Algorithms for Dynamic Right-Sizing in Data Centers

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1 Introduction

TODO: Hardware prices vs. energy costs in data centres and purpose of this paper (offline algorithm, approximation algorithm,...).

1.1 Model description

We want to address the issue of above-mentioned ever-growing energy consumption by examing a scheduling problem that commonly arises in data centres. More specifically, we consider a model consisting of a fixed amount of homogeneous servers denoted by $m \in \mathbb{N}$ and a fixed amount of time slots denoted by $T \in \mathbb{N}$. In turn, each server possesses two power states, i.e. each server is either powered on (*active state*) or powered off (*sleep state*).

Need to describe what a time slot means?

Better name than sleep state?

For any time slot $t \in [T]$, we have a *mean arrival rate* denoted by λ_t, i.e. the amount of expected load to process in time slot t. We expect the arrival rates to be normalised such that each server $i \in [m]$ can handle a load between zero and one in any time slot. We denote the assigned load for server i in time slot t by $\lambda_{i,t} \in [0,1]$. Consequently, for any time slot t, we expect an arrival rate between zero and m, i.e. $\lambda_t \in [0,m]$; otherwise, the servers would not be able to process the given load in time.

The incurring costs of a single machine can be described by the sum of the machine's *(power state) switching costs*, specified by $\beta \in \mathbb{R}_{\geq 0}$, as well as its *operating costs*, specified by $f : [0,1] \to \mathbb{R}$. We assume that a sleeping server does not generate any costs. Note that $f(0)$ describes the costs generated by an idle server, not a sleeping one; in particular, $f(0)$ may be non-zero. Further, we assume convexity for f. This may seem like a notable restriction at first, but it indeed captures the behaviour of most modern server models. Since we are dealing with homogeneous servers, β and f are the same for all machines.

We want to stress that f may not exclusively consider energy costs. For example, f may also allow for costs incurred by delays, such as lost revenue caused by users waiting for their responses. Similarly, β may also allow for delay costs, wear and tear costs or the like. [1]

For convenience, we assume all machines sleeping at time $t = 0$ and force all machines to sleep after the scheduling process, i.e. at times $t > T$. This justifies the consolidation of power up and power down costs into β because it allows us to model both costs as being incurred when powering up a server; that is, a model with power up costs $\beta_\uparrow$ and power down costs $\beta_\downarrow$ can be simply transferred to our model by setting $\beta \coloneqq \beta_\uparrow + \beta_\downarrow$.

1.2 Problem statement

Using above definitions, we can define the input of our model by setting $\mathcal{I} \coloneqq (m, T, \Lambda, \beta, f)$ where $\Lambda = (\lambda_1, \ldots, \lambda_T)$ is the sequence of arrival rates. We will subsequently identify a problem instance by its input $\mathcal{I}$. Naturally, given a problem instance $\mathcal{I}$, we want to schedule our servers in such a way that we minimise the sum of incurred costs while warranting that we are processing the given loads in time.

For this, consider for each server $i \in [m]$ the sequence of its states S_i and the sequence of its assigned loads L_i; that is

$$
\begin{aligned}
S_i &\coloneqq (s_{i,1}, \ldots, s_{i,T}) \in \{0,1\}^T \\
L_i &\coloneqq (\lambda_{i,1}, \ldots, \lambda_{i,T}) \in [0,1]^T
\end{aligned}
$$

where $s_{i,t} \in \{0,1\}$ denotes whether server i at time t is sleeping (0) or active (1). Recall that we assume all machines sleeping at times $t \notin [T]$; consequently, for $t \notin [T]$ and $i \in [m]$, we have $s_{i,t} = 0$.

Using these sequences S_i and L_i, we can now define the sequence of all state changes and the sequence of all assigned loads:

good sentence? $\in (\{0,1\}^m)^T$

$$
\begin{aligned}
\mathcal{S} &\coloneqq (S_1, \ldots, S_m) \\
\mathcal{L} &\coloneqq (L_1, \ldots, L_m)
\end{aligned}
$$

We will subsequently call a pair $\Sigma \coloneqq (\mathcal{S}, \mathcal{L})$ a *schedule*. Finally, we are ready to define our problem statement.

Given an input $\mathcal{I}$, our goal is to find a schedule Σ that satisfies the following optimisation:

Definition of $c(\Sigma)$ too hidden?

$$
\text{minimise} \qquad c(\Sigma) \coloneqq \underbrace{\sum_{t=1}^{T}\sum_{i=1}^{m}\big(f(\lambda_{i,t}) * s_{i,t}\big)}_{\text{operating costs}} + \underbrace{\beta * \sum_{t=1}^{T}\sum_{i=1}^{m} \min\{0, s_{i,t} - s_{i,t-1}\}}_{\text{switching costs}} \tag{1}
$$

$$
\text{subject to} \qquad \sum_{i=1}^{m}(\lambda_{i,t} * s_{i,t}) = \lambda_t, \quad \forall t \in [T] \tag{2}
$$

We call a schedule *feasible* if it satisfies (2) and *optimal* if it satisifes (1) and (2).

2 Preliminaries

In this section, we conduct the prepratory work needed for our algorithms and proofs.

This sounds strange... What should be written here?

Proposition 2.1. *Given a problem instance $\mathcal{I}$ and a feasible schedule Σ, there exists a feasible schedule Σ' such that*

(i) $c(\Sigma') \leq c(\Sigma)$ *and*

(ii) Σ' *never powers on and shuts down servers at the same time slot, i.e.* Σ' *satisfies the following formula:*

$$
\forall t \in [T]\Big[\big(\forall i \in [m](s_{i,t} - s_{i,t-1} \geq 0)\big) \vee \big(\forall i \in [m](s_{i,t} - s_{i,t-1} \leq 0)\big)\Big] \tag{3}
$$

Proof. Let $\Sigma = (\mathcal{S}, \mathcal{L})$ be a feasible schedule for $\mathcal{I}$. We give a procedure that repeatedly modifies Σ such that it satisfies (3) and reduces or retains its costs.

Let $t \in [T]$ be the first time slot falsifying (3). If there does not exist such a t, we are done. Otherwise, we can obtain $i, j \in [m]$ such that $s_{i,t} - s_{i,t-1} = 1$ and $s_{j,t} - s_{j,t-1} = -1$. WLOG we may assume $i < j$.

First, since all servers are sleeping at time $t = 0$, we have

$$s_{k,1} - s_{k,0} = s_{k,1} - 0 = s_{k,1} \geq 0, \quad \forall k \in [m]$$

Thus, we may assume $t > 1$. Now consider the state sequences S_i and S_j:

$$\begin{aligned} S_i &= (s_{i,1}, \ldots, s_{i,t-1} = 0, s_{i,t} = 1, \ldots, s_{i,T}) \\ S_j &= (s_{j,1}, \ldots, s_{j,t-1} = 1, s_{j,t} = 0, \ldots, s_{j,T}) \end{aligned}$$

We modify S_i and S_j by swapping their states for time slots $\geq t$, that is we set

$$\begin{aligned} S'_i &\coloneqq (s_{i,1}, \ldots, s_{i,t-1} = 0, s_{j,t} = 0, \ldots, s_{j,T}) \\ S'_j &\coloneqq (s_{j,1}, \ldots, s_{j,t-1} = 1, s_{i,t} = 1, \ldots, s_{i,T}) \end{aligned}$$

Similarly, we need to swap the assigned loads:

$$\begin{aligned} L'_i &\coloneqq (\lambda_{i,1}, \ldots, \lambda_{i,t-1}, \lambda_{j,t}, \ldots, \lambda_{j,T}) \\ L'_j &\coloneqq (\lambda_{j,1}, \ldots, \lambda_{j,t-1}, \lambda_{i,t}, \ldots, \lambda_{i,T}) \end{aligned}$$

Finally, we construct a new schedule $\Sigma' \coloneqq (\mathcal{S}', \mathcal{L}')$ given by

$$\begin{aligned} \mathcal{S}' &\coloneqq (S_1, \ldots, S'_i, \ldots, S'_j, \ldots, S_T) \\ \mathcal{L}' &\coloneqq (L_1, \ldots, L'_i, \ldots, L'_j, \ldots, L_T) \end{aligned}$$

We now want to verify that Σ' is a feasible schedule, that is Σ' satisfies (2). For time slots $< t$ the schedules Σ' and Σ still coincide. For time slots $\geq t$ we only changed the order of summation (2). Thus, Σ' is feasible.

Further, Σ' and Σ coincide in their operating costs; however, their switching costs differ in that there are no switching costs at time slot t for server i using Σ'. As we assume $\beta \geq 0$, we conclude $c(\Sigma') \leq c(\Sigma)$

Moreover, we decreased the amount of bad spotsat time slot t concerning (3). Hence, by repeating described process on Σ', we obtain a terminating procedure that returns a schedule satisfying the conditions. ∎

bad spots? better description?

Corollary 2.2. *Given a problem instance $\mathcal{I}$, there exists an optimal schedule Σ satisfying (3).*

Proof. Let Σ be an optimal schedule for $\mathcal{I}$. Applying proposition 2.1 to Σ yields the result. ∎

Next, we want to consider the sequence of active servers. For this, let $\mathcal{X}$ denote the sequence of sums of active servers at each time slot t, i.e.

$$\mathcal{X} \coloneqq \left(x_1 = \sum_{i=1}^{m} s_{i,1}, \ldots, x_T = \sum_{i=1}^{m} s_{i,T}\right) \in \{0, \ldots, m\}^T$$

Recall that we assume all machines sleeping at times $t \notin [T]$; consequently, for $t \notin [T]$ we have $x_t = 0$.

The next proposition poses the cornerstone of our subsequent works.

Proposition 2.3 (Equal load sharing)**.** *Given $x_t \in \mathbb{N}$ active servers at time slot t, an arrival rate $\lambda_t \in [0, x_t]$, and a convex cost function f, a most cost-efficient and feasible scheduling strategy is to assign each active server a load of λ_t / x_t.*

This is sounds awkwardly formulated, doesn't it?

Proof. Let Σ be an arbitrary, feasible schedule using x_t servers at time slot t, and let A be its set of active servers at time slot t, that is $A := \{i \in [m] \mid s_{i,t} = 1\}$. Consider the operating costs of Σ given by

$$\sum_{i=1}^{m} \big(f(\lambda_{i,t}) * s_{i,t}\big) = \sum_{i \in A} f(\lambda_{i,t})$$

Since Σ is feasible (see constraint (2)), we have

$$\sum_{i \in A} \lambda_{i,t} = \lambda_t$$

Thus, we can obtain weights $\mu_1, \ldots, \mu_{x_t} \in [0,1]$ that relate $\lambda_{i,t}$ and λ_t for $i \in A$ such that

$$\sum_{i=1}^{x_t} \mu_i = 1 \quad \text{and} \quad \sum_{i \in A} f(\lambda_{i,t}) = \sum_{i=1}^{x_t} f(\mu_i \lambda_t)$$

In particular, we have $\sum_{i=1}^{x_t} \mu_i \lambda_t = \lambda_t$. Using these weights, we now consider the operating costs of a schedule Σ^* that equally distributes λ_t to its x_t active servers:

$$\sum_{i=1}^{x_t} f\Big(\frac{\lambda_t}{x_t}\Big) = x_t * f\Big(\frac{\lambda_t}{x_t}\Big) = x_t * f\Big(\sum_{i=1}^{x_t} \frac{\mu_i \lambda_t}{x_t}\Big)$$

Using Jensen's inequality[1] and the fact that $\sum_{i=1}^{x_t} (1/x_t) = 1$, we can give an upper bound for the costs :

style (footnote) okay?

$$x_t * f\Big(\frac{\lambda_t}{x_t}\Big) \le x_t \sum_{i=1}^{x_t} \frac{1}{x_t} f(\mu_i \lambda_t) = \frac{x_t}{x_t} \sum_{i=1}^{x_t} f(\mu_i \lambda_t) = \sum_{i=1}^{x_t} f(\mu_i \lambda_t) = \sum_{i \in A} f(\lambda_{i,t})$$

Hence, the operating costs of Σ^* give a lower bound for the operatings costs of Σ, and the claim follows. $\square$

[1]For convex $f : \mathbb{R} \to \mathbb{R}$, arbitrary $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and $x_1, \ldots, x_n \in [0,1]$ satisfying $\sum_{i=1}^{n} x_i = 1$ we have:

$$f\Big(\sum_{i=1}^{n} x_i \lambda_i\Big) \le \sum_{i=1}^{n} x_i f(\lambda_i)$$

Proposition 2.3 tells us that there always exists an optimal schedule that equally distributes its arrival rate to its active servers at any time slot. As a result, we can restrict ourselves in finding such an optimal schedule. Together with corollary 2.2, this allows us to subsequently identify an optimal schedule by its sequence of active servers $\mathcal{X}$. Moreover, we are now able to simplify our optimisation conditions (1) and (2).

For this, given a problem instance $\mathcal{I}$, we define the operating costs function $c_{op}(x, \lambda)$ that describes the costs incurred by equally distributing λ on x active servers using f:

$$c_{op} : \{0, \ldots, m\} \times [0, m] \to \mathbb{R}, \quad c_{op}(x, \lambda) = \begin{cases} 0, & \text{if } x = 0 \\ x * f(\lambda/x), & \text{if } x \neq 0 \wedge \lambda \leq x \\ \infty, & \text{otherwise} \end{cases}$$

Further, we define the switching costs function $c_{sw}(x_{t-1}, x_t)$ describing the costs that incur by changing the amount of active server x_{t-1} to x_t:

$$c_{sw}(x_{t-1}, x_t) \coloneqq \beta * \max\{0, x_t - x_{t-1}\}$$

Lastly, we can define the costs function $c(x_{t-1}, x_t, \lambda_t)$ that describes the incurring costs for a single time step using an equal distribution of loads:

$$c(x_{t-1}, x_t, \lambda_t) \coloneqq c_{op}(x_t, \lambda_t) + c_{sw}(x_{t-1}, x_t)$$

The optimisation conditions for a schedule Σ now simplify to one single minimalisation:

$$\text{minimise} \quad c(\Sigma) = \sum_{t=1}^{T} c(x_{t-1}, x_t, \lambda_t)$$

3 Optimal offline scheduling

In this section, we derive an optimal offline algorithm based on our preliminary work. We reduce our problem specified $\mathcal{I}$ to a shortest path problem of a level structured graph G. We then use a dynamic programming approach to find a shortest path of G and thereby an optimal schedule for $\mathcal{I}$ in pseudo-polynomial time.

3.1 Graph for an optimal schedule

Let $\mathcal{I}$ be a problem instance. Thanks our preliminary work, we know that there exists an optimal schedule which is identifiable by its sequence of active servers $\mathcal{X}$. In order to find this sequence $\mathcal{X}$, consider the weighted, level structured graph G defined as follows:

In order to warrant that there are at least $\lceil \lambda_t \rceil$ active servers $\forall t \in [T-1]$, we define an auxiliary function which calculates the costs for handling an arrival rate λ with x active servers: Then, $\forall t \in [T-2]$ and $i, j \in \{0, \ldots, m\}$, we add edges from (t, i) to $(t+1, j)$ with weight. Finally, for $0 \leq i \leq m$ we add edges from $(0, 0)$ to $(1, i)$ with weight $d(0, i, \lambda_1)$ and from $(T-1, i)$ to $(T, 0)$ with weight $d(i, 0, \lambda_T) = 0$.

$$
\begin{aligned}
V &\coloneqq \{v_{i,t} \mid i \in \{0,\dots,m\}, t \in \{0,\dots,T+1\}\} \\
E &\coloneqq \{(v_{i,t}, v_{j,t+1}) \mid i,j \in \{0,\dots,m\}, t \in \{0,\dots,T\}, v_{i,t}, v_{j,t+1} \in V\} \\
c_G(v_{i,t}, v_{j,t+1}) &\coloneqq c(i,j,\lambda_{t+1}) \\
G &\coloneqq (V, E, c_G)
\end{aligned}
$$

For any possible amount of active servers i and any time slot t we add a node $v_{i,t}$. Moreover, we add a start node $v_{0,0}$ as well as an end node $v_{0,T+1}$. Next, we connect all nodes with their direct successors with respect to time.

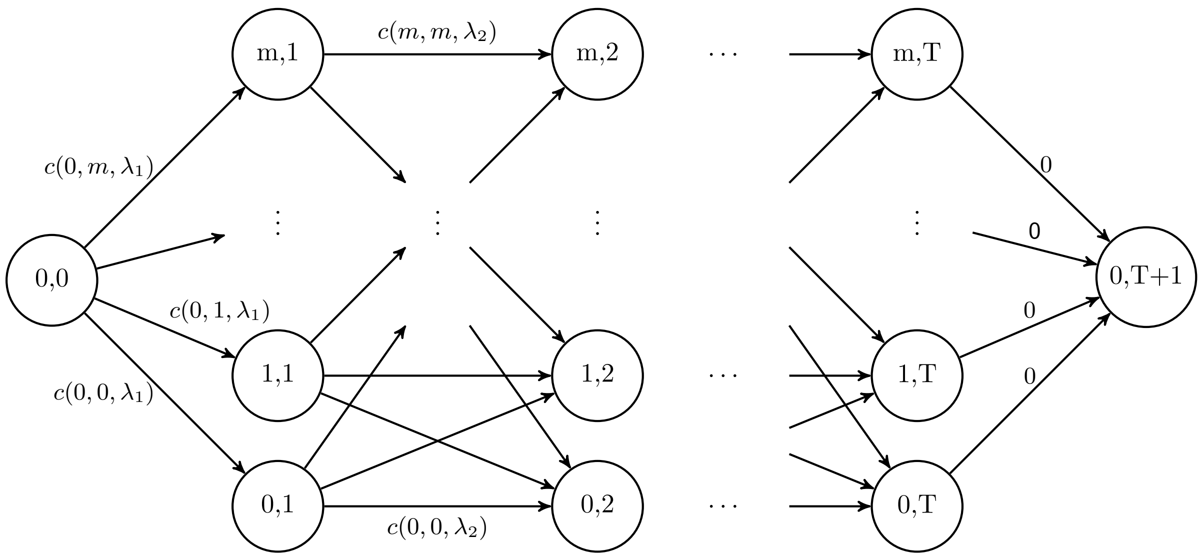


Figure 1: Level structured graph for an optimal offline algorithm.

Note: All edges from (t,i) to $(t+1,j)$ have weight $d(i,j,\lambda_{t+1})$

Proposition 3.1. *Any given optimal schedule $\mathcal{X}$ corresponds to a shortest path P from $(0,0)$ to $(T,0)$ with $costs(\mathcal{X}) = costs(P)$ and vice versa.*

Proof.

"⇒": We construct a feasible path in our graph from $\mathcal{X}$ as follows:

$$
\begin{aligned}
&\text{First set} && e_t \coloneqq \Big(\big(t,\mathcal{X}(t)\big), \big(t+1,\mathcal{X}(t+1)\big)\Big), && \forall t \in \{0,\dots,T-1\} \\
&\text{then set} && P \coloneqq (e_0,\dots,e_{T-1})
\end{aligned}
$$

As each edge e_t in our graph has weight $d\big(\mathcal{X}(t-1),\mathcal{X}(t),\lambda_t\big)$, it corresponds to the costs of switching from $\mathcal{X}(t-1)$ to $\mathcal{X}(t)$ servers and processing λ_t with $\mathcal{X}(t)$ active servers. Hence, it directly follows that P is a shortest path of the graph with $costs(P) = costs(\mathcal{X})$.

"$\Leftarrow$": Let $P = \big((0,0) = v_0, \ldots, v_T = (T,0)\big)$ with $v_t \in \big\{(t,i) \mid 0 \leq i \leq m\big\}$ be a shortest path of the graph.
We can construct an optimal schedule from P by setting $\mathcal{X} \coloneqq \big(v_0(1), \ldots, v_T(1)\big)$
By definition (2) it is guaranteed that P only traverses edges such that there are enough active servers $\forall t \in [T]$. Therefore, the created schedule is feasible. Its optimality directly follows from the definition of the edges' weights and so does the equality $costs(\mathcal{X}) = costs(P)$.

□

3.2 A pseudo-polynomial minimum cost algorithm

Algorithm 1 Calculate costs for m homogeneous servers

Require: Convex cost function f, $\lambda_0 = \lambda_T = 0$, $\forall t \in [T-1] : \lambda_t \in [0, m]$

```
function SCHEDULE(m, T, β, λ_1, ..., λ_{T−1})
    if T < 2 then return
    let p[2 ... T − 1, m] and M[1 ... T − 1, m] be new arrays
    for j ← 0 to m do
        M[1, j] ← d(0, j, λ_1)
    for t ← 1 to T − 2 do
        for j ← 0 to m do
            opt ← ∞
            for i ← 0 to m do
                M[t + 1, j] ← M[t, i] + d(i, j, λ_{t+1})
                if M[t + 1, j] < opt then
                    opt ← M[t + 1, j]
                    p[t + 1, j] ← i
            M[t + 1, j] ← opt
    return p and M
```

Algorithm 2 Extract schedule for m homogeneous servers

```
function EXTRACT(m, p, M, T)
    let x[0 ... T] be a new array
    x[0] ← x[T] ← 0
    if T < 2 then return x          ▷ Trivial solution
    x[T − 1] ← arg min_{0≤i≤m} {M[T − 1, i]}
    for t ← T − 2 to 1 do
        x[t] ← p[t + 1, x[t + 1]]
    return x
```

3.2.1 Runtime analysis

The algorithm visits every vertex and every edge of the graph exactly once. As the number of vertices is bounded by $\mathcal{O}(Tm)$ and the number of edges is bounded by $\mathcal{O}(Tm^2)$ the running time is given by:

$$\mathcal{O}(Tm + Tm^2) = \mathcal{O}(Tm^2)$$

As we need $\log_2(m)$ bits to encode m, the running time is polynomial in the numeric value of the input but exponential in the length of the input. Hence, the algorithm is pseudo-polynomial.

3.2.2 A memory optimized algorithm

TODO: use only array with size 2m

4 A polynomial 4-approximation algorithm for monotonically increasing convex f

We consider a modification of the problem discussed in chapter 3. Assuming that f is convex and monotonically increasing, we can modify our algorithm to obtain a polynomial time 4-approximation algorithm.

4.1 Graph for a 4-optimal schedule

We modify our graph from chapter 3.1 to the reduce the number of vertices. For this, we stop adding m vertices for each timestep, but use vertices that approximate the number of active servers instead. First, let $b \coloneqq \lceil \log_2(m) \rceil$. We add vertices $(t, 0)$ and $(t, 2^i), \forall t \in [T-1], 0 \leq i \leq b$. All edges and weights are added analogous to chapter 3.1.

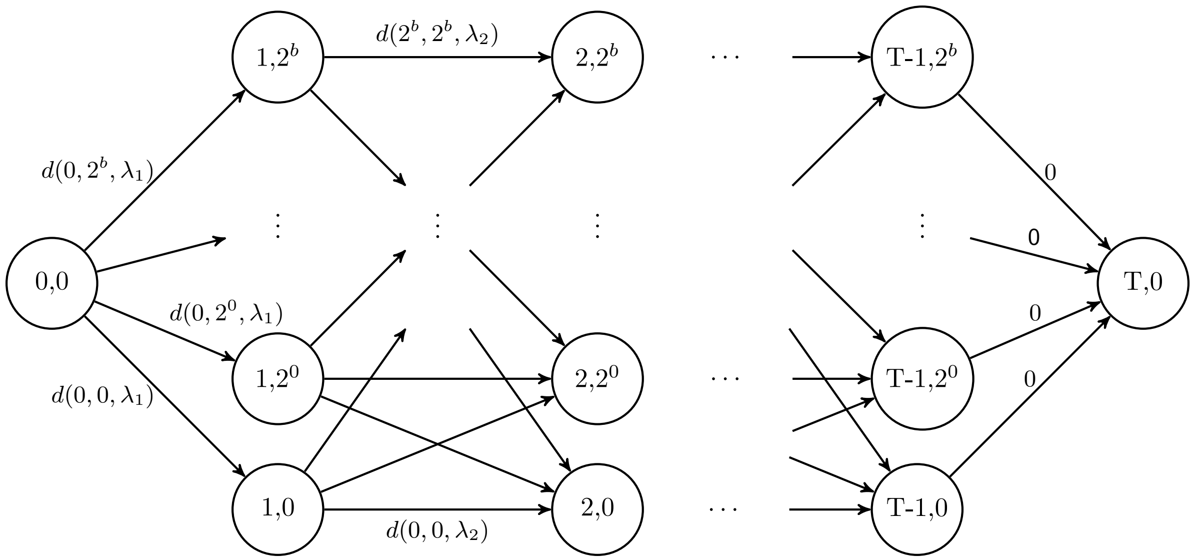


Figure 2: Graph for a 4-approximation algorithm

Definition 4.1. Let $\mathcal{X} = (x_0, \ldots, x_T)$ be a schedule and $t > 0$.
We say that $\mathcal{X}$ changes its **state** at time t if

$$x_t \neq x_{t-1}$$

and that $\mathcal{X}$ changes its **2-state** at time t if

$$x_t = 0 \quad \text{or} \quad x_t \notin \left(2^{\lfloor \log_2(x_{t-1}) \rfloor}, 2^{\lceil \log_2(x_{t-1}) \rceil}\right)$$

Proposition 4.2.

1. *Any given optimal schedule $\mathcal{X}$ can be transformed to a 4-optimal schedule $\mathcal{X}'$ which corresponds to a path P from $(0,0)$ to $(T,0)$ with $costs(\mathcal{X}') = costs(P)$.*

2. *Any shortest path P from $(0,0)$ to $(T,0)$ corresponds to a 4-optimal schedule $\mathcal{X}$ with $costs(P) = costs(\mathcal{X})$.*

Proof.

1. Assume we have an optimal schedule identified by $\mathcal{X} = (x_0, \ldots, x_T)$. For $0 \leq t < T$ we inductively set:

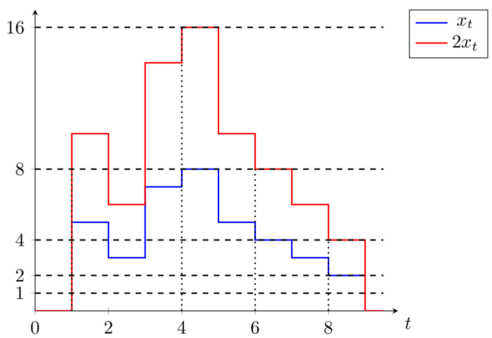
$$x'_0 \coloneqq 0, \qquad x'_{t+1} \coloneqq \begin{cases} \min\left\{2^{\lfloor \log_2(2x_{t+1}) \rfloor}, 2^b\right\}, & \text{if } 0 < x_t \leq x_{t+1} \\ 2^{\lceil \log_2(2x_{t+1}) \rceil}, & \text{if } 0 < x_{t+1} < x_t \text{ and } x'_t \geq 4x_{t+1} \\ x'_t, & \text{if } 0 < x_{t+1} < x_t \text{ and } x'_t < 4x_{t+1} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Then let $\mathcal{X}' \coloneqq (x'_0, \ldots, x'_T)$ be the modified sequence of active servers. Notice that $x_t \leq x'_t \leq 4x_t$ holds as x'_t is at most the smallest power of two larger than $2x_t$ which

implies that $\mathcal{X}'$ is feasible.
We can now construct a feasible path in our graph from $\mathcal{X}'$ as follows:

$$\begin{aligned} &\text{First set} && e_t := \Big(\big(t, \mathcal{X}'(t)\big), \big(t+1, \mathcal{X}'(t+1)\big)\Big), && \forall t \in \{0, \dots, T-1\} \\ &\text{then set} && P := (e_0, \dots, e_{T-1}) \end{aligned}$$

By the definition of the edges' weights it follows that $costs(\mathcal{X}') = costs(P)$.
Next, let $(t_0 = 0, t_1, \dots, t_n = 0)$ be the sequence of times where the optimal schedule $\mathcal{X}$ changes its 2-state. Notice that the modified schedule $\mathcal{X}'$ changes its state only at times t_i and that $2x_{t_i} \leq x'_{t_i}$ holds (TODO: only if not discrete but continous time steps). This can be seen exemplarily in figure 3 by obvserving that $\mathcal{X}'$ changes its state only if $\mathcal{X}$ crosses or touches a bordering power of two.



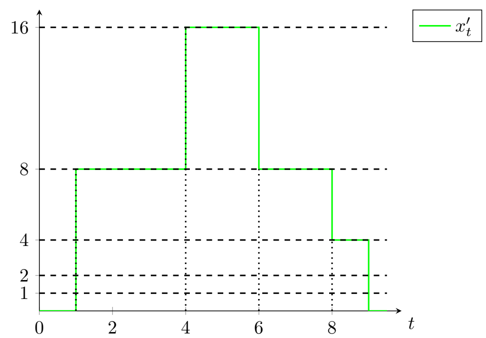


Figure 3: Adaption of an optimal schedule

For this reason, we now only have to consider the fraction of costs of $\mathcal{X}'$ and $\mathcal{X}$ between time steps t_{i-1} and t_i

$$\frac{costs(\mathcal{X}', t_{i-1}, t_i)}{costs(\mathcal{X}, t_{i-1}, t_i)} \tag{5}$$

For $x_{t_i} = 0$ it follows from (**??**) that $costs(\mathcal{X}', t_{i-1}, t_i) = costs(\mathcal{X}, t_{i-1}, t_i) = 0$. Hence, we can restrict ourselves to $0 < t_i < T$ with $x_{t_i} \neq 0$. The costs incurred by $\mathcal{X}'$ are given by

$$\begin{aligned} costs(\mathcal{X}', t_{i-1}, t_i) &= \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + x'_{t_i} f(\lambda_{t_i}/x'_{t_i}) && \text{by } (\mathbf{??}) \\ &\leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x'_{t_i}) && \text{by } (4) \\ &\leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i}) && \text{f monotonically increasing} \\ \implies \quad costs(\mathcal{X}', t_{i-1}, t_i) &\leq \beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i}) && (6) \end{aligned}$$

and the costs of $\mathcal{X}$ by

$$costs(\mathcal{X}, t_{i-1}, t_i) = \beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i}) \tag{7}$$

W.l.o.g. we may assume $x_{t_i} f(\lambda_{t_i}/x_{t_i}) > 0$, otherwise the claim follows trivially. (TODO: is it really trivial?)

(i) $\underline{x_{t_i} \leq x_{t_{i-1}}}$: From (4) it follows that $x'_{t_i} \leq x'_{t_{i-1}}$. Thus, we can simplify (5):

$$\begin{aligned}
\frac{costs(\mathcal{X}', t_{i-1}, t_i)}{costs(\mathcal{X}, t_{i-1}, t_i)} &\leq \frac{\beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (6),(7)} \\
&= \frac{4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{x_{t_i} f(\lambda_{t_i}/x_{t_i})} && (x_{t_i} \leq x_{t_{i-1}} \text{ and } x'_{t_i} \leq x'_{t_{i-1}}) \\
&= 4
\end{aligned}$$

(ii) $\underline{x_{t_i} > x_{t_{i-1}}}$: From (4) it follows that $x'_{t_i} \geq x'_{t_{i-1}}$. Thus, we can simplify (5):

$$\begin{aligned}
\frac{costs(\mathcal{X}', t_{i-1}, t_i)}{costs(\mathcal{X}, t_{i-1}, t_i)} &\leq \frac{\beta \max\{0, x'_{t_i} - x'_{t_{i-1}}\} + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta \max\{0, x_{t_i} - x_{t_{i-1}}\} + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (6),(7)} \\
&= \frac{\beta(x'_{t_i} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && (x_{t_i} > x_{t_{i-1}} \text{ and } x'_{t_i} \geq x'_{t_{i-1}}) \\
&= \frac{\beta(\min\{2^{\lfloor \log_2(2x_{t_i}) \rfloor}, 2^b\} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by (4)} \\
&\leq \frac{\beta(2^{\lfloor \log_2(2x_{t_i}) \rfloor} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\
&\leq \frac{\beta(2x_{t_i} - x'_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\
&\leq \frac{\beta(2x_{t_i} - 2x_{t_{i-1}}) + 4x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} && \text{by } (2x_{t_{i-1}} \leq x'_{t_{i-1}}) \\
&\leq 4 \; \frac{\frac{1}{2}\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})}{\beta(x_{t_i} - x_{t_{i-1}}) + x_{t_i} f(\lambda_{t_i}/x_{t_i})} \\
&\leq 4
\end{aligned}$$

From (i) and (ii) it follows:

$$costs(\mathcal{X}') \leq 4costs(\mathcal{X})$$

2. From 1 we obtain that we can construct a 4-optimal path P' from any optimal schedule. Now, let P be a shortest path. We have $costs(P) \leq costs(P') < \infty$, and since every path P with $costs(P) < \infty$ corresponds to a feasible schedule $\mathcal{X}$ with $costs(P) = costs(\mathcal{X})$, $\mathcal{X}$ must also be at least 4-optimal.

□

References

[1] Minghong Lin, Adam Wierman, Lachlan L. H. Andrew, and Eno Thereska. Dynamic right-sizing for power-propotional data centers. *IEEE/ACM Transactions on Networking (TON)*, 21:1378–1391, 2013.

Appendix

Below, we give an overview of just given definitions and conventions commonly referred to in our paper:

Good idea to have an appendix?

- Input:
  - $m \in \mathbb{N} \ldots$ number of homogeneous servers
  - $T \in \mathbb{N} \ldots$ number of time slots
  - $\lambda_1, \ldots, \lambda_T \in [0, m] \ldots$ arrival rates
  - $\Lambda := (\lambda_1, \ldots, \lambda_T) \ldots$ sequence of arrival rates
  - $\beta \in \mathbb{R}_{\geq 0} \ldots$ switching costs of a server
  - $f : [0, 1] \to \mathbb{R} \ldots$ convex operating costs function of a server
  - $\mathcal{I} := (m, T, \Lambda, \beta, f) \ldots$ input of a problem instance
- Problem statement:
  - $s_{i,t} \ldots$ state of server i at time t, i.e. sleeping (0) or active(1)
  - $S_i := (s_{i,1}, \ldots, s_{i,T}) \in \{0, 1\}^T \ldots$ sequence of states for server i
  - $\lambda_{i,t} \in [0, 1] \ldots$ assigned load for server i at time t
  - $L_i := (\lambda_{i,1}, \ldots, \lambda_{i,T}) \in [0, 1]^T \ldots$ sequence of assigned loads for server i
  - $\mathcal{S} := (S_1, \ldots, S_m) \ldots$ sequence of all state changes
  - $\mathcal{L} := (L_1, \ldots, L_m) \ldots$ sequence of all assigned loads
  - $\Sigma := (\mathcal{S}, \mathcal{L}) \ldots$ schedule for a problem instance $\mathcal{I}$
- Miscellaneous:
  - $x_t \ldots$ number of active servers at time t
  - $\mathcal{X} := (x_1, \ldots, x_T) \ldots$ sequence of number of active servers
- Conventions:
  - $\lambda_t = 0$ for all $t \notin [T]$, i.e. there is no load before and after the scheduling process
  - $s_{i,t} = 0$ for all $t \notin [T]$, i.e. all servers are powered down before and after the scheduling process