# MIDTERM, FRIDAY, 2022-02-25

*You must justify all your answers to recieve full credit.*

**Question 1:** Consider the following Boolean expression:

$$E(A,B,C) \equiv (A \rightarrow B) \wedge (B \rightarrow C)$$

**(A)** Find the negation $\neg E(A,B,C)$ – in your answer all three Boolean variables $(A,B,C)$ and any of the operator(s) $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus)$ can appear, but all the negations should be only applied to variables (rather than to expressions in parentheses).

**(B)** Assume that Boolean variable is assigned $B = \texttt{True}$. Express $E(A, \texttt{True}, C)$ and simplify this expression. After simplification it only contains the remaining variables, but no references to the Boolean constants `True` or `False`.

**Question 2:** Describe the given sets using the set-builder notation (in the following form $\{x \in \ldots \mid \ldots\}$), where one can specify the "universe", which encloses all the elements $x$ in this set, and some predicate – a property which must be satisfied by elements in this universe $x$ in order to belong to this set.

**(A)** $S_1 = \{-100, -98, -96, -94, \ldots, 94, 96, 98, 100\}$

**(B)** $S_2 = \{1, 81, 121, 361, 441, \ldots, 1681\}$ (full squares up to 2000 ending with digit "1").

**Question 3:**

**(A)** Find a rational number $a \in \mathbb{Q}$ for which the equation $x + \dfrac{1}{x} = a$ has a rational root $x$. Express this root as $x = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^{+}$.

**(B)** Find a rational number $b \in \mathbb{Q}$ for which the equation $x + \dfrac{1}{x} = b$ has an irational root $x$. Prove that the root is irrational. (You can use the fact that the square root of an integer which is not a full square is an irrational number.)

**Question 4:** Find a closed interval of real numbers $[a,b] \subseteq \mathbb{R}$, which satisfies the following properties:

- $10 \leq a < b \leq 15$.
- Function $f(x) = \sin x$ is a bijection from $[a;b]$ to the interval $[-1;1]$.

Also prove that for your values $a$ and $b$ the function $f(x) = \sin x$ is bijective mapping from $[a;b]$ to $[-1;1]$ using the definitions of surjective and injective functions.

**Question 5:** Define a binary relation on the set of plane points with integer coordinates: $(x,y) \in \mathbb{Z}^2$.

$$R = \Big\{ \big((x_1,y_1),(x_2,y_2)\big) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ \mid\ \big(2 \mid (x_1 - x_2) \wedge 3 \mid (y_1 - y_2)\big) \vee \big(3 \mid (x_1 - x_2) \wedge 2 \mid (y_1 - y_2)\big) \Big\}$$

In other words, either the difference of $x$-coordinates of both points in this relation is divisible by 2 and the difference of $y$-coordinates is divisible by 3, or the other way round (the difference of $x$-coordinates is divisible by 3 and the difference of $y$-coordinates is divisible by 2)

**(A)** Is the relation $R$ reflexive?

**(B)** Is the relation $R$ symmetric?

**(C)** Is the relation $R$ transitive?

**(D)** Is the relation $R$ equivalence?

**Question 6:** Consider the following binary relation $R \subseteq \{1,2,3,4,5\} \times \{1,2,3,4,5\}$ which is defined as a set of pairs $(a,b)$:

$$R = \{(1,1),(1,2),(2,1),(2,5),(3,4),(4,3)\}.$$

**(A)** Create the matrix $M_R$ for this binary relation.

**(B)** Draw the matrix $M_{R^t}$ for the transitive closure of this binary relation $R^t$.

**Question 7:**

**(A)** Run Euclidean algorithm to find $\gcd(1000, 711)$.

**(B)** Find an integer solution $x, y \in \mathbb{Z}$ for the following Bézout's identity:

$$1000x + 711y = 0.$$

**(C)** Find an integer number that is divisible by 711 and its decimal notation ends with these three digits: `001`.

**Question 8:**

**(A)** A sequence of natural numbers $a_0, a_1, a_2, \ldots$ is defined by this recurrence:

$$\begin{cases} a_0 = 1, \\ a_{k+1} = (10 \cdot a_k) \bmod 2624, \text{ for every } k \in \mathbb{N}. \end{cases}$$

Compute the first 12 members of this sequence.

**(B)** Consider the definition of an eventually periodic sequence:

$$\exists M \in \mathbb{N}\, \exists T \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(T > 0 \land (n \geq M \rightarrow a_n = a_{n+T})\big)$$

Show that the given sequence $a_n$ matches the definition of an eventually periodic sequence. Find the smallest natural numbers $M$ and also $T > 0$ in this definition that would make it true for the sequence $a_n$.

**(C)** Consider the fraction $\dfrac{1}{2624}$. It is a rational number – therefore it can be expressed as a repeating/periodic decimal. Identify the *repetend* (also called the *period*) – the sequence of digits that repeats itself infinitely. Identify the *prefix* (the digits preceding the repetend).

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**Note:** For example, the fraction $1/44 = 0.02272727272727272\ldots = 0.02\overline{27}$ has two-digit repetend $\overline{27}$ and the prefix: 02.

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