PH108 : Basics of Electricity & Magnetism : Problem Set 1

Only * problems are to be solved in the tut session

1.* Calculate the curl and divergence of the following vector functions. If the curl turns out to be zero, construct a scalar function ϕ of which the vector field is the gradient:

   (a) $F_x = x + y$; $F_y = -x + y$; $F_z = -2z$
   (b) $G_x = 2y$; $G_y = 2x + 3z$; $G_z = 3y$
   (c) $H_x = x^2 - z^2$; $H_y = 2$; $H_z = 2xz$

2. Calculate the Laplacian of the following functions:

   (a) $T_a = x^2 + 2xy + 3z + 4$
   (b) $T_b = \sin(x)\ \sin(y)\ \sin(z)$
   (c) $T_c = e^{-5x}\ \sin(4y)\ \cos(3z)$
   (d) $\vec{v} = x^2\hat{\imath} + 3xz^2\hat{\jmath} - 2xz\hat{k}$

3.* Test the Stokes theorem for the vector $\vec{v} = xy\,\hat{\imath} + 2yz\,\hat{\jmath} + 3zk\,\hat{k}$ using a triangular area with vertices at (0,0,0), (0,2,0) and (0,0,2).

4. Compute the unit normal vector $\hat{n}$ to the ellipsoidal surfaces defined by constant values of $\Phi(x, y, z) = V\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$. What is $\hat{n}$ when $a = b = c$?

5.* A force defined by $\vec{F} = A(y^2\hat{\imath} + 2x^2\hat{\jmath})$ is exerted on a particle which is initially at the origin of the co-ordinate system. A is a positive constant. We transport the particle on a triangular path defined by the points (0,0,0), (1,0,0), (1,1,0) in the counterclockwise direction.

   (a) How much work does the force do when the particle travels around the path? Is this a conservative force?
   (b) The particle is placed at rest right at the origin. Is this a stable situation? Give any argument (mathematical, physical, intuitive) to justify the stability (or instability) of this situation.

6. The area bounded by the curve $r = 2R\cos\theta$ has a surface charge density $\sigma(r,\theta) = \sigma_0 \frac{r}{R} \sin^4\theta$. What is the total amount of charge?

7.* Suppose that the height of a certain mountain (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 + 14x + 10y + 40),$$

   where x is the distance (in km) east, y the distance north of the closest town.

   (a) Where is the top of the mountain located, and how high is it?
   (b) How steep is the slope (in feet per km) at a point 1 km north and 1 km east of the town? In what direction is the slope steepest, at that point?

8. The gradient operator ∇ behaves like a vector in " some sense". For example, divergence of a curl $(\nabla.(\nabla \times \vec{A}) = 0)$ for any $\vec{A}$, may suggest that it is just like $\vec{A}.(\vec{B} \times \vec{C})$ being zero if any two vectors are equal. Prove that $\nabla \times \nabla \times \vec{F} = \nabla(\nabla.\vec{F}) - \nabla^2\vec{F}$. To what extent does this look like the well known expansion of $\vec{A} \times \vec{B} \times \vec{C}$?