

Discrete Quiz 4

Question 1. Define the universe U to be all possible remainders when we divide by 360: $\{0, 1, 2, \dots, 359\}$. Also define 3 subsets in this universe:

$$\begin{cases} K_2 &= \{x \in U \mid x \text{ divisible by } 2\}, \\ K_3 &= \{x \in U \mid x \text{ divisible by } 3\}, \\ K_5 &= \{x \in U \mid x \text{ divisible by } 5\}, \end{cases}$$

Denote by Φ the subset of U containing all numbers that are mutually prime with 360 (no common divisors greater than 1): $\Phi = \{1, 7, 11, 13, \dots, 359\}$. Which set equality is valid regarding the subset Φ :

- (A) $\Phi = (K_2 \cup K_3 \cup K_5)$
- (B) $\Phi = (K_2 \cap K_3 \cap K_5)$
- (C) $\Phi = (\overline{K_2} \cup \overline{K_3} \cup \overline{K_5})$
- (D) $\Phi = (\overline{K_2} \cap \overline{K_3} \cap \overline{K_5})$
- (E) $\Phi = (\overline{K_2 \cap K_3} \cup \overline{K_2 \cap K_5} \cup \overline{K_3 \cap K_5})$

Pick your answer as a single letter like this: G

Question 2. Find the size of the set you constructed in the previous example.

Write your answer as a single non-negative integer like this: 17

Question 3. We have the following sets:

A is the set of all finite sequences of even positive positive numbers (such as $(6, 22, 10, 14, 2, 6)$, and so on)

B is the set of all infinite nondecreasing lists of even positive numbers (such as $(40 \leq 40 \leq 42 \leq 46 \leq \dots)$, and so on)

C is the set of all infinite nonincreasing lists of even positive numbers (such as $(64 \geq 58 \geq 58 \geq 54 \geq \dots)$, and so on).

Clearly, all three sets are infinite. Determine their cardinalities - which list of cardinalities is equal to the list $(|A|, |B|, |C|)$?

- (A) $(|\mathbb{N}|, |\mathbb{N}|, |\mathbb{N}|)$. (B) $(|\mathbb{N}|, |\mathbb{R}|, |\mathbb{N}|)$. (C) $(|\mathbb{N}|, |\mathbb{N}|, |\mathbb{R}|)$.
- (D) $(|\mathbb{N}|, |\mathbb{R}|, |\mathbb{R}|)$. (E) $(|\mathbb{R}|, |\mathbb{R}|, |\mathbb{R}|)$.

Pick your answer as a single letter like this: G

Question 4. Let $f(x) = (x^2) \bmod 11$. Find the set $f(S)$ if $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Write the list of elements of $f(S)$ as a sorted list like this: 1, 2, 3

Question 5. How many 2-element sets are there in the powerset $\mathcal{P}(\{A, B, C, D, E\})$?

Write your answer as a non-negative integer like this: 17

Question 6. Given two sets $A = \{x, y\}$ and $B = \{x, \{x\}\}$, check, if statements are true or false:

- (A) $x \subseteq B$.
- (B) $\emptyset \in \mathcal{P}(B)$.
- (C) $\{x\} \subseteq A - B$.
- (D) $|\mathcal{P}(A)| = 4$.

Write your answer as a sorted list of letters (which are true) like this: A, B, C, D

Question 7. We define functions $g : A \rightarrow A$ and $f : A \rightarrow A$, where $A = \{1, 2, 3, 4\}$ by listing all argument-value pairs: $f = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$, $g = \{(1, 3), (2, 1), (3, 4), (4, 2)\}$. Find the value pairs for the function $(f \circ g)^{-1}$.

Write your answer as a comma-separated list like this: (1, 1), (2, 2), (3, 3), (4, 4)

Question 8. Find the value of this infinite sum: $1 - 1/3 + 1/9 - 1/27 + 1/81 - \dots$

Write your answer as a simple fraction: P/Q

Question 9. It is known that the function $f(n) = n^3 + 88n^2 + 3$ is in $O(n^3)$ - its asymptotic growth is as fast as the growth of the function $g(n) = n^3$. $\exists C \in \mathbb{Z}^+ \exists n_0 \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+$,

$(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ Find the smallest positive integer C that would satisfy the above definition, and for your C find the smallest possible n_0 .

Write your answer (C, n_0) as a pair of two numbers like this: 17, 17

Question 10. "Big O notation" allows to arrange functions according to their growth rate for large n . Identify, which list of functions is such that the first element of this list is in the big-O of the next element of that list and so on. (Intuitively, the first element in the list is the slowest growing function, the last element is the fastest growing one.)

- (1) $\log(n^{10})$, (2) $(\log n)^2$, (3) $\log \log n$, (4) $n \log n$,
- (5) $\log(n!)$, (6) $\log 2^n$.

Write your answer as a comma-separated list like this: 1, 2, 3, 4, 5, 6

Question 11. Digits of all rational numbers P/Q in $(0; 1)$ are eventually periodic: they infinitely repeat some group of digits (the period) starting from some place. For example, the fraction $11/205 = 0.05(36585)$ has period of 5 digits and a pre-period "05" of just two digits. Find the predicate logic expression that tells that sequence of digits $d(1), d(2), d(3), \dots$ is eventually periodic (it may have pre-period of any length, including length zero).

- (A) $\exists N \in \mathbb{Z}^+ \exists T \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+$,
 $(n \geq N - 1 \rightarrow d(n) = d(n + T))$.
- (B) $\exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+ \exists T \in \mathbb{Z}^+$,
 $(n \geq N - 1 \rightarrow d(n) = d(n + T))$.
- (C) $\forall n \in \mathbb{Z}^+ \exists N \in \mathbb{Z}^+ \exists T \in \mathbb{Z}^+$,
 $(n \geq N - 1 \rightarrow d(n) = d(n + T))$.
- (D) $\forall n \in \mathbb{Z}^+ \forall N \in \mathbb{Z}^+ \exists T \in \mathbb{Z}^+$,
 $(n \geq N - 1 \rightarrow d(n) = d(n + T))$.

Pick your answer as a single letter like this: G

Answers

Question 1. Answer (D).

Any number that is mutual prime with $360 = 2^3 \cdot 3^2 \cdot 5$ is not divisible by any of the primes 2, 3, 5. And also vice versa. This set is expressed as intersection of the complements:

$$\Phi = (\overline{K_2} \cap \overline{K_3} \cap \overline{K_5})$$

Question 2. Answer: 96.

$$\begin{aligned} |\Phi| &= 360 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{5}\right) = \\ &= 360 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 96. \end{aligned}$$

In the above formula we start with all 360 elements; then we throw out one half (all that are divisible by 2); then from the remaining ones we throw out one third (all that are divisible by 3); finally from the remaining numbers we throw out one fifth (all that are divisible by 5). Since divisibility by 2 does not affect divisibility by 3 and 5 (they are independent), all the ratios can be multiplied.

Another solution: Since we know the sizes of each set of numbers divisible by 2, 3, 5:

$$|K_2| = 180, |K_3| = 120, |K_5| = 72.$$

We can express their union by *inclusion-exclusion principle*:

$$\begin{aligned} |K_2 \cup K_3 \cup K_5| &= |K_2| + |K_3| + |K_5| - \\ &- |K_2 \cap K_3| - |K_2 \cap K_5| - |K_3 \cap K_5| + |K_2 \cap K_3 \cap K_5| = \\ &= 180 + 120 + 72 - 60 - 36 - 24 + 12 = 264. \end{aligned}$$

We then apply De Morgan's law to find the count of all elements that are *outside* that union of $K_2 \cup K_3 \cup K_5$:

$$|\overline{K_2} \cap \overline{K_3} \cap \overline{K_5}| = |\overline{K_2 \cup K_3 \cup K_5}| = 360 - 264 = 96.$$

Question 3. Answer: B.

- A (the set of all finite sequences of even natural numbers can be enumerated with numbers from \mathbb{N}). You can encode every such sequence in a finite alphabet of 13 symbols, using just digits, commas and parentheses. For example, (6, 22, 10, 14, 2, 6). The shortest encoding is (1) –it consists of just three symbols: two parentheses and a digit. There can be only finite number of such lists of length 3; we sort the all lexicographically (i.e. in some alphabetical order), and assign them numbers. After that we enumerate all lists writeable with 4 symbols (sorted lexicographically) and so on. Eventually all the sequences will be sorted.

- B (the set of all infinite nondecreasing sequences of even numbers) has cardinality \mathbb{R} . You can repeat the diagonalization argument: Assume from the contrary that the elements from B can be enumerated: we get infinitely many infinite sequences b_1, b_2, \dots . Then take the first element from b_1 (and pick some even number that is bigger than that); then take the second element from b_2 (and pick some even number that is bigger than that; plus it is bigger than all the previously picked numbers), and so on.

You can also encode any subset $A \subseteq \mathbb{N}$ as such sequence (simply arrange all the elements in increasing order and multiply them by 2 to get even numbers). We get that B has at least as many elements as $\mathcal{P}(\mathbb{N})$.

- C (the set of all nonincreasing infinite sequences can be enumerated). Since the sequence is non-increasing, it can have only finitely many places where it actually decreases; since natural numbers cannot decrease infinitely. We can encode all the “constant runs” of the sequence as pairs:

$$\begin{aligned} &(64, 58, 58, 54, 50, 50, 50, 50, 2, 2, \dots) \rightarrow \\ &\rightarrow ((64, 1), (58, 2), (54, 1), (50, 4), (2, \infty)). \end{aligned}$$

As we saw before, all the finite sequences that are encoded in an alphabet of 14 symbols (10 digits, 2 parentheses, commas and infinity) can be enumerated.

Question 4. Answer: 0, 1, 3, 4, 5, 9.

We can square each number, compute the remainder and sort the results (and eliminate duplicates).

Question 5. Answer: 6.

The set $\{\{A, B\}, C, D, E\}$ has 4 elements (A, B are always glued together). There are 6 ways to select two out of four elements. (Can be computed as a binomial coefficient $C_4^2 = \frac{4!}{2!2!}$ or simply by listing all the 6 pairs.

Question 6. Answer: B, D.

x cannot be a subset of A (since it is not a set itself). $\{x\}$ is not a subset of $A - B = \{y\}$.

Question 7.

Answer: (1, 3), (2, 2), (3, 4), (4, 1).

We first compute $f \circ g$ (to get $(f \circ g)(x) = f(g(x))$ we first apply g , then f):

$$f \circ g = \{(1, 4), (2, 2), (3, 1), (4, 3)\}.$$

The inverse happens, if we switch the order in all these pairs (4 maps back to 1 etc.)

$$(f \circ g)^{-1} = \{(1, 3), (2, 2), (3, 4), (4, 1)\}.$$

Question 8. Answer: $3/4$.

The sum of the infinite geometrical progression is $b_1/(1 - q)$. In our case:

$$\frac{1}{1 - (-1/3)} = \frac{1}{4/3} = \frac{3}{4}.$$

Question 9. Answer: $2, 88$. Clearly, $f(n) = |n^3 + 88n^2 + 3|$ cannot be smaller than $C \cdot |n^3|$, if $C = 1$, because $88n^2$ is always positive and makes $f(n)$ larger than simply n^3 .

If we take $C = 2$, then the inequality starts to hold for all $n > 88$. It is possible to prove that for such n :

$$\begin{aligned} n^3 + 88n^2 + 3 &= n^2(n + 88) + 3 = \\ &= n^2(n + n) + n^2(88 - n) + 3 = 2n^3 + n^2(88 - n) + 3 \leq 2n^3. \end{aligned}$$

The last inequality is true, since $n^2(88 - n) + 3 < 0$ for any $n > 88$.

Question 10. Answer: $3, 1, 2, 6, 4, 5$ (or $3, 1, 2, 6, 5, 4$).

Logarithm of a logarithm is a very slowly growing function; $\log n^{10}$ is just equal to 10 times $\log n$. $(\log n)^2 = \log^2 n$ is slightly faster than a logarithm.

Finally $\log 2^n$ is simply n ; but both $\log(n!)$ and $n \log n$ grow slightly faster than n ; they are "Big-O" of each other:

$$n \log n \text{ is in } O(\log n!);$$

$$\log n! \text{ is in } O(n \log n).$$

It does not matter, in which order we list them.

To verify all these claims, you need to prove various limits:

$$\lim_{n \rightarrow \infty} \frac{\log \log n}{\log n} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{\log n}{(\log n)^2} = 0,$$

and so on. Most of these limits are easy to find (L'Hospital's Rule and so on). With $\log n!$ you might need to use integrals to estimate the sum of $\log 1 + \log 2 + \dots + \log n$.

Note. Unless noted otherwise, all logarithms in our course are base 2.

Question 11. Answer: A. Clearly the N and T should not depend on n ; so they are the first quantifiers.

(B) describes a sequence of digits where some digit repeats itself infinitely often (which is true for any sequence of digits).

(C) describes the set of all sequences; one can always pick N that is larger than n , then the condition is trivially true.

(D) describes a sequence where each digit appears infinitely often.