Contest Problem Set 12300

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Power Question 12300

The following ten questions are worth 10 points each [where a question is broken into multiple parts, the point values for each part are indicated]. To receive full credit the presentation must be legible, orderly, clear, and concise. In general, proofs are not required and will earn no points except on questions that ask for proofs. Derivations are not necessary but may be helpful in assigning partial credit if your final answer is not 100% accurate.

Any question that asks for a function or expression in terms of one or more variables requires a closed-form expression to receive full credit; expressions that involve summation notation or recursion are worth half credit. Throughout the Power Question, the variables n and k are assumed to be integers greater than 1.

The pages submitted for credit should be numbered in consecutive order at the top of each page in what your team considers to be proper sequential order. Please write on one side of the answer papers only.

1. (a) How many total digits are in the base-10 written representation of all integers from 1 to 100, inclusive? Do not include leading zeros. [2 points]

   (b) How many total digits are in the base-10 written representation of all integers from 1 to 1000, inclusive? Do not include leading zeros. [3 points]

   (c) How many total digits are in the base-10 written representation of all integers from 1 to 10000, inclusive? Do not include leading zeros. [5 points]

2. (a) How many total zeros are in the base-10 written representation of all integers from 1 to 100, inclusive? Do not include leading zeros. [2 points]

   (b) How many total zeros are in the base-10 written representation of all integers from 1 to 1000, inclusive? Do not include leading zeros. [3 points]

   (c) How many total zeros are in the base-10 written representation of all integers from 1 to 10000, inclusive? Do not include leading zeros. [5 points]

3. How many total digits are in the base-10 written representation of all integers from 1 to 10^n, inclusive? Do not include leading zeros. Express your answer in terms of n.

4. Prove your answer to the previous question.

5. How many total zeros are in the base-10 written representation of all integers from 1 to 10^n, inclusive? Do not include leading zeros. Express your answer in terms of n.

6. For each of the digits 1 through 9, how many total copies of that digit are in the base-10 written representation of all integers from 1 to 10^n, inclusive? Do not include leading zeros. Express your answer(s) in terms of n.

7. Prove your answer to the previous question.

8. (a) How many total zeros are in the base-2 written representation of all integers from 1 to 2^n, inclusive? Do not include leading zeros. Express your answer in terms of n. [5 points]

   (b) How many total ones are in the base-2 written representation of all integers from 1 to 2^n, inclusive? Do not include leading zeros. Express your answer in terms of n. [5 points]

9. For each of the digits 0 through $k-1$, how many total copies of that digit are in the base-k written representation of all integers from 1 to k^n, inclusive? Do not include leading zeros. Express your answer(s) in terms of n and k.

10. Prove or disprove: The number of zeros used to write all integers in base k from 1 to k^n, inclusive, without leading zeros equals the number of total digits used to write all integers in base k from 1 to k^{n-1}, inclusive, without leading zeros. If it helps, you may use the formulas you found in questions 3 or 6 without additional proof; any other formulas you derived in earlier questions must be proven as part of your answer to this question.

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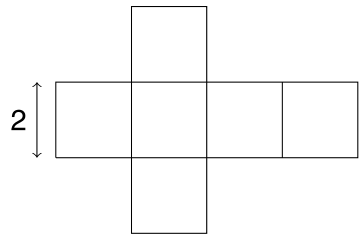
Select E only if you cannot determine a uniquely correct answer between A, B, C, and D.

1. (A) (B) (C) (D) (E)	11. (A) (B) (C) (D) (E)	21. (A) (B) (C) (D) (E)
2. (A) (B) (C) (D) (E)	12. (A) (B) (C) (D) (E)	22. (A) (B) (C) (D) (E)
3. (A) (B) (C) (D) (E)	13. (A) (B) (C) (D) (E)	23. (A) (B) (C) (D) (E)
4. (A) (B) (C) (D) (E)	14. (A) (B) (C) (D) (E)	24. (A) (B) (C) (D) (E)
5. (A) (B) (C) (D) (E)	15. (A) (B) (C) (D) (E)	25. (A) (B) (C) (D) (E)
6. (A) (B) (C) (D) (E)	16. (A) (B) (C) (D) (E)	26. (A) (B) (C) (D) (E)
7. (A) (B) (C) (D) (E)	17. (A) (B) (C) (D) (E)	27. (A) (B) (C) (D) (E)
8. (A) (B) (C) (D) (E)	18. (A) (B) (C) (D) (E)	28. (A) (B) (C) (D) (E)
9. (A) (B) (C) (D) (E)	19. (A) (B) (C) (D) (E)	29. (A) (B) (C) (D) (E)
10. (A) (B) (C) (D) (E)	20. (A) (B) (C) (D) (E)	30. (A) (B) (C) (D) (E)

1. The net shown below is of a 3D figure where all the faces are squares. The side lengths are marked as shown. What is the volume of the 3D figure formed by the net?



(A) 12 (B) 8 (C) 2 (D) 4 (E) Other

2. Kyle's teacher assigns him to do all odd-numbered problems from Problem 1 to Problem 19 inclusive. How many problems must Kyle do for homework?

(A) 10 (B) 19 (C) 20 (D) 8 (E) Other

3. Andrew wants to sign up for a special cup where his monsters can fight other particpants' monsters. The only monsters that can participate in the cup are monsters whose ID number is at most 151 and whose power level is at most 1500. The table below lists the ID number and power level of his six favorite monsters. How many of these monsters can participate in the cup?

Name	**ID Number**	**Power Level**
Blasty	9	1472
Big Ounce	6	1792
Ferno	391	1373
Holmes	25	969
Pirouette	648	1675
Sarpal	131	1550

(A) 4 (B) 5 (C) 3 (D) 1 (E) Other

4. Of the five descriptions below, which describe quadrilaterals that CANNOT exist?
   I. A parallelogram with exactly 3 obtuse angles
   II. A rhombus with exactly 4 right angles
   III. A trapezoid with at least 2 angles with equal measure
   IV. A rectangle with no right angles
   V. A quadrilateral with all angles of different measure

(A) I, IV (B) V (C) I, V (D) IV, V (E) Other

5. A cluster from a specific protein consists of a manganese ion (which has charge $+2$) bound to three neutral (meaning they have have charge 0) histidine residues, a neutral water molecule, and an aspartate residue. Given that the aspartate residue has charge -1, what is the total charge of all the components of the cluster?

(A) -1 (B) 1 (C) -2 (D) 0 (E) Other

6. Chelina is currently at Page 13 of a humanities text but hasn't read the page yet. She starts reading on Monday but has to finish Page 100 by Friday of the same week. She plans to read the same number of pages every day until the day before the due date, but because the discussion is on Friday, she does not plan on reading any pages on Friday. How many pages does Chelina need to read every day in order to reach the goal?

(A) 21 (B) 25 (C) 20 (D) 22 (E) Other

7. Each of the side lengths of a certain quadrilateral is either 5 or 22, and its perimeter is an odd number. What is the perimeter of the quadrilateral?

(A) 39 (B) 37 (C) 69 (D) 45 (E) Other

8. Consider the equation $0.11x + 0.13 = 2$. What is the least positive integer that can be added to the right hand side of the equation so that the solution for x also increases by an integer amount?

(A) 7 (B) 11 (C) 100 (D) 1 (E) Other

9. The side lengths of a rectangle with area 25 are whole numbers, and the diagonals of this rectangle are not perpendicular. What is the perimeter of this rectangle?

(A) 20 (B) 100 (C) 25 (D) 52 (E) Other

10. Given that $f(x) = ax^2 + bx$, $f(1) = 3$, and $f(2) = 4$, find the value of b.

(A) 3 (B) -1 (C) 4 (D) 5 (E) Other

11. Freya is flying 2700 miles from San Francisco (with time zone GMT minus 8 hours) to Boston (time zone GMT minus 5 hours). Her flight takes off from San Francisco at 10:05 PM on Tuesday. If she arrives in Boston after 7:05 AM (Boston time) on Wednesday, she will be jet-lagged. Compute the minimum speed, in miles per hour, that the plane must travel for Freya to avoid being jet-lagged.

(A) 300 (B) 600 (C) 450 (D) 900 (E) Other

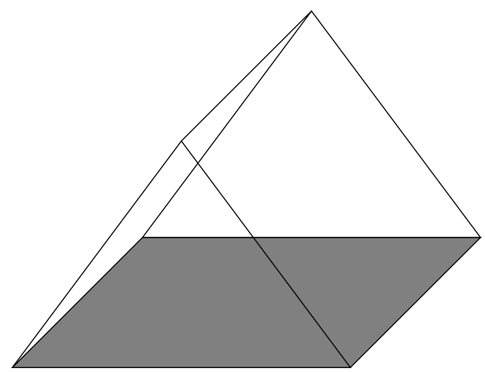
12. An infused drink is made by putting some combination of sliced fruits in a dispenser filled with water. If the choices of fruits to include are peaches, pomelos, and persimmons, how many different kinds of infused drinks can be made? (At least one type of fruit must be used.)

(A) 15 (B) 8 (C) 4 (D) 3 (E) Other

13. For some integers a and b, the polynomial $\left(4x^{13} - 6x^{45}\right)\left(7x^{36} + 10x^{23}\right) + ax^b$ has degree less than 80. Compute $100a + b$.

(A) 3494 (B) 3681 (C) 4281 (D) 1683 (E) Other

14. A camping tent is in the shape of a triangular prism, as shown below. The shaded face is a square with area 28 square meters, and the height of the prism (measured above the square base) is 3 meters. What is the surface area of the entire camping tent (all four “walls”” and the floor), in square meters?



(A) $28 + 22\sqrt{7}$ (B) $20 + 16\sqrt{7}$ (C) $28 + 24\sqrt{7}$ (D) $24 + 20\sqrt{7}$ (E) Other

15. If $\left\lfloor \frac{500+\frac{1}{n}}{0.05} \right\rfloor - \left\lfloor \frac{500}{0.05} \right\rfloor = 1$, then n must lie in the interval $(a, b]$, where a and b are real numbers. Compute $a + b$.

(A) 23 (B) 30 (C) 15 (D) 25 (E) Other

16. A capsule machine has 9 toys, where six are standard toys while three are rare toys. Ellie picks 5 toys at random from the capsule machine. What is the probability that she gets all the rare toys?

(A) $\frac{5}{9}$ (B) $\frac{1}{3}$ (C) $\frac{5}{42}$ (D) $\frac{1}{126}$ (E) Other

17. Given that $\cos x + \frac{1}{\cos x} = -2$, what is the value of $(\sin x)^{12} + (\cos x)^{18} + (\sin x \cdot \cos x)^{15}$? (It may help to note that $\sin^2 x + \cos^2 x = 1$ for all x.)

(A) 0 (B) -1 (C) -33344 (D) 33344 (E) Other

18. Sally enters two odd numbers into a calculator and computes their sum. She reveals that one of the odd numbers is divisible by 45 and the other number is divisible by 9. What is the largest number that must evenly divide the two odd numbers’ sum?

(A) 45 (B) 6 (C) 27 (D) 18 (E) Other

19. Given that $x + \frac{1}{x} = -\frac{5}{2}$, compute the greatest integer less than or equal to the least possible value of $x^7 + \frac{1}{x^9}$.

(A) -511 (B) -513 (C) -128 (D) -129 (E) Other

20. Triangle ABC has $AB = 6$, $BC = 7$, $CA = 8$. Transversal line $\overline{DE}$ is parallel to $\overline{AB}$, with D and E lying on $\overline{BC}$ and $\overline{CA}$ respectively. In addition, $DE = CD + CE - 3$. Compute the difference in perimeters between quadrilateral $ABDE$ and triangle DCE.

(A) 14 (B) 12 (C) 11 (D) 13 (E) Other

21. The quantity $\sqrt{48{,}400{,}000}$ can be written in the form $a\sqrt{b}$, where a and b are positive integers; this form is not necessarily the *simplest* radical form (in that b may contain the square of a prime number in its prime factorization). Compute the number of possibilities for the ordered pair (a, b).

(A) 40 (B) 28 (C) 24 (D) 32 (E) Other

22. Kana's class has some number of students, and she considers picking 0 or 1 student. The number of ways she can do so is less than 2% of the number of ways to pick *any* number of students (without respect to order). What is the smallest possible number of students in Kana's class?

(A) 8 (B) 10 (C) 11 (D) 12 (E) Other

23. Compute the value of $\log_{(9-4\sqrt{5})}(38 + 17\sqrt{5})$.

(A) $-\frac{3}{2}$ (B) $\frac{5}{3}$ (C) $\frac{4}{3}$ (D) $-\frac{5}{2}$ (E) Other

24. In the wave world, there are 7 countries connected by a number of wave roads. Each wave road links two countries, and the electric country is the country with 4 wave roads leading out, the most of all the countries. Georgia found that she can find a route only on wave roads such that she can start at one country, visit all other countries exactly once, and returning to the starting country, while only being at the starting country at the beginning and at the end. What is the least possible number of wave roads in this wave world?

(A) 8 (B) 7 (C) 9 (D) 11 (E) Other

25. In triangle $\triangle XYZ$, all three side lengths are distinct even integers. If the internal bisector of $\angle X$ intersects $\overline{YZ}$ at M and $\frac{YM}{MZ} = \frac{1}{6}$, compute the smallest possible perimeter of $\triangle XYZ$.

(A) 50 (B) 42 (C) 40 (D) 48 (E) Other

26. For how many positive integers $p \leq 1000$ is $p^{600} - 1$ a multiple of 1001?

(A) 720 (B) 600 (C) 360 (D) 840 (E) Other

27. Let x and y be positive real numbers such that $x + xy = 20y - y^2$ and $y + xy = 22x - x^2$. Compute $x + y$.

(A) $21 + \sqrt{2}$ (B) $21 - \sqrt{2}$ (C) $19 - \sqrt{2}$ (D) $20 - \sqrt{2}$ (E) Other

28. Parallelogram $ABCD$ has side lengths $AB = 22$ and $BC = 24$, and diagonal $AC = 28$. Point E is on $\overline{AB}$, and F is the intersection of $\overline{CE}$ and $\overline{BD}$. If $DF = 4BF$, what is the positive difference between the perimeters of $\triangle ACE$ and $\triangle BCE$?

(A) 14 (B) $\frac{25}{2}$ (C) 15 (D) $\frac{29}{2}$ (E) Other

29. If $\log_{10}(x) - 2023 = 0$, and $\log_{10}(x^x) - 2023 = k$, what is the sum of the digits of k?

(A) 18196 (B) 18207 (C) 18218 (D) 18217 (E) Other

30. Captain Joe's space cruiser is confined to the coordinate plane with integer x- and y-coordinates between 0 and 6, inclusive. He may only move between points using a revolutionary technique called *wormholing*, which takes him from one point to another such that his y-coordinate increases. Let N be the number of distinct ways Captain Joe can travel from $(0, 0)$ to $(6, 6)$ using only a series of wormholes. Find the remainder when N is divided by 1000.

(A) 776 (B) 768 (C) 401 (D) 807 (E) Other

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Problems 1 & 2

1. Steve brought 100 students to Kay's Cafe to get hamburgers for lunch on a field trip, which can either be gluten free or not gluten free. The regular hamburgers cost \$5.50, while the gluten free hamburgers cost \$7.50. However, one-fourth of the students have a gluten allergy, so they must get gluten free hamburgers. How many dollars is the positive difference between the minimum and maximum cost of giving lunch to all the participants?

1.

2. An integer m is chosen at random between -3 and 3, inclusive, and an integer b is chosen at random between -2 and 2, inclusive. What is the probability that the x-intercept of the line $y = mx + b$ exists and its x-coordinate is positive? Express your answer as a common fraction.

2.

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Problems 3 & 4

3. A cylinder has radius 13 and height 37. A plane perpendicular to the circular faces of the cylinder slices through the cylinder while passing through the centers of the circular faces. Compute the perimeter of the resulting cross section.

3.

4. Rebecca is doing a math problem where her answer is of the form $\frac{p}{50}$, where $\gcd(p, 50) = 1$ and p is a positive integer less than or equal to 50. She then needs to turn in a "converted" answer of $p + 50$. Compute the sum of all possible final "converted" answers.

4.

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Problems 5 & 6

5. Let m and c be integers chosen at random from the intervals $0 \leq m \leq 9$ and $0 \leq c \leq 399$. Compute the probability that a height of m meters and c centimeters is equivalent to a height of at least 8 meters. Express your answer as a common fraction.

5.

6. A plane figure composed of four equilateral triangles joined along 3 of their edges can be folded up along those edges into a regular tetrahedron. Given that the outer perimeter of this figure is 18, what is the square of the volume of the tetrahedron? Express your answer as a common fraction.

6.

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Problems 7 & 8

7. Let a and b be rational numbers such that $4^a 9^b = 648$ and $(6^a)^b = 216$. Compute $9^a 4^b$.

7.

8. A spider rests at $(0, 0)$ on the two-dimensional Cartesian plane. Each second, it moves either 1 unit in the positive x-direction or 1 unit in the positive y-direction. There are two barriers of infinite length along the lines with equations $y = x - 2.9$ and $y = x + 2.9$, which the spider cannot cross. The spider moves until it lands on the point $(45, 45)$, after which it stops. Let A be the number of unique paths the spider could take. Compute the remainder when A is divided by 100.

8.

Team Round 12300

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1.	2.	3.	4.	5.
6.	7.	8.	9.	10.

1. Abigail draws line segment $\overline{AB}$. She then draws a circle with radius AB and center A and a circle with radius AB and center B, and then she marks the two intersections of the circles as points C and D. She then draws $\overline{CD}$ and marks the intersection of $\overline{AB}$ and $\overline{CD}$ as point X. Compute the degree measure of $\angle AXC$.

2. Cindy thinks of a number. She writes down that number, the result when the number is rounded up to the nearest ten, and the result when the number is rounded up to the nearest hundred. Finally, she adds up the three written numbers and gets a result of 1737. What number was Cindy thinking of?

3. Let $f(x) = ax^2 + bx$ for some positive integer constants a and b. If $f(20) - f(23) = 20a - 23b$, compute $\frac{a}{b}$. Express your answer as a common fraction.

4. At a party with 20 people, 5 of them are introverted and the other 15 of them are extroverted. Each of the 15 extroverted people shakes hands exactly once with every other person. However, the introverted people do not take part in any additional handshakes. In total, how many handshakes take place?

5. Let n be a positive integer such that the sum of the digits of $1 + 2 + 3 + \cdots + n$ is a multiple of 9. Compute the sum of the nine smallest values of n.

6. Points X, Y, and Z lie in the coordinate plane with y-coordinates of 20, 19, and 10, respectively. Given that $\overline{XY} \perp \overline{YZ}$ and the difference between the x-coordinates of X is Y is 2, compute the area of $\triangle XYZ$. Express your answer as a common fraction.

7. In a soccer game, the teams do an unconventional penalty shoot-out, where each team has one of their players make 5 attempts to score goals. So far, Riley has scored one goal out of two attempts, while his opponent's team has scored 3 goals out of 5 attempts. In each of the remaining attempts, the probability that Riley scores a goal is $\frac{2}{3}$, and these attempts are independent from each other. Compute the probability that Riley scores at least as many goals as the opposing team over 5 attempts. Express your answer as a common fraction.

8. The polynomial $(x^2 + 2x + 2)(x^2 - 2x + 2)(x^2 + k)$ has 6 complex roots. When these roots are plotted in the complex plane, the points create a convex hexagon with area 14. Compute the sum of all possible real values of k.

9. Let x be a real number such that $\left(\cot\left(\frac{\pi}{2} - x\right)\right)^2 = 1$. Compute the number of possible values of x with an absolute value less than 100.

10. Define $d(n)$ to be equal to the number of positive divisors of an integer n. Let S be the set of all positive integers n such that the product of $d(n-2)$, $d(n)$, and $d(n+3)$ is equal to 36. Compute the sum of the six smallest elements of S.

Relay Round 12300

Relay 1

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Answer 1-1		Answer 1-2		Answer 1-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12300

(1-1) Let $T = 10$. Let K be the number of prime numbers p such that p is a factor of $(T+1)!$, but p^2 is not. Find the remainder when $K+11$ is divided by 100.

(1-2) Let $T = TNYWR$. Let x be the positive solution to the equation $x = \sqrt{(T+2)\sqrt{(T+3)\sqrt{(T+2)\sqrt{(T+3)x}}}}$. Let $K = x^3$. Find the remainder when $K+14$ is divided by 100.

(1-3) Let $T = TNYWR$. Let K be the number of integers in the domain of the function $f(x) = \sqrt{-2x} + \frac{1}{\sqrt{T-x^2}}$. Find the remainder when $K+13$ is divided by 100.

Relay Round 12300

Relay 2

Name ______________________

Name ______________________

Name ______________________

School ______________________

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Answer 2-1		Answer 2-2		Answer 2-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12300

(2-1) Let $T = 34$. Let $f(x)$ be a polynomial of the smallest possible positive degree with integer coefficients and a leading coefficient of 1 such that one of its roots is $x = (T+1)+\sqrt{T+2}$. Let K be the absolute value of the sum of the coefficients of this polynomial. Find the remainder when $K+36$ is divided by 100.

(2-2) Let $T = TNYWR$. Let N be the number of positive factors of $T+1$. Let K be the area of a rectangle with sides of length $T+1$ and N. Find the remainder when $K+70$ is divided by 100.

(2-3) Let $T = TNYWR$. A $T \times 2T$ rectangle is inscribed in a semicircle with one side on the diameter of the semicircle. The smallest possible area of the semicircle is $K\pi$. Find the remainder when $K+19$ is divided by 100.

Relay Round 12300

Relay 3

Name ______________________

Name ______________________

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School ______________________

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Answer 3-1		Answer 3-2		Answer 3-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12300

(3-1) Let $T = 64$. Let K be the $(T+1)$st digit after the decimal point in the decimal expansion of $\frac{1}{17}$. Find the remainder when $K + 92$ is divided by 100.

(3-2) Let $T = TNYWR$. Then $\sin(30T^\circ) + \cos(30T^\circ) = \frac{a+\sqrt{b}}{c}$, where a, b, and c are integers (with b nonnegative), and $|c|$ is as small as possible. Let $K = a + b + c$. Find the remainder when $K + 9$ is divided by 100.

(3-3) Let $T = TNYWR$. Let $V = 2T + 1$. The integers $\{1, 2, \ldots, V^2\}$ are placed in the cells of a $V \times V$ checkerboard, one integer per cell. Each row, column, and diagonal is then added up, and the sums are themselves added up. Let M be the greatest possible value of the result. Let K be the remainder when M is divided by V. Find the remainder when $K + 23$ is divided by 100.

Relay Round 12300

Relay 4

Name ______________________

Name ______________________

Name ______________________

School ______________________

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Answer 4-1		Answer 4-2		Answer 4-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12300

(4-1) Let $T = 20$. Let K be the greatest composite number less than $4(T+2)^2$. Find the remainder when $K + 69$ is divided by 100.

(4-2) Let $T = TNYWR$. The graph of the equation $x^2 - 4T^2 = 2(x+2T)(y-5T)$ consists of two lines in the coordinate plane. Let K be the y-coordinate of the point where these two lines intersect. Find the remainder when $K + 31$ is divided by 100.

(4-3) Let $T = TNYWR$. Let U be the sum of all $T+1$-digit palindromes, and let K be the remainder when U is divided by 100. Find the remainder when $K + 58$ is divided by 100.

Relay Round 12300

Relay 5

Name ______________________

Name ______________________

Name ______________________

School ______________________

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Answer 5-1		Answer 5-2		Answer 5-3	
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6
7	7	7	7	7	7
8	8	8	8	8	8
9	9	9	9	9	9

Be sure to fill in your answer to each question by fully darkening the appropriate number bubbles in the area provided. You may also write the digits in the boxes above the number bubbles, but in the event of a discrepancy what is bubbled in will count as your official answer.

Relay Round 12300

(5-1) Let $T = 38$. Let K be the coefficient of the x^2 term in the expansion of $(1 + 0x)(1 + x)(1 + 2x)(1 + 3x)\cdots(1 + Tx)$. Find the remainder when $K + 65$ is divided by 100.

(5-2) Let $T = TNYWR$. Let $f(x) = |200 - 2x|$, and let $K = f^{T+1}(T)$ (in other words, start with T and apply the function $T + 1$ times). Find the remainder when $K + 23$ is divided by 100.

(5-3) Let $T = TNYWR$. A car averages 60 miles per hour on a trip from Kansas City to Omaha and N miles per hour on the return trip, with an average speed for the entire round trip of $T + 1$ miles per hour. Let $K = a + b$, where a and b are relatively prime positive integers such that $N = \frac{a}{b}$. Find the remainder when $K + 63$ is divided by 100.

Answers 12300

Sprint Round

1. B	11. C	21. C
2. A	12. E	22. E
3. E	13. C	23. A
4. A	14. A	24. C
5. B	15. B	25. A
6. D	16. C	26. A
7. E	17. E	27. B
8. B	18. D	28. C
9. D	19. B	29. B
10. C	20. C	30. B

Target Round

1. 150
2. $\frac{12}{35}$
3. 126
4. 1500
5. $\frac{7}{20}$
6. $\frac{81}{8}$
7. 432
8. 62

Team Round

1. 90
2. 567
3. $\frac{20}{149}$
4. 180
5. 220
6. $\frac{45}{4}$
7. $\frac{20}{27}$
8. 0
9. 128
10. 451

Relay Round

(1-1)	13	(K = 2)	(1-2)	14	(K = 3600)	(1-3)	17	(K = 4)
(2-1)	76	(K = 40)	(2-2)	78	(K = 308)	(2-3)	3	(K = 6084)
(3-1)	92	(K = 0)	(3-2)	11	(K = 2)	(3-3)	45	(K = 22)
(4-1)	4	(K = 1935)	(4-2)	43	(K = 12)	(4-3)	58	(K = 0)
(5-1)	96	(K = 265031)	(5-2)	11	(K = 88)	(5-3)	86	(K = 23)

1. (a) For 1 to 100 there are $\boxed{192}$ total digits. [2 points]

   (b) For 1 to 1000 there are $\boxed{2893}$ total digits. [3 points]

   (c) For 1 to 10000 there are $\boxed{38894}$ total digits. [5 points]

   All answers are obtained from the formula derived in question 9.

2. (a) For 1 to 100 there are $\boxed{11}$ total zeros. [2 points]

   (b) For 1 to 1000 there are $\boxed{192}$ total zeros. [3 points]

   (c) For 1 to 10000 there are $\boxed{2893}$ total zeros. [5 points]

   All answers are obtained from the formula derived in question 9.

3. The total number of digits is $\boxed{n(10^n+1)-\dfrac{10^n-10}{9}}$. See question 4 for more detail.

4. See question 9 for a more general proof; this formula is then derived by substituting $k=10$ into the formula for the total number of digits.

5. The total number of zeros is $\boxed{(n-1)(10^{n-1}+1)-\dfrac{10^{n-1}-10}{9}}$. This expression can be obtained either by substituting $k=10$ into the relevant formula in question 9 or by applying the theorem in question 10 to the formula in question 3.

6. There are $\boxed{n(10^{n-1})}$ copies of each of the digits 2 through 9 and $\boxed{n(10^{n-1})+1}$ copies of the digit 1. See question 7 for the derivation.

7. From question 3 there are $n(10^n+1)-\frac{10^n-10}{9}$ total digits, and from question 5 there are $(n-1)(10^{n-1}+1)-\frac{10^{n-1}-10}{9}$. This leaves $n(10^n+1)-\frac{10^n-10}{9}-(n-1)(10^{n-1}+1)+\frac{10^{n-1}-10}{9}=9n(10^{n-1})+1$ nonzero digits. Let x be the number of times the digit 2 appears; by symmetry the digits 3 through 9 each appear x times as well. The digit 1 appears $x+1$ times though, because its single appearance in 10^n gets added to the x appearances it has in common with the other digits. There are thus $9x+1$ nonzero digits, and when we equate that with $9n(10^{n-1})+1$ from above we get $x=n(10^{n-1})$ so there are $n(10^{n-1})$ copies of each of the digits 2 through 9 and $n(10^{n-1})+1$ copies of the digit 1.

8. Out of a total of $n(2^n+1)-2^n+2$ digits, $\boxed{(n-1)(2^{n-1}+1)-2^{n-1}+2}$ of them are zeros and therefore $\boxed{n(2^{n-1})+1}$ of them are ones. The number of total digits and zeros are obtained from the formulas derived in question 9.

9. First, let's calculate the total number of digits in the base-k written representation of all integers from 1 to k^n. Note that there is a single number (k^n) with $n+1$ digits, and for each number j from 1 to n inclusive there are k^j-k^{j-1} base-k numbers with j digits (the largest number with j digits is k^j-1 and the smallest number with j digits is k^{j-1}). Summing over all possible values of j gives us $S=\sum_{j=1}^{n} j(k^j-k^{j-1})$. Now to

get a closed form we take $kS = \sum_{j=1}^{n} j(k^{j+1} - k^j) = \sum_{j=2}^{n+1} (j-1)(k^j - k^{j-1}) = \sum_{j=2}^{n+1} j(k^j - k^{j-1}) - \sum_{j=2}^{n+1} (k^j - k^{j-1})$.

Note that the second half of that last sum telescopes to $\sum_{j=2}^{n+1} (k^j - k^{j-1}) = k^{n+1} - k$. Now we have $S - kS = \sum_{j=1}^{n} j(k^j - k^{j-1}) - \sum_{j=2}^{n+1} j(k^j - k^{j-1}) + k^{n+1} - k$ which simplifies to $1(k^1 - k^{1-1}) - (n+1)(k^{n+1} - k^{n+1-1}) + k^{n+1} - k$ because all but two terms in the sums cancel out. Simplifying further we get $S - kS = n(1-k)(k^n + 1) + k^n - k$. Dividing both sides by $(1-k)$ and adding $n+1$ for the number of digits in k^n (remember S only calculates the number of digits for numbers that have up to n digits) we get the total number of digits is $S + (n+1) = n(k^n + 1) - \frac{k^n - k}{k-1}$.

Next we calculate the total number of zeros in the base-k written representation of all integers from 1 to k^n. We proceed by casework on the number of digits. Remember from above that for each number j from 1 to n inclusive there are $k^j - k^{j-1}$ base-k numbers with j digits, giving us a total of $j(k^j - k^{j-1})$ digits among the numbers with j digits. Of these digits, $k^j - k^{j-1}$ are leading digits (each of the $k^j - k^{j-1}$ j-digit numbers has one leading digit), leaving the total number of non-leading digits to be $j(k^j - k^{j-1}) - k^j + k^{j-1} = (j-1)(k^j - k^{j-1})$. By symmetry, each of the k distinct digits will show up an equal number of times, so we divide by k to get the number of zeros among the j-digit numbers (again, for j from 1 to n inclusive; we'll add the $n+1$ digit number later) to be $(j-1)(k^{j-1} - k^{j-2})$. Summing over all possible values of j gives us $S = \sum_{j=1}^{n} (j-1)(k^{j-1} - k^{j-2})$, which should look surprisingly similar to our calculation of the total number of digits (keep that in mind for question 10). $S = \sum_{j=0}^{n-1} j(k^j - k^{j-1})$ by reindexing, and $S = \sum_{j=1}^{n-1} j(k^j - k^{j-1})$ because the $j = 0$ term is 0. Using the same techniques we used in the previous paragraph, we get $S = (n-1)k^{n-1} - \frac{k^{n-1}-1}{k-1}$, and when we add the n zeros from the $n+1$-digit number k^n we get a total number of zeros of $\boxed{(n-1)(k^{n-1} + 1) - \frac{k^{n-1} - k}{k-1}}$.

Finally we calculate the total number of each digit from 1 to $k-1$. Note that there are $n(k^n + 1) - \frac{k^n - k}{k-1}$ total digits and $(n-1)(k^{n-1} + 1) - \frac{k^{n-1}-k}{k-1}$ zeros for a total of $n(k^n + 1) - \frac{k^n - k}{k-1} - (n-1)(k^{n-1} + 1) + \frac{k^{n-1}-k}{k-1} = (k-1)n(k^{n-1}) + 1$ nonzero digits. If $k = 2$ then this is the number of ones and we are done. If $k > 2$, let x be the number of times the digit 2 appears; by symmetry each of the digits greater than 1 appear x times as well. The digit 1 appears $x+1$ times though, because its single appearance in k^n gets added to the x appearances it has in common with the other digits. There are thus $(k-1)x + 1$ nonzero digits, and when we equate that with $(k-1)n(k^{n-1}) + 1$ from above we get $x = n(k^{n-1})$ so there are $\boxed{n(k^{n-1})}$ copies of each of the digits greater than 1 and (regardless of whether $k = 2$ or not) $\boxed{n(k^{n-1}) + 1}$ copies of the digit 1.

10. We seek to prove that the number of zeros used to write all integers in base k from 1 to k^n equals the number of total digits used to write all integers in base k from 1 to k^{n-1}. As derived in question 9, the number of zeros used to write all integers in base k from 1 to k^n is $(n-1)(k^{n-1} + 1) - \frac{k^{n-1}-k}{k-1}$ and the number of total digits used to write all integers in base k from 1 to k^{n-1} is $(n-1)(k^{n-1} + 1) - \frac{k^{n-1}-k}{k-1}$. As these two expressions are equal, the conjecture is true for all values of n.

Solutions 12300

Sprint Round Solutions

1. The 3D figure is a cube with a side length of 2, so the volume is $2 \cdot 2 \cdot 2 = \boxed{8}$.

2. One way to solve is by listing the numbers – 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. Another way is note that if we add one to each of the problem numbers, we get the list of even numbers from 2 to 20 inclusive. Either way, we find that Kyle must do $\boxed{10}$ problems for homework.

3. Each monster needs to have an ID number and a qualifying power level in order to participate. By checking the entries while making sure we read the instructions, we find that Blasty, Big Ounce, Holmes, and Sarpal have a qualifying ID number, but only Blasty and Holmes also have a qualifying power level. Therefore, only $\boxed{2}$ of Andrew's monsters can participate in the cup.

4. Solving this problem requires considering the definition of each term as well as the angle properties of each. The first description is not possible as parallelograms can only have at most 2 obtuse angles (we can see this when we attempt to sketch one). The fourth description is not possible as all rectangles by default have four right angles. All other descriptions can be sketched by hand – a rhombus with four right angles (which is a square), a trapezoid with two right angles, and another trapezoid for the last description. In summary, $\boxed{\text{I and IV}}$ can not be made.

5. The total charge of the cluster is the sum of the charges of all of the individual molecules in the cluster. The histidine residues and water molecule are neutral, so they do not contribute any charge. Therefore, the total charge is $2 - 1 = \boxed{1}$.

6. Chelina has to read $100 - 13 + 1 = 88$ pages in four days, so she needs to read $\frac{88}{4} = \boxed{22}$ pages every day in order to reach the goal.

7. Either there is 1 side of length 5 and 3 sides of length 22, or 3 sides of length 5 and 1 side of length 22. The latter case is impossible due to the Triangle Inequality, as $5 + 5 + 5 < 22$, but the former case is possible. So the perimeter is $5 + 22 + 22 + 22 = \boxed{71}$.

8. We can solve the original equation to get $x = 17$. If we add a positive integer a to the right hand side, then the new equation is $0.11x + 0.13 = 2 + a$, which has solution $x = \frac{1.87+a}{0.11} = 17 + \frac{100a}{11}$. For x to increase by an integer amount, $\frac{100a}{11}$ must be a positive integer, so the smallest positive value of a is $\boxed{11}$.

9. The two side lengths of the rectangle multiply to 25, so either both of them are 5, or one of them is 1 and the other is 25. However, a square has perpendicular diagonals, so we eliminate the first possibility. So, the perimeter of the rectangle is $2(1+25) = \boxed{52}$.

10. From $f(1) = 3$, we obtain $a + b = 3$, and from $f(2) = 4$, we obtain $4a + 2b = 4$. Multiplying the first equation by 4 gives us $4a + 4b = 12$, and subtracting the second equation from this new equation gives us $2b = 8$. Dividing both sides by 2 gives our final answer of $b = \boxed{4}$.

11. The flight can take no longer than $9 - 3 = 6$ hours, so it must travel at least $\frac{2700}{6} = \boxed{450}$ miles per hour.

12. For each of the three fruits, we can choose to include it or not, for a count of $2^3 = 8$ different drinks. We subtract off the one case where none of the fruits are included (which results in plain water) to get a total count of $8 - 1 = \boxed{7}$ different kinds of infused drinks.

13. By multiplying the polynomial, we find that $\left(4x^{13} - 6x^{45}\right)\left(7x^{36} + 10x^{23}\right) = 28x^{49} + 40x^{36} - 42x^{81} - 60x^{68}$, and there is only one term where the exponent of x is greater than 80. In order for $\left(4x^{13} - 6x^{45}\right)\left(7x^{36} + 10x^{23}\right) + ax^b$ to have degree less than 80, we must have $-42x^{81} + ax^b = 0$. Then $a = 42$ and $b = 81$, so $100a + b = \boxed{4281}$.

14. The lateral surface area is the sum of the two left/right rectangular faces (with dimensions of the tent's slant height and the base length), the front/back triangular faces, and the base. The base has area 28 (and hence side length $2\sqrt{7}$), with the slant height being 4 by the Pythagorean Theorem. Then each left/right face has area $4 \cdot \sqrt{28}$; doubling that gives $16\sqrt{7}$. Finally, the front and back faces add to $6\sqrt{7}$, giving a total surface area of $\boxed{28 + 22\sqrt{7}}$.

15. The difference simplifes to $\left\lfloor \frac{20}{n} \right\rfloor = 1$, which is satisfied whenever $1 \leq \frac{20}{n} < 2$, or $10 < n \leq 20$. Then $a = 10$ and $b = 20$, so $a + b = \boxed{30}$.

16. There are $\binom{9}{5} = 126$ ways in total to choose the toys. If Ellie gets all the rare toys, then the remaining two toys must be standard toys, and there are $\binom{6}{2} = 15$ ways to choose those. In total, the probability is $\boxed{\dfrac{5}{42}}$.

17. Note that $(\cos(x))^2 + 2\cos(x) + 1 = 0$, so $\cos(x) = -1$. This means that $\sin(x) = 0$, so $0^{12} + (-1)^{18} + (0 \cdot 1)^{15} = \boxed{1}$.

18. The sum of two odd numbers must be even, and thus a multiple of 2. Additionally, since both odd numbers are divisible by 9, their sum is also divisible by 9, so it is a multiple of 18. To show that 18 is the largest number that must evenly divide the sum, we consider the sums $45 + 9 = 54$ and $45 + 27 = 72$, and since the greatest common divisor of 54 and 72 is 18, we conclude that $\boxed{18}$ is the largest possible number that must divide the sum.

19. The equation $x+\frac{1}{x}=-\frac{5}{2}$ can be multiplied through by x to obtain $x^2+\frac{5}{2}x+1=0$, i.e. $2x^2+5x+2=0$, or $x=\frac{-5\pm3}{4}=-\frac{1}{2},-2$. The smaller value of $x^7+\frac{1}{x^9}$, then, is $-\frac{1}{128}-512$, whose floor is $\boxed{-513}$.

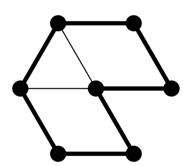
20. Note that $\triangle DCE$ is similar to $\triangle ABC$, so given that $AB=\frac{2}{5}(BC+CA)$, $CD+CE=\frac{DE}{6}\cdot 15$. Hence, $DE=\frac{5}{2}DE-3$, or $DE=2$. It follows that $\triangle DCE$ is simply $\triangle ABC$ scaled down by a factor of 3. Thus, the perimeter of $ABDE$ is 18 and the perimeter of DCE is 7, so the answer is $\boxed{11}$.

21. Squaring both sides, we get that $a^2b=48,400,000=2200^2\cdot 10$. Note that every perfect square factor of $2200^2\cdot 10$ corresponds to a solution where a^2 is that factor. So, we want to find the number of perfect square factors of $2200^2\cdot 10=(11\cdot 5^2\cdot 2^3)^2\cdot 2\cdot 5=11^2\cdot 5^5\cdot 2^7$, which is $\left(\frac{2}{2}+1\right)\left(\frac{4}{2}+1\right)\left(\frac{6}{2}+1\right)=\boxed{24}$.

22. Let n be the number of students. The number of ways to pick 0 or 1 student is just $\binom{n}{0}+\binom{n}{1}=1+n$, while the total is 2^n (this is the well known sum of binomial coefficients identity; provable by a direct application of the Binomial Theorem). Hence $50(1+n)<2^n$, or $n\geq\boxed{9}$.

23. Note that $9-4\sqrt{5}=(\sqrt{5}-2)^2$. Also note that $\frac{38+17\sqrt{5}}{2+\sqrt{5}}=9+4\sqrt{5}=(2+\sqrt{5})^2$ by radical conjugation, so $38+17\sqrt{5}=(2+\sqrt{5})^3$. Since $(\sqrt{5}-2)(\sqrt{5}+2)=1$, we have $9-4\sqrt{5}=a^2$ and $38+17\sqrt{5}=\frac{1}{a^3}$, where $a=\sqrt{5}-2$. Thus, $\log_{a^2}a^{-3}=\boxed{-\dfrac{3}{2}}$.

24. We can consider a graph where there are 7 vertices corresponding to the 7 countries and the edges correspond to wave roads. Georgia's route can be considered a *Hamiltonian cycle*, which is a cycle containing every vertex, without repeating any edges or vertices. This cycle would have 7 edges, and normally, the degree of each vertex in a cycle would be 2. However, the electric country has a total of 4 wave roads leading out, which correspond to 4 edges. This means there are 2 unused edges, so the graph must have at least $7+2=\boxed{9}$ edges. Below is a graph where the bolded lines can represent the path.



25. By the angle bisector theorem, $\frac{XY}{XZ}=\frac{1}{6}$. If $XY=2$, then $XZ=12$, and by the triangle inequality, $10<YZ<14$. However, the question states that the side lengths must be distinct even integers, so YZ cannot be 12. We must scale up the lengths of XY and XZ by a factor of 2, which then allows $YZ=22$. Thus, the smallest possible perimeter of the triangle is $4+24+22=\boxed{50}$.

26. By the Chinese Remainder Theorem, we require that $p^{1000}\equiv 1 \bmod 7, 1 \bmod 11, 1 \bmod 13$. By Fermat's Little Theorem, we know that, if p is relatively prime to 7, then $p^6\equiv 1 \bmod 7$, and similarly for $p^{10}\equiv 1 \bmod 11$ and $p^{12}\equiv 1 \bmod 13$. Then $p^{\text{lcm}(6,10,12)}=p^{60}\equiv p^{600}\equiv 1 \bmod 1001$, so long as p is not a multiple of 7, 11, or 13. The number of such integers under 1000 is given by $\phi(1001)=1001\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}=\boxed{720}$.

27. Divide the first and second equations by y and x respectively to get that $\frac{x+xy}{y} = \frac{x}{y} + x = 20 - y$ and $\frac{y+xy}{x} = \frac{y}{x} + y = 22 - x$. Then note that $\frac{x}{y} = 20 - x - y = 20 - (x+y)$ and $\frac{y}{x} = 22 - x - y = 22 - (x+y)$. We have $\frac{y}{x} - \frac{x}{y} = 2$; let $\frac{x}{y} = a$. Then $\frac{y}{x} = \frac{1}{a}$, so we obtain the equation $\frac{1}{a} = a + 2$, which becomes the quadratic equation $a^2 + 2a - 1 = 0$. Using the quadratic formula, we get $a = -1 \pm \sqrt{2}$, but recall that a is positive since x and y are both positive. It follows that $a = -1 + \sqrt{2}$, so $x + y = 20 - a = 20 - (-1 + \sqrt{2}) = \boxed{21 - \sqrt{2}}$.

28. Notice that $\angle BEF \cong \angle FCD$ and $\angle EBF \cong \angle CDF$ because they are alternate interior angles, so $\triangle EBF \sim \triangle CDF$ by AA Similarity. Since $FD = 4 \cdot BF$, we have $22 = CD = 4BE$, so $BE = \frac{11}{2}$. Then $AE = AB - BE = 22 - \frac{11}{2} = \frac{33}{2}$. Now let $EC = x$. The perimeter of $\triangle ACE$ is $\frac{33}{2} + 28 + x$, and the perimeter of $\triangle BCE$ is $\frac{11}{2} + 22 + x$, so the positive difference between the perimeters is $\frac{33}{2} - \frac{11}{2} + 28 - 24 + x - x = \boxed{15}$.

29. From the first equation, we have $x = 10^{2023}$, so $\log_{10}(x^x) = 10^{2023}\log_{10}(10^{2023}) = 2023 \cdot 10^{2023}$ and $k = 2023 \cdot (10^{2023} - 1)$. Noting that $10^n - 1$ in base 10 is just n nines, we obtain that the digits of k consist of 2022, followed by 2019 nines, followed by 7977. The sum of digits of k is therefore $6 + 9 \cdot 2019 + 30 = 9 \cdot 2023 = \boxed{18207}$.

30. For each "row" of points which lie on the lines $y = 1$, $y = 2$, $y = 3$, $y = 4$, and $y = 5$, there are 8 ways to either choose a point in the row of points, or ignore the row altogether. Each combination of choices leads to a set of points which creates a unique path from $(0, 0)$ to $(6, 6)$ using wormholes, so the number of ways is $8^5 = 32768$. Thus, the answer is $\boxed{768}$.

Target Round Solutions

1. The minimum cost comes from buying $\frac{1}{4}$ gluten-free subs and $\frac{3}{4}$ regular subs, which is $25 \cdot \$7.50 + 75 \cdot \$5.50$. The maximum cost comes from buying all gluten-free subs, which is $100 \cdot \$7.50$. The positive difference is then $|100 \cdot \$7.50 - (25 \cdot \$7.50 + 75 \cdot \$5.50)| = 75 \cdot \$7.50 - 75 \cdot \$5.50 = 75 \cdot \$2 = \boxed{\$150}$.

2. The x-coordinate of the x-intercept is $-\frac{b}{m}$, where $m \neq 0$. In order for this value to be positive, one of b and m must be negative and the other positive. If $b < 0 < m$, then there are $2 \cdot 3 = 6$ options. If $m < 0 < b$, then there are $3 \cdot 2 = 6$ options. As there are $7 \cdot 5 = 35$ total ways to pick b and m, the probability is $\boxed{\dfrac{12}{35}}$.

3. The cross section of the cylinder would be a rectangle. The length of the rectangle is the height of the cylinder, which is 37, and the width of the rectangle is the diameter of the circular face, which is $13 \cdot 2 = 26$. Therefore, the perimeter of the cross section is $2 \cdot (37 + 26) = \boxed{126}$.

4. The possible values of p are those not sharing a common factor larger than 1 with 50, meaning those that are multiples of neither 2 nor 5. The sum of the multiples of 2 from 1 to 100 is $2 + 4 + 6 + \cdots + 50 = 650$, and the sum of the multiples of 5 from 1 to 100 is $5 + 10 + 15 + \cdots + 50 = 275$. The multiples of 10 (which sum to 150) must be subtracted, however, giving a sum of $650 + 275 - 150 = 775$ of the numbers from 1 to 100 that are multiples of 2 or 5. It follows that the sum of all possible p (the numbers that are *not* multiples of 2 or 5) is $1275 - 775 = 500$. To count the number of values of p, we use Euler's totient function on 100, to get $\phi(50) = 50\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 20$. Thus, there are 20 values of p, meaning we sum 20 different values of $p + 50$. The p's sum to 500, and the 50's sum to $20 \cdot 50 = 1000$, so the total sum is $500 + 1000 = \boxed{1500}$.

5. The height of m meters and c centimeters is equal to $100m + c$ centimeters, which must be at least 800 centimeters. This is only possible if $m \geq 5$. If $m = 5$, $c \geq 300$; if $m = 6$, $c \geq 200$; if $m = 7$, $c \geq 100$; and if $8 \leq m \leq 9$, c can be anything. Hence the number of choices is $100 + 200 + 300 + 400 + 400 = 1400$, out of 4000 total, for a probability of $\boxed{\dfrac{7}{20}}$.

6. The tetrahedron's net consists of 6 segments around the perimeter whose lengths add up to 18, so the tetrahedron's side length is 3. Then its volume is $\frac{1}{6\sqrt{2}} \cdot 3^3 = \frac{9\sqrt{2}}{4}$ (to derive this volume, calculate the area of one equilateral triangle face, then calculate the tetrahedron's vertical height using the Pythagorean theorem. The volume is equal to one-third of the product of the base and height). The square of the volume is $\boxed{\dfrac{81}{8}}$.

7. Since a and b are rational, the exponents in $4^a 9^b$ must match the exponents in the prime factorization of 648. We have $4^a 9^b = 648 = 2^3 \times 3^4 = 4^{\frac{3}{2}} 9^2$, so $a = \frac{3}{2}$ and $b = 2$. These two values also satisfy $(6^a)^b = (6^{\frac{3}{2}})^2 = 6^3 = 216$, therefore, $9^a 4^b = 9^{\frac{3}{2}} 4^2 = 27 \times 16 = \boxed{432}$.

8. Let us calculate the number of paths from $(0,0)$ to $(0,1)$ and $(1,0)$, then $(1,2)$ and $(2,1)$, and in general, $(k,k+1)$ and $(k+1,k)$. There is one path to each of $(0,1)$ and $(1,0)$. From there, there are three ways to get from one of $(0,1)$ or $(1,0)$ to $(1,2)$ or $(2,1)$. In general, there are three ways to get from one of $(k-1,k)$ or $(k,k-1)$ to $(k,k+1)$ or $(k+1,k)$. Each step triples the number of paths from $(0,0)$ to the next points. So the number of paths from $(0,0)$ to $(44,45)$ is 3^{44}, and the number of paths from $(0,0)$ to $(45,44)$ is 3^{44}, so the number of paths from $(0,0)$ to $(100,100)$ is $2 \cdot 3^{44}$. By Euler's Totient Theorem $3^{40} \equiv 1 \bmod 100$, so $2 \cdot 3^{44} \equiv 2 \cdot 3^4 \equiv 2 \cdot 81 \equiv \boxed{62} \bmod 100$.

Team Round Solutions

1. Abigail is essentially doing a geometric construction where she is drawing the perpendicular bisector of $\overline{AB}$. To show that $\overline{CD}$ is indeed a perpendicular bisector, we find that $\triangle ABC$ and $\triangle ABD$ are equilateral triangles, making $\triangle CAD$ and $\triangle CBD$ both isosceles triangles and $\angle CAD = \angle CBD = 120°$. In fact, from SAS Congruency, $\triangle CAD \cong \triangle CBD$, so $\angle ACD = \angle BCD = 30°$. Then from SAS Congruency, $\triangle ACX \cong \triangle BCX$, so $\angle AXC = \angle BXC$. As AB is a line, we find that $\angle AXC = \boxed{90}^\circ$.

2. Since $1737 < 3 \cdot 1000$, Cindy's number is at most three digits long. Let Cindy's number be $\overline{ABC}$, where A, B, and C are digits. Then $\overline{ABC} + \overline{A(B+1)0} + \overline{(A+1)00} = 300A + 100 + 20B + 10 + C = 300A + 20B + C + 110 = 1737$. Since $300A + 20B + 110$ is divisible by 10, $C = 7$. Plugging in and dividing both sides by 20 gives $15A + B = 81$. Now we can divide 81 by 15 to obtain A and B, as A is the quotient and B is the remainder (since B is a digit and thus less than 15). The quotient is 5 and the remainder is 6, so $A = 5$ and $B = 6$. Therefore, Cindy's number is $\boxed{567}$.

3. We have $f(20) = 400a + 20b$ and $f(23) = 529a + 23b$, so $f(20) - f(23) = -129a - 3b$. Setting this equal to $20a - 23b$ and simplifying yields $149a = 20b$, or $\frac{a}{b} = \boxed{\dfrac{20}{149}}$.

4. In the crowd of 15 extroverted people, there are $\binom{15}{2} = 105$ handshakes. Between the crowd of 15 extroverted people and the crowd of 5 introverted people, there are $15 \cdot 5 = 75$ handshakes, but none amongst the introverted group. This makes the total $105 + 75 = \boxed{180}$ handshakes.

5. Note that the sum of the digits of $1 + 2 + 3 + \cdots + n$ is a multiple of 9 if and only if $1 + 2 + 3 + \cdots + n$ is a multiple of 9, so we use this condition instead. Writing $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, we have $n(n+1) \equiv 0 \bmod 9$, so we have $n \equiv 0, 8 \bmod 9$. Thus, the nine smallest possible values of n are $8, 9, 17, 18, 26, 27, 35, 36, 44$. Summing these yields $\boxed{220}$.

6. Mark points A and B such that A and B share the same x-coordinate as X and Z, respectively, and A and B share the same y-coordinate as Y. Observe that $\angle XYA + \angle ZYB = 90^\circ$ and $\angle YZB + \angle ZYB = 90^\circ$, so $\angle XYA \cong \angle YZB$. By AA Similarity, $\triangle XYA \sim \triangle YZB$. Then, since $\frac{XA}{YA} = \frac{YB}{ZB} = 2$, we have that $YB = 9$ and $ZB = \frac{9}{2}$. By the Pythagorean Theorem, $XY = \sqrt{5}$ and $YZ = \frac{9}{2}\sqrt{5}$, so the area of $\triangle XYZ$ is $\frac{1}{2} \cdot \sqrt{5} \cdot \frac{9}{2}\sqrt{5} = \boxed{\frac{45}{4}}$.

7. In order for Riley's team to not have less goals than his team's opponent, he must score at least 2 goals of the 3 attempts left. The probability that all of the remaining attempts are successful is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$. As for the probability that Riley makes exactly 2 goals of the 3 attempts left, there are 3 ways that can happen (based on when he misses), so the probability that Riley makes exactly 2 goals after 3 attempts is $3 \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{9}$. Therefore, the probability that Riley's team does not have less goals than his opponent's team in the end is $\frac{4}{9} + \frac{8}{27} = \boxed{\frac{20}{27}}$.

8. Using the quadratic formula, we find that the roots are $x = \pm i \pm 1, \pm\sqrt{-k}$ if k is negative and $x = \pm i \pm 1, \pm i\sqrt{k}$ if k is positive. Regardless of the sign of k, the hexagon is made up of a square with side length 2 and two triangles that both have base 2 and height $\sqrt{\pm k} - 1$, whichever sign makes the radicand positive. The area of the hexagon is then $4 + 2(\sqrt{\pm k} - 1) = 14$, and solving, $k = \pm 36$. The sum of these possible values of k is $-36 + 36 = \boxed{0}$.

9. Since $\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$, we have $\tan(x) = \pm 1$, so $x = \pi k + \frac{\pi}{4}, \pi k + \frac{3\pi}{4}$, for k an integer. Thus, there are 2 values of x for every increment of π. As $31\pi \leq 100 < 32\pi$, there are 124 x-values between -31π and 31π, plus an additional 4 values, since $\left(31 + \frac{3}{4}\right)\pi < 100$. In total, we have $\boxed{128}$ values.

10. Since $36 = 2^2 \cdot 3^2$, one of $d(n-2)$, $d(n)$, or $d(n+3)$ is odd, which means one of $n-2$, n, or $n+3$ is a square. It is helpful to note that $d(n-2)$, $d(n)$, and $d(n+3)$ equal (in some order) 3, 3, 4 or 2, 2, 9 or 2, 3, 6. It is also helpful to note that $d(k) = 2$ if and only if k is prime and $d(k) = 3$ if and only if k is a square of a prime. For the 3, 3, 4 case, note that the perfect squares must differ by 2, 3, or 5, and so the only value of n that works is if $n = 6$. For the 2, 2, 9 case, since there are two primes, we must have $n+3$ be an even perfect square, and we find that $n = 193$ can work. For the 2, 3, 6 case, the perfect square must be of a prime, and should the perfect square be odd, it can only equal $n-2$ or n since $n-2$ and n have the opposite parity of $n+3$. Then we can find that the other values of n less than 193 that work are $9, 25, 49, 169$. Thus, the sum of the first six values of n is $6+9+25+49+169+193 = \boxed{451}$.

Relay Round Solutions

(1-1) If $p > T+1$, then p does not divide $(T+1)!$ because $(T+1)!$ does not have any factors of p. If $p \leq \frac{T+1}{2}$, then we also have $2p \leq T+1$, so $(T+1)!$ would be divisible by p^2. However, if $\frac{T+1}{2} < p \leq T+1$, then p does divide $(T+1)!$ but not p^2 because $2p > (T+1)!$. Therefore, K is the number of primes greater than $\frac{T+1}{2}$ and less than or equal to $T+1$, and K can be manually counted since $T+1 \leq 100$.

(1-2) Converting the square roots to rational exponents, we have $x = (T+2)^{\frac{1}{2}}(T+3)^{\frac{1}{4}}(T+2)^{\frac{1}{8}}(T+3)^{\frac{1}{16}}x^{\frac{1}{16}} = (T+2)^{\frac{5}{8}}(T+3)^{\frac{5}{16}}x^{\frac{1}{16}}$. Then raising both sides to the 16th power results in $x^{16} = (T+2)^{10}(T+3)^5x$. Moving everything to the left side and factoring yields $x(x^{15} - (T+2)^{10}(T+3)^5) = 0$. Since $x > 0$, $x = ((T+2)^{10}(T+3)^5)^{\frac{1}{15}} = \sqrt[3]{(T+2)^2(T+3)}$, and $K = x^3 = (T+2)^2(T+3)$.

(1-3) We will analyze each function's domain intervals in $f(x)$ separately and take the intersection of these intervals to obtain the domain of $f(x)$. The domain of $\sqrt{-2x}$ is $(-\infty, 0]$, and the domain of $\frac{1}{\sqrt{T-x^2}}$ is $(-T, T)$. This means that the domain of $f(x)$ is $(-\infty, 0] \cap (-T, T) = (-T, 0]$. If T is not a perfect square, there are $K = \left\lfloor \sqrt{T} \right\rfloor - 0 + 1 = \left\lfloor \sqrt{T} \right\rfloor + 1$ integers within this domain. Otherwise, there are $K = \left(\left\lfloor \sqrt{T} \right\rfloor - 1\right) + 1 = \left\lfloor \sqrt{T} \right\rfloor$ integers within this domain.

(2-1) Note that the sum of the coefficients of a polynomial is when $x = 1$ is plugged into the resulting polynomial. If $T+2$ is a perfect square, then $\sqrt{T+2}$ is a perfect square and we have the linear equation $x - ((T+1) + \sqrt{T+2})$, giving us $K = \left|1 - ((T+1) + \sqrt{T+2})\right|$. Otherwise, $x = (T+1) + \sqrt{T+2}$ is a radical solution and so its conjugate $x = (T+1) - \sqrt{T+2}$ must also be a solution. Then we have the quadratic equation $(x-((T+1)+\sqrt{T+2}))(x-((T+1)-\sqrt{T+2})) = x^2-(2T+2)x+((T+1)^2-(T+2))$. The sum of the coefficients of the quadratic is $1-(2T+2)+(T^2+2T+1-T-2) = T^2-T-2 = (T-2)(T+1)$, so $K = |(T-2)(T+1)|$.

(2-2) To find the number of positive factors T has, find the prime factorization of the number. Let this prime factorization be $a^b \times c^d \times e^f$.... Then the number of positive factors of this number is the product $N = (b+1)(d+1)(f+1)\cdots$. Brute force with this part of the problem is also possible. The area of the rectangle with side lengths N and $T+1$ is $K = N \times (T+1)$.

(2-3) There are two cases to consider: either the side length T or the side length $2T$ lies on the diameter of this semicircle. If the side length T lies on the diameter of the circle, then the length of the radius would be $r = \sqrt{(2T)^2 + (\frac{T}{2})^2} = \sqrt{4T^2 + \frac{T^2}{4}} = \frac{T\sqrt{17}}{2}$. If the side length $2T$ lies on the diameter of the circle, then the length of the radius would be $r = \sqrt{(T)^2 + (T)^2} = \sqrt{2T^2} = T\sqrt{2}$. Since $\sqrt{2} < \frac{\sqrt{17}}{2}$, the rectangle that has side length $2T$ on the semicircle's diameter will produce the smallest radius and the smallest area. This area is $K = \frac{1}{2}(T\sqrt{2})^2 = T^2$, meaning that $K = T^2$.

(3-1) Since the decimal expansion of $\frac{1}{17} = 0.05882352941176470588\ldots$ contains 16 digits, the Tth digit in the decimal expansion of $\frac{1}{17}$ is the same digit is the (T mod 16)th digit. Using this observation will help find the value of K quicker.

(3-2) Realize that $\sin(30T^\circ)$ and $\cos(30T^\circ)$ have a period of 360° and $30(12) = 360$. This means that $\sin(30T^\circ) + \cos(30T^\circ) = \sin(30(T \bmod 12)^\circ) + \cos(30(T \bmod 12)^\circ) = \frac{a+\sqrt{b}}{c}$. Evaluating the trig functions and following the conditions for a, b, c properly will give the correct value of K.

(3-3) Drawing out smaller squares of odd side length and generalizing for a $(2T+1) \times (2T+1)$ square for bigger values of T, we see that there are $(2T+1)^2 - (2T)^2 = 4T+1$ diagonal cells counted three times in the sum, $(2T)^2$ non-diagonal cells counted two times in the sum, and add an extra $(2T+1)^2$ to account for the fact that the middle cell is counted four times. Since the middle cell has to contain the largest number, the diagonal cells has to contain the next $4T$ largest numbers, and the non-diagonal cells has to contain the least $(2T)^2$ numbers within the numbers 1 to $(2T+1)^2$, the maximum sum of all of the rows, columns, and diagonals, using proper summation formulas and simplifying, is $16T^4 + 48T^3 + 44T^2 + 22T + 4$. We seek to find $16T^4 + 48T^3 + 44T^2 + 22T + 4 \bmod (2T+1)$ by taking powers of of the modulus $(2T+1)$ reducing down the polynomial of degree 4 to a polynomial of degree 1. Taking $(2T+1)^4$ and subtracting that from the preivous number, we now have to find $16T^3 + 20T^2 + 14T + 3 \bmod (2T+1)$. Taking $(2T+1)^3$ and subtracting that from the previous number, we now have to find $-(4T^2 - 2T - 1) \bmod (2T+1)$. Taking $(2T+1)^2$ and subtracting that from the previous number, we find that $6T + 2 \bmod (2T+1)$, which simplifies down to $2T \bmod (2T+1)$. Since $2T < 2T+1$, the remainder is $K = 2T$.

(4-1) Since $4(T+2)^2 = (2(T+2))^2$, $(2(T+2))^2 - 1$ will be the largest composite number less than $4(T+2)^2$ because $(2(T+2))^2 - 1$ will have a factorization of $(2(T+2)+1)(2(T+2)-1)$ by the difference of squares. This means that $K = 4(T+2)^2 - 1$.

(4-2) Moving $x^2 - 4T^2$ to the right side and factoring, we get $(x+2T)(2(y-5T) - (x-2T)) = 0$. This yields the lines $x = -2T$ and $y = \frac{1}{2}x + 4T$. Solving for the y-coordinate of the intersection point, we get that $K = y = \frac{1}{2}(-2T) + 4T = 3T$.

(4-3) The approaches differ depending on the value of $T+1$. Notice how in some cases, we could primarily focus on the value of the tens and the ones digit as we will only care about the remainder when K is divided by 100.

- If $T+1 = 1$, then the sum is $1 + 2 + \cdots + 9 = \frac{10 \cdot 9}{2} = 45$.
- If $T+1 = 2$, then the sum is $11 + 22 + \cdots + 99 = \frac{110 \cdot 9}{2} = 495$, which has a remainder of 95 wen divided by 100.
- If $T+1 = 3$, then there are 10 palindromes for each ones and leading digit and 9 palindromes for each tens digit, for a sum of $10(101+202+\cdots+909)+9(10+20+\cdots+90) = 1100(1+2+\cdots+9)$, which has a remainder of 0 when divided by 100.
- If $T+1 = 4$, then there are 10 palindromes for each ones and leading digit and 9 palindromes for each tens and hundreds digit, for a sum of $10(1001+2002+\cdots+9009)+9(110+220+\cdots+990) = 11000(1+2+\cdots+9)$, which has a remainder of 0 when divided by 100.
- If $T+1 > 4$, then there are $10^{\lfloor \frac{T}{2} \rfloor}$ palindromes for each ones and leading digit and $9 \cdot 10^{\lfloor \frac{T}{2} \rfloor - 1}$ palindromes for each tens and hundreds digit, for a sum of $10^{\lfloor \frac{T}{2} \rfloor} \cdot 45 \cdot (10^T + 1) + 9 \cdot 10^{\lfloor \frac{T}{2} \rfloor - 1} \cdot 45 \cdot (10^{T-1} + 10^{T-2} + \cdots + 10^1)$, which equals $45 \cdot 10^{\lfloor \frac{T}{2} \rfloor} \cdot (10^T + 10^{T-1})$, which has a remainder of 0 when divided by 100.

Use the right case to determine the value of K.

(5-1) The coefficient of the x^2 term is the sum of the products of all possible pairs of two distinct values from the set $S = \{0, 1, 2, \ldots, T\}$. For small values of T, this can be computed by hand, but for larger values, it can be very time-consuming. We notice that $\sum\limits_{i=0}^{T}\sum\limits_{j=0}^{T} ij = 2K + \sum_{i=1}^{T} i^2$ since we only count each pair of elements from S once, and the elements must be distinct ($i \neq j$). We rearrange to get $K = \dfrac{1}{2} \cdot (\sum\limits_{i=0}^{T}\sum\limits_{j=0}^{T} ij - \sum\limits_{i=1}^{T} i^2)$. We know that $\sum\limits_{i=0}^{T}\sum\limits_{j=0}^{T} ij = \sum\limits_{i=0}^{T} i \cdot \sum\limits_{j=0}^{T} j = (\frac{T\cdot(T+1)}{2})^2$ and $\sum_{i=1}^{T} i^2 = \frac{T(T+1)(2T+1)}{6}$, so $K = \frac{1}{2}((\frac{T(T+1)}{2})^2 - \frac{T(T+1)(2T+1)}{6}) = \frac{1}{2}(\frac{T^2(T+1)^2}{4} - \frac{T(T+1)(2T+1)}{6}) = \frac{T\cdot(T+1)\cdot(T-1)\cdot(3T+2)}{24}$.

(5-2) Plug in the value of T into $f(x)$ and quickly evaluate as many terms as possible until a pattern emerges between some or all of the generated numbers. Then deduce the value of $f^{T+1}(T)$ from the patterns observed from the particular value of T to find the correct value of K.

(5-3) Let d be the distance from Kansas City to Omaha in miles. From the distance-rate-time formula, the time from Kansas City to Omaha is $\frac{d}{60}$ hours, and the time for the return trip is $\frac{d}{N}$ hours. Thus, the average speed for the entire round trip is equal to $\frac{2d}{\frac{d}{60}+\frac{d}{N}} = \frac{2}{\frac{1}{60}+\frac{1}{N}} = T + 1$. Solving for $\frac{1}{N}$ results in $\frac{1}{N} = \frac{119-T}{60(T+1)}$, so $N = \frac{60(T+1)}{119-T}$. Plugging in T and simplifying (if necessary) will yield the correct value of K.

