Discrete Sample Quiz 3

Question 1. *Definition.* A Boolean formula is a *Conjunctive Normal Form (CNF)*, if it is a conjunction of one or more clauses, where a clause is a disjunction of literals. Each literal is either a variable ($u, v, \ldots$) or its negation ($\neg u, \neg v, \ldots$).

Identify, which is a CNF computing the following truth table for Boolean expression $E(a, b, c)$:

a	b	c	$E(a,b,c)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

In the test you will have to pick between several long expressions like this one:

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c) \wedge \ldots$$

Question 2. The following CNF is given:

$$E = (a \vee \neg c) \wedge (b \vee \neg b).$$

Find 2 Boolean expressions equivalent to E:
(A) $a \to \neg b$,
(B) $a \to c$,
(C) $\neg c \to \neg a$,
(D) $(a \wedge c) \to b$,
(E) $c \to a$,
(F) $a \to (c \to b)$,
(G) $\neg a \to \neg c$.

Question 3. We have six statements about Python programs $p \in \mathcal{P}$ that convert inputs $i \in \mathbb{Z}^+$ into results $r \in \mathbb{Z}^+$. A Python program may either loop indefinitely or halt (i.e. it eventually stops).

1. Some programs return the correct result for all possible inputs and they never loop indefinitely.
2. For any program one can find another program such that it returns the same result for the same inputs as the first one (and also loops indefinitely, if the first program does the same).
3. There is a program that only loops indefinitely for at most finitely many inputs (or maybe none at all), but for all other inputs it produces the correct result.
4. There is at least one Python program that always halts, and for sufficiently large inputs it produces the correct result, but it may err for some small-size inputs.
5. For a program to produce a correct result for some input i it is strictly necessary to halt.
6. A Python program always produces exactly one result for the given input provided that it halts.

We use these 3 predicates:
$A(p_1, i_2, r_3)$ is true iff Python program $p_1 \in \mathcal{P}$ receives input $i_2 \in \mathbb{Z}^+$ and outputs result $r_3 \in \mathbb{Z}^+$.
$H(p_1, i_2)$ is true iff program $p_1 \in \mathcal{P}$ receives input i_2 and halts (i.e. does not loop indefinitely).
$C(i_1, r_2)$ is true iff for input i_1 the correct result is r_2.

Please sort these answers to the five English sentences above.
Write your answer as a comma-separated list: For example, `F,E,D,C,B,A` tells that the 1st statement is **(F)**, the 2nd one is **(E)**, ..., the last one is **(A)**.

(A) $\forall p \in \mathcal{P}\ \forall i \in \mathbb{Z}^+\ \forall r \in \mathbb{Z}^+,$
$(A(p,i,r) \wedge C(i,r) \ \to\ H(p,i)).$

(B) $\forall p_1 \in \mathcal{P}\ \exists p_2 \in \mathcal{P}\ \forall i \in \mathbb{Z}^+\ \exists r \in \mathbb{Z}^+,$
$((\neg H(p_1,i) \wedge \neg H(p_2,i)) \vee (A(p_1,i,r) \leftrightarrow A(p_2,i,r)))$

(C) $\exists p \in \mathcal{P}\ \exists N \in \mathbb{Z}^+\ \forall i \in \mathbb{Z}^+\ \forall r \in \mathbb{Z}^+,$
$((i \leq N \wedge \neg H(p,i)) \vee (A(p,i,r) \wedge C(i,r))).$

(D) $\forall p \in \mathcal{P}\ \forall i \in \mathbb{Z}^+\ \forall r_1 \in \mathbb{Z}^+\ \forall r_2 \in \mathbb{Z}^+,$
$(H(p,i) \wedge A(p,i,r_1) \wedge A(p,i,r_2) \ \to\ r_1 = r_2).$

(E) $\exists p \in \mathcal{P}\ \exists N \in \mathbb{Z}^+\ \forall i \in \mathbb{Z}^+,$
$(H(p,i)\ \wedge\ (A(p,i,r) \wedge i > N \ \to\ C(i,r))).$

(F) $\exists p \in \mathcal{P}\ \forall i \in \mathbb{Z}^+\ \forall r \in \mathbb{Z}^+,$
$(H(p,i)\ \wedge\ (A(p,i,r) \to C(i,r)).$

Question 4. There is a set of 4 students $S = \{s_1, s_2, s_3, s_4\}$ and a set of 2 chairs $C = \{c_1, c_2\}$. Find, how many such functions $f : S \to C$ exist, how many of them are injective, surjective and bijective.

Fill in your answer:

All functions $S \to C$	...
Injective functions $S \to C$	...
Surjective functions $S \to C$	...
Bijective functions $S \to C$	...

Question 5. There is a predicate $S(x, y, z)$ defined for triplets of positive integers, $S : \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \to \{\mathtt{T}, \mathtt{F}\}$. $S(x, y, z)$ is true iff $x \cdot y = z$.
Express these statements about positive integers using only $S(x, y, z)$, Boolean operations and quantifiers.

(a) $x/y = z$,

(b) $x = 1$,

(c) $x = y$,

(d) x is divisible by y (i.e. $y \mid x$).

(e) x has odd number of positive divisors.

(f) x is not a prime.

Discussion on the Coq Lab

Lemma 1: For all propositions a, $\neg\neg a \rightarrow a$.
Proof:

- Assume that $\neg\neg a$ is true.
- We sort two cases by "classic" axiom (Excluded Middle): either a or $\neg a$ must be true.
- If a is true, we are happy.
- Otherwise $\neg a$ is true (along with $\neg\neg a$ obtained before). This is a contradiction.
- Therefore a must be true in all cases (it is either trivial, or a contradiction).

Lemma 2: For all propositions a and b, $\neg(a \rightarrow b) \rightarrow a$.
Proof:

- Assume that $\neg(a \rightarrow b)$ is true.
- We have to prove that a is true. By Lemma 1, we will prove instead that $\neg nega$ is true, then it will also imply a.
- We will assume that $\neg a$ is true, and attempt to get a contradiction (this means that $\neg\neg a$ must be true).
- Let's prove now that $a \rightarrow b$ is true - this would be an immediate contradiction with $\neg(a \rightarrow b)$.
- To prove $a \rightarrow b$, assume that a is true and let's prove b. But earlier we assumed that $\neg a$.
- a and $\neg a$ cannot be simultaneously true. This is a contradiction.

Peirce Lemma: For all propositions a and b, $((a \rightarrow b) \rightarrow a) \rightarrow a$.

Hint. Just use Lemma 1 and 2 for this. And also the "classic" axiom: Sort 2 cases when $(a \rightarrow b)$ or $\neg(a \rightarrow b)$ are true.

Lemma 4: For all propositions a and b, $(\neg b \rightarrow \neg a) \rightarrow (a \rightarrow b)$.
This is the opposite direction from a well-known contrapositive ($\neg b \rightarrow a$ and $a \rightarrow b$ mean the same thing.)

Hint. Use "classic" axiom (Excluded middle) on b.

Lemma 5: For all propositions a, b, c, d, e,

$$((((a \rightarrow b) \rightarrow (\neg c \rightarrow \neg d)) \rightarrow c) \rightarrow e) \rightarrow \neg a \rightarrow (d- > e).$$

Hint. Indeed, assume that $(((a \rightarrow b) \rightarrow \neg c \rightarrow \neg d) \rightarrow c) \rightarrow e$; also assume $\neg a$ and d. Then you can prove $(((a \rightarrow b) \rightarrow \neg c \rightarrow \neg d) \rightarrow c)$ which is similar to what you need.

After all this, you can do Sample20 from the Coq lab.