SEPARATION OF VARIABLES

Look for solutions that are products of functions, each of which depends on only one of the coordinates.

Two dimensional system (Cartesian coordinates)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

If $V(x, y) = X(x)Y(y)$ then

$$\frac{1}{X}\frac{d^2 X}{d x^2} + \frac{1}{Y}\frac{d^2 Y}{d y^2} = 0$$

$$\frac{1}{X}\frac{d^2 X}{d x^2} = -\frac{1}{Y}\frac{d^2 Y}{d y^2} = k^2 \quad (const)$$

SEPARATION OF VARIABLES

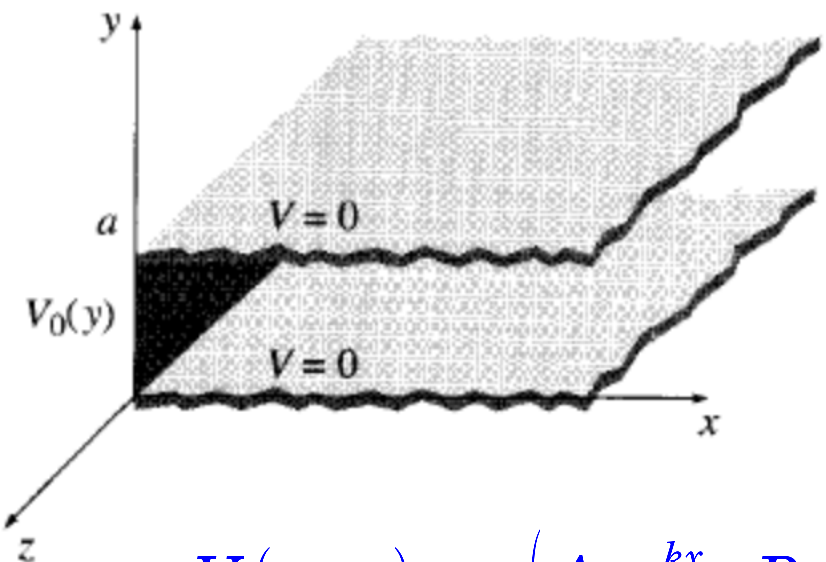
$$\frac{d^2 X}{d x^2} = k^2 X \quad \Rightarrow \quad X(x) = A e^{kx} + B e^{-kx}$$

$$\frac{d^2 Y}{d y^2} = -k^2 Y \quad \Rightarrow \quad Y(y) = C \sin ky + D \cos ky$$

- Role of x and y can be interchanged. X(x) can be sinusoidal and Y(y) can be exponential.
- A, B, C, and D are constants to be determined by the boundary conditions of the problem.
- We can in principle, have an infinite set of solutions depending on the allowed values of k.
- Laplace's equation is linear, hence a linear combination of solutions is also a solution.

$$V(x, y) = \sum_{\text{allowed k}} \left(A_k e^{kx} + B_k e^{-kx} \right) \left(C_k \sin ky + D_k \cos ky \right)$$

SEPARATION OF VARIABLES – 2D CARTESIAN



Boundary conditions:

$(i)\quad V = 0 \quad \text{when} \quad y = 0$

$(ii)\quad V = 0 \quad \text{when} \quad y = a$

$(iii)\quad V = V_0(y) \quad \text{when} \quad x = 0$

$(iv)\quad V \to 0 \quad \text{as} \quad x \to \infty$

$$V(x,y) = \left(A_k e^{kx} + B_k e^{-kx}\right)\left(C_k \sin ky + D_k \cos ky\right)$$

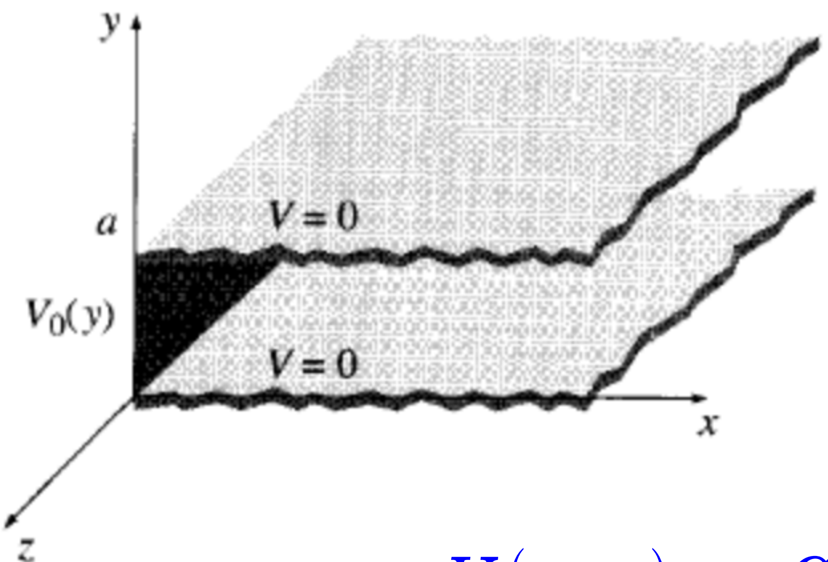
$$V \to 0 \quad \text{as} \quad x \to \infty \quad \Rightarrow \quad A = 0$$

$$V(x,y) = e^{-kx}\left(C_k \sin ky + D_k \cos ky\right)$$

$$V = 0 \quad \text{when} \quad y = 0 \quad \Rightarrow \quad D = 0$$

$$V(x,y) = C_k e^{-kx} \sin ky$$

SEPARATION OF VARIABLES – 2D CARTESIAN



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$(iv)\quad V \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty$

$$V(x,y) = C_k e^{-kx} \sin ky$$

$$V = 0 \quad \text{when} \quad y = a \quad \Rightarrow \quad \sin ka = 0 \quad \Rightarrow \quad k = \left(\frac{n\pi}{a}\right) \text{ for } n = 1,2,3,\ldots$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$V = V_0(y) \quad \text{when} \quad x = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

SEPARATION OF VARIABLES - 2D CARTESIAN

$$\sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \int_0^a V_0(y)\sin(n'\pi y/a)\,dy$$

$$\sin(n\pi y/a)\sin(n'\pi y/a) = -\frac{1}{2}\left[\cos\frac{(n+n')\pi y}{a} - \cos\frac{(n-n')\pi y}{a}\right]$$

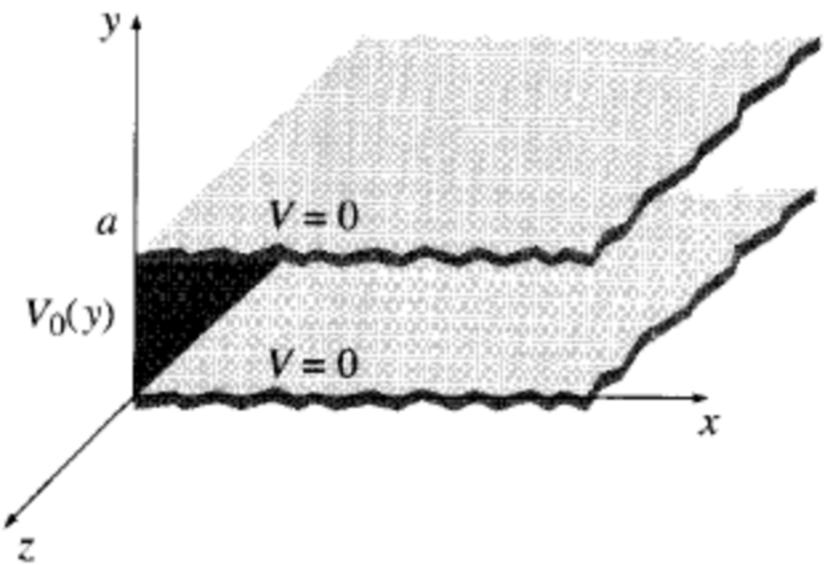
$$\int_0^a \cos(m\pi y/a)\,dy = 0 \quad \forall \ \text{integer}\, m$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \frac{a}{2}\delta(n-n')$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \frac{a}{2}C_{n'}$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

SEPARATION OF VARIABLES – 2D CARTESIAN



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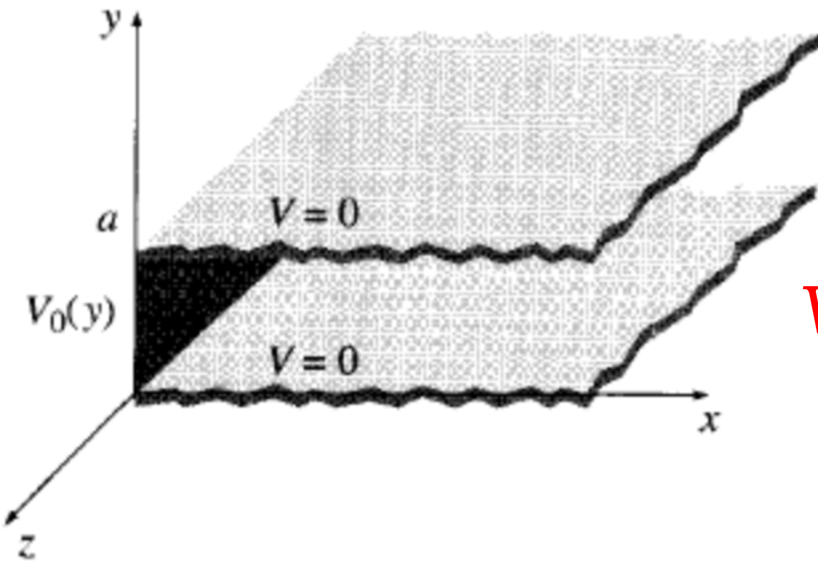
$(iv)\quad V \to 0 \quad \text{as} \quad x \to \infty$

$$V(x,y) = \sum_{n=1}^{\infty}\left[\frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy\right]e^{-n\pi x/a}\sin(n\pi y/a)$$

If $V_0(y) = V_0$,

$$C_n = \frac{2V_0}{a}\int_0^a \sin(n\pi y/a)\,dy = \frac{4V_0}{n\pi} \qquad n = 1,3,5,\ldots$$

SEPARATION OF VARIABLES – 2D CARTESIAN

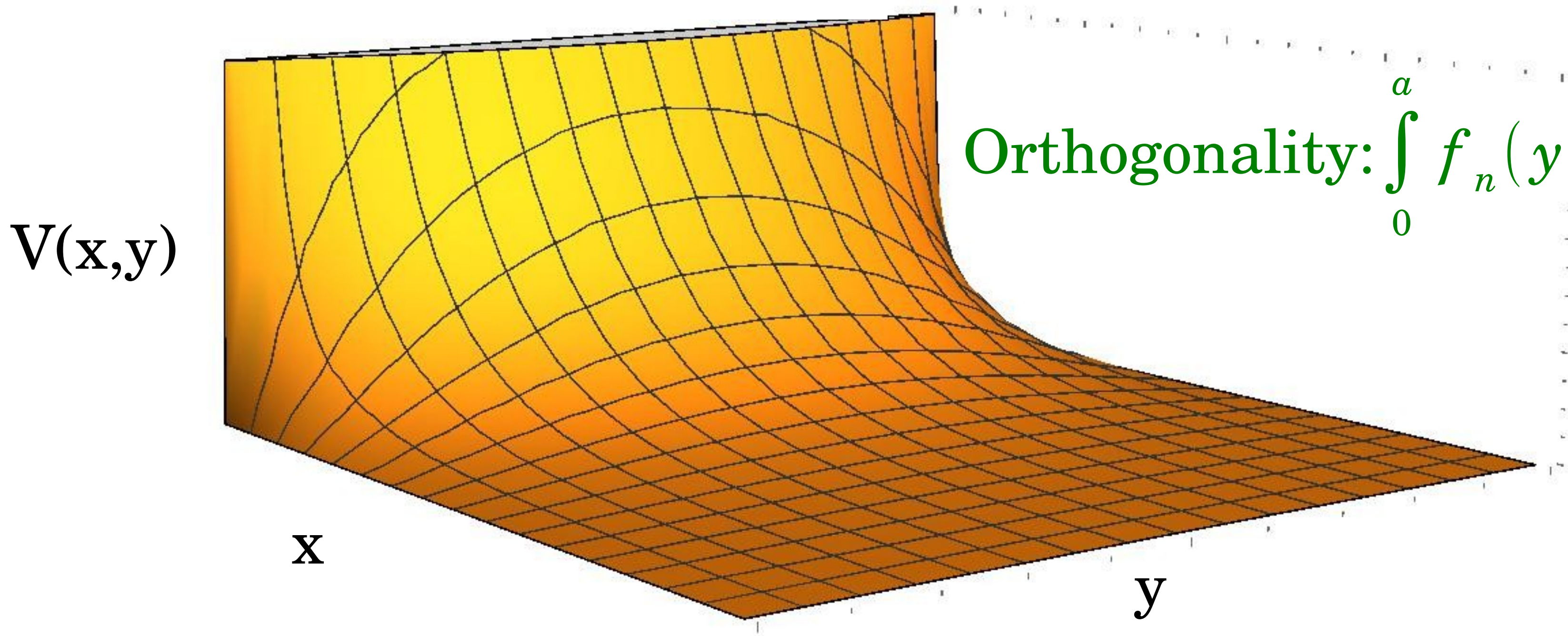


If, $V_0(y) = V_0$

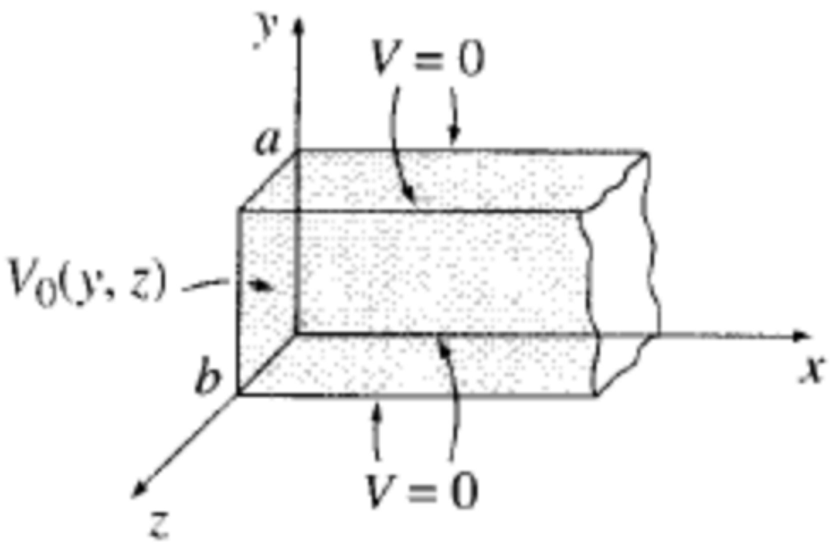
$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}e^{-n\pi x/a}\sin(n\pi y/a)$$

Completeness: $g(y)=\sum_{n=1}^{\infty} C_n f_n(y)$

Orthogonality: $\int_0^a f_n(y) f_{n'}(y)\,dy=\delta_{n,n'}$



SEPARATION OF VARIABLES – 3D CARTESIAN



Boundary conditions:

$(i)\;\; V = 0 \quad \text{when} \quad y = 0$

$(ii)\;\; V = 0 \quad \text{when} \quad y = a$

$(i)\;\; V = 0 \quad \text{when} \quad z = 0$

$(ii)\;\; V = 0 \quad \text{when} \quad z = b$

$(iii)\;\; V = V_0(y, z) \quad \text{when} \quad x = 0$

$(iv)\;\; V \to 0 \quad \text{as} \quad x \to \infty$

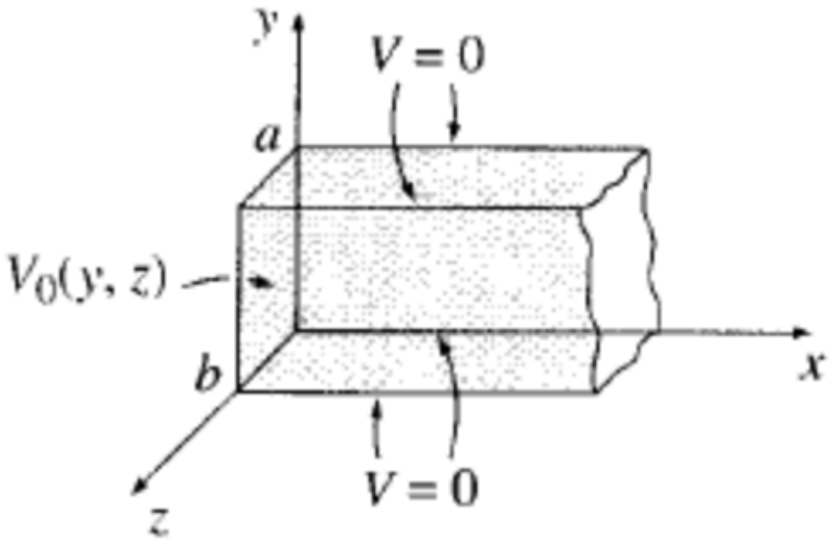
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If $V(x, y) = X(x)Y(y)Z(z)$ then

$$\frac{d^2 X}{d x^2} = C_1 X \qquad \frac{d^2 Y}{d y^2} = C_2 Y \qquad \frac{d^2 Z}{d z^2} = C_3 Z$$

with $\quad C_1 + C_2 + C_3 = 0$

SEPARATION OF VARIABLES – 3D CARTESIAN



Boundary conditions:

$(i)\quad V=0 \quad \text{when} \quad y=0$

$(ii)\quad V=0 \quad \text{when} \quad y=a$

$(i)\quad V=0 \quad \text{when} \quad z=0$

$(ii)\quad V=0 \quad \text{when} \quad z=b$

$(iii)\quad V=V_0(y,z) \quad \text{when} \quad x=0$

$(iv)\quad V\rightarrow 0 \quad \text{as} \quad x\rightarrow\infty$

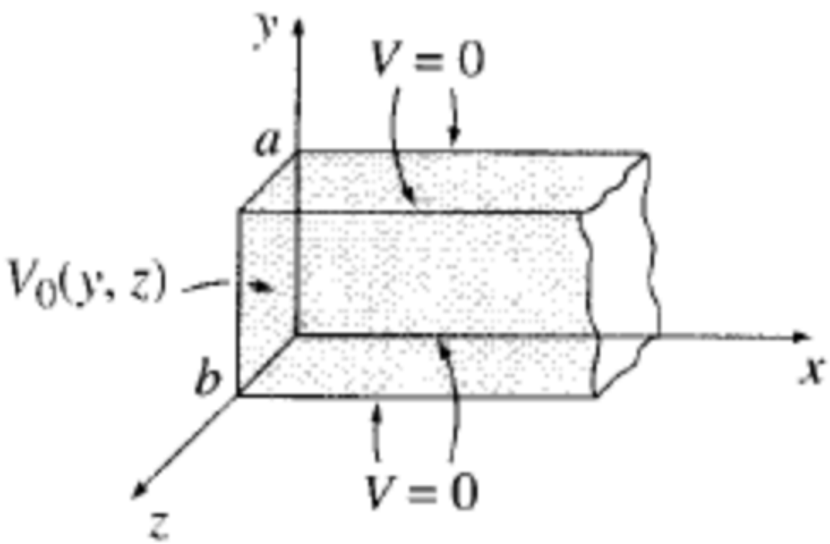
$$\frac{d^2X}{dx^2}=(k^2+l^2)X \quad \Rightarrow \quad X(x)=Ae^{\sqrt{k^2+l^2}\,x}+Be^{-\sqrt{k^2+l^2}\,x}$$

$$\frac{d^2Y}{dy^2}=-k^2Y \quad \Rightarrow \quad Y(y)=C\sin ky+D\cos ky$$

$$\frac{d^2Z}{dz^2}=-l^2Z \quad \Rightarrow \quad Z(z)=E\sin lz+F\cos lz$$

$$A=0, \quad D=0, \quad F=0, \quad k=\frac{n\pi}{a}, \quad l=\frac{m\pi}{b}$$

SEPARATION OF VARIABLES – 3D CARTESIAN



Boundary conditions:

$(i)\quad V = 0 \quad \text{when} \quad y = 0$

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$(iii)\quad V = V_0(y,z) \quad \text{when} \quad x = 0$

$(iv)\quad V \to 0 \quad \text{as} \quad x \to \infty$

$$V(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\, e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a)\sin(m\pi z/b)$$

$$C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\sin(m\pi z/b)\,dy\,dz$$

If, $V_0(y,z) = V_{0,}$

$$V(x,y) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5,\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\sin(m\pi z/b)$$

SEPARATION OF VARIABLES – 3D SPHERICAL

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

If the problem has azimuthal symmetry, V is independent of ϕ

In this case, let: $V(r,\theta) = R(r)P(\theta)$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = -\frac{1}{P}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) = k$$

The radial equation is $r^2 \frac{d^2 R}{d r^2} + 2r\frac{dR}{dr} - l(l+1)R = 0$

Let, $R = A r^n$

$$n(n-1) + 2n - l(l+1) = 0$$

$$n = l \,,\, -(l+1)$$

$$R(r) = A r^l + \frac{B}{r^{l+1}}$$

SEPARATION OF VARIABLES – 3D SPHERICAL

The angular equation is, $\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial P}{\partial\theta}\right)+l(l+1)P=0$

$$\Rightarrow \quad \frac{d^2P}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dP}{d\theta}+l(l+1)P=0$$

Let, $x=\cos\theta$, $\quad \Rightarrow \quad dx=-\sin\theta\, d\theta \quad \Rightarrow \quad d\theta=-\dfrac{dx}{\sqrt{1-x^2}}$

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

The solution to this equation are the **Legendre Polynomials,**

$$\boldsymbol{P_l(\cos\theta)}$$

LEGENDRE POLYNOMIALS

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

Let, $P(x)=\sum_{n=0}^{\infty} a_n x^n$

$$\frac{dP}{dx}=\sum_{n=1}^{\infty} n\,a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$\frac{d^2P}{dx^2}=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Substituting back in the equation,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}2n\,a_n x^n+\sum_{n=0}^{\infty}l(l+1)a_n x^n \;=\; 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}2n\,a_n x^n+\sum_{n=0}^{\infty}l(l+1)a_n x^n \;=\; 0$$

LEGENDRE POLYNOMIALS

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n\,a_n + l(l+1)a_n\right]x^n = 0$$

Equating coefficients of powers of x,

$$\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n\,a_n + l(l+1)a_n\right] = 0 \qquad \forall\, n = [0,\infty]$$

This defines a recursion relation among the coefficients,

$$a_{n+2} = \frac{n(n+1)-l(l+1)}{(n+2)(n+1)}a_n = -\frac{(l+n+1)(l-n)}{(n+2)(n+1)}a_n$$

$$a_2 = -\frac{l(l+1)}{1.2}a_0$$

$$a_3 = -\frac{(l+2)(l-1)}{1.2.3}a_1$$

$$a_4 = \frac{(l+3)(l+1)l(l-2)}{1.2.3.4}a_0$$

$$a_5 = \frac{(l+4)(l+2)(l-1)(l-3)}{1.2.3.4.5}a_1$$

a_0 and a_1 can be arbitrarily chosen

Odd and even powers do not mix

LEGENDRE POLYNOMIALS

$$a_{2m} = (-1)^m \frac{(l+2m-1)(l+2m-3)\ldots(l+1)l(l-2)\ldots(l-2m+2)}{(2m)!} a_0$$

$$a_{2m+1} = (-1)^m \frac{(l+2m)(l+2m-2)\ldots(l+2)(l-1)(l-3)\ldots(l-2m+1)}{(2m+1)!} a_1$$

$$P(x) = a_0 y_1(x) + a_1 y_2(x)$$

$$y_1 = 1 + \sum_{m=1}^{\infty} \left(\frac{a_{2m}}{a_0} \right) x^{2m} \qquad y_2 = x + \sum_{m=1}^{\infty} \left(\frac{a_{2m+1}}{a_1} \right) x^{2m+1}$$

y_1 and y_2 are solutions of the Legendre equation

If l is a non-negative integer, then either y_1 terminates, or y_2 terminates!

$y_1(x)$ terminates when $l = 2m$ (even integer)

$y_2(x)$ terminates when $l = 2m+1$ (odd integer)

LEGENDRE POLYNOMIALS

$$a_{2m} = (-1)^m \frac{(l+2m-1)(l+2m-3)\ldots(l+1)l(l-2)\ldots(l-2m+2)}{(2m)!} a_0$$

$$a_{2m+1} = (-1)^m \frac{(l+2m)(l+2m-2)\ldots(l+2)(l-1)(l-3)\ldots(l-2m+1)}{(2m+1)!} a_1$$

$$y_1 = 1 + \sum_{m=1}^{\infty} \left(\frac{a_{2m}}{a_0} \right) x^{2m} \qquad y_2 = x + \sum_{m=1}^{\infty} \left(\frac{a_{2m+1}}{a_1} \right) x^{2m+1}$$

$$l=0: \; y_1(x) = 1$$

$$l=2: \; y_1(x) = 1 - 3x^2$$

$$l=4: \; y_1(x) = 1 - 10x^2 + \frac{35}{3} x^4$$

$$l=1: \; y_2(x) = x$$

$$l=3: \; y_2(x) = x - \frac{5}{3} x^3$$

$$l=5: \; y_2(x) = x - \frac{14}{3} x^3 + \frac{21}{5} x^4$$

Together with the condition that $P_N(1) = 1$,
these define the Legendre Polynomials

SEPARATION OF VARIABLES - 3D SPHERICAL

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Rodrigues Formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

Legendre Polynomials:

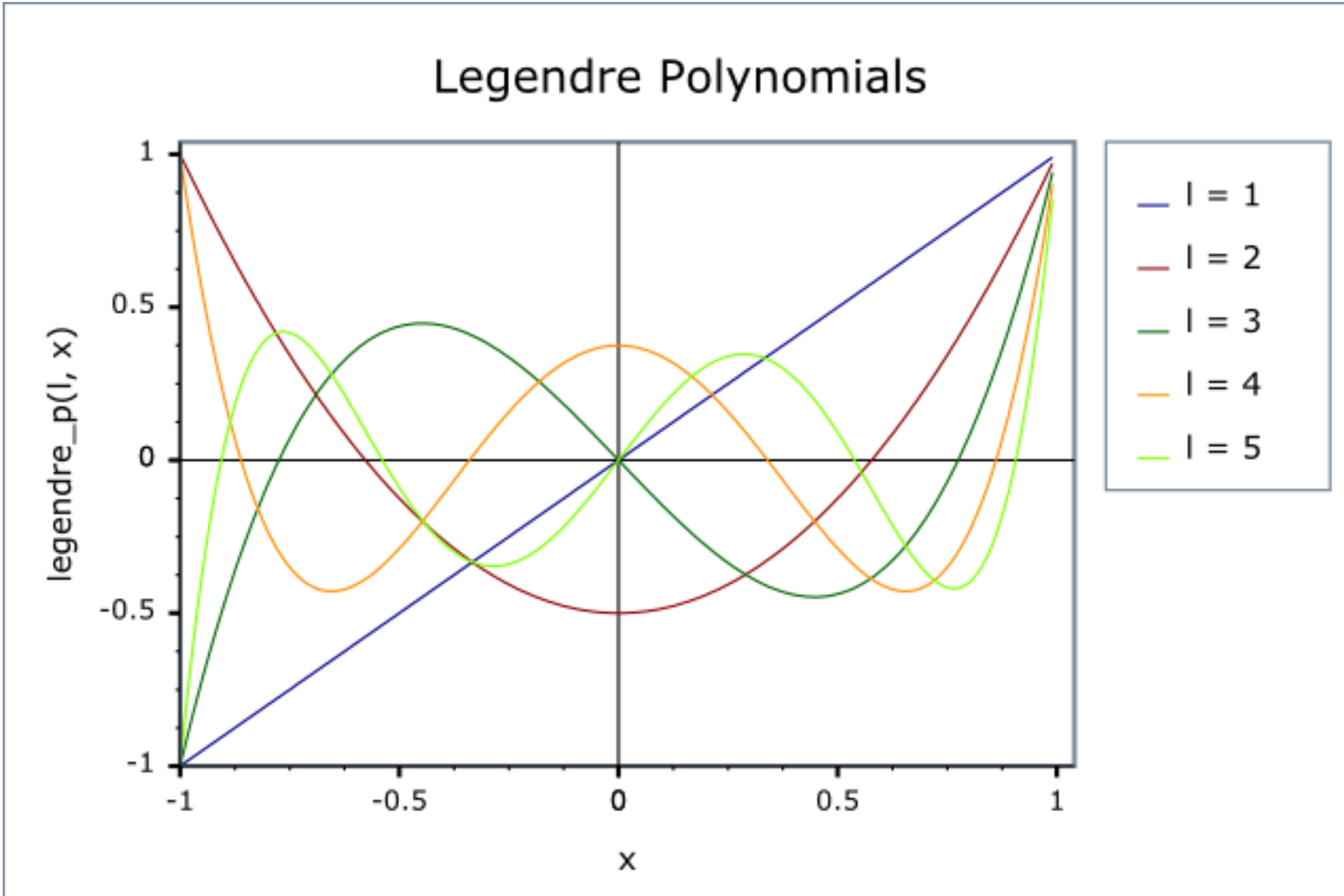
$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\delta_{nm}$$

Use orthogonality
to find expansion co-effs…

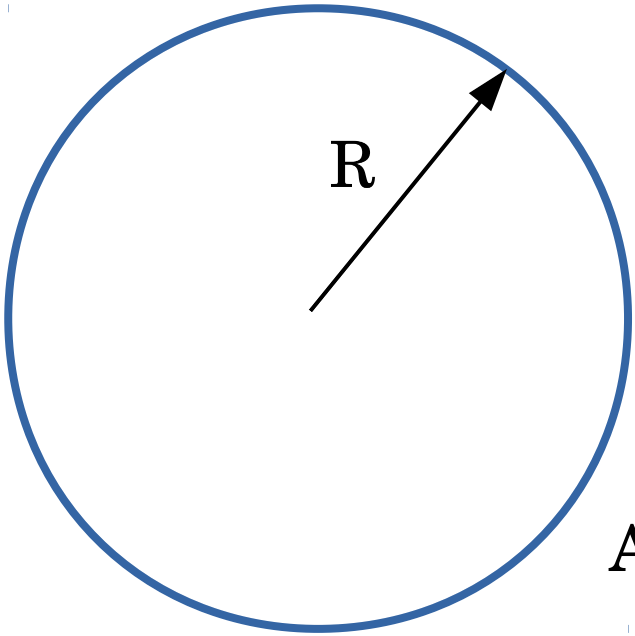
General Solution, $V(r, \theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

Potential specified on the surface of a sphere
Solution inside should not have 1/r type terms
Solution outside should not have r type terms
Use exapansion in Legendre Polynomials to find the coefficients.

SEPARATION OF VARIABLES – 3D SPHERICAL



SEPARATION OF VARIABLES - 3D SPHERICAL



Hollow sphere of radius R

Potential on the surface $V_0(\theta)$

Find potential inside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

At $r=R$, $V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$

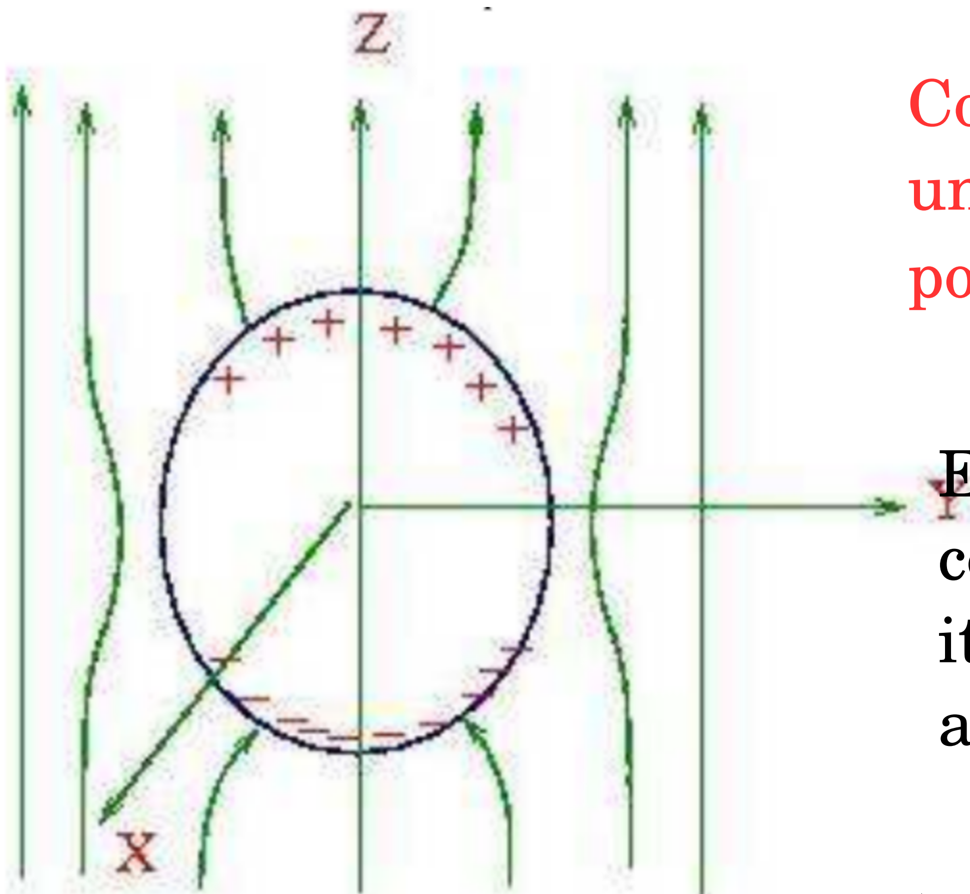
$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

If, $V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)]$

$$A_0 = k/2, \qquad A_1 = -k/2R$$

$$V(r,\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

SPHERICAL SYMMETRY + SEPARATION



Conducting sphere of radius R in a uniform electric field **E**. What is the potential outside the sphere?

$$\boldsymbol{E} = E_0 \hat{\boldsymbol{z}}$$

Electric field induces charges on the conductor, which modifies the field itself near the sphere. The field far away from the sphere is unperturbed.

$$\boldsymbol{E}\,|_{r\to\infty} = E_0 \hat{\boldsymbol{z}}$$

$$V(r,\theta)\,|_{r\to\infty} = -E_0 z + C = -E_0 r\cos\theta + C$$

Sphere is an equipotential (set it zero) , $C=0$

If we assume a separable form for the potential, the general solution is,

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

SPHERICAL SYMMETRY + SEPARATION

The general solution must match the potential at infinity.

As $r \to \infty$, the B_l terms in the general solution go to zero.

$$V(r,\theta)|_{r\to\infty} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta = -E_0 r P_1$$

$$A_0 = 0, \quad A_1 = -E_{0,} \quad A_{2,3,\ldots} = 0$$

$$V(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Boundary condition on surface of conductor, $V(R,\theta) = 0$

$$B_0 = 0, \quad \frac{B_1}{R^2} = E_0 R, \quad B_{2,3,\ldots} = 0$$

$$V(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\cos\theta$$

SPHERICAL SYMMETRY + SEPARATION

$$V(r,\theta) = -E_0\left(1-\frac{R^3}{r^3}\right)r\cos\theta$$

$$\boldsymbol{E} = -\nabla . V = -\left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)V(r,\theta)$$

$$\boldsymbol{E} = \hat{r}\,E_0\left(1+\frac{2R^3}{r^3}\right)\cos\theta-\hat{\theta}\,E_0\left(1-\frac{R^3}{r^3}\right)\sin\theta$$

$$\boldsymbol{E}.\hat{\boldsymbol{r}}\,|_{r=R} = \frac{\sigma}{\epsilon_0} \quad \Rightarrow \quad \sigma = 3\epsilon_0 E_0\cos\theta$$

$$Q_{\text{upper}} = 3\epsilon_0 E_0 R^2\int_0^{2\pi} d\phi\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = 3\pi\epsilon_0 E_0 R^2$$

$$Q_{\text{lower}} = -Q_{\text{upper}} = -3\pi\epsilon_0 E_0 R^2$$

WHEN DOES SEPARATION WORK?

The potential or charge density is specified on the boundaries of some region, and we wish to find the potential in the interior.

Will separation of variables always work? **YES AND NO!**

Is there a general algorithm that predicts when separation will work? **NO!**

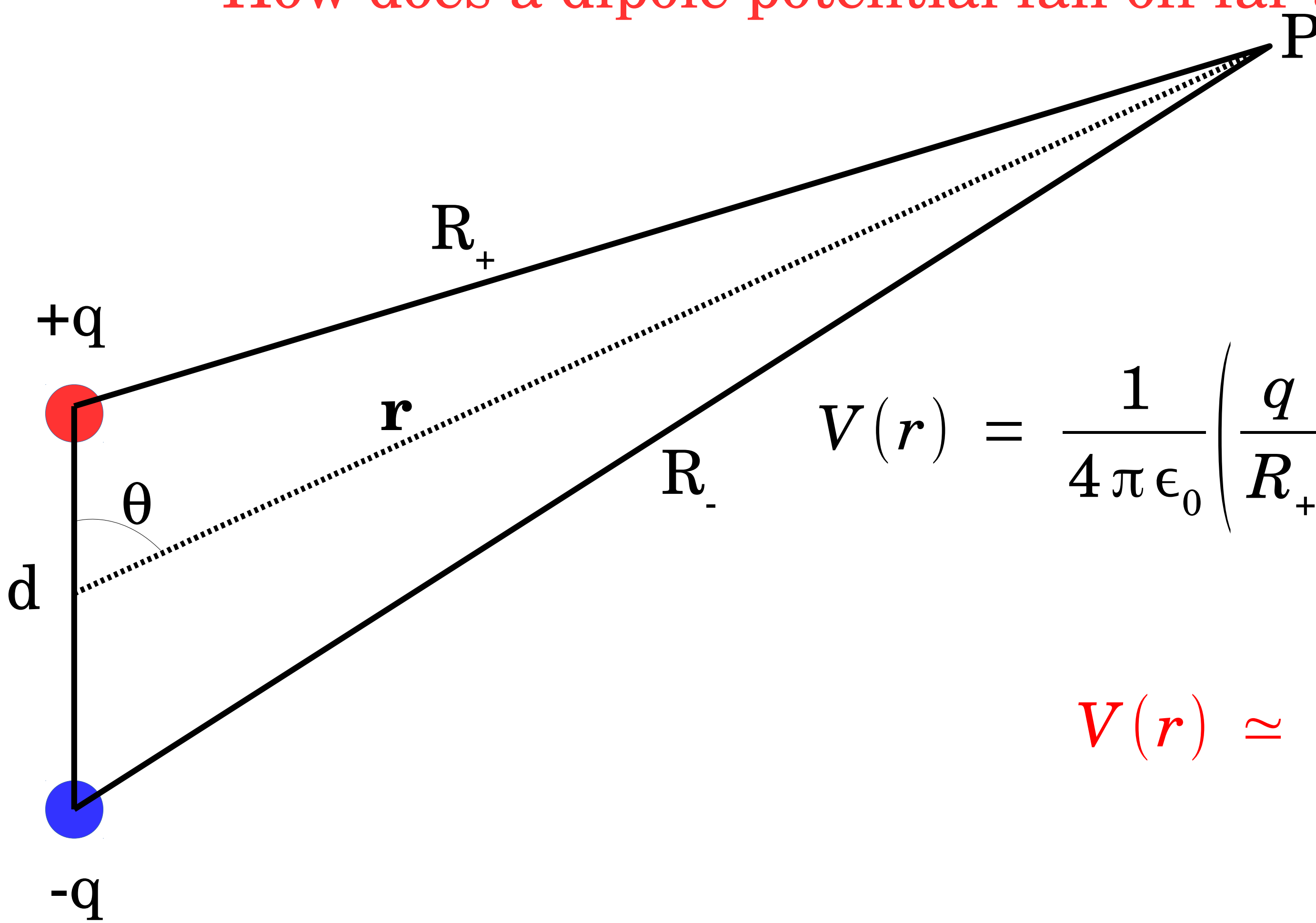
Generally, existence of symmetry in the problem allows for separation to work. In particular, if the boundary conditions are themselves separable, separation of variables should work!

POTENTIALS AT LARGE DISTANCES

How does a charge distribution look from far away?

If a charge distribution has a net charge Q, the potential at large distance falls off as, $V = Q/4\pi\epsilon_0 r$

How does a dipole potential fall off far away?

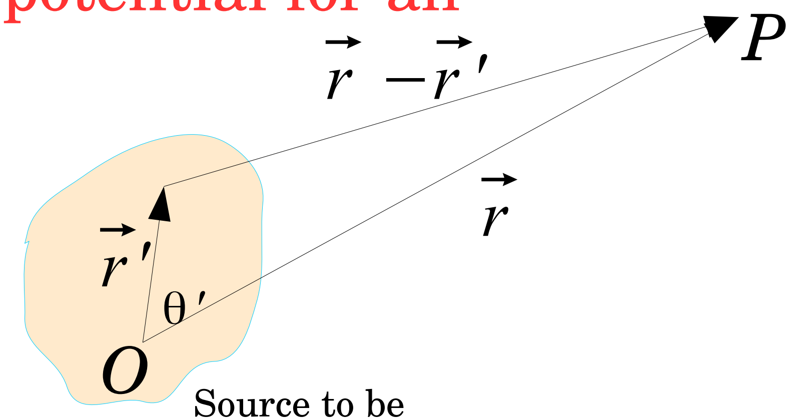


$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_+} - \frac{q}{R_-}\right) \simeq \frac{1}{4\pi\epsilon_0}\frac{q\,d\cos\theta}{r^2}$$

$$V(r) \simeq \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}$$

POTENTIALS AT LARGE DISTANCES

Can we say something about the potential for an arbitrary charge distribution?



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

For points far away, $r \gg r'$

so we expand in a power series in $\dfrac{r'}{r}$

$$|\vec{r} - \vec{r}\,'| = \left[r^2 + r'^2 - 2rr'\cos\theta'\right]^{\frac{1}{2}} = r\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{1}{r}\left[1+\epsilon\right]^{-\frac{1}{2}} \quad \text{with} \quad \epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right]$$

POTENTIALS AT LARGE DISTANCES

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r}-2\cos\theta'\right)^2 \right.$$

$$\left. -\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r}-2\cos\theta'\right)^3+\ldots\right]$$

Collecting powers of (r'/r)

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\frac{3\cos^2\theta'-1}{2}\right.$$

$$\left. +\left(\frac{r'}{r}\right)^3\frac{5\cos^3\theta'-3\cos^2\theta'}{2}\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

MULTIPOLE EXPANSION OF THE POTENTIAL

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int d^3r'\left[\rho(r')r'^n P_n(\cos\theta')\right]$$

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int d^3r'\rho(r')\right.$$

Monopole term

$$+\frac{1}{r^2}\int d^3r'\left[\rho(r')\,r'\cos\theta'\right]$$

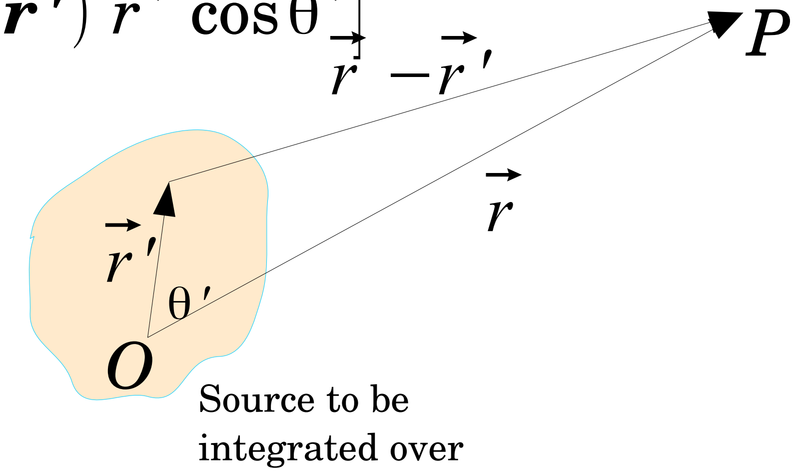
Dipole term

$$\left.+\frac{1}{r^3}\int d^3r'\left[\rho(r')\,r'^2\,\frac{3\cos^2\theta'-1}{2}\right]+\ldots\right]$$

Quadrupole term

THE DIPOLE TERM

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int d^3r'\left[\rho(\boldsymbol{r}')\, r'\cos\theta'\right]$$



$$r'\cos\theta' = \hat{\boldsymbol{r}}.\boldsymbol{r}'$$

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\boldsymbol{r}}.\int \boldsymbol{r}'\rho(\boldsymbol{r}')\,d^3r'$$

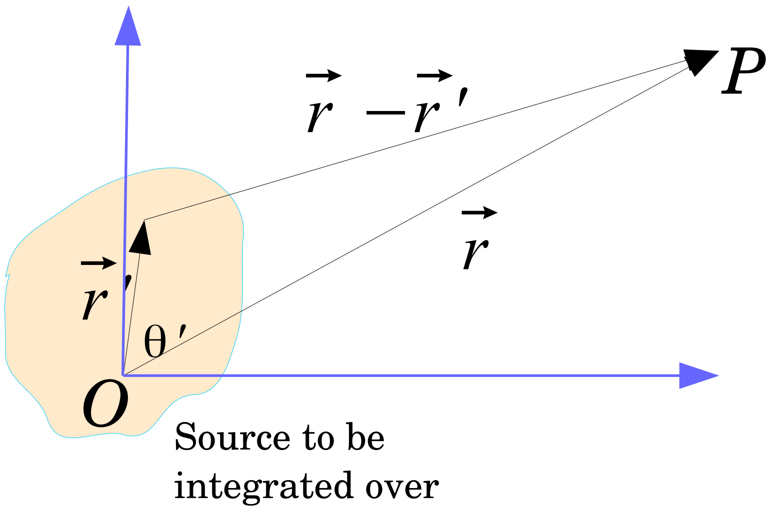
Define the dipole moment as, $\boldsymbol{p} \equiv \int \boldsymbol{r}'\rho(\boldsymbol{r}')\,d^3r'$

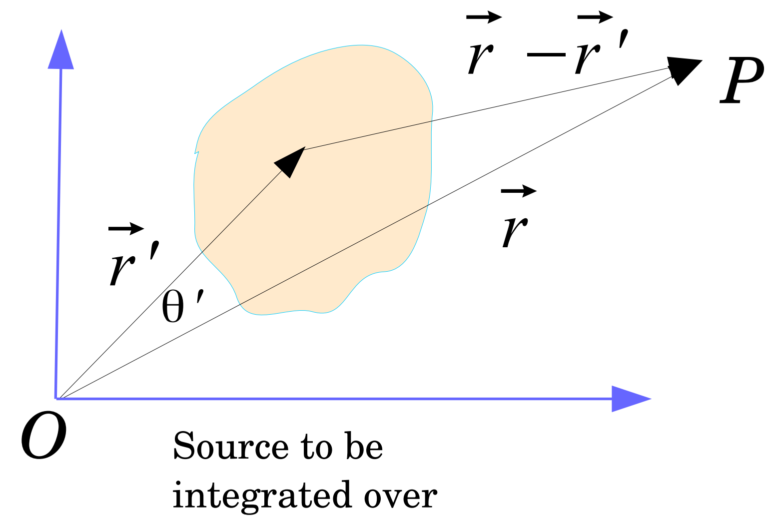
$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}.\hat{\boldsymbol{r}}}{r^2}$$

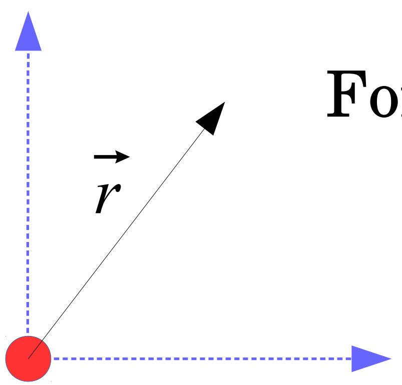
For the physical dipole, $\rho(\boldsymbol{r}') = q\,\delta^3(\boldsymbol{r}' - \boldsymbol{r}_+) - q\,\delta^3(\boldsymbol{r}' - \boldsymbol{r}_-)$

$$\boldsymbol{p} = q(\boldsymbol{r}_+ - \boldsymbol{r}_-) = q\,\boldsymbol{d}$$

ORIGIN OF COORDINATES IN MULTIPOLE EXPNS.

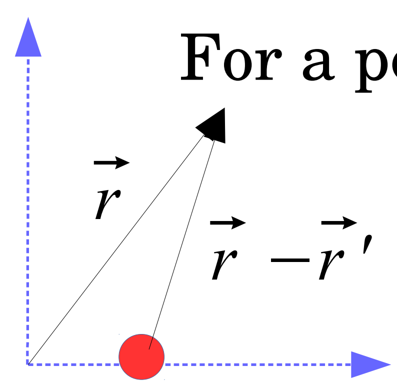






For a point charge at the origin, $V_{\text{tot}} = V_{\text{mono}} = \dfrac{q}{4\pi\epsilon_0 r}$

$$V_{\text{dip}} = V_{\text{quad}} = \ldots = 0$$



For a point charge away from the origin, $V_{\text{tot}} = \dfrac{q}{4\pi\epsilon_0 |r - r'|}$

$$V_{\text{mono}} = \frac{q}{4\pi\epsilon_0 r}$$

$$V_{\text{dip}},\ V_{\text{quad}},\ \ldots \neq 0$$

ORIGIN OF COORDINATES IN MULTIPOLE EXPNS.

The lowest order term in the multipole expansion is independent of the origin. The higher order terms are NOT necessarily so.

How does the dipole moment change under a shift of origin?

Suppose we displace the origin by a vector $\boldsymbol{a}$,

The new dipole moment, $\tilde{\boldsymbol{p}} = \int \tilde{\boldsymbol{r}}'\rho(\boldsymbol{r}')d^3r' = \int (\boldsymbol{r}'-\boldsymbol{a})\rho(\boldsymbol{r}')d^3r'$

$$= \int \boldsymbol{r}'\rho(\boldsymbol{r}')d^3r' - \boldsymbol{a}\int \rho(\boldsymbol{r}')d^3r'$$

$$= \boldsymbol{p} - Q\boldsymbol{a}$$

If the total charge, $Q = 0$, the dipole moment is independent of the origin, $\tilde{\boldsymbol{p}} = \boldsymbol{p}$