Quiz 11 (Graphs)

Question 1. Let $G = (V, E)$ be a graph, where V is the set of all positive divisors of 144 (including 1 and 144 itself). Two different vertices d_1, d_2 are connected by an edge iff one of the numbers divides another ($d_1 \mid d_2$ or $d_2 \mid d_1$). Find the number of vertices $|V|$ and the number of edges $|E|$ in this graph.
Write two comma-separated integers.

Question 2. How long is the longest simple circuit in W_{20}? (A simple circuit is a circular path that may visit vertices multiple times, but does not contain any edge more than once.)
Write a positive integer.

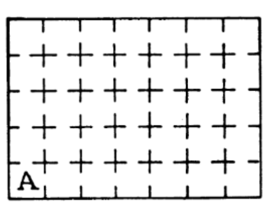
Question 3. Let G be a planar connected graph with 60 vertices, each vertex has degree 3. How many regions are there in G?
Write a positive integer.

Question 4. This is an adjacency matrix for some graph:

$$M_G = \left(\begin{array}{ccccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right).$$

It is known that G is a planar graph. Find the number of vertices $|V|$, number of edges $|E|$ and the number of regions $|R|$ for this graph.
Write 3 comma-separated integers.

Question 5 (Dudeney2016, Prob.434), "536 Puzzles".



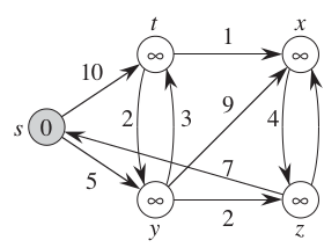
A prisoner currently is in the cell "A". He has to visit each prison cell no more than once and return back to the cell "A". What is the largest number of prison cells that can be visited in this way?
Write a positive integer.
Note. You may also want to prove to yourself that the number is the largest possible.

Question 6 There is a bipartite graph $G = (V, E)$ with exactly $|V| = 17$ vertices. (A graph is *bipartite*, if the set of vertices V can be split into two parts X, Y so that all edges are between a vertex in X and a vertex in Y.) Find the largest possible number of vertices in such a graph.
Write a positive integer.

Question 7 Verify, if these statements are true. A simple undirected graph is called a *cubic* graph, if every vertex has degree 3.
(A) There exists a cubic graph with 7 vertices.
(B) There exists a cubic graph with 6 vertices that is not isomorphic to $K_{3,3}$.
(C) There exists a cubic graph with 8 edges.
Write a sequence of 3 comma-separated letters (e.g. `T,T,T` or `F,F,F`).

Question 8. Verify, if these statements are true:
(A) There exists a simple directed graph with indegrees $0, 1, 2, 4, 5$ and outdegrees $0, 3, 3, 3, 3$. (A graph is *simple*, if it is not a *multigraph* – there is no more than one edge (u, v) for any vertices u, v.)
(B) There exists a connected undirected simple planar graph with 5 regions and 8 vertices, each vertex has degree 3.
(C) There exists a connected undirected simple planar graph with 8 regions and 6 vertices, each region is surrounded with 3 edges.
Write a sequence of 3 comma-separated letters (e.g. `T,T,T` or `F,F,F`).

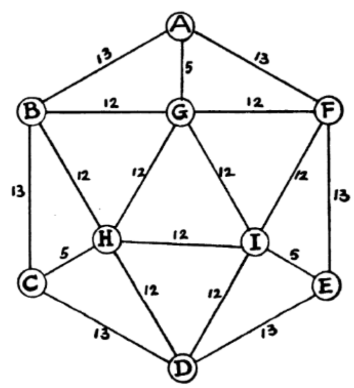
Question 9. Use Dijkstras Algorithm to find the shortest paths from the source vertex s to all other vertices t, x, y, z. The length of a path is obtained by adding the weights of the directed edges.



Write 4 comma-separated numbers – the shortest paths to the vertices t, x, y, z respectively.
Note. Dijkstra's algorithm (Rosen2019, p.747) initializes the set of vertices S that we know the shortest paths to (initially it only contains the source vertex $S = \{s\}$; the distance from s to itself is 0; initialize the distances to all the other vertices to ∞). At every step consider all the edges that go from the set S to $\overline{S}$, i.e. to the vertices where we still do not know the shortest paths. Update all the shortest paths (if crossing from the set S to $\overline{S}$ finds a shorter path than ∞ or the currently known minimum length, then decrease the estimate for this vertex). Finally, add the minimum

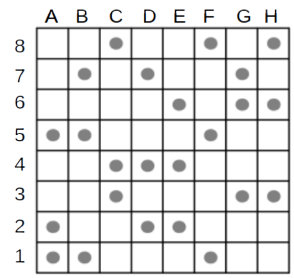
vertex from $\overline{S}$ to S. Repeat the steps until all vertices are added to S and all the shortest path estimates have reached their smallest values.

Question 10 (Dudeney2016, Prob.423), "536 Puzzles". A man starting from the town A, has to inspect all the roads shown from town to town. Their respective lengths, 13, 12, and 5 miles are all shown. What is the shortest possible route he can adopt, ending his journey wherever he likes?



Write an integer – the length of the shortest route.
Note. This graph obviously has no Euler path (since there are more than 2 vertices with odd degrees). The problem is to find a path that is likely **not** simple (uses the same edge several times), but that includes every edge shown and the total of weights is minimal.

Question 11
Somebody placed 24 chess rooks on a 8×8 chessboard as shown in the picture (each horizontal and each vertical has exactly 3 rooks).



We imagine that this chess-board defines a bipartite graph between the set of all verticals $X = \{A, B, C, D, E, F, G, H\}$ and the set of all horizontals $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Any rook defines an edge between these two sets. For example, the rook $C8$ defines an edge $(C, 8)$.

Find a subset of verticals $V \subseteq X$ such that $|V| = 3$, but the neighbor set has size $|N(V)| = 5$.

Write 3 comma-separated letters in your answer (the vertices from V). It is sufficient to write just one possible answer, if there are many.
Note 1. For example, the answer `F, G, H` does not work, since the set of vertices $\{\mathtt{F}, \mathtt{G}, \mathtt{H}\} \subseteq X$ is neighboring with a set of six vertices $\{1, 3, 5, 6, 7, 8\} \subseteq Y$, i.e. the rooks on these three verticals attack six horizontals, but not five.

Note 2. For the condition of the Hall's marriage theorem we need the inequality $|V| \leq |N(V)|$ for **every** $V \subseteq X$. You could prove to yourself that it is always satisfied (also for all the other placements of 24 rooks where each horizontal and each vertical has 3 rooks).
Note 3. Interpret for yourself what does a "perfect matching" between the sets X and Y mean in this subject-area with a chessboard and rooks.