Maxwell's equations: Laws of electrodynamic fields.

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Electrostatic {no time varying fields}

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$
$$\vec{\nabla}\times\vec{E} = 0$$

Using Helmholtz' theorem

$$\vec{E} = -\nabla U + \vec{\nabla}\times\vec{W},\qquad \vec{E} = -\nabla\left(\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho}{|r-r'|}\,dv'\right)$$

$\vec{W} = 0$,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho}{r^2}\,\hat{r}\,dv'$$

$\downarrow$

Coulomb's law.

Coulomb's law for continuous charge distribution

Surface charge ($\sigma(\vec{r})$ = surface charge density)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,ds$$

σ = surface charge density,

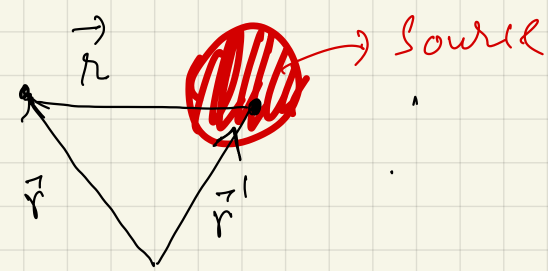
Line charge (λ = line charge density)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}}\,\hat{\mathscr{r}}\,ds$$

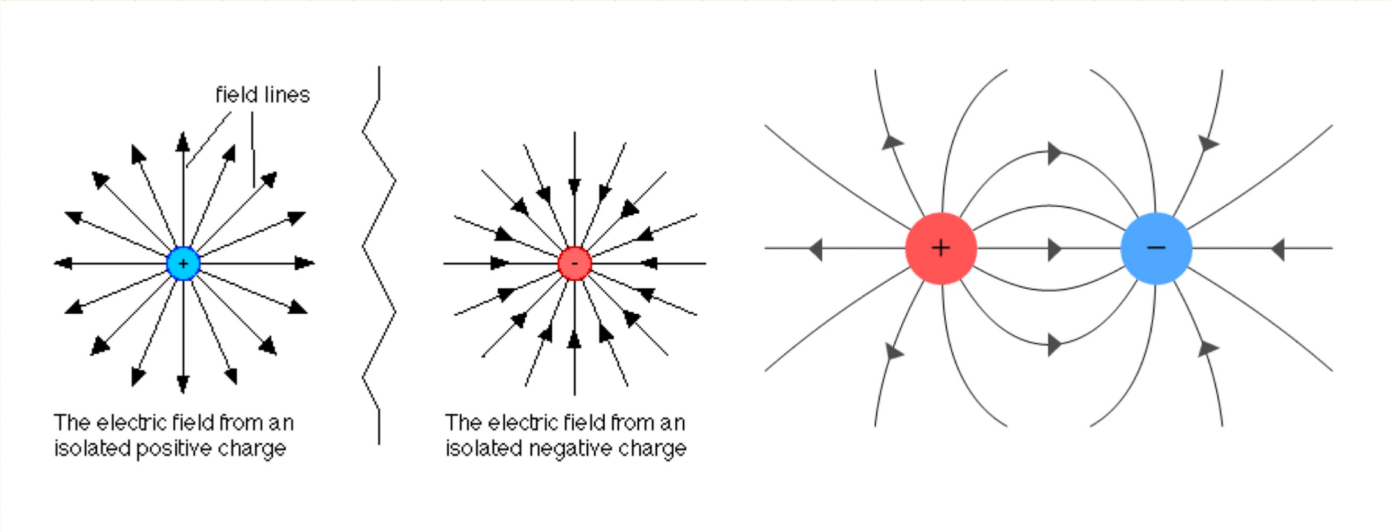
Collection of discrete charges

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i}\,\hat{\mathscr{r}}_i$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$



Field lines.



Originates at (+)

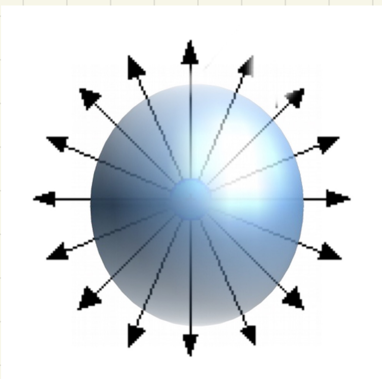
Ends at (−)

Strength indicated by the density

They can never cross.

Flux : $\oint \vec{E} \cdot d\vec{s}$ on a closed surface. (due to a point charge)

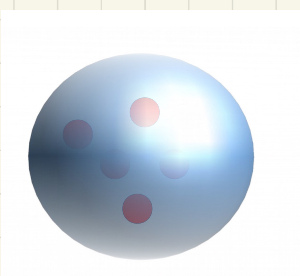
$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\,\hat{r}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}\right)$$



$$= \frac{q}{\varepsilon_0}$$

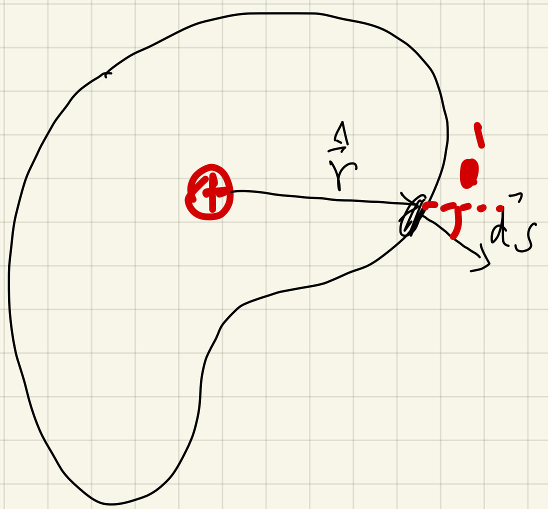
Collection of charges.

$$\vec{E} = \sum_{i}^{n} \vec{E}_i$$ principle of superposition.



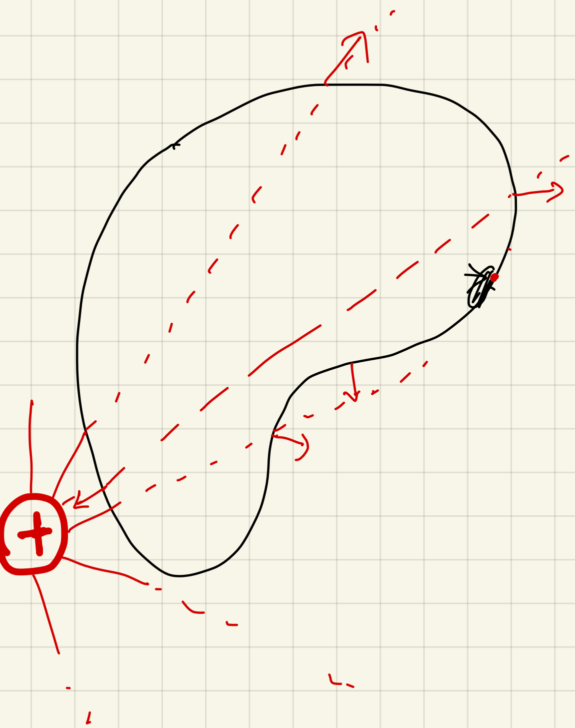
$$\oint \vec{E} \cdot d\vec{a} = \sum_i \frac{q_i}{\varepsilon_0} = \frac{1}{\varepsilon_0} Q_{ecl.}$$

Arbitrary surface.



$$\vec{E}\cdot d\vec{s} = \frac{q}{4\pi\varepsilon_0}\int \frac{\hat{r}\cdot d\vec{s}}{r^2} = \frac{q}{4\pi\varepsilon_0}\int \underbrace{\frac{|ds|\cos\theta'}{r^2}}_{d\Omega}$$

Projection of a surface onto a unit sphere.



Outside charge: the flux will be cancelled.

Coulomb's law. for continuous charges.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{r^2} \hat{r} \, d\tau'$$

$$\left.\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \right\} \to \text{Maxwell's equation.}$$

$$\vec{E} = -\frac{1}{4\pi} \int \nabla \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) \underset{\downarrow \; \rho/\varepsilon_0}{D(\vec{r}\,')} \, d\tau' \xrightarrow[(\vec{\nabla} \times \vec{E} = 0)]{\text{Helmholtz}} \text{Coulomb's law}$$

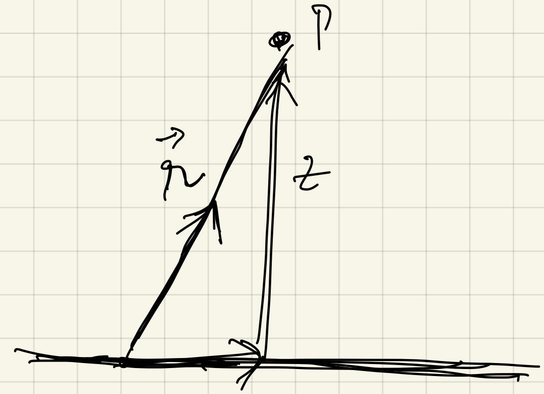
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\int_V \vec{\nabla} \cdot \vec{E} \, d\tau = \int \frac{\rho}{\varepsilon_0} d\tau = \frac{Q_{encl}}{\varepsilon_0}$$

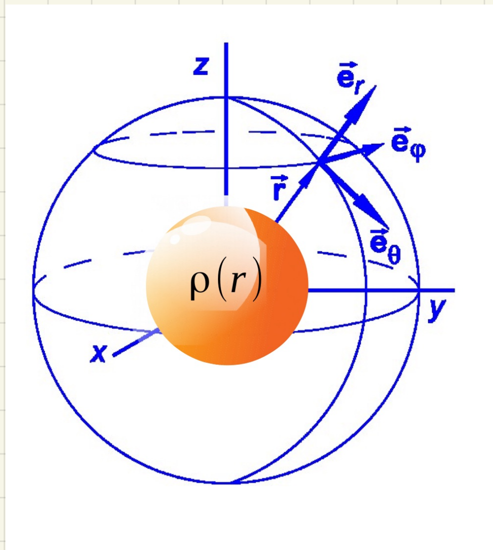
$$\left. \oint \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\varepsilon_0} \right\} \text{Gauss's theorem}$$

Symmetric considerations

i) Spherically symmetric charge distribution



$$E_\theta,\ E_\varphi = 0$$

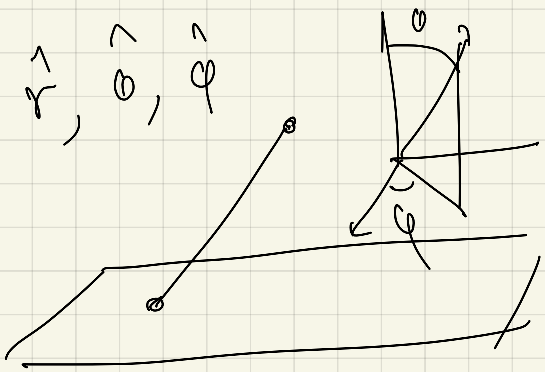


Apply Gauss' law.

$$\oint \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\int \rho(r)\, r^2 \sin\theta \, d\theta\, d\varphi\, dr$$
$$= Q/\varepsilon_0$$

$\vec{E} = E\hat{r}$, taking a spherical surface.

$\hat{r}$, $\hat{\theta}$, $\hat{\varphi}$

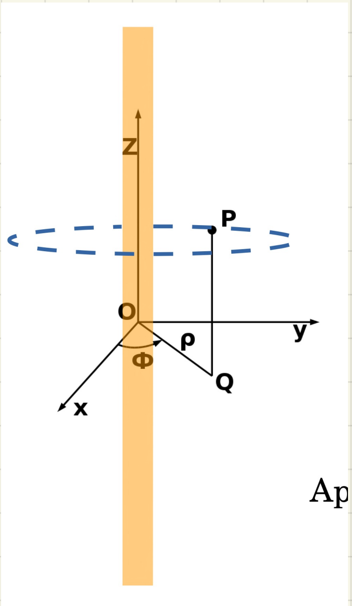


$$E\cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$$Q = \frac{4}{3}\pi R^3 \rho$$

ii) Long narrow wire



Gaussian surface: cylinder.

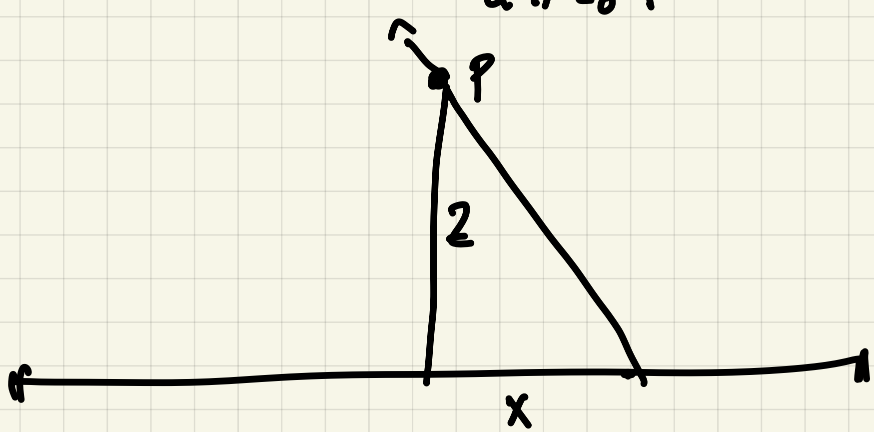
$$\vec{E} = E \hat{\rho}$$

$$\frac{\lambda L}{\varepsilon_0} = \int E \rho \, d\varphi \, dz$$

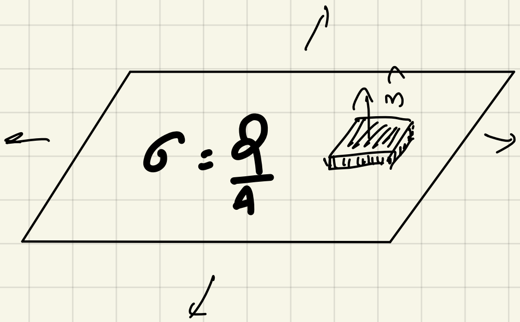
$Q = \lambda l$

$$\frac{\lambda L}{\varepsilon_0} = 2\pi \rho L E$$

$$E = \frac{\lambda}{2\pi \varepsilon_0 \rho} = \underline{\underline{\frac{1}{4\pi \varepsilon_0} \left(\frac{2\lambda}{\rho} \right)}}.$$



Infinite sheet of charge.

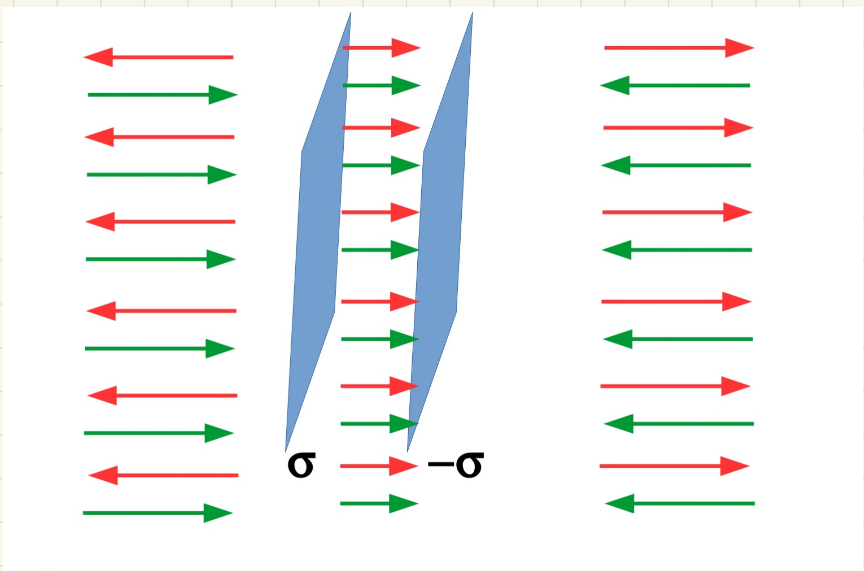


Infinite sheet.

Gaussian surface. 'pill box.' $\vec{E} = E\hat{n}$

$$\int \vec{E}\cdot da = 2AE = \frac{\sigma A}{\varepsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$ (Independant of distance).



Summary.

1. Electrostatic potentials

2. Electrostatic boundary condition

3. Energy of charge distribution.

In electrostatics,

$$\bar{\nabla} \cdot \bar{E} = \rho/\varepsilon_0$$

$$\bar{\nabla} \times \bar{E} = 0$$

$$\vec{F} = q\vec{E}$$

Since $\bar{\nabla} \times \bar{E} = 0$,

$\vec{E} = -\nabla\bar{V}$ (always true).

$$V(r) = -\int_0^r \vec{E} \cdot d\vec{l}$$

$$V(b) - V(a) = -\int_0^a \vec{E} \cdot dl + \int_0^b \bar{E} \cdot dl$$

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot dl = \int_a^b (\nabla V) \, da.$$

$$\bar{E} = -\nabla V$$

Scalar, follows superposition principle.

→ Potential due to a point charge q

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}\int \frac{q}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$r = |\vec{r} - \vec{r}'|$$

Collection of charges

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}$$

Continuous charge distribution.

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\,d\tau'}{r} \quad \Big\}\ \text{volume}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')\,ds'}{r} \quad \text{or} \quad \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')\,dl'}{r}$$

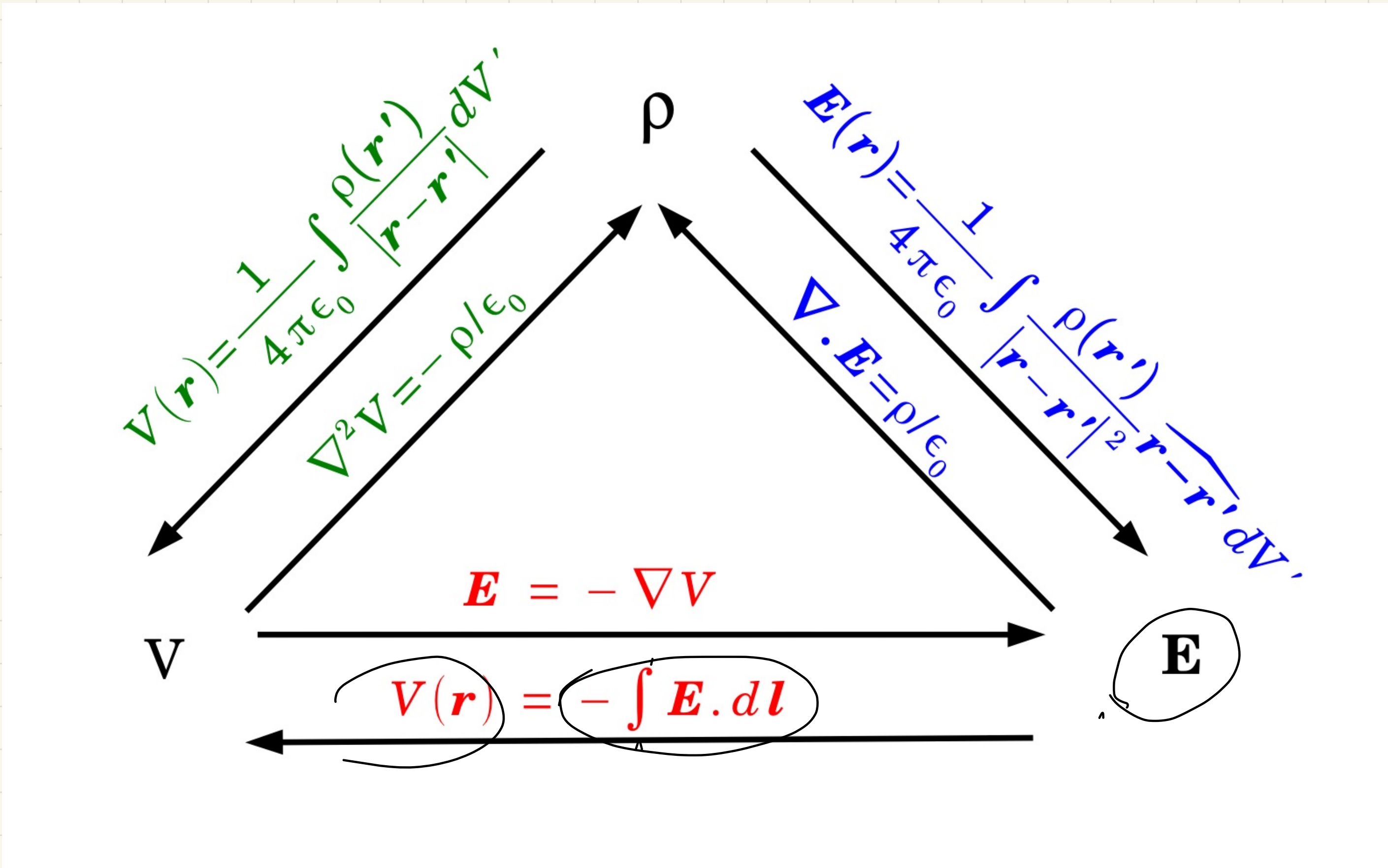
$$\left.\begin{array}{l} \vec{E} = -\vec{\nabla} V \\ \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = -\vec{\nabla}\cdot(\vec{\nabla} V) \end{array}\right\} \rightarrow \nabla^2 V = -\rho/\varepsilon_0$$

Poisson's equation.

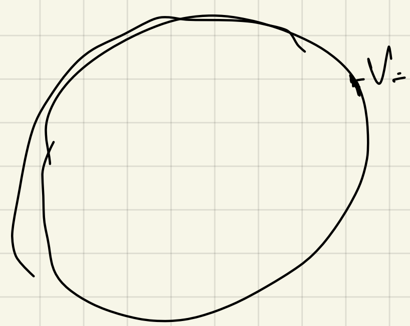
In regions where $\rho = 0$,

$\nabla^2 V = 0$, Laplace's equation.

Relations between ρ, V, and $\vec{E}$



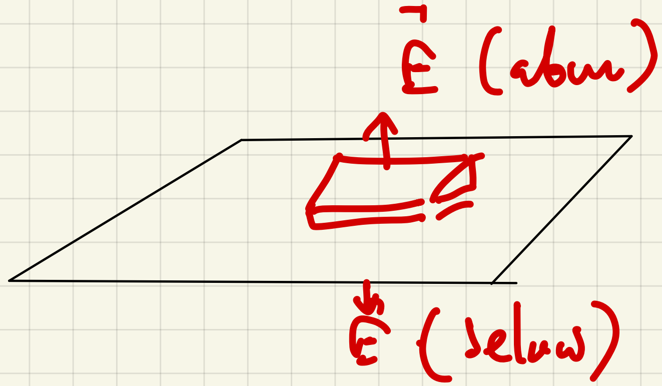
$\frac{\partial V}{\partial \hat{n}} = \vec{E}_{\perp}$



$\nabla^2 V = 0$

$\nabla^2 V = \rho/\epsilon_0$

Electrostatic boundary conditions.



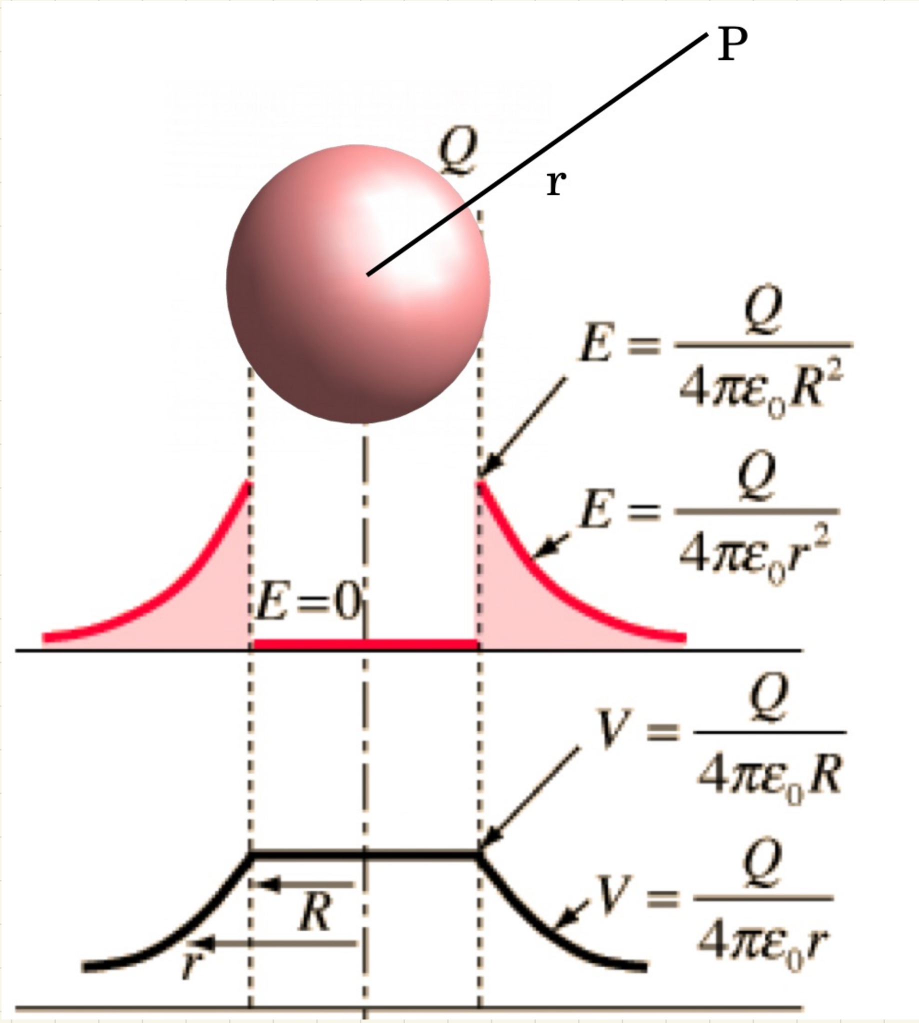
$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$E_{\perp}^{(above)} - E_{\perp}^{(below)} = \frac{\sigma}{\varepsilon_0}$$

$$E_{\parallel}^{(above)} - E_{\parallel}^{(below)} = 0 .$$

$$\Delta V = V(\text{above}) - V(\text{below}) = -\int_b^a E \cdot dl$$

as $dl \to 0$, $V(\text{above}) = V(\text{below})$.



Potential inside a hollow sphere.

$$V\,(\text{outside}) = \frac{R^2\sigma}{\varepsilon_0 r}$$

$$V\,(\text{inside}) = \frac{R\sigma}{\varepsilon_0}$$

Work done to move a charge.

$\longrightarrow$ From a to b.

$$W = \int_a^b \vec{F} \cdot d\vec{l}$$

test.
q

$\vec{F} = q\vec{E}$

to move, we need to work against $\vec{F}$.

$$W = -q\int_a^b \vec{E} \cdot d\vec{l}$$

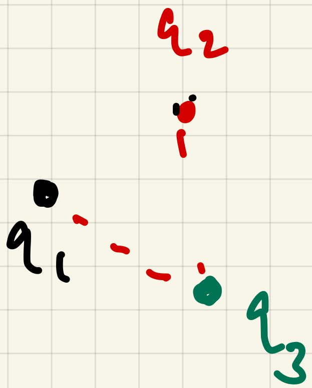
Source

when $\vec{\nabla} \times \vec{E} = 0$, $E = -\nabla V$

$$W = -q\int_a^b -\nabla V \cdot dl = q\left[V(b) - V(a)\right]$$

$$V(b) - V(a) = \frac{W}{q} \quad \Big\} \text{ Work done / unit charge.}$$

Work done to assemble point-charge distributions.



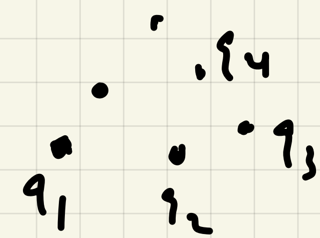
$$W_2 = q_2 V_1 = \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}}$$

$$W_3 = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \quad \ldots$$

Total work done $= \dfrac{1}{2}\dfrac{1}{4\pi\epsilon_0}\displaystyle\sum_{i=1}^{N}\sum_{j\neq i=1}^{N}\frac{q_i q_j}{r_{ij}}$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{j\neq i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$



$$\frac{1}{2}\left(\sum_i q_i V(\vec{r}_i)\right).$$

Continuous charge.

$$W = \frac{1}{2}\sum_i q_i V(r)$$

In continuous system, $\frac{1}{2}\int \rho\, V(r)\, d\tau.$

$$\rho = \varepsilon_0 (\bar{\nabla}\cdot\vec{E}) \Rightarrow \frac{\varepsilon_0}{2}\int (\bar{\nabla}\cdot\vec{E})\, V\, d\tau.$$

$$\nabla\cdot(V\vec{E}) = (\bar{\nabla}\cdot\vec{E})V + \vec{E}\cdot\nabla\bar{V}$$

$$W = \frac{\varepsilon_0}{2}\left[\int \nabla\cdot(V\vec{E})\, d\tau - \int \vec{E}\cdot\nabla\bar{V}\, d\tau\right]$$

$$\int \nabla\cdot(V\vec{E})\, d\tau = \oint V\vec{E}\cdot ds.$$

$\Downarrow$

0. (for all space).

$$W = \frac{\varepsilon_0}{2}\int E^2\, d\tau$$

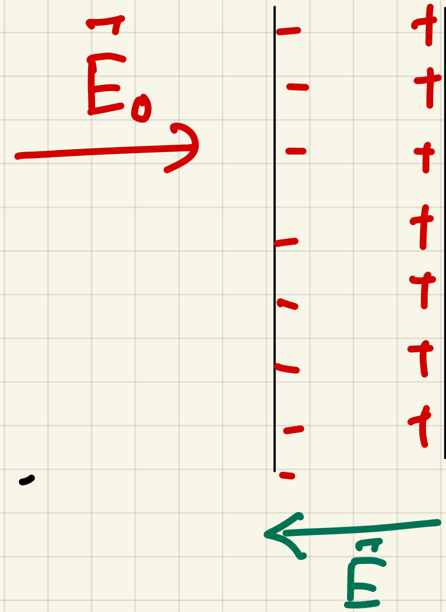
Lecture - 9

1) Conductors.

2) Electrostatic pressure.

3) Capacitors

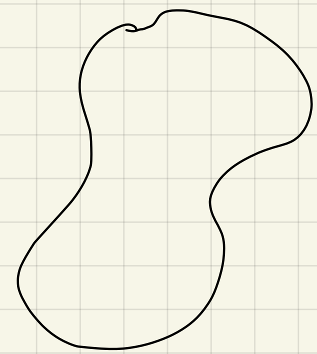
Conductors



No field inside the conductor.

1) $E = 0$ inside the conductor.

2) $E_{\parallel}$ on the surface is zero.



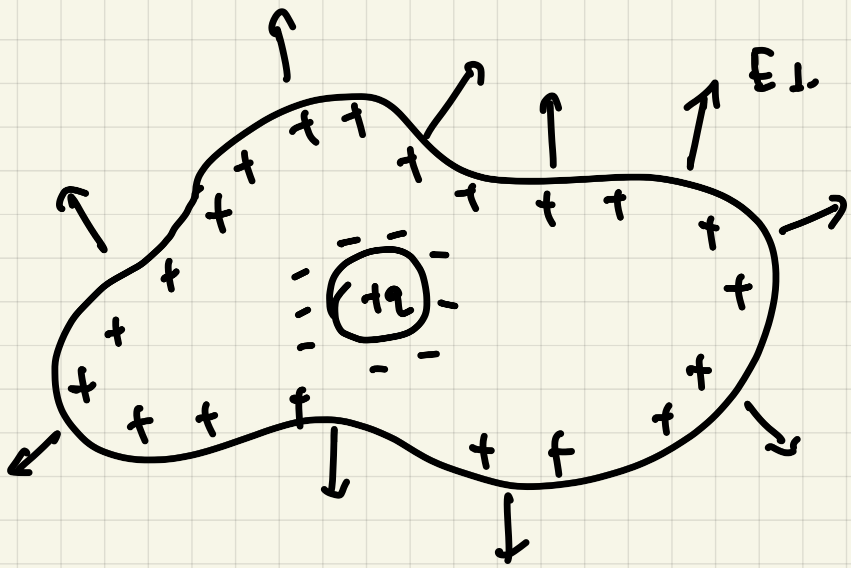
3) $\vec{E}$ is $\perp$ to the surface. $\frac{\sigma}{\varepsilon_0}\hat{n}$

$$E^{\perp}_{outside} - E^{\perp}_{inside} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$E^{\perp}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} \qquad \frac{\partial V}{\partial n} = (\vec{\nabla} V \cdot \hat{n})$$

Electrostatic pressure



$\vec{E}_\perp$ above $= \frac{\sigma}{\varepsilon_0}\hat{n}$

$E_\perp$ below $= 0$

$\vec{E}_\parallel$ above $= \vec{E}_\parallel$ below $= 0$.

→ Charges are equally distributed on surface.

$\vec{E}$ is discontinuous at the surface.

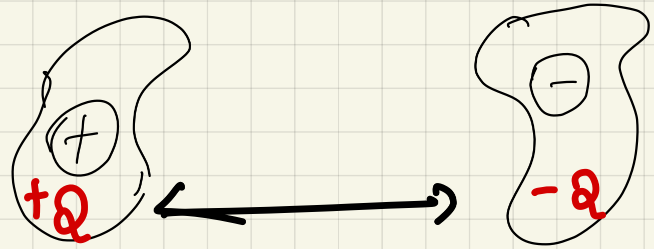
$$\vec{E}_{surface} = \frac{1}{2}\left(\underbrace{\vec{E}_{above}}_{\frac{\sigma}{\varepsilon_0}\hat{n}} + \underbrace{\vec{E}_{below}}_{0}\right)$$

$$\vec{E}_{surf} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$d\vec{F} = \left(\frac{\sigma}{2\varepsilon_0}\right)(\sigma\, dA)\,\hat{n}$$

$$P = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon_0}{2}E_{above}^2$$

Capacitors



$$V = V^+ - V^- = -\int_{-}^{+} \vec{E}\cdot d\vec{l}$$

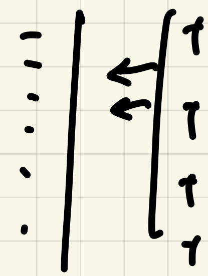
V^+ $$-\int_{\infty}^{+} E\cdot dl$$

V^- $$-\int_{-\infty}^{-} \vec{E}\cdot dl$$

For a given distance, $V \propto Q$

$\frac{Q}{V} = C$ Capacitance. (Farad F).

Parallel plate $C = \frac{A\varepsilon_0}{d}$



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V = \frac{Q d}{A\varepsilon_0}$$

Spherical. $V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$, $C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$