1. $\vec{F}(\vec{r}) = \dfrac{\Psi(\rho)}{\rho^n}\hat{\rho}$

$\vec{\nabla}\cdot\vec{F}(\vec{r}) = 4$

$\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left[\rho\left(\dfrac{\Psi(\rho)}{\rho^n}\right)\right] = 4$

$\Rightarrow \dfrac{\partial}{\partial\rho}\left[\rho\left(\dfrac{\Psi(\rho)}{\rho^n}\right)\right] = 4\rho$

$\Rightarrow \dfrac{\Psi(\rho)}{\rho^{n-1}} = 2\rho^2 + C$

$\Rightarrow \boxed{\Psi(\rho) = \rho^{n-1}\left[C + 2\rho^2\right]}$ I

$|\vec{F}(\rho_0, \phi_0, z_0)| = 1$

$\dfrac{\Psi(\rho_0)}{\rho_0^n} = 1 \Rightarrow \rho_0^{n-1}\left[C + 2\rho_0^2\right] = \rho_0^n$

$\Rightarrow \boxed{C = \rho_0(1 - 2\rho_0)}$ II

$$\boxed{\Psi(\rho) = \rho^{n-1}\left[\rho_0 + 2(\rho^2 - \rho_0^2)\right]}$$ III

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2 a For $\xi = \xi_0$

$$\left.\begin{aligned} x &= a(\xi_0^2 - \eta^2) \\ y &= 2a\xi_0\eta \\ z &= z \end{aligned}\right\}$$

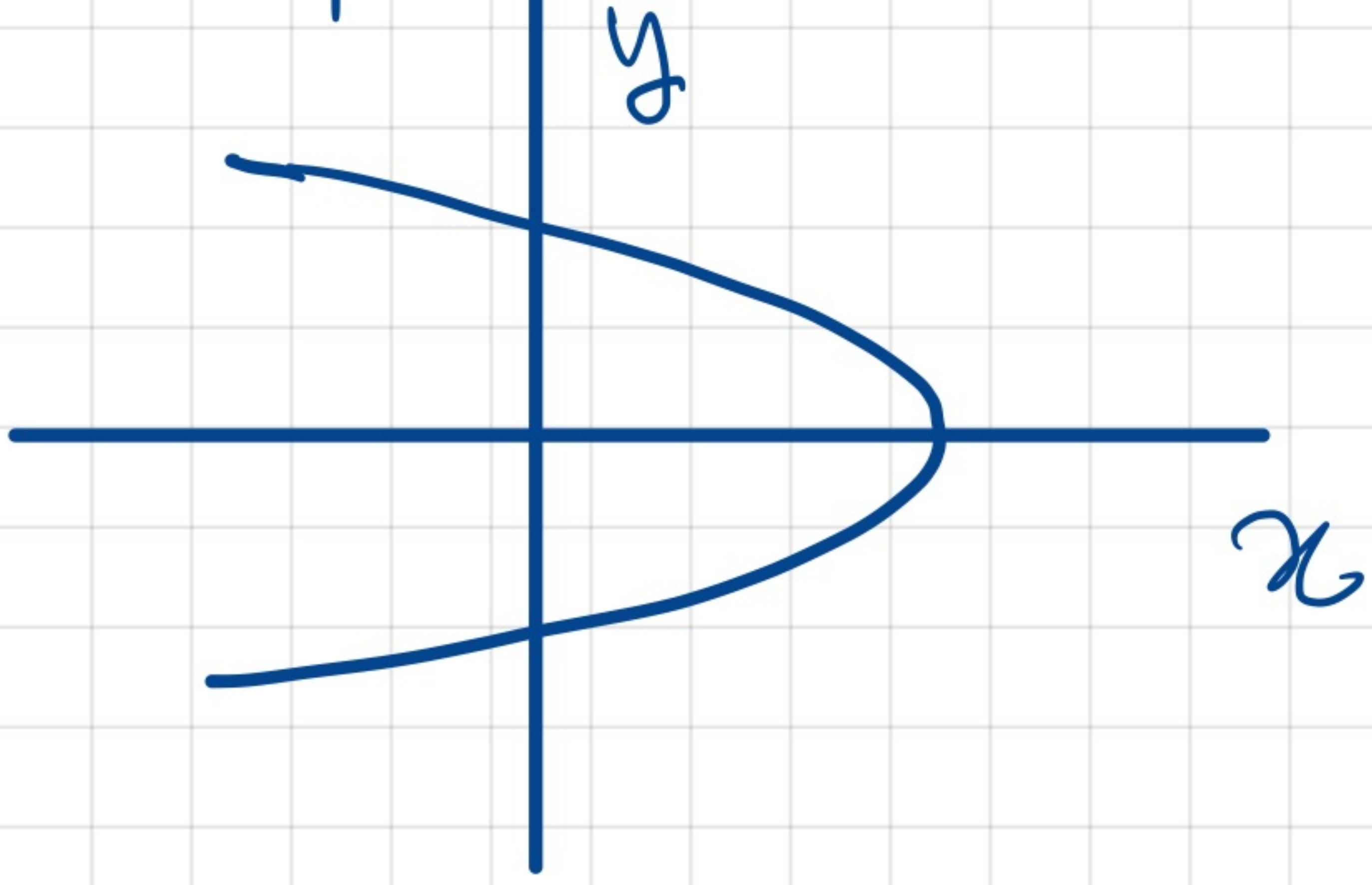
$\Rightarrow x = a\left(\xi_0^2 - \dfrac{y^2}{4a^2\xi_0^2}\right)$

$\Rightarrow y^2 = 4a^2\xi_0^2(a\xi_0^2 - x)$

$$\boxed{y^2 = 4k_2(k_1 - x)}$$

$k_1 = a\xi_0^2 > 0$

$k_2 = ak_1 > 0$



(a) For $\eta = \eta_0$

$$\left.\begin{aligned} x &= a(\xi^2 - \eta_0^2) \\ y &= 2a\xi\eta_0 \\ z &= z \end{aligned}\right\}$$

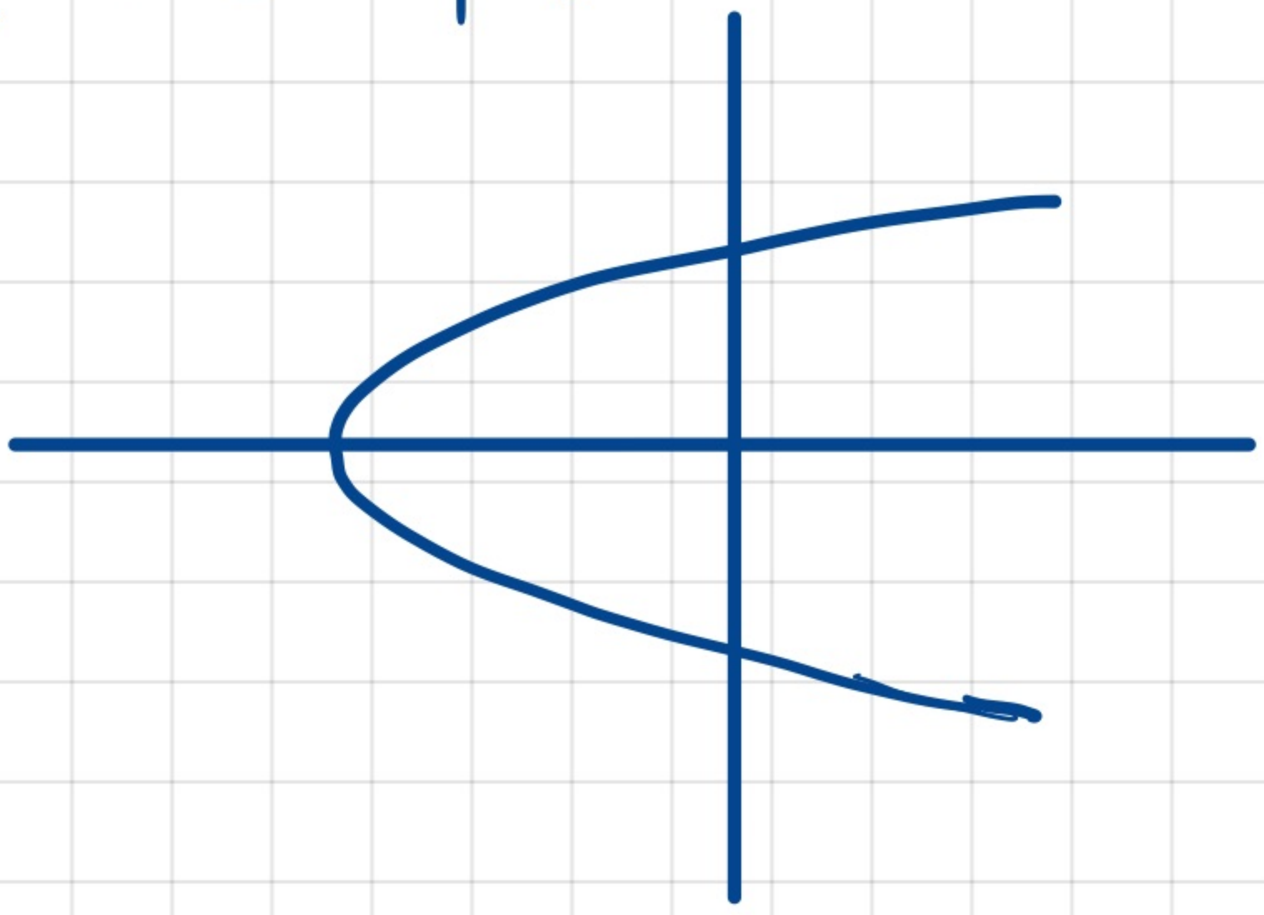
$$\Rightarrow x = a\left(\frac{y^2}{4a^2\eta_0^2} - \eta_0^2\right)$$

$$\Rightarrow y^2 = 4a^2\eta_0^2(a\eta_0^2 + x)$$

$$\boxed{y^2 = 4k_2(k_1 + x)}$$

$k_1 = a\eta_0^2 > 0$

$k_2 = ak_1 > 0$



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = a(\xi^2 - \eta^2)\hat{x} + 4a\xi\eta\,\hat{y} + z\hat{z}$$

(b) $h_\xi = \left|\frac{\partial \vec{r}}{\partial \xi}\right| = \left|2a\xi\,\hat{x} + 4a\eta\,\hat{y}\right| = 2a\sqrt{\xi^2 + \eta^2}$

(c) $h_\eta = \left|\frac{\partial \vec{r}}{\partial \eta}\right| = \left|-2a\eta\,\hat{x} + 4a\xi\,\hat{y}\right| = 2a\sqrt{\xi^2 + \eta^2}$

(d) $h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$

(e) $$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_\xi h_\eta h_z}\left[\frac{\partial}{\partial \xi}(h_\eta h_z F_\xi) + \frac{\partial}{\partial \eta}(h_\xi h_z F_\eta) + \frac{\partial}{\partial z}(h_\xi h_\eta F_z)\right]$$

$$\vec{F} = \left(\sqrt{\xi^2 + \eta^2}\right)\hat{\eta} \Rightarrow \boxed{F_\xi = F_z = 0,\ F_\eta = \sqrt{\xi^2 + \eta^2},\ h_z = 1}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{4a^2(\xi^2 + \eta^2)}\left[\frac{\partial}{\partial \eta}\left\{2a(\xi^2 + \eta^2)\right\}\right] = \frac{\eta}{a(\xi^2 + \eta^2)}$$

(3) $\vec{A}(r,\theta,\phi) = \hat{\phi}\left(\frac{100\mu_0}{4\pi}\right)\left(\frac{A_0 \sin\theta}{r}\right) e^{-r/b}$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{100\mu_0 A_0}{4\pi}\left[\frac{2\cos\theta}{r^2}\hat{r} + \frac{b\sin\theta}{r}\hat{\theta}\right] e^{-r/b}$$

$$\boxed{\begin{aligned} B_r &= \left(\frac{100\mu_0 A_0}{4\pi}\right)\cdot\left(\frac{2\cos\theta}{r^2}\right) e^{-r/b} \\ B_\theta &= \left(\frac{100\mu_0 A_0}{4\pi}\right)\cdot\left(\frac{\sin\theta}{br}\right) e^{-r/b} \\ B_\phi &= 0 \end{aligned}}$$

Evaluate at (x_0, y_0, z_0)

$$\left.\begin{aligned} x_0 &= r_0 \sin\theta_0 \cos\phi_0 \\ y_0 &= r_0 \sin\theta_0 \sin\phi_0 \\ z_0 &= r_0 \cos\theta_0 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} r_0 &= \sqrt{x_0^2 + y_0^2 + z_0^2} \\ \theta_0 &= \cos^{-1}(z_0/r_0) \\ \phi_0 &= \tan^{-1}(y_0/x_0) \end{aligned}}$$

(a) $$\left.\frac{4\pi B_r}{\mu_0 A_0}\right|_{(r_0,\theta_0,\phi_0)} = 100\left(\frac{2\cos\theta_0}{r_0^2}\right) e^{-r_0/b} = 100\left(\frac{2z_0}{r_0^3}\right) e^{-r_0/b}$$

(b) $$\left.\frac{4\pi B_\theta}{\mu_0 A_0}\right|_{(r_0,\theta_0,\phi_0)} = 100\left(\frac{b\sin\theta_0}{r_0}\right) e^{-r_0/b} = 100\left(\frac{b\sqrt{(x_0^2+y_0^2)}}{r_0}\right) e^{-r_0/b}$$

(c) $$\left.\frac{4\pi B_\phi}{\mu_0 A_0}\right|_{(r_0,\theta_0,\phi_0)} = 0$$

(d) $$\frac{1}{\mu_0 A_0}\iint_S \vec{B}\cdot ds = \frac{1}{\mu_0 A_0}\iint(\vec{\nabla}\times\vec{A})\cdot d\vec{s} = I$$

For the cylinder, this is equivalent to (using Stokes' theorem)

$$\frac{1}{\mu_0 A_0}\oint \vec{A}(r=R, \theta=\pi/2, \phi)\cdot d\vec{l} = \frac{100}{4\pi}\oint \frac{e^{-R/b}}{R}\cdot R\,d\phi$$

$$\boxed{I = 100\left[\frac{1}{2} e^{-R/b}\right]}$$

(4) $f(x) = x^2 - (a+b)x + ab$

Roots: $x = \dfrac{(a+b) \pm \sqrt{(a+b)^2 - 4ab}}{2}$

$$\boxed{x = \frac{1}{2}\left[(a+b) \pm (a-b)\right] = \{a, b\}}$$

$f'(x) = \{2x - (a+b)\}$

$f'(a) = (a-b)$, $f'(b) = -(a-b)$

$$\delta(f(x)) = \frac{\delta(x-a)}{|f'(a)|} + \frac{\delta(x-b)}{|f'(b)|}$$

Here we have $a > 0$ and $b < 0$.

(I) $\displaystyle\int_{-\infty}^{0} g(x)\,\delta(f(x))\,dx = g(b)/|f'(b)|$

(II) $\displaystyle\int_{0}^{\infty} g(x)\,\delta(f(x))\,dx = g(a)/|f'(a)|$

Ex: $\displaystyle\int_{0}^{\infty} x^2\,\delta(x^2 - 4x - 21)\,dx$

$a = 7$, $b = -3$, $g(x) = x^2$

$|f'(a)| = |f'(b)| = 4$

$\displaystyle\int_{-\infty}^{0} g(x)\,\delta(f(x))\,dx = 9/10 = 0.9$

$\displaystyle\int_{0}^{\infty} g(x)\,\delta(f(x))\,dx = 49/10 = 4.9$

(5) (a) Count the number of field lines (N_A or N_B) originating / approaching each charge (q_A or q_B)

$|q_A| \propto N_A$ and $|q_B| \propto N_B$

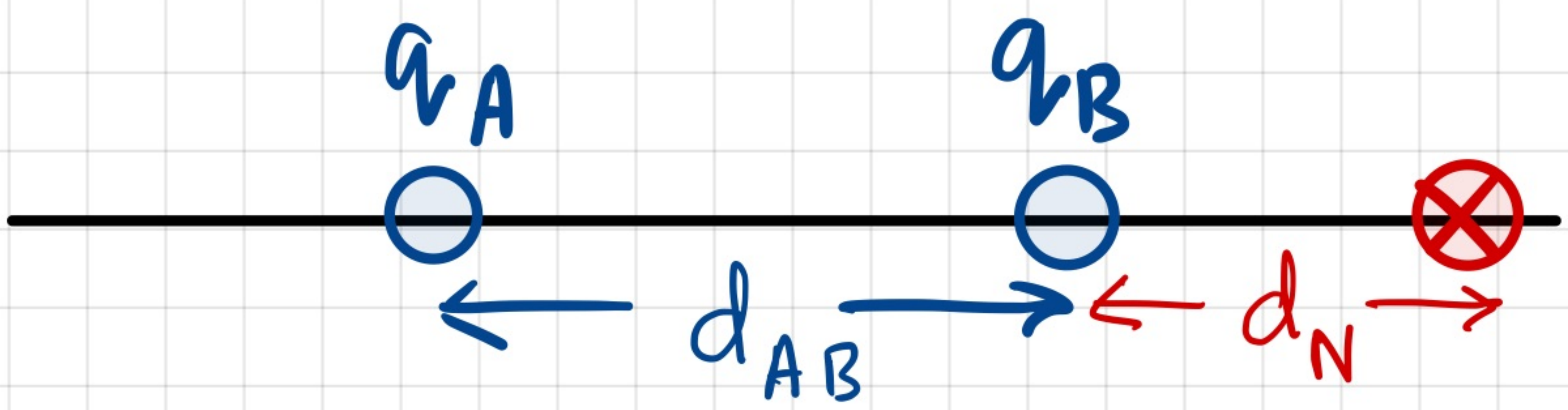
(a) $|q_A| > |q_B|$ if $N_A > N_B$ and vice-versa

(b) $$|q_A| / |q_B| = \frac{N_A}{N_B} = n$$

(c) Let d_{AB} = distance between q_A & q_B

Here $q_A = nq$ and $q_B = -q$

For $n > 1$



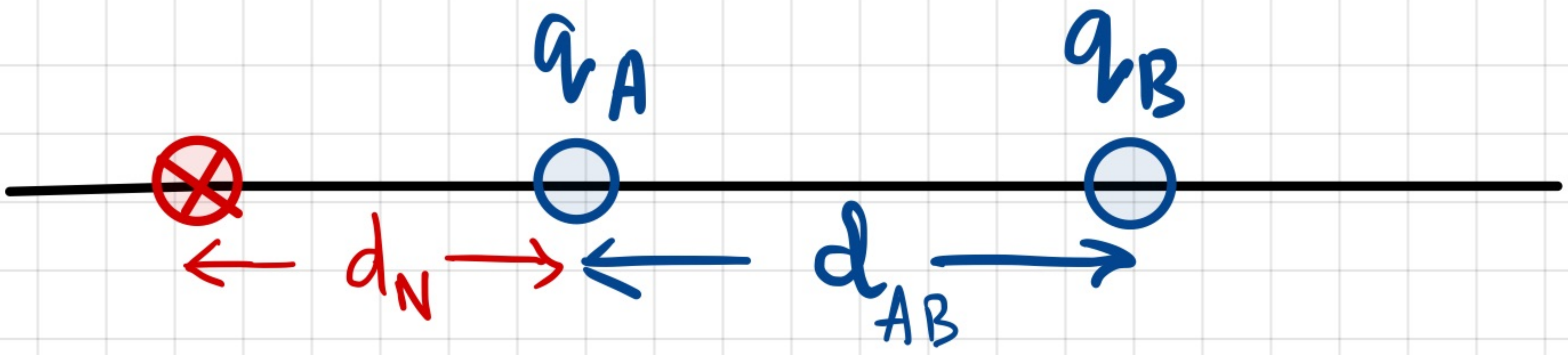
$$\frac{1}{4\pi\varepsilon_0}\left[\frac{nq}{(d_N + d_{AB})^2} - \frac{q}{d_N^2}\right] = 0$$

$$n d_N^2 - (d_N + d_{AB})^2 = 0$$

$$d_N^2 - \frac{2d_{AB}}{(n-1)} d_N - \frac{d_{AB}^2}{(n-1)} = 0$$

$$\boxed{d_N = \frac{d_{AB}}{(n-1)}\left[1 + \sqrt{n}\right]}$$

For $n < 1$



For $n < 1$

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{n q}{d_N^2} - \frac{q}{(d_N + d_{AB})^2}\right] = 0$$

$$n(d_N + d_{AB})^2 - d_N^2 = 0$$

$$d_N^2 - \left(\frac{2 n d_{AB}}{(1-n)}\right) d_N - \left(\frac{n}{1-n}\right) d_{AB}^2 = 0$$

$$d_N = \left(\frac{n}{1-n}\right) d_{AB}\left[1 + \frac{1}{\sqrt{n}}\right]$$

Distance from q_B

$$(d_{AB} + d_N) = \left\{\left(\frac{n}{1-n}\right)\left(1 + \frac{1}{\sqrt{n}}\right) + 1\right\} d_{AB}$$

$$\Rightarrow \quad \boxed{d_{AB} + d_N = \frac{d_{AB}}{(1-n)}\left[1 + \sqrt{n}\right]}$$