Foundations of Mathematics and the Foundational Crisis

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Overview

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4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

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- Desire for a universal and consistent system

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- And then came Russell. . .

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Answer: $R \in R \iff R \notin R$, a contradiction!

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- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism

Source: `geopolicraticus.tumblr.com/post/142561195372`

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- Regarded as "the outstanding example of an unreadable masterpiece."
- Nonetheless, many adopted it as a new foundation.

Principia Mathematica

$*54{\cdot}43.\quad \vdash :.\ \alpha, \beta \in 1 \,.\supset :\ \alpha \cap \beta = \Lambda \,.\equiv .\ \alpha \cup \beta \in 2$

Dem.

$\vdash .\ *54{\cdot}26 \,.\supset \vdash :.\ \alpha = \iota`x \,.\, \beta = \iota`y \,.\supset :\ \alpha \cup \beta \in 2 \,.\equiv .\ x \neq y \,.$

$[*51{\cdot}231] \qquad\qquad \equiv .\ \iota`x \cap \iota`y = \Lambda \,.$

$[*13{\cdot}12] \qquad\qquad \equiv .\ \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash .\ (1) \,.\, *11{\cdot}11{\cdot}35 \,.\supset$

$\qquad \vdash :.\ (\exists x, y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.\supset :\ \alpha \cup \beta \in 2 \,.\equiv .\ \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash .\ (2) \,.\, *11{\cdot}54 \,.\, *52{\cdot}1 \,.\supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Principia Mathematica's infamous proof of $1 + 1 = 2$. It was not until page 379 that this proof was possible.

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- The existence of any mathematical object ist equivalent to the possibility of its construction, according to Brouwer.
  - $\Rightarrow$ No antinomies as paradoxical sets cannot be constructed.
- Consequently, some assumptions of classical logic must be rejected.

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- *"Taking this tertium non datur (law of excluded middle) from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists."*
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- Consequently, only a few scholars adhered to intuitionism.

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- Mathematics must not justify the existence of its objects as its objects are just meaningless shapes.

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- The dream of a complete and consistent mathematical system
- However, this dream just stayed a dream...

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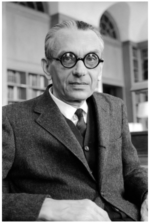
- Brouwer, in a state of frustration and despair, subsequently stopped publishing intuitionistic articles.
- Optimism for a complete and consistent formal system grew. . .

The End of the Crisis

- . . . but then came Gödel.



Source: `newyorker.com/tech/elements/waiting-for-godel`

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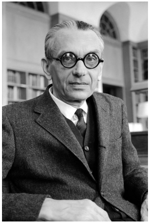
- …but then came Gödel.
- In 1931, he proved that there is no sufficiently strong, complete and consistent formal system.



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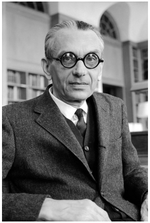

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Theorem (First Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic is incomplete.

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  - Zermelo-Fraenkel set theory (ZFC) as established foundation.
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- Justification of foundations is often regarded philosophical work.

The Next Mathematical Crisis?

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- Some see it as an inevitable enrichment; others face it with large distrust.
- Are we part of the next crisis?

The End

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