Discrete Sample Quiz 8

Question 1 What is the probability that the sum of the numbers on two dice is even when they are rolled? (Express your answer as P/Q.)

Question 2
(A) What is the probability to get sequence "`Heads`, `Heads`, `Heads`, `Heads`, `Heads`, `Heads`" when tossing a coin?
(B) What is the probability to get sequence "`Heads, Tails, Heads, Tails, Heads, Tails, Heads, Tails`" when tossing a coin?
(C) What is the probability to get even number of "Heads" when tossing a coin 6 times?

Question 3 Assume that there is the following gambling game involving dice (cube with six outcomes: $1, 2, 3, 4, 5, 6$ with equal probabilities). Player A guesses a number n between 1 and 6 (for example $n = 3$). Then he rolls a dice three times. We have the following four outcomes (events):
E_1: If one of the dice rolls equals Player's A number n, then Player A wins 1 euro.
E_2: If two of the dice rolls equal Player's A number n, then Player A wins 2 euros.
E_3: If all three dice rolls equal Player's A number n, then Player A wins 3 euros.
E_4: If none of the dice rolls equals n, then Player A loses 1 euro.
(A) Find the probability for each event E_1, E_2, E_3, E_4.
(B) Find the expected value of the money that Player A wins or loses during a single round of this game. (One round = three dice rolls as described above). Use this formula – a weighted sum of the euros multiplied by the respective probabilities:

$$p(E_1) \cdot 1 + p(E_2) \cdot 2 + p(E_3) \cdot 3 + p(E_4) \cdot (-1).$$

Question 4 Find a probability that choosing a random month during a leap year, it would have exactly 5 Sundays.

Question 5 Suppose you and a friend each choose at random an integer between 1 and 8, inclusive. For example, some possibilities are $(3, 7)$, $(7, 3)$, $(4, 4)$, $(8, 1)$, where your number is written first and your friends number second. Find the following probabilities:
(A) p(you pick 5 and your friend picks 8).
(B) p(sum of the two numbers picked is < 4).
(C) p(both numbers match).
(D) p(the sum of the two numbers is a prime).
(E) p(your number is greater than your friends number).

Question 6 In a card-game "Zolīte" there are 26 cards: four Aces (worth 11 points each); four Kings (worth 4 points each); four Queens (worth 3 points each); four Jacks (worth 2 points each); four cards "10" (worth 10 points each); and also four "9", one "8" and one "7" (they are worth 0 points).
A single trick (*stiķis*) in this game consists of any three cards (you can assume that all the C_{26}^3 tricks have equal probabilities.)
(A) Let X be the random variable that expresses the number of points of a single card out of the 26 cards. Find the expected value $E(X)$.
(B) Find the variance $V(X)$.
(C) Let Y be the random variable that expresses the total number of points for all three cards c_1, c_2, c_3 in a random trick. Find the expected value $E(Y)$.
(D) Is $E(Y)$ exactly three times larger than $E(X)$?

Question 7 You create a random bit string of length five (all 32 bit strings are equally probable). Consider these events:
E_1: the bit string chosen begins with 1;
E_2: the bit string chosen ends with 1;
E_3: the bit string chosen has exactly three 1s.
(A) Find $p(E_1 \mid E_3)$.
(B) Find $p(E_3 \mid E_2)$.
(C) Find $p(E_2 \mid E_3)$.

Question 8 A tiny ant travels in 3D space. It starts in the point $(0; 0; 0)$ and makes 9 steps altogether: With equal probability each step can be parallel to either x, y or z axes (incrementing the corresponding coordinate).
(A) What is the probability that the ant has taken at least one step in the direction of each coordinate axis?
(B) What is the probability that the ant has reached the point $(3; 3; 3)$?

Question 9 There were 5 people who left their hats in a lobby as they entered some building. After that there was an electricity blackout; in the total darkness everybody exited the building and grabbed a random hat. All $5! = 120$ permutations of hats are considered equally likely. Find the probability that nobody got his/her own hat.
You may want to define five events: E_1 means that the 1st person got his/her hat; E_2 – the 2nd person got his/her hat, etc. Then you can apply the inclusion/exclusion principle:

$$\begin{aligned}
&p(E_1 \cup E_2 \cup E_3 \cup E_3 \cup E_4 \cup E_5) = \\
&= p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - \\
&- p(E_1 \cap E_2) - p(E_1 \cap E_3) - \ldots - p(E_4 \cap E_5) + \\
&+ p(E_1 \cap E_2 \cap E_3) + \ldots + p(E_3 \cap E_4 \cap E_5) - \\
&- p(E_1 \cap E_2 \cap E_3 \cap E_4) - \ldots - p(E_2 \cap E_3 \cap E_4 \cap E_5) + \\
&+ p(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5).
\end{aligned}$$