Algorithm 1 Optimal schedule for $m = 2$ homogeneous servers

Require: Convex cost function f, $\lambda_0 = \lambda_T = 0$, $\forall t \in [T-1] : \lambda_t \in (0, 2]$

1: **function** SCHEDULE($T, \beta, \lambda_1, \ldots, \lambda_{T-1}$)

2:     **if** $T < 2$ **then return** $(0, 0)$

3:     $m_{1,1}.prev \leftarrow m_{1,2}.prev \leftarrow 0$

4:     $m_{1,1}.costs \leftarrow$ COSTS$(\lambda_1, 1) + \beta$ ▷ Costs using 1 server at $t = 1$

5:     $m_{1,2}.costs \leftarrow$ COSTS$(\lambda_1, 2) + 2 * \beta$ ▷ Costs using 2 servers at $t = 1$

6:     **for** $t \leftarrow 2$ to $T - 1$ **do**

7:         $m_{t,1}.prev \leftarrow prev \leftarrow$ OPTIMAL$(t-1)$ ▷ Shortest path using 1 machine

8:         $m_{t,1}.costs \leftarrow m_{t-1,prev} +$ COSTS$(\lambda_t, 1)$

9:         $acc \leftarrow$ COSTS$(\lambda_t, 2)$ ▷ Shortest path using 2 machines

10:         $m_{t,2}.costs \leftarrow m_{t-1,2}.costs + acc$

11:         $acc \leftarrow m_{t-1,1}.costs + \beta + acc$

12:         **if** $acc < m_{t,2}.costs$ **then**

13:             $m_{t,2}.costs \leftarrow acc$

14:             $m_{t,2}.prev \leftarrow 1$

15:         **else**

16:             $m_{t,2}.prev \leftarrow 2$

17:     $x_0 \leftarrow x_T \leftarrow 0$ ▷ Extract shortest path from calculations

18:     $prev \leftarrow$ OPTIMAL$(T-1)$

19:     **for** $t \leftarrow T - 1$ to 1 **do**

20:         $x_t \leftarrow prev$

21:         $prev \leftarrow m_{t,prev}.prev$

22:     **return** $(x_0, \ldots, x_T)$

23:

24: **function** OPTIMAL(t)

25:     **if** $m_{t,1}.costs < m_{t,2}.costs$ **then**

26:         **return** 1

27:     **return** 2

28:

29: **function** COSTS(λ, x)

30:     **if** $x < \lambda$ **then**

31:         **return** ∞ ▷ Too few servers

32:     **return** $x * f(\lambda / x)$

Sketch for proof of correctness. We show that the algorithm calculates the correct costs and returns an optimal schedule $\forall T \in \mathbb{N}_{\geq 2}$ using induction.

Basis: For $T = 2$ it holds $\lambda_0 = \lambda_2 = x_0 = x_T = 0$ and $\lambda_1 \in (0, 2]$. Therefore, the costs for the optimal schedule are $min\big\{f(\lambda_1) + \beta, f(\lambda_1/2) + 2 * \beta\big\}$. These costs are calculated at line 4-5 and minimized at line 18.

Inductive step: Assuming the algorithm delivers a correct result for $T \in \mathbb{N}_{\geq 2}$, we show that it delivers a correct result for $T + 1$.
Firstly, we recognize that the executed statements of the algorithm for T and $T + 1$ are the same up to the last iteration of the for-loop at line 6. Hence, using the induction hypothesis, we can assume that $m_{1,\{1,2\}}$ up to $m_{T-1,\{1,2\}}$ are calculated correctly at this point.

In order to calculate the minimum costs for using 1 machine at time T, we simply need to find $min\{m_{T-1,1}, m_{T-1,2}\} + f(\lambda_T)$. This step is handled at line 7-8.
In order to calculate the minimum costs for using 2 machines at time T, we need to find $min\{m_{T-1,1} + \beta, m_{T-1,2}\} + f(\lambda_T/2)$. This step is handled at line 9-16.
Note: We do not need to consider possibilities where 2 servers are active but only 1 server is used. This follows from the convexity of f:

$$\begin{aligned} f\left(\frac{0+\lambda_t}{2}\right) = & \quad f\left(\frac{\lambda_t}{2}\right) \overset{convexity}{\leq} \frac{f(0)+f(\lambda_t)}{2} \\ \Leftrightarrow & \quad \underbrace{2 * f\left(\frac{\lambda_t}{2}\right)}_{\text{using 2 servers}} \leq \underbrace{f(0)+f(\lambda_t)}_{\text{2 active servers, using only 1}} \end{aligned}$$

□