

Foundations of Mathematics and the Foundational Crisis

Kevin Kappelmann

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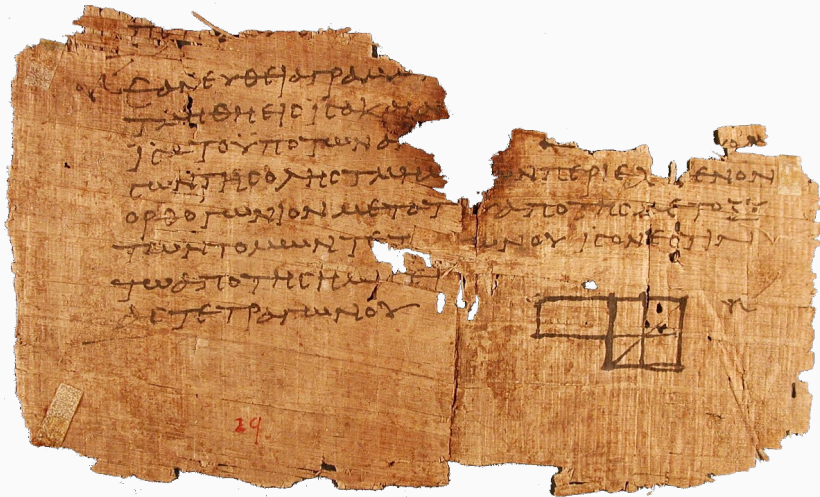
Technical University of Munich

Overview

1. Causes of the Crisis
2. The Foundational Crisis
 - 2.1 Logicism
 - 2.2 Intuitionism
 - 2.3 Formalism
 - 2.4 End of the Crisis
3. Aftermath and Prospects

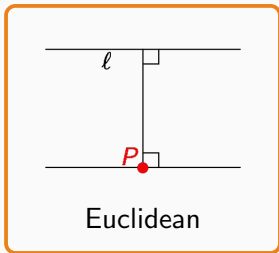
Causes of the Crisis

Euclid's Elements – A Work of Timeless Certainty



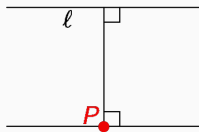
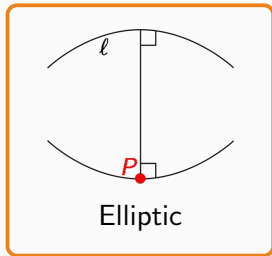
Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ , there exists
one line through P parallel to ℓ .



Euclid's Elements – A Work of Timeless Certainty

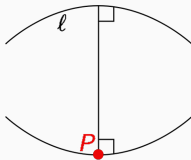
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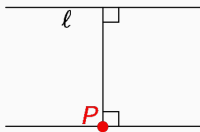
Euclidean

Euclid's Elements – A Work of Timeless Certainty

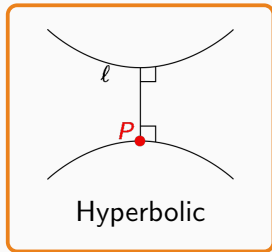
Given a line ℓ and a point P not lying on ℓ , there exist
at least two lines through P parallel to ℓ .



Elliptic



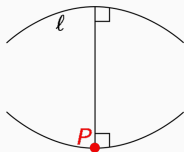
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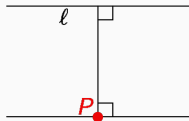
Hyperbolic

Euclid's Elements – A Work of Timeless Certainty

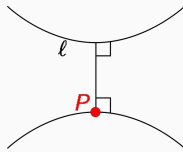
Given a line ℓ and a point P not lying on ℓ , there exist
? many lines through P parallel to ℓ .



Elliptic



Euclidean



Hyperbolic

Which axioms represent the truth?

A Search for Foundations

Axiomatisation of systems in the late 19th century

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- Arithmetic of natural numbers by Peano

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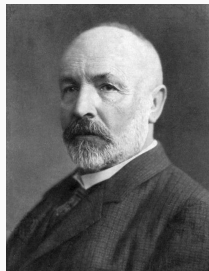
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Desire for a universal and consistent system

Cantor's Set Theory

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set."

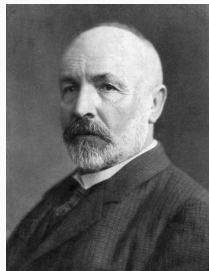
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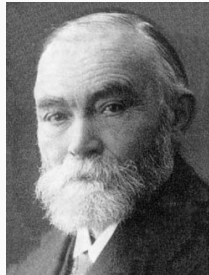
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Certainly universal, but fairly naive

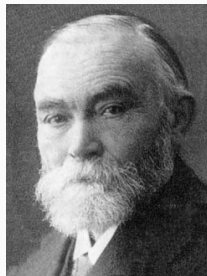
Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.



Frege's Logic System

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Just before finishing his work, he received a letter from Russell. . .

Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

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Answer: $R \in R \iff R \notin R$, a contradiction!

The Begin of the Crisis

“Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.”

– Gottlob Frege

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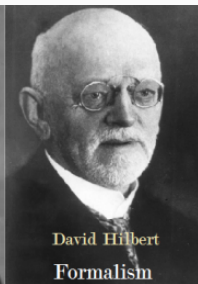
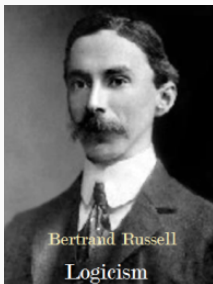
A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism



The Foundational Crisis

Logicism

A Foundation Made of Logic

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A Foundation Made of Logic

- Russell and Whitehead revisited Frege's idea of reducing mathematics to logic.
- Only fundamentally logical laws as axioms
 - Justify the use of the axioms



- Type theory to avoid antinomies

Principia Mathematica

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Principia Mathematica

- Type theory to avoid antinomies
- Difficulties in explaining some axioms
- Regarded as “the outstanding example of an unreadable masterpiece”

Principia Mathematica's infamous proof of $1 + 1 = 2$

***54.43.** $\vdash :: \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54.26 . \supset \vdash :: \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51.231] $\equiv . \iota'x \cap \iota'y = \Lambda .$

[*13.12] $\equiv . \alpha \cap \beta = \Lambda$ (1)

$\vdash . (1) . *11.11.35 . \supset$

$\vdash :: (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda$ (2)

$\vdash . (2) . *11.54 . *52.1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

The Foundational Crisis

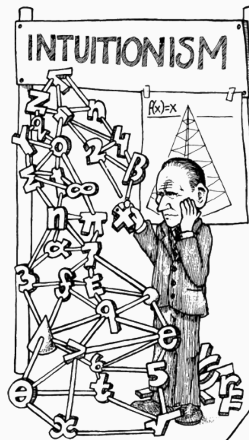
Intuitionism

Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.

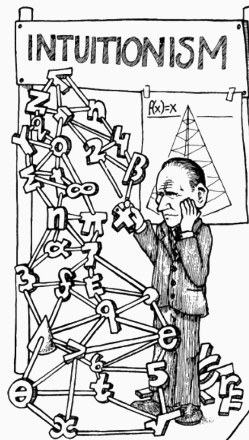
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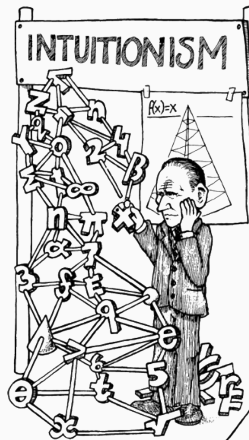
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- The existence of an object is equivalent to the possibility of its construction.



Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.
 - The existence of an object is equivalent to the possibility of its construction.
- ⇒ Some assumptions of classical logic must be rejected.



$$P \vee \neg P \equiv ?$$

Intuitionists reject the *law of excluded middle*:

For any proposition P , either P or its negation is true.

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If it is rational, our statement is proved.

If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. □

What a Hassle!

“Taking the law of excluded middle from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists.”

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Only a few scholars adhered to intuitionism.

The Foundational Crisis

Formalism

Mathematics as a Symbolic Game

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Mathematics as a Symbolic Game

- Mathematics shall be based on meaningless symbols and syntactic operations.
- No need to justify the existence of objects
- The system's consistency must be verified.

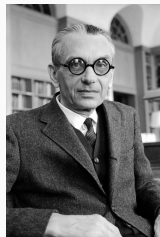


The Foundational Crisis

End of the Crisis

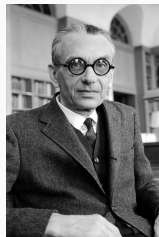
The Incompleteness of Mathematics

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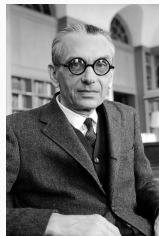


Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

The Incompleteness of Mathematics

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Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

Theorem (Second Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out cannot prove its own consistency.

Aftermath and Prospects

Modern Mathematics

To this day, formalism lays the foundation of mathematics.

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- Zermelo-Fraenkel set theory (ZFC) as established foundation

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- Zermelo-Fraenkel set theory (ZFC) as established foundation

Most mathematicians do not deal with foundational research.

Digitalisation of mathematics

Digitalisation of mathematics

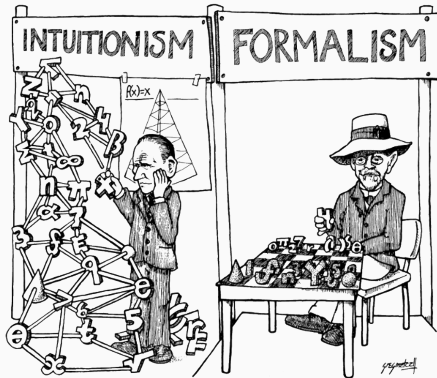
- Some see it as an inevitable enrichment; others face it with distrust.

Digitalisation of mathematics

- Some see it as an inevitable enrichment; others face it with distrust.
- Can we trust proofs by computers?

Are we part of the next
mathematical crisis?

Thanks for your attention! Any questions?



References I



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