Examples Uncertainty

Axioms of Probability

Example 1

a) Would it be rational for an agent to hold the following three beliefs:

1) $P(A) = 0.4$ 2) $P(B) = 0.3$ 3) $P(A \vee B) = 0.5$

?

b) ~~If~~ If so, what range of probabilities would be rational for the agent to hold for $P(A \wedge B)$?

Solution

a) The probability assignment of a set of propositions (e.g. A, B, $C \vee D$ ...) $\boxed{\text{MUST BE}}$ consistent with the axioms of probability.

Let us define the $\boxed{\text{atomic events}}$

$A, B \Rightarrow$ boolean propositions

$\Downarrow$

There should be 4 atomic events

A	B	Atomic Events	Probabilities of atomic events
F	F	e_1	d
F	T	e_2	c
T	F	e_3	b
T	T	e_4	a

We can represent this also as:

	B	¬B
A	a	b
¬A	c	d

$\Rightarrow$ There we have the following equalities:

$P(A) = a + b = 0.4$

$P(B) = a + c = 0.3$

$P(A \vee B) = a + b + c = 0.5$

and also $P(TRUE) = a + b + c + d = 1$

$\hookrightarrow$ 4 equations with 4 variables

let us solve the equations:

$\left.\begin{array}{l} a + b = 0.4 \\ a + b + c = 0.5 \end{array}\right\} \Rightarrow \boxed{c = 0.1}$

$a + c = 0.3 \rightarrow \boxed{a = 0.2}$

$a + b = 0.4 \rightarrow \boxed{b = 0.2}$

$a + b + c + d = 1 \quad d = 1 - (0.1 + 0.2 + 0.2)$

$\boxed{d = 0.5}$

Now let us address question $\boxed{a}$

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

$P(A \vee B) \rightsquigarrow 0.5$, $P(A) \rightsquigarrow a+b=0.4$, $P(B) \rightsquigarrow a+c=0.3$, $P(A \wedge B) \rightsquigarrow a=0.2$

Is it true that $0.5 = 0.4 + 0.3 - 0.2$

YES

let us address question $\boxed{b}$ → the range of probabilites of $P(A \wedge B)$

Remember:

	B	$\neg B$
A	a	b
$\neg A$	c	d

(1) $P(A) = a + b = 0.4$

(2) $P(B) = a + c = 0.3$

(3) $P(A \vee B) = a + b + c \overset{?}{=} 0.7$

(4) $P(A \wedge B) = a$

let us assume another value

From (1) & (3) ⇒ $c = 0.3$

in (2) ⇒ $a = 0$ ⇒ $P(\neg A) = 1$

in (1) $b = 0.4$

We also have:

$$a+b+c+d=1$$

$$\Downarrow$$

$$d=1-(a+b+c)=1-(0+0.4+0.3)=0.3$$

$$d=0.3$$

$$\Downarrow$$

$$P(\neg A \wedge \neg B)=d=0.3$$

But we had $P(A \wedge B)=0=a$

$$\Downarrow$$

$$P(\neg(A\wedge B))=1-0=1=P(\neg A \vee \neg B)=$$

$$=\underbrace{P(\neg A)}_{1}+\underbrace{P(\neg B)}_{\substack{1-0.4=\\=0.6}}-\underbrace{P(\neg A \wedge \neg B)}_{0.3}$$

Therefore $1=1+0.6-0.3 \leftarrow$ Impossible

$$\Downarrow$$

Irrational.

Let us see what are the possible values for $P(A\vee B)$

If $P(A)=a+b=0.4$
$P(B)=a+c=0.3$ $\Big\} \Rightarrow 2a+b+c=0.7$
$P(A\vee B)=a+b+c=x$
and $a+b+c+d=1$

$$a=0.7-x$$

but $0 \le a \le 1$

$$\Downarrow$$

$x=0.7-P(A\wedge B)$

$$0\le 0.7-x\le 1 \Rightarrow x \le 0.7$$

How about the smallest value of x?

$$x = P(A \vee B) \geqslant 0$$

Is it rational to have $x = 0$ NO

$P(A) = a + b = 0.4$

$P(B) = a + c = 0.3$

$P(A \vee B) = \underbrace{a + b + c}_{x} \stackrel{?}{=} 0$ But $P(A \vee B) \geqslant 0.4$

$\Downarrow$

Therefore if $x = 0.4$

$\Downarrow$

$c = 0 \Rightarrow a = 0.3$

$b = 0.1$

$d = 1 - 0.4 = 0.6$

$d = 0.6$

Then by applying the axioms of probability

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

$x = 0.4$ $\quad a + b = 0.4$ $\quad a + c = 0.3$ $\quad a = 0.3$

$$0.4 = 0.4 + 0.3 - 0.3 \quad ✓$$

Final answer to (b)

$$0.4 \leq P(A \vee B) < 0.7$$

Example 2

Consider an intelligent agent with two binary random variables:

S (study)

G (passing grade)

The agent holds the beliefs:

$$P(s) = 0.6 \qquad P(g) = 0.6$$

What is the range of probability for $P(s \wedge g)$ when the agent knows $P(\neg s \wedge \neg g) < 0.3$

Solution

G \ S	s	$\neg s$
g	a_1	a_2
$\neg g$	a_3	a_4

$\Rightarrow$

(1) $a_1 + a_2 = 0.6$

(2) $a_1 + a_3 = 0.6$

(3) $a_4 < 0.3$

The agent also knows:

(4) $a_1 + a_2 + a_3 + a_4 = 1$

If we add (1) & (2) $\Rightarrow$ $2a_1 + a_2 + a_3 = 1.2$ (5)

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 1 - a_4 \\ (3) \quad a_4 &< 0.3 \end{aligned}\right\}$$

$$\Downarrow$$

$$a_1 + a_2 + a_3 > 1 - 0.3 = 0.7$$

If we substract (5) from (4)

$$2a_1 + a_2 + a_3 = 1.2$$
$$a_1 + a_2 + a_3 > 0.7$$
$$\overline{\qquad a_1 > 0.5 \qquad}$$

but $a_1 = \underline{P(s1f) > 0.5}$

Examples using Conditional Probabilities

Example 1 Given the random variables:

$A = \{$apple, orange, kiwi, cherry$\}$

$B = \{$pie, cake, mousse$\}$

with the joint probability table:

B \ A	apple	orange	kiwi	cherry
pie	0.32	0.08	0.09	0.01
cake	0.01	0.15	0.05	0.09
mousse	0.001	0.03	0.019	0.15

Compute the conditional probabilities:

1. $P(\text{pie} \mid \neg\text{apple})$
2. $P(\text{cake} \mid \text{orange} \vee \text{cherry})$

Solution

1. $P(\text{pie} \mid \neg\text{apple}) = \dfrac{P(\text{pie}, \neg\text{apple})}{P(\neg\text{apple})} =$

$$= \frac{0.08 + 0.09 + 0.01}{1 - (0.32 + 0.01 + 0.001)} = \frac{0.18}{1 - 0.331} =$$

$$= \frac{0.18}{0.669} = 0.269$$

2. $P(\text{cake} \mid \text{orange} \vee \text{cherry}) =$

$$\frac{P(\text{cake}, \text{orange} \vee \text{cherry})}{P(\text{orange} \vee \text{cherry})} =$$

$$= \frac{0.15 + 0.09}{\underbrace{0.08 + 0.15 + 0.03}_{\text{orange}} + \underbrace{0.01 + 0.09 + 0.15}_{\text{cherry}}} =$$

$$= \frac{0.24}{0.51} = 0.470$$

BUY a calculator

Example #2 (Conditional Probabilities)

We have two random variables: A and B.

If
$$\frac{P(A=F,\ B=T)}{P(A=T,\ B=T)} = 2$$

a) What is $P(A=T\ /\ B=T)$?

Solution

We know $P(\neg A, B) = 2\,P(A, B)$

But $P(A|B) = \dfrac{P(A,B)}{P(B)}$ (1)

$$P(\neg A|B) = \frac{P(\neg A, B)}{P(B)} \quad (2)$$

From (1) & (2) $\Rightarrow$ $\dfrac{(2)}{(1)}$
$$\frac{P(\neg A|B)}{P(A|B)} = \frac{P(\neg A, B)}{P(A, B)} = 2$$

$$2\,P(A|B) = P(\neg A|B)$$
$$= 1 - P(A|B)$$

$$P(A|B) = \frac{1}{3}$$

b) If $P(A|B) = 3\,P(B|A)$
and $P(A) = 0.123$
what is $P(B) = ?$

Solution

We know that by applying Bayes Rule

$$P(A|B) = \frac{P(AB)}{P(B)} \Rightarrow P(AB) = P(A|B)P(B)$$

$$= P(B|A)P(A)$$

$$\underbrace{P(A|B)}_{=\,3P(B|A)}P(B) = P(B|A)P(A)$$

$$\Downarrow$$

$$3P(B|A)\,P(B) = P(B|A)\,P(A)$$

$$\Downarrow$$

$$3P(B) = P(A)$$

$$\Downarrow$$

$$P(B) = \frac{P(A)}{3} = \frac{0.123}{3} =$$

$$= 0.041$$

$$P(B) = 0.041$$

Example #3 (Conditional Probabilities)

You are in a funhouse of mirrors with your friends. Everyone looks in the mirrors and see themselves as thin or fat. You tell your best friend that you see yourself as thin. Given the distorting mirrors, discrimination between looking thin and being thin is 60% reliable. Is it possible to know from the mirror if a stranger is thin?

Solution let us use two random variables:

H - for thin people

LH - for looking thin in the mirror

Given that mirrors distort, we can formulate conditional probabilities that capture this property of mirrors:

$$\left.\begin{array}{l} P(LH \mid H) = 0.6 \\ P(\neg LH \mid \neg H) = 0.6 \end{array}\right\} \text{discrimination between } \underbrace{\text{looking thin}}_{LH} \text{ and } \underbrace{\text{being thin}}_{H}$$

How do we represent the probability that a stranger is thin? (if he looks thin in the mirror)

$$\Rightarrow \boxed{P(H|LH)} \propto \underbrace{P(LH\,|\,H)}_{} \times P(H)$$

$$= 0.6 \times P(H)$$

also

$$P(\neg H\,|\,LH) \propto \underbrace{P(LH\,|\,\neg H)}_{\substack{1-P(\neg LH|\neg H)\\ =1-0.6=0.4}} \times P(\neg H)$$

$$= 0.4 \times P(\neg H)$$

but we do not know $P(H) \Rightarrow$

we cannot compute

$P(H|LH)$ or $P(\neg H|LH)$!

If we know that everyone is thin $\Rightarrow P(H)=1$

$$\text{then } P(H|LH) = \frac{P(H, LH)}{\underbrace{P(LH)}_{\text{No}}} = 1$$

the condition does not matter!

If we know that 9 out of 10 people are fat $\Rightarrow$ then 1 out of 10 are thin!

$$\Rightarrow P(H) = 0.1$$

In that case: $P(H \mid LH) \propto P(LH \mid H) \times P(H) =$

$$= 0.6 \times 0.1 = 0.06$$

and $P(\neg H \mid LH) \propto \underbrace{P(LH \mid \neg H)}_{1-0.6=\,=0.4} \times \underbrace{P(\neg H)}_{1-0.1=\,=0.9} =$

$$= 0.4 \times 0.9 = 0.36$$

We need to compute the exact probabilities:

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{LH} \\
 & & T & F \\ \cline{3-4}
\multirow{2}{*}{H} & T & a & b \\ \cline{3-4}
 & F & c & d \\ \cline{3-4}
\end{array}$$

$a = 0.06$ | $b = 1-a$

$c = 0.36$ | $d = 1-c$

$\Downarrow$

$$P(H \mid LH) = \frac{a}{a+c} = \frac{0.06}{0.06+0.36} =$$

$$= 0.142857\,1$$

$$P(\neg H \mid LH) = \frac{c}{a+c} = \frac{0.36}{0.06+0.36} =$$

$$=$$

similarly

$\Downarrow$

$$P(H \mid \neg LH) = \frac{b}{b+d} \qquad P(\neg H \mid \neg LH) = \frac{d}{b+d}$$

Naive Bayesian Reasoning Example

A PhD student is ready to graduate and applies for faculty positions. She worked on research and published papers. Use Naive Bayesian Reasoning to decide if she will become a Professor, given the following knowledge:

1. 35% of PhD students become faculty
2. Among all faculty 75% work on research and 60% publish papers
3. 40% of the PhD graduates that did not become faculty work on research and 20% publish papers.

Solution

Let us define the random variables:

Research .- a boolean random variable for working on research

Publish - a boolean random variable for the fact that they publish

Faculty - a boolean random variable for becoming faculty

We know that:

$P(\text{Faculty}) = 0.35$

$P(\text{Research} \mid \text{Faculty}) = 0.75$ — $P(\text{Research} \mid \neg\text{Faculty}) = 0.4$

$P(\text{Publish} \mid \text{Faculty}) = 0.6$ — $P(\text{Publish} \mid \neg\text{Faculty}) = 0.2$

To answer whether "she will become Faculty" we need to evaluate:

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) = ?$$

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) = \frac{P(\text{Research}, \text{Publish} \mid \text{Faculty}) \times P(\text{Faculty})}{P(\text{Research}, \text{Publish})}$$

* we assume that Research and Publish are conditionally independent

(you do research because you like it, not because you only want to publish, and you publish, not only your research, but sometimes surveys!)

$$P(\text{Research}, \text{Publish} \mid \text{Faculty}) = P(\text{Research} \mid \text{Faculty}) \times P(\text{Publish} \mid \text{Faculty})$$

Then:

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) = \frac{P(\text{Research} \mid \text{Faculty}) \times P(\text{Publish} \mid \text{Faculty}) \times P(\text{Faculty})}{P(\text{Research}, \text{Faculty})}$$

$$= \frac{0.75 \times 0.6 \times 0.35}{P(\text{Research}, \text{Faculty})}$$

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) = \frac{0.1575}{P(\text{Research}, \text{Publish})}$$

--- we do not know it!!!

However, we know:

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) + P(\neg\text{Faculty} \mid \text{Research}, \text{Publish}) =$$

$$= 1$$

We compute:

$$P(\neg\text{Faculty} \mid \text{Research}, \text{Publish}) =$$

$$= \frac{P(\text{Research}, \text{Publish} \mid \neg\text{Faculty}) \times P(\neg\text{Faculty})}{P(\text{Research}, \text{Publish})}$$

$$= \frac{\overset{0.4}{P(\text{Research} \mid \neg\text{Faculty})} \times \overset{0.2}{P(\text{Publish} \mid \neg\text{Faculty})} \times \overset{(1-0.35)}{P(\neg\text{Faculty})}}{P(\text{Research}, \text{Publish})}$$

$$= \frac{0.4 \times 0.2 \times 0.65}{P(\text{Research}, \text{Publish})} = \frac{0.24}{P(\text{Research}, \text{Publish})}$$

Now, we can compute $P(\text{Research}, \text{Publish})$:

(Because $P(\text{Faculty} \mid \text{Research}, \text{Publish}) + P(\neg\text{Faculty} \mid \text{Research}, \text{Publish}) = 1$

$\Downarrow$

$$\frac{0.1575}{P(\text{Research}, \text{Publish})} + \frac{0.24}{P(\text{Research}, \text{Publish})} = 1 \Rightarrow$$

$$P(\text{Research}, \text{Publish}) = 0.1575 + 0.24 = 0.3975$$

Now we can evaluate:

$$P(\text{Faculty} \mid \text{Research}, \text{Publish}) = \frac{0.1575}{0.3975} =$$

$$\boxed{= 0.3962}$$

The answer!