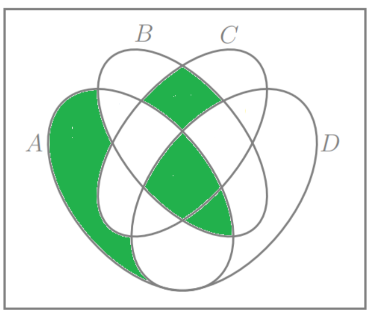
HOMEWORK 03, DUE BY 2022-02-04

Question 1:

(A) Let A, B, C, D be arbitrary sets. Prove or disprove the following set identity using membership tables or known set identities (Rosen2019, p.136).

$$(A \cap B \cap C \cap \overline{D}) \cup (\overline{A} \cap C) \cup (\overline{B} \cap C) \cup (C \cap D) = C.$$

(B) The image below shows a Venn diagram for arbitrary sets A, B, C, D. A region in this diagram is painted green. Write a set expression for this region using A, B, C, D and also set union, set intersection, and set complement operations.



(C) Consider the universe $\mathbb{R}$ of all real numbers. Find the values for the infinite unions and intersections – write the sets X, Y, Z without using union or intersection symbols.

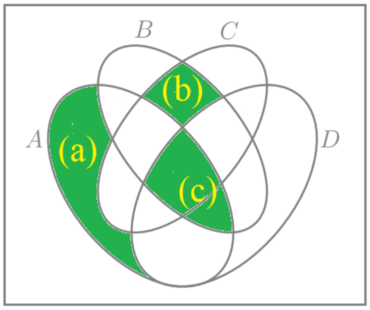
$$X = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}; \frac{3n+1}{2n}\right], \quad Y = \bigcap_{n=0}^{\infty} \left(-\frac{1}{n^2}; \frac{3n+1}{2n}\right), \quad Z = \bigcap_{n=1}^{\infty} \left[\log_2 n; +\infty\right).$$

Answer:

(A) We apply set identities – after every line we specify the name of identity that leads to the next line:

$$\begin{aligned}
& (A \cap B \cap C \cap \overline{D}) \cup (\overline{A} \cap C) \cup (\overline{B} \cap C) \cup (C \cap D) = && \text{Distributive law} \\
= \; & (A \cap B \cap C \cap \overline{D}) \cup \Big((\overline{A} \cup \overline{B} \cup D) \cap C \Big) = && \text{Commutative law} \\
= \; & \Big((A \cap B \cap \overline{D}) \cap C \Big) \cup \Big((\overline{A} \cup \overline{B} \cup D) \cap C \Big) = && \text{Distributive law} \\
= \; & \Big((A \cap B \cap \overline{D}) \cup (\overline{A} \cup \overline{B} \cup D) \Big) \cap C = && \text{De Morgan's Law} \\
= \; & \Big((A \cap B \cap \overline{D}) \cup \overline{(A \cup B \cup \overline{D})} \Big) \cap C = && \text{Complement law} \\
= \; & U \cap C = && \text{Identity law} \\
= \; & C. &&
\end{aligned}$$

(B) Consider three shaded areas marked (a), (b), (c) in the image:



a. Elements from A, but not in any other set: $A \cap \overline{B} \cap \overline{C} \cap \overline{D}$,

b. Elements from $B \cap C$ (but outside the other sets): $\overline{A} \cap B \cap C \cap \overline{D}$,

c. Elements from the intersection of A, B, D (in fact, two adjacent regions): $A \cap B \cap D$.

The ultimate answer is the union of all these areas:

$$(A \cap \overline{B} \cap \overline{C} \cap \overline{D}) \cup (\overline{A} \cap B \cap C \cap \overline{D}) \cup (A \cap B \cap D).$$

Note: The method of building this set expression closely resembles that of a Disjunctive Normal Form (DNF), which was similarly looking for different ways how to satisfy the Boolean function (and then connecting them all together with disjunctions).

(C) Write out the first terms for union X:

$$X = \left[1; 2\right] \cup \left[\frac{1}{2}; \frac{7}{4}\right] \cup \left[\frac{1}{3}; \frac{5}{3}\right] \cup \ldots$$

Both sequences $1/n$ and $(3n+1)/2n = 1.5 + \frac{1}{2n}$ are decreasing and approaching their limits (0 and 1.5 respectively). The union of all these intervals does not include 0, since it does not belong to any interval, but it is possible to get $1/n$ as close to 0 as we want.

On the other hand, the very first interval $[1; 2]$ contains the right endpoint, and all the subsequent right endpoints are smaller than that. We conclude that the infinite union contains all points from 0 (exclusive) to 2 (inclusive). We can write this set as a half-open interval:

$$X = \left(0; 2\right].$$

Note: The set Y is not well defined. Indeed, for $n = 0$ the interval cannot be computed, there is division by 0. Everyone who noticed that should get a full pass for this subitem. Unfortunately, there was a typo.

Just for fun let us consider a similar set Y', which is well defined:

$$Y' = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n^2}; \frac{3n+1}{2n}\right),$$

Write out the first terms for this set intersection:

$$Y' = \left(-\frac{1}{1}; \frac{2}{1}\right) \cap \left(-\frac{1}{4}; \frac{7}{4}\right) \cap \left(-\frac{1}{9}; \frac{5}{3}\right) \cap \ldots$$

The sequence of left endpoints grows and converges to 0, the sequence of right endpoints converges to 1.5. This sequence of intervals is shrinking, so the infinite intersection will contain the points 0 and 1.5 themselves (as they belong to **all** the intersecting intervals). We conclude that the infinite intersection of open intervals is a closed interval:

$$Y' = [0; \frac{3}{2}]$$

Just as before, write out the first few terms of the intersection Z:

$$Z = [0; +\infty) \cap [1; +\infty) \cap [\log_2 3; +\infty) \cap [2; +\infty) \cap \ldots$$

We observe that the right endpoint is always the same $+\infty$, but the left endpoint slowly grows. It turns out that its limit is also ∞. For this reason there is no real number that belongs to all these intersecting intervals. We get the following:

$$Z = \varnothing.$$

Question 2: Let (a_n) denote a sequence of natural numbers: $a_0, a_1, \ldots$ (it is infinite and starts with the term a_0). Write the expressions with predicates and quantifiers to formalize the following properties of some sequence (a_i). The formula should be true for all the sequences that match the description and false for all other sequences.

(A) Every value $x \in \mathbb{N}$ appears in (a_i) exactly twice.

(B) If some value $x \in \mathbb{N}$ appears in (a_i) at all, then it must appear infinitely often.

(C) The sequence (a_i) coincides with some arithmetic progression *almost everywhere*. In other words, there is an arithmetic progression such that the terms of a_i equal the respective terms of that arithmetic progression (with a possible exception of a finite number of terms).

Note: All the quantifiers should specify their domain; the formulas can use all 4 arithmetic operations, equalty and inequality predicates ($a = b$, $a < b$, $a \leq b$), Boolean connectors and sequence terms a_i.

Answer:

(A) Write the conjunction of these two statements: (1) Every natural number n appears (at least) twice – in two different sequence locations a_i and a_j. (2) If this number n appears in some location a_k then its index k must equal either i or j.

$$\forall n \in \mathbb{N}\, \exists i \in \mathbb{N}\, \exists j \in \mathbb{N} \Big(a_i = n \wedge a_j = n \wedge i \neq j \bigwedge \forall k \in \mathbb{N} \big(a_k = n \rightarrow (k = i \vee k = j) \big) \Big).$$

(B) Write the following statement: for all natural n, i, if $a_i = n$, then for any (arbitrarily large) M one can find k such that $k > M$ and $a_k = n$. (This means, that there does not exist the largest index M such that $a_M = n$ – for each index one can find an even larger index k such that a_k also equals n.)

$$\forall n \in \mathbb{N}\, \forall i \in \mathbb{N}\, \Big(a_i = n \rightarrow \forall M \in \mathbb{N}\, \exists k \in \mathbb{N}\, \big(k > M \wedge a_k = n \big) \Big).$$

(C) Write the following statement: There exists an integer b_0 (perhaps even negative!) – the initial member of the arithmetic series. There also exists a natural number d – the difference of the arithmetic series. Then there is some (sufficiently large) natural number M such that for every $k > M$ we must have a_k equal to the k-th member of the arithmetic series.

$$\exists b_0 \in \mathbb{Z}\, \exists d \in \mathbb{N}\, \exists M \in \mathbb{N}\, \forall k \in \mathbb{N} \big(k > M \rightarrow a_k = b_0 + k \cdot d \big).$$

Question 3: Let $\mathcal{B}$ be the set of all infinite sequences of bits (digits `0` and `1`). All sequences $(b_i) \in \mathcal{B}$ have their terms enumerated starting from index zero: $b_0, b_1, b_2, \ldots$.

(A) Let $\mathcal{B}_1$ be the subset of $\mathcal{B}$ – it contains only those sequences that satisfy $b_{2k} = 0$ (equal to `0`-bit in all even positions). Show that this set has the same cardinality as the set of all real numbers: $|\mathcal{B}_1| = |\mathbb{R}|$

(B) Let $\mathcal{B}_2$ be the subset of $\mathcal{B}$ containing only those sequences that have no more than finitely many bits equal to `1` (or even none bits `1` at all). Show that this set is infinite and countable: $|\mathcal{B}_2| = |\mathbb{N}|$

Answer:

(A) First prove that $|\mathcal{B}_1| \leq |\mathbb{R}|$ (there exists an injection from $\mathcal{B}_1$ to $\mathbb{R}$). Indeed, every infinite sequence of bits $x = b_0, b_1, b_2, \ldots$ can map to a real number $f(x)$

$$f(x) = 0.(b_0, b_1, b_2, \ldots)_{10} \in [0; 1]$$

It is the a number from the interval $[0; 1]$ with decimal notation having only digits 0 and 1. For example, $f(0, 0, 0, 0, 0, \ldots)$ (the sequence of all zeroes) maps to a real number $x = 0$. But the sequence $0, 1, 0, 1, 0, \ldots$ (all ones) via this function maps to a real number $1/99 = 0.010101\ldots$. There are no collisions, because two different bit sequences would generate real numbers that differ in some position (and we do not need to worry about the digit 9 periods such as $0.99999\ldots$ being equal to $1.00000\ldots$ – rare cases when the same real number has two decimal representations).

Next, we prove that $|\mathbb{R}| \leq |\mathcal{B}_1|$. Namely, every real number $x \in \mathbb{R}$ can map to $g(x) \in \mathcal{B}_1$ via injective function g. Informally speaking, we encode every real number $x \in \mathbb{R}$ to a bit sequence, where every even bit equals to zero. This function needs to be collision-free: $x_1 \neq x_2$ should imply that $g(x_1) \neq g(x_2)$ – different real numbers should map to different bit sequences.

To construct such function g, write the real number as an infinite decimal fraction (using minus sign if necessary). For example, the number $-17\frac{8}{11} \in \mathbb{R}$ has the following decimal expansion:

$$-17\frac{8}{11} = -17.727272727272\ldots$$

We only need twelve symbols to write all real numbers (10 digits, the minus sign and the decimal point). Twelve symbols can be encoded as finite bit sequences (also bit sequences that have the bit 0 in all even places). For example, you can use the following encoding table:

In Decimal Notation	As a Bit Sequence
0	00000000
1	00000001
2	00000100
3	00000101
4	00010000
5	00010001
6	00010100
7	00010101
8	01000000
9	01000001
0	01000100
−	01000101
.	01010000

If you do not want to reinvent binary codes, just use the ASCII code table https://bit.ly/33fu0Tb. Just make sure that you fill in all even positions with 0s. The important thing about the encoding is that every character has fixed-length code (in our case it uses 8 bits) and all the even positions in this code are 0s. Then the number

$-17\frac{8}{11} = -17.727272727272$ from the above example maps to the following sequence:

$$g\left(-17\frac{8}{11}\right) = \texttt{01000101.00000001.00010101.01010000.00010101.00000100.00010101.00000100}\ldots$$

The bit sequence only contains 0s and 1s (dots are added between the 8-bit codewords for better readability).

Once we have both injective functions $f\colon \mathcal{B}_1 \to \mathbb{R}$ and also $g\colon \mathbb{R} \to \mathcal{B}_1$, apply Schröder–Bernstein theorem to obtain that $|\mathcal{B}_1| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |\mathcal{B}_1|$ implies $|\mathcal{B}_1| = |\mathbb{R}|$.

Note: None of the functions $f\colon \mathcal{B}_1 \to \mathbb{R}$ or $g\colon \mathbb{R} \to \mathcal{B}_1$ was a bijection – there were many elements in the codomain that were not mapped to (so these are not surjections). This technical task (to create a bijection out of two injections going in the opposite directions) was handled by Schröder–Bernstein theorem.

(B) To show that $|\mathcal{B}_2| = |\mathbb{N}|$ we can build a bijection directly. For every sequence in $|\mathcal{B}_2|$ define its size to be the smallest index k such that $\forall n \geq k\ (b_n = 0)$. Observe that there are no more than 2^k sequences of size k, since we can arbitrarily choose the first k bits – $b_0, b_1, \ldots, b_{k-1}$.

We now enumerate the elements from $|\mathcal{B}_2|$ by increasing size (and for each size order the sequences lexicographically – if the first difference between two sequences is in some bit at the location i, then the sequence with $b_i = 0$ precedes the sequence with $b_j = 1$.

Let us list the first few sequences in this enumeration (the lexicographically ordered part of each sequence has length of exactly the size, and it is shown in blue):

Enumeration Order#	Size	Bit sequence from $\mathcal{B}_2$
0	0	00000000...
1	1	10000000...
2	2	01000000...
3	2	11000000...
4	3	00100000...
5	3	01100000...
6	3	10100000...
7	3	11100000...

This enumeration not only assigns some sequence from $\mathcal{B}_2$ to every number $\mathbb{N}$, but also does not skip any sequences; each sequence receives a unique number.

Example: For example the sequence 1011100000... has size 5. This means that it is preceded by one sequence of size 0, one sequence of size 1, two sequences of size 2, four sequences of size 3, eight sequences of size 4, and also all those sequences of size 5 lexicographically preceding 10111 – there are ten such sequences.

Therefore, 1011100000... maps to the natural number $1 + 1 + 2 + 4 + 8 + 10 = 26$. Therefore our bijection has value: $f(1011100000\ldots) = 26$

Question 4: Let $S = \{1, \ldots, 18\}$ be the set of all positive integers up to 18. Define a relation $R \in S \times S$:

$$R = \big\{(a, b) \in S \times S \mid b - a \in \{-6, 0, 13\}\big\}.$$

Let R^* be the transitive closure of R. Prove or disprove the following properties of R^*:

(A) Is R^* a symmetric relation?

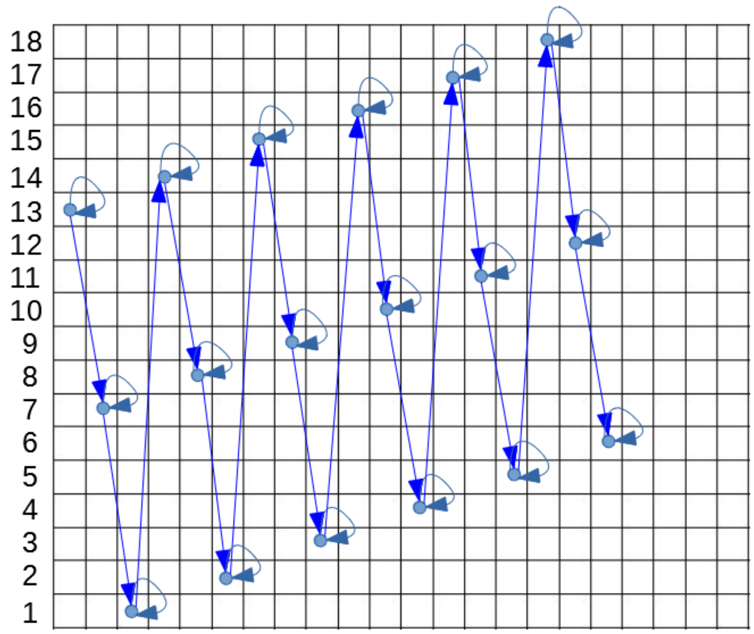
(B) Is R^* an antisymmetric relation?

(C) Is R^* a partial order or a total order relation?

(D) Does there exist an element $a \in S$ such that $(a, b) \in R^*$ for every $b \in S$?

Answer:

We start by constructing the transitive closure relation R^* before answering any questions about it. The original relation R is shown in the image below. The elements in S are represented by levels in this grid and the blue arrows show the relation – how is it possible to travel between the levels.



Since $6 + 13 = 19$, there does not exist any $a \in S$ that can travel both upwards and downwards. For this reason, every element in S has at most one outbound arrow. So the R (if it is represented as a digraph) will contain just one long path which starts in $13 \in S$ and ends in $6 \in S$. All the elements of S are on this path.

If we build a transitive closure for R, then $(a, b) \in R^*$ whenever b is reachable from a. To find out, which elements are in relation R^* with the others, list the elements in the order of blue arrows. Then $(a, b) \in R^*$ iff element b is same as a or to the right of a.

$$13,\ 7,\ 1,\ 14,\ 8,\ 2,\ 15,\ 9,\ 3,\ 16,\ 10,\ 4,\ 17,\ 11,\ 5,\ 18,\ 12,\ 6.$$

(A) The relation R^* is not symmetric, since $(13, 7) \in R^*$, but $(7, 13) \notin R^*$: It is possible to travel by blue arrows from 13 to 7, but not in the opposite direction (there are no arrows that enter 13).

(B) The relation is R^* antisymmetric. In fact, $(a, b) \in R^*$ means that $(b, a) \notin R^*$ (since the blue arrows only flow in one direction – it is not even possible to have $(a, a) \in R^*$). For this reason $(a, b) \in R^* \wedge (b, a) \notin R^*$ always implies $a = b$, since the condition is impossible.

(C) A partial order relationship must be reflexive, antisymmetric and transitive – see the definition in (Rosen2019, p.650). The relation R^* is antisymmetric. It is also transitive (since it is a transitive closure of another relation). It is also reflexive since every element $a \in S$ is in relation R with itself, since $|a - a| = 0 \in \{-6, 0, 13\}$ And so the R^* must also be reflexive, since it preserves all these self-relations.

The relation R^* is also a total order – see the definition in (Rosen2019, p.651). For R^* every two elements are comparable (one must precede another in the path built from the blue arrows). The only case when $(a, b) \in R^*$ and $(b, a) \in R^*$ is when both numbers are equal.

In this total order R^*, number $13 \in S$ is the *smallest* element, but $6 \in S$ is the *largest* element.

Question 5: Let L denote a linear equation $Ax + By + C = 0$, where A, B, C are real numbers, but x, y denote unknown variables. We require that $|A| + |B| > 0$ – these coefficients should not be 0 at the same time. Let $\mathcal{L}$ denote the set of all such equations with various constants A, B, C.

Define a relation $R \subseteq \mathcal{L} \times \mathcal{L}$: Two equations L_1 ($A_1x + B_1y + C_1 = 0$) and L_2 ($A_2x + B_2y + C_2 = 0$) are in relation R iff the system of the two equations

$$\begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2x + B_2y + C_2 = 0 \end{cases}$$

does not have a single solution – either it has infinitely many solutions or none at all. (If you interpret $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ as line equations in the coordinate plane, then L_1 and L_2 are in the relation R iff these two lines $L_1, L_2 \in \mathcal{L}$ are either identical or parallel.)

(A) Prove that the relation R is an equivalence relation.

(B) Define a subset $S \subseteq \mathcal{L}$ containing exactly one representative from each equivalence class defined by this relation. Describe how to find that representative for any equation $L \in \mathcal{L}$. It is sufficient to describe the way how you can bring every line equation $Ax + By + C = 0$ to a single standard form - so that every two equivalent line equations would have the same standard form. What kind of standard form you choose is up to you.

Answer:

(A)

Algebraic solution: The system of two linear equations has infinitely many (or zero) solutions iff it is possible to multiply one equation by some number C and subtract from another one – so that coefficients for variables x and y all cancel out. If we multiply by a constant C the 2nd equation and subtract from the first we get:

$$\begin{cases} A_1 = C \cdot A_2 \\ B_1 = C \cdot B_2 \end{cases}$$

Swap sides in the last equality as $C \cdot B_2 = B_1$ and multiply with the first equality:

$$A_1 \cdot C \cdot B_2 = C \cdot A_2 \cdot B_1.$$

If $C \neq 0$ we get $A_1B_2 = A_2 \cdot B_1$ (and even if $C = 0$, then $A_1 = B_1 = 0$, and we still have $A_1 \cdot B_2 = A_2 \cdot B_1$).

Relation R is *reflexive* for any line $L1 : A_1x + B_1 + C_1 = 0$, since $A_1 \cdot B_1 = A_1 \cdot B_1$).

Relation R is *symmetric*, since $A_1 \cdot B_2 = A_2 \cdot B_1$ equality still holds, if we swap A_1 by A_2 and also B_1 with B_2.

Relation R is *transitive*. Indeed, assume that there are three equations $L_1 : A_1x + B_1y + C_1 = 0$, $L_1 : A_2x + B_2y + C_2 = 0$, and $L_3 : A_3x + B_3y + C_3 = 0$. Assume that $(L_1, L_2) \in R$ and also $(L_2, L_3) \in R$. Rewrite this as two algebraic equalities on the coefficients:

$$\begin{cases} A_1 \cdot B_2 = A_2 \cdot B_1 \\ A_2 \cdot B_3 = A_3 \cdot B_2 \end{cases}$$

Let us show that $(L_1, L_3) \in R$ as required by transitivity – we must verify the equality: $A_1 \cdot B_3 = A_3 \cdot B_1$.

Case1: If $B_2 = 0$, then definitely $A_2 \neq 0$ (both coefficients are not simultaneously 0 since it was given that $|A_2| + |B_2| > 0$. But then both expressions in the above system $A_2 \cdot B_1 = A_2 \cdot B_3 = 0$ (and consequently $B_1 = B_3 = 0$). This proves that $A_1 \cdot B_3 = A_3 \cdot B_1$.

Case2: If $B_2 \neq 0$, then we can multiply both sides of the equality we should prove by this B_2:

$$\begin{array}{l} A_1 \cdot B_3 = A_3 \cdot B_1 \text{ if and only if } A_1 \cdot B_3 \cdot B_2 = A_3 \cdot B_1 \cdot B_2. \\ A_1 \cdot B_3 \cdot B_2 = A_3 \cdot B_1 \cdot B_2 \text{ if and only if } (A_1 \cdot B_2) \cdot B_3 = (A_3 \cdot B_2) \cdot B_1. \\ (A_1 \cdot B_2) \cdot B_3 = (A_3 \cdot B_2) \cdot B_1 \text{ if and only if } (A_2 \cdot B_1) \cdot B_3 = (A_2 \cdot B_3) \cdot B_1 \end{array}$$

The above equalities regroup multiplications (and during the last step also used the known equalities from our system). The very last equality $(A_2 \cdot B_1) \cdot B_3 = (A_2 \cdot B_3) \cdot B_1$ is true (it multiplies the same numbers). Therefore $A_1 \cdot B_3 = A_3 \cdot B_1$ – the original equality is also true. Thus $(L_1, L_3) \in R$.

Geometric solution: Geometrically, $Ax+By+C=0$ is a line equation (including lines that are parallel to x and y axes. The fact that two line equations form a system that does not have a single solution (but has infinitely many solutions or none at all) means that such two lines are either parallel or identical. In other words – they are both drawn at the same angle w.r.t. x axis in this coordinate system.

Reflexivity of R means that each line is identical to itself: $R(L, L)$.

Symmetry means that if L_1 is parallel (or identical) to L_2, then L_2 is also parallel or identical to L_1.

Transitivity means that if one line is parallel to to the second, and the second is parallel to the third, then the first must be parallel to the third (namely, all their angles to the x axis are the same).

(B)

Let us describe how to create the single representative $L \in S$ for the entire equivalence class of all lines pointing in the same direction. Since the parameter C does not affect the relation at all, for the representative $L \in S$ we pick $C = 0$ (i.e. require that the lines in this set S all go through the origin $(0, 0)$.

Even with $C = 0$ we still have many line equations pointing in the same direction with mutually proportional coefficients (such as $x + y = 0$, $3x + 3y = 0$ or $(-17)x + (-17)y = 0$). If the coefficient $A \neq 0$, we divide the equation by it – the line does not change (so it stays equivalent to the original line); and now the coefficient $A = 1$.

There is one exception – lines which are parallel to the x axis (where $A = 0$). In such cases we divide the equation by B – so that the coefficient before y becomes 0. For each line in S, the coefficient C is always zero; and either $A = 1$ or $B = 1$ (if A was 0). Consequently, we can define the set S as follows:

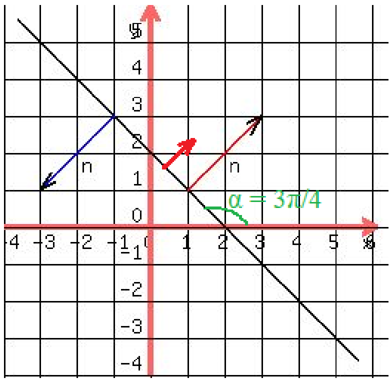
$$S = \left\{ Ax + By + C = 0 \;\mid\; |A| + |B| > 0 \wedge C = 0 \wedge (A = 1 \vee (A = 0 \wedge B = 1)) \right\}.$$

(B alternative) In practice checking if a real number A equals 0 or does not equal 0 might be hard (if A is very close to 0 so that it could be a rounding error).

For this reason it is sometimes popular to rewrite line equations so as to make the expression $A^2 + B^2 = 1$. Consider the angle α to the x axis (two lines are equivalent w.r.t. R if this angle is the same). The possible angles are $\alpha \in [0; \pi)$ – they can be anywhere between 0° and 180° (non-inclusive). Now define the set S^* containing exactly one representative from each equivalence class.

$$S^* = \left\{ \sin\alpha \cdot x - \cos\alpha \cdot y = 0 \;\mid\; \alpha \in [0; \pi) \right\}.$$

Such line equations are called *line equations in the normal form*. The name *normal form* follows from the concept of *normal* (a vector that is perpendicular to the given line $Ax + By + C = 0$ and has coordinates (A, B)). For example, the line $x + y - 2 = 0$ shown in the image has various normals $(2, 2)$, $(-2, -2)$, $(1, 1)$, $(1/\sqrt{2}, 1/\sqrt{2})$ (or any other vector perpendicular to it).



To find the element from S^* that is equivalent to the line $x + y - 2 = 0$ use $\alpha = 135^\circ = \frac{3\pi}{4}$ – it is the angle this line has with the x axis. Write the equation from S^* – a parallel line that goes through $(0;0)$:

$$\sin\frac{3\pi}{4}x - \cos\frac{3\pi}{4} + 0 = 0 \text{ which is same as } \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0.$$

Question 6: Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ Denote by $f^{(n)}$ the n-fold composition of f with itself: $f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

In the examples below we will use the n-fold composition for two real-valued functions f and g.

(A) Let $f(x) = \frac{1}{2} \cdot \left(x + \frac{2}{x}\right)$. Define the following sequence:

$$a_n = \begin{cases} 1, & \text{if } n = 0, \\ f^{(n)}(1), & \text{if } n > 0. \end{cases}$$

(B) Let $g(x) = 3.847 \cdot x(1-x)$. Define the following sequence:

$$b_n = \begin{cases} 0.5, & \text{if } n = 0, \\ g^{(n)}(0.5), & \text{if } n > 0. \end{cases}$$

For each of the sequences describe their behavior as $n \to \infty$ (their *asymptotic behavior*) – is there a limit for the sequence itself or are there limits for any subsequence(s) of it. You may need computer simulation to find out this behavior.

Your answer for both (A) and (B) should describe the asymptotic behavior in English and also its formalization using predicates and quantifiers – the quantifiers can be over either domain $\mathbb{N}$ or $\mathbb{R}$. If necessary, you can introduce constants to denote the limits in your predicate formulas.

Answer:

(A) Applying the same function to get new members of a sequence is named a *recurrent sequence*. Let us prove that the sequence a_n has limit $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

We compute the first few terms of this sequence:

$$\frac{1}{1}, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \frac{886731088897}{627013566048}, \ldots$$

Assuming that this sequence has a limit L, apply the limit to both sides of the equality:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \to \quad L = \frac{1}{2}\left(L + \frac{2}{L}\right) \quad \to \quad 2L = L + \frac{2}{L}.$$

Ultimately we get $L^2 = 2$. So, if the limit of a_n exists, it must be $\sqrt{2}$. (The negative solution $L = -\sqrt{2}$ does not work, since all members $a_n > 0$.)

Let us prove that the sequence indeed approaches this limit.

Let $x_1 = 3$, $x_2 = 17$, $x_3 = 577$, $x_4 = 665857$ and so on – be the numerators for the rational numbers a_n (when represented as irreducible fractions). And similarly, $y_1 = 2$, $y_2 = 12$, $y_3 = 408$, $y_4 = 470832$ and so on – be the denominators of these fractions.

We can prove the following relationship for each $n \geq 1$ (known as as Pells equation). (See more about it https://bit.ly/3rHpuq6).

$$x_n^2 - 2y_n^2 = 1.$$

Indeed, we can check that $3^2 - 2 \cdot 2^2 = 1$, $17^2 - 2 \cdot 12^2 = 289 - 288 = 1$ and so on. Assume that this relation holds for (x_n, y_n); prove that it also holds for (x_{n+1}, y_{n+1}). We have the following relation (this shows how a_{n+1} is computed from a_n):

$$\frac{x_{n+1}}{y_{n+1}} = \frac{1}{2}\left(\frac{x_n}{y_n} + \frac{2y_n}{x_n}\right) = \frac{x_n^2 + 2y_n^2}{2x_n y_n}.$$

Set $x_{n+1} = x_n^2 + 2y_n^2$ and $y_{n+1} = 2x_n y_n$. (We will later see that the fraction x_{n+1}/y_{n+1} is irreducible, so these equalities are indeed true.

Now prove that $x_{n+1}^2 - 2y_{n+1}^2 = 1$. Plug in and do some algebra:

$$(x_n^2 + 2y_n^2)^2 - 2(2x_n y_n)^2 = x_n^4 + 4x_n^2 y_n^2 + 4y_n^4 - 8x_n^2 y_n^2 = (x_n^2 - 2y_n^2)^2 = 1^2 = 1.$$

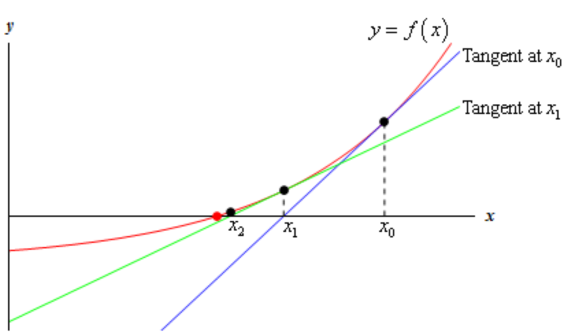
As we see x_{n+1}, y_{n+1} do not have any common factors (otherwise the expression on the right hand side cannot be 1). Moreover, they also satisfy the Pells equation.

Since x_n and y_n are both growing sequences (that follows from the equations how they are computed), we can divide by y_n^2 and find the limit ($n \to \infty$) in the Pells equation:

$$x_n^2 - 2y_n^2 = 1 \quad \to \quad \left(\frac{x_n}{y_n}\right)^2 - 2 = \frac{1}{y_n^2}.$$

We see that $(x_n/y_n)^2 - 2 = 0$ as $n \to \infty$. Therefore (x_n/y_n) in the limit goes to $\sqrt{2}$.

Note: If you do not like the Pells equation, you can also apply Newtons tangent method to solve the equation $f(x) = x^2 - 2 = 0$. This method (and its convergence to $\sqrt{2}$) is shown in https://bit.ly/3HK8btW. Here is the graphical interpretation of the sequence that approximates $\sqrt{2}$. This is one of the popular *numerical methods* to compute square roots on a computer.



Description with Quantifiers: The sequence limit in calculus means that for every (arbitrarily small) $\varepsilon > 0$ there is a natural number M such that all the sequence terms after this number differ from the limit ($\sqrt{2}$ in our case) by less than ε. Express this formally:

$$\forall \varepsilon \in \mathbb{R}\, \exists M \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n > M \rightarrow |a_n - \sqrt{2}| < \varepsilon).$$

(B) The sequence b_n is generated by recursion $b_{n+1} = 3.847 \cdot b_n(1 - b_n)$. That equation $b_{n+1} = r \cdot b_n(1 - b_n)$ is named *Logistic map* (it generates so called *bifurcation diagrams* – pictures where small change of parameter r may lead to a chaotic behavior of the sequence or to the doubling of the number of the values its subsequences can reach in the limit). Read more about them here: https://bit.ly/3HKszuR.

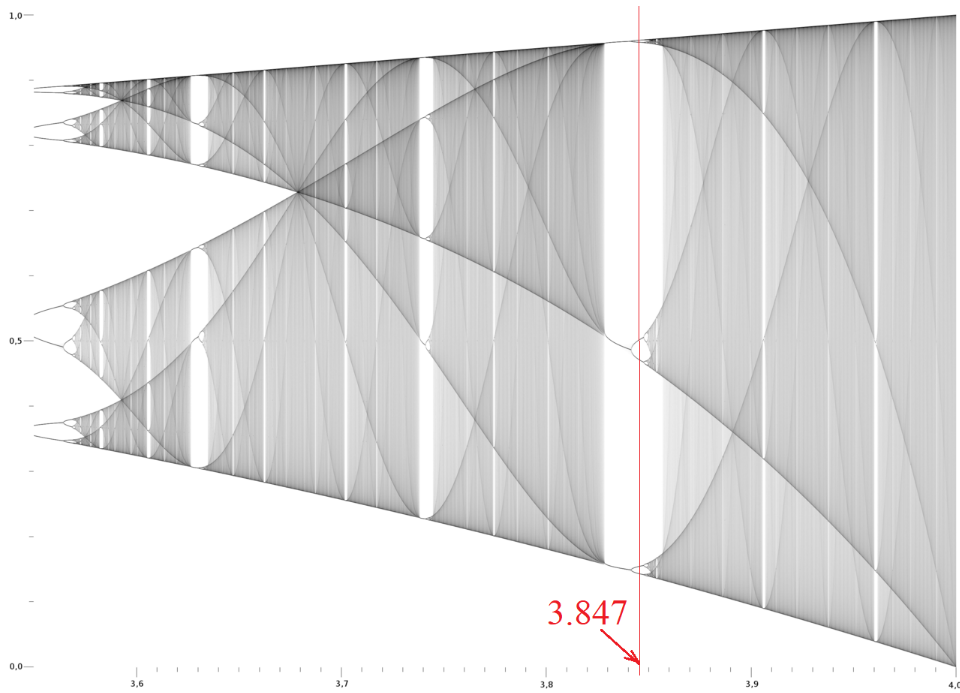


Fig. 1: Attractor (in this case 6 limit values) that occur in the bifurcation equation, if $r = 3.847$.

Simulation of this sequence on a computer (after a few thousand steps) leads to approaching the following 6 values:

$$\begin{aligned}
b_{6k} &\approx 0.504636213252 = L_0\\
b_{6k+1} &\approx 0.961667310761 = L_1\\
b_{6k+2} &\approx 0.141813092689 = L_2\\
b_{6k+3} &\approx 0.468188130392 = L_3\\
b_{6k+4} &\approx 0.957856855050 = L_4\\
b_{6k+5} &\approx 0.155292234788 = L_5
\end{aligned}$$

We can formalize the existence of exactly six limit values (for subsequences of b_n) in the following way: There exist the attractor values $L_0, \ldots, L_5$ such that for each ε and for each $i \in \{0, 1, 2, 3, 4, 5\}$ there is a natural M such that b_{6n+i} differs from L_i by less than ε. Let us formalize this:

$$\exists L_0, L_1, L_2, L_3, L_4, L_5 \in \mathbb{R}\, \forall \varepsilon \in \mathbb{R}\, \exists M \in \mathbb{N}\, \forall n \in \mathbb{N}\, \forall i \in \mathbb{N}\, \left(i \leq 5 \rightarrow |b_{6n+i} - L_i| < \varepsilon\right).$$

Note: An *attractor* for a numeric sequence (or actually any other dynamic system) is a set of states towards which this system approaches – for a wide range of the initial states. In our case the initial value $b_0 = 0.5$ could be changed to any other number from the interval $(0; 1)$ – and the six limit values would be the same.

```python
def f(x):
    return 3.847*x*(1-x)

x = 0.5
for i in range(1, 60020):
    x = f(x)
    if i > 60000:
        print('b[{}] = {}'.format(i, x))
```