Foundations of Mathematics and the Foundational Crisis

Kevin Kappelmann

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Technical University of Munich

Overview

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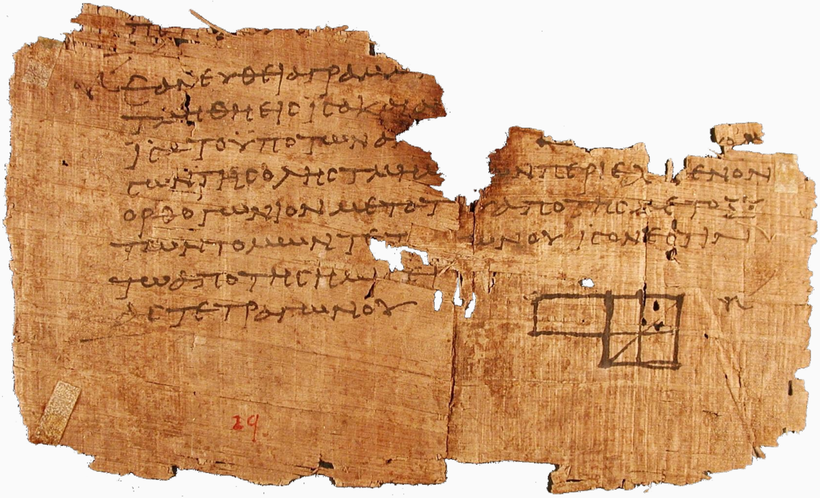
2. The Foundational Crisis
   - 2.1 Logicism
   - 2.2 Intuitionism
   - 2.3 Formalism
   - 2.4 End of the Crisis
3. Aftermath and Prospects

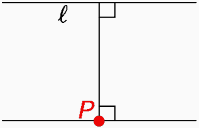
Causes of the Crisis

Euclid's Elements – A Work of Timeless Certainty



Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exists one line through P parallel to ℓ.



Euclidean

Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exists
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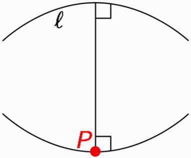
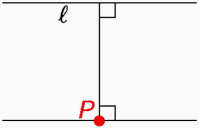


Elliptic



Euclidean

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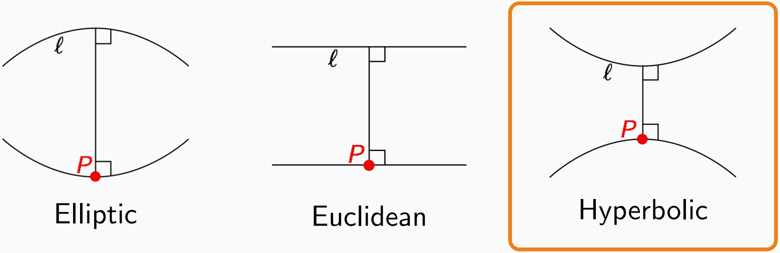
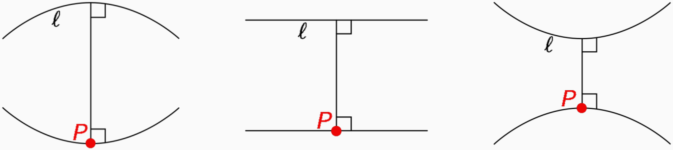
Given a line ℓ and a point P not lying on ℓ, there exist at least two lines through P parallel to ℓ.



Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exist

? many lines through P parallel to ℓ.



Elliptic

Euclidean

Hyperbolic

Which axioms represent the truth?

A Search for Foundations

Axiomatisation of systems in the late 19th century

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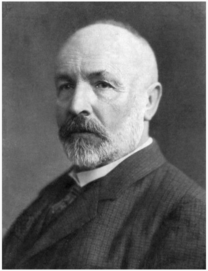
Desire for a universal and consistent system

Cantor's Set Theory

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set."

– Georg Cantor



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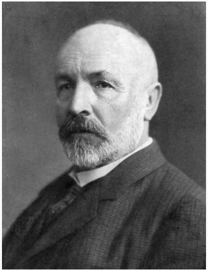
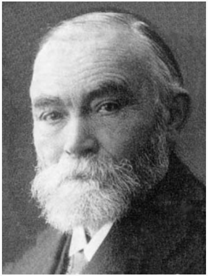


Certainly universal, but fairly naive

Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.



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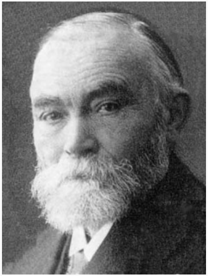

Just before finishing his work, he received a letter from Russell…

Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

$$R := \{X \mid X \notin X\}$$

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Answer: $R \in R \iff R \notin R$, a contradiction!

The Begin of the Crisis

"Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished."

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A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism

The Foundational Crisis

Logicism

A Foundation Made of Logic

- Russell and Whitehead revisited Frege's idea of reducing mathematics to logic.

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- Only fundamentally logical laws as axioms
    - Justify the use of the axioms

Principia Mathematica

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- Regarded as “the outstanding example of an unreadable masterpiece”

Principia Mathematica

Principia Mathematica's infamous proof of $1+1=2$

$*54\cdot43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$$
\begin{aligned}
&\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y . \\
&[*51\cdot231] \qquad\qquad\qquad\qquad \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda . \\
&[*13\cdot12] \qquad\qquad\qquad\qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1) \\
&\vdash . (1) . *11\cdot11\cdot35 . \supset \\
&\qquad \vdash :. (\exists x, y) . \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2) \\
&\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash . \text{Prop}
\end{aligned}
$$

From this proposition it will follow, when arithmetical addition has been defined, that $1+1=2$.

The Foundational Crisis

Intuitionism

Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.

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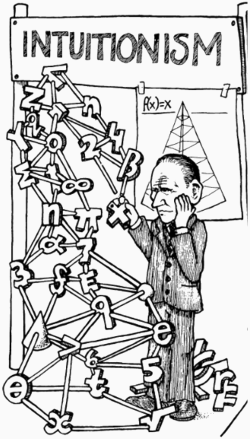

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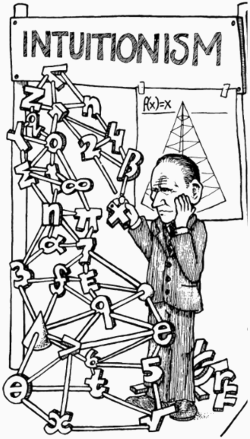

Proofs with Real Evidence

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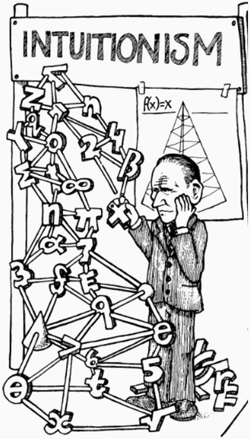
- The existence of an object is equivalent to the possibility of its construction.

$\Rightarrow$ Some assumptions of classical logic must be rejected.



$P \vee \neg P \equiv$?

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If it is rational, our statement is proved.
If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. $\square$

What a Hassle!

> *"Taking the law of excluded middle from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists."*
>
> – *David Hilbert*

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Only a few scholars adhered to intuitionism.

The Foundational Crisis

Formalism

Mathematics as a Symbolic Game

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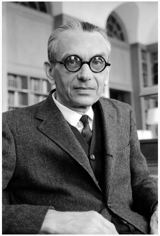
- Mathematics shall be based on meaningless symbols and syntactic operations.
- No need to justify the existence of objects
- The system's consistency must be verified.

The Foundational Crisis

End of the Crisis

The Incompleteness of Mathematics

In 1931, Gödel ended the crisis with his two *incompleteness theorems*.



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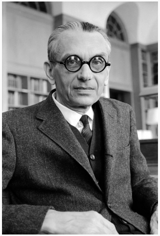

Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

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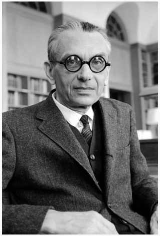

Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

Theorem (Second Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out cannot prove its own consistency.

Aftermath and Prospects

Modern Mathematics

To this day, formalism poses the foundation of mathematics.

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- Zermelo-Fraenkel set theory (ZFC) as established foundation

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Most mathematicians do not deal with foundational research.

Mathematics $\cup$ Computer Science

Digitalisation of mathematics

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- Some see it as an inevitable enrichment; others face it with distrust.

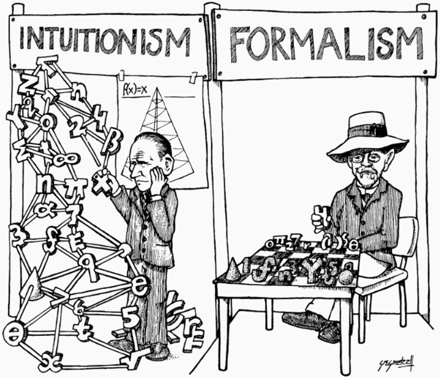
Mathematics $\cup$ Computer Science

Digitalisation of mathematics

- Some see it as an inevitable enrichment; others face it with distrust.
- Can we trust proofs by computers?

Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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