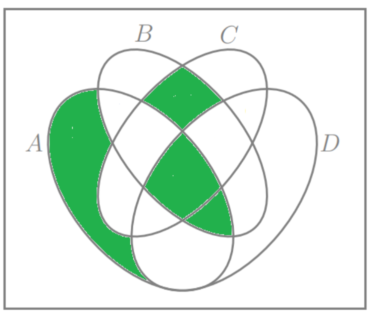
HOMEWORK 03, DUE BY 2022-02-04

Question 1:

(A) Let A, B, C, D be arbitrary sets. Prove or disprove the following set identity using membership tables or known set identities (Rosen2019, p.136).

$$(A \cap B \cap C \cap \overline{D}) \cup (\overline{A} \cap C) \cup (\overline{B} \cap C) \cup (C \cap D) = C.$$

(B) The image below shows a Venn diagram for arbitrary sets A, B, C, D. A region in this diagram is painted green. Write a set expression for this region using A, B, C, D and also set union, set intersection, and set complement operations.



(C) Consider the universe $\mathbb{R}$ of all real numbers. Find the values for the infinite unions and intersections – write the sets X, Y, Z without using union or intersection symbols.

$$X = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}; \frac{3n+1}{2n} \right], \quad Y = \bigcap_{n=0}^{\infty} \left(-\frac{1}{n^2}; \frac{3n+1}{2n} \right), \quad Z = \bigcap_{n=1}^{\infty} \left[\log_2 n; +\infty \right).$$

Question 2: Let (a_n) denote a sequence of natural numbers: $a_0, a_1, \ldots$ (it is infinite and starts with the term a_0). Write the expressions with predicates and quantifiers to formalize the following properties of some sequence (a_i). The formula should be true for all the sequences that match the description and false for all other sequences.

(A) Every value $x \in \mathbb{N}$ appears in (a_i) exactly twice.

(B) If some value $x \in \mathbb{N}$ appears in (a_i) at all, then it must appear infinitely often.

(C) The sequence (a_i) coincides with some arithmetic progression *almost everywhere*. In other words, there is an arithmetic progression such that the terms of a_i equal the respective terms of that arithmetic progression (with a possible exception of a finite number of terms).

Note: All the quantifiers should specify their domain; the formulas can use all 4 arithmetic operations, equalty and inequality predicates ($a = b$, $a < b$, $a \leq b$), Boolean connectors and sequence terms a_i.

Question 3: Let $\mathcal{B}$ be the set of all infinite sequences of bits (digits `0` and `1`). All sequences $(b_i) \in \mathcal{B}$ have their terms enumerated starting from index zero: $b_0, b_1, b_2, \ldots$.

(A) Let $\mathcal{B}_1$ be the subset of $\mathcal{B}$ – it contains only those sequences that satisfy $b_{2k} = 0$ (equal to `0`-bit in all even positions). Show that this set has the same cardinality as the set of all real numbers: $|\mathcal{B}_1| = |\mathbb{R}|$

(B) Let $\mathcal{B}_2$ be the subset of $\mathcal{B}$ containing only those sequences that have no more than finitely many bits equal to `1` (or even none bits `1` at all). Show that this set is infinite and countable: $|\mathcal{B}_2| = |\mathbb{N}|$

Question 4: Let $S = \{1, \ldots, 18\}$ be the set of all positive integers up to 18. Define a relation $R \in S \times S$:

$$R = \left\{(a, b) \in S \times S \ \mid\ b - a \in \{-6, 0, 13\}\right\}.$$

Let R^* be the transitive closure of R. Prove or disprove the following properties of R^*:

(A) Is R^* a symmetric relation?

(B) Is R^* an antisymmetric relation?

(C) Is R^* a partial order or a total order relation?

(D) Does there exist an element $a \in S$ such that $(a, b) \in R^*$ for every $b \in S$?

Question 5: Let L denote a linear equation $Ax + By + C = 0$, where A, B, C are real numbers, but x, y denote unknown variables. We require that $|A| + |B| > 0$ – these coefficients should not be 0 at the same time. Let $\mathcal{L}$ denote the set of all such equations with various constants A, B, C.

Define a relation $R \subseteq \mathcal{L} \times \mathcal{L}$: Two equations L_1 ($A_1x + B_1y + C_1 = 0$) and L_2 ($A_2x + B_2y + C_2 = 0$) are in relation R iff the system of the two equations

$$\begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2x + B_2y + C_2 = 0 \end{cases}$$

does not have a single solution – either it has infinitely many solutions or none at all. (If you interpret $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ as line equations in the coordinate plane, then L_1 and L_2 are in the relation R iff these two lines $L_1, L_2 \in \mathcal{L}$ are either identical or parallel.)

(A) Prove that the relation R is an equivalence relation.

(B) Define a subset $S \subseteq \mathcal{L}$ containing exactly one representative from each equivalence class defined by this relation. Describe how to find that representative for any equation $L \in \mathcal{L}$. It is sufficient to describe the way how you can bring every line equation $Ax + By + C = 0$ to a single "standard form" - so that every two equivalent line equations would have the same standard form. What kind of "standard form" you choose is up to you.

Question 6: Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ Denote by $f^{(n)}$ the n-fold composition of f with itself: $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

In the examples below we will use the n-fold composition for two real-valued functions f and g.

(A) Let $f(x) = \frac{1}{2} \cdot \left(x + \frac{2}{x}\right)$. Define the following sequence:

$$a_n = \begin{cases} 1, & \text{if } n = 0, \\ f^{(n)}(1), & \text{if } n > 0. \end{cases}$$

(B) Let $g(x) = 3.847 \cdot x(1 - x)$. Define the following sequence:

$$b_n = \begin{cases} 0.5, & \text{if } n = 0, \\ g^{(n)}(0.5), & \text{if } n > 0. \end{cases}$$

For each of the sequences describe their behavior as $n \to \infty$ (their *asymptotic behavior*) – is there a limit for the sequence itself or are there limits for any subsequence(s) of it. You may need computer simulation to find out this behavior.

Your answer for both (A) and (B) should describe the asymptotic behavior in English and also its formalization using predicates and quantifiers – the quantifiers can be over either domain $\mathbb{N}$ or $\mathbb{R}$. If necessary, you can introduce constants to denote the limits in your predicate formulas.