Validity and Satisfiability

Decide which sentences are
- valid
- satisfiable
- un(satisfiable)
- neither

a. light ⇒ light

<u>Answer</u> Consider the Truth Table.

Light	light ⇒ light	¬light ∨ light
F	T	T
T	T	T

it is valid!

b. Sun ⇒ light

<u>Answer</u> Consider the Truth Table:

Sun	Light	Sun ⇒ light	¬Sun ∨ light
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

it is satisfiable!

c. $\underbrace{((\text{Cloudy} \vee \text{Rain}) \Rightarrow \text{Dark})}_{A} \Leftrightarrow \underbrace{\overbrace{(\text{Cloudy} \Rightarrow \text{Dark})}^{B1} \vee \overbrace{(\text{Rain} \Rightarrow \text{Dark})}^{B2}}_{B}$.

S : $A \Leftrightarrow B$

$B = B1 \vee B2$

Answer

Consider the truth Table:

A: $\neg(\text{Cloudy} \vee \text{Rainy}) \vee \text{Dark}$

B1: $\neg\text{Cloudy} \vee \text{Dark}$

B2: $\neg\text{Rain} \vee \text{Dark}$

Cloudy	Rain	Dark	A	B1	B2	B	S
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	T	F	T	T	T
T	F	T	T	T	T	T	T
T	T	F	F	F	F	F	T
T	T	T	T	T	T	T	T

valid

d. $\neg(\text{fat} \vee \text{happy} \vee (\text{fat} \Rightarrow \text{happy}))$

<u>Answer</u> Consider the Truth Table

fat	happy	fat ∨ happy x	fat ⇒ happy y	¬fat ∨ happy	S ¬(x∨y)
F	F	F	T	T	F
F	T	T	T	T	F
T	F	T	F	F	F
T	T	T	T	T	F

unsatisfiable

Entailment Example

Prove that sentence S_1 entails sentence S_2

where: S_1: $\underbrace{(Tall \Rightarrow Handsome)}_{S_A} \land \underbrace{(\neg Tall \land Stinny \Rightarrow Stinny)}_{S_B}$

S_2: $Tall \land Stinny \Rightarrow Handsome$.

$S_1 = S_A \land S_B$

Answer: Consider the Truth Table.

S_A: $\neg Tall \lor Handsome$

S_B: $Tall \lor \neg Stinny \lor Stinny$

S_2: $\neg Tall \lor \neg Stinny \lor Handsome$

Tall	Stinny	Handsome	S_A	S_B	S_1	S_2
F	F	F	T	F	F	T
F	F	T	T	F	F	T
F	T	F	T	F	F	T
F	T	T	T	F	F	T
T	F	F	F	T	F	T
T	F	T	T	T	T	T
T	T	F	F	T	F	F
T	T	T	T	T	T	T

S_1 entails S_2 True because S_2 is in the two models where $S1$ is true.

Same entailment, a different answer, using $\boxed{\text{inference rules}}$ instead of model checking.

We want to prove that $S_1 \models S_2$,
with S_1: $(Tall \Rightarrow Handsome) \land (\neg Tall \land Skinny \Rightarrow Skinny)$

S_2: $Tall \land Skinny \Rightarrow Handsome$.

Answer:

- let us use the inference rule: $(a \Rightarrow b) \equiv \neg a \lor b$

$S_1 \models S_2$ becomes:

$(\neg Tall \lor Handsome) \land (\underline{\neg(\neg Tall \land Skinny)} \lor Skinny)$
$\models (\underline{\neg(Tall \land Skinny)} \lor Handsome)$

- let us use the de Morgan inference rule:

$(\neg Tall \lor Handsome) \land (Tall \lor \underline{\neg Skinny \lor Skinny}) \models$
$(\neg Tall \lor \neg Skinny \lor Handsome$.

- But $\neg Skinny \lor Skinny = T$

$(\neg Tall \lor Handsome) \land \underline{(Tall \lor T)} \models (\neg Tall \lor \neg Skinny \lor Handsome)$

Tall $\lor$ T shall always be TRUE

- But

$(\neg Tall \lor Handsome) \models (\neg Tall \lor Handsom \lor \neg Skinny)$

because if $X = \neg Tall \lor Handsome$, this is $X \models X \lor \neg Skinny$

therefore, whenever X is True, $X \lor \neg Skinny$ shall <u>also</u> be true.

CNF Translation

Consider the sentence:

$$(Unicorn \Rightarrow Mythical) \Leftrightarrow (Immortal \lor Imaginary)$$

translate this sentence in CNF

Answer

1. Eliminate $\Leftrightarrow$ using the inference rule: $\alpha \Leftrightarrow \beta \equiv (\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$

$$((Unicorn \Rightarrow Mythical) \Rightarrow (Immortal \lor Imaginary)) \land$$
$$((Immortal \lor Imaginary) \Rightarrow (Unicorn \Rightarrow Mythical))$$

2. Eliminate $\Rightarrow$ using the inference rule $\alpha \Rightarrow \beta \equiv \neg\alpha \lor \beta$

$$(\neg(\neg Unicorn \lor Mythica) \lor (Imaginary \lor Immortal)) \land$$
$$(\neg(Immortal \lor Imaginary) \lor (\neg Unicorn \lor Mythical))$$

3. Push $\neg$ in: using the deMorgan inference rules

$$((Unicorn \land \neg Mythical) \underline{\lor} (Immortal \lor Imaginary)) \land$$
$$((\neg Imortal \land \neg Imaginary) \underline{\lor} (\neg Unicorn \lor Mythical))$$

4. Apply the distributivity inference rules.

$$\underbrace{(Unicorn \lor Immortal \lor Imaginary)}_{D_1} \land \underbrace{(\neg Mythical \lor Immortal \lor Imaginary)}_{D_2} \land \underbrace{(\neg Imortal \lor \neg Unicorn \lor Mythical)}_{D_3} \land \underbrace{(\neg Imaginary \lor \neg Unicorn \lor Mythical)}_{D_4}$$

$D_1 \land D_2 \land D_3 \land D_4$

Conjunctions

D_1, D_2, D_3 & D_4 are in Normal Form (only disjunctions)