Foundations of Mathematics and the Foundational Crisis

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Overview

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- Desire for a universal and consistent system

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- And then came Russell. . .

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Answer: $R \in R \iff R \notin R$, a contradiction!

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- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism

Source: `geopolicraticus.tumblr.com/post/142561195372`

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- Chief work: Principia Mathematica

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- Nonetheless, many adopted it as a new foundation.

Principia Mathematica

$\ast$**54·43.** $\vdash :.\ \alpha, \beta \in 1 \,.\supset:\, \alpha \cap \beta = \Lambda \,.\equiv.\, \alpha \cup \beta \in 2$

Dem.

$\vdash . \ast 54{\cdot}26 . \supset \vdash :.\ \alpha = \iota`x \,.\, \beta = \iota`y \,.\supset:\, \alpha \cup \beta \in 2 \,.\equiv.\, x \neq y \,.$

$[\ast 51{\cdot}231] \qquad \equiv .\, \iota`x \cap \iota`y = \Lambda \,.$

$[\ast 13{\cdot}12] \qquad \equiv .\, \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . \ast 11{\cdot}11{\cdot}35 . \supset$

$\vdash :.\ (\exists x, y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.\supset:\, \alpha \cup \beta \in 2 \,.\equiv.\, \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . \ast 11{\cdot}54 . \ast 52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Principia Mathematica's infamous proof of $1 + 1 = 2$. It was not until page 379 that this proof was possible.

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- The existence of any mathematical object is equivalent to the possibility of its construction, according to Brouwer.
  - $\Rightarrow$ No antinomies as paradoxical sets cannot be constructed.
- Consequently, some assumptions of classical logic must be rejected.

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Proof. It is known that $\sqrt{2}$ is irrational. Let us consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, our statement is proved. If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. $\square$

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- *"Taking this tertium non datur (law of excluded middle) from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists."*

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- Consequently, only a few scholars adhered to intuitionism.

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- Mathematics must not justify the existence of its objects as its objects are just meaningless shapes.

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- The dream of a complete and consistent mathematical system

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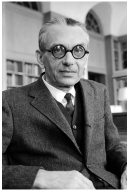
- Optimism for a complete and consistent formal system grew...

The End of the Crisis

- . . . but then came Gödel.



Source: `newyorker.com/tech/elements/waiting-for-godel`

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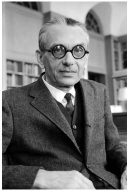

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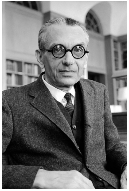

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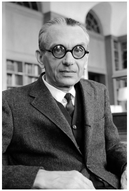

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Theorem (First Incompleteness Theorem)

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Theorem (Second Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.

Modern Mathematics

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  - Zermelo-Fraenkel set theory (ZFC) as established foundation
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- Justification of foundations is often regarded philosophical work.

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- Can we trust proofs by computers?
- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

The End

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