Foundations of Mathematics - Grundlagenkrise

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Preface

Although — or maybe even because — mathematics is likely the most well-conceived and exact science of mankind, it has not been free of scepticism and controversies. In its heart, mathematics is dealing with the discovery of unchangeable, sempiternal truths. For this, we are using rigorous proofs. But what is it that we call a proof?

One might define a proof as a coherent chain of logical arguments leading from a set of premises to a conclusion. This definition, however, raises many new questions. What deserves to be called coherent? When is an argument logical? Do all humans follow the same rules of logic? Which premises are there to begin with?

Some of these difficult questions were essential during a heated phase in the beginning of the 20th century: the foundational crisis of mathematics. In this period, some of the greatest mathematicians tried to give different explanations for many of named questions. As a result of longstanding debates and rigorous work, we received the sophisticated foundations of modern mathematics.

Owing to this achievements, contemporary mathematicians are able to concentrate on the extension of mathematics rather than paying attention to its foundations. However, due to the invention of the computer in the late 20th century, a new field of proof theory has arisen. Computer scientists and mathematicians began to develop proof assistants and automated theorem provers. While former is designed to verify proofs typed by humans, latter acts on a fully automatic basis.

In particular automatic theorem provers gave raise to new questions regarding mathematical proofs. Does a computer generated proof have the same credibility as a proof written by a person? What is a proof and how can I persuade somebody that I am right? Does it suffice to understand every isolated step of it, or do I need to understand its holistic idea? Must it be accepted by some, a few, or even just one person? It seems like we are facing a new foundational crisis.

Nevertheless, these controversies are not focused by this paper, but rather they are subjects for debate in the seminar "Formal Proof in Mathematics and Computer Science" offered by the Chair for Logic and Verification at the Technical University of Munich in 2017. This paper shall start off the seminar by giving a brief overview of the foundations of mathematics with a focus on the foundational crisis in the 20th century.

Contents

1 Historical introduction and controversies

While one might think that mathematics as a subject about absolute certainty ought to be free of controversies, history taught us otherwise. In fact, it has been subjected to fiery disputations since ancient Greece. Before diving into the most famous controversy in the history of mathematics, namely the *foundational crisis* (in German *Grundlagenkrise*), we want to give a brief historical overview focusing on the foundations and some of the most well-known controversies[1] of mathematics.

1.1 Ancient mathematics

The history of mathematics is coined by a series of abstractions. Add two apples to three other apples, and you get five apples. Add two bananas to three other bananas, and you get five bananas. We abstract our results and derive: two plus three equals five. The concept of numbers might be the first mathematical abstraction in history. Unsurprisingly, arithmetics and geometry - owing to their inuitive nature - were the first developed branches of mathematics.

First evidence for more complex mathematics dates back to around 2400 BC. Egyptians and Babylonians used basic arithmetics and geometry for trading, taxation, building and construction, and land measurement. Mathematics by this time, however, was just regarded as an applied tool to solve practical problems rather than an exact science. Heuristics, pictures and vague analogies justified the use of given formulas rather than rigorous proofs.

The idea of demonstrating conclusions using coherent arguments, and thereby the development of the notion of a proof, was one of the great achievements in ancient Greece. Beginning with Thales in 600 BC, mathematics as a means of the exploration of perpetual truths began to raise. To no surprise, proofs where focused on arithmetics and geometry, which both seemed to consist of unquestionable truths.

In 500 BC the Pythagoreans did not only invent one of the most famous theorems in mathematics, but to our larger interest experienced the first noteworthy mathematical controversy. They were in firm belief that all numbers were commensurable; that is, for every pair of non-zero numbers a and b, the ratio $\frac{a}{b}$ is a rational number. It allegedly was the work of Hippasus who dismissed this ideal by proofing the existence of irrational numbers. Legend has it that Hippasus was sentenced to death by drowning for the discovery of this unbearable truth. What remains as a fact, however, is that Pythagorean mathematics changed drastically after this discovery.

Meanwhile, the Greek philosopher Zeno of Elea devised four paradoxes, now commonly known as Zeno's paradoxes, which mainly deal with the illusion of motion. The first paradox, referred to as Achilles and the tortoise, can be recounted as follows:

> Achilles and the Tortoise want to conduct a race. Achilles gives the Tortoise a head start since he clearly is the faster runner. But by the time Achilles has reached the point where the Tortoise started, the slow individual will have moved on a few steps to a new position. When Achilles again reaches this new position,

[1] We emphasise the smaller extent and impact of various mathematical disputations by using the word *controversy* as opposed to *crisis*.

> the labouring Tortoise will have moved on again. Each time Achilles reaches the point where the Tortoise was, the cunning reptile will always have moved a little way ahead; hence, it will always hold a lead.

Readers that took a foundational calculus class may already be able to refute this paradox. The solution demands for the calculus of infinitesimals and the concept of the continuum chiefly developed in the 17th century. Up to that time, no man was able to give Zeno a satisfying answer.

We want to forestall that paradoxes indeed turned out to not only pose a fruitful, philosophical subject of conversation but also play an important part in the history of mathematics. By questioning the truth of the seemingly indisputable, they offer the possibilty to reveal the inconsistency of a mathematical system at its core. We shall say more about this later.

Finally, we ought to name the most influential mathematical work of ancient Greece if not even of all time[2]: Euclid's Elements. The collection, consisting of 13 books, was written around 300 BC by the Greek mathematician Euclid in Alexandria. The books cover Euclidian geometry ("the geometry of our world") as well as elementary number theory, albeit not in an algebraic but geometric way. The intriguing thing, besides its exceptional volume, is its axiomatic, deductive treatment of mathematics. To this day, mathematicians most commonly use systems axiomatised in the same fashion as Euclid did over 2000 thousand years ago. Reading a proof from Euclid's Elements feels suprisingly similar to reading a proof from a modern textbook.

> "At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world." [6]
>
> — Bertrand Russell

Euclid's Elements was manifested as a work of timeless certainty. Nobody could doubt that! Or at least, so had been thought for a long time. The discovery of non-Euclidian geometry, though not shaking the consistency of Euclid's work, questioned the exclusive existence of a geometry as postulated by Euclid. We will discuss this in more detail in section 2.1.

1.2 An infinitely small crisis

The Greeks, albeit discussing the possibility of the continuum, conducted a "static" way of mathematics in the sense that subjects of interest were mainly those that deal with stationary objects, e.g. arithmetic and geometry. In addition, it was widely belived that objects, in particular time and space, were only finitely divisible. This kind of mathematics, however, is incapable of finding adequate answers to Zeno's paradoxes (see section 1.1). It was not until the middle of the 17th century that Leibniz as well as Newton independently developed the tools for infinitesimal calculus capable of solving the mistery of infinite divisibilty.

Infinitesimal calculus, these days simply known as calculus, is the study of continuous change. Although Leibniz and Newton were able to calculate the derivatives and integrals

[2] Up to the middle of the 19th century, Euclid's Elements is said to had been having a wide circulation rivaled only by the bible.

of functions using a notion of “infinite small quantities”, they were not able to deliver an elaborated foundation for their used methods. Rather, they used heuristic principles, such as the law of continuity[3], as justification which gave raise to valid scepticism and critques most notably by Bishop Berkeley in 1734. He addressed the uncertainty and ominosity of the calculus derived by Leibniz and Newton in his book “The Analyst”. For this, he satirically compared the use of “infinite small quantities” and its vague justification with critiques common to religion. His book subtitled:

> “A DISCOURSE Addressed to an Infidel MATHEMATICIAN. WHEREIN It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith”

In addition to this uncertainty, when Newton and Leibniz first published their results, there was great controversy over which mathematician deserved credit. These factors ultimately caused what we shall call the second controversy of mathematics which divided English-speaking mathematicians from continental European mathematicians for many years[4].

2 The foundational crisis

While controversies like that of the Pythagoreans or the infinitesimal calculus certainly influenced the development of modern mathematics, they do not deserve to be called a proper crisis, since they were restricted to a small tract of mathematics or did not cause an exceptional impact on mathematical foundations.

The foundational crisis of mathematics started in 1902 with Russell’s paradox and ended in 1931 with an intriguing twist by Kurt Gödel and his incompleteness theorems. We will firstly examine the causes that lead to this milestone of mathematics and then proceed to chronologically discuss involved persons, their representated schools of thought, and important events.

2.1 Causes

In 19th century, Euclid’s Elements had been established as a mathematical showcase for more than 2000 years due to its axiomatic, deductive treatment of mathematics. The first book began with his five postulates[5] of geometry:

> “Let the following be postulated:
>
> 1. To draw a straight line from any point to any point.
> 2. To produce [extend] a finite straight line continuously in a straight line.
> 3. To describe a circle with any centre and distance [radius].

[3] the principle that ”whatever succeeds for the finite, also succeeds for the infinite”

[4] Despite its critiques, infinitesimal calculus kept being used as a successful means of calculation. Its sound foundations were formalised 150 years later by Cauchy and Weierstrass.

[5] Euclid used the word *postulate* instead of *axiom*.

4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Whereas postulates 1–4 seem fairly reasonable and understandable, many mathematicians became curious about Euclid's fifth postulate. The postulate, which to this day is known as the parallel postulate, seems more complicated than its four predecessors and troubled many geometers for at least a thousand years. Many believed it could be proved as a theorem from the first four postulates, but all attempts to do so failed.

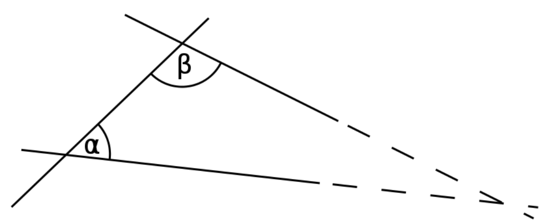

Figure 1: Visualisation of the parallel postulate from [3, Wikipedia]. If the sum of the interior angles α and β is less than $180°$, the two straight lines, produced indefinitely, meet on that side.

In circa 1813, Carl Friedrich Gauß worked out that in fact the fifth postulate is independent of the other postulates; that is to say, the parallel postulate as well as its negation can be added to the first four postulates without causing any inconsistencies. While the first case gives us the geometry as postulated by Euclid, the latter revealed a new kind of geometry commonly referred to as non-Euclidian geometry.

Though Gauß did not publish his finding, it was rediscovered just a few years later. The announcement had a large impact. Suddenly, Euclid's Elements, and with it, the certainty of mathematics, began to totter; mathematicians became greatly sceptical of known systems — Which axioms pose the truth? Which will cause harm? Which are dispensible and which are not?

What followed in the late 19th century is a series of rigorous axiomatisation of mathematical systems. Giuseppe Peano axiomatised the arithmetics of natural numbers, Moritz Pasch and David Hilbert modernised the foundations of geometry, and Gottlob Frege introduced an axiomatised predicate logic in his revolutionising "Begriffsschrift" (German for, roughly, "concept-script").

While all these efforts certainly consolidated the trust in given systems, there were some major drawbacks. For one, every system demanded its own foundation and thereby a new set of axioms. Further, albeit it was tried to justify defined axioms, the desire of complete certainty was not met due to the lack of consistency proofs. Thus, it was seeked for a consistent-proofed, universal system of which all branches of mathematics could then be deduced.

One idea of a universal foundation was given by Georg Cantor around 1880. He introduced a non-formalized kind of set theory, now referred to as naive set theory, in which sets and operations on sets are described with natural language.

> "A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought—which are called elements of the set." [1]
>
> — Georg Cantor

Evidently, such an informal description is not adequate for a study of consistency as it leaves room for ambiguities. Nontheless, many mathematicians took pleasure in Cantor's set theory and used it as a foundation for different systems.

Frege tried to address the issue of a consistent, universal system in his "Grundgesetze der Arithmetik" (German for "fundamental laws of arithmetics"). He was trying to give a common foundation by reducing all of mathematics, in particular Cantor's set theory, to pure logic, removing all mathematical symbolism. His first efforts had not been well received; however, it caught the attention of Bertrand Russell. Just as Frege was finishing his final work on his *Grundgesetze*, he received a letter from Russell, which should consequently pose the origin of the foundational crisis. Cantor's set theory as well as Frege's work were casualties of *Russell's paradox* discovered in 1901:

> Let us consider *the set of all sets that are not members of themselves* denoted by R. In formal notation:
>
> $$R \coloneqq \{X \mid X \notin X\}$$
>
> By Cantor's definition, R is indeed a well-defined set. We are now interested in whether R is a member of itself, that is if $R \in R$ holds. Let us assume that this is indeed the case. Then, by definition of R, it follows that R must not be a member of itself, i.e. $R \notin R$, which contradicts our assumption. Now, let us assume the contrary, namely $R \notin R$. Then, again by definition of R, we observe that R must be a member of itself, i.e $R \in R$. Hence, we conclude
>
> $$R \in R \iff R \notin R$$
>
> which clearly is a contradiction and thus shows the inconsistency of Cantor's set theory.

Though, Frege's work did not contain a definition of sets, it was not difficult for Russell to transfer the paradox to the system constructed by Frege. At the last minute, Frege hastily wrote an appendix in which he explained the issue and proposed a solution by restricting the antinomy-causing axioms of the system; however, this was to no avail and consequently caused Frege utter dismay.

> "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion." [4]
>
> — Gottlob Frege

It was beyond all doubt that Frege's undertaking was dismissed and, even more shocking, the established foundation of mathematics, Cantor's naive set theory, was shattered; mathematics was broken at its core. A new, consistent foundation had to be found; a Herculean task, as it turned out, which was tackled by three different schools of thought which we shall discuss next.

2.2 Logicism — a foundation made of logic

Mathematics as an extension of logic; this is the central idea of logicism. Although Frege backed up from his endeavours, his pursued idea, the reduction of mathematics to pure logic, was subsequently revisited from Russell and Alfred Whitehead in their famous three-volume work *Principia Mathematica* published in 1910–1913.

After the discovery of his paradox, Russell began to study the causes that lead to those kind of antinomies[6]. This was an important task because Russell needed to protect his future work to avoid the bitter fate as experienced by Frege. He concluded that "In all the contradictions there is a common characteristic, which we may describe as self-reference or reflexiveness" [5, p. 224]. As a means of defence, he invented the idea of type theories.

In general, a type theory is an extension of a formal system which assigns every object a type. Operations on objects can then be restricted in dependency on involved types.[7] In simple terms, the type theory in Principia Mathematica uses natural numbers as types. A set of type n is then only allowed to contain members of types at most $n - 1$. Naturally, this system is not susceptible to antinomies like Russell's as it completely eliminates self-referentiality.

Russell, however, recognised that his type theory caused major limitations which made the logical system incapable of representing all of mathematics. In addition, Russell's and Whitehead's attempts in proving the existence of an infinitude of objects using their logical framework failed. Thus, they decided to patch up their system by introducing the *axiom of reducibility* and the *axiom of infinity*; however, these two axioms arguably do not fit the bill.

The former overcomes the limitations caused by Russell's type theory in a fairly specific and technical way. As a result, it was widely critised that the axiom is too ad hoc and stipulated just in order to attain a desired result. The latter axiom, on the other hand, guarantees the existence of an infinite amount of objects; but what makes this a logical necessity? Is the universe infinite? Are there an infinite number of atoms?

While Russell's and Whitehead's general intention had been widely advocated, Principia Mathematica was exposed to ever-growing criticisms due to mentioned axioms. They do not seem to be logically self-evident and hence contradict with logicism's central idea. Furthermore, albeit posing a perfection of coherency and precision, the books were known as being illegible due to extremly fine-grained, incremental definitions and rather unusual syntactical notations and have been described as "the outstanding example of an unreadable masterpiece" [2].

Nontheless, the critiques did not completely debase Russell's and Whitehead's work, but many adopted it as a new mathematical foundation up to 1931 — the time Gödel once and for all dismissed Russell's and Whitehead's intentions. Others, however, were fairly dissatisfied. Among them were the advocates of formalism which we shall discuss next.

[6] There were found antinomies similar to Russel's at the same time, which we are not going to discuss (such as the Burali-Forti paradox or Richard's paradox). [5]

[7] For instance, in type-safe programming languages, such as Standard ML or Java, it is not permitted to compare values of different types (e.g. integers and strings).

$*54\cdot43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51\cdot231] \qquad \equiv . \iota' x \cap \iota' y = \Lambda .$

$[*13\cdot12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Figure 2: Principia Mathematica's infamous proof of $1 + 1 = 2$.

2.3 Formalism — mathematics as a symbol modifier

In contrast to logicists, formalists strongly support the autonomy of mathematics. Mathematics does not need to justify the existence of its objects, as its objects are of a purely syntactical nature. It is based on symbols and axioms which describe syntacical operations on them. In other words, mathematics in its heart is just a symbol modifier. This approach is justified by its success.

2.4 Intuitionism — proofs with real evidence

2.5 The end of the crisis — Gödel's incompleteness theorems

3 Aftermath and prospects

Formalism:
Only restriction: Operations are not allowed to lead to a cotradiction -¿ proof system consistent. This needs a new field of studies: meta-mathematic. Hilbertprogram ist necessary (justfied by anomalies like Russel's). Hilbert axiomatised geometry, Zermelo axiomatised Cantor's set theory (which was not regarded as a foundation of mathematics at first as Zermelo said that its consistency proof is a distant prospect; after extensions from Fraenkel and Skolem this changed and replaced Princ. Math. as foundation).

Intutionism:
Math is not just made of symbols, and it is not existent in a world idependent of mankind, but it is only existent in humans' minds produced by intuition. That creation process is neither of a symbolic kind nor is it reducible to pure logic. Symbols and logic are merely representations used for these mental creations. Formalism is a void game without meaning and mathematical operations (transfinite set theory? Ugh… potential vs. actual infinities). Important restriction: No law of excluded middle! (use example https://en.wikipedia.org/wiki/Law_of_excluded_middle#example). Main advocate was Brouwer who developed a set theory without named law. (more accepted after world war??)

In 1922 Weyl published a paper (after meeting Brouwer) which stated "Brouwer - this is revolution". Weyl was a student of Hilbert. Hilbert was not amused and compared Brouwer's and Weyl's intentions as an attempted coup that will fail. Hilbert then resumed his metamathematical work which he had paused.

The following years were characterised by many articles which spread the dispute between the both schools of thought. Intuitionism was hard to define (spreaded across many papers from Brouwer and written in dutch) and formalism was not fully developed yet but still under development. Hilbert's followers were extremly active and gained many supporters. 1925 proof by Ackermann (student of Hilbert) about the consistency of law of excluded middle. Confidence in formalism grew. Meanwhile, criticism for Intuitionism grew as its limitations would harm the applicability of mathematics. Even Weyl agreed to that in 1924.

In 1928 German mathematicans were allowed to attend the international congress for the first time after WW1. However, they were not allowed to vote and Bouwer solidarised with them and called on the Germans to boycott the congress. Nevertheless, most Germans attended the congress including Hilbert. He presented his foundational programm and Bouwer was not able to discount Hilbert as he did not attended the congress. A few days afterwards, Hilbert as one of the major publishers of the prestigous mathematical magazine "Mathematischen Annalen" decided to exclude Brouwer as a co-publisher without conducting a referendum. This lead to confusion and disputes but in the end Hilbert's wish came true and Brouwer was excluded. After this incident, Brouwer stopped publishing articles dealing with Intuitionism. As a result, intutionism lost its public attention.

Formalism seemed to be the winner. Goedel's incompleteness theorem destroyed the hope of a proofed consistent formal system. Nevertheless, modern mathematics is build on formalism. What axioms do I need at least in order to proof this theorem are current questions of formalism. The justification of the axioms are often regarded philosophical work and are often not regarded by mathematicians.

Topic overview:
At the beginning of the 20th century it became clear that mathematics needed formal foundations, which was partly caused by discovering various paradoxes. A couple of solutions were proposed including Hilbert's program of formalism. The crisis was partially solved by adopting the new foundations of set theory although Hilbert's ambitions were not completely fulfilled due to Gödel. The student should cover main events and players from this story and accompany the presentation by the most important technical details.

Quotes:

- „Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können." -– David Hilbert: Über das Unendliche, Mathematische Annalen 95 (1926), S. 170
- Dieses Tertium non datur (satz ausgeschlossenem dritten) dem Mathematiker zu nehmen, waere etwa, wie wenn man dem Astronomen das Fernrohr oder dem Boxer den Gebrauch der Faeuste untersagen wollte." – David Hilbert: Die Grundlagen der Mathe-

matik, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, 6. Band (1928), S. 80

- Theory, by Hrbacek and Jech, we read on page 54 : "Axiom of Infinity. An inductive (i.e. infinite) set exists." Compare this against the axiom of God as presented by Maimonides (Mishneh Torah, Book 1, Chapter 1): The basic principle of all basic principles and the pillar of all the sciences is to realize that there is a First Being who brought every existing thing into being. Mathematical axioms have the reputation of being self- math experience page 171

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