Foundations of Mathematics and the Foundational Crisis

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June 13, 2017

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Overview

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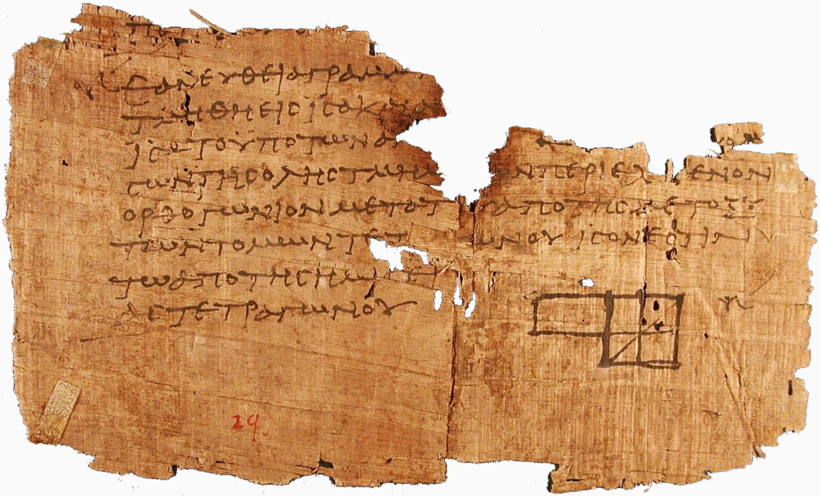
2. The Foundational Crisis
    2.1 Logicism
    2.2 Intuitionism
    2.3 Formalism
    2.4 End of the Crisis
3. Modern Mathematics

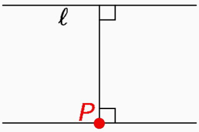
Causes of the Crisis

Euclid's Elements – A Work of Timeless Certainty



Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exists
one line through P parallel to ℓ.



Euclidean

Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exists
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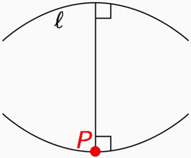
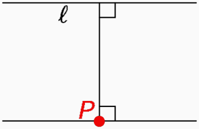


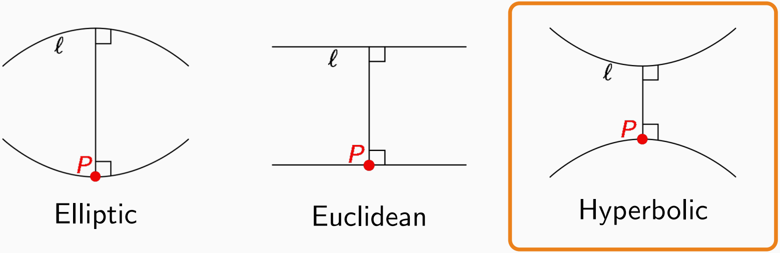
Elliptic



Euclidean

Euclid's Elements – A Work of Timeless Certainty

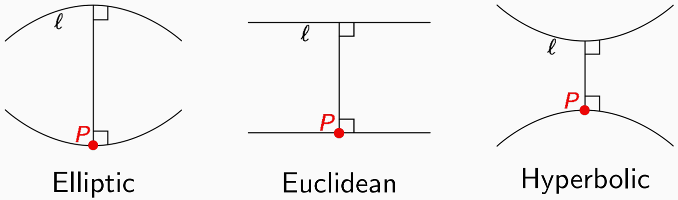
Given a line ℓ and a point P not lying on ℓ, there exist
at least two lines through P parallel to ℓ.



Euclid's Elements – A Work of Timeless Certainty

Given a line ℓ and a point P not lying on ℓ, there exist

? many lines through P parallel to ℓ.



Which axioms represent the truth?

A Search for Foundations

Axiomatisation of systems in the late 19th century

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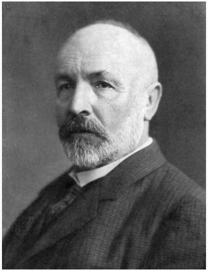
Desire for a universal and consistent system

Cantor's Set Theory

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought – which are called elements of the set."

– Georg Cantor



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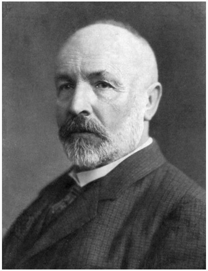
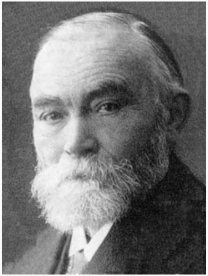


Certainly universal, but fairly naive

Frege's Logic System

Frege tried to build a consistent foundation by reducing mathematics to logic.



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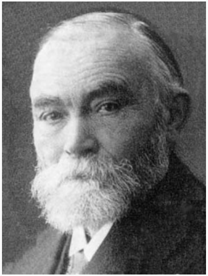

Just before finishing his work, he received a letter from Russell...

Russell's Paradox

Consider *the set of all sets that are not members of themselves*:

$$R := \{X \mid X \notin X\}$$

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Answer: $R \in R \iff R \notin R$, a contradiction!

The Begin of the Crisis

"Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished."

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A new foundation of mathematics had to be found.

The Foundational Crisis

The Three Schools of Thought

Three schools of thought tried to establish a new foundation.

- Logicism
- Intuitionism
- Formalism

The Foundational Crisis

Logicism

A Foundation Made of Logic

- Russell and Whitehead revisited Frege's idea of reducing mathematics to logic.

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- Russell and Whitehead revisited Frege's idea of reducing mathematics to logic.
- Only fundamentally logical laws as axioms
    - Justify the use of the axioms

Principia Mathematica

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- Regarded as “the outstanding example of an unreadable masterpiece”

Principia Mathematica

Principia Mathematica's infamous proof of $1 + 1 = 2$

$$*54{\cdot}43. \quad \vdash :. \alpha, \beta \in 1 \,.\, \supset : \alpha \cap \beta = \Lambda \,.\, \equiv \,.\, \alpha \cup \beta \in 2$$

Dem.

$$\vdash \,.\, *54{\cdot}26 \,.\, \supset \vdash :. \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset : \alpha \cup \beta \in 2 \,.\, \equiv \,.\, x \neq y \,.$$

$$[*51{\cdot}231] \qquad \equiv \,.\, \iota`x \cap \iota`y = \Lambda \,.$$

$$[*13{\cdot}12] \qquad \equiv \,.\, \alpha \cap \beta = \Lambda \qquad (1)$$

$$\vdash \,.\, (1) \,.\, *11{\cdot}11{\cdot}35 \,.\, \supset$$

$$\vdash :. (\exists x, y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset : \alpha \cup \beta \in 2 \,.\, \equiv \,.\, \alpha \cap \beta = \Lambda \qquad (2)$$

$$\vdash \,.\, (2) \,.\, *11{\cdot}54 \,.\, *52{\cdot}1 \,.\, \supset \vdash \,.\, \text{Prop}$$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

The Foundational Crisis

Intuitionism

Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.

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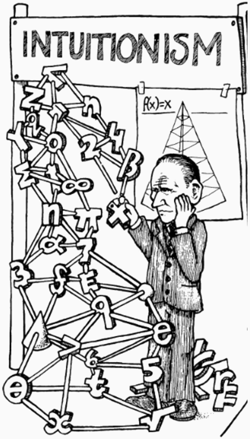
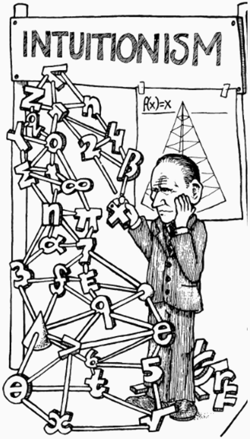


Proofs with Real Evidence

- Mathematics is a constructive process conducted by humans.
- The existence of an object is equivalent to the possibility of its construction.



Proofs with Real Evidence

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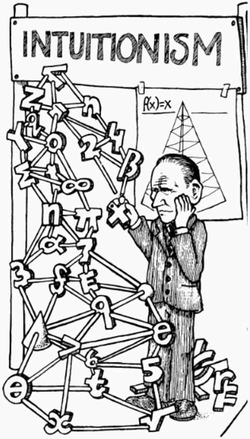
- The existence of an object is equivalent to the possibility of its construction.

$\Rightarrow$ Some assumptions of classical logic must be rejected.



$P \vee \neg P \equiv$ **?**

Intuitionists reject the *law of excluded middle*:

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Proof. It is known that $\sqrt{2}$ is irrational. Let us consider the number $\sqrt{2}^{\sqrt{2}}$.
If it is rational, our statement is proved.
If it is irrational, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ proves our statement. $\square$

What a Hassle!

"Taking the law of excluded middle from the mathematician would be the same as, say, denying the astronomer his telescope and the boxer the use of his fists."

– David Hilbert

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Only a few scholars adhered to intuitionism.

The Foundational Crisis

Formalism

Mathematics as a Symbolic Game

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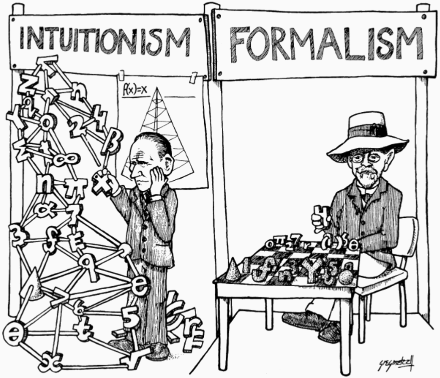
- Mathematics shall be based on meaningless symbols and syntactic operations.
- No need to justify the existence of objects
- The system's consistency must be verified.


PRINCIPIA
LOGIC

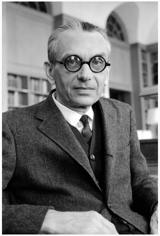

INTUITIONISM
FORMALISM
f(x)=x

The Foundational Crisis

End of the Crisis

The Incompleteness of Mathematics

In 1931, Gödel ended the crisis with his two *incompleteness theorems*.



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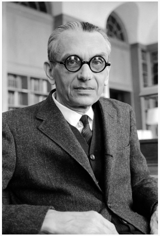

Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

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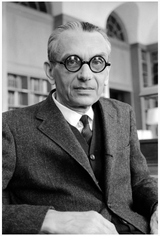

Theorem (First Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out is incomplete.

Theorem (Second Incompleteness Theorem)

Any consistent formal system within which a certain amount of elementary arithmetic can be carried out cannot prove its own consistency.

Modern Mathematics

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To this day, formalism lays the foundation of mathematics.

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- Zermelo-Fraenkel set theory (ZFC) as established foundation

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Most mathematicians do not deal with foundational research.

Mathematics $\cup$ Computer Science

Digitalisation of mathematics

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- Some see it as an inevitable enrichment; others face it with distrust.

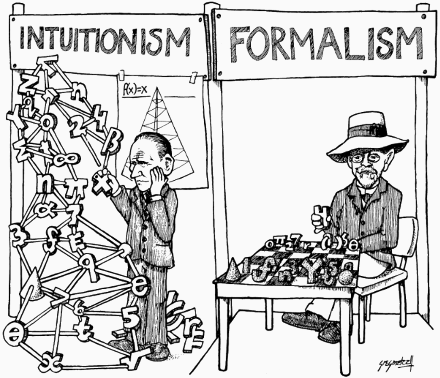
Mathematics $\cup$ Computer Science

Digitalisation of mathematics

- Some see it as an inevitable enrichment; others face it with distrust.
- Can we trust proofs by computers?

Are we part of the next mathematical crisis?

Thanks for your attention! Any questions?



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- Cantor: `upload.wikimedia.org/wikipedia/commons/e/e7/Georg_Cantor2.jpg`
- Frege: `nndb.com/people/523/000179983/gottlob-frege-2-sized.jpg`
- Schools of Thought: `geopolicraticus.tumblr.com/post/142561195372`

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