Uzdevumi 2020.g. 17. janvāra nodarbībai

Uzdevums 1.1: Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

Uzdevums 1.2: Fix an integer $k \geq 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leq m' < m$ that is coprime to m. The first player who cannot move anymore loses.
An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is n, and bad otherwise.
Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

Uzdevums 1.3: Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Uzdevums 1.4: Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.