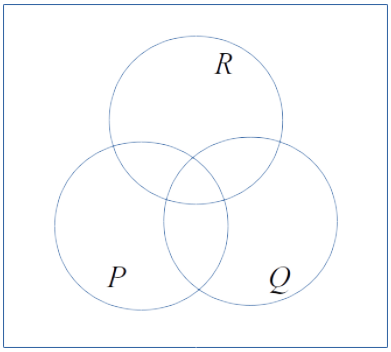
Midterm, 2020-02-17

Question 1 (Boolean Expressions).
Consider Boolean expression:

$$E_0 = (p \to q \to r) \wedge (q \to r \to p) \wedge (r \to p \to q)$$



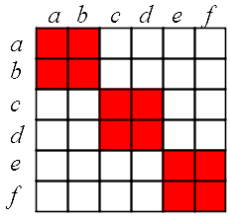
(A) Copy the Venn diagram's circles in your solution and shade those regions in the diagram that make E_0 true (being inside each circle P, Q, R means that the respective variable p, q, r is true; being outside the circle means that the variable is false).

(B) In the truth table of E_0 how many entries are `True`?
(*Note.* Building the truth table is optional. Regardless whether you build one or not, you should justify your answer.)

(C) Rewrite the Boolean expression E_0 into an equivalent one, using only conjunctions ($\wedge$) and negations ($\neg$).

Assume that implication ($\to$) is right-associative and conjunction ($\wedge$) has higher precedence than implication.

Question 2 (Nested Quantifiers).
Verify, if the following predicate/quantifier expressions are true for the given predicate. The predicate P is defined on $A \times A$, where $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}\}$. Predicate $P(\mathtt{a}, \mathtt{b})$ is true iff the square on row `a` and column `a` is shaded (and the predicate $P(\mathtt{a}, \mathtt{b})$ is false, if that square is white).



(A) Does the predicate P satisfy the logic formula:

$$\forall i \in A,\ P(i, i).$$

(B) Does the predicate P satisfy the logic formula:

$$\forall i \in A,\ \forall j \in A,\ P(i, j) \to P(j, i).$$

(C) Does the predicate P satisfy the logic formula:

$$\forall i, j, k \in A,\ P(i, j) \wedge P(j, k) \to P(i, k).$$

(D) Does the predicate P satisfy the logic formula:

$$\forall i, j \in A,\ P(i, j) \vee P(j, i).$$

(E) Does the predicate P satisfy the logic formula:

$$\forall i \in A,\ \exists j \in A,\ P(i, j).$$

Question 3 (Estimate with Big-O Notation).
Define the sequence $S(n)$ as a sum of squares from 1^2 to n^2:

$$S(n) = \sum_{i=1}^{n} i^2.$$

We have $S(1) = 1^2 = 1$, $S(2) = 1^2 + 2^2 = 5$, $S(3) = 1^2 + 2^2 + 3^2 = 14$, and so on.

(A) Is the function $S(n)$ in $O(n^1)$? Is it in $O(n^2)$? Is it in $O(n^3)$? Is it in $O(n^4)$? Explain your reasoning.

(B) Pick any one of the notations from the previous items ($g(n)$ is either $O(n^1)$, or $O(n^2)$, or $O(n^3)$, or $O(n^4)$). Check the definition of Big-O notation: Find the *witness*: the value k and the constant C such that the absolute value of $S(n)$ does not exceed $C \cdot |g(n)|$ for all $n > k$.

Question 4 (Chinese Remainder Theorem).
Consider the following system of three congruences:

$$\begin{cases} x \equiv 1 \pmod 5, \\ x \equiv 2 \pmod 7, \\ x \equiv 3 \pmod 9. \end{cases}$$

(A) Does it have a solution? Will it have solution, even if we replace $1, 2, 3$ with other numbers on the right sides of the equation.

(B) Find an arithmetic progression (what is its first member A, difference B) where all members satisfy the first two congruences from the system.

(C) Find an arithmetic progression (what is its first member C, difference D) where all members satisfy all three congruences in the system.

Note. Arithmetic progression is an infinite sequence where every next member can be obtained by adding the same number (the difference) to the previous one. For example,

$$A,\ A+B,\ A+2B,\ A+3B,\ldots$$

is an arithmetic progression with the first member A and the difference B.

Question 5 (Binary notation).
Somebody has written two binary fractions on the board: α is infinite, β is finite (just 6 digits after the point):

$$\begin{cases} \alpha = 0.(011110)_2 = 0.011110011110011110\ldots_2 \\ \beta = 0.011110_2. \end{cases}$$

(A) Express the number β as a sum of some negative powers of 2; namely, show how to add up some of the numbers

$$\{2^{-1}, 2^{-2}, 2^{-3}, \ldots\}$$

to get β.

(B) Express β as an irreducible fraction `P/Q`; write this in the regular decimal notation.

(C) Write the product $64_{10} \cdot \alpha = 1000000_2 \cdot \alpha$ in the binary notation.

(D) Express α as an irreducible fraction `P/Q` in decimal notation.

Question 6 (Truth-tellers and Liars). Among the people A, B, C one is a truth-teller, the other two are liars. Every person (A, B, and C) has a closed box in front of himself/herself. Exactly one of the boxes has a candy inside. A, B, C know everything about each other and the location of candy.
Someone else (person D) approaches all of them. D knows, who are people A,B, and C (it is written on their name-cards), but D does not know anything about their lying behavior or the location of the candy. D is allowed to ask YES/NO questions to one or more people.

(A) Can D find out who has the candy by asking three questions?

(B) Can D find out who has the candy by asking two questions?

(C) Can D find out who has the candy by asking one question?

Justify your answers (by construction or by showing that it is impossible).

Question 7 (Time Complexity of Truth Tables).
Assume that there is a Boolean expression E with n variables:

$$E = E(a_1, a_2, \ldots, a_n).$$

The expression E contains $2n$ Boolean operators (such as $\neg$, $\wedge$, $\vee$). Variables $a_1, a_2, \ldots, a_n$ can independently take values `True` or `False`.
Consider the following algorithm to find, if E is a tautology by building the truth table. We will either find a false value, or establish that all values were true (in this case E is a tautology).

(1) For each assignment of n truth values to $a_1, \ldots, a_n$:
(2) For each of the $2n$ Boolean operators in E:
(3) Compute the value of that Boolean operator
(4) If E has value `False`:
(5) Return "E is not a tautology."
(6) If E has value `True`:
(7) Continue loop on Line **(1)**.
(8) Return "E is a tautology."

(A) Find the worst-case runtime $T(n)$ for this algorithm as an expression of n. (Assume that evaluating one Boolean operator $\neg$, $\wedge$, $\vee$ takes 1 unit of time.)

(B) Find a function $g(n)$ such that $T(n)$ is in $O(g(n))$.

Question 8 (About Rational and Irrational).
We denote two real numbers by p and q. Prove or disprove statements about the rational and irrational numbers.

(A) If $p + q$ is rational, then either both p, q are rational, or both are irrational.

(B) If pq is rational, then either both p, q are rational, or both are irrational.

(C) If p^2 and q^2 are both rational, then the product $(p + q)(p - q)$ is rational.

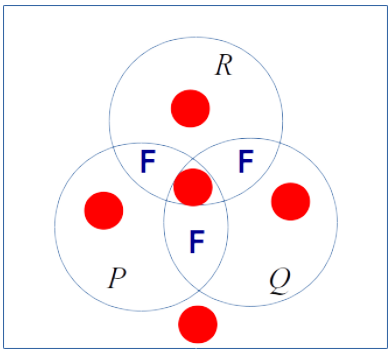
(D) If p^3 and p^5 are both rational, then p is rational.

(E) If pq and $p + q$ are both rational, then p and q are both rational.

Answers

Question 1

(A) The only way to have $p \to q \to r = p \to (q \to r)$ evaluate to `False` is p = `True` and $(q \to r)$ = `False` (i.e. q = `True` and r = `False`).
If we consider also $q \to r \to p$ and $r \to p \to q$, there are two more ways to get the whole expression E_0 false. Each of these ways means that exactly two variables are `True` and the third one is `False`. All the other areas make the expression true. Shaded areas look like this (red circles). There are just 3 regions where the expression E_0 evaluates to false (blue letters "F").



(B) There are exactly $8-3=5$ entries in the truth table which make the expression true (there are only 3 ways to make the expression true, as shown in **(A)**).
(C) Let us just transform one subexpression:

$$\begin{aligned} &p \to (q \to r) = \\ =&\neg p \vee (q \to r) = \\ =&\neg p \vee (\neg q \vee r) = \\ =&\neg p \vee \neg q \vee r = \\ =&\neg(p \wedge q \wedge \neg r). \end{aligned}$$

If we combine all 3 subexpressions (where p, q, r switch their order), we get a longer conjuction for E_0:

$$\neg(p \wedge q \wedge \neg r) \wedge \neg(q \wedge r \wedge \neg p) \wedge \neg(r \wedge p \wedge \neg q).$$

Question 2

(A) Yes, $\forall i \in A,\ P(i,i)$ is true (we say that the 2-argument predicate is *reflexive*): all the squares $P(i,i)$ on the diagonal of the values are shaded.

(B) Yes, $\forall i \in A, \forall j \in A,\ P(i,j) \to P(j,i)$ is true (we say that the 2-argument predicate is *symmetric*): the squares in the table of $P(i,j)$ are symmetric against the diagonal of the values: $P(i,j)$ is shaded iff $P(j,i)$ is shaded.

(C) Yes, $\forall i,j,k \in A,\ P(i,j) \wedge P(j,k) \to P(i,k)$ is true (we say that the 2-argument predicate is *transitive*). Notice that $P(i,j)$ can be true iff $i = j$ or (i,j) is one of the pairs of neighbors: $(\mathtt{a},\mathtt{b})$, or $(\mathtt{c},\mathtt{d})$, or $(\mathtt{e},\mathtt{f})$. $P(j,k)$ is also true; so j,k also belong to the same pair. Therefore i and k also belong to the same pair.

(D) No, $\forall i,j \in A,\ P(i,j) \vee P(j,i)$ is false. For example, neither $P(\mathtt{a},\mathtt{c})$ nor $P(\mathtt{c},\mathtt{a})$ are true.

(E) Yes, $\forall i \in A,\ \exists j \in A,\ P(i,j)$ is true: On each row i there is at least one shaded square.

Question 3 (A) $S(n)$ is in $O(n^3)$ (and therefore it is also in $O(n^4)$). On the other hand, $S(n)$ is not in $O(n^2)$ or $O(n^1)$.
Let us show that $S(n)$ is not in $O(n^2)$. Assume from the contrary that there are numbers k and C such that for each $n > k$:

$$|S(n)| = |1^2 + 2^2 + \ldots + n^2| \le C \cdot |n^2|.$$

Since all numbers there are positive, we drop the absolute values. We pick any even number $n > k$ which also satisfies $n > 8C$. In this case:

$$\begin{aligned} &S(n) = 1^2 + 2^2 + \ldots + n^2 > \\ >&\left(\frac{n}{2}+1\right)^2 + \left(\frac{n}{2}+2\right)^2 + \ldots + n^2 > \\ >&\frac{n}{2} \cdot \left(\frac{n}{2}\right)^2 \ge \frac{n}{8} \cdot n^2 \ge Cn^2. \end{aligned}$$

(In these inequalities, we first drop the first half of the sum; then replace each term with $(n/2)^2$ and we still can prove that the sum is more than Cn^2 for the given constant C.)
Such n can always be found (no matter what k and C are used). Therefore $S(n)$ can never satisfy $|S(n)| \le C \cdot n^2$.
Since n is even smaller than n^2, $S(n)$ is not in $O(n)$ either.
(B) Let us prove that $S(n)$ is in $O(n^3)$. Let us pick the *witness* to check the definition of the Big-O Notation: $k = 1, C = 1$.

$$\begin{aligned} &1^2 + 2^2 + \ldots + n^2 \le \\ \le\ &n^2 + n^2 + \ldots + n^2 = n \cdot n^2 = 1 \cdot n^3. \end{aligned}$$

Question 4 (A) Yes, the system will always have solution (even if you replace the numbers $1, 2, 3$ by any other integers). Chinese Remainder theorem only needs that the modules ($5, 7, 9$ in our case) are mutually prime.

(B) The numbers satisfying $x \equiv 1 \pmod 5$ make this arithmetic progression:

$$1, 6, 11, 16, 21, 26, 31, \ldots$$

Number 16 in this progression is also congruent to 2 (modulo 7). The next number would be $16 + 35 = 51$ (because adding $5 \cdot 7 = 35$ does not change the remainders, when we divide by 5 or by 7.
We conclude that the following arithmetic progression satisfies the top two congruences (remainder 1 modulo 5 and remainder 2 modulo 7):

$$16, 51, 86, 121, 156, 191, \ldots$$

Answer: $A = 16$, $B = 35$.
(C) Observe that the first member of this arithmetic sequence ($A = 16$) gives the remainder 7 when divided by 9. The difference ($B = 35$) is congruent to 8 and also to -1 modulo 9.
We conclude that by adding four differences, the result is

$$16 + 4 \cdot 35 \equiv 16 + 4 \cdot (-1) \equiv 12 \equiv 3 \pmod 9.$$

The first number from the progression (item (B)) that is also congruent to 3 modulo 9 is 156. We can add $5 \cdot 7 \cdot 9 = 315$ to this number, and all the remainders will stay the same:

$$156, 471, 786, 1101, 1416, 1731, 2046, \ldots$$

Answer: $C = 156$, $D = 315$.

Question 5 (A) $\beta = 0.011110_2 = 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5}$.
(B) $\beta = \frac{8+4+2+1}{2^5} = \frac{15}{32}$.
(C) $64\alpha = 1000000_2 \cdot \alpha =$
$= 011110.011110011110011110\ldots_2$ (to multiply by $2^6 = 64$ we shift the point six positions to the right).
(D) Subtract α from 64α: We get 011110_2 (because all the digits after the point are the same – they cancel out). We get the equation:

$$64\alpha - \alpha = 63\alpha = 011110_2 = 30_{10}$$

Therefore $\alpha = \frac{30}{63} = \frac{10}{21}$. We could also get the same answer by finding the sum of an infinite geometric progression:

$$30 \cdot \left(\frac{1}{64} + \frac{1}{64^2} + \frac{1}{64^3} + \ldots \right).$$

Question 7 (A) In order to get the estimate of time complexity of the algorithm, consider just the first three lines, where all the computations take place:

(1) For each assignment of n truth values to $a_1, \ldots, a_n$:
(2) For each of the $2n$ Boolean operators in E:
(3) Compute the value of that Boolean operator
The outer loop on Line 1 repeats 2^n times as there are exactly 2^n ways to assign true/false to n variables. The inner loop on Line 2 repeats $2n$ times (once for every operation you have to evaluate in the expression). Line 3 takes just 1 unit of time (this was given in the exercise). The time complexity $T(n)$ is the product $(2n) \cdot 2^n$; this (worst case) happens whenever the expression is a tautology. If it is not a tautology, this algorithm will terminate earlier (and it will take less time).

(B) This $T(n) = 2n \cdot 2^n$ is in $O(2n \cdot 2^n)$, since any function is in the Big-O of itself (we can take $k = 1$ and $C = 1$). If we want, we can drop the multiplier 2 to simplify it slightly: $T(n)$ is in $O(n \cdot 2^n)$ (in this case $k = 1$ and $C = 2$).
Answer: $T(n)$ is in $O(g(n))$ where $g(n) = n \cdot 2^n$.

Question 8 (A) True: "If $p + q$ is rational, then either both p, q are rational, or both are irrational."
From the contrary, if p is rational and q is irrational, then $(p + q) - p = q$ should be rational, which is a contradiction. Same thing happens, if p is irrational and q is rational.

(B) False: "If pq is rational, then either both p, q are rational, or both are irrational."
We can take $p = 0$ and $q = \sqrt{2}$; then $pq = 0$ is rational, also p is rational, but q is irrational.

(C) True: "If p^2 and q^2 are both rational, then the product $(p + q)(p - q)$ is rational."
Denote the two rational numbers by $\alpha = p^2$ and $\beta = q^2$. Then their difference is also a rational number: $\alpha - \beta = p^2 - q^2 = (p + q)(p - q)$.

(D) True: "If p^3 and p^5 are both rational, then p is rational."
If $p = 0$ then all p, p^3 and p^5 are rational.
If $p \neq 0$, then denote the non-zero rational numbers $\alpha = p^3$ and $\beta = p^5$. Then their ratio $\frac{\beta}{\alpha} = \frac{p^5}{p^3} = p^2$ is also rational. Finally, if we divide $\alpha = p^3$ by the rational p^2, we get that also p is rational (as a fraction of two rational numbers). Shortly: $p = p^1 = p^{2 \cdot 3 - 5} = \frac{\alpha \cdot \alpha}{\beta}$.

(E) False: "If pq and $p+q$ are both rational, then p and q are both rational."
Consider two irrational numbers $p = 1 + \sqrt{2}$ and $q = 1 - \sqrt{2}$. Then $p+q = 2$ and $pq = (1+\sqrt{2})(1-\sqrt{2}) = 1 - 2 = -1$, i.e. the sum and the product are both rational numbers.