Alternative Homework 2: Error Correction Codes

Note. This is a parody of MIT OCW content. See https://ocw.mit.edu/terms/. The original assignments and related materials can be retrieved from https://bit.ly/3dabHyG and https://bit.ly/36xvx4e.

Question 1. Alice designed a $(5, 2, 3)$ code for 2-bit chunks of data ($k = 2$) that uses 5-bit code words ($n = 5$) and permits single-error correction because the minimum Hamming distance is 3 ($d = 3$). The first two bits of each code word directly transmit the data bits being coded; the other three are added for error protection. Unfortunately, her dog chewed up her notebook and destroyed part of the codebook (the part shown with question marks below). You are asked to reconstruct a suitable code.

Input		Codebook				
00	→	0	0	?	?	?
01	→	0	1	?	?	?
10	→	1	0	?	?	?
11	→	1	1	0	0	?

Table 1. *The Codebook used by Alice.*

(A) Find just one way to rebuild the codebook – showing just one way how to code each of the four input strings.

(B) Of the 32 possible bit strings the decoder might encounter, how many are legal codes?

(C) Some of these 32 bit strings have Hamming distance exactly = 1 from a legal code and may therefore be corrected to the nearest legal value under the assumption they were produced by a single error. How many?

(D) Other 5-bit codes have a Hamming distance greater than one from any legal code, and therefore can only be produced by multiple errors. How many?

(E) Your boss thinks it costs more to transmit a 1 than a 0. Is it possible to reduce the number of 1s used in your codebook? (Only the question marks can change their bit values; everything else in the codebook should stay as it is.)

Question 2. Let

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(A) Find a matrix G of a largest possible size and full column rank such that $G \cdot H$ is a matrix consisting of all 0s, where all operations are carried out modulo 2.

(B) What can you say about the minimum distance of the code generated by G, i.e. what are the possible Hamming distances between $\mathbf{x}_1^T \cdot G$ and $\mathbf{x}_2^T \cdot G$, where $\mathbf{x}_1^T, \mathbf{x}_2^T \in \{0, 1\}^4$ are any 4-bit sequences represented as row vectors.

Note. We call a matrix G of a full column rank, if all columns there are linearly independent: namely, for any non-empty subset of columns in G, their total cannot be a vector of all 0s (modulo 2).

Question 3. You are designing a product that uses a Rectangular Code of the sort discussed in Chapter 4 of the notes (see Page 48 in https://bit.ly/2M5ptGR) to assure the correctness of a critical byte of information that is being sent over a noisy channel. In your design every chunk of 9 payload bits is protected by a rectangular code. The design uses seven extra error-protection bits to protect the payload of eight data bits ($D0, \ldots, D8$), and can correct single errors and detect double errors.

D0	D1	D2	PR0
D3	D4	D5	PR1
D6	D7	D8	PR2
PC0	PC1	PC2	P0

Table 2. *Your Error Detection Code.*

One of your colleagues, Ben, has reviewed your design and suggests that you change it so that you do not transmit the parity bits $PR0$ and $PC0$, just the four bits associated with the other rows and columns ($PR1$, $PR2$, and $PC1$, $PC2$), and the total parity bit ($P0$). He says that will be equally effective and more efficient.

(A) Is Ben's efficiency higher? What is his code rate? What is yours?

(B) Can all single errors be corrected with Ben's design? Either briefly explain why, or provide a counter example.

(C) Can any double errors be corrected with Ben's design? Either briefly explain why not, or provide an example.

(D) Can all double errors be detected and identified as double errors with Ben's design? Either briefly explain why, or provide a counter example.