

## Discrete Sample Quiz 4

**Question 1.** We define three sets in the universe  $U$  of integer numbers between 1 and 70 (inclusive):

$$\begin{cases} K_2 &= \{x \in U \mid 2 \mid x\}, \\ K_5 &= \{x \in U \mid 5 \mid x\}, \\ K_7 &= \{x \in U \mid 7 \mid x\}, \end{cases}$$

Find the size of the following sets. Here  $|X|$  denotes the number of elements in a finite set (BTW,  $|X|$  is also the notation for the cardinality of an infinite set).

*Note.* It was not intentional, but vertical bar:  $|$  in this exercise happens to be used in three different ways: It is a separator when defining sets  $K_2$ ; it is used to denote divisibility; it also denotes set cardinality.

$$\begin{array}{ll} |K_2 \cup K_5| & \dots \\ |K_2 \cap K_7| & \dots \\ |\overline{K_7}| & \dots \\ |\overline{K_2 \cap K_5}| & \dots \\ |K_2 \cap K_5 \cap K_7| & \dots \\ |K_2 \cap K_5 \cap \overline{K_7}| & \dots \\ |K_2 \cup K_5 \cup K_7| & \dots \\ |\overline{K_2 \cup K_5 \cup K_7}| & \dots \end{array}$$

**Question 2.** Find a counterexample to refute the following predicate expression:

$$\begin{aligned} &(\exists x \in U, P(x)) \wedge (\exists x \in U, Q(x)) \rightarrow \\ &\rightarrow \exists x \in U, (P(x) \wedge Q(x)). \end{aligned}$$

Here  $P(x)$  is true iff  $P$  is a full square (a square of some integer number),  $Q(x)$  is true iff  $x$  is divisible by 5, and  $U$  is the set of all integers from the interval  $[120; 130]$ .

*Note.* The three  $x$ 's in this formula refer to three unrelated (local) variables. If it looks confusing, you can rewrite it like this:

$$\begin{aligned} &(\exists x_1 \in U, P(x_1)) \wedge (\exists x_2 \in U, Q(x_2)) \rightarrow \\ &\rightarrow \exists x_3 \in U, (P(x_3) \wedge Q(x_3)). \end{aligned}$$

(A) Identify the variables which you need to pick for your counter-example.

(B) Pick the values for these variables to make the above statement false.

**Question 3.** Determine the cardinality of the following sets (some finite number? equal to  $|\mathbb{N}|$ ? equal to  $|\mathcal{P}(\mathbb{N})|$ ? equal to  $|\mathbb{R}|$ ? equal to  $|\mathcal{P}(\mathbb{R})|$ ?)

- (A) The set of positive real numbers from  $(0; 1)$  with decimal representation containing only digits 0 and 1?
- (B) The set of positive real numbers from  $(0; 1)$  with decimal representation containing only digits 0 and 1 (and it is known that the number of 0s is finite)?
- (C) The set of positive real numbers from  $(0; 1)$  that are fully periodic decimal fractions with a period of 2020 digits?

(D) The set of positive real numbers from  $(0; 1)$  that have decimal representation without any digits "9"?

(E) The set of all irrational  $x \in (0; 1)$  such that  $x^3$  is rational?

(F) Ordered pairs of real numbers  $x_1, x_2$  such that  $x_1, x_2 \in (0; 1)$ .

(G) Finite sequences of real numbers of any length:  $x_1, \dots, x_n$  (where  $n$  can take different values from  $\mathbb{Z}$ )?

(H) Infinite sequences of real numbers:  $\{x_n\}$ :  $x_1, x_2, x_3, \dots$

**Question 4.** Let  $f(x) = \left\lfloor \frac{x^3}{3} \right\rfloor$ . Find  $f(S)$  if  $S$  is:

(A)  $S = \{2, 1, 0, 1, 2, 3\}$ .

(B)  $S = \{0, 1, 2, 3, 4, 5\}$ .

(C)  $S = \{1, 5, 7, 11\}$ .

Is function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  injective? Is it surjective? If it is not, mention counterexamples to show this.

**Question 5.** Determine, if the given set is a power-set of some other set. If yes, which one?

(A)  $\{\emptyset, \{\emptyset\}, \{a\}, \{\{a\}\}, \{\{\{a\}\}\}, \{\emptyset, a\}, \{\emptyset, \{a\}\}, \{\emptyset, \{\{a\}\}\}, \{a, \{a\}\}, \{a, \{\{a\}\}\}, \{\{a\}, \{\{a\}\}\}, \{\emptyset, a, \{a\}\}, \{\emptyset, a, \{\{a\}\}\}, \{\emptyset, \{a\}, \{\{a\}\}\}, \{a, \{a\}, \{\{a\}\}\}, \{\emptyset, a, \{a\}, \{\{a\}\}\}\}$ .

(B)  $\{\emptyset, \{a\}\}$ .

(C)  $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ .

(D)  $\{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}$ .

(E)  $\{\emptyset, \{a, \emptyset\}\}$ .

**Question 6.** Given two sets  $A = \{x, y\}$  and  $B = \{x, \{x\}\}$ , check, if statements are true or false:

(A)  $x \subseteq B$ .

(B)  $\emptyset \in \mathcal{P}(B)$ .

(C)  $\{x\} \subseteq A - B$ .

(D)  $|\mathcal{P}(A)| = 4$ .

**Question 7.** We define functions  $g : A \rightarrow A$  and  $f : A \rightarrow A$ , where  $A = \{1, 2, 3, 4\}$  by listing all argument-value pairs:

$$g = \{(1, 4), (2, 1), (3, 1), (4, 2)\}, \quad f = \{(1, 3), (2, 2), (3, 4), (4, 1)\}.$$

Find these functions by listing their argument/value pairs (or establish that they do not exist).

- (A) Find  $f \circ g$ .  
 (B) Find  $g \circ f$ .  
 (C) Find  $g \circ g$ .  
 (D) Find  $g \circ (g \circ g)$ .  
 (E) Find  $f^{-1}$ .  
 (F) Find  $g^{-1}$ .

**Question 8.** Find these sums:

- (A)  $1/4 + 1/8 + 1/16 + 1/32 + \dots$   
 (B)  $2 + 4 + 8 + 16 + 32 + \dots + 2^{28}$ .  
 (C)  $2 - 4 + 8 - 16 + 32 - \dots - 2^{28}$ .  
 (D)  $1 - 1/2 + 1/4 - 1/8 + 1/16 - \dots$

**Question 9.** Find an appropriate  $O(g(n))$  for each function  $f(n)$  defined below (pick your  $g(n)$  to be the slowest growing among the functions such that  $f(n)$  is in  $O(g(n))$ ).

- (A)  $f(n) = 1^2 + 2^2 + \dots + n^2$ .  
 (B)  $f(n) = \frac{3n - 8 - 4n^3}{2n - 1}$ .  
 (C)  $f(n) = \sum_{k=1}^n k^3$ .  
 (D)  $f(n) = \frac{6n + 4n^5 - 4}{7n^2 - 3}$ .  
 (E)  $f(n) = \sum_{k=2}^n k \cdot (k - 1)$ .  
 (F)  $f(n) = 3n^2 + 8n + 7$

**Question 10.** For the given functions, find an optimal  $O(g(x))$ ; find  $C$  and  $n_0$  (from the definition  $|f(x)| < C \cdot |g(x)|$  as long as  $x > n_0$ ).

- (A)  $f(x) = 3x^4 + \log_2 x^8$ .

(B)  $f(x) = \sum_{k=1}^n (k^3 + k)$ .

(C)  $f(n) = (x + 2) \log_2(x^2 + 1) + \log_2(x^3 + 1)$ .

(D)  $f(n) = n^3 + \sin n^7$ .

**Question 11.** This is a Python fragment; variable  $n$  can become very large;  $t$  is some fixed parameter. Denote by  $f(n)$  the number of operations depending on the variable  $n$ , where an operation is an addition or a multiplication, or raising to the power 2. Find the slowest growing  $g(n)$  so that  $f(n)$  is in  $O(g(n))$ .

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sum = 0
t = 2
for i in range(1, n+1):
    for j in range(1, n+1):
        sum += (i*t + j*t + 1)**2
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**Question 12.** There are two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  defined for all real numbers and taking real values. Find, which predicate logic expressions describe a statement that is logically equivalent to the English sentence “The function  $f(n)$  is in  $O(g(n))$ ”.

*Note.* There may be multiple correct answers.

- (A)  $\forall n \in \mathbb{R} \exists n_0 \in \mathbb{R} \exists C \in \mathbb{R},$   
 $(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ .  
 (B)  $\exists n_0 \in \mathbb{R} \forall n \in \mathbb{R} \exists C \in \mathbb{R},$   
 $(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ .  
 (C)  $\exists n_0 \in \mathbb{R} \exists C \in \mathbb{R} \forall n \in \mathbb{R},$   
 $(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ .  
 (D)  $\exists n_0 \in \mathbb{R} \exists C \in \mathbb{R} \forall n \in \mathbb{R},$   
 $(n > n_0 \rightarrow f(n) \leq C \cdot |g(n)|)$ .  
 (E)  $\exists n_0 \in \mathbb{R} \exists C \in \mathbb{R} \forall n \in \mathbb{R},$   
 $(n > n_0 \rightarrow |f(n)| \leq C \cdot g(n))$ .  
 (F)  $\exists n_0 \in \mathbb{R} \exists C \in \mathbb{R} \forall n \in \mathbb{R},$   
 $(n \geq n_0 \rightarrow |f(n)| < C \cdot |g(n)|)$ .  
 (G)  $\exists n_0 \in \mathbb{Z}^+ \exists C \in \mathbb{Z}^+ \forall n \in \mathbb{R},$   
 $(n > n_0 \rightarrow |f(n)| \leq C \cdot |g(n)|)$ .