Summary - Lecture 4.

1) Gradient, divergence, curl in curvilinear coordinates. $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$

$$\vec{\nabla} T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{u}_3$$

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(v_1 h_2 h_3) + \frac{\partial}{\partial u_2}(v_2 h_1 h_3) + \frac{\partial}{\partial u_3}(v_3 h_1 h_2)\right].$$

$$\begin{aligned}\vec{\nabla} \times \vec{v} = {} & \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial u_2}(h_3 v_3) - \frac{\partial}{\partial u_3}(h_2 v_2)\right)\hat{u}_1 + \\ & \frac{1}{h_1 h_3}\left(\frac{\partial}{\partial u_3}(h_1 v_1) - \frac{\partial}{\partial u_1}(h_3 v_3)\right)\hat{u}_2 + \\ & \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial u_1}(h_2 v_2) - \frac{\partial}{\partial u_2}(h_1 v_1)\right)\hat{u}_3\end{aligned}$$

Displacement vector in curvilinear coordinates:

$$\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}.$$

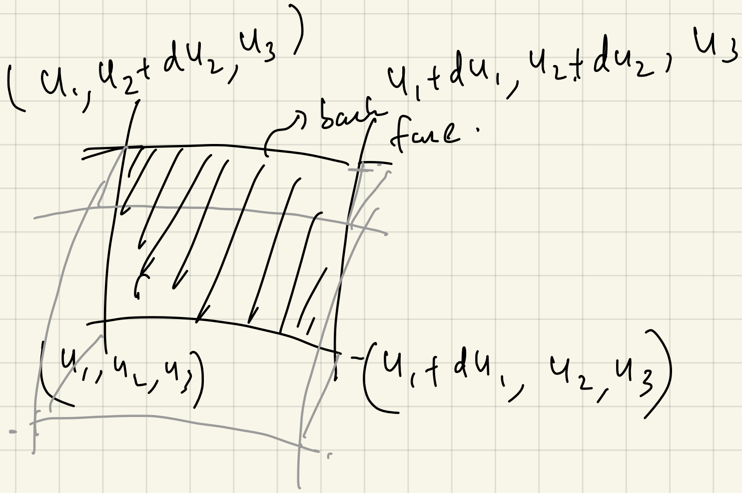
$$\vec{dr} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3 = \sum_i h_i du_i \hat{u}_i$$

Define $f \equiv f(u_1, u_2, u_3)$ or $f(u_i)$

$$df = f(u_i + du_i) - f(u_i) = (\vec{\nabla} f \cdot \vec{dr})$$

$$= (\nabla f_1)(h_1 du_1) + (\nabla_2 f) h_2 du_2 + (\nabla_2 f) h_3 du_3$$

$$\nabla_2 f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}, \quad \nabla_2 f = \frac{1}{h_2}\frac{\partial f}{\partial u_2}, \quad \nabla_3 f = \frac{\partial f}{\partial u_3}$$



$$d\vec{a}_{12} = -\left(h_1\, du_1\, \hat{u}_1\right) \times \left(h_2\, du_2\, \hat{u}_2\right)$$

$$d\vec{a}_{12} = -h_1 h_2\, du_1\, du_2\, \hat{u}_3$$

$$\vec{v} = \{v_1, v_2, v_3\}.$$

$$\text{Flux of } \vec{v}\ (\text{back}) = \vec{v}\cdot d\vec{a}_{12}(u_3) = -\left(v_3 h_1 h_2\, du_1\, du_2\right)_{u_3}$$

$$\text{flux of } \vec{v}\ (\text{front}) = \vec{v}\cdot d\vec{a}_{12}(u_3+du_3) = \left(v_3 \cdot h_1 h_2\, du_1\, du_2\right)_{u_3+du_3}$$

Net flux over $d\vec{a}_{12}$

$$= \left(v_3 h_1 h_2\, du_1\, du_2\right)_{u_3+du_3} - \left(v_3 h_1 h_2\, du_1\, du_2\right)_{u_3}$$

$$= \frac{\partial}{\partial u_3}\left(h_1 h_2 v_3\right) du_1\, du_2\, du_3$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}\left(v_3 h_1 h_2\right) . \, d\tau.$$

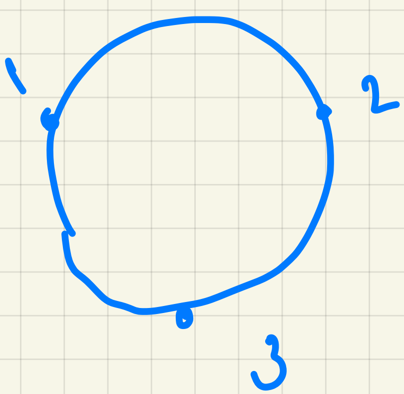
$$d\tau = h_1 h_2 h_3\, du_1\, du_2\, du_3$$

Total flux:

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_3}\left(v_3 h_1 h_2\right) + \frac{\partial}{\partial u_1}\left(v_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(v_2 h_3 h_1\right)\right] d\tau$$

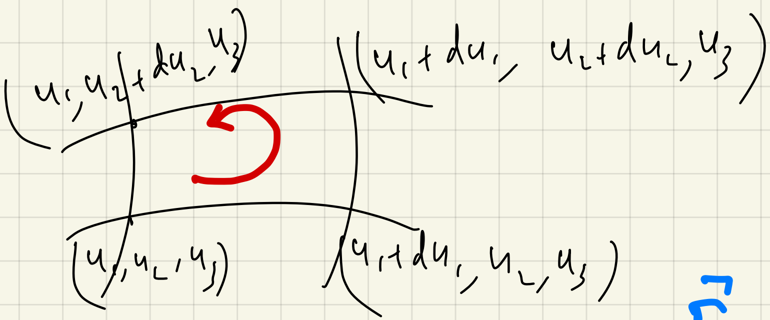
$$\oint_S \vec{v}\cdot d\vec{a} = \int_V \frac{1}{h_1 h_2 h_3}\left[\quad\right] d\tau . = \int_V \left(\vec{\nabla}\cdot\vec{v}\right) d\tau .$$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\underbrace{\left(v_1 h_2 h_3\right)}_{I} + \frac{\partial}{\partial u_2}\left(v_2 h_3 h_1\right) + \frac{\partial}{\partial u_3}\left(v_3 h_1 h_2\right)\right]$$



Cyclic permutations:

Curl in curvilinear coordinates:



$$\vec{F} = F_1 \hat{u}_1 + F_2 \hat{u}_2 + F_3 \hat{u}_3$$

Bottom side.

$$\vec{F}\cdot d\vec{l} = (F_1 h_1 du_1)_{u_2}$$

Top side:

$$-(F_1 h_1)_{u_2+du_2}\, du_1$$

Top and bottom together,

$$-\frac{\partial}{\partial u_2}(F_1 h_1)\, du_1\, du_2 .$$

From left and right side, $\frac{\partial}{\partial u_1}(F_2 h_2)\, du_1\, du_2$

$$\text{Net integral} = \left[\frac{\partial}{\partial u_1}(F_2 h_2) - \frac{\partial}{\partial u_2}(F_1 h_1)\right] du_1\, du_2$$

$$= \frac{1}{h_1 h_2}\underbrace{\left[\quad\right]}_{(\vec{\nabla}\times\vec{F})_3}\hat{u}_3 \cdot \left(h_1 du_1\, \hat{u}_1 \times h_2 du_2\, \hat{u}_2\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 F_2) - \frac{\partial}{\partial u_2}(h_1 F_1)\right]\hat{u}_3 +$$

$$\frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2)\right]\hat{u}_1 +$$

$$\frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 F_1) - \frac{\partial}{\partial u_1}(h_3 F_3)\right]\hat{u}_2$$

$$\int_{\text{small loop}} \vec{F} \cdot d\vec{l} = (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}$$

$$\oint \vec{F} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$u_1 = r,\ u_2 = \theta,\ u_3 = \varphi.$

Special case:

$\vec{V} = \dfrac{\hat{r}}{r^2}$ $\qquad \vec{\nabla}\cdot\vec{V} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(v_1 h_2 h_3\right) + \dfrac{\partial}{\partial u_2}\left(v_2 h_1 h_3\right) + \dfrac{\partial}{\partial u_3}\left(v_3 h_1 h_2\right)\right].$

$\vec{V} = \left\{V_r, V_\theta, V_\varphi\right\} = \left\{\dfrac{1}{r^2}, 0, 0\right\}.$

spherical polar: $u_1 = r,\ u_2 = \theta,\ u_3 = \varphi$

$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[V_r\, r^2 \sin\theta\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right).$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = 0\ ?$$

Integrate $\vec{v}$ on a spherical surface of radius $\vec{R}$.

$$\oint \vec{v}\cdot d\vec{r} = \int_0^{2\pi}\int_0^{\pi} \frac{1}{R^2} R^2 \sin\theta \, d\theta \, d\varphi = 4\pi$$

But, $\int \nabla\cdot\vec{v}\, d\tau = 0$?

$\nabla\cdot\vec{v} = 0$ for $r \neq 0$

$$\int_{-\varepsilon}^{\varepsilon} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(x) = 1$$

or

$$\int_a^b f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r}) = 4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\int\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) d\vec{a} = 4\pi \underbrace{\int \delta^3(\vec{r})\,d\tau}_{1} = 4\pi$$