

1 Introduction

1.1 Input and conventions

TODO: rewrite in nice paragraph

Input:

- $m \in \mathbb{N}$: Number of homogeneous servers
- $T \in \mathbb{N}$: Number of time steps
- $\beta \in \mathbb{R}_{\geq 0}$: Power up costs
- $\lambda_0, \dots, \lambda_T \in [0, m]$: Arrival rates

Notations:

- Let $\lambda_{i,t}$ be the assigned arrival rate at time t for server i
- Let x_t be the number of active servers at time t
- Let $\mathcal{X} := (x_0, \dots, x_T)$ be the sequence of active servers
- If $A = (a_0, \dots, a_n)$ is a tuple with $n + 1$ entries, we write $A(i)$ for the i -th component of A with $0 \leq i \leq n$

Requirements:

- Convex cost function f
- Power down costs are w.l.o.g. equal to 0
- $\lambda_0 = \lambda_T = 0$ and $\mathcal{X}(0) = \mathcal{X}(T) = 0$, i.e. all servers are powered down at $t = 0$ and $t = T$

1.2 Preliminaries

Definition 1.1. Let $\mathcal{X} = (x_0, \dots, x_T)$ be the sequence of active servers of a schedule. We call a schedule and its sequence **feasible** if

$$\forall t \in \{0, \dots, T\} : x_t \geq \lambda_t$$

We call a feasible schedule **optimal** if its incurred costs are minimal under all feasible schedules.

Lemma 1.2. *Given a convex cost function f , x active servers and an arrival rate λ , the optimal strategy is to assign each server a load of λ/x .*

Proof. $\forall x \in \mathbb{N}, \mu_i \in [0, 1] : \sum_{i=1}^x \mu_i = 1 :$

$$\begin{aligned} f\left(\frac{\lambda}{x}\right) &= f\left(\sum_{i=1}^x \frac{\mu_i \lambda}{x}\right) \\ \implies f\left(\frac{\lambda}{x}\right) &\leq \sum_{i=1}^x \frac{1}{x} f(\mu_i \lambda) && \text{by Jensen's inequality} \\ \iff x f\left(\frac{\lambda}{x}\right) &\leq \sum_{i=1}^x f(\mu_i \lambda) \end{aligned}$$

□

Lemma 1.2 allows us to uniquely identify an optimal schedule by its sequence of numbers of active servers \mathcal{X} .

Definition 1.3. Define the minimum costs function of a feasible sequence \mathcal{X} between time steps t and t' with $0 \leq t < t' \leq T$ as

$$\text{costs}(\mathcal{X}, t, t') := \beta \max\{0, x_{t'} - x_t\} + x_{t'} f(\lambda_{t'}/x_{t'}) \quad (1)$$

Then the minimum costs of a feasible sequence \mathcal{X} at time $0 < t \leq T$ are given by

$$\text{costs}(\mathcal{X}, t-1, t) := \text{costs}(\mathcal{X}, t) = \underbrace{\beta \max\{0, x_t - x_{t-1}\}}_{\text{power up costs}} + x_t f(\lambda_t/x_t)$$

and the total costs by

$$\text{costs}(\mathcal{X}) := \sum_{t=1}^T \beta \max\{0, x_t - x_{t-1}\} + x_t f(\lambda_t/x_t)$$

2 Optimal scheduling for m homogeneous servers

TODO: introduction text

2.1 Graph for an optimal schedule

We construct a directed acyclic graph as follows:

$\forall t \in [T-1]$ and $i, j \in \{0, \dots, m\}$ we add vertices (t, i) modelling the number of active servers at time t . Moreover, we add vertices $(0, 0)$ and $(T, 0)$ for our initial and final state respectively.

In order to warrant that there are at least $\lceil \lambda_t \rceil$ active servers $\forall t \in [T-1]$, we define an auxiliary function which calculates the costs for handling an arrival rate λ with x active servers:

$$c(x, \lambda) := \begin{cases} 0, & \text{if } x = 0 \\ x f(\lambda/x), & \text{if } x \neq 0 \wedge \lambda \leq x \\ \infty, & \text{otherwise} \end{cases} \quad (2)$$

Then, $\forall t \in [T-2]$ and $i, j \in \{0, \dots, m\}$, we add edges from (t, i) to $(t+1, j)$ with weight

$$d(i, j, \lambda_{t+1}) := \beta \max\{0, j - i\} + c(j, \lambda_{t+1}) \quad (3)$$

Finally, for $0 \leq i \leq m$ we add edges from $(0, 0)$ to $(1, i)$ with weight $d(0, i, \lambda_1)$ and from $(T-1, i)$ to $(T, 0)$ with weight $d(i, 0, \lambda_T) = 0$.

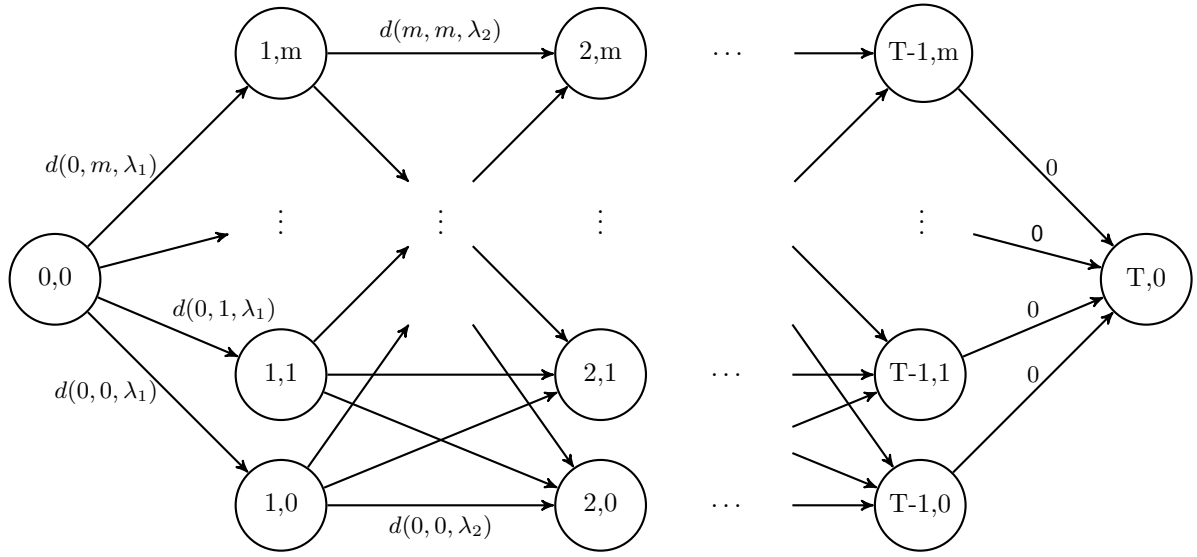


Figure 1: Graph for optimal schedule algorithm.

Note: All edges from (t, i) to $(t + 1, j)$ have weight $d(i, j, \lambda_{t+1})$

Proposition 2.1. Any given optimal schedule \mathcal{X} corresponds to a shortest path P from $(0, 0)$ to $(T, 0)$ with $\text{costs}(\mathcal{X}) = \text{costs}(P)$ and vice versa.

Proof.

“ \Rightarrow ”: We construct a feasible path in our graph from \mathcal{X} as follows:

$$\begin{aligned} \text{First set} \quad e_t &:= \left((t, \mathcal{X}(t)), (t + 1, \mathcal{X}(t + 1)) \right), & \forall t \in \{0, \dots, T - 1\} \\ \text{then set} \quad P &:= (e_0, \dots, e_{T-1}) \end{aligned}$$

As each edge e_t in our graph has weight $d(\mathcal{X}(t - 1), \mathcal{X}(t), \lambda_t)$, it corresponds to the costs of switching from $\mathcal{X}(t - 1)$ to $\mathcal{X}(t)$ servers and processing λ_t with $\mathcal{X}(t)$ active servers. Hence, it directly follows that P is a shortest path of the graph with $\text{costs}(P) = \text{costs}(\mathcal{X})$.

“ \Leftarrow ”: Let $P = ((0, 0) = v_0, \dots, v_T = (T, 0))$ with $v_t \in \{(t, i) \mid 0 \leq i \leq m\}$ be a shortest path of the graph.

We can construct an optimal schedule from P by setting $\mathcal{X} := (v_0(1), \dots, v_T(1))$

By definition (2) it is guaranteed that P only traverses edges such that there are enough active servers $\forall t \in [T]$. Therefore, the created schedule is feasible. Its optimality directly follows from the definition of the edges' weights and so does the equality $\text{costs}(\mathcal{X}) = \text{costs}(P)$.

□

2.2 A pseudo-polynomial minimum cost algorithm

Algorithm 1 Calculate costs for m homogeneous servers

Require: Convex cost function f , $\lambda_0 = \lambda_T = 0$, $\forall t \in [T - 1] : \lambda_t \in [0, m]$

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1: function SCHEDULE( $m, T, \beta, \lambda_1, \dots, \lambda_{T-1}$ )
2:   if  $T < 2$  then return
3:   let  $p[2 \dots T - 1, m]$  and  $M[1 \dots T - 1, m]$  be new arrays
4:   for  $j \leftarrow 0$  to  $m$  do
5:      $M[1, j] \leftarrow d(0, j, \lambda_1)$ 
6:   for  $t \leftarrow 1$  to  $T - 2$  do
7:     for  $j \leftarrow 0$  to  $m$  do
8:        $opt \leftarrow \infty$ 
9:       for  $i \leftarrow 0$  to  $m$  do
10:         $M[t + 1, j] \leftarrow M[t, i] + d(i, j, \lambda_{t+1})$ 
11:        if  $M[t + 1, j] < opt$  then
12:           $opt \leftarrow M[t + 1, j]$ 
13:           $p[t + 1, j] \leftarrow i$ 
14:         $M[t + 1, j] \leftarrow opt$ 
15:   return  $p$  and  $M$ 

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Algorithm 2 Extract schedule for m homogeneous servers

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1: function EXTRACT( $m, p, M, T$ )
2:   let  $x[0 \dots T]$  be a new array
3:    $x[0] \leftarrow x[T] \leftarrow 0$ 
4:   if  $T < 2$  then return  $x$  ▷ Trivial solution
5:    $x[T - 1] \leftarrow \arg \min_{0 \leq i \leq m} \{M[T - 1, i]\}$ 
6:   for  $t \leftarrow T - 2$  to  $1$  do
7:      $x[t] \leftarrow p[t + 1, x[t + 1]]$ 
8:   return  $x$ 

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2.2.1 Runtime analysis

The algorithm visits every vertex and every edge of the graph exactly once. As the number of vertices is bounded by $\mathcal{O}(Tm)$ and the number of edges is bounded by $\mathcal{O}(Tm^2)$ the running time is given by:

$$\mathcal{O}(Tm + Tm^2) = \mathcal{O}(Tm^2)$$

As we need $\log_2(m)$ bits to encode m , the running time is polynomial in the numeric value of the input but exponential in the length of the input. Hence, the algorithm is pseudo-polynomial.

2.2.2 A memory optimized algorithm

TODO: use only array with size $2m$

3 A polynomial 4-approximation algorithm for monotonically increasing convex f

We consider a modification of the problem discussed in chapter 2. Assuming that f is convex and monotonically increasing, we can modify our algorithm to obtain a polynomial time 4-approximation algorithm.

3.1 Graph for a 4-optimal schedule

We modify our graph from chapter 2.1 to the reduce the number of vertices. For this, we stop adding m vertices for each timestep, but use vertices that approximate the number of active servers instead. First, let $b := \lceil \log_2(m) \rceil$. We add vertices $(t, 0)$ and $(t, 2^i)$, $\forall t \in [T-1], 0 \leq i \leq b$. All edges and weights are added analogous to chapter 2.1.

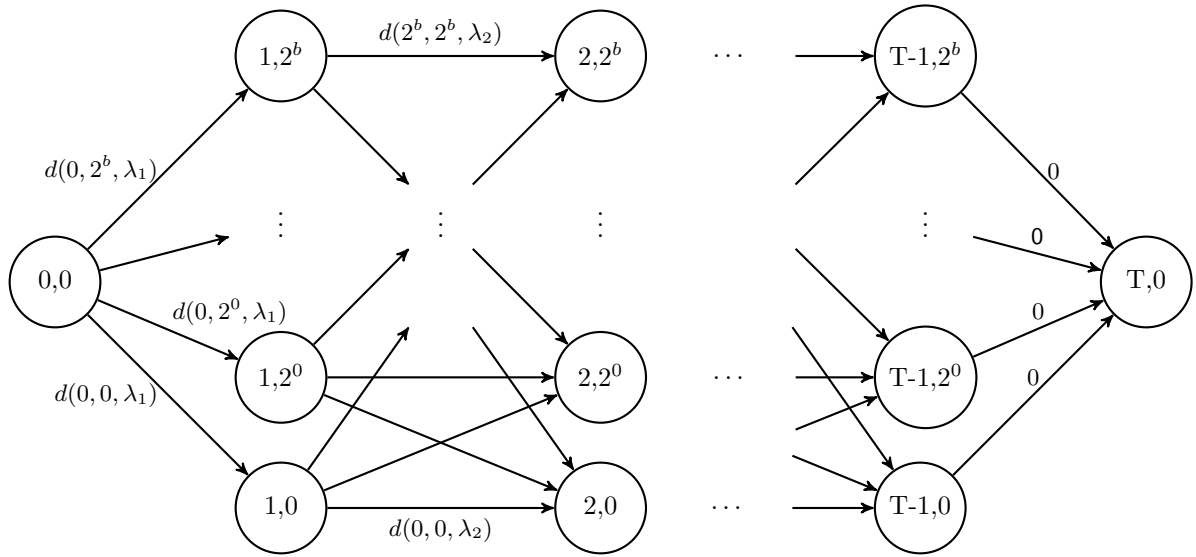


Figure 2: Graph for a 4-approximation algorithm

Definition 3.1. Let $\mathcal{X} = (x_0, \dots, x_T)$ be a schedule and $t > 0$. We say that \mathcal{X} changes its **state** at time t if

$$x_t \neq x_{t-1}$$

and that \mathcal{X} changes its **2-state** at time t if

$$x_t = 0 \quad \text{or} \quad x_t \notin (2^{\lfloor \log_2(x_{t-1}) \rfloor}, 2^{\lceil \log_2(x_{t-1}) \rceil})$$

Proposition 3.2.

1. Any given optimal schedule \mathcal{X} can be transformed to a 4-optimal schedule \mathcal{X}' which corresponds to a path P from $(0, 0)$ to $(T, 0)$ with $\text{costs}(\mathcal{X}') = \text{costs}(P)$.
2. Any shortest path P from $(0, 0)$ to $(T, 0)$ corresponds to a 4-optimal schedule \mathcal{X} with $\text{costs}(P) = \text{costs}(\mathcal{X})$.

Proof.

1. Assume we have an optimal schedule identified by $\mathcal{X} = (x_0, \dots, x_T)$. For $0 \leq t < T$ we inductively set:

$$x'_0 := 0, \quad x'_{t+1} := \begin{cases} \min\{2^{\lceil \log_2(2x_{t+1}) \rceil}, 2^b\}, & \text{if } 0 < x_t \leq x_{t+1} \\ 2^{\lceil \log_2(2x_{t+1}) \rceil}, & \text{if } 0 < x_{t+1} < x_t \text{ and } x'_t \geq 4x_{t+1} \\ x'_t, & \text{if } 0 < x_{t+1} < x_t \text{ and } x'_t < 4x_{t+1} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

Then let $\mathcal{X}' := (x'_0, \dots, x'_T)$ be the modified sequence of active servers. Notice that $x_t \leq x'_t \leq 4x_t$ holds as x'_t is at most the smallest power of two larger than $2x_t$ which implies that \mathcal{X}' is feasible.

We can now construct a feasible path in our graph from \mathcal{X}' as follows:

$$\begin{aligned} \text{First set} \quad e_t &:= \left((t, \mathcal{X}'(t)), (t+1, \mathcal{X}'(t+1)) \right), & \forall t \in \{0, \dots, T-1\} \\ \text{then set} \quad P &:= (e_0, \dots, e_{T-1}) \end{aligned}$$

By the definition of the edges' weights it follows that $\text{costs}(\mathcal{X}') = \text{costs}(P)$.

Next, let $(t_0 = 0, t_1, \dots, t_n = 0)$ be the sequence of times where the optimal schedule \mathcal{X} changes its 2-state. Notice that the modified schedule \mathcal{X}' changes its state only at times t_i and that $2x_{t_i} \leq x'_{t_i}$ holds (TODO: only if not discrete but continuous time steps). This can be seen exemplarily in figure 3 by observing that \mathcal{X}' changes its state only if \mathcal{X} crosses or touches a bordering power of two.

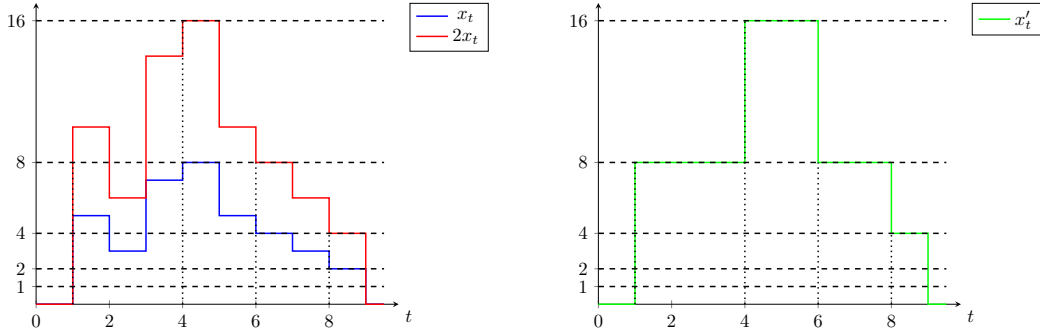


Figure 3: Adaption of an optimal schedule

For this reason, we now only have to consider the fraction of costs of \mathcal{X}' and \mathcal{X} between time steps t_{i-1} and t_i

$$\frac{\text{costs}(\mathcal{X}', t_{i-1}, t_i)}{\text{costs}(\mathcal{X}, t_{i-1}, t_i)} \quad (5)$$

For $x_{t_i} = 0$ it follows from (1) that $\text{costs}(\mathcal{X}', t_{i-1}, t_i) = \text{costs}(\mathcal{X}, t_{i-1}, t_i) = 0$. Hence, we can restrict ourselves to $0 < t_i < T$ with $x_{t_i} \neq 0$. The costs incurred by \mathcal{X}' are given by

$$\text{costs}(\mathcal{X}', t_{i-1}, t_i) = \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + x'_{t_i} f(\lambda_{t_i}/x'_{t_i}) \quad \text{by (1)}$$

$$\leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x'_{t_i}) \quad \text{by (4)}$$

$$\leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i}) \quad \text{f monotonically increasing}$$

$$\Rightarrow \text{costs}(\mathcal{X}', t_{i-1}, t_i) \leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i}) \quad (6)$$

and the costs of \mathcal{X} by

$$\text{costs}(\mathcal{X}, t_{i-1}, t_i) = \beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i}) \quad (7)$$

W.l.o.g. we may assume $x_{t_i} f(\lambda_{t_i}/x_{t_i}) > 0$, otherwise the claim follows trivially. (TODO: is it really trivial?)

(i) $x_{t_i} \leq x_{t_{i-1}}$: From (4) it follows that $x'_{t_i} \leq x'_{t_{i-1}}$. Thus, we can simplify (5):

$$\begin{aligned} \frac{\text{costs}(\mathcal{X}', t_{i-1}, t_i)}{\text{costs}(\mathcal{X}, t_{i-1}, t_i)} &\leq \frac{\beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (6),(7)} \\ &= \frac{4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{x_{t_i} f(\lambda_{t_i}/x_{t_i})} && (x_{t_i} \leq x_{t_{i-1}} \text{ and } x'_{t_i} \leq x'_{t_{i-1}}) \\ &= 4 \end{aligned}$$

(ii) $x_{t_i} > x_{t_{i-1}}$: From (4) it follows that $x'_{t_i} \geq x'_{t_{i-1}}$. Thus, we can simplify (5):

$$\begin{aligned} \frac{\text{costs}(\mathcal{X}', t_{i-1}, t_i)}{\text{costs}(\mathcal{X}, t_{i-1}, t_i)} &\leq \frac{\beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (6),(7)} \\ &= \frac{\beta(x'_{t_i} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && (x_{t_i} > x_{t_{i-1}} \text{ and } x'_{t_i} \geq x'_{t_{i-1}}) \\ &= \frac{\beta(\min\{2^{\lfloor \log_2(2x_{t_i}) \rfloor}, 2^b\} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (4)} \\ &\leq \frac{\beta(2^{\lfloor \log_2(2x_{t_i}) \rfloor} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\ &\leq \frac{\beta(2x_{t_i} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\ &\leq \frac{\beta(2x_{t_i} - 2x_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by } (2x_{t_{i-1}} \leq x'_{t_{i-1}}) \\ &\leq 4 \frac{\frac{1}{2}\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\ &\leq 4 \end{aligned}$$

From (i) and (ii) it follows:

$$\text{costs}(\mathcal{X}') \leq 4\text{costs}(\mathcal{X})$$

2. From 1 we obtain that we can construct a 4-optimal path P' from any optimal schedule. Now, let P be a shortest path. We have $\text{costs}(P) \leq \text{costs}(P') < \infty$, and since every path P with $\text{costs}(P) < \infty$ corresponds to a feasible schedule \mathcal{X} with $\text{costs}(P) = \text{costs}(\mathcal{X})$, \mathcal{X} must also be at least 4-optimal.

□