HOMEWORK 01, DUE BY 2022-01-13

Question 1: Consider the following cubic equation: $x^3 + 2x^2 - 4x - 3 = 0$.

(A) It is known that this equation has an integer root r. Guess that root. Divide the cubic polynomial with $(x - r)$, where r is replaced by the root you guessed.

(B) Find all roots of this cubic equation.

(C) Factorize the polynomial $P(x) = x^3 + 2x^2 - 4x - 3$: Express it as a product $(x - a)(x - b)(x - c)$, where a, b, c are real numbers.

Question 2: Consider the following truth table computing a Boolean function $f(x_1, x_2, x_3)$ of three variables:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
`false`	`false`	`false`	`true`
`false`	`false`	`true`	`false`
`false`	`true`	`false`	`false`
`false`	`true`	`true`	`false`
`true`	`false`	`false`	`false`
`true`	`false`	`true`	`true`
`true`	`true`	`false`	`true`
`true`	`true`	`true`	`true`

(A) Build a DNF given the truth table. Write all terms as conjunctions of exactly three variables or their negatons.

(B) Apply Boolean identities to create a shorter DNF.

(C) Write an expression for the Boolean function $f(x_1, x_2, x_3)$ using only implication and negation operations.

Question 3: Consider the following argument: *"If I gain weight, then I get depressed, and if I get depressed, then I eat too much. If I eat too much, then I get lazy, and if I get lazy, then I don't exercise. If I don't exercise, then I gain weight. Therefore, I will gain weight."*

(A) Define atomic propositions used in this argument (such propositions state simple facts, they do not contain "if … then" constructs or negations). Denote every atomic proposition by a letter.

(B) Write all the sentences using the atomic propositions you defined in the previous step. Every sentence becomes a Boolean expression. Write these sentences into a single column – all of them except the last one are hypotheses. The last one (following the word "Therefore…" is the conclusion).

(C) Is this argument valid? To check its validity, you could use e.g. the *Indirect truth table method* – try to make the conclusion false, and then see, which values the variables in the atomic propositions should be assigned certain values. If you can conclude that in order to make the conclusion false you need to make some hypothesis false as well, then the argument is valid. (On the other hand, if you can assign values so that all hypotheses are true, but the conclusion is still false, then the argument is not valid.)

Question 4:

(A) Prove that $\log_{12} 2$ is irrational.

(B) Does there exist a positive integer $a \in \mathbb{Z}_{>0}$ such that $\log_{12} a$ is rational fraction p/q that is not an integer? (Either prove that such a cannot exist or show a counterexample.)

Question 5: Lidl supermarket chain sells identical cookies in two kinds of packages: package #1 contains 5 cookies, package #2 contains 13 cookies.

(A) Use the well-ordering principle to show that every amount of cookies $n \geq 48$ can be bought by selecting zero or more packages of either type.

(B) Show that 47 cookies cannot be bought in this manner.

Question 6 (Supplementary Task):

Introduction: NAND gates are devices with two inputs x and y and one output z that compute the negation of conjunction: $z = \neg(x \wedge y)$ ("not (x and y)" also known as NAND operation). Denote NAND by $x \uparrow y$. It has the following truth table:

x	y	$z = x \uparrow y$
`false`	`false`	`true`
`false`	`true`	`true`
`true`	`false`	`true`
`true`	`true`	`false`

It is possible to express other Boolean operations using just the NAND ($\uparrow$) operation.

- Negation: $\neg x \equiv x \uparrow x$.
- Conjunction: $x \wedge y \equiv \neg(x \uparrow y) \equiv (x \uparrow y) \uparrow (x \uparrow y)$.
- Disjunction: $x \vee y \equiv \neg(\neg x \wedge \neg y) \equiv (\neg x \uparrow \neg y) \equiv ((x \uparrow x) \uparrow (y \uparrow y))$.
- Implication: $x \rightarrow y \equiv \neg x \vee y \equiv x \uparrow (y \uparrow y)$.

Problem: Assume that you need to use only NAND gates to compute XOR for two, three or four variables:

(A) $x_1 \oplus x_2$

(B) $x_1 \oplus x_2 \oplus x_3$

(C) $x_1 \oplus x_2 \oplus x_3 \oplus x_4$

Draw the circuits for these three expressions (A), (B), (C). If there are multiple solutions, select the circuit which uses as few NAND gates as possible. Justify your answers – why do the circuits compute the given expressions and why are they minimal.

Note: In a circuit you can reuse the output of the same NAND gate as input several times. The following image shows how $x_1 \wedge x_2$ can be computed using just 2 NAND gates. On the other hand, the formula $x_1 \wedge x_2 \equiv (x_1 \uparrow x_2) \uparrow (x_1 \uparrow x_2)$ contains three NAND operations, but there are two identical subexpressions $(x_1 \uparrow x_2)$.