate about the z-axis

$$\oint \boldsymbol{E} . d\boldsymbol{l} = 0 \quad \Rightarrow \quad E_\phi = 0$$

$E_z = 0$ Why?

Flip about z-axis

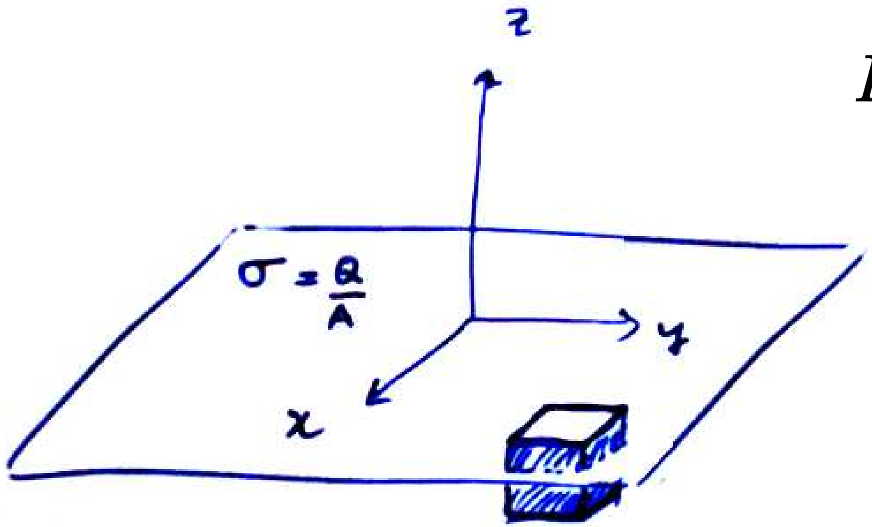
Nothing distinguishes z from -z

$$\Rightarrow \quad E_z = 0$$

Apply Gauss's Law: $E_\rho . 2\pi\rho = \frac{1}{\epsilon_0}\lambda$

GAUSS'S LAW + SYMMETRY - SURFACE

Consider an infinite sheet of charge with a surface charge density σ



$E_{//}\ (E_x, E_y) = 0$ Why?

Rotate the sheet about any point

Translate by any in-plane vector

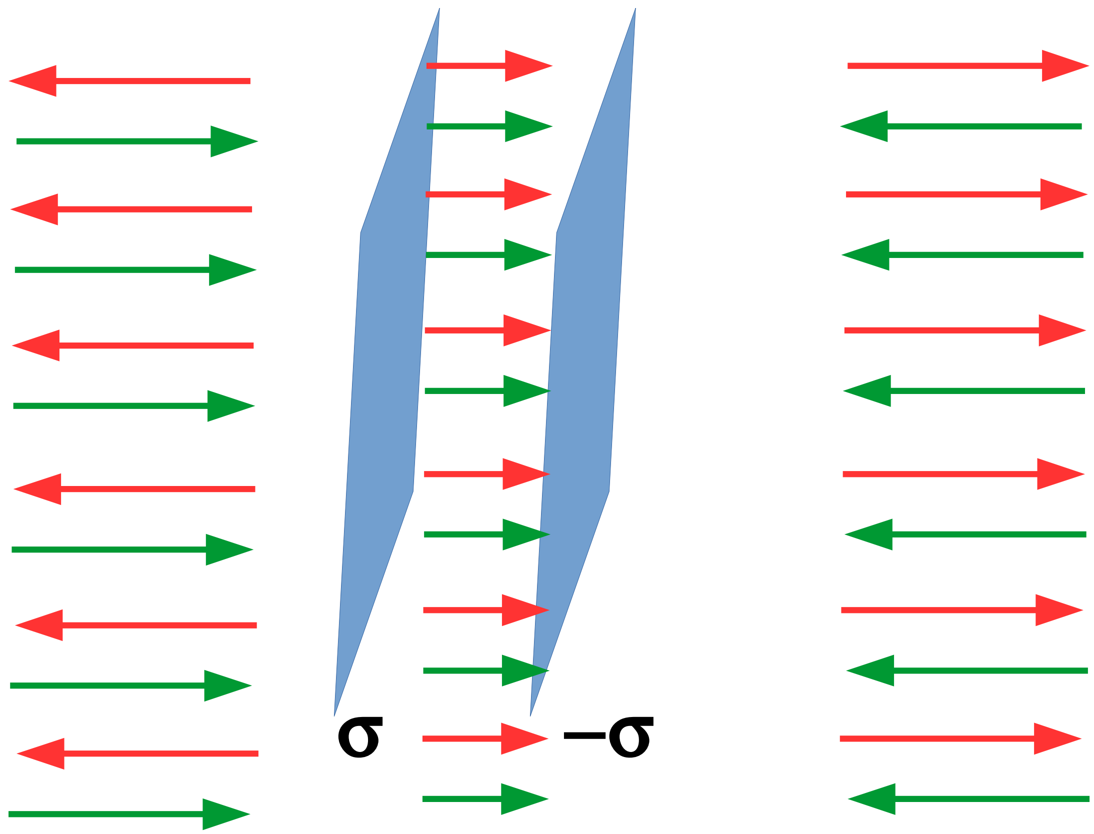
Field cannot change $\Rightarrow$ $E_{//} = 0$

Apply Gauss's Law: $$\int \boldsymbol{E}.d\boldsymbol{a} = 2A|\boldsymbol{E}| = \frac{1}{\epsilon_0}\sigma A$$

$$\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}}$$

GAUSS'S LAW + SYMMETRY - SURFACE

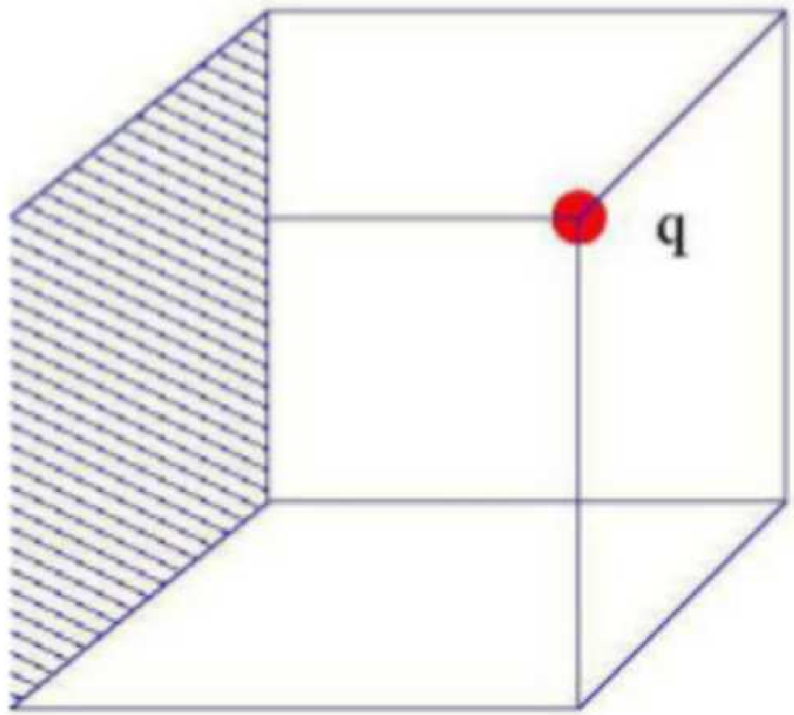
Consider two parallel plates with equal and opposite charge densities $\pm\sigma$. What is the electric field?

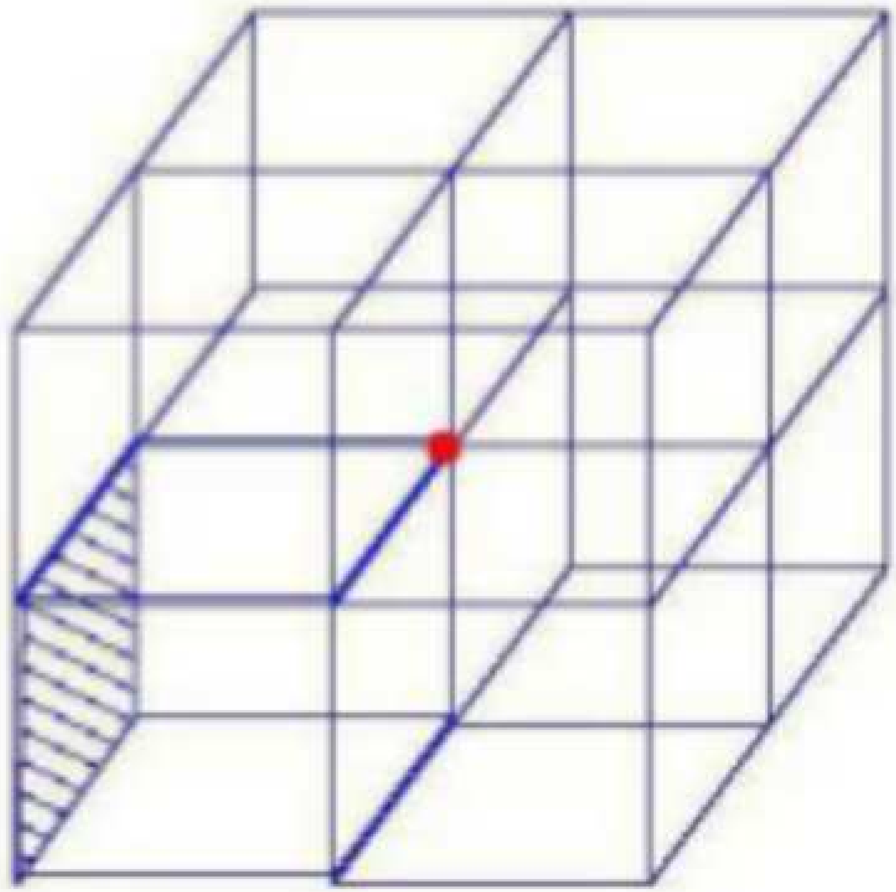


$$C = \frac{\epsilon_0 A}{d}$$

$$\boldsymbol{E} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} + \dfrac{\sigma}{2\epsilon_0}\hat{\boldsymbol{n}} = \dfrac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

GAUSS'S LAW + SYMMETRY





What is the flux of the electric field through the shaded face?

$$\frac{q}{24\,\epsilon_0}$$

THE ELECTRIC POTENTIAL

Gauss's Law is always true.

It may not always be useful!

If we can take advantage of the symmetries of a problem, Gauss's Law can be a very powerful tool.

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E} = -\nabla V$$

$$V(\boldsymbol{r}) \equiv \text{Electric Potential}$$

$$\nabla . \boldsymbol{E} = \rho/\epsilon_0 \quad \Rightarrow \quad \boxed{\nabla^2 V = -\rho/\epsilon_0}$$ Poisson's Equation

In regions of no charge $\boxed{\nabla^2 V = 0}$ Laplace Equation