Foundations of Mathematics and the Foundational Crisis

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Overview

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4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

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- Desire for a universal and consistent system

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- And then came Russell. . .

Russell's Paradox

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Answer: $R \in R \iff R \notin R$, a contradiction!

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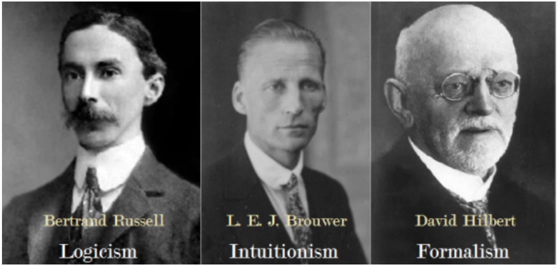
- Systems founded on Cantor's set theory were on shaky ground.
- A new foundation of mathematics had to be found.

The Three Schools of Thought

- Three schools of thought tried to establish a new foundation.
  - Logicism
  - Intuitionism
  - Formalism



Source: `geopolicraticus.tumblr.com/post/142561195372`

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- Chief work: Principia Mathematica

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- Difficulties in explaining some axioms
  - Axiom of reducibility
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- Regarded as "the outstanding example of an unreadable masterpiece."
- Nonetheless, many adopted it as a new foundation.

Principia Mathematica

$*54{\cdot}43.\quad \vdash :.\ \alpha, \beta \in 1\ .\supset :\ \alpha \cap \beta = \Lambda\ .\equiv .\ \alpha \cup \beta \in 2$

Dem.

$$\vdash .\ *54{\cdot}26\ .\supset \vdash :.\ \alpha = \iota`x\ .\ \beta = \iota`y\ .\supset :\ \alpha \cup \beta \in 2\ .\equiv .\ x \neq y\ .$$

$$[*51{\cdot}231] \qquad \equiv .\ \iota`x \cap \iota`y = \Lambda\ .$$

$$[*13{\cdot}12] \qquad \equiv .\ \alpha \cap \beta = \Lambda \qquad (1)$$

$$\vdash .\ (1)\ .\ *11{\cdot}11{\cdot}35\ .\supset$$

$$\vdash :.\ (\exists x, y)\ .\ \alpha = \iota`x\ .\ \beta = \iota`y\ .\supset :\ \alpha \cup \beta \in 2\ .\equiv .\ \alpha \cap \beta = \Lambda \qquad (2)$$

$$\vdash .\ (2)\ .\ *11{\cdot}54\ .\ *52{\cdot}1\ .\supset \vdash .\ \mathrm{Prop}$$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Principia Mathematica's infamous proof of $1 + 1 = 2$. It was not until page 379 that this proof was possible.

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- Mathematics is a constructive process conducted by humans.
- The existence of any mathematical object ist equivalent to the possibility of its construction, according to Brouwer.
  - $\Rightarrow$ No antinomies as paradoxical sets cannot be constructed.
- Consequently, some assumptions of classical logic must be rejected.

$P \vee \neg P \equiv ?$

- Intuitionists reject the *law of excluded middle*: $\vdash P \vee \neg P$
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Proof. It is known that $\sqrt{2}$ is irrational. Let us consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, our statement is proved. If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. $\square$

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- Consequently, only a few scholars adhered to intuitionism.

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- Mathematics must not justify the existence of its objects as its objects are just meaningless shapes.
- Hilbert, as one of the most renowned mathematicians of given time, was the driving force of formalism.

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  2. Prove the system's consistency with metamathematical reasoning.

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The End

Georg Cantor.
Beiträge zur Begründung der transfiniten Mengenlehre.
Mathematischen Annalen, 46:481, 1895.

Philip J. Davis, Reuben Hersh, and Elena Anne Marchisotto.
The Mathematical Experience.
Modern Birkäuser Classics, 2012.

Gottlob Frege.
Grundgesetze der Arithmetik, volume 2.
1902.

David Hilbert.
Die Grundlagen der Mathematik.
Abhandlungen aus dem Mathematischen Seminar der Hamburger Universität, page 80, 1928.