Final Exam

RBS, Discrete Structures

2020-04-23

Problem 1

By U we denote the set of all positive integers between 1 and 120. This is our *universe* in which we define several subsets:

$$\begin{cases} A = \{x \in U \ \mid \ 2 \mid x\}, \\ B = \{x \in U \ \mid \ 3 \mid x\}, \\ C = \{x \in U \ \mid \ 5 \mid x\}, \\ X = \{x \in U \ \mid \ 2 \mid x \ \vee \ 3 \mid x\}, \\ Y = \{x \in U \ \mid \ (3 \mid x \ \wedge \ 5 \mid x) \ \vee \neg(2 \mid x)\}. \end{cases}$$

(A) Express X using the sets A, B, C (using set union $V \cup W$, set intersection $V \cap W$, set complement $\overline{V}$ operations).
(B) Express Y using the sets A, B, C in a similar way.
(C) Find $|X|$ - the size of the set X.
(D) Find $|Y|$ - the size of the set Y.

Problem 2

Let A and B be sets with sizes $|A| = 8$ and $|B| = 5$ and $|A \cap B| = 3$.

Calculate the largest and the smallest possible values for each of the following set sizes:

(A) $|A \cup B|$.
(B) $|A \times (B \times B)|$.
(C) $|\mathcal{P}(\mathcal{P}(A \cap B))|$ - the powerset of a powerset of $A \cap B$.
(D) $|A \oplus B|$ - the symmetric difference of the sets A and B.

Problem 3

Consider the following recurrent sequence:

$$\begin{cases} a_0 = 3 \\ a_1 = 4 \\ a_{n+2} = 5a_{n+1} - 6a_n, \ \text{if } n \geq 0 \end{cases}$$

Assume that b_n is another sequence satisfying the recurrence rule

$$b_{n+2} = 5b_{n+1} - 6b_n, \ \text{if } n \geq 0$$

(The first two members b_0, b_1 are not known.)

(A) Write the first 6 members of this sequence $(a_0, \dots, a_5)$.
(B) Write the characteristic equation for this sequence.
(C) Write the general expression for an arbirary sequence b_n satisfying the recurrent expression as a sum of two geometric progressions (you can leave unknown coefficients in your answer; just explain which ones they are).
(D) Write the formula to compute a_n (that would satisfy the initial conditions $a_0 = 3$ and $a_1 = 4$).

Problem 4

Consider this code snippet in Python:

```
n = 1000
sum = 0
for i in range(1, n*n+1):
    for j in range(1,i+1):
        sum += i % j
```

And a similar one in R:

```
n <- 1000
sum <- 0
for (i in 1:(n*n)) {
    for (j in 1:i) {
        sum <- sum + i %% j
    }
}
```

(A) Explain in human language what this algorithm does.
(B) Denote by $f(n)$ the number of times the variable `sum` is incremented. Write the Big-O-Notation for $f(n)$. Find a function $g(n)$ such that $f(n)$ is in $O(g(n))$. (If there are multiple functions, pick the one with the slowest growth.)
(C) Express the function $f(n)$ precisely - how many times `sum` is incremented in terms of variable n.

Problem 5

Let A be the set of all positive divisors of the number 120 (including 1 and 120 itself).
(A) What is the multiplication of all numbers in the set A?
(B) Express this number as the product of prime powers.

Problem 6

Define the following binary relationship on the set of integer numbers $\mathbb{Z}$:
We say that aRb (numbers $a, b \in \mathbb{Z}$ are in the relation R) iff

$$\begin{cases} a - b \equiv 0 \pmod{11} \\ a - b \equiv 0 \pmod{12} \\ a - b \equiv 0 \pmod{13} \end{cases}$$

Item	Statement	True or False?
(A)	R is reflexive	
(B)	R is symmetric	
(C)	R is antisymmetric	
(D)	R is transitive	
(E)	aRb iff $a = b$	

For all items where you answered `FALSE`, specify a counterexample (values for some numbers that would make the condition true, but the conclusion false). If the statement was true, write "none".

(A) counterexample ___

(B) counterexample ___

(C) counterexample ___

(D) counterexample ___

(E) counterexample ___

Problem 7

Four people A, B, C, D each has his own hat. After the meeting they leave their building in a hurry, everyone grabs some hat at random so that all 4! permutations of the hats have equal probabilities.

Let the random variable X denote the number of hats that were picked up correctly. (For example, if the hat assignment is this: $(A \to A, B \to B, C \to D, D \to C)$, then $X = 2$, because two people got their own hats.)

(A) Find $E(X)$ - the expected value of X.
(B) Find $V(X)$ - the variance of X.

Problem 8

There was a crooked man who had a crooked 1 euro coin. On lucky days it would flip the *heads* with probability $p = \frac{2}{3}$, and the *tails* with probability $p = \frac{1}{3}$, but on unlucky days it was the opposite ($p(\texttt{heads}) = \frac{1}{3}$, but $p(\texttt{tails}) = \frac{2}{3}$). There were equal probabilities of $\frac{1}{2}$ for lucky and unlucky days.

One morning he flipped the coin 5 times and altogether got three *heads* and two *tails*.

Let us introduce the following events:

- E (evidence): Five coin tosses result in three *heads* and two *tails*.
- H (hypothesis): The current day is lucky.

(A) Find $P(E|H)$ - the conditional probability of E given that the day is lucky.
(B) Find $P(E|H) \cdot P(H)$ - the probability that the day is lucky and E happens.
(C) Find $P(E|\overline{H})$ - the conditional probability of E given that the day is not lucky.
(D) Find $P(E|\overline{H}) \cdot P(\overline{H})$ - the probability that the day is unlucky and E happens.
(E) Find $P(E)$ - as the sum of two probabilities (E happened on a lucky day and also E happened on unlucky day).
(F) Find the conditional probability $P(H|E)$ - the likelyhood that the croocked man has a lucky day, given that the event E has happened.

Problem 9

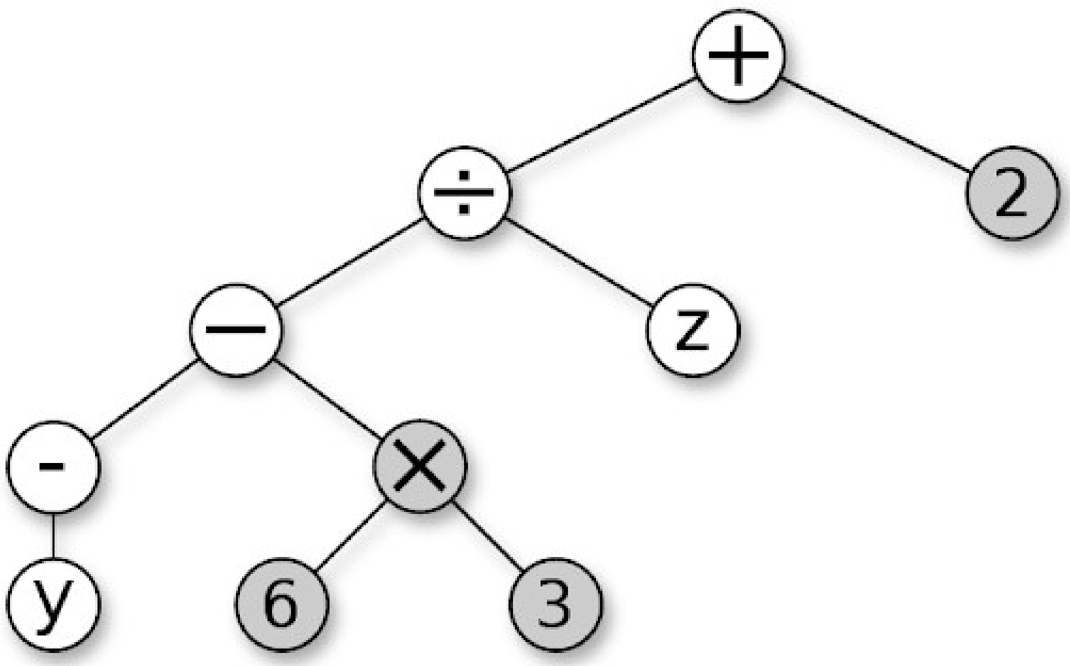

Figure: A syntax tree for an expression

The syntax tree describes an algebraic expression (please note the difference between the unary minus that flips the value of the variable y and the binary minus that subtracts the two subexpressions: $-y$ and 6×3).

(A) Write the preorder DFS traversal of this tree.
(B) Write the inorder DFS traversal of this tree.
(C) Write the postorder DFS traversal of this tree.

Note. In all 3 answers denote the unary minus with the tilde sign ~, but the regular/binary minus with –.

Problem 10

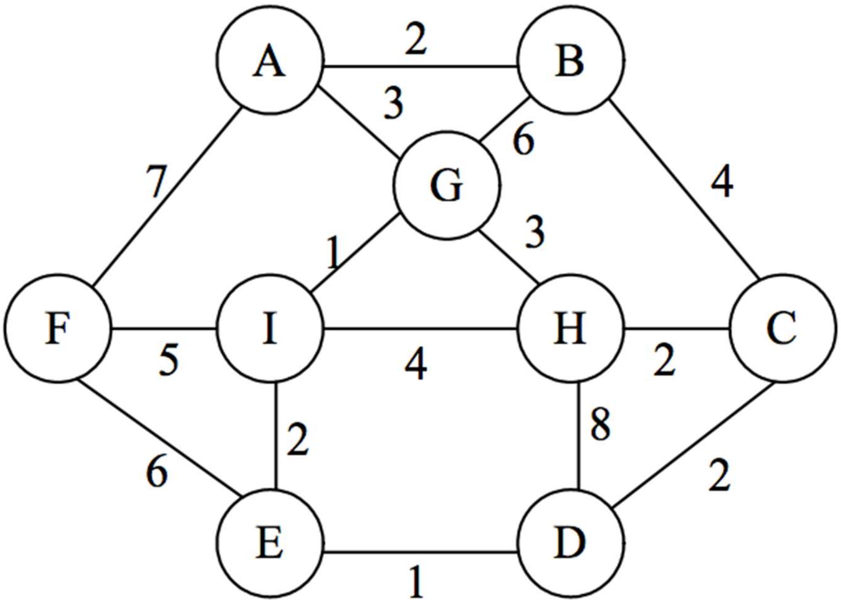

Figure: Graph with 9 vertices

Run the Prim's algorithm on the following weighted graph, start growing the tree from the vertex *I*.

Step	Newly Added Edge
Step 1	
Step 2	
Step 3	
Step 4	
Step 5	
Step 6	
Step 7	
Step 8	

What is the total weight of the obtained Minimum Spanning Tree? _____

Answers

Problem 1

(A) $X = A \cup B$ (Boolean OR means set union)
(B) $Y = (B \cap C) \cup \overline{A}$ (Boolean and means set intersection; negation means set complement)
(C) $|X| = |A| + |B| - |A \cap B| = 120 + 60 - 20 = 160$ (principle of inclusion-exclusion).
(D) $|Y|$ is all odd numbers and also four even numbers divisible by 15 (30,60,90,120). The total is $60 + 4 = 64$.

Problem 2

In all the answers the largest and the smallest value are equal, because we know exactly how the two sets intersect; how many elements belong to just one of the sets A, B, and how many elements belong to the both sets.

(A) $|A \cup B| = |A| + |B| - |A \cap B| = 8 + 5 - 3 = 10$ (the principle of inclusion-exclusion).
(B) $|A \times (B \times B)| = 8 \cdot 5 \cdot 5 = 200$ (Cartesian product has size that is the product of all participant sets: one can combine three elements from the sets A, B and B in this many ways).
(C) $2^{2^3} = 2^8 = 256$ (the number of elements in the powerset of any set X can be obtained by raising 2 to the power $|X|$).
(D) $|A \oplus B| = (8 - 3) + (5 - 3) = 7$ (we remove the common elements from both A and B).

Problem 3

(A) $a_0 = 3$,
$a_1 = 4$,
$a_2 = 5 \cdot 4 - 6 \cdot 3 = 2$,
$a_3 = 5 \cdot 2 - 6 \cdot 4 = -14$,
$a_4 = 5 \cdot (-14) - 6 \cdot 2 = -82$,
$a_5 = 5 \cdot (-82) - 6 \cdot (-14) = -326$,
$a_6 = 5 \cdot (-326) - 6 \cdot (-82) = -1138$.

(B) The characteristic equation is obtained, if we try to find a_n in the form of a geometric progression r^n:
$r^{n+2} = 5r^{n+1} - 6r^n$, or
$r^2 - 5r + 6 = 0$.
It has two roots: $r_1 = 2, r_2 = 3$.

(C) The general form of the expression for any iterative sequence b_n satisfying the relationship $b_{n+2} = 5b_{n+1} - 6b_n$ is as follows:

$$b_n = A \cdot 2^n + B \cdot 3^n,$$

where A, B are two constants that depend on the two initial values of the sequence b_n.

(D) We need to solve a system of two equations, to ensure that the formula $a_n = A \cdot 2^n + B \cdot 3^n$ has correct values for $n = 0$ and $n = 1$. We get the following system:

$$\begin{cases} A + B = 3, \\ 2A + 3B = 4. \end{cases}$$

Substitute $B = 3 - A$ into the second equation. We get that $2A + 9 - 3A = 4$ and $A = 5$. We also get that $B = -2$. Therefore the exact formula to calculate the sequence a_n is this:

$$a_n = 5 \cdot 2^n - 2 \cdot 3^n, \text{ where } n \geq 0.$$

This actually works, if we plug in the values calculated in **(A)** for $n = 0, \ldots, 6$.

Problem 4

(A) The algorithm takes all numbers i from 1 to n^2 and divides them by all the smaller numbers $j < i$, and adds up all the obtained remainders.

(C) The outer loop is repeated n^2 times. The inner loop is repeated $1 + 2 + 3 + \cdots + n^2$ times. This is an arithmetic progression. The sum of an arithmetic progression is the arithmetic mean of the first and the last member multiplied by the number of members:

$$f(n) = \frac{1 + n^2}{2} \cdot n^2 = \frac{n^4 + n^2}{2}.$$

(B) $f(n)$ is in $O(n^4)$. Therefore we can take $g(n) = n^4$. We can pick another $g(n)$ that is multiplied by some nonzero constant (such as $\frac{n^4}{2}$ or $17n^4$ or anything else - that also counts as a valid answer).
Certainly, $f(n)$ is also in $O(n^k)$ for any $k > 4$, but the function $g(n) = n^4$ is the slowest growing.

Problem 5

(A) If expressed as a product of two positive integers $120 = ab$, one of the divisors a or b would be smaller than $\sqrt{120} \approx 11$, and the other one would be bigger. We can easily list all the ways to express 120 as a product of two integers:

$$1 \cdot 120 = 2 \cdot 60 = 3 \cdot 40 = 4 \cdot 30 = 5 \cdot 24 = 6 \cdot 20 = 8 \cdot 15 = 10 \cdot 12,$$

and there are no other factorizations, since all the divisors less than 11 are already listed.
Multiplying them all together would give

$$(120)^8 = 429981696000000000$$

(B) As a product of prime factors:

$$(120)^8 = (2^3 \cdot 3 \cdot 5)^8 = 2^{24} \cdot 3^8 \cdot 5^8.$$

Problem 6

Item	Statement	True or False?
(A)	R is reflexive	TRUE
(B)	R is symmetric	TRUE
(C)	R is antisymmetric	FALSE
(D)	R is transitive	TRUE
(E)	aRb iff $a = b$	FALSE

(A) Counterexample: None
(B) Counterexample: None
(C) Consider counterexample $a = 0, b = 11 \cdot 12 \cdot 13 = 1716$.
While it is true that aRb and bRa, nevertheless $a \neq b$.
(D) Counterexample: None
(E) Counterexample is same as in **(C)**: $a = 0, b = 1716$.

Problem 7

- For 1 of 24 permutations $X = 4$ (all hats stay in place),
- For 0 permutations $X = 3$ (it is not possible for exactly three hats to stay in place, because then the 4th hat also returns to its owner),
- For 6 of 24 permutations $X = 2$ (there are $\binom{4}{2} = 6$ ways how to pick 2 hats that stay in place; and the remaining two hats can switch places only in one way),
- For 8 of 24 permutations $X = 1$ (there are $\binom{4}{1} = 4$ ways how to pick 1 hat that stays in place; and the remaining three hats can rotate in two ways).
- For the remaining $24 - (1 + 6 + 8) = 9$ permutations $X = 0$ (no hats stay in place).

(A) $E(X) = \frac{1}{24} \cdot 4 + \frac{6}{24} \cdot 2 + \frac{8}{24} \cdot 1 = 1$. This means that the expected number of hats that stay in place is exactly 1.
(B) For all 24 permutations, subtract the value $E(X) = 1$ from every hat experiment outcome. To make addition faster, we group the terms by their value (one value $X = 4$, six values $X = 2$ and so on):

$$V(X) = \frac{(4 - E(X))^2 + 6 \cdot (2 - E(X))^2 + 8 \cdot (1 - E(X))^2 + 9 \cdot (0 - E(X))^2}{24}$$

$$= \frac{3^2 + 6 \cdot 1^2 + 8 \cdot 0^2 + 9 \cdot (-1)^2}{24} = \frac{24}{24} = 1.$$

Therefore, $V(X) = 1$ (variance also equals 1, but the unit of measurement is not hats but ``hats squared'').

Problem 8

(A) $P(E|H)$ is the outcome of the Binomial distribution: There are $n = 5$ coin-toss experiments; the probability of success for any single experiment is $p = \frac{2}{3}$ (since we know that the day is lucky and hypothesis H holds). Therefore,

$$P(E|H) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{80}{243}$$

(B) $P(E|H) \cdot P(H) = \frac{80}{243} \cdot \frac{1}{2} = \frac{40}{243}$, since $P(H) = \frac{1}{2}$ (the *a priori* probability of a lucky day is exactly 1/2).

(C) $P(E|\overline{H})$ is the outcome of the Binomial distribution: Again, there are $n = 5$ coin-toss experiments, but now the probability of a single experiment is just $p = \frac{1}{3}$. Therefore,

$$P(E|\overline{H}) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 = \frac{40}{243}$$

(D) $P(E|\overline{H}) \cdot P(\overline{H}) = \frac{40}{243} \cdot \frac{1}{2} = \frac{20}{243}$.

(E) We can compute $P(E)$ as the sum of two mutually incompatible events: event E can happen either on a lucky day or on an unlucky day:

$$P(E) = P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H}) = \frac{40}{243} + \frac{20}{243} = \frac{60}{243}.$$

(F) Use Bayes formula:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H})} = \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{\frac{40}{243}}{\frac{60}{243}} = \frac{2}{3}.$$

Bayes formula is intuitive: It shows the proportion of the subcase $P(E|H) \cdot P(H)$ (i.e. event E hapens on a lucky day) out of the whole probability $P(E) = P(E|H) \cdot P(H) + P(E|\overline{H}) \cdot P(\overline{H})$ (i.e. event E happens either on a lucky or unlucky day).

Problem 9

(A) $+ : - \sim \mathrm{y} \times 6\ 3\ \mathrm{z}\ 2$,
(B) $\mathrm{y} \sim\ -\ 6 \times 3 : \mathrm{z} + 2$,
(C) $\mathrm{y} \sim 6\ 3\ \times - \mathrm{z} : 2\ +$.

Note: In inorder traversal (**(B)**) we first visit the first subtree (e.g., y), and only then the parent node (e.g., unary minus ~). See (Rosen2019, p.811).

Problem 10

We start from vertex I. At every step we grow the tree by a single edge (so that it stays connected and the newly added edge has the smallest possible weight).

Step	Newly Added Edge
Step 1	$IG, w = 1$
Step 2	$IE, w = 2$
Step 3	$ED, w = 1$
Step 4	$DC, w = 2$
Step 5	$CH, w = 2$
Step 6	$GA, w = 3$
Step 7	$AB, w = 2$
Step 8	$IF, w = 5$

The total weight of all added edges (same as the total weight of the MST) is 18.

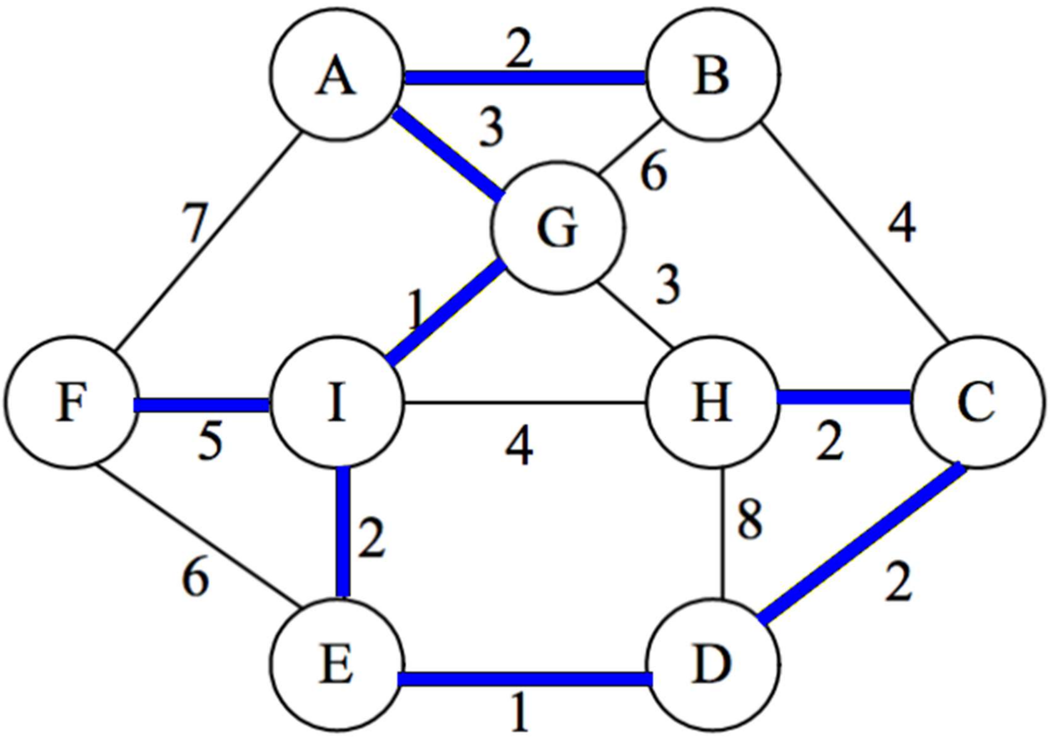

Figure: MST edges shown in blue