

# Foundations of Mathematics and the Foundational Crisis

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# Overview

- 1 Causes of the Crisis
- 2 The Foundational Crisis
  - Logicism
  - Intuitionism
  - Formalism
  - Peak and End
- 3 Aftermath and Prospects

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  - Arithmetic of natural numbers by Peano
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- Desire for a universal and consistent system

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- And then came Russell...

# Russell's Paradox

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Answer:  $R \in R \iff R \notin R$ , a contradiction!



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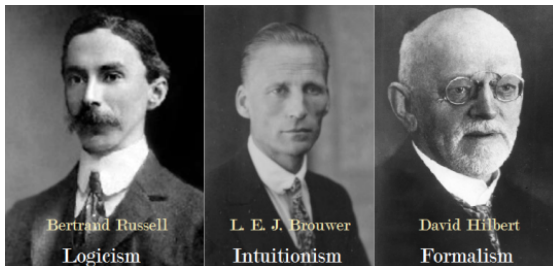
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- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

# The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism



Source: [geopolicraticus.tumblr.com/post/142561195372](https://geopolicraticus.tumblr.com/post/142561195372)

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- Chief work: Principia Mathematica

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  - Axiom of reducibility
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- Regarded as “the outstanding example of an unreadable masterpiece”
- Nonetheless, many adopted it as a new foundation.

# Principia Mathematica

\*54.43.  $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

*Dem.*

$\vdash . *54.26 . \supset \vdash :. \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51.231] \qquad \qquad \qquad \equiv . \iota'x \cap \iota'y = \Lambda .$

$[*13.12] \qquad \qquad \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11.11.35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11.54 . *52.1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that  $1 + 1 = 2$ .

Principia Mathematica's infamous proof of  $1 + 1 = 2$ . It was not until page 379 that this proof was possible.

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- The existence of any mathematical object is equivalent to the possibility of its construction, according to Brouwer.
  - ⇒ No antinomies since paradoxical sets cannot be constructed
- Consequently, some assumptions of classical logic must be rejected.

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- *“Taking this tertium non datur (law of excluded middle) from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists.”*  
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- Consequently, only a few scholars adhered to intuitionism.

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- Mathematics does not need to justify the existence of its objects since its objects are just meaningless shapes.

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  - 1 Formalise a system that is able to derive all of mathematics using syntactical operations
  - 2 Prove the system's consistency with metamathematical reasoning
- The dream of a complete and consistent mathematical system

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- Optimism for a complete and consistent formal system grew. . .

# The End of the Crisis

- ...but then came Gödel.



Source: [newyorker.com/tech/elements/  
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## Theorem (First Incompleteness Theorem)

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## Theorem (First Incompleteness Theorem)

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## Theorem (Second Incompleteness Theorem)

*Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.*

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- The justification of foundations is often regarded philosophical work.



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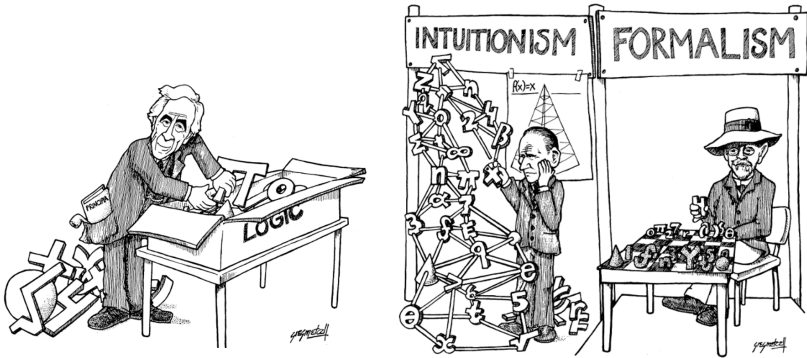
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- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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