Foundations of Mathematics and the Foundational Crisis

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Overview

Causes of the Crisis

Euclid’s Elements – A Work of Timeless Certainty

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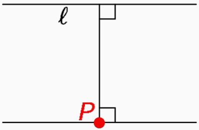
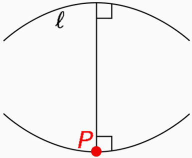
Given a line ℓ and a point P, there exists
one line through P parallel to ℓ.



Euclidean

Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P, there exists
no line through P parallel to ℓ.



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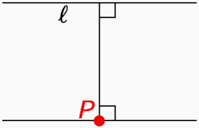

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Given a line ℓ and a point P, there exist
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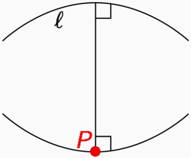

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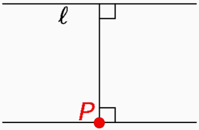

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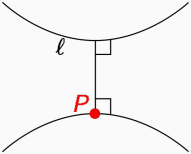
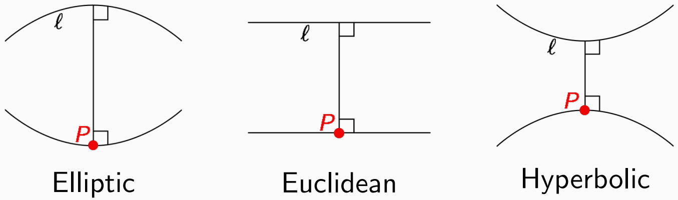


Hyperbolic

Euclid's Elements – A Work of Timeless Certainty

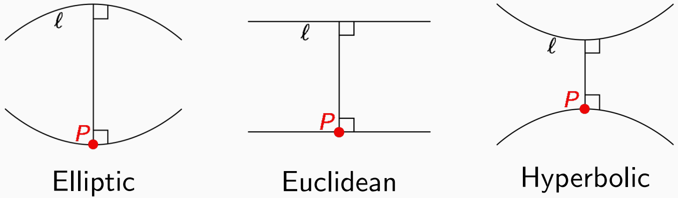
Given a line ℓ and a point P, there exist

? many lines through P parallel to ℓ.



Euclid's Elements – A Work of Timeless Certainty

Which axioms represent the truth?



A Search for Foundations

Axiomatisation of systems in the late 19th century

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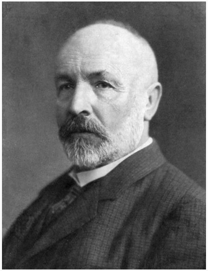
Desire for a universal and consistent system

Cantor's Set Theory

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set."

– Georg Cantor



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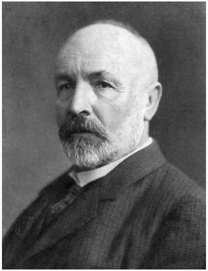
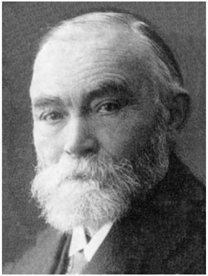


Certainly universal, but fairly naive.

Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.



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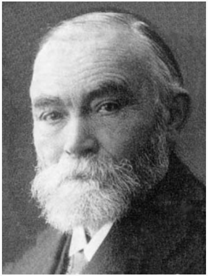

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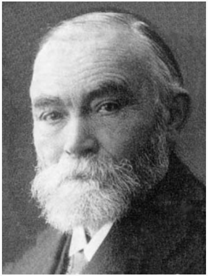

Then one day, just before finishing his work, he received a letter from Russell. . .

Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

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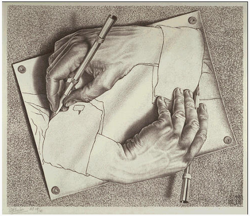
Answer: $R \in R \iff R \notin R$, a contradiction!

The Begin of the Crisis

Self-referentiality broke Cantor's and Frege's system.



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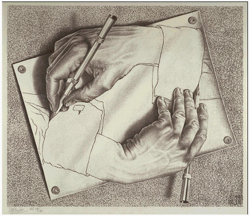

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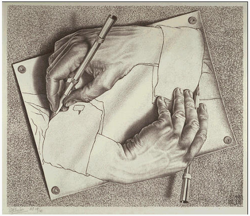

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A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism

The Foundational Crisis

Logicism

A Foundation Made of Logic

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  - Justifications that used axioms are self-evident truths

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- Regarded as “the outstanding example of an unreadable masterpiece”

Principia Mathematica

Prcinipia Mathematica's infamous proof of $1 + 1 = 2$

$*$**54·43.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota‘x . \beta = \iota‘y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$$

$$[*51{\cdot}231] \qquad \equiv . \iota‘x \cap \iota‘y = \Lambda .$$

$$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$$

$$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$$

$$\vdash :. (\exists x, y) . \alpha = \iota‘x . \beta = \iota‘y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$$

$$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

The Foundational Crisis

Intuitionism

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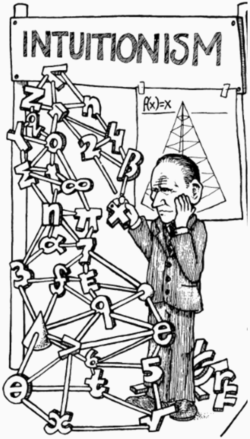

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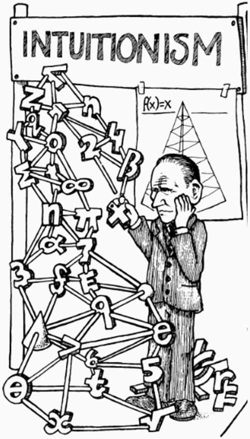

Proofs with Real Evidence

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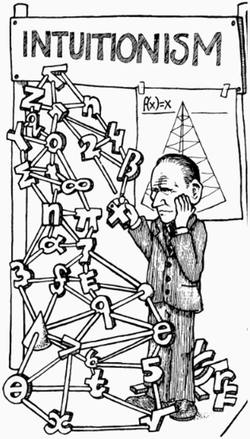
- Consequently, some assumptions of classical logic must be rejected.



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Only a few scholars adhered to intuitionism.

The Foundational Crisis

Formalism

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- Mathematics shall be based on symbols and axioms that describe syntactic operations on symbols.
- Mathematics does not need to justify the existence of its objects since its objects are just meaningless shapes.

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  1. Formalise a system that is able to derive all of mathematics using syntactical operations
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- The dream of a complete and consistent mathematical system

The Foundational Crisis

Peak and End

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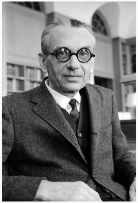
- Optimism for a complete and consistent formal system grew…

The End of the Crisis

- . . . but then came Gödel.



Source: `newyorker.com/tech/elements/waiting-for-godel`

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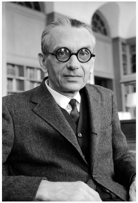
- …but then came Gödel.
- In 1931, he proved that there is no sufficiently strong, complete, and consistent formal system.



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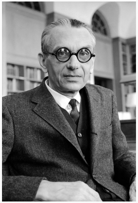

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Theorem (First Incompleteness Theorem)

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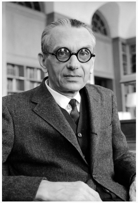

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Theorem (First Incompleteness Theorem)

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Theorem (Second Incompleteness Theorem)

Any consistent formal system rich enough to contain a formalisation of recursive arithmetic cannot prove its own consistency.

Aftermath and Prospects

Modern Mathematics

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  - Zermelo-Fraenkel set theory (ZFC) as established foundation
- Most modern mathematicians do not deal with foundational research but try to extend a specific branch of mathematics.
- The justification of foundations is often regarded philosophical work.

The Next Crisis?

- Digitisation of mathematics

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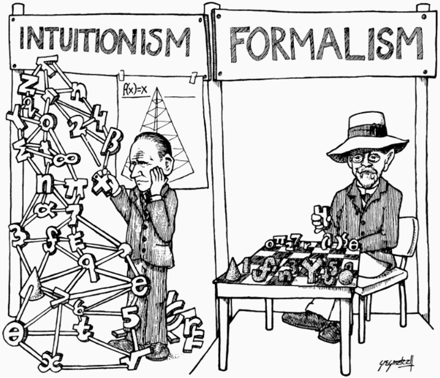
- Digitisation of mathematics
- Can we trust proofs by computers?
- Some see it as an inevitable enrichment; others face it with distrust.
- Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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