Contest Problem Set 12200

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Power Question 12200

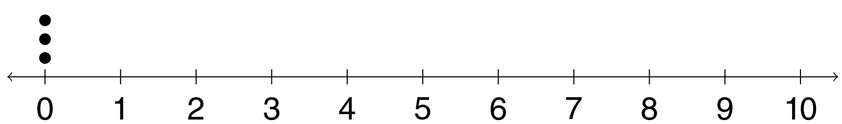
This Power Question deals with *chip firing*. *Chips* are placed at integer points on the number line (in Part One) or lattice points in the plane (in Part Two). The chips are indistinguishable, except when noted otherwise (for instance, in Problem 2), and more than one chip can be at the same location. Chips can be *fired* according to rules given in each part, causing one or more chips to move to new locations.

The following 10 questions are worth 10 points each.

Part One: Multitude Amplification

Chips lie at integer coordinates on a number line and they can be fired under the following rule: if there are c chips at point n, then a single one of these chips can be fired forward a distance of at most c. In other words, the chip moves to point $n' \in \mathbb{N}$ where $n < n' \leq n + c$.

As an example, consider the three chips at point 0 on the number line below. Any one of them can be fired forward up to 3 units. However, once one chip is fired forward, the remaining two chips at point 0 can be fired at most 2 units forward, not 3 units.



For an ordered pair of positive integers (c, d), suppose that c chips begin at point 0 on the number line. Then, let $m(c, d)$ be the minimum number of fires needed to move all of the chips to point d on the number line.

1. Compute the following:
   (a) $m(1, 10)$
   (b) $m(2, 2)$
   (c) $m(10, 2)$
   (d) $m(2, 5)$
   (e) $m(3, 5)$

2. Suppose we label the c chips $s_1, s_2, \ldots, s_c$ and fire chips under the condition that of all chips at a given point on the number line, only the chip of maximal index can be fired. For example, if point 3 on the number line contains chips $\{s_1, s_2, s_5\}$, then only s_5 can be fired. Prove that for any $1 \leq k \leq c$, chip s_k must be fired at least $\left\lceil \frac{d}{k} \right\rceil$ times to reach point d. (The notation $\lceil x \rceil$ denotes the least integer greater than or equal to x.)

3. For any ordered pair of positive integers (c, d), prove in general that $m(c, d) \geq \sum_{k=1}^{c} \left\lceil \frac{d}{k} \right\rceil$. That is, prove that this is true, disregarding the labeling condition imposed in Problem 2.

Part Two: Hall of Mirrors

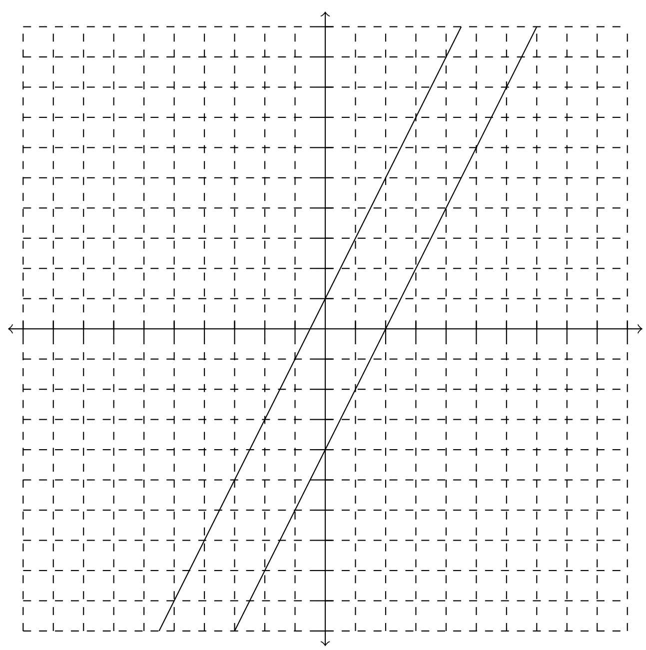
In the *Mirror Problem*, chips lie on lattice points on the *xy*-plane. Chips can only be fired in the following way: if $c \geq 2$ chips lie at (x, y), then two of those chips can be fired simultaneously, one moving to $(x+1, y)$ and the other to $(x, y+1)$. Given two relatively prime positive integers a and b, suppose there are two "mirrors", represented by the parallel lines $\mathcal{M}_1 : ay - bx = a^2$ and $\mathcal{M}_2 : ay - bx = -b^2$, forming an infinite hallway. Let $f(x, y) = ay - bx$. The mirrors interact with the chips in the following way:

(i) If a chip is fired to a point (x, y) where $f(x, y) > a^2$, then it instead moves to $(x+b, y-a)$.

(ii) If a chip is fired to a point (x, y) where $f(x, y) \leq -b^2$, then it instead moves to $(x-b, y+a)$.

You can think of these chips as being "relocated" by the mirrors, entering one and exiting out the other.

As an example, if $(a, b) = (1, 2)$, mirrors will be the solid lines shown in the diagram below (tick marks are 1 unit). If there are two chips at $(2, 1)$ and the two chips are fired simultaneously, then one chip ends at $(2, 2)$ but the other chip ends at $(1, 2)$ (as $(3, 1)$ lies outside the boundary, the *x*-coordinate is decreased by 2 while the *y*-coordinate is increased by 1).



In other words, chips must remain strictly between the two mirrors or exactly on mirror $\mathcal{M}_1$. We also require that all chips begin within these boundaries.

Call a position *stable* if no chips can be fired. From some starting positions, it is possible to reach a stable position; that is, there exists a finite sequence of fires after which no more fires are possible. From other starting positions, chips can be "fired indefinitely": no finite sequence of fires yields a stable position. For example, with $a = b = 1$ and two chips starting at the origin, the chips can be fired indefinitely.

Our end goal is to prove the following claim: for any two relatively prime integers a and b, there exists an integer c such that, given c chips starting at the origin, no stable position can be reached.

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4. In each part, a and b are given, as well as the initial positions of some chips; these chips are fired until no more fires are possible. Find the positions of the chips after no more can be fired.
   (a) $(a, b) = (1, 2)$; 2 chips start at $(1, 2)$
   (b) $(a, b) = (1, 2)$; 3 chips start at $(0, 1)$
   (c) $(a, b) = (2, 3)$; 2 chips start at $(2, 0)$
   (d) $(a, b) = (2, 3)$; 2 chips start at $(0, 2)$ and 1 chip starts at $(3, 1)$

5. For the following pairs (a, b) and c chips starting at the origin, compute the set of positions of the chips once the chips can no longer be fired, or state (without proof) that they can be fired indefinitely. For example, $(a, b) = (2, 1)$ and $c = 2$ results in $\{(0, 1), (0, 2)\}$.
   (a) $(a, b) = (2, 3)$ and $c = 4$
   (b) $(a, b) = (4, 3)$ and $c = 8$
   (c) $(a, b) = (1, 2)$ and $c = 4$

6. Prove that $h(x, y) = ax + by$ remains invariant under mirror interactions. That is, if a chip is fired to a point (x, y) where it is relocated by the mirrors to position (x', y'), then $h(x, y) = h(x', y')$.

7. Prove that for any stable position and positive integer n, at most one chip lies on the line $ax+by = n$.

Now, suppose we start with c chips at the origin and eventually reach a stable position. For each integer n, let $v(n)$ be the number of chips that move to (or start at) a point on the line $h(x, y) = ax + by = n$ at some time during the process. Note that $v(n) = 0$ if it is not possible to have a chip on the line $ax+by = n$, and $v(0) = c$.

8. Prove that if $n \geq \max(a, b)$, then $v(n) = \left\lfloor \frac{v(n-a)}{2} \right\rfloor + \left\lfloor \frac{v(n-b)}{2} \right\rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer function.

9. Use the result of Problem 8 to show that $v(ap + bq) > \dfrac{v(0)}{2^{p+q}} - 2$ for all ordered pairs (p, q) of nonnegative integers.

10. Finally, we will prove the original claim. Suppose $c = 2^{a+b}$, and $a > b$. (The same argument works for $b > a$, and $a = b = 1$ is a special case; you may ignore these.)
    (a) Prove that if a stable position were to be reached, then $v(n) > 2$ for all integers n with $(a - 1)(b - 1) \leq n < ab$.
    (b) Prove that there exists no finite sequence of fires that can produce a stable position.

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Select E only if you cannot determine a uniquely correct answer between A, B, C, and D.

1. (A) (B) (C) (D) (E)	11. (A) (B) (C) (D) (E)	21. (A) (B) (C) (D) (E)
2. (A) (B) (C) (D) (E)	12. (A) (B) (C) (D) (E)	22. (A) (B) (C) (D) (E)
3. (A) (B) (C) (D) (E)	13. (A) (B) (C) (D) (E)	23. (A) (B) (C) (D) (E)
4. (A) (B) (C) (D) (E)	14. (A) (B) (C) (D) (E)	24. (A) (B) (C) (D) (E)
5. (A) (B) (C) (D) (E)	15. (A) (B) (C) (D) (E)	25. (A) (B) (C) (D) (E)
6. (A) (B) (C) (D) (E)	16. (A) (B) (C) (D) (E)	26. (A) (B) (C) (D) (E)
7. (A) (B) (C) (D) (E)	17. (A) (B) (C) (D) (E)	27. (A) (B) (C) (D) (E)
8. (A) (B) (C) (D) (E)	18. (A) (B) (C) (D) (E)	28. (A) (B) (C) (D) (E)
9. (A) (B) (C) (D) (E)	19. (A) (B) (C) (D) (E)	29. (A) (B) (C) (D) (E)
10. (A) (B) (C) (D) (E)	20. (A) (B) (C) (D) (E)	30. (A) (B) (C) (D) (E)

1. Melody gives away pencils to classrooms, and each box of pencils has 12 pencils. In the first building, Melody gives away 12 boxes of pencils, and in the second building, Melody gives away 18 boxes of pencils. How many total pencils did Melody give away in both buildings?

(A) 30 (B) 288 (C) 144 (D) 240 (E) Other

2. Compute the product $1\frac{1}{2} \cdot 2\frac{1}{3} \cdot 3\frac{1}{7}$.

(A) 11 (B) 10 (C) $6\frac{41}{42}$ (D) 9 (E) Other

3. Jon is running a local fair where the ticket price is \$2 per child and \$5 per adult. If there are ten times as many adults who bought tickets for the fair as there are children, and the total revenue from tickets one day is \$780, how many adults bought tickets for the fair?

(A) 140 (B) 39 (C) 150 (D) 144 (E) Other

4. At a store, an N95 mask costs \$5, while a KN95 mask costs \$2. Bella has \$16 to spend on masks for her family of four people, and she must buy one mask per person. What is the greatest number of N95 masks she can buy?

(A) 2 (B) 4 (C) 3 (D) 0 (E) Other

5. A cube has side length 2. What is the positive numerical difference between the sum of the lengths of its edges and its surface area?

(A) 12 (B) 0 (C) 6 (D) 18 (E) Other

6. Jeremiah's plant is 9 inches tall at noon on Thursday. For the next two weeks, the plant grows by 1 inch every 48 hours. Jeremiah plans on cutting the plant once the plant reaches 15 inches tall. On what day of the week should Jeremiah cut the plant?

(A) Friday (B) Wednesday (C) Tuesday (D) Thursday (E) Other

7. The Dream Kingdom has 86 members in its parliament, and in order to pass a law, at least 75% of the parliament must vote “Yes”. In a recent vote, a food law did not pass, and would have passed if and only if at least 3 more members had voted “Yes”. How many members voted “Yes” in that vote?

(A) 64 (B) 63 (C) 66 (D) 62 (E) Other

8. What is the probability that a randomly chosen positive factor of 140 is not a multiple of 5?

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{1}{5}$ (E) Other

9. The zeroes of the function $f(x) = x^2 - 60x + 896$ are a and b, where a and b are integers and $a \leq b$. Compute $10a + b$.

(A) 366 (B) 294 (C) 348 (D) 312 (E) Other

10. Square $WXYZ$ has side length 12. Point P is inside the square such that the area of WPY is 18. What is the largest possible area of quadrilateral $WPYZ$?

(A) 54 (B) 68 (C) 90 (D) 72 (E) Other

11. When read from left to right, the 16th, 17th and 18th digits after the decimal point of Euler’s number e are 2, 3, and 5. What is the 16th digit after the decimal point of $3e$, when read from left to right?

(A) 6 (B) 3 (C) 0 (D) 9 (E) Other

12. Amee asks a select number of students and teachers if they like pineapple on pizza, and she records the results. Amee then selects someone she interviewed at random. The probability that she selects a teacher is $\frac{1}{3}$, and the probability that she selects someone who does not like pineapple on pizza is $\frac{5}{6}$, and the probability that she selects a teacher, given that she selects someone who does not like pineapple on pizza, is $\frac{9}{25}$. What is the probability that Amee selects someone who is a teacher or someone who does not like pineapple on pizza?

(A) $\frac{7}{10}$ (B) $\frac{8}{15}$ (C) $\frac{13}{15}$ (D) $\frac{17}{30}$ (E) Other

13. A fully-pumped wall ball is a sphere with diameter 12 inches. Jake kicks away a deflated wall ball, which only holds 62.5% of the volume of a fully-pumped wall ball. What is the volume of that deflated wall ball (in cubic inches)?

(A) 288π (B) 20π (C) 180π (D) 108π (E) Other

14. Compute $12^{10} + 5 \cdot 12^8 \cdot 16^2 + 10 \cdot 12^6 \cdot 16^4 + 10 \cdot 12^4 \cdot 16^6 + 5 \cdot 12^2 \cdot 16^8 + 16^{10}$.

(A) 120000000000 (B) 800000000000 (C) 10240000000000 (D) 400000000000 (E) Other

15. Dawsen plans on flipping a coin 19 times. The probability that exactly 3 heads are flipped is given by $\frac{n}{2^{19}}$, where n is a positive integer. Compute n.

(A) 1141 (B) 5814 (C) 3360 (D) 82782 (E) Other

16. On a remote island, there are 50 people, each of whom is either a truth-teller (and only says true statements) or a liar (and only says false statements). Each person on the island makes one statement describing each other person on the island, saying whether that person is a truth-teller or liar, for a total of 49 statements per person. Compute the maximum possible number of these statements that can call the other person a liar.

(A) 4901 (B) 1250 (C) 5000 (D) 2500 (E) Other

17. A hexagon has five right angles and one angle measuring 270°. Its side lengths, in some order, are 6, 7, 8, 9, 13, and 17. What is the greatest possible area of the hexagon?

(A) 173 (B) 158 (C) 167 (D) 179 (E) Other

18. Say a positive integer is *deciduous* if, each time consecutive digits in the number do not increase from left to right, the digits always decrease by at least 5. For example, 49 and 92 are both deciduous, but 66 and 85 are not. Compute the number of two-digit deciduous positive integers.

(A) 45 (B) 69 (C) 51 (D) 55 (E) Other

19. Quadrilateral $ABCD$ is a square with side length 50. Triangles ABE, BCF, CDG, and DAH are all congruent isosceles triangles such that $AE = EB = 65$, and for each of the four isosceles triangles, each plane it resides on is perpendicular to the plane of $ABCD$ and E, F, G, H are all on the same side of $ABCD$. Additionally, mark points I, J, K, L such that EI, FJ, GK, HL are the altitudes of the respective isosceles triangles. Compute the volume of solid $EFGHIJKL$.

(A) 15000 (B) 75000 (C) $15000\sqrt{2}$ (D) $37500\sqrt{2}$ (E) Other

20. A square table can fit up to four chairs around it, one for each edge. Meri adjoins 70 of these tables together to form one rectangular table. The maximum number of chairs that can fit around this large table is N, one for each edge of a small table comprising the sides of the large table. Compute the sum of all possible values of N.

(A) 576 (B) 254 (C) 288 (D) 280 (E) Other

21. A rectangular prism has edge lengths a, b, and c. Three new rectangular prisms are formed, each by doubling a different edge length of the original prism. The new prisms have surface areas of 18, 20, and 22. Compute the surface area of the original prism.

(A) 12 (B) $\frac{121}{25}$ (C) $\frac{34}{5}$ (D) $\frac{66}{5}$ (E) Other

22. Compute the sum of all positive integers $n \leq 10$ for which $10^{2^n} - 1$ is divisible by at least one of $10^n - 1$ and $10^n + 1$.

(A) 7 (B) 4 (C) 12 (D) 55 (E) Other

23. For a matrix $M = \begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$, the determinant of M^{128} equals 2^n, where n is a positive integer. Compute the value of n.

(A) 7 (B) 64 (C) 128 (D) 1 (E) Other

24. Consider a regular octagon. Some of the sides are painted red, but all of the sides painted red are consecutive. If rotations of the octagon are considered different, then how many different colorings have at least 2 red sides?

(A) 45 (B) 48 (C) 42 (D) 56 (E) Other

25. On the complex plane, if the complex number z is rotated by 30° counterclockwise about the origin, the result is equal to $z^{\frac{3}{2}}$. Assuming $|z| = 1$, what is the sum of the real and imaginary parts of z? (If $x = a + bi$ where a and b are real, then $|x| = \sqrt{a^2 + b^2}$.)

(A) $\frac{1+\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) 0 (D) $\frac{\sqrt{3}-1}{2}$ (E) Other

26. Lucas plans on getting pet creatures for an exhibition, and he wants to be able to evenly put the creatures in teams of 4. He has a total of 15 different creatures to choose from. In how many ways (including not getting any creatures at all) can Lucas choose creatures such that the number of creatures chosen is divisible by 4?

(A) 8256 (B) 8512 (C) 8192 (D) 34048 (E) Other

27. Let s_k be the combined surface area of k^3 unit cubes, and let S_k be the surface area of the large cube formed when all of those k^3 unit cubes are glued together. Given that $\displaystyle\sum_{k=1}^{n}(s_k - S_k) = 2100$, compute n.

(A) 7 (B) 5 (C) 9 (D) 8 (E) Other

28. For a positive integer n, the value $\prod_{k=1}^{100}(n + k)$ equals the product of all integers from $n + 1$ to $n + 100$ inclusive. Given that $n \leq 100$ and this product ends in exactly 25 zeroes when written in base-10, compute the sum of the possible values of n.

(A) 4775 (B) 4750 (C) 4800 (D) 4700 (E) Other

29. How many ways are there to divide a group of 8 different marbles into one or more unlabeled groups so that each group has an equal number of marbles?

(A) 96 (B) 115 (C) 76 (D) 86 (E) Other

30. Call a nondecreasing sequence of positive integers (a_n) *budding* if all of its elements are less than or equal to 5 and *developed* if the median of its elements is greater than or equal to 5. Let s_1, s_2, s_3, s_4, s_5 be a nondecreasing sequence of five positive integers, with $s_5 \leq 10$. Compute the number of such sequences that are either budding or developed but not both.

(A) 1382 (B) 1397 (C) 1442 (D) 1412 (E) Other

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Problems 1 & 2

1. Nico received a jar of 80 jellybeans as a gift. The colors of the jellybeans are red, green, and blue. Upon eating 7 red jellybeans, 9 blue jellybeans, and 13 green jellybeans from the jar, Nico observed that there are an equal number of remaining jellybeans of each color. How many green jellybeans were originally in the jar before Nico ate any?

1.

2. Suppose 50% of 50% of 80% of A is 30% of 40% of 60% of B. What percent of $\sqrt{B}$ is $\sqrt{A}$?

2.

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Problems 3 & 4

3. On a multiple choice practice problem with 5 answer choices and only 1 right answer choice, Daniel guesses an answer choice at random. If Daniel gets the question wrong, he selects a different answer choice at random, continuing until he gets it right, and never guessing the same answer choice more than once. Compute the expected number of guesses Daniel must make in order to get the question right.

3.

4. A circle is inscribed in an isosceles triangle with side lengths 10, 10, and 12. To the nearest whole percent, what percent of the area inside the triangle lies outside the circle?

4.

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Problems 5 & 6

5. A quadratic polynomial $P(x)$ with leading coefficient 1 satisfies $P(0) = 12$ and $P(1) = 8$. Determine the value of n such that $P(3n) = 9P(n)$. Express your answer as a common fraction.

5.

6. Andrew's frog can hop 91 spaces forward or backward or 117 spaces forward or backward on a number line. If his frog starts at -2 on the number line, what is the sum of the twenty smallest positive numbers that Andrew's frog can eventually reach?

6.

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Problems 7 & 8

7. Hardworking Hudson works 40 hours per week (Monday-Friday). Each weekday, he works an integer number of hours between 7 and 10, inclusive. He also never works fewer hours than he did the day before. In how many different ways can he distribute his work hours between the five weekdays?

7.

8. Let w, x, y, and z be distinct complex numbers such that $|w| = |x| = |y| = |z|$, w and y both have real part a, and x and z both have imaginary part b. Let $v = a + bi$. Given that $|v - w| = 28$, $|v - x| = 14$, and $|v - y| = 18$, compute $|v - z|$.

8.

Team Round 12200

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10.

1. A right triangle has one leg of length x, an area of x, and a hypotenuse of length $2\sqrt{10}$. Compute x.

2. Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \sqrt{x^2 + 4}$. What is the value of $f\left(100 + \frac{1}{100}\right) + g\left(100 - \frac{1}{100}\right)$?

3. Addison plays three games, and the probability that he wins in each of them is $\frac{1}{2}$. Addison gets two tokens if he wins the first game, three tokens if he wins the second game, and four tokens if he wins the third game, and nothing for any games he loses. What is the expected value of tokens that Addison gets after playing the three games? Express your answer as a common fraction.

4. Candice rotates her pencil by 90 degrees clockwise about one end, and finds that the total distance traveled by the midpoint was 24 centimeters. To the nearest centimeter, how many centimeters long is Candice's pencil?

5. Dante is saving for a new game console, which costs \$349.99. On Sunday, November 28, Dante sets aside \$100 from his savings to be spent on the new console. Every weekday (excluding Saturday and Sunday), Dante works at a pizza shop from 4 PM to 6 PM and earns \$15 per hour. After 6 PM on that day, Dante collects the money he earns and sets aside half of his daily earnings to be spent on the new game console. December N (where N is a whole number) is the first day where Dante will have enough allocated money to buy the new game console. Determine the value of N.

6. Keneya's figurine of a stop motion video is 7 centimeters high initially. The difference in elevation between the figurine and a ceiling that stays at the same place in the entire video halves every second, and Keneya's figurine is 22 centimeters high after 4 seconds. Compute the height of the ceiling in centimeters.

7. Juliet baked 100 cookies and wants to share them evenly among $p \geq 1$ people, not including herself, so that she has no more than 2 cookies left over. Compute the sum of all possible values of p.

8. An arithmetic sequence has first term a and common difference 2, where a is a positive integer. The first ten terms in the sequence have product P_1. Another arithmetic sequence has first term $a + 1$ and common difference 2, and its first ten terms have product P_2. If $P_1 P_2$ ends in exactly ten zeros when written in base 10, let S the sum of all possible values of a less than 5^7. Compute the smallest integer greater than $\frac{S}{5^6}$.

9. Define the *fractional part* of a real number r as $\{r\} = r - \lfloor r \rfloor$, where $\lfloor r \rfloor$ is the greatest integer less than or equal to r. For example, $\{5.5\} = 0.5$. How many real values of $r \geq 1$ satisfy the equation $\{r\} + \left\{\frac{1}{r}\right\} = \frac{6}{5}$?

10. Triangle ABC has $AB = 13$, $BC = 14$, $CA = 15$. The circle with diameter $\overline{CA}$ passes through point P_1 on $\overline{AB}$ and point P_2 on $\overline{BC}$. The circle with diameter $\overline{P_1P_2}$ passes through point P_3 on $\overline{BP_1}$ and point P_4 on $\overline{BP_2}$. We continue in this fashion, constructing circles with diameter $\overline{P_{2i-1}P_{2i}}$ passing through point P_{2i+1} on $\overline{BP_{2i-1}}$ and point P_{2i+2} on $\overline{BP_{2i}}$. Let K_i be the area of $\triangle BP_{2i+1}P_{2i+2}$. Compute $\sum\limits_{i=0}^{\infty}(-1)^i K_i$.

Express your answer as a common fraction.

Relay Round 12200

Relay 1

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Answer 1-1		Answer 1-2		Answer 1-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12200

(1-1) Let $T = 50$. Let K be the number of ways to form a T-member committee from a group of $T+3$ people. Find the remainder when $K+91$ is divided by 100.

(1-2) Let $T = TNYWR$. Let K be the product of the roots of the polynomial $x^2 - Tx + 6$. Find the remainder when $K+33$ is divided by 100.

(1-3) Let $T = TNYWR$. Let J be the remainder when T is divided by 11. Let $K = 7^J + 6^J$. Find the remainder when $K+62$ is divided by 100.

Relay Round 12200

Relay 2

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Answer 2-1		Answer 2-2		Answer 2-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round
12200

(2-1) Let $T = 46$. Let K be the smallest triangular number greater than T. Find the remainder when $K + 98$ is divided by 100.

(2-2) Let $T = TNYWR$. A line ℓ_1 with slope $\frac{1}{3}$ and a line ℓ_2 with slope $\frac{1}{5}$ intersect at the point $(17, 76)$ in the coordinate plane. Let P and Q be the points where the line $y = T$ intersects the lines ℓ_1 and ℓ_2, respectively. Let K be the distance PQ. Find the remainder when $K + 77$ is divided by 100.

(2-3) Let $T = TNYWR$. Let K be the remainder when $(T + 1)^{T+1}$ is divided by 11. Find the remainder when $K + 30$ is divided by 100.

Relay Round 12200

Relay 3

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Answer 3-1		Answer 3-2		Answer 3-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round
12200

(3-1) Let $T = 52$. Let D be the distance between the x- and y-intercepts of the function $y = x^2 - 2\sqrt{T}x + T$. Let $K = \frac{D^2}{2}$. Find the remainder when $K + 7$ is divided by 100.

(3-2) Let $T = TNYWR$. Let K be the minimum number of people in a room needed to guarantee that at least $T + 1$ people have the same birthday, assuming that no one is born on February 29th. Find the remainder when $K + 45$ is divided by 100.

(3-3) Let $T = TNYWR$. Let K be the smallest positive integer greater than $T - 20$ that is both a triangular number and a square number. Find the remainder when $K + 93$ is divided by 100.

Relay Round
12200

Relay 4

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Answer 4-1		Answer 4-2		Answer 4-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12200

(4-1) Let $T = 1$. Consider a prism that has rectangular lateral faces and $(T+3)$-sided polygons for bases. Let K be the total number of diagonals of all faces of such a prism. Find the remainder when $K+40$ is divided by 100.

(4-2) Let $T = TNYWR$. Let a and b be natural numbers such that $\frac{T+1}{T+2} < \frac{a}{b} < \frac{T+2}{T+3}$. Let K be the smallest possible value of $a+b$. Find the remainder when $K+76$ is divided by 100.

(4-3) Let $T = TNYWR$. Let K be the number of integer triples (a, b, c), in degrees, where $0 \leq a, b, c \leq T$ and $\sin a + \sin b + 1 - \sin^2 c = \cos a + \cos b + \cos^2 c$. Find the remainder when $K+32$ is divided by 100.

Relay Round 12200

Relay 5

Name ______________________

Name ______________________

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School ______________________

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Answer 5-1		Answer 5-2		Answer 5-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12200

(5-1) Let $T = 10$. Let K be the number of digits in the base-21 representation of 3^T. (Note that $\log_{21} 7 \approx 0.64$.) Find the remainder when $K + 78$ is divided by 100.

(5-2) Let $T = TNYWR$. Let $\triangle ABC$ have side lengths $AB = T + 2$, $AC = T + 3$, and $BC = T + 4$. Let R be the circumradius and r be the inradius of this triangle. Let $K = 6Rr$. Find the remainder when $K + 39$ is divided by 100.

(5-3) Let $T = TNYWR$. Let $f(x) = |200 - 2x|$ and let $K = f^{T+1}(T)$ (in other words, start with T and apply the function $T + 1$ times). Find the remainder when $K + 64$ is divided by 100.

Answers
12200

Sprint Round

1. E
2. A
3. C
4. A
5. B
6. C
7. D
8. B
9. D
10. C
11. E
12. C
13. C
14. C
15. E
16. B
17. A
18. C
19. B
20. C
21. A
22. E
23. C
24. E
25. A
26. A
27. E
28. B
29. E
30. D

Target Round

1. 30
2. 60(%)
3. 3
4. 41(%)
5. $\frac{16}{5}$
6. 2690
7. 5
8. 36

Team Round

1. 6
2. 200
3. $\frac{9}{2}$
4. 31
5. 21
6. 23
7. 540
8. 95
9. 3
10. $\frac{1050}{97}$

Relay Round

(1-1)	17	(K = 23426)	(1-2)	39	(K = 6)	(1-3)	67	(K = 164305)
(2-1)	53	(K = 55)	(2-2)	23	(K = 46)	(2-3)	35	(K = 5)
(3-1)	85	(K = 1378)	(3-2)	71	(K = 31026)	(3-3)	18	(K = 1225)
(4-1)	52	(K = 12)	(4-2)	92	(K = 216)	(4-3)	95	(K = 8463)
(5-1)	82	(K = 4)	(5-2)	63	(K = 7224)	(5-3)	72	(K = 8)

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Power Question 12200

1. (a) The answer is $\boxed{10}$. As there is only one chip, firing that chip can only move one space at a time. The chip needs to moved 10 spaces forward, so the minimum possible value is 10.

   (b) The answer is $\boxed{3}$. We can fire one of the chips from 0 to 2. But for the other chip remaining at point 0 on the number line, we need 2 fires as that chip can only fire at most 1 unit. We know this is the optimum because we need at least 2 fires to move 2 chips, but doing it in 2 moves requires firing both chips 2 units forward, which cannot happen. So the fires are as follows: $\{0,0\} \to \{0,2\} \to \{1,2\} \to \{2,2\}$ (where the numbers denote the positions of each of the two chips at each step).

   (c) The answer is $\boxed{11}$. We can fire nine of the chips two units forward as before the chips are fired, there are at least 2 chips at point 0. However, the remaining chip needs two fires as that chip can only go forward 1 unit at a time.

   (d) The answer is $\boxed{8}$. The total number of spaces to fire is $5 \cdot 2 = 10$. So to minimize the number of fires, we need to maximize the number of times a chip gets fired 2 units forward. This means the strategy is to fire a chip 2 spaces then fire the other chip 1 space two times so that a chip can be fired 2 spaces again. So the fires are as follows: $\{0,0\} \to \{0,2\} \to \{1,2\} \to \{2,2\} \to \{2,4\} \to \{3,4\} \to \{4,4\} \to \{4,5\} \to \{5,5\}$.

   (e) The answer is $\boxed{10}$. The optimal sequence of fires is as follows: $\{0,0,0\} \to \{0,0,3\} \to \{0,2,3\} \to \{1,2,3\} \to \{2,2,3\} \to \{2,3,4\} \to \{3,3,4\} \to \{3,4,5\} \to \{4,4,5\} \to \{4,5,5\} \to \{5,5,5\}$.

2. Suppose at some point chip s_k is fired from a pile P, meaning at the time of firing, the chips in P are a subset of $\{s_1, s_2, \ldots, s_k\}$. This is because s_k can only be fired if it is the chip of maximal index. Because of this, chip s_k is fired a distance at most $|P| \leq k$ units forward. Therefore, chip s_k must be fired at least $\frac{d}{k}$ times to reach point d. Because a chip can only be fired an integer number of times, we achieve that s_k must be fired at least $\left\lceil \frac{d}{k} \right\rceil$ as desired.

3. First, consider that given any pile of chips, the outcome is the same regardless of which chip we choose to fire, i.e. the chips are all indistinguishable. Because the chips are indistinguishable, the outcome is only dependent on the distance fired, not which chip is fired. Therefore, we can impose the labelling proposed in Problem 2 and fire under the given condition, without affecting how many total fires are required. By the result of Problem 2, each chip s_k is fired at most a distance k meaning it takes at least $\left\lceil \frac{d}{k} \right\rceil$ fires to move chip k to point d. Thus, the number of fires required to move all chips $\{s_1, s_2, \ldots, s_c\}$ to point d is at least $\displaystyle\sum_{k=1}^{c} \left\lceil \frac{d}{k} \right\rceil$.

4. (a) The final positions of the chips are $\boxed{(1,3),(2,2)}$. This happens after firing one time, and note that $(1,3)$ is exactly on line $\mathcal{M}_1$, so it does not have to move. As there are now at most one chip in each space, no more fires can be done.

   (b) The final positions of the chips are $\boxed{(0,1),(1,1),(2,1)}$. Firing initially sends a chip on $(0,1)$

to $(0,2)$ and a chip on $(0,1)$ to $(1,1)$, but since $(0,2)$ lies above the line $y-2x=1$, that chip gets moved to $(0+2,2-1)=(2,1)$.

(c) The final positions of the chips are $\boxed{(0,2),(2,1)}$. Firing initially puts sends a chip on $(2,0)$ to $(3,0)$ and a chip on $(2,0)$ to $(2,1)$, but the chip on $(3,0)$ lies on the line $2y-3x=-9$, so it gets moved to $(3-3,0+2)=(0,2)$. As there are now at most one chip in each space, no more fires can be done.

(d) The final positions of the chips are $\boxed{(1,2),(3,2),(1,3)}$. Firing the two chips at $(0,2)$ initially sends one to $(0,3)$ and one to $(1,2)$, but since $(0,3)$ lies above the line $2y-3x=4$, that chip gets moved to $(0+3,3-2)=(3,1)$. Now there are two chips on $(3,1)$, so we can fire there to send one chip to $(3,2)$ and one chip to $(4,1)$. But the chip at $(4,1)$ lies below the line $2y-3x=-9$, so that chip gets sent to $(4-3,1+2)=(1,3)$. As there is now at most one chip in each space, no more fires can be done.

5. In the following solutions, we use the notation $\{(x_1,y_1):c_1,\ldots,(x_n,y_n):c_n\}$ to denote the position with c_i chips at (x_i,y_i) for each index i.

(a) $\boxed{\{(2,0),(2,1),(0,2),(1,2)\}}$ The firing sequence is as follows:

$$\begin{aligned}
&\{(0,0):4\}\\
\to&\{(1,0):2,(0,1):2\}\\
\to&\{(2,0):1,(1,1):2,(0,2):1\}\\
\to&\{(2,0):1,(2,1):1,(1,2):1,(0,2):1\}.
\end{aligned}$$

(b) $\boxed{\{((2,1),(1,2),(0,3),(0,4),(3,1),(1,3),(3,2),(2,3)\}}$ The firing sequence is as follows:

$$\begin{aligned}
&\{(0,0):8\}\\
\to&\{(1,0):4,(0,1):4\}\\
\to&\{(2,0):2,(1,1):4,(0,2):2\}\\
\to&\{(0,4):1,(2,1):3,(1,2):3,(0,3):1\}\\
\to&\{(0,4):1,(2,1):1,(1,2):1,(0,3):1,(3,1):1,(2,2):2,(1,3):1\}\\
\to&\{(0,4):1,(2,1):1,(1,2):1,(0,3):1,(3,1):1,(1,3):1,(3,2):1,(2,3):1\}.
\end{aligned}$$

(c) $\boxed{\text{Indefinitely Fired}}$. We can see that three chips with two at $(1,1)$ and one at $(2,1)$ can be fired indefinitely. After the sequence

$$\begin{aligned}
&\{(1,1):2,(2,1):1\}\\
\to&\{(2,1):2,(1,2):1\}\\
\to&\{(1,2):2,(2,2):1\}\\
\to&\{(1,3):1,(2,2):2\}\\
\to&\{(1,3):2,(2,3):1\}\\
\to&\{(2,3):2,(3,3):1\},
\end{aligned}$$

we see that all three chips have simply shifted up two units and to the right by one unit, i.e. from $(1,1)$ to $(2,3)$ and $(2,1)$ to $(3,3)$. Because the resulting position maintains the same "orientation" with respect to the mirrors, this cycle will repeat indefinitely. The initial position with four chips at the origin produces the sequence of fires

$$\begin{aligned} &\{(0,0):4\} \\ \to &\{(1,0):2,(0,1):2\} \\ \to &\{(0,1):1,(1,1):2,(2,1):1\}, \end{aligned}$$

which contains the set of positions described above. Therefore, by ignoring the chip at $(0,1)$, we can conclude that the chips can be indefinitely fired.

6. Note that the only two possible transformations are $(x,y) \mapsto (x+b, y-a)$ and $(x,y) \mapsto (x-b, y+a)$. For each of these cases, we have $h(x+b, y-a) = a(x+b) + b(y-a) = ax+by = h(x,y)$, and $h(x-b, y+a) = a(x-b) + b(y+a) = ax+by = h(x,y)$, as desired.

7. Note that any line of the form $ax+by=n$ is perpendicular to both mirrors, so the maximum distance between two chips a line $ax+by$ is strictly bounded above by the distance between the two mirrors (since chips in a stable position can lie on $\mathcal{M}_1$ but not $\mathcal{M}_2$). The distance between the two mirrors is $\sqrt{a^2+b^2}$.

   Because a stable position cannot contain two chips at the same position, for the sake of contradiction, suppose we have two chips at different positions (p,q) and (p',q') that both lie on the line $ax+by=n$. Without loss of generality assume $p \geq p'$. Then we have $ap+bq=n$ and $ap'+bq'=n$, and subtracting yields $a(p-p') = b(q'-q)$. Because at least one of $p-p'$ and $q'-q$ is nonzero, the other must be nonzero as well. Because a and b are relatively prime, $p-p'$ must be a positive multiple of b, and similarly, $q'-q$ must be a positive multiple of a. However, this means that the distance between the two chips is $\sqrt{(p-p')^2+(q'-q)^2} \geq \sqrt{a^2+b^2}$, a contradiction because the distance between the two chips must be less than $\sqrt{a^2+b^2}$.

   Thus, in any stable position and positive integer n, at most one chip lies on the line $ax+by=n$.

8. We first prove that the order of firing chips does not matter. If a fire $\mathcal{F}$ is made from (p,q) moving two chips to $(p+1,q)$ and $(p,q+1)$, then the point in time at which $\mathcal{F}$ is made will decrease the number of chips at (p,q) by 2 and increase the number at $(p+1,q)$ and $(p,q+1)$ each by 1. Therefore, any ordering of the same set of fires (provided they are valid with at least 2 chips at the given position) will produce the same stable position.

   By Bezout's Theorem, since $gcd(a,b)=1$, we can state that there are infinitely many lattice points on the line $ax+by=n$ for any integer n, and by incrementing by $(b,-a)$ or $(-b,a)$, there is exactly one such lattice point (p,q) between $\mathcal{M}_1$ and $\mathcal{M}_2$ or on $\mathcal{M}_1$.

   Then, note that whenever a chip moves, the function h applied to the coordinates of that chip is strictly nondecreasing, because either a will increase by 1, or b will increase by 1, or (by the result of Problem 6) h will not change as the chip is moved by the mirrors. Therefore, a chip fired from line $h(x,y) = n$ will never revisit the same line. Furthermore, realize that any chip at (p,q), on the line $h(x,y) = h(p,q)$ where (p,q) is not the origin, must have been fired from either of the lines $h(x,y) = h(p-1,q)$ or $h(x,y) = h(p,q-1)$. More specifically, a chip will visit the line $h(x,y) = h(p,q)$ exactly once for every fire from $h(x,y) = h(p-1,q)$ and $h(x,y) = h(p,q-1)$. Because the final position is stable, the number of fires from any (x,y) must equal $\left\lfloor \frac{v(ax+by)}{2} \right\rfloor$, noting that the only way for a chip to leave a point is to be fired, and all points must end with at most one chip in order for the final position to be stable. Therefore, we find that the number of visits to a line $h(x,y) = n$ is $v(n) = v(ap+bq) = \left\lfloor \frac{v(a(p-1)+bq)}{2} \right\rfloor + \left\lfloor \frac{v(ap+b(q-1))}{2} \right\rfloor = \left\lfloor \frac{v(n-a)}{2} \right\rfloor + \left\lfloor \frac{v(n-b)}{2} \right\rfloor$, as desired.

9. Note that for any integer x, $\left\lfloor \frac{x}{2} \right\rfloor \geq \frac{x}{2} - \frac{1}{2}$, meaning that

$$v(n) = \left\lfloor \frac{v(n-a)}{2} \right\rfloor + \left\lfloor \frac{v(n-b)}{2} \right\rfloor \geq \frac{1}{2}\left(\frac{v(n-a)}{2} + \frac{v(n-b)}{2}\right) - 1$$

We can then prove the result by inducting on $p+q$. For the base case, we must have $p = q = 0$, meaning $v(0) > v(0) - 2$, which is clearly true. Then, assume for some positive integer k, for all ordered pairs (p,q) where $p+q = k-1$, we have $v(ap+bq) > \frac{v(0)}{2^{k-1}} - 2$. To prove the result, we first note that because k is nonzero, for any $p+q = k$, at least one of $p-1, q-1$ is nonnegative meaning at least one of $(p-1,q), (p,q-1)$ is an ordered pair of nonnegative integers. Without loss of generality, suppose this is $(p-1,q)$, i.e. $p \geq 1$ and $q \geq 0$. Noting $v(n) \geq 0$ for all n, we get

$$\begin{aligned} v(ap+bq) &\geq \frac{1}{2}\left(\frac{v(a(p-1)+bq)}{2} + \frac{v(ap+b(q-1))}{2}\right) - 1 \\ &> \frac{1}{2}\left(\frac{v(0)}{2^{(p-1)+q}} - 2 + 0\right) - 1 \\ &= \frac{v(0)}{2^{p+q}} - 2, \end{aligned}$$

proving the claim.

10. (a) Suppose $(a-1)(b-1) \leq n < ab$. By the Chicken McNugget Theorem, because $ab-a-b < n$, there exist nonnegative integers p and q with $ap+bq = n$. Because $n < ab$, it follows that $p < b$ and $q < a$. Therefore, by the result of Problem 9,

$$v(n) = v(ap+bq) > \frac{v(0)}{2^{p+q}} - 2 \geq \frac{v(0)}{2^{(a-1)+(b-1)}} - 2.$$

Substituting $v(0) = c = 2^{a+b}$, we obtain $v(n) > \frac{2^{a+b}}{2^{a+b-2}} - 2 = 2$, as desired.

(b) Now we argue by strong induction that $v(n) \geq 2$ for all $n > ab-a-b$. For the base case, part (a) shows that $v(n) \geq 2$ for all n with $ab-a-b < n < ab$.

Now, assume that $v(n) \geq 2$ for all n with $ab-a-b < n < (ab-a-b)+k$ for some $k \geq a+b$. It suffices to prove that $v(ab-a-b+k) \geq 2$. By the result of Problem 8,

$$v(ab-a-b+k) = \left\lfloor \frac{v(ab-2a-b+k)}{2} \right\rfloor + \left\lfloor \frac{v(ab-a-2b+k)}{2} \right\rfloor.$$

Because $k \geq a+b$, it follows that $ab-2a-b+k$ and $ab-a-2b+k$ are both greater than $ab-a-b$, so the inductive hypothesis applies, and therefore $v(ab-2a-b+k) \geq 2$ and $v(ab-a-2b+k) \geq 2$. Thus

$$v(ab-a-b+k) \geq \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor = 2,$$

as desired, which shows that $v(n) \geq 2$ for all $n > ab-a-b$.

However, there are infinitely many n for which $v(n) > 0$, and it is impossible for chips to visit infinitely many positions in a finite number of moves. Therefore, it is not possible to achieve a stable position given 2^{a+b} chips starting at the origin.

Solutions
12200

Sprint Round Solutions

1. Melody gives away $12 \cdot 12$ pencils in the first building and $12 \cdot 18$ pencils in the second building. We could multiply each part out and add to get the final result, but a faster route is to note that $12 \cdot 12 + 12 \cdot 18 = 12 \cdot (12 + 18) = 12 \cdot 30$. This is equivalent to saying that Melody gave away a total of 30 boxes in the day, which each box having 12 pencils. Thus, Melody gave away $12 \cdot 30 = \boxed{360}$ pencils in both buildings.

2. Writing each mixed fraction as an improper fraction turns the product into $\frac{3}{2} \cdot \frac{7}{3} \cdot \frac{22}{7}$. Notice that everything cross-cancels except for the 22 in the numerator and the 2 in the denominator. Therefore the product is just $\frac{22}{2} = \boxed{11}$.

3. Let the number of children who bought tickets be c. Then the revenue from tickets is equal to $2c + 5(10c) = 52c = 780$, so $c = 15$. Thus, a total of $\boxed{150}$ adults bought tickets to the fair.

4. Bella cannot buy four N95 masks because it costs \$20, which is greater than \$16. Bella cannot buy three N95 masks because it costs \$15, resulting in \$1 left, which is too little to buy another mask. Bella can buy two N95 masks and two KN95 masks for a cost of \$14, so the greatest number of N95 masks Bella can buy is $\boxed{2}$.

5. Since a cube has 12 edges, the sum of the lengths of its edges is $12 \cdot 2 = 24$. It has six faces, so its surface area is $6 \cdot 2^2 = 24$. The positive numerical difference between the sum of the lengths of its edges and its surface area is $24 - 24 = \boxed{0}$.

6. Notice that 24 hours is 1 full day, so 48 hours is 2 full days. The sprout is 9 inches tall on Thursday. With some basic skip-counting, we find that the sprout grows 6 inches after 12 more days, and so the day when Jeremiah's plant is 15 inches tall is $\boxed{\text{Tuesday}}$.

7. The number of members that need to vote "Yes" to pass a law is 75% of 86 rounded up, which is 65. Since the vote was three members short, the number of members that voted "Yes" is $65 - 3 = \boxed{62}$.

8. To find the probability, we determine the number of factors that are not a multiple of 5 and divided that by the total number of factors of 140. One way to count is by listing out all positive factors of 140 and counting the factors that do not have a units digit of 5 or 0. Another way to count is to observe from the prime factorization of 140 (which is $2^2 \cdot 5 \cdot 7$) that there are $3 \cdot 2 \cdot 2 = 12$ total factors and $3 \cdot 2 = 6$ factors that are not a multiple of 5. Our desired probability is $\frac{6}{12} = \boxed{\frac{1}{2}}$.

9. Completing the square by adding $\left(\frac{60}{2}\right)^2 = 900$ to both sides gives us $x^2 - 60x + 900 + 896 = 900$. The expression $x^2 - 60x + 900$ is a perfect square trinomial that can be factored as $(x-30)^2$. We are left with the equation $(x-30)^2 = 900 - 896 = 4$. The roots of this equation are 28 and 32, so $a = 28$ and $b = 32$. We see that $10(28) + 32 = \boxed{312}$.

10. Note that triangle WPY can either be inside or outside of triangle WYZ, which has area $\frac{12^2}{2} = 72$. If WPY is inside WYZ, then the area of $WPYZ$ is $72 - 18 = 54$. If WPY is outside WYZ, then the area of $WPYZ$ is $72 + 18 = 90$. The larger of these two possible areas is $\boxed{90}$.

11. We know that all digits to the right of the 18th digit must be at least a 0 and at most a 9. Now consider a number greater than 235 and less than 236. Three times that number would be greater than 705 and less than 708. Then we find that while the 18th digit of $3e$ is at least 5 and at most 8, the 16th digit of $3e$ must be $\boxed{7}$.

12. If the probability that Amee selects a teacher given that she selects someone who does not like pineapple on pizza is $\frac{9}{25}$, then we find that $\frac{9}{25}$ of all the people who does not like pineapple on pizza are teachers. The probability that she selects a teacher who does not like pineapple on pizza is $\frac{5}{6} \cdot \frac{9}{25} = \frac{3}{10}$. Then by the Inclusion-Exclusion Principle, the probability that she selects a teacher or someone who does not like pineapple on pizza is $\frac{1}{3} + \frac{5}{6} - \frac{3}{10} = \boxed{\dfrac{13}{15}}$

13. The diameter of the sphere has length 12 inches, so the radius of the sphere has length 6 inches. From the volume of a sphere formula, the volume of a fully-pumped wall ball is $\frac{4}{3}\pi \cdot 6^3 = 288\pi$ cubic inches. Since $62.5\% = \frac{5}{8}$, the volume of the deflated wall ball is $\frac{5}{8} \cdot 288\pi = \boxed{180\pi}$ cubic inches.

14. Observe that all numbers in the sum are from the Binomial Theorem, and the expression can be rewritten as $(12^2 + 16^2)^5 = (20^2)^5 = 20^{10}$. Computing the exponent results in $\boxed{10240000000000}$.

15. There are $\binom{19}{3} = \frac{19 \cdot 18 \cdot 17}{3 \cdot 2 \cdot 1}$ ways to order 3 heads and 16 tails. Each of these results has probability $\frac{1}{2^{19}}$. Thus, the probability of getting exactly 3 heads after 19 flips is $\binom{19}{3} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{16} = \frac{969}{2^{13}}$, so $n = \boxed{969}$.

16. Each truth-teller must call each liar a liar, and each liar must call each truth-teller a liar. If there are t truth-tellers and $50 - t$ liars, then there will be $2t(50 - t)$ instances of liar-calling. This is maximized when $t = 50 - t$, or when $t = 25$. This gives $\boxed{1250}$ instances of liar-calling in total.

17. Note that with the given angles, the hexagon is shaped like an L. If we align the L parallel to coordinate axes, the two shorter horizontal sides add up to the longest horizontal side, and the two shorter vertical sides add up to the longest vertical side. Since $6 + 7 = 13$ and $8 + 9 = 17$ (and there is no side length of 4), 13 and 17 are the longer side lengths, and the sides of length 13 and 17 cannot be parallel. The area of the hexagon can be computed by taking the area of the rectangle with side lengths 13 and 17, and then deleting a smaller rectangle whose side lengths are chosen from one of 6 or 7 and one of 8 or 9. The smallest rectangle that can be deleted is one with side lengths 6 and 8, so the greatest possible area is $13 \cdot 17 - 6 \cdot 8 = \boxed{173}$.

18. Each two-digit number can be written in the form $10a+b$, where a is the tens digit and b is the ones digit. There are two cases to consider: $a<b$ and $a>b$ (note that $a \neq b$). All cases where $a<b$ work, except the one in which $a=0$, so there are $\binom{9}{2}=36$ such numbers from picking two different digits. As for the case where $a>b$, we use casework on the tens digit. There are 5 numbers with $a=9$, 4 numbers with $a=8$, 3 numbers with $a=7$, 2 numbers with $a=6$, and 1 number with $a=5$. In total, there are $36+5+4+3+2+1=\boxed{51}$ two-digit deciduous numbers.

19. By the Pythagorean Theorem, the topmost vertices of each isosceles triangle lie $\sqrt{65^2-25^2}=60$ units above the plane of the square. Solid $EFGHIJKL$ is a prism whose base $IJKL$ is a square whose diagonal length is the side length of the original square $ABCD$. Thus, $EF=\frac{50}{\sqrt{2}}=25\sqrt{2}$, the base area is $625\cdot 2=1250$, and the prisms's volume is $1250\cdot 60=\boxed{75000}$.

20. If the large table consists of t tables to one side and $\frac{70}{t}$ tables to the other, then it will be able to accommodate $2\left(t+\frac{70}{t}\right)=\frac{2t^2+140}{t}$ chairs around its perimeter. We want to sum this expression over all factors of t of 70 that are less than $\sqrt{70}$ (lest we overcount). The resulting sum is $\frac{2\cdot 1^2+140}{1}+\frac{2\cdot 2^2+140}{2}+\frac{2\cdot 5^2+140}{5}+\frac{2\cdot 7^2+140}{7}=142+74+38+34=\boxed{288}$.

21. From the given information, we have $2(2ab+2ac+bc)=18$, $2(2ab+ac+2bc)=20$, and $2(ab+2ac+2bc)=22$. Summing these together yields $2(5ab+5ac+5bc)=60$, or $2ab+2ac+2bc=\boxed{12}$.

22. The expressions 10^n-1 and 10^n+1 have similar forms, where 10^n-1 has all n digits 9s and 10^n+1 has $n+1$ total digits with $n-1$ digits being 0s between two 1s. As such, we can experiment to find patterns, and we can suspect that 10^n-1 divides 10^m-1 if and only if n divides m and that 10^n+1 divides 10^m-1 if and only if $2n$ divides m. In fact, if k is the remainder if m is divided by n, then 10^m-1 divided by 10^n-1 leaves a remainder of 10^k-1 and 10^m-1 divided by 10^n+1 leaves a remainder of $(-1)^{\frac{m-k}{n}}10^k-1$. The expression 10^k-1 equals zero when $k=0$, and the expression $(-1)^{\frac{m-k}{n}}10^k-1$ equals zero when $k=0$ and $\frac{m}{n}$ is even. Since 2^n only has a prime factor of 2, the values of n are exactly the powers of 2 (including $n=1$), and the sum of those integers is $1+2+4+8=\boxed{15}$.

23. Notice that $\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}=ad-bc$ and $\det\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)=\det\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2\end{bmatrix}=(ad-bc)^2$. It turns out that the determinant is multiplicative. Since $\det M=5\cdot 4-6\cdot 3=2$, we find that $\det M^{128}=(\det M)^{128}=2^{128}$, so $n=\boxed{128}$.

24. Note that for each number of red side lengths that are at least 2 and less than 8, there are 8 ways to place the red side lengths. If the number of red side lengths is equal to 8, then there is only one coloring (the entire perimeter). Thus, we get $8\cdot 6+1=\boxed{49}$ possible colorings.

25. Let $z=\cos\theta+i\sin\theta$. By de Moivre's formula, we have $z^{\frac{3}{2}}=\cos\left(\frac{3}{2}\theta\right)+i\sin\left(\frac{3}{2}\theta\right)=\cos(\theta+30^\circ)+i\sin(\theta+30^\circ)$. We want $\frac{3}{2}\theta=\theta+30^\circ$, therefore $\theta=60^\circ$. We have $\Re(z)=\cos(60^\circ)=\frac{1}{2}$ and $\Im(z)=\sin(60^\circ)=\frac{\sqrt{3}}{2}$, thus their sum is $\boxed{\dfrac{1+\sqrt{3}}{2}}$.

26. Lucas can choose 0, 4, 8, or 12 creatures, so the number of ways for Lucas to choose creatures is $\binom{15}{0}+\binom{15}{4}+\binom{15}{8}+\binom{15}{12}$. We could evaluate the expression manually, but we can also note that this is the sum of every fourth coefficient in the polynomial $f(x)=(1+x)^{15}$. By using the roots of unity filter, we can reimagine the sum as $\frac{f(1)+f(i)+f(-1)+f(-i)}{4}$, where $i=\sqrt{-1}$. Then $f(1)=2^{15}$, $f(-1)=0$, $f(i)=(1+i)^{15}=-128i+128$, and $f(-i)=(1-i)^{15}=128i+128$. Then the wanted sum is $\frac{2^{15}+128+128}{4}=\boxed{8256}$.

27. We have $s_n=6n^3$ and $S_n=6n^2$, so we end up with $\sum_{k=1}^{n}(6k^3-6k^2)=6\left(\frac{n(n+1)}{2}\right)^2-6\left(\frac{n(n+1)(2n+1)}{6}\right)=\frac{3}{2}n^2(n+1)^2-n(n+1)(2n+1)=2100$. Multiplying both sides by 2 and factoring results in $n(n+1)(3n+2)(n-1)=4200$. We could try doing Rational Root Theorem, but a faster way is to note that the prime factorization of 4200 is $2^3\cdot3\cdot5^2\cdot7$, allowing us to narrow our search to smaller numbers near multiples of 7. If $n=6$, then $n(n+1)(3n+2)(n-1)=6\cdot7\cdot20\cdot5=4200$, so $n=\boxed{6}$.

28. Note that any 100 consecutive integers will contain exactly 20 multiples of 5 and 4 multiples of 25, covering 24 of the required factors. Since any number in the 100 consecutive integers that is not a multiple of 125 will not generate another factor of 5 above the existing 24, we need a multiple of 125 to be one of the integers in the product, so $25\le n\le100$. Since 100 consecutive integers cannot contain 2 multiples of 125, nor can they contain a multiple of 625 if $n\le100$, we confirm that all these values of n would result in $\prod_{k=1}^{100}(n+k)$ having exactly 25 terminating zeroes, so the desired sum is $25+26+\cdots+100=\frac{125\cdot76}{2}=\boxed{4750}$.

29. The marbles can be split into either 1, 2, 4, or 8 groups. There is just one way to split the marbles into 1 group. For 2 groups, there are $\binom{8}{4}=70$ ways to split the marbles into two distinct groups, so there are $\frac{70}{2}=35$ ways to split them into two unlabeled groups. For 4 groups, there are $\binom{8}{2}\binom{6}{2}\binom{4}{2}=2520$ ways to split the marbles into four distinct groups, so there are $\frac{2520}{4!}=105$ ways to split the marbles into four unlabeled groups. Finally, there is just one way to split the marbles into 8 groups. The total number of ways is $1+35+105+1=\boxed{142}$.

30. We approach by complementary counting, where we count the sequences that are neither budding nor developed and the sequences that are both budding and developed. In order for a nondecreasing sequence to be neither budding nor developed, its median must be less than or equal to 4, and it must contain at least one term strictly greater than 5. Since (a_n) has 5 terms, the third term is the median. There are 3 cases: (1) $a_4<5<a_5$, (2) $5\le a_4<a_5$, and (3) $5<a_4=a_5$. Using sticks-and-stones, we find that the number of sequences in (1) is $\binom{7}{4}\cdot\binom{5}{1}=175$. In (2), there are $\binom{6}{3}\cdot\binom{6}{2}=300$ sequences, and there are $\binom{6}{3}\cdot5=100$ sequences in (3). Adding these up, we find that there are $175+300+100=575$ nondecreasing five-term sequences that are neither budding nor developed.

As for the sequences that are both budding and developed, the median must equal 5, so $a_3=a_4=a_5=5$. If $a_1=a_2$, then there are 5 possibilities, and if $a_1<a_2$, then there are $\binom{5}{2}=10$ possibilities. Therefore, there are 15 nondecreasing sequences of five positive integers all less than or equal to 10 that are both budding and developed. In total, the number of nondecreasing sequences of five positive integers all less than or equal to 10 that are either budding or developed but not both is thus $2002-575-15=\boxed{1412}$.

Target Round Solutions

1. After Nico ate the jellybeans, there were $80 - 7 - 9 - 13 = 51$ jellybeans remaining. Since there were an equal number of red, green, and blue jellybeans, there were $\frac{51}{3} = 17$ green jellybeans. Before he ate any, the number of green jellybeans was $17 + 13 = \boxed{30}$.

2. We have $\frac{5}{10} \cdot \frac{5}{10} \cdot \frac{8}{10} \cdot A = \frac{3}{10} \cdot \frac{4}{10} \cdot \frac{6}{10} \cdot B$. Solving for A and simpifying yields $A = \frac{9}{25}B$, then taking the square root results in $\sqrt{A} = \frac{3}{5}\sqrt{B}$. Therefore, $\sqrt{A}$ is $\boxed{60\%}$ of $\sqrt{B}$.

3. Daniel can get the answer right first try with a probability of $\frac{1}{5}$. He can get the answer right second try with a probability of $\frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}$. From there, we can find that it will take Daniel k attempts with probability $\frac{1}{5}$ for each value of k from 1 to 5 inclusive, so the expected value is just $\frac{1+2+3+4+5}{5} = \boxed{3}$.

4. By finding the altitude to the side of length 12, we find that the isosceles triangle is made up of two 6-8-10 triangles, so the area of the isosceles triangle is 48. Then from using the formula $A = rs$ (where A is the triangle's area, r is the inradius, and s is half the triangle's perimeter), we get that $r = 3$, making the area of the inscribed circle 9π. Then the area outside the inscribed circle is $48 - 9\pi$, so the desired percentage is $100\% \cdot \frac{48-9\pi}{48} \approx \boxed{41\%}$.

5. Let $P(x) = x^2 + bx + c$; then $P(0) = 12$ gives $c = 12$ and $P(1) = 8$ gives $b = -5$. Then $(3n)^2 - 5(3n) + 12 = 9(n^2 - 5n + 12)$. Expanding and simplifying both sides results in $9n^2 - 15n + 12 = 9n^2 - 45n + 108$, and solving for n yields $n = \boxed{\dfrac{16}{5}}$.

6. Notice that $91 = 13 \cdot 7$ and $117 = 13 \cdot 9$, so the distance between any two spaces that Andrew's frog can land on must be a multiple of 13. In fact, if Andrew's frog hops 91 spaces forward four times then hops 117 spaces backward three times, then Andrew's frog ends up $91 \cdot 4 - 117 \cdot 3 = 13$ spaces past the original spot. Then the twenty smallest positive numbers that Andrew's frog can hop on are $11, 24, 37, 50, \ldots, 258$, and the sum is $\frac{11+258}{2} \cdot 20 = \boxed{2690}$.

7. This is equivalent to finding the number of integer solutions to the equation $m + t + w + r + f = 40$, with $7 \le m \le t \le w \le r \le f \le 10$. By subtracting 7 from each of m, t, w, r, f, we obtain the equivalent equation $(m-7) + (t-7) + (w-7) + (r-7) + (f-7) = 5$ with $0 \le (m-7) \le (t-7) \le (w-7) \le (r-7) \le (f-7) \le 3$. From here, we count the number of tuples $(m-7, t-7, w-7, r-7, f-7)$ that satisfy this through casework. If $(f-7) = 3$, then we can have either $(r-7) = 2$ and everything else 0 or $(w-7) = (r-7) = 1$. If $(f-7) = 2$, then we can have either $(r-7) = 2$ and $(w-7) = 1$ or $(t-7) = (w-7) = (r-7) = 1$. Finally, if $(f-7) = 1$, then $(m-7) = (t-7) = (w-7) = (r-7) = (f-7) = 1$. In total, the number of tuples that satisfy the constraints is $\boxed{5}$.

8. Plot the complex numbers w, x, y, z on the complex plane, and label them as points W, X, Y, Z respectively. Since they all have the same magnitude, they all lie on the same circle centered about the origin. The line WY contains all complex numbers with real part a, and the line XZ contains all complex numbers with imaginary part b. Since $v = a + bi$, point V lies on both of these lines, so it is the intersection of WY and XZ. We are given the lengths $VW = |v - w| = 28$, $VX = |v - x| = 14$, and $VY = |v - y| = 18$, and we are trying to find the length $VZ = |v - z|$. This is a straightforward application of Power of a Point: $VW \cdot VY = VX \cdot VZ$. Plugging in lengths and solving gives $|v - z| = VZ = \frac{28 \cdot 18}{14} = \boxed{36}$.

Team Round Solutions

1. The other leg has length 2, since the area of a right triangle is half the product of its leg lengths. Then $2^2 + x^2 = (2\sqrt{10})^2 = 40$, so $x = \boxed{6}$.

2. For convenience, let $a = 100$. We obtain $f\left(a + \frac{1}{a}\right) = \sqrt{\left(a + \frac{1}{a}\right)^2 - 4} = \sqrt{\left(a - \frac{1}{a}\right)^2} = \left|a - \frac{1}{a}\right|$. Furthermore, $g(x) = \sqrt{x^2 + 4}$ similarly implies that $g\left(a - \frac{1}{a}\right) = \left|a + \frac{1}{a}\right|$. It follows that the desired expression comes out to $\left|100 - \frac{1}{100}\right| + \left|100 + \frac{1}{100}\right| = \boxed{200}$.

3. By linearity of expectation, the expected number of tokens Addison earns is the sum divided by 2, which is $\boxed{\frac{9}{2}}$. We can also manually verify by casework on which games he wins.

4. The midpoint moves half the distance of the tip, or essentially the length of a 45° circular arc with radius equal to the full length of the pencil. If this arc has length 24, then the full 360° arc has length $\frac{360}{45} \cdot 24 = 192$. This is the circumference of a circle with radius $\frac{96}{\pi} \approx \boxed{31}$ centimeters, which is the length of the pencil to the nearest centimeter.

5. There are 2 hours from 4 PM to 6 PM, so Dante earns $\$15 \cdot 2 = \$30$ per weekday. However, Dante only allocates half the money for the new game console, so each day, Dante's money for the new game console increases by $\frac{\$30}{2} = \$15$. Because Dante starts with \$100 allocated, he still needs to earn \$249.99, and since $\frac{249.99}{15}$ is greater than 16 and less than 17, it would take Dante seventeen weekdays to earn that money. After counting up from November 28, we see that Dante will have enough money allocated on December 21, so $N = \boxed{21}$.

6. Let x be the height of the ceiling in centimeters. The distance between the ceiling and the figurine halves every second, so the difference will be $\left(\frac{1}{2}\right)^4(x-7)$ after 4 seconds. The difference is also equal to $x-22$. Setting these equal to each other, we have $\frac{1}{16}(x-7) = x-22$, and solving gives $x = \boxed{23}$.

7. Juliet may share either 98, 99, or all 100 cookies. The factors of 98 are 1, 98, 2, 49, 7, 14. The factors of 99 are 1, 99, 3, 33, 9, 11. The factors of 100 are 1, 100, 2, 50, 4, 25, 5, 20, 10. With careful addition (where we remember to only count 1 and 2 once), we obtain a sum of $217 + 155 + 168 = \boxed{540}$.

8. The product P_1 is $a(a+2)(a+4)\cdots(a+18)$, while the product P_2 is $(a+1)(a+3)(a+5)\cdots(a+19)$. Thus, $P_1P_2 = a(a+1)(a+2)\cdots(a+19)$, which must contain exactly one factor of 5^{10} (but not 5^{11}), by Legendre's formula. Since this expression contains 20 terms, and 20 is a multiple of 5, it does not matter what we pick as the value of a mod 5. To have one factor of 5^{10}, we must have a multiple of 5^7, as the other three terms can only correspond to 5^1s in the prime factorization of P_1P_2 (since there cannot be two terms with powers of at least 5^2 in their factorizations). Thus, one of a through $a+19$ must be a multiple of 5^7. Since we are looking only for $a < 5^7$, we can have $a = 5^7 - 19$ through $a = 5^7 - 1$, the sum of which is $19 \cdot 5^7 - 190$. Finally, the smallest integer greater than $\frac{S}{5^6}$ is $19 \cdot 5 = \boxed{95}$.

9. If r is an integer (because $\{r\} = 0$ in that case) or if $\left\{\frac{1}{r}\right\} \le \frac{1}{5}$, the equation cannot be satisfied, since $\{r\}$ is always between 0 and 1. Thus, we need r to be a non-integer less than 5, but at least 1. Additionally, because $\{r\}$ increases faster than $\left\{\frac{1}{r}\right\}$ decreases, the function $f(r) = \{r\} + \left\{\frac{1}{r}\right\}$ is strictly increasing and continuous for r in the range $k \le r < k+1$ for any positive integer $k \le 4$. Furthermore, note that if $k \le 9$, then $\frac{1}{k} \le \{r\} + \left\{\frac{1}{r}\right\} < 1 + \frac{1}{k+1}$, and all values in this interval are possible. From the inequality, there are only solutions for r where the corresponding interval contains $\frac{6}{5}$, which gives $\boxed{3}$ solutions. (Alternatively, let $r = n + x$ where $x = \{r\}$, note that $x + \frac{1}{n+x} = \frac{6}{5}$, and find the values of n for which the resulting quadratic in x has a solution in the interval $[0, 1)$.)

10. First, note that $\triangle ABC$ can be split into a 5-12-13 and 9-12-15 triangle and that $[ABC] = 84$. Next, we can use knowledge of circumcircles, trigonometry, and Power of a Point to determine the length of BP_i, where i is an integer. In particular, the circle with diameter AC is the circumcircle of $\triangle AP_2C$, so $\angle AP_2C = 90^\circ$. Since $\cos\angle ABP_2 = \frac{5}{13}$, $BP_2 = 5$. Then, from Power of a Point, $BP_1 = \frac{70}{13}$. By using similar steps, we find that $BP_4 = \frac{350}{169}$ and $BP_3 = \frac{25}{13}$. Notice that $\triangle BP_3P_4 \sim \triangle BAC$, so we can conclude that each triangle has side lengths equal to $\frac{25}{169}$ times the side lengths of the triangle *two* steps before it. Therefore, $K_{i+2} = \frac{5^4}{13^4}K_i$. Additionally, the area of $\triangle BP_1B_2$ is equal to $\frac{1}{2} \cdot \frac{70}{13} \cdot 5 \cdot \sin\angle ABC = \frac{2100}{169} = 84 \cdot \frac{25}{169}$. Thus, $\displaystyle\sum_{i=0}^{\infty}(-1)^i K_i = 84 \cdot \frac{25}{169} - 84 \cdot \left(\frac{25}{169}\right)^2 + 84 \cdot \left(\frac{25}{169}\right)^3 - \cdots = \boxed{\frac{1050}{97}}$.

Relay Round Solutions

(1-1) The number of T-member committees that can be formed from a group of $T+3$ people is $K = \binom{T+3}{T}$.

(1-2) Let p and q be the roots of the polynomial $x^2 - Tx + 6$. Then $(x-p)(x-q) = x^2 - (p+q)x + pq$. Matching corresponding terms give that the product of the roots $K = pq = 6$, regardless of the value of T.

(1-3) Plugging in the value of T and using the fact that $6^T \bmod 100$ cycles through $06, 36, 16, 96, 76, 56, 36, 16, 96, \ldots$ and $7^T \bmod 100$ cycles through $07, 49, 43, 01, 07, 49, \ldots$, we can find the desired value of $K \bmod 100$. (We do not need to find the exact value of K. We only need to find $K \bmod 100$ to calculate $K + n \bmod 100$, where n is the given integer in the problem.)

(2-1) The nth triangular number is known to be $\frac{n(n+1)}{2}$. We see to solve for the smallest value of $K = \frac{n(n+1)}{2}$ such that $K = \frac{n(n+1)}{2} > T$ and $n(n+1) > 2T$. Since n and $n+1$ are close together, either n or $n+1$ must be equal to $\lfloor(\sqrt{2T})\rfloor$. Evaluating the floor function and testing the two values of n will reveal the one correct value of K.

(2-2) Since we are given a slope and a point, we can write the equation of each line ℓ and m in point-slope form. Line ℓ will have equation $y - 76 = \frac{1}{3}(x-17)$, so $y = \frac{1}{3}(x-17)+76$. Since this line intersects with the line $y = T$, we have that $T = \frac{1}{3}(x-17)+76$. Then we can solve for x in terms of T to get $x = 3(T-76)+17$. Similarly, line m will have equation $y - 76 = \frac{1}{5}(x-17)$, so $y = \frac{1}{5}(x-17)+76$. Since this line intersects with the line $y = T$, we can solve for x in terms of T to be $x = 5(T-76)+17$. The difference between these two x-values is $|(5(T-76)+17) - (3(T-76)+17)| = |2(T-76)| = K$, the distance PQ.

(2-3) Use modular arithmetic mod11 to find the desired remainder. The procedures for finding this quantity may vary depending on the value of T.

(3-1) Factoring the given equation, we get $y = (x - \sqrt{T})^2$, so the x-intercept of this function is $(\sqrt{T}, 0)$, and the y-intercept of this function is $(0, T)$. The distance between these two coordinates is $D = \sqrt{(T-0)^2 + (0-\sqrt{T})^2} = \sqrt{T^2+T}$. Thus, $K = \frac{\sqrt{(T^2+T)^2}}{2} = \frac{T(T+1)}{2}$.

(3-2) Since no one is born on February 29th by the given assumption, we assume that there are 365 days in the year. By the Pigeonhole Principle, the minimum number of people in a room needed to guarantee that at least $T+1$ people have the same birthday is $K = 365T + 1$.

(3-3) The numbers that are both triangular and a perfect square include 1, 36, and 1225. Choose the smallest of these three values that is greater than $T - 20$.

(4-1) From the problem statement, we seek to find the total number of face diagonals within a $(T+3)$-gonal prism. There are $2(T+3)$-gons as bases and $T+3$ rectangles connecting the two bases to form the prism. This means that there are $K = \frac{2((T+3)(T+3-3)}{2}) + 2(T+3) = (T+2)(T+3)$ face diagonals.

(4-2) For all positive integers p, q, r, s, $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$; the fraction in the middle is also known as the mediant operator. The mediant gives the fraction of smallest denominator between $\frac{p}{q}$ and $\frac{r}{s}$ if $qr - ps = 1$. Because $(T+2)^2 - (T+1)(T+3) = 1$, the smallest possible fraction that lies strictly between $\frac{T+1}{T+2}$ and $\frac{T+2}{T+3}$ is $\frac{a}{b} = \frac{T+1+T+2}{T+2+T+3} = \frac{2T+3}{2T+5}$. Therefore, $K = 4T+8$.

(4-3) Notice that $1 - \sin^2 c = \cos^2 c$, meaning that c is independent of a and b. Simplifying the given expression, we get $\sin a + \sin b = \cos a + \cos b$. This equality is only satisfied when $a + b = 90$ since $\sin a = \cos(90 - a) = \cos b$. If $0 \leq a, b, c \leq 44$, then there are $K = 0$ integer triples (a, b, c) that would satisfy the original equation since there would be no possible pairs (a, b) with $a + b = 90$. If $45 \leq a, b, c \leq 90$, then there are $K = (2T-89)(T+1)$ solutions to the original equation, where $2T-89$ represents the number of solutions there are to $a+b = 90$ where $45 \leq a, b, c \leq 90$ (e.g. when $T = 45$, there is only 1 solution: $(45, 45)$; when $T = 46$, there are 3 solutions: $(44, 46), (45, 45), (46, 44)$; when $T = 47$, there are 5 solutions: $(43, 47), (44, 46), (45, 45), (46, 44), (47, 43)$, etc.). If $91 \leq a, b, c \leq 99$, then there are $K = 91(T+1)$ solutions to the original equation. The 91 represents the number of solutions to $a + b = 90$, but since the given interval does not include negative numbers, the only possible pairs (a, b) have $0 \leq a, b \leq 90$, so there are $90 - 0 + 1 = 91$ pairs (a, b) and $T + 1$ possible values for c.

(5-1) The number of digits in the base-b representation of n^T is $\lfloor(\log_b n^T)\rfloor + 1$. We seek to find $K = \lfloor(\log_{21} 3^T)\rfloor + 1$. Given the approximation $\log_{21} 7 \approx 0.64$, and $\log_{21} 3 + \log_{21} 7 = 1$, it follows that $\log_{21} 3 \approx 0.36$. This means that $K = \lfloor(0.36T)\rfloor + 1$.

(5-2) Consider two formulas for the area of a triangle: $\frac{(AB)(BC)(AC)}{4R}$ and rs, where a, b, and c are the side lengths of $\triangle ABC$, s is the semiperimeter of the triangle, R is the circumradius of the triangle, and r is the inradius of the triangle. Setting these area equations equal to each other, we see that

$$\begin{aligned}
\frac{(AB)(BC)(AC)}{4R} &= r\left(\frac{AB+BC+AC}{2}\right) \\
\frac{(T+2)(T+3)(T+4)}{4R} &= r\left(\frac{T+2+T+3+T+4}{2}\right) \\
\frac{(T+2)(T+3)(T+4)}{4R} &= r\left(\frac{3(T+3)}{2}\right) \\
(T+2)(T+3)(T+4) &= 6Rr(T+3) \\
6Rr &= (T+2)(T+4).
\end{aligned}$$

Therefore, $K = 6Rr = (T+2)(T+4)$.

(5-3) Plug in the value of T into $f(x)$ and quickly evaluate as many terms as possible until a pattern emerges between some or all of the generated numbers. Then, deduce the value of $f^{T+1}(T)$ from the patterns observed from the particular value of T to find the correct value of K.

