d) $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$ → use partial sum

a) construct algorithm to approximate π within ε.

provide a theoretical upperbound ε on the error of the approximation.

Recall: consider $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$. let S denote the sum of this series and S_k be the corresponding sequence of partial sum. For any integer $N \geq 1$, the remainder R_n satisfies: $|R_N| = |S - S_n| \leq |S_{n+1} - S_N| = b_{n+1}$

So, the error of approximating the infinite series by the N^{th} partial sum S_N is the magnitude at most the size of the next term b_{N+1}.

Here → $\alpha_n = \frac{4}{2n+1}$. So when we approximate π with the partial sum: $\pi_N = 4\sum_{0}^{N} \frac{(-1)^n}{2n+1}$

Our theoretical upper bound: $|\pi - \pi_N| \leq \frac{4}{2(N+1)+1} = \frac{4}{2N+3}$

So for general ε → plug $N=0$, $S=0$

b) For $\varepsilon = 5 \cdot 10^{-3}$: $\frac{4}{2N+3} \leq 5\cdot 10^{-3} \Rightarrow \frac{1}{2N+3} \leq \frac{5\cdot 10^{-3}}{4} \Rightarrow 2N+3 \geq 800$

$\Rightarrow 2N \geq 793$

$\Rightarrow N \geq 398.5$

So we will add up 399 terms.

c) Plot ε vs N of (a) on a log-log scale $\left[\varepsilon_N = \frac{4}{2N+4}\right]$

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
$= \ln 1 - \ln b$

- What is the slope of the line as N increases?

Plot the actual error (b) as a function of N on the same axes.

Use exact value of matlab for π. actual_error $= |\pi_N - \pi|$ pi

- How does the actual error compare to the estimated threshold?

Both plots are straight lines as N increases, with a negative slope. As they are both linear, the convergence is also linear (1st order). We also observe as N increases the actual error remains bellow our estimated threshold, which is confirmed both numerically and by partial sum theorem.

2) Determine which one approaches 1 faster. Confirm numerically by determining the rate of convergence of each sequence. → Taylor Expansion

$$\lim_{x\to 0} \frac{\sin x^2}{x^2} \quad . \text{ Let } y = x^2 : \lim_{y\to 0} \frac{\sin y}{y} = 1$$

Rate of convergence

$|f(x) - L| \leq k \cdot |g(x)|$

$\lim_{x\to 0} f(x)$

$$\lim_{x\to 0} \frac{(\sin x)^2}{x^2} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 1^2 = 1$$

Taylor Expansion of $\sin(x^2) = x^2 - \frac{x^2}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} \approx x^2 - \frac{x^6}{6}$

Divide by $x^2 \approx 1 - \frac{x^4}{6} \rightarrow$ rate of convergence $O(x^4)$

Taylor Expansion of $(\sin x)^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} \approx x^2 - \frac{x^4}{3}$

Divide by $x^2 \approx 1 - \frac{x^2}{3} \rightarrow$ rate of convergence : $O(x^2)$

Since the rate of convergence of $\frac{\sin(x^2)}{x^2}$ is $O(x^4)$ it will reach it faster than $\left(\frac{\sin x}{x}\right)^2$ which has $O(x^2)$ rate of convergence.

Simulating the sequences on MATLAB, we can see that the green plot which is $\frac{\sin x^2}{x^2}$ reduces its error much faster than the magenta plot which is $\left(\frac{\sin x}{x}\right)^2$.

3) (a) $f_c(x) = x^2 - 12x + c$.

- What are the zeros of f_c with $c = 35$

$$f(x) = x^2 - 12x + 35 = 0 \Rightarrow x_{1,2} = \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 1 \cdot 35}}{2 \cdot 1} = \frac{12 \pm \sqrt{4}}{2} = \frac{12 \pm 2}{2} \begin{cases} x_1 = 7 \\ x_2 = 5 \end{cases}$$

- if $c = 35 - 10^{-2} = 35 - 0.01 = 34.99$

$$f(x) = x^2 - 12x + 34.99 = 0 \Rightarrow x_{1,2} = \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 1 \cdot (34.99)}}{2 \cdot 1} = \frac{12 \pm \sqrt{4.04}}{2}, \quad \sqrt{4.40} \approx 2.00998$$

So, $x_1' \approx 7.00499$ & $x_2' \approx 4.99501$

- Relative to the size of the change in c, how big is the change in the zeros of the function?

$$\left.\begin{aligned} \Delta x_1 &= |x_1 - x_1'| \approx |7 - 7.00499| \approx 0.005 \\ \Delta x_2 &= |x_2 - x_2'| \approx |5 - 4.99501| \approx 0.005 \end{aligned}\right\}$$

So a small change in the constant term leads to a small change in the roots.

(b) $g_c(x) = x^2 - 10x + c$

▶ what are the zeros when $c=25$?

$$g(x) = x^2 - 10x + 25 = 0 \Rightarrow x_{1,2} = \frac{10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 25}}{2} = \frac{10}{2} = 5 \Rightarrow \boxed{x_{1,2} = 5}$$ double root.

▶ What are the zeros if $c = 25 - 10^{-2} = 25 - 0.01 = 24.99$

$$g(x) = x^2 - 10x + 24.99 = 0 \Rightarrow x_{1,2} = \frac{10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 24.99}}{2 \cdot 1} = \frac{10 \pm \sqrt{0.04}}{2} \quad , \sqrt{0.04} = 0.2$$

$\Rightarrow x_1' = 5.1 \quad , \quad x_2 = 4.9$

▶ Relative to the size of the change in c, how big is the change in the zeros of the function?

$\Delta x_1 = |x_1 - x_1'| = 0.1 \quad \& \quad \Delta x_2 = |x_1 - x_2| = 0.1$

A small change in the constant term creates a much larger shift in the roots of the equation.

(c) Compare the nature of the roots of f & g and how might this contribute to the relative changes in the zeros for each function?

The f(x) equation has two simple distinct roots, a small change in the coefficient, which is just slightly moving the graph up or down, won't lead to any significant changes. The roots will move slightly. However, for g(x) that has one double root we notice a significant change with just a small shift. This happens because the double root is the point where the function is tangent to the x-axis, it isn't crossing it. So a small disturbance here leads from one double root to two distinct roots.

Therefore, we can derive that the multiplicity of a root affects how sensitive it is to change.

4) Identify potential round-off error:

GIVEN: $a, b, c \in \mathbb{R}$ coefficients

STEP 1: calculate $disc = \sqrt{b^2 - 4ac}$

STEP 2: calculate $root1 = \dfrac{-2c}{b + disc}$

STEP 3: calculate $root2 = -\dfrac{b}{a} - root1$

OUTPUT: root1 & root 2

STEP 1: if the values b^2 and $4ac$ are almost equal we could lose significant digits, leading to large roundoff error.

STEP 2: if the denominator $b + disc$ is a very small number, almost zero, it can lead to either division with zero which is impossible or if $b^2 \approx 4ac$, it still carries the roundoff error from above. In either case root1 can be inaccurate.

STEP 3: if a very small again it can lead to division with zero and if the previous cases also happen, the error is still carried and magnified, leading to an inaccurate root2 too.

5) (a) Plot the function $f(x) = \ln(1+x) - \cos x - x + 1$ over $-5 \cdot 10^{-6} \leq x \leq 5 \cdot 10^{-6}$ using 1001 uniformly spaced point in x.

We see a plot that looks very noisy $\rightarrow$ cancellation error

(b) Reformulate f to avoid cancellation and plot it again.

In order to reformulate f we will apply Taylor expansion about $x=0$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

Substitute back :

$$f(x) = \cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \frac{x^4}{4} - \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} - \cancel{x} + \cancel{1} = \frac{x^3}{3} - \overset{6}{\frac{x^4}{4}} - \frac{x^4}{24}$$

$$= \frac{x^3}{3} - \frac{7x^4}{24}$$

c) In the original plotting of the function we can see that the cancellation error occurs when x is very small because all terms are nearly equal, hence the noisy graph, which likely means that the computed values are at the floating point roundoff error. The reformulated function using Taylor Expansion about $x=0$ avoids these errors and results in a much nicer plot.

(6) (a) Determine third-degree Taylor polynomial and associated remainder term for the function $f(x) = \sqrt{1+x}$. Use $x_0 = 0$.

$\hookrightarrow f(x) = (1+x)^{1/2}$

Taylor Series: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(k+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

$$\left.\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2} \\ f''(x) &= -\frac{1}{4}\cdot(1+x)^{-3/2} \\ f''(x) &= \frac{3}{8}\cdot(1+x)^{-5/2} \end{aligned}\right\}$$

at $x_0 = 0$: $f(0) = 1$

$f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

$f'''(0) = \frac{3}{8}$

So the third-degree Taylor Polynomial:

$$f(x) = 1 + \frac{1}{2}x - \frac{1}{2!\cdot 4}\cdot x^2 + \frac{3}{3!\cdot 8}\cdot x^3$$

$$\Rightarrow f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

Remainder of 3rd term : $f^{(4)}(x) = \frac{-15}{4!}\cdot(1+x)^{-7/2}$

$$\frac{f^4(\xi)}{4!}x^4 = -\frac{15}{24}\cdot(1+x)^{-7/2}$$

So remainder: $\frac{-15}{24\cdot 16}\cdot(1+\xi)^{-7/2}\cdot x^4$

Third-degree Taylor Polynomial and remainder:

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{15}{384}\cdot(1+\xi)^{-7/2}\cdot x^4$$

b) approximate $\sqrt{1.5}$ with (a), compute the theoretical error bound associated with the approximation. Compare the theoretical error bound with the actual error.

So, to approximate $\sqrt{1.5} \Rightarrow \sqrt{1+x} = \sqrt{1.5} \Rightarrow x = 0.5$

↳ we evaluate at $x=0.5$

$$f(0.5) = 1 + \frac{0.5}{2} - \frac{0.5^2}{8} + \frac{0.5^3}{16} = 1.2265625$$

From the Taylor Expansion, we are guaranteed that the approximation theoretical error will be the remainder.

Error $= |R_3(x)|$. If we plug in $x=0.5$:

$$|R_3(0.5)| \leq \frac{15}{16} \cdot \frac{1}{4!} \cdot (0.5)^4 = \frac{15}{384} \cdot (0.5)^4 \approx 2.44 \cdot 10^{-3}$$

So this is the upper theoretical error bound.

Actual Error: $|f(0.5) - \text{true\_value}|$ $\quad (\sqrt{1.5} \approx 1.224744871)$

$$= |1.2265625 - 1.224744871| \approx 1.82 \cdot 10^{-3}$$

According to our computations, the actual error is less than the theoretical approximation error. This is most likely due to the theoretical error being a "safe" error measure. We saw the same in the exercise with the partial sums.

(c) Compute the limit & determine rate of convergence:

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$$

Rate of convergence

$|f(x) - L| \le k \cdot |g(x)|$

$\lim_{x\to 0} f(x)$

We'll apply Taylor's Expansion about $x=0$:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16} \cdot \sqrt{1+\xi}$$

$$\Rightarrow 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{x^3}{16} \cdot \sqrt{1+\xi} - 1 - \frac{1}{2}x = -\frac{1}{8}x^2 + \frac{x^3}{16} \cdot \sqrt{1+\xi}$$

$$\lim_{x\to 0} \frac{-\frac{1}{8}x^2 + \frac{1}{16}x^3 \cdot \sqrt{1+\xi}}{x^2} = \lim_{x\to 0} -\frac{1}{8} + \frac{x}{16} \cdot \sqrt{1+\xi} = -\frac{1}{8}$$

$$\left|f(x) + \frac{1}{8}\right| = \left|\frac{x}{16} \cdot \underbrace{\sqrt{1+\xi}}_{\le 1}\right| \le \frac{1}{16} \cdot x$$

Therefore, the rate of convergence $O(x)$ which is linear.

Reminder :

$$\cos x = 1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{720}x^6 \cdot \cos\xi = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)$$

Taylor series about $x=0$

$$f(x) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \cdot \cos\xi - 1 + \frac{1}{2}x^2}{x^2}$$

$$= \frac{1}{24} - \frac{1}{720}x^2 \cdot \cos\xi$$

$\lim_{x\to 0} f(x) = \frac{1}{24}$, $|f(x) - l| = \left|f(x) - \frac{1}{24}\right| = \left|-\frac{1}{720} \cdot x^2 \underbrace{\cos\xi}_{\le 1}\right| \le \underbrace{\frac{1}{720}}_{k} \cdot$ ...

7) $Ax=b$, $A=\begin{bmatrix} 3.4 & 2.8 \\ 8.0 & 6.6 \end{bmatrix}$, $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $b=\begin{bmatrix} 3.4 \\ 8.0 \end{bmatrix}$

a) Determine analytic solution (How do we know its unique? (the problem is well posed))

$$Ax=b \Rightarrow \begin{bmatrix} 3.4 & 2.8 \\ 8.0 & 6.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 8.0 \end{bmatrix} \Rightarrow \begin{cases} 3.4x_1 + 2.8x_2 = 3.4 & (1) \\ 8.0x_1 + 6.6x_2 = 8.0 & (2) \end{cases} \Rightarrow$$

$\Rightarrow (2)$ $8x_1 = 8 - 6.6x_2 \Rightarrow x_1 = 1 - 0.825x_2$. Substitute back:

$$3.4(1-0.825x_2) + 2.8x_2 = 3.4 \Rightarrow 3.4 - 2.80500x_2 + 2.8x_2 = 3.4$$

$$\Rightarrow -0.005x_2 = 0 \Rightarrow \boxed{x_2 = 0}$$

$(1) \Rightarrow$ $3.4x_1 + 2.8x_2 = 3.4 \Rightarrow \boxed{x_2 = 1}$

So solution: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The solution is unique if $\det A \neq 0$:

$\det A = (3.4)\cdot(6.6) - (2.8)(8.0) = 22.44 - 22.4 = 0.04$ $\rightsquigarrow$ so the system has a unique solution

$\longrightarrow$ the problem is well-posed.

b) Solve the system with $b' = \begin{bmatrix} 3.41 \\ 8.0 \end{bmatrix}$.

$$Ax=b' \Rightarrow \begin{bmatrix} 3.4 & 2.8 \\ 8.0 & 6.6 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3.41 \\ 8.0 \end{bmatrix} \Rightarrow \begin{cases} 3.4x_1' + 2.8x_2' = 3.41 \\ 8.0x_1' + 6.6x_2' = 8.0. \end{cases}$$

$\Rightarrow x_1' = 1 - 0.825x_2'$. Substitute back: $3.4\cdot(1-0.825x_2') + 2.8x_2' = 3.41$

$$\Rightarrow 3.4 - 2.80500x_2' + 2.8x_2' = 3.41$$

$$\Rightarrow -0.005x_2' = 0.01$$

$$\Rightarrow \boxed{x_2' = -2}$$

$$x_1' = 1 - 0.825\cdot(-2) \Rightarrow \boxed{x_1' = 2.65}$$

So new solution after a small shift: $x' = \begin{bmatrix} 2.65 \\ -2 \end{bmatrix}$

(c) comment on the conditioning of the problem. What is det A and for what value will the problem be ill-posed?

$$\|A\|_\infty = \max(|3.4|+|2.8|, |8.0|+|6.6|) = \max(6.2, 14.6) = 14.6$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{0.04} \cdot \begin{pmatrix} 6.6 & -2.8 \\ -8.0 & 3.4 \end{pmatrix} = \begin{pmatrix} 165 & -70 \\ -200 & 85 \end{pmatrix}$$

$$\|A^{-1}\|_\infty = \max(|165|+|-70|, |-200|+|85|) = \max(235, 285) = 285$$

Condition number for

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = (14.6)\cdot(285) = 4161$$

The condition number is a measure we use to detect the matrix's sensitivity to small changes in its input. It shows how error can be amplified in the end:

↳ because the condition number is so large we can say that it is extremely sensitive to pertrubations, as we also saw on (b).

↳ The $\det A = 0.04$ is a very small value, almost close to 0 which creates the ill-condition of the problem so even small changes can highly effect the final result, or lead quickly to round-off error.