Tilted Tray Simulation Report

Suyog Kapsikar

1 Mathematical Modelling

We can model the given problem using a simple yet effective stochastic process called **Markov Chain**. We can use concept of **stationary distribution**. Stationary distribution of i^{th} state tells us in long time, what is probability of system being in i^{th} state.

Let us start with a simple case of 2 induction stations as shown in below picture 2. We can focus on only

Figure 1:

one tray at a time and we need to observe the system at only at the induction zone. Let us call this point as the transition epoch of the system. At a given transition epoch a tray can be in two states

0: Tray is empty. Implies that the tray has dropped an item in one of the chutes from last transition epoch (induction zone that the tray had visited, say zone 1) and some new item will be placed on tray. This newly placed item might be dropped into into one of the chutes before the next the induction zone (zone 2) or after the next induction zone (between zone 2 and 1). *In-short the currently placed item is yet to pass the whole of conveyor belt.*

1: Tray is not empty. Implies that the currently placed item has passed over half of the tray. It will be dropped into one of the chutes before next induction zone with certainty

If the system (a given tray) is in state 0

- It **will remain in state 0** ***with probability 0.5*** as item can be dropped into one of the coming 1000 chutes before next induction zone or 1000 after that.
- It **will go to state 1** ***with probability 1***.

Below is the Morkov Chain. Each time the chain visits state 0, we can say an item is deliver into the chute

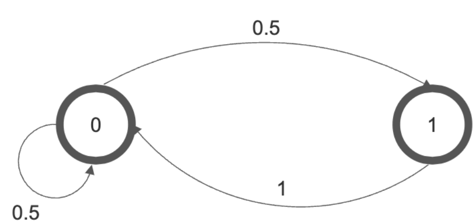

Figure 2: Markov Chain for 2 induction zone

by the tray. Then we can calculate stationary distribution of state 0 and multiply by that with number of epochs to get the expected number of item delivered per hr. Stationary distribution is given by $\pi = P\pi$ where P is the transition probability matrix given by $\begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$. After solving for π_0, we get $\pi_0 = \frac{2}{3}$. We can make below reasonable assumption before calculating expected number if items sorted in an hr

1. At the beginning all the trays are empty. An item is put on tray when tray crosses an induction zone for the first time
2. An item is equally likely to go for any of the 2000 orders
3. Induction zones are exactly opposite to each other, after 1000 chutes

$$\text{Expected number of items sorted per hr} = \frac{2}{3} * (10000 - 500) * 2 = 12666.67$$

500 have been subtracted from 10000 as expected number of movements each tray needs before reaching an induction zone is 500. 2 has been multiplied as each epoch is made of half of full belt rotation.

For n induction zones

Inspired from above maths, we can also model for n induction zone. Along with above assumptions, we will assume that number of induction zones divide 2000 in equal numbers.

We can define states if Markov chain as below:

- 0: Tray can travel full belt from the point of observation
- 1: Tray can travel $\frac{n-1}{n}$ part of the belt
- k: Tray can travel $\frac{n-k}{k}$ part of belt
- $n-1$: Tray can travel $\frac{1}{n}$ part of the belt

Below is the pictorial markov chain and corresponding transition probability matrix

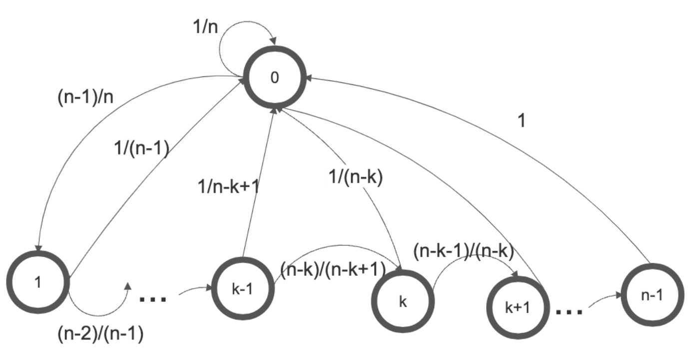

Figure 3: Markov Chain for n induction zone

$$P = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & ... & k+1 & ... & n-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ ... \\ k \\ ... \\ n-1 \end{array} & \begin{pmatrix} \frac{1}{n} & \frac{n-1}{n} & 0 & ... & 0 & ... & 0 \\ \frac{1}{n-1} & 0 & \frac{n-2}{n-1} & ... & 0 & ... & 0 \\ \frac{1}{n-2} & 0 & 0 & ... & 0 & ... & 0 \\ & & & & & & \\ \frac{1}{n-k} & 0 & 0 & ... & \frac{n-k-1}{n-k} & ... & 0 \\ & & & & & & \\ 1 & 0 & 0 & ... & 0 & ... & 0 \end{pmatrix} \end{array}$$

After not so complex but a lot of math we can solve $\pi = P\pi$ to get $\pi_0 = \frac{2}{n+1}$

$$\text{Expected number of items sorted per hr} = \frac{2}{n+1} * (10000 - \frac{1000}{n}) * n$$

$\frac{1000}{n}$ have been subtracted from 10000 as expected number of movements each tray needs before reaching an induction zone is $\frac{1000}{n}$. n has been multiplied as each epoch is made of $\frac{1}{n}^{th}$ of full belt rotation.

2 System Simulation and comparison with theoretical results

Following algorithm has been used to simulate the system.

Algorithm 1 Simulation for 1 hour

Require: $n \geq 0$
Ensure: $2000 \mod n = 0$
  Create 2000 tray objects with attributes CurrentLocation and CarryOrder
  Initial Condition: all the trays are uniquely at one of 2000 locations and carry no item
  $k \leftarrow 0$
  $inductionLoc \leftarrow [\,]$
  while $k \neq 2000$ **do**
    $k \leftarrow k + \frac{2000}{n}$
    $inductionLoc \leftarrow [inductionLoc, k]$
  end while
  $SortedItems = 0$
  for *movements* from 1 to 10000 **do**
    for *trays* from 1 to 2000 **do**
      if $tray.currentLocation \in inductionLoc$ **then**
        if $tray.CarryOrder = Null$ **then**
          $tray.CarryOrder =$ Random Integer between $[1, 2000]$ ▷ This is item assignment to empty tray
        end if
      end if
      $tray.currentLocation \leftarrow tray.currentLocation + 1$
      if $tray.currentLocation = tray.CarryOrder$ **then**
        $tray.CarryOrder = Null$ ▷ Item has been dropped into Chute
        $SortedItems \leftarrow SortedItems + 1$
      end if
    end for
  end for

Above is the algorithm for simulation of 1 hour. Above system w simulated over number of sample paths in python to take take average of *SortedItems* to claim it falls close to 'Expected number of items sorted per hr'.

Below are the results in comparison with theoretical results:

n	Simulation Expectation	Gain as per Simulation	SamplePaths	Theoretical Expectation	Gain as per Markov Chain Model
1	9000	-		9000.00	-
2	12438.49	38.21%	100	12666.67	40.74%
4	15202.48	68.92%	100	15600.00	73.33%
5	15897.1	76.63%	100	16333.33	81.48%
8	17033.63	89.26%	100	17555.56	95.06%
10	17444.56	93.83%	100	18000.00	100.00%
16	18120.04	101.33%	100	18705.88	107.84%
20	18350.16	103.89%	100	18952.38	110.58%

Some points to note:

- For 1 induction station the capacity in denoted as 9000 instead of 10000 because of assumption of initial condition, i.e., all they trays are empty at the beginning
- Capacity derived using simulation is lesser than that of mathematical model as stationary distribution is converged in long time. 10000 movements are not sufficient enough time for this convergence. If System is simulated for longer time of 10hrs or so, the gap between theoretical and simulation expectations narrows.
- Even if theoretical results differ slightly from real-time simulation, elegant models can quickly give expected benefits for complex systems.