

Foundations of Mathematics and the Foundational Crisis

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June 10, 2017

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2. The Foundational Crisis
 - 2.1 Logicism
 - 2.2 Intuitionism
 - 2.3 Formalism
 - 2.4 Peak and End
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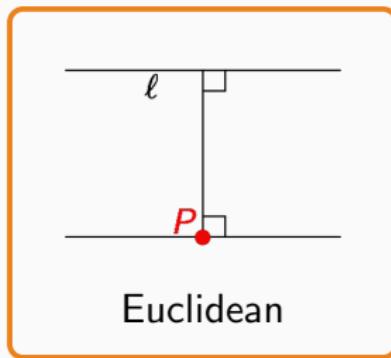
Causes of the Crisis

Euclid's Elements – A Work of Timeless Certainty



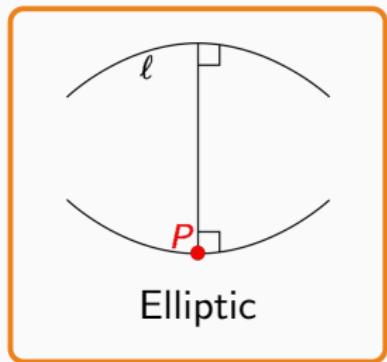
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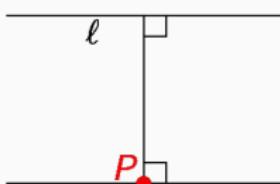


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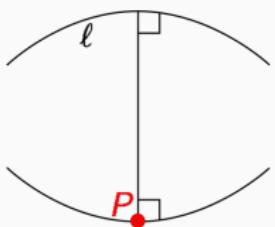
Elliptic



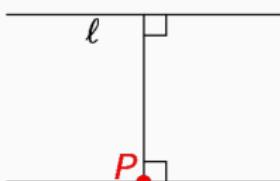
Euclidean

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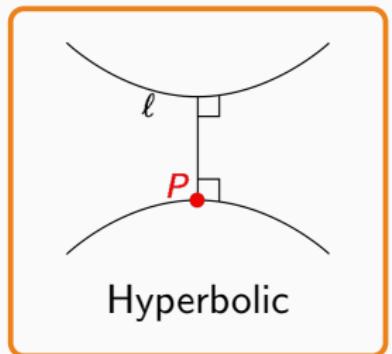
Given a line ℓ and a point P , there exist at least two lines through P parallel to ℓ .



Elliptic



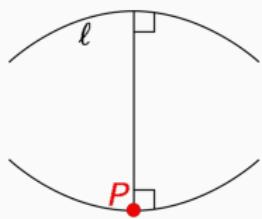
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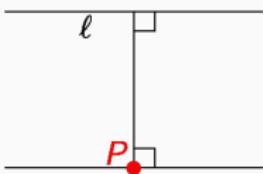
Hyperbolic

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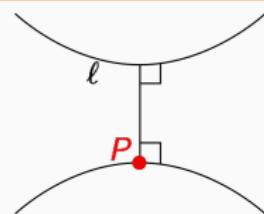
Given a line ℓ and a point P , there exist
? many lines through P parallel to ℓ .



Elliptic



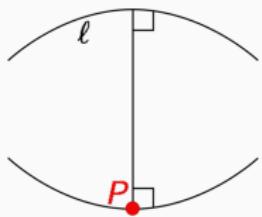
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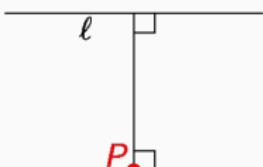
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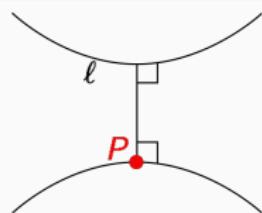
Which axioms represent the truth?



Elliptic



Euclidean



Hyperbolic

A Search for Foundations

Axiomatisation of systems in the late 19th century

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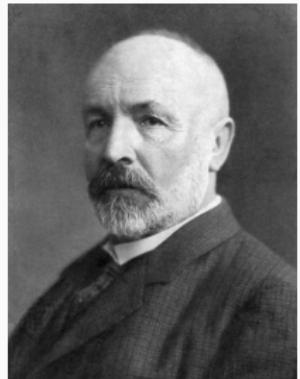
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Desire for a universal and consistent system

Cantor's Set Theory

“A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set.”

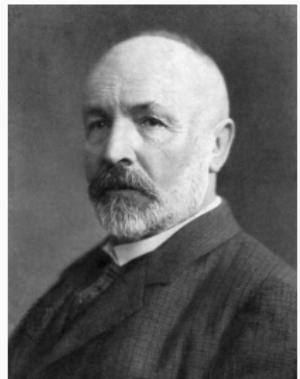
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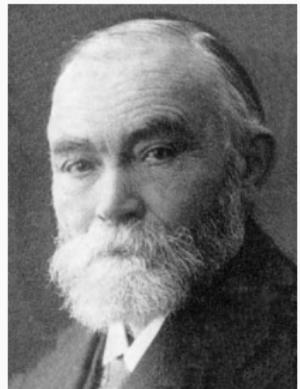
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Certainly universal, but fairly naive.

Frege's Logic System

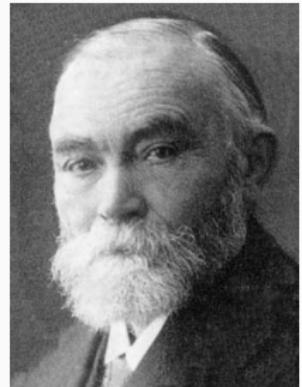
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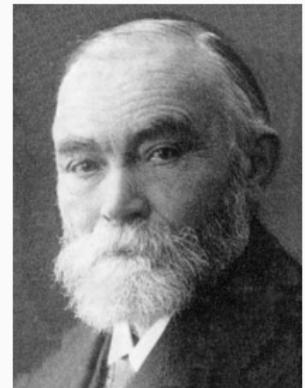
- Sophisticated work, but not well received



Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.

- Sophisticated work, but not well received



Then one day, just before finishing his work, he received a letter from Russell...

Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

$$R := \{X \mid X \notin X\}$$

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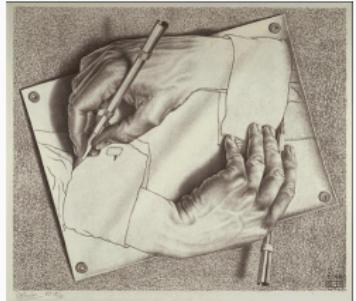
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Question: Is R a member of itself? That is, does $R \in R$ hold?

Answer: $R \in R \iff R \notin R$, a contradiction!

The Begin of the Crisis

Self-referentiality broke Cantor's and Frege's system.



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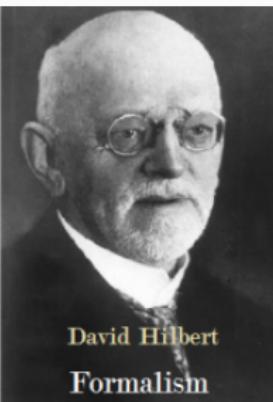
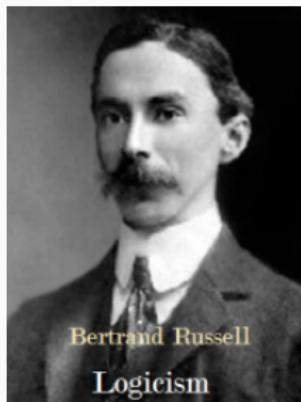
A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism



The Foundational Crisis

Logicism

A Foundation Made of Logic

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- Only fundamentally logical laws are used as axioms.
 - Justifications that used axioms are self-evident truths



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- Type theory to avoid antinomies
- Difficulties in explaining some axioms
 - Axiom of reducibility
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- Regarded as “the outstanding example of an unreadable masterpiece”

Principia Mathematica

Principia Mathematica's infamous proof of $1 + 1 = 2$

*54·43. $\vdash \alpha, \beta \in 1. \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54·26. \supset \vdash \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51·231] $\equiv . \iota'x \cap \iota'y = \Lambda .$

[*13·12] $\equiv . \alpha \cap \beta = \Lambda \quad (1)$

$\vdash . (1) . *11·11·35 . \supset$

$\vdash : (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (2) . *11·54 . *52·1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

The Foundational Crisis

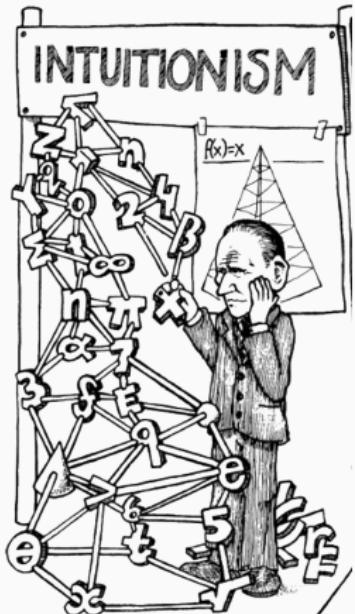
Intuitionism

Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.

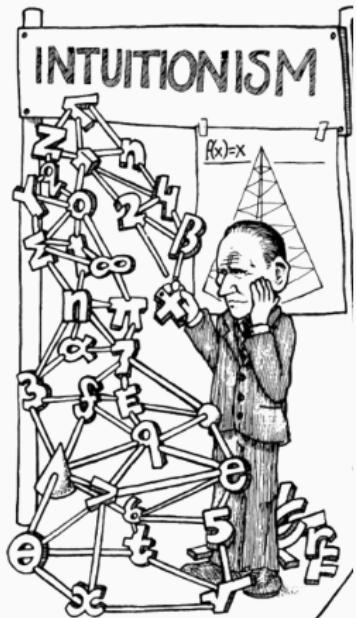
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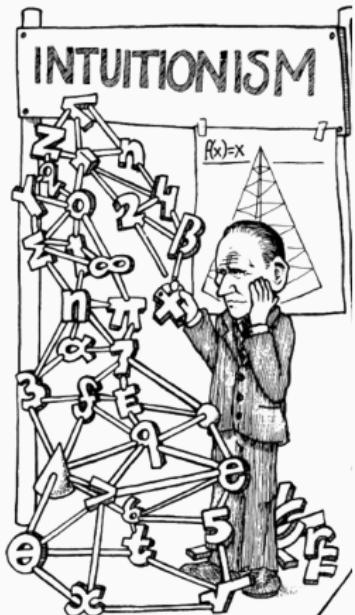
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Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.
- The existence of an object is equivalent to the possibility of its construction.
- Consequently, some assumptions of classical logic must be rejected.



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For any proposition P , either P or its negation is true.

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Only a few scholars adhered to intuitionism.



The Foundational Crisis

Formalism

Mathematics as a Symbolic Game

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- Mathematics shall be based on symbols and axioms that describe syntactic operations on symbols.
- Mathematics does not need to justify the existence of its objects since its objects are just meaningless shapes.



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 1. Formalise a system that is able to derive all of mathematics using syntactical operations
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- The dream of a complete and consistent mathematical system

The Foundational Crisis

Peak and End

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- Optimism for a complete and consistent formal system grew... .

The End of the Crisis

- ... but then came Gödel.



Source: [newyorker.com/tech/elements/
waiting-for-godel](http://newyorker.com/tech/elements/waiting-for-godel)

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- ... but then came Gödel.
- In 1931, he proved that there is no sufficiently strong, complete, and consistent formal system.



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Theorem (First Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic is incomplete.

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Theorem (First Incompleteness Theorem)

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Theorem (Second Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.

Aftermath and Prospects

Modern Mathematics

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Modern Mathematics

- To this day, formalism poses the foundation of mathematics.
 - Zermelo-Fraenkel set theory (ZFC) as established foundation
- Most modern mathematicians do not deal with foundational research but try to extend a specific branch of mathematics.
- The justification of foundations is often regarded philosophical work.

The Next Crisis?

- Digitisation of mathematics

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- Can we trust proofs by computers?

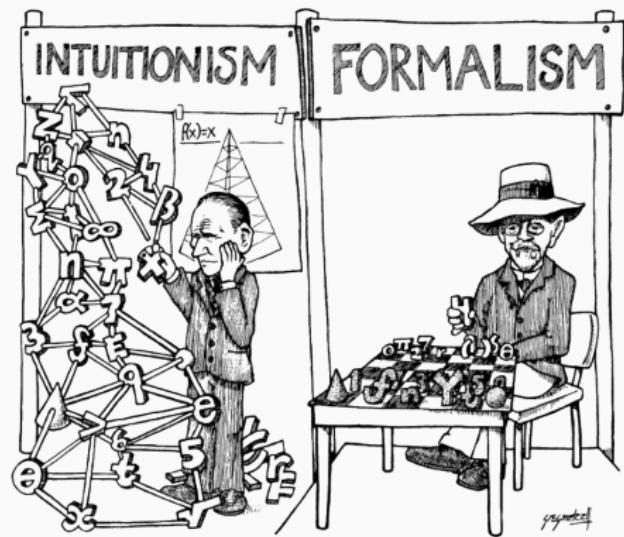
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The Next Crisis?

- Digitisation of mathematics
- Can we trust proofs by computers?
- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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