Integrate $\vec{v}$ on a spherical surface of radius $\vec{R}$.

$$\oint \vec{v} \cdot dr = \int_0^{2\pi}\int_0^{\pi} \frac{1}{R^2} R^2 \sin\theta \, d\theta \, d\varphi = 4\pi$$

But, $\int \nabla \cdot \vec{v} \, d\tau = 0$?

$\underline{\nabla \cdot \vec{v} = 0 \text{ for } r \neq 0}$

$$\int_{-\varepsilon}^{\varepsilon} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(x) = 1$$

or

$$\int_a^b f(x)\,\delta(x - x_0)\,dx = f(x_0)$$

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r}) = 4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\int \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\vec{a} = 4\pi \underbrace{\int \delta^3(\vec{r})\,d\tau}_{1} = 4\pi$$

$$x \frac{d}{dx}\left(\delta(x)\right) = ?$$

$$\int_{-\infty}^{\infty} x \frac{d}{dx}\left(\delta(x)\right) = \cancel{x\,\delta(x)}\Big|_{-\infty}^{\infty} - \int \delta(x)\,dx .$$

0

$$\Rightarrow \quad x \frac{d}{dx}\delta(x) = -\delta(x)$$

$$\text{or} \quad \frac{d}{dx}\left(\delta(x)\right) = -\frac{\delta(x)}{x} .$$ } More Singular.

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

$$\delta(f(x))$$

Roots of $f(x) = x_1, x_2 \ldots x_n . \{x_i\}$

$$f(x) = f(x_i + x - x_i)$$

$$= f(\cancel{x_i}) + (x - x_i) f'(x)\Big|_{x = x_i} + O\left((x-x_i)^2\right)$$

0

$$f(x) = (x - \alpha_i)\, f'(x)$$

$$\delta(f(x)) = \sum_i \delta\big((x - \alpha_i)\, f'(\alpha_i)\big).$$

$$\delta(f(x)) = \sum_i \frac{\delta(x - \alpha_i)}{|f'(\alpha_i)|}$$

Helmholtz theorem.

$$\vec{\nabla} \cdot \vec{F} = D(\vec{r})$$
$$\vec{\nabla} \times \vec{F} = \vec{C}(\vec{r})$$
$$\vec{\nabla} \cdot \vec{C} = 0$$

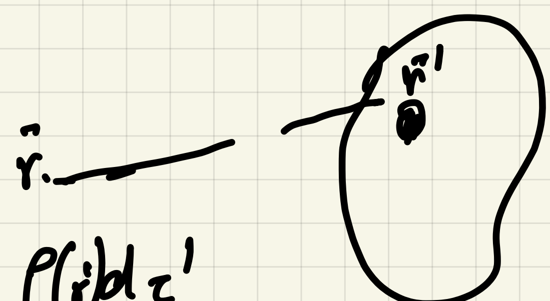
$$\vec{F} = -\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r})^{*}$$

*Conditions apply.

$$U = \frac{1}{4\pi} \int \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'$$

$$\vec{W} = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'$$

Example of such integrals.



$$V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\vec{r}')\, d\tau'}{|r - r'|}$$

$$\vec{F} = -\vec{\nabla} U + \vec{\nabla} \times \vec{w}$$

$$\vec{\nabla} \cdot \vec{F} = -\nabla^2 U$$

$$= -\frac{1}{4\pi} \int \nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} D(\vec{r}')\, d\tau' = -\frac{1}{4\pi} \int -4\pi\, D(\vec{r}')\, \delta(\vec{r} - \vec{r}')$$

$$= D(\vec{r})$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{\nabla} \times \vec{w})$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{w}) - \underset{\substack{\downarrow \\ C}}{\nabla^2 \vec{w}}$$

$$\vec{\nabla} \cdot \vec{w} = \frac{1}{4\pi} \int \nabla \left(\frac{1}{|r - r'|} \right) \cdot \vec{C}(\vec{r}')\, d\tau'$$

$$\nabla \left(\frac{1}{|r - r'|} \right) = -\nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

$$\underline{\vec{\nabla} \cdot \vec{w}} = -\frac{1}{4\pi} \int \left(\nabla' \left(\frac{1}{|r - r'|} \right) \cdot \vec{C}(\vec{r}') \right) d\tau'$$

$$= -\frac{1}{4\pi} \int \vec{\nabla}' \left(\frac{1}{|r - \vec{r}'|} \cdot \vec{C}(\vec{r}') \right) d\tau' - \int \frac{1}{|r - \vec{r}'|} \underset{\substack{\downarrow \\ 0}}{\left(\vec{\nabla}' \cdot \vec{C}(\vec{r}') \right.} d\tau$$

$\bar{U}$ & $\bar{W}$ exists only if D and C vanish faster than $\left(\frac{1}{r^2}\right)$

If the divergence D(**r**) and the curl **C(r)** of a vector function **F(r)** are specified, and if they both go to zero faster than 1/r^2 as r goes to infinity, and if **F(r)** itself goes to zero as r goes to infinity, then **F(r)** is uniquely given by

$$\boldsymbol{F} = -\nabla U + \nabla \times \boldsymbol{W}$$

Uniqueness?

$$\vec{F} = -\bar{\nabla} U + \bar{\nabla} \times \vec{W} + \vec{G}$$

$$\nabla \cdot \vec{G} = 0$$
$$\bar{\nabla} \times \bar{G} = 0.$$
and $\vec{G} = 0$ at $r \to \infty$