Foundations of Mathematics and the Foundational Crisis

Kevin Kappelmann

Technical University of Munich

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Overview

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- Desire for a universal and consistent system

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- And then came Russell...

Russell's Paradox

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Answer: $R \in R \iff R \notin R$, a contradiction!

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- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism

Source: `geopolicraticus.tumblr.com/post/142561195372`

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- Chief work: Principia Mathematica

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  - Axiom of reducibility
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- Regarded as "the outstanding example of an unreadable masterpiece"
- Nonetheless, many adopted it as a new foundation.

Principia Mathematica

$*54\cdot43$. $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51\cdot231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13\cdot12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Principia Mathematica's infamous proof of $1 + 1 = 2$. It was not until page 379 that this proof was possible.

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- The existence of any mathematical object is equivalent to the possibility of its construction, according to Brouwer.
  - $\Rightarrow$ No antinomies since paradoxical sets cannot be constructed
- Consequently, some assumptions of classical logic must be rejected.

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Proof. It is known that $\sqrt{2}$ is irrational. Let us consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, our statement is proved. If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. $\square$

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- *"Taking this tertium non datur (law of excluded middle) from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists."*
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- Consequently, only a few scholars adhered to intuitionism.

Mathematics as a Symbolic Game

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- Mathematics shall be based on symbols and axioms that describe syntactical operations on symbols.
- Mathematics does not need to justify the existence of its objects since its objects are just meaningless shapes.

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- The dream of a complete and consistent mathematical system

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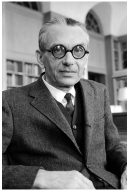
- Brouwer, in a state of frustration and despair, subsequently stopped publishing intuitionistic articles.
- Optimism for a complete and consistent formal system grew...

The End of the Crisis

- . . . but then came Gödel.



Source: `newyorker.com/tech/elements/waiting-for-godel`

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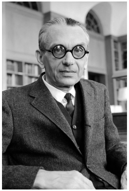
- In 1931, he proved that there is no sufficiently strong, complete, and consistent formal system.



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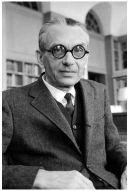

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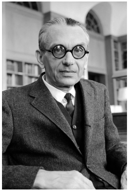

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Theorem (First Incompleteness Theorem)

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Theorem (Second Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.

Modern Mathematics

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  - Zermelo-Fraenkel set theory (ZFC) as established foundation
- Most modern mathematicians do not deal with foundational research but try to extend a specific branch of mathematics.
- The justification of foundations is often regarded philosophical work.

The Next Crisis?

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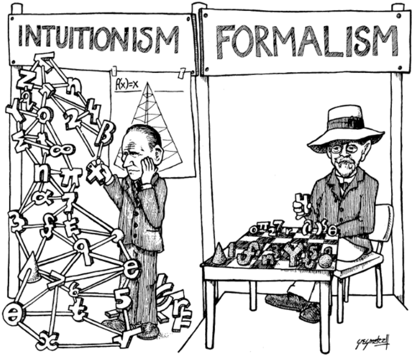
- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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