Homework 3

Discrete Structures
RBS

Submit a PDF to the "Homework 3" folder by 2020-03-16.

Problem 1 (Rosen2019, #33, p.444) – *After 6.4.*
Prove that if n is a positive integer, then $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$.

Problem 2 (Rosen2019, #48, p.456) – *After 6.5.*
A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

Problem 3 (Rosen2019, #16, p.464) – *After Ch.6.*
Show that in any set of $n+1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

Problem 4 (Rosen2019, #33, p.465) – *After Ch.6.*
How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of `01`.

Problem 5 (Miller2014, Exercise1.18 `https://bit.ly/2TfZErQ`) *The Theory and Applications of Benford's Law. Steven J. Miller (editor).*
Compute the values of this function $f(x) = \left|x^2 \cdot \tan x\right|$ for all integers $x \in \{1, \ldots, 100000\}$. Record the very first digit that appears in every value $f(x)$.
(A) What is the ratio of the digit 1 among these 10^5 digits (empirical probability)?
(B) What is the theoretical ratio of the first digit 1 predicted by the Benford's law?
Note. Benford's Law is routinely checked by people who falsify the results of elections or otherwise fabricate large amounts of data. Generating digits with the uniform random distribution (where each digit has the same chance to appear) would create data sets that look highly artificial when statistically examined.

Problem 6 (Rosen2019, #23, p.503) – *After 7.3.*
Suppose that E_1 and E_2 are the events that an incoming mail message contains the words w_1 and w_2, respectively. Assuming that E_1 and E_2 are independent events and that $(E_1 \mid S)$ and $(E_2 \mid S)$ are independent events, where S is the event that an incoming message is spam, and that we have no prior knowledge regarding whether or not the message is spam, show that

$$p(S \mid E_1 \cap E_2) =$$

$$= \frac{p(E_1 \mid S) \cdot p(E_2 \mid S)}{P(E_1 \mid S) \cdot P(E_2 \mid S) + P(E_1 \mid \overline{S}) \cdot P(E_2 \mid \overline{S})}.$$

Problem 7 (Rosen2019, #39, p.519) – *After 7.4.*
Suppose that the number of aluminum cans recycled in a day at a recycling center is a random variable with an expected value of 50000 and a variance of 10000.
(A) Use Markov's inequality (Exercise 37) to find an upper bound on the probability that the center will recycle more than 55000 cans on a particular day.
(B) Use Chebyshev's inequality to provide a lower bound on the probability that the center will recycle 40000 to 60000 cans on a certain day.

Problem 8 (Rosen2019, #15, p.522) – *After Ch.7.*
Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than n is not divisible by either p or q?

Problem 9 (Rosen2019, #22, p.523) – *After Ch.7.*
Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.
(A) Find the probability that a particular ball lands in a specified bin.
(B) What is the expected number of balls that land in a particular bin.
(C) What is the expected number of balls tossed until a particular bin contains a ball?
(D) What is the expected number of balls tossed until all bins contain a ball?
Hint: Let X_i denote the number of tosses required to have a ball land in the ith bin once $i-1$ bins contain a ball. Find $E(X_i)$ and use the linearity of expectations.

Problem 10 (Rosen2019, #30, p.524) – *After Ch.7.*
Use Chebyshev's inequality to show that the probability that more than 10 people get the correct hat back when a hatcheck person returns hats at random does not exceed $1/100$ no matter how many people check their hats.
Hint. See Example 6, (Rosen2019, p.507) about the random hat assigning experiment and Exercise 43, (Rosen2019, p.520) about the fixed elements in a random permutation.