

Discrete Quiz 8

Question 1 Two chess queens are placed on two different places in a 4×4 chess-board. Assume that all the $\binom{16}{2}$ possibilities how they are placed have equal probabilities. Find a probability that one queen *attacks* the other. (Two queens attack each other, if they are located on the same horizontal, vertical or diagonal). Write your answer as a rational fraction: P/Q

Question 2 Assume that you are generating 10-bit sequences (a string of 0's and 1's). All the 2^{10} sequences have equal probabilities. Find the probability of an event that the 10-bit sequence does NOT contain two consecutive 0's anywhere. Write your answer as a rational fraction: P/Q

Question 3 Two players have 4 cubic dices. Instead of the usual numbers, their faces have the following numbers on their faces:

Dice A: 4, 4, 4, 4, 0, 0;

Dice B: 3, 3, 3, 3, 3, 3;

Dice C: 6, 6, 2, 2, 2, 2;

Dice D: 5, 5, 5, 1, 1, 1.

In a single round Players Alice and Bob randomly select two of the dices (with equal probabilities they can select any of the six pairs - (A, B) , or (A, C) , or (A, D) , or (B, C) , or (B, D) , or (C, D)). There can be three outcomes:

Outcome 1: They have selected "opposite dices" - pairs (A, C) or (B, D) . In this case the payoff is zero (nobody pays anything to the other).

Outcome 2: The dices win in the "clockwise manner" ($A > B$ or $B > C$ or $C > D$ or $D > A$) - then Alice wins 1 euro.

Outcome 3: The dices win in the "counter-clockwise manner" ($B > A$ or $C > B$ or $D > C$ or $A > D$) - then Bob wins 1 euro.

(Note. The expression $A > B$ means that the number that rolled out on the the dice A was larger than the number on dice B ; but $B > A$ denotes the opposite event.)

Find the expected value - how much money Alice is expected to win in a single round of such a game.

Write your answer as a rational fraction: P/Q

For example, if the expected win for Alice is 0.10 EUR, then write 1/10. If Alice is expected on average to lose 0.10 EUR per one round of this game, then write -1/10.

Question 4 For every year we count the number of Friday's that fall on the 13th date of some month (such as Friday, March 13, 2020). Denote this count by X – it is your random variable. Find the expected value and the variance of X . Round them to the nearest thousandth. Write your answer as two comma-separated numbers:

D.DDD, D.DDD.

Question 5 What is the probability that a randomly chosen positive integer between 1 and 600 is not divisible by either 6 or 10?

Write your answer as a rational fraction: P/Q

Question 6 A chip factory *Intel* adds one toy animal to every bag of chips. There are three sorts of animals - Alligators, Bears or Cats (each one appears with probability $p = 1/3$). Find the expected number of the chip bags one needs to purchase to collect all three animals. Write your answer as a rational fraction: P/Q

Question 7 You create a random bit string of length five (all 32 bit strings are equally probable). Consider these events:

E_1 : the bit string chosen begins with 1;

E_3 : the bit string chosen has exactly three 1s.

(A) Find $p(E_1 \mid E_3)$.

(B) Find $p(E_3 \mid E_1)$.

Write your answer as a comma-separated rational fractions P1/Q1, P2/Q2

Answers

Question 1. Answer: 19/30

To make the counting easier, let us assume that both queens are distinguishable (we can have Q_1 and Q_2 as white and black queen). Then the ways to place them are $16 \cdot 15$ – which is twice the number of combinations $\binom{16}{2} = \frac{15 \cdot 16}{1 \cdot 2}$.

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Q x x x   x Q x x   x x x .
x x . .   x x x .   x Q x x
x . x .   . x . x   x x x .
x . . x   . x . .   . x . x

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Queen Q_1 can be placed in a corner of the 4×4 square (4 chances out of 16), on an edge (8 chances out of 16) or near the center (4 chances out of 16) – see the pictures above. In the first two cases queen Q_1 attacks 9 places (out of 15). In the last case it attacks 11 places (out of 15). The ultimate probability that Q_1 attacks Q_2 is

$$\frac{4}{16} \cdot \frac{9}{15} + \frac{8}{16} \cdot \frac{9}{15} + \frac{4}{16} \cdot \frac{11}{15} = \frac{152}{16 \cdot 15} = \frac{19}{30}.$$

We could also count the mutual attack positions in the original problem (when Q_1 and Q_2 are indistinguishable). But we would get the same result, since attacking positions are symmetric (if Q_1 attacks Q_2 , then Q_2 attacks Q_1).

Question 2. Answer: 9/64

We can count all the sequences that do not contain two consecutive 0's: <https://bit.ly/2TPwMaj>. If we denote by $f_0(n)$ the count of all n -bit sequences that do not contain two consecutive 0s, we can prove that

$$f_0(0) = 1, f_0(1) = 2, f_0(2) = 3, f_0(3) = 5, \dots$$

For arbitrary n we get $f_0(n) = F_{n+2}$, where F_n (0, 1, 1, 2, 3, ...) is the Fibonacci sequence.

The 12th member of the Fibonacci sequence is $F_{12} = 144$. Therefore the proportion of 10-bit sequences equals $\frac{144}{1024} = \frac{9}{64}$.

Question 3. Answer: 2/9

There is $1/3$ probability to pick (A, C) or (B, D) (payoff is 0 in this case).

Regarding the other four pairs, the probabilities are the following:

- A wins B with probability $2/3$.
- B wins C with probability $2/3$.
- C wins D with probability $\frac{1}{3} \cdot \frac{1}{1} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$: **Either** we have $C = 6$ (probability $1/3$); and it wins with probability 1. **Or** we have $C = 2$ (probability $2/3$) and it wins with probability $1/2$ (when-ever $D = 1$).

- D wins A with probability $\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$: **Either** we have $D = 5$ (probability $1/2$); and it wins with probability 1. **Or** we have $D = 1$ (probability $1/2$) and it wins with probability $1/3$ (when-ever $A = 0$).

For each pair (A, B) , (B, C) , (C, D) , or (D, A) the expected payoff for Alice is

$$\frac{2}{3}(+1 \text{ EUR}) + \frac{1}{3}(-1 \text{ EUR}) = \frac{1}{3} \text{ EUR}.$$

The expected probability for Alice to win in a single round can be obtained as a sum, where we multiply probabilities for each pair $((A, C), (B, D), (A, B), (B, C), (C, D), \text{ or } (D, A))$ with their respective payoffs (either 0 or $1/3$ euro):

$$\frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} = \frac{2}{9}.$$

Question 4. Answer: 1.714, 0.490

Assume that each year (and therefore each month) can start on each weekday with exactly the same probability (that is $1/7$). Every month has date 13, so there is a $1/7$ probability that this date will happen on Friday.

(A) If X is the number of months having 13th date on Friday, then $E(X)$ must be $12/7 = 1.714$. It is not because the weekdays that start months in a single year are independent (which they are not!), but rather because every 28-year cycle should have the same number of each type of month (January, February, etc.) starting on each weekday: exactly 4 Januaries starting on Monday, exactly 4 Januaries starting on Tuesday, etc. ($4 + 4 + \dots + 4 = 28$). Only those Januaries that start on Sundays will have 13th date on Friday.

(B) Let us create a table reflecting the number of Fridays on 13th depending on whether the year is the leap year and the weekday it starts.

Year	Non-leap	Leap
Sun	2 (Jan, Oct)	3 (Jan, Apr, Jul)
Mon	2 (Apr, Jul)	2 (Sep, Dec)
Tue	2 (Sep, Dec)	1 (Jun)
Wed	1 (Jun)	2 (Mar, Nov)
Thu	3 (Feb, Mar, Nov)	2 (Feb, Aug)
Fri	1 (Aug)	1 (May)
Sat	1 (May)	1 (Oct)

The probability to get non-leap year starting on a certain weekday is $\frac{3}{4} \cdot \frac{1}{7} = \frac{3}{28}$. The probability to get leap year starting on a certain weekday is $\frac{1}{28}$.

We find the variance $V(X)$ using its definition:

$$\begin{aligned}
 V(X) &= \sum (x_i - E(X))^2 p(x_i) = \\
 &= \frac{3}{28} \left(\left(3 - \frac{12}{7} \right)^2 + 3 \left(2 - \frac{12}{7} \right)^2 + 3 \left(1 - \frac{12}{7} \right)^2 \right) +
 \end{aligned}$$

$$+ \frac{1}{28} \left(\left(3 - \frac{12}{7} \right)^2 + 3 \left(2 - \frac{12}{7} \right)^2 + 3 \left(1 - \frac{12}{7} \right)^2 \right) =$$

$$= \frac{1}{7} \cdot \frac{9^2}{7^2} + \frac{3}{7} \cdot \frac{2^2}{7^2} + \frac{3}{7} \cdot \frac{(-5)^2}{7^2} = \frac{168}{7^3} = \frac{24}{49} \approx 0.490.$$

Note. In fact, the assumption that a year starts with each weekday with a probability $1/7$ is (slightly) false. <https://bit.ly/2vo0YQt> explains that each 400 year cycle in Gregorian calendar repeats the same weekdays. Namely, the calendar for year 1600 is identical to the calendar of year 2000; year 1620 starts on the same weekday as year 2020, and so on.

To verify this, notice that exactly 97 of all 400 years are leap years: (**either** years divisible by 4, but not with 100, **or** years divisible by 400. During this 400 year cycle the number of days:

$$303 \cdot 365 + 97 \cdot 366 \equiv 303 \cdot 1 + 97 \cdot 2 \equiv 497 \equiv 0 \pmod{7}.$$

Since this number is divisible by 7, we should count the proportions of the weekdays within one 400 year cycle.

Question 5. Answer: 23/30

Let U be the (universe) set of all numbers between 1 and 600. Define three more sets:

$$A = \{i \in U \mid i \text{ is divisible by } 2\},$$

$$B = \{i \in U \mid i \text{ is divisible by } 3\},$$

$$C = \{i \in U \mid i \text{ is divisible by } 5\},$$

Numbers $x \in \overline{A \cap B} = \overline{A} \cup \overline{B}$ are not divisible by 6. Numbers $x \in \overline{A \cap C} = \overline{A} \cup \overline{C}$ are not divisible by 10. Numbers in the intersection are not divisible either by 6 or by 10:

$$(\overline{A \cup B}) \cap (\overline{A \cup C}) = \overline{A} \cup (\overline{B \cap C}) = \overline{A} \cup \overline{B \cup C}.$$

Inclusion-exclusion principle tells that the number of elements in this union:

$$|\overline{A \cup B \cup C}| = |\overline{A}| + |\overline{B \cup C}| - |\overline{A \cap B \cup C}|.$$

We can compute:

- $|\overline{A}| = (1/2) \cdot 600 = 300$... not divisible by 2.
- $|\overline{B \cup C}| = 600 - |B \cup C| = 600 - (|B| + |C| - |B \cap C|) = 600 - (200 + 120 - 40) = 320$... not divisible either by 3 or 5.
- $|\overline{A \cap B \cup C}| = 160$.

Let us return to the original question: counting the elements in $\overline{A \cup B \cup C}$:

$$|\overline{A \cup B \cup C}| = 300 + 320 - 160 = 460.$$

All these numbers are not divisible either by 10 or by 6, so their proportion is $\frac{460}{600} = \frac{23}{30}$.

Question 6. Answer: 11/2

When you buy the first bag of chips, you necessarily get a toy animal you did not have before. Waiting time for this is always 1.

Assume that you already have one toy; then it might take $x_1 = 1, x_2 = 2, x_3 = 3$, etc. more bags to find a different toy animal. The respective probabilities of these events are $p_1 = \frac{2}{3}, p_2 = \frac{1}{3} \cdot \frac{2}{3}, p_3 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$, etc. These probabilities make a *geometric distribution*. The expected waiting time is the sum $x_1 p_1 + x_2 p_2 + x_3 p_3 + \dots$:

$$S = 1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} + 3 \cdot \frac{1^2}{3^2} \cdot \frac{2}{3} + \dots$$

To find the value of S , we multiply it by 3:

$$3S = 3 \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 \cdot \frac{1^2}{3^2} \cdot \frac{2}{3} \dots$$

If we subtract 2nd from the 1st and bring the factor $2/3$ to the front:

$$3S - S = \frac{2}{3} \left(4 + \left(3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{3} \right) + \left(4 \cdot \frac{1^2}{3^2} - 3 \cdot \frac{1}{3^2} \right) + \dots \right).$$

$$2S = \frac{2}{3} \left(4 + \frac{1}{3} + \frac{1}{3^2} + \dots \right)$$

We get that $3S = 4\frac{1}{2}$ and $S = \frac{3}{2}$. I.e. we expect to get the second toy after 1.5 bags of chips.

Once we have 2 toys, the final toy can also be found by summation and infinite geometrical progression. The expected waiting time is 3 (bags of chips).

The total waiting time for all 3 toys is $\frac{3}{3} + \frac{3}{2} + \frac{3}{1} = \frac{11}{2}$.

Question 7. Answer: $3/5, 3/8$

The conditional probability of E_1 given E_3 is defined like this:

$$p(E_1 \mid E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)}.$$

The conditional probability of E_3 given E_1 is defined similarly:

$$p(E_3 \mid E_1) = \frac{p(E_1 \cap E_3)}{p(E_1)}.$$

- Out of 32 bit sequences there are 6 sequences that are in $E_1 \cap E_3$: They start with 1 and also contain exactly three 1's. Indeed, the first bit is 1, and there are $\binom{4}{2}$ ways to select the remaining two bits that equal 1.
- Out of 32 bit sequences there are 16 sequences that are in E_1 . They start with 1.
- Out of 32 bit sequences there are 10 sequences that are in E_3 . There are $\binom{5}{3}$ ways to select three bits that equal 1.

Compute both conditional probabilities:

$$p(E_1 \mid E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)} = \frac{6/32}{10/32} = \frac{3}{5}.$$

$$p(E_3 \mid E_1) = \frac{p(E_1 \cap E_3)}{p(E_1)} = \frac{6/32}{16/32} = \frac{3}{8}.$$