

Foundations of Mathematics and the Foundational Crisis

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- But it is not the first controversy in mathematical history...

Overview

1 History of Mathematics

- First steps
- Ancient Greece

2 Foundational Crisis

- Causes
- Logicism
- Formalism
- Intuitionism
- Peak and End

3 Aftermath and Prospects

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- Maths is about abstraction.
- Numbers were the first abstraction.

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- Maths served as an applied tool.
 - No proofs, but heuristics and pictures

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- 4 That all right angles are equal to one another.
- 5 That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.”

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 - Predicate logic by Frege
- Desire for universal and consistent system

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Here comes the Russell

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The End



Georg Cantor.

Beiträge zur Begründung der transfiniten Mengenlehre.

Mathematischen Annalen, 46:481, 1895.