Examples of Skolemization

Example #1 Convert in CNF:

Every philosopher writes at least one book.

In FOL:

$\forall x [Philo(x) \Rightarrow \exists y (Book(y) \wedge Write(x,y))]$

Conversion into CNF:

STEP1: Eliminate $\Rightarrow$

$\forall x [\neg Philo(x) \vee \exists y (Book(y) \wedge Write(x,y))]$

STEP2: Move $\neg$ inwards Same expression

STEP 3: Eliminate $\exists$

in the FOL axiom:

$\forall x [\neg Philo(x) \vee \exists y (Book(y) \wedge Write(x,y))]$

need to eliminate

variable y is in the scope of x

Skolemize need a Skolem function

$y = g(x)$

STEP4: ~~Eliminate quantifiers~~

~~$\forall x$~~ $[\neg Philo(x) \vee (Book(g(x)) \wedge Write(x, g(x))]$

STEP5: Eliminate quantifiers

$\neg Philo(x) \vee (Book(g(x)) \wedge Write(x, g(x))$

STEP6: Distribute $\wedge$ over $\vee$:

$(\neg Philo(x) \vee Write(x, g(x)) \wedge (\neg Philo(x) \vee Book(g(x)))$

Example #2 Convert in CNF:

All students of a philosopher read one of their teachers book.

in FOL:

$$\forall x \forall y \,[\, Philo(x) \wedge StudentOf(y,x) \Rightarrow \exists z \,[\, Book(z) \wedge Write(x,z) \wedge Read(y,z) \,]_2 \,]_1$$

x: teacher/philosopher, y: student.

Conversion into CNF:

STEP 1: Eliminate $\Rightarrow$

$$\forall x \forall y \,[\, \neg (Philo(x) \wedge StudentOf(y,x)) \vee \exists z \,[\, Book(z) \wedge Write(x,z) \wedge Read(y,z) \,]_2 \,]_1$$

STEP 2: Move $\neg$ inwards:

$$\forall x \forall y \,[\, \neg Philo(x) \vee \neg StudentOf(y,x) \vee \exists z \,[\, Book(z) \wedge Write(x,z) \wedge Read(y,z) \,]_2 \,]_1$$

STEP 3: Eliminate $\exists$ quantifiers $\Rightarrow$ z is in the scope of $\forall$ quantified variables x and y

$\Downarrow$

Skolem function $z = f(x,y)$

STEP 4: Skolemize

$$\forall x \forall y \,[\, \neg Philo(x) \vee \neg StudentOf(y,x) \vee (Book(f(x,y)) \wedge Write(x, f(x,y)) \wedge Read(y, f(x,y))) \,]_1$$

Step 5 : Eliminate all quantifiers.

$\neg Philo(x) \vee \neg StudentOf(y,x) \vee$
$\vee (Book(f(x,y)) \wedge Write(x,f(x,y)) \wedge$
$\wedge Read(y,f(x,y)))$

Step 6 : Distribute $\wedge$ over $\vee$:

$[\neg Philo(x) \vee \neg StudentOf(y,x) \vee Book(f(x,y))]$
$\wedge [\neg Philo(x) \vee \neg StudentOf(y,x) \vee Write(x,f(x,y))]$
$\wedge [\neg Philo(x) \vee \neg StudentOf(y,x) \vee Read(y,f(x,y))]$

Rules of Skolemization

Rule 1 $\forall x (\ldots \exists y \ldots)$

$\Downarrow$

$y = f(x)$

$\forall x (\forall y \ldots (\forall z (\ldots \exists s \ldots)$

$\Downarrow$

$S = f(x, y, z)$

Rule 2 There are no universally quantified variables in which the $\exists$ quantified variable is

Example $(\exists P(x) \Rightarrow Q(y)) \vee \exists z\, S(x,z)]$

Replace the variable with a constant!

$\theta = \{x/A ; z/B\}$

$\Rightarrow (P(A) \Rightarrow Q(y)) \vee S(A,B)$

Rule 3 There are multiple $\exists$ quantifiers that need to be eliminated and they are in the scope of universally quantified variables!

Example #3

$$\forall x \Big[\big(P(x) \Rightarrow \exists y \; Q(x,y) \big) \wedge \forall z \big[R(x,z) \Rightarrow \exists w \; S(x,z,w) \big] \Big]$$

Need to eliminate 2 $\exists$ quantifiers:

variable $y \rightarrow$ in scope of x

$w \rightarrow$ in scope of x and z

$\Downarrow$

$y = f_1(x)$

$w = f_2(x,z)$

$\Downarrow$

$$\forall x \Big[\big(P(x) \Rightarrow Q(x, f_1(x)) \big) \wedge \forall z \big[R(x,z) \Rightarrow S(x, z, f_2(x,z)) \big] \Big]$$

Example #4 let us transform in CNF:

In each country, every citizen has some rights and obligations and none is above the law.

let us first encode in FOL:

$\forall c \forall z [Country(c) \wedge Citizen of(c, z) \Rightarrow \exists r, o [Rights(z, r) \wedge Obligation(z, o) \wedge \neg Above Law(z)]]$

Now let us transform in CNF.

STEP 1: Eliminate $\Rightarrow$

$\forall c \forall z [\neg (Country(c) \wedge Citizen Of(c, z)) \vee \exists r, o [Rights(z, r) \wedge Obligatin(z, o) \wedge \neg Above Law(z)]]$

STEP 2: More $\neg$ inwards

$\forall c \forall z (\neg Country(c) \vee \neg Citizen Of(c, z) \vee \exists r \exists o [Rights(z, r) \wedge Obligatin(z, o) \wedge \neg Above Law(z)])$

STEP 3: Eliminate $\exists$

There are two variables $\Rightarrow$ 2 Skolem functions and both are within the scope of 2 universally quantified variables

$r = f_1(c, z)$

$o = f_2(c, z)$

STEP 4: Skolemize

$$\forall c \forall z [\neg \text{Country}(c) \vee \neg \text{CitizenOf}(c,z) \vee (\text{Rights}(z, f_1(c,z)) \wedge \text{Obligation}(z, f_2(c,z))) \wedge \neg \text{Aborelaw}(z)]$$

STEP 5: Eliminate $\forall$

$$\neg \text{Country}(c) \vee \neg \text{CitizenOf}(c,z) \vee (\text{Rights}(z, f_1(c,z)) \wedge \text{Obligation}(z, f_2(c,z))) \wedge \neg \text{Aborelaw}(z))$$

STEP 6: Distribute $\vee$ over $\wedge$

$$[\neg \text{Country}(c) \vee \neg \text{CitizenOf}(c,z) \vee \text{Rights}(z, f_1(c,z))] \wedge [\neg \text{Country}(c) \vee \neg \text{CitizenOf}(c,z) \vee \text{Obligation}(z, f_2(c,z))] \wedge [\neg \text{Country}(c) \vee \neg \text{CitizenOf}(c,z) \vee \neg \text{Aborelaw}(z)]$$

Example #5

Transform in CNF:

For every mother and father and their child, the first day of school is unique. The memory will last a lifetime.

A. Translate in FOL

$$\forall m \forall f \forall c [mother(m) \wedge father(f) \wedge ChildOf(m,c) \wedge ChildOf(f,c) \Rightarrow \exists d (FirstDaySchool(c,d) \wedge Memory\_last(c,d,Lifetime) \wedge Memory\_last(m,d,Lifetime) \wedge (Memory\_last(f,d,Lifetime))]$$

CNF transformation:

STEP1 : Eliminate $\Rightarrow$

$$\forall m \forall f \forall c [\neg (mother(m) \wedge father(f) \wedge ChildOf(m,c) \wedge ChildOf(f,c)) \vee \exists d (FirstDaySchool(c,d) \wedge Memory\_last(c,d,Lifetime) \wedge Memory\_last(m,d,Lifetime) \wedge Memory\_last(f,d,Lifetime))]$$

STEP2 More $\neg$ inwards.

$$\forall m \forall f \forall c [\neg mother(m) \vee \neg father(f) \vee \neg ChildOf(m,c) \vee \neg ChildOf(f,c)) \vee \exists d (FirstDaySchool(c,d) \wedge Memory\_last(c,d,Lifetime) \wedge Memory\_last(m,d,Lifetime) \wedge Memory\_last(f,d,Lifetime))]$$

STEP 3: Eliminate ∃

$\forall m \forall f \forall c [\neg mother(c) \vee \neg father(f) \vee \neg childOf(m,c) \vee$
$\vee \neg childOf(f,c) \vee \exists d [FirstDaySchool(c,d) \wedge$
$\wedge Memorylast(c,d,lifetime) \wedge Memorylast(m,d,lifetime)$
$\wedge Memorylast(f,d,lifetime)]]$

The existentially quantified variable d is in the scope of 3 universally quantified variables =)

$d = F(m,f,c)$

STEP 4: Skolemize

$\forall m \forall f \forall c [\neg mother(c) \vee \neg father(f) \vee \neg childOf(m,c) \vee$
$\vee \neg childOf(f,c) \vee (FirstDaySchool(c, F(m,f,c)) \wedge$
$\wedge Memorylast(c, F(m,f,c), lifetime) \wedge$
$\wedge Memorylast(m, F(m,f,c), lifetime) \wedge$
$\wedge Memorylast(f, F(m,f,c), lifetime)$

STEP 5: Eliminate ∀

$\neg mother(c) \vee \neg father(f) \vee \neg childOf(m,c) \vee$
$\vee childOf(f,c) \vee [FirstDaySchool(c, F(m,f,c)) \wedge$
$\wedge Memorylast(c, F(m,f,c), lifetime) \wedge$
$\wedge Memorylast(m, F(m,f,c), lifetime) \wedge$
$\wedge Memorylast(f, F(m,f,c), lifetime)]$

STEP 6: Distribute $\wedge$ over $\vee$:

$[\neg mother(c) \vee \neg father(f) \vee \neg ChildOf(m,c) \vee \neg ChildOf(f,c) \vee FirstDaySchool(c, F(m,f,c))] \wedge$

$[\neg mother(e) \vee \neg father(f) \vee \neg ChildOf(m,c) \vee \neg ChildOf(f,c) \vee MemoryLast(c, F(m,f,c))] \wedge$

$[\neg mother(c) \vee \neg father(f) \vee \neg ChildOf(m,c) \vee \neg ChildOf(f,c) \vee MemoryLast(m, F(m,f,c))] \wedge$

$[\neg mother(c) \vee \neg father(f) \vee \neg ChildOf(m,c) \vee \neg ChildOf(f,c) \vee MemoryLast(f, F(m,f,c))]$

IMPORTANT!

A sentence in CNF

$(\underbrace{\neg A \vee B \vee C}_{clause\,1}) \wedge (\underbrace{A \vee \neg C \vee D}_{clause\,2}) \wedge \underbrace{(A \vee D)}_{clause\,3}$

We write KBs in clause form

1. clause 1
2. clause 2
3. clause 3