N3. Define the sequence $a_0, a_1, a_2, \dots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Solution 1. Call a nonnegative integer *representable* if it equals the sum of several (possibly 0 or 1) distinct terms of the sequence. We say that two nonnegative integers b and c are *equivalent* (written as $b \sim c$) if they are either both representable or both non-representable.

One can easily compute

$$S_{n-1} := a_0 + \cdots + a_{n-1} = 2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3.$$

Indeed, we have $S_n - S_{n-1} = 2^n + 2^{\lfloor n/2 \rfloor} = a_n$ so we can use the induction. In particular, $S_{2k-1} = 2^{2k} + 2^{k+1} - 3$.

Note that, if $n \geqslant 3$, then $2^{\lceil n/2 \rceil} \geqslant 2^2 > 3$, so

$$S_{n-1} = 2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3 > 2^n + 2^{\lfloor n/2 \rfloor} = a_n.$$

Also notice that $S_{n-1} - a_n = 2^{\lceil n/2 \rceil} - 3 < a_n$.

The main tool of the solution is the following claim.

Claim 1. Assume that b is a positive integer such that $S_{n-1} - a_n < b < a_n$ for some $n \geqslant 3$. Then $b \sim S_{n-1} - b$.

Proof. As seen above, we have $S_{n-1} > a_n$. Denote $c = S_{n-1} - b$; then $S_{n-1} - a_n < c < a_n$, so the roles of b and c are symmetrical.

Assume that b is representable. The representation cannot contain a_i with $i \geqslant n$, since $b < a_n$. So b is the sum of some subset of $\{a_0, a_1, \dots, a_{n-1}\}$; then c is the sum of the complement. The converse is obtained by swapping b and c. □

We also need the following version of this claim.

Claim 2. For any $n \geqslant 3$, the number a_n can be represented as a sum of two or more distinct terms of the sequence if and only if $S_{n-1} - a_n = 2^{\lceil n/2 \rceil} - 3$ is representable.

Proof. Denote $c = S_{n-1} - a_n < a_n$. If a_n satisfies the required condition, then it is the sum of some subset of $\{a_0, a_1, \dots, a_{n-1}\}$; then c is the sum of the complement. Conversely, if c is representable, then its representation consists only of the numbers from $\{a_0, \dots, a_{n-1}\}$, so a_n is the sum of the complement. □

By Claim 2, in order to prove the problem statement, it suffices to find infinitely many representable numbers of the form $2^t - 3$, as well as infinitely many non-representable ones.

Claim 3. For every $t \geqslant 3$, we have $2^t - 3 \sim 2^{4t-6} - 3$, and $2^{4t-6} - 3 > 2^t - 3$.

Proof. The inequality follows from $t \geqslant 3$. In order to prove the equivalence, we apply Claim 1 twice in the following manner.

First, since $S_{2t-3} - a_{2t-2} = 2^{t-1} - 3 < 2^t - 3 < 2^{2t-2} + 2^{t-1} = a_{2t-2}$, by Claim 1 we have $2^t - 3 \sim S_{2t-3} - (2^t - 3) = 2^{2t-2}$.

Second, since $S_{4t-7} - a_{4t-6} = 2^{2t-3} - 3 < 2^{2t-2} < 2^{4t-6} + 2^{2t-3} = a_{4t-6}$, by Claim 1 we have $2^{2t-2} \sim S_{4t-7} - 2^{2t-2} = 2^{4t-6} - 3$.

Therefore, $2^t - 3 \sim 2^{2t-2} \sim 2^{4t-6} - 3$, as required. □

Now it is easy to find the required numbers. Indeed, the number $2^3 - 3 = 5 = a_0 + a_1$ is representable, so Claim 3 provides an infinite sequence of representable numbers

$$2^3 - 3 \sim 2^6 - 3 \sim 2^{18} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots .$$

On the other hand, the number $2^7 - 3 = 125$ is non-representable (since by Claim 1 we have $125 \sim S_6 - 125 = 24 \sim S_4 - 24 = 17 \sim S_3 - 17 = 4$ which is clearly non-representable). So Claim 3 provides an infinite sequence of non-representable numbers

$$2^7 - 3 \sim 2^{22} - 3 \sim 2^{82} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots .$$

Solution 2. We keep the notion of representability and the notation S_n from the previous solution. We say that an index n is *good* if a_n writes as a sum of smaller terms from the sequence $a_0, a_1, \ldots$. Otherwise we say it is *bad*. We must prove that there are infinitely many good indices, as well as infinitely many bad ones.

Lemma 1. If $m \geqslant 0$ is an integer, then 4^m is representable if and only if either of $2m+1$ and $2m+2$ is good.

Proof. The case $m = 0$ is obvious, so we may assume that $m \geqslant 1$. Let $n = 2m+1$ or $2m+2$. Then $n \geqslant 3$. We notice that

$$S_{n-1} < a_{n-2} + a_n.$$

The inequality writes as $2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3 < 2^n + 2^{\lfloor n/2 \rfloor} + 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$, i.e. as $2^{\lceil n/2 \rceil} < 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1} + 3$. If $n \geqslant 4$, then $n/2 \leqslant n-2$, so $\lceil n/2 \rceil \leqslant n-2$ and $2^{\lceil n/2 \rceil} \leqslant 2^{n-2}$. For $n = 3$ the inequality verifies separately.

If n is good, then a_n writes as $a_n = a_{i_1} + \cdots + a_{i_r}$, where $r \geqslant 2$ and $i_1 < \cdots < i_r < n$. Then $i_r = n-1$ and $i_{r-1} = n-2$, for if $n-1$ or $n-2$ is missing from the sequence $i_1, \ldots, i_r$, then $a_{i_1} + \cdots + a_{i_r} \leqslant a_0 + \cdots + a_{n-3} + a_{n-1} = S_{n-1} - a_{n-2} < a_n$. Thus, if n is good, then both $a_n - a_{n-1}$ and $a_n - a_{n-1} - a_{n-2}$ are representable.

We now consider the cases $n = 2m+1$ and $n = 2m+2$ separately.

If $n = 2m+1$, then $a_n - a_{n-1} = a_{2m+1} - a_{2m} = (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+1$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then $2^{2m} < a_{2m}$, so 2^{2m} is a sum of some distinct terms a_i with $i < 2m$. It follows that $a_{2m+1} = a_{2m} + 2^{2m}$ writes as a_{2m} plus a sum of some distinct terms a_i with $i < 2m$. Hence $2m+1$ is good.

If $n = 2m+2$, then $a_n - a_{n-1} - a_{n-2} = a_{2m+2} - a_{2m+1} - a_{2m} = (2^{2m+2} + 2^{m+1}) - (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+2$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then, as seen in the previous case, it writes as a sum of some distinct terms a_i with $i < 2m$. Hence $a_{2m+2} = a_{2m+1} + a_{2m} + 2^{2m}$ writes as $a_{2m+1} + a_{2m}$ plus a sum of some distinct terms a_i with $i < 2m$. Thus $2m+2$ is good. □

Lemma 2. If $k \geqslant 2$, then 2^{4k-2} is representable if and only if 2^{k+1} is representable.

In particular, if $s \geqslant 2$, then 4^s is representable if and only if 4^{4s-3} is representable. Also, $4^{4s-3} > 4^s$.

Proof. We have $2^{4k-2} < a_{4k-2}$, so in a representation of 2^{4k-2} we can have only terms a_i with $i \leqslant 4k-3$. Notice that

$$a_0 + \cdots + a_{4k-3} = 2^{4k-2} + 2^{2k} - 3 < 2^{4k-2} + 2^{2k} + 2^k = 2^{4k-2} + a_{2k}.$$

Hence, any representation of 2^{4k-2} must contain all terms from a_{2k} to a_{4k-3}. (If any of these terms is missing, then the sum of the remaining ones is $\leqslant (a_0 + \cdots + a_{4k-3}) - a_{2k} < 2^{4k-2}$.) Hence, if 2^{4k-2} is representable, then $2^{4k-2} - \sum_{i=2k}^{4k-3} a_i$ is representable. But

$$2^{4k-2} - \sum_{i=2k}^{4k-3} a_i = 2^{4k-2} - (S_{4k-3} - S_{2k-1}) = 2^{4k-2} - (2^{4k-2} + 2^{2k} - 3) + (2^{2k} + 2^{k+1} - 3) = 2^{k+1}.$$

So, if 2^{4k-2} is representable, then 2^{k+1} is representable. Conversely, if 2^{k+1} is representable, then $2^{k+1} < 2^{2k} + 2^k = a_{2k}$, so 2^{k+1} writes as a sum of some distinct terms a_i with $i < 2k$. It follows that $2^{4k-2} = \sum_{i=2k}^{4k-3} a_i + 2^{k+1}$ writes as $a_{4k-3} + a_{4k-4} + \cdots + a_{2k}$ plus the sum of some distinct terms a_i with $i < 2k$. Hence 2^{4k-2} is representable.

For the second statement, if $s \geqslant 2$, then we just take $k = 2s-1$ and we notice that $2^{k+1} = 4^s$ and $2^{4k-2} = 4^{4s-3}$. Also, $s \geqslant 2$ implies that $4s-3 > s$. □

Now $4^2 = a_2+a_3$ is representable, whereas $4^6 = 4096$ is not. Indeed, note that $4^6 = 2^{12} < a_{12}$, so the only available terms for a representation are $a_0, \dots, a_{11}$, i.e., 2, 3, 6, 10, 20, 36, 72, 136, 272, 528, 1056, 2080. Their sum is $S_{11} = 4221$, which exceeds 4096 by 125. Then any representation of 4096 must contain all the terms from $a_0, \dots, a_{11}$ that are greater that 125, i.e., 136, 272, 528, 1056, 2080. Their sum is 4072. Since $4096 - 4072 = 24$ and 24 is clearly not representable, 4096 is non-representable as well.

Starting with these values of m, by using Lemma 2, we can obtain infinitely many representable powers of 4, as well as infinitely many non-representable ones. By Lemma 1, this solves our problem.