Discrete Sample Quiz 9

Question 1 (Rosen7e, Ch.8, Q4). Denote by a_n the number of ways to go down an n-step staircase if you go down 1, 2, or 3 steps at a time. (In other words, a_n denotes the number of ways how n can be expressed as a sum of 1, 2, 3, if the order in the sum matters.)

Define a_n as a recurrent sequence. Assume that the sequence starts from a_1 .

Question 2 (Rosen7e, Ch.8, Q11). A vending machine accepts only \$1 coins, \$1 bills, and \$2 bills. Let a_n denote the number of ways of depositing n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

- (A) Find a recurrence relation for a_n and give the necessary initial condition(s).
- **(B)** Find an explicit formula for a_n by solving the recurrence relation in part **(A)**.

Question 3 (Rosen7e, Ch.8, Q16-Q20). For each item solve the recurrence relation (characteristic equation or simply guess the pattern for the terms):

(A)
$$a_n = a_{n-2}$$
, $a_0 = 2$, $a_1 = -1$.

(B)
$$a_n = 2a_{n-1} + 2a_{n-2}, a_0 = 0, a_1 = 1.$$

(C)
$$a_n = 3na_{n-1}, a_0 = 2$$
.

(D)
$$a_n = a_{n-1} + 3n$$
, $a_0 = 5$.

(E)
$$a_n = 2a_{n-1} + 5$$
, $a_0 = 3$.

Question 4 (Rosen7e, Ch.8, Q24) Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r + 2)(r + 4)^2 = 0$.

- (A) Describe the form for the general solution to the recurrence relation.
- **(B)** Define a recurrent sequence that leads to this characteristic equation.

Question 5 (Rosen7e, Ch.8, Q27) The Catalan numbers C_n count the number of strings of n pluses (+) and n minuses (-) with the following property: as each string is read from left to right, the number of pluses encountered is always at least as large as the number of minuses.

- (A) Verify this by listing these strings of lengths 2, 4, and 6 and showing that there are C_1 , C_2 , and C_3 of these, respectively.
- **(B)** Explain how counting these strings is the same as counting the number of ways to correctly parenthesize strings of variables.

Note. Catalan numbers

$$C_0 = 1$$
, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$,...

are defined as
$$C_n = \frac{1}{n+1} \binom{2n}{n}$$
.

Question 6 (Rosen7e, Ch.8, Q28, Q29)

- (A) What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} 4a_{n-2} + F(n)$ have when $F(n) = 2^n$?
- **(B)** What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} 4a_{n-2} + F(n)$ have when $F(n) = n2^n$?

Question 7 (Rosen7e, Ch.8, Q32). Consider the recurrence relation $a_n = 2a_{n-1} + 3n$.

- (A) Write the associated homogeneous recurrence relation.
- **(B)** Find the general solution to the associated homogeneous recurrence relation.
- (C) Find a particular solution to the given recurrence
- **(D)** Write the general solution to the given recurrence relation.
- (E) Find the particular solution to the given recurrence relation when $a_0 = 1$.

Question 8 (Rosen7e, Ch.8, Q37). Suppose f(n) = f(n/3) + 2n, f(1) = 1. Find f(27).

Question 9 (Rosen7e, Ch.8, Q53-Q63) Find the coefficient of x^8 in the power series of each of the function:

(A)
$$(1 + x^2 + x^4)^3$$
.

(B)
$$(1 + x^2 + x^4 + x^6)^3$$
.

(C)
$$(1 + x^2 + x^4 + x^6 + x^8)^3$$
.

(D)
$$(1 + x^2 + x^4 + x^6 + x^8 + x^{10})^3$$
.

(E)
$$(1+x^3)^{12}$$
.

(F)
$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)$$
.

(G)
$$1/(1-2x)$$
.

(H)
$$x^3/(1-3x)$$
.

(I)
$$1/(1-x)^2$$
.

(J)
$$x^2/(1+2x)^2$$
.

(K)
$$1/(1-3x^2)$$
.

Answers

Question 1. For staircases with n = 1, 2, 3 steps we can enumerate all possible sums and count them.

1-step staircases: 1.

2-step staircases: 1 + 1, 2.

3-step staircases: 1 + 1 + 1, 1 + 2, 2 + 1, 3.

$$\begin{cases} a_1 = 1, \\ a_2 = 2, \\ a_3 = 4, \\ a_n = a_{n-1} + a_{n-2} + a_{n-3}, \ n \ge 4. \end{cases}$$

Why an can be expressed as sum of a_{n-1} , a_{n-2} , a_{n-3} ? Let us consider the first movement downstairs. It can be either 1 step down (then the remaining steps can be made in a_{n-1} different ways), or 2 steps down (then the remaining steps can be made in a_{n-2} different ways), or 3 steps down (then the remaining steps can be made in a_{n-3} different ways).

All these options do not intersect (since any moving downstairs starts with either 1, 2 or 3), therefore we simply add all these ways together: $a_{n-1} + a_{n-2} + a_{n-3}$.

Question 2.

(A) Let us find the initial conditions (values a_1 and a_2) and the recurrence relation (how to express a_n via the the two previous values).

$$\left\{ \begin{array}{l} a_1=2,\\ a_2=5,\\ a_n=2a_{n-1}+a_{n-2},\ n\geq 3. \end{array} \right.$$

The initial conditions can be verified by noticing that 1 dollar can be paid in two ways (a coin or a bill); 2 dollars can be paid in five ways (either one \$2 bill or any of the following: coin+coin, coin+bill, bill+coin, bill+bill).

If $n \ge 3$, then a_n either starts by a \$1 coin (plus paying the remaining sum in a_{n-1} different ways) or by a \$1 bill (plus paying the remaining sum in a_{n-1} different ways), or by a \$2 bill (plus paying the remaining sum in a_{n-2} different ways). By adding together these mutually exclusive options we get

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} = 2a_{n-1} + a_{n-2}$$
.

Question 3.

(A) Characteristic equation: $r^2 - 1 = 0$. The roots are $r_1 = -1$ and $r_2 = 1$. So, the general form of this sequence is

$$a_n = A \cdot (-1)^n + B \cdot 1^n.$$

To get $a_0 = 2$ and $a_1 = -1$, we need $A = \frac{3}{2}$ and $B = \frac{1}{2}$. Therefore,

$$a_n = \frac{3}{2} \cdot (-1)^n + \frac{1}{2}.$$

This expression would generate our alternating sequence

$$2, -1, 2, -1, 2, -1, \ldots$$

(B) The characteristic equation is

$$r^2 - 2r - 2 = 0.$$

And the roots of this square equation are $r_{1,2} = 1 \pm \sqrt{3}$. We need to find constants A and B such that the formula

$$a_n = A \cdot r_1^n + B \cdot r_2^n$$

is correct and we get $a_0 = 0$ and $a_1 = 1$. By substituting n = 0 and n = 1 we get this system:

$$\begin{cases} A + B = 0 \\ A(1 + \sqrt{3}) + B(1 - \sqrt{3}) = 1 \end{cases}$$

We get that B = -A and $2A\sqrt{3} = 1$. Therefore $A = \frac{1}{2\sqrt{3}}$ and $B = -\frac{1}{2\sqrt{3}}$. The formula to compute a_n is therefore this:

$$a_n = \frac{1}{2\sqrt{3}} \left(1 + \sqrt{3}\right)^n - \frac{1}{2\sqrt{3}} \left(1 - \sqrt{3}\right)^n.$$

By plugging into this formula numbers from [0; 10], we can compute the first members of the sequence:

(C) a_n can be obtained from $a_0 = 2$, if we multiply n times by 3 (i.e. by 3^n), and also by $1 \cdot 2 \cdot \ldots \cdot n = n!$. Therefore, we get

$$a_n = 2 \cdot 3^n \cdot n!$$

(D) a_n can be obtained from $a_0 = 5$ by adding some members of an arithmetic progression:

$$a_n = a_0 + (3 + 6 + \dots + (3n)) = 5 + \frac{3n(n+1)}{2}.$$

Question 4

(A) $a_n = A \cdot (-2)^n + B \cdot (-4)^n + C \cdot n \cdot (-4)^4$. (B) $(r+2)(r+4)^2 = (r+2)(r^2+8r+16) = r^3+8r^2+16r+2r^2+16r+32 = r^3+10r^2+32r+32$. From that cubic equation we can get this recurrent relationship:

$$a_n = -10a_{n-1} - 32a_{n-2} - 32a_{n-3}.$$

Initial conditions (values for the first three members, say, a_1 , a_2 , a_3 can be selected in any way).