Foundations of Mathematics and the Foundational Crisis

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Overview

Causes of the Crisis

Euclid's Elements – A Work of Timeless Certainty

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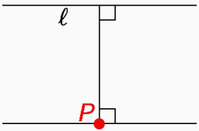
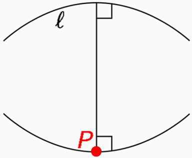
Given a line ℓ and a point P, there exists
one line through P parallel to ℓ.



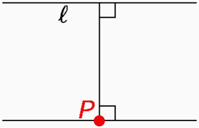
Euclidean

Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P, there exists
no line through P parallel to ℓ.



Elliptic



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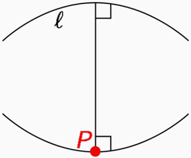
Given a line ℓ and a point P, there exist
at least two lines through P parallel to ℓ.



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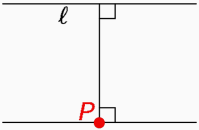
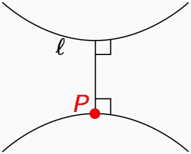


Euclidean

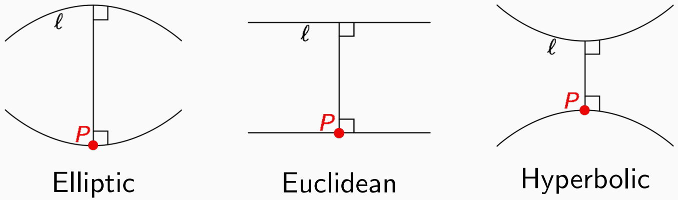


Hyperbolic

Euclid's Elements – A Work of Timeless Certainty

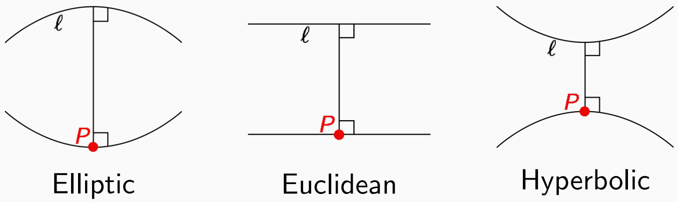
Given a line ℓ and a point P, there exist

? many lines through P parallel to ℓ.



Euclid's Elements – A Work of Timeless Certainty

Which axioms represent the truth?



Search for Foundations

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Desire for a universal and consistent system

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- And then came Russell...

Russell's Paradox

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Answer: $R \in R \iff R \notin R$, a contradiction!

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- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism

Source: `geopolicraticus.tumblr.com/post/142561195372`

The Foundational Crisis

Logicism

A Foundation Made of Logic

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- Chief work: Principia Mathematica

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  - Axiom of reducibility
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- Regarded as “the outstanding example of an unreadable masterpiece”
- Nonetheless, many adopted it as a new foundation.

Principia Mathematica

$$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$$

Dem.

$$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$$

$$[*51{\cdot}231] \qquad \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda .$$

$$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$$

$$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$$

$$\vdash :. (\exists x, y) . \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$$

$$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Principia Mathematica's infamous proof of $1 + 1 = 2$. It was not until page 379 that this proof was possible.

The Foundational Crisis

Intuitionism

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- The existence of any mathematical object is equivalent to the possibility of its construction, according to Brouwer.
  - $\Rightarrow$ No antinomies since paradoxical sets cannot be constructed
- Consequently, some assumptions of classical logic must be rejected.

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- Consequently, only a few scholars adhered to intuitionism.

The Foundational Crisis

Formalism

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- Mathematics shall be based on symbols and axioms that describe syntactic operations on symbols.
- Mathematics does not need to justify the existence of its objects since its objects are just meaningless shapes.

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- The dream of a complete and consistent mathematical system

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Peak and End

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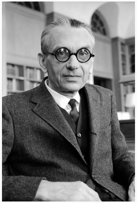
- Optimism for a complete and consistent formal system grew…

The End of the Crisis

- . . . but then came Gödel.



Source: `newyorker.com/tech/elements/waiting-for-godel`

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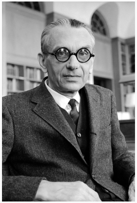
- In 1931, he proved that there is no sufficiently strong, complete, and consistent formal system.



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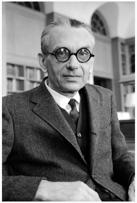

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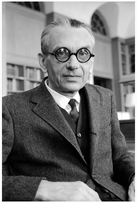

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Theorem (First Incompleteness Theorem)

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Theorem (Second Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.

Aftermath and Prospects

Modern Mathematics

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    - Zermelo-Fraenkel set theory (ZFC) as established foundation
- Most modern mathematicians do not deal with foundational research but try to extend a specific branch of mathematics.
- The justification of foundations is often regarded philosophical work.

The Next Crisis?

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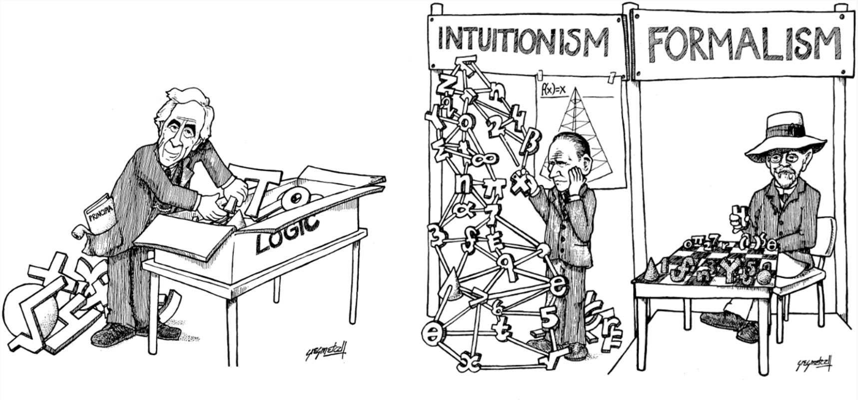
- Can we trust proofs by computers?
- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



Source: `maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1980/0025570x.di021111.02p0048m.pdf`

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Image sources I

- Euclid's Elements: `math.ubc.ca/~cass/Euclid/papyrus/tha.jpg`
- Geometries: `upload.wikimedia.org/wikipedia/commons/thumb/7/78/Noneuclid.svg/2000px-Noneuclid.svg.png`