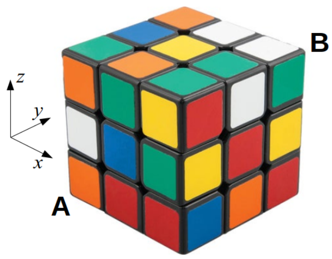
Discrete Sample Quiz 7

Question 1 (Traversing Rubik's Cube). A tiny ant wants to travel between two opposite vertices of a $3 \times 3 \times 3$ cube. It can crawl on the surface of the cube (and even inside it); but any path should consist from exactly 9 unit-length steps, where each step is parallel to one of the coordinate axes (x, y or z). Find the number of paths that the ant could take as it crawls from the vertex A to the vertex B.



Write your answer as an integer number equal to the number of paths from A to B satisfying the condition.

Question 2 (Permutations with Repetition).
(A) Find the number of ways (denoted by N) one can arrange the letters in the 6-letter word `OLIVIA`. Both letters "I" are not distinguishable (if they switch places, it is still the same arrangement).
(B) Assume that somebody writes out all the permutations of the word `OLIVIA` in the *lexicographic ordering* (ordered as in a dictionary: sorted by the 1st letter; if the 1st letter is the same, then by the 2nd letter, etc.). For example, the first permutation: w_1 = `AIILOV` and the last one is w_N = `VOLIIA`. Find the permutations $w_{60}, w_{61}, w_{62}, w_{63}$.
Write your answer as a comma-separated list (first the total number of permutations, then the 60th,61st,62nd,63rd permutation).
N,$w_{60}, w_{61}, w_{62}, w_{63}$.

Question 3 (Raising to a high power). Joe wrote the decimal notation of the following number:

$$X = \left(10^{10} + 1\right)^{10}.$$

After that, Joe erased the last 50 digits of the number X, and got the number Y. Finally, Joe erased all the digits of Y, except the last ten digits, and got a sequence of digits Z.
(A) Write all the digits of Z (if it has leading zeroes, write them as well).
(B) Express Z as C_n^k ("n choose k"), where $k \leq n$.
Write your answer like this:
`dddddddddd,choose(n,k)`.

Question 4 (Combinations with Repetition). You have unlimited number of jellybeans – they can have any of these four colors: red, orange, green, yellow. The jellybeans of each color are identical. Solve the following: express your answers as a binomial coefficients "n choose k", where $k \leq n$.
(A) How many collections of 20 jellybeans can you make? (The order of jellybeans in the collection does not matter. Those jellybean collections that have the same number of each color are considered identical.)
(B) How many collections of 20 jellybeans can you make, if there has to be at least one jellybean of each color.
Write your answer in a form:
`choose(n1,k1),choose(n2,k2)`
Replace `n1,k1,n2,k2` with appropriate integers.

Question 5 (Ordering your combinations). Somebody has written out all the combinations, how to choose $k = 4$ months out of a set of $n = 12$ months:
{`JAN`, `FEB`, `MAR`, `APR`, `MAY`, `JUN`, `JUL`, `AUG`, `SEP`, `OCT`, `NOV`, `DEC`}.

All these four-month combinations are written in a sorted (so that a month appearing earlier in a year is always written first), and furthermore – all these combinations are arranged in increasing order. Months are ordered in this way: `JAN` < `FEB` < ... < `DEC`.
The very first combination of months is `JAN`, `FEB`, `MAR`, `APR`, the last one is `SEP`, `OCT`, `NOV`, `DEC`.
Write a comma-separated list of the four-months that is in the 100th place in this list.

Question 6: Pennies and jars. Assume that you have 50 pennies and three jars (these jars are initially labeled A, B, and C).
(A) In how many ways can you put the pennies in the jars, assuming that the pennies and the jars are distinguishable?
(B) In how many ways can you put the pennies in the jars, assuming that the pennies are identical, but jars are distinguishable?
(C) Assume that we remove the labels (A,B,C) from the jars; and the jars become indistinguishable. In how many ways can you put the pennies in the jars, assuming that both the pennies and the jars are indistinguishable?

Question 7 (Necklace with colored beads). Assume that we have a circular necklace with 2012 equally spaced beads. Exactly 7 of the beads are red, the remaining ones are white. How many necklaces there are? (Necklaces that is obtainable from each other by rotating the circle by some angle $\frac{2\pi}{2012}k$ ($k = 1, \ldots, 2011$) are considered identical.)
Note. A more general question was asked in CGMO (China Girls Mathematical Olympiad), see Problem 8, https://bit.ly/2TcfHH5.