

# Foundations of Mathematics and the Foundational Crisis

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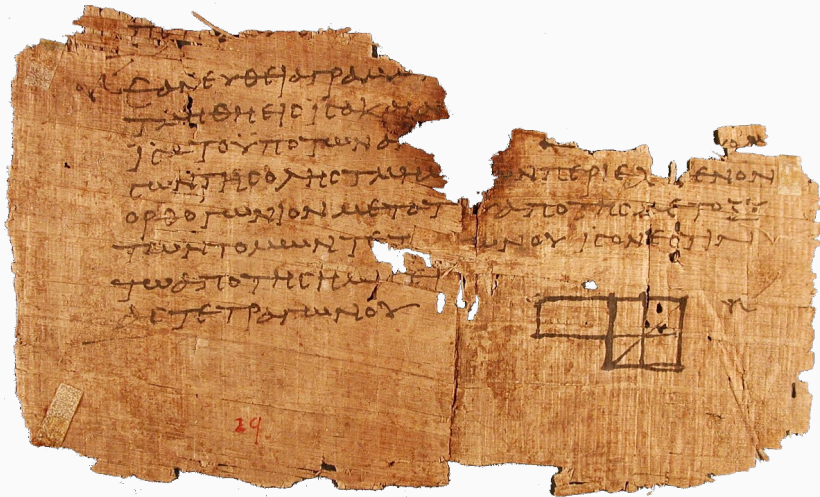
# Overview

1. Causes of the Crisis
2. The Foundational Crisis
  - 2.1 Logicism
  - 2.2 Intuitionism
  - 2.3 Formalism
  - 2.4 End of the Crisis
3. Modern Mathematics

## Causes of the Crisis

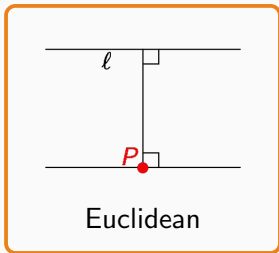
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# Euclid's Elements – A Work of Timeless Certainty



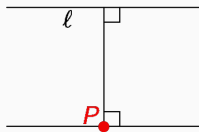
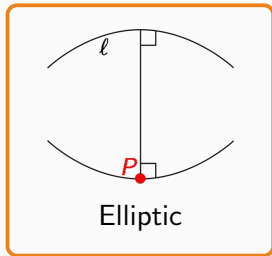
# Euclid's Elements – A Work of Timeless Certainty

Given a line  $\ell$  and a point  $P$  not lying on  $\ell$ , there exists  
one line through  $P$  parallel to  $\ell$ .



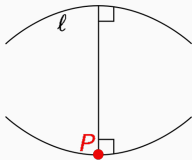
# Euclid's Elements – A Work of Timeless Certainty

Given a line  $\ell$  and a point  $P$  not lying on  $\ell$ , there exists  
no line through  $P$  parallel to  $\ell$ .

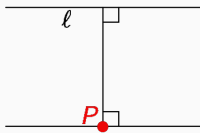


# Euclid's Elements – A Work of Timeless Certainty

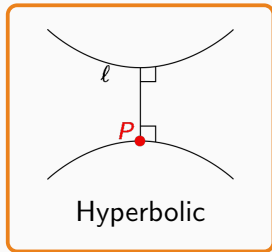
Given a line  $\ell$  and a point  $P$  not lying on  $\ell$ , there exist  
at least two lines through  $P$  parallel to  $\ell$ .



Elliptic



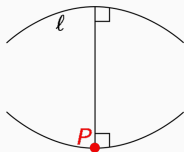
Euclidean



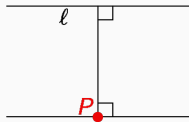
Hyperbolic

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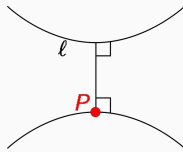
Given a line  $\ell$  and a point  $P$  not lying on  $\ell$ , there exist  
? many lines through  $P$  parallel to  $\ell$ .



Elliptic



Euclidean



Hyperbolic



Which axioms represent the truth?

# A Search for Foundations

Axiomatisation of systems in the late 19th century

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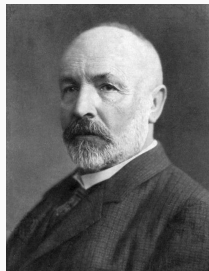
- Arithmetic of natural numbers by Peano
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Desire for a universal and consistent system

# Cantor's Set Theory

*"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set."*

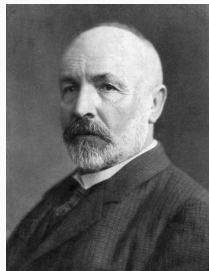
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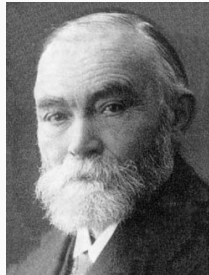


Certainly universal, but fairly naive



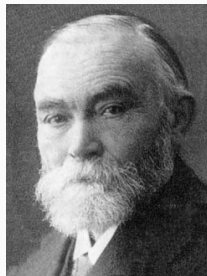
# Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.



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Just before finishing his work, he received a letter from Russell. . .

# Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

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Question: Is  $R$  a member of itself? That is, does  $R \in R$  hold?

Answer:  $R \in R \iff R \notin R$ , a contradiction!

## The Begin of the Crisis

*“Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished.”*

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A new foundation of mathematics had to be found.

# The Foundational Crisis

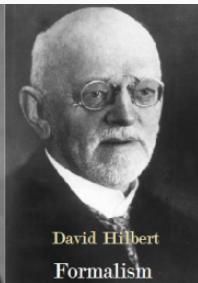
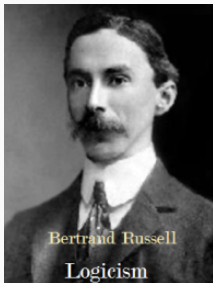
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# The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism



# The Foundational Crisis

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Logicism

## A Foundation Made of Logic

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# A Foundation Made of Logic

- Russell and Whitehead revisited Frege's idea of reducing mathematics to logic.
- Only fundamentally logical laws as axioms
  - Justify the use of the axioms



- Type theory to avoid antinomies

# Principia Mathematica

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- Difficulties in explaining some axioms



# Principia Mathematica

- Type theory to avoid antinomies
- Difficulties in explaining some axioms
- Regarded as “the outstanding example of an unreadable masterpiece”

## Principia Mathematica's infamous proof of $1 + 1 = 2$

**\*54.43.**  $\vdash :: \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

*Dem.*

$\vdash . *54.26 . \supset \vdash :: \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[\*51.231]  $\equiv . \iota'x \cap \iota'y = \Lambda .$

[\*13.12]  $\equiv . \alpha \cap \beta = \Lambda$  (1)

$\vdash . (1) . *11.11.35 . \supset$

$\vdash :: (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda$  (2)

$\vdash . (2) . *11.54 . *52.1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that  $1 + 1 = 2$ .

# The Foundational Crisis

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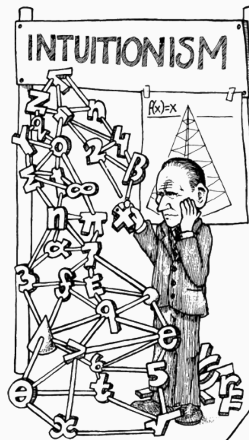
**Intuitionism**

# Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.

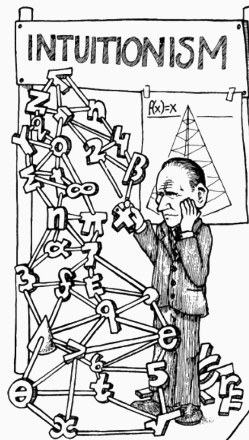
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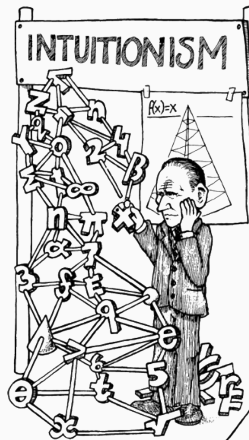
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# Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.
  - The existence of an object is equivalent to the possibility of its construction.
- ⇒ Some assumptions of classical logic must be rejected.



$$P \vee \neg P \equiv ?$$

Intuitionists reject the *law of excluded middle*:

*For any proposition  $P$ , either  $P$  or its negation is true.*



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If it is irrational,  $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$  proves our statement. □

# What a Hassle!

*“Taking the law of excluded middle from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists.”*

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Only a few scholars adhered to intuitionism.

# The Foundational Crisis

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**Formalism**

# Mathematics as a Symbolic Game

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# Mathematics as a Symbolic Game

- Mathematics shall be based on meaningless symbols and syntactic operations.
- No need to justify the existence of objects
- The system's consistency must be verified.



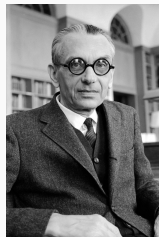
# **The Foundational Crisis**

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**End of the Crisis**

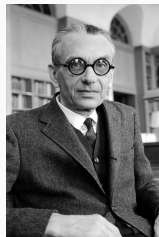
# The Incompleteness of Mathematics

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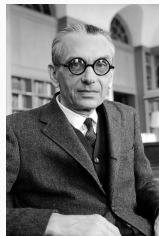


## Theorem (First Incompleteness Theorem)

*Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.*

# The Incompleteness of Mathematics

In 1931, Gödel ended the crisis with his two *incompleteness theorems*.



## **Theorem (First Incompleteness Theorem)**

*Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.*

## **Theorem (Second Incompleteness Theorem)**

*Any consistent formal system within which a certain amount of elementary arithmetic can be carried out cannot prove its own consistency.*

# Modern Mathematics

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# Modern Mathematics

To this day, formalism lays the foundation of mathematics.

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Most mathematicians do not deal with foundational research.

Digitalisation of mathematics

## Digitalisation of mathematics

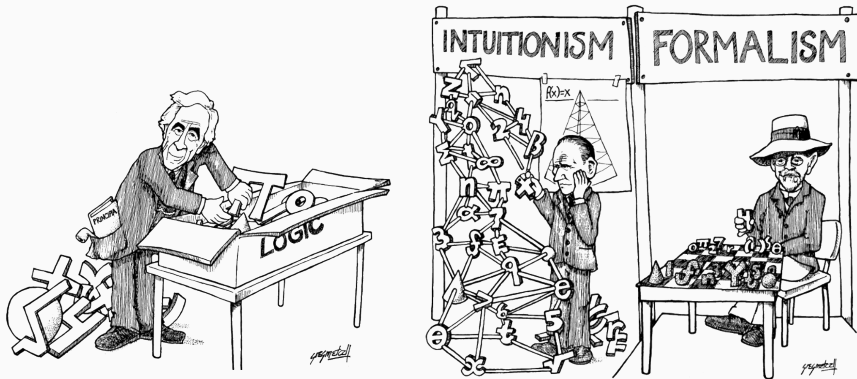
- Some see it as an inevitable enrichment; others face it with distrust.

## Digitalisation of mathematics

- Some see it as an inevitable enrichment; others face it with distrust.
- Can we trust proofs by computers?

Are we part of the next  
mathematical crisis?

Thanks for your attention! Any questions?





# References I



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- Geometries: `upload.wikimedia.org/wikipedia/commons/thumb/7/78/Noneuclid.svg/2000px-Noneuclid.svg.png`
- Cantor: `upload.wikimedia.org/wikipedia/commons/e/e7/Georg_Cantor2.jpg`
- Frege: `nndb.com/people/523/000179983/gottlob-frege-2-sized.jpg`
- Schools of Thought:  
`geopolicraticus.tumblr.com/post/142561195372`

## Image Sources II

- Schools of Thought (comic): [maa.org/sites/default/files/pdf/upload\\_library/22/Allendoerfer/1980/0025570x.di021111.02p0048m.pdf](http://maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1980/0025570x.di021111.02p0048m.pdf)
- Gödel: [newyorker.com/tech/elements/waiting-for-godel](http://newyorker.com/tech/elements/waiting-for-godel)