Foundations of Mathematics and the Foundational Crisis

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  - $\Rightarrow$ Controversies
- But it is not the first controversy in mathematical history...

Overview

1 Causes of the crisis

2 The Foundational Crisis
- Logicism
- Formalism
- Intuitionism
- Peak and End

3 Aftermath and Prospects

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4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

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- Desire for universal and consistent system

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- And then came Russell...

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Answer: $R \in R \iff R \notin R$, a contradiction!

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- Systems founded on Cantor's set theory were on shaky ground

Logicism

Formalism

Intuitionism

The End

Georg Cantor.
Beiträge zur Begründung der transfiniten Mengenlehre.
Mathematischen Annalen, 46:481, 1895.

Gottlob Frege.
Grundgesetze der Arithmetik, volume 2.
1902.