

## Discrete Quiz 8

**Question 1** Two chess queens are placed on two different places in a  $4 \times 4$  chess-board. Assume that all the  $\binom{16}{2}$  possibilities how they are placed have equal probabilities. Find a probability that one queen *attacks* the other. (Two queens attack each other, if they are located on the same horizontal, vertical or diagonal). Write your answer as a rational fraction: P/Q

**Question 2** Assume that you are generating 10-bit sequences (a string of 0's and 1's). All the  $2^{10}$  sequences have equal probabilities. Find the probability of an event that the 10-bit sequence does NOT contain two consecutive 0's anywhere. Write your answer as a rational fraction: P/Q

**Question 3** Two players have 4 cubic dices. Instead of the usual numbers, their faces have the following numbers on their faces:

**Dice A:** 4, 4, 4, 4, 0, 0;

**Dice B:** 3, 3, 3, 3, 3, 3;

**Dice C:** 6, 6, 2, 2, 2, 2;

**Dice D:** 5, 5, 5, 1, 1, 1.

In a single round Players Alice and Bob randomly select two of the dices (with equal probabilities they can select any of the six pairs -  $(A, B)$ , or  $(A, C)$ , or  $(A, D)$ , or  $(B, C)$ , or  $(B, D)$ , or  $(C, D)$ ). There can be three outcomes:

**Outcome 1:** They have selected "opposite dices" - pairs  $(A, C)$  or  $(B, D)$ . In this case the payoff is zero (nobody pays anything to the other).

**Outcome 2:** The dices win in the "clockwise manner" ( $A > B$  or  $B > C$  or  $C > D$  or  $D > A$ ) - then Alice wins 1 euro.

**Outcome 3:** The dices win in the "counter-clockwise manner" ( $B > A$  or  $C > B$  or  $D > C$  or  $A > D$ ) - then Bob wins 1 euro.

(Note. The expression  $A > B$  means that the number that rolled out on the the dice  $A$  was larger than the number on dice  $B$ ; but  $B > A$  denotes the opposite event.)

Find the expected value - how much money Alice is expected to win in a single round of such a game.

Write your answer as a rational fraction: P/Q

For example, if the expected win for Alice is 0.10 EUR, then write 1/10. If Alice is expected on average to lose 0.10 EUR per one round of this game, then write -1/10.

**Question 4** For every year we count the number of Friday's that fall on the 13th date of some month (such as Friday, March 13, 2020). Denote this count by  $X$  – it is your random variable. Find the expected value and the variance of  $X$ . Round them to the nearest thousandth. Write your answer as two comma-separated numbers:

D.DDD, D.DDD.

**Question 5** What is the probability that a randomly chosen positive integer between 1 and 600 is not divisible by either 6 or 10?

Write your answer as a rational fraction: P/Q

**Question 6** A chip factory *Intel* adds one toy animal to every bag of chips. There are three sorts of animals - Alligators, Bears or Cats (each one appears with probability  $p = 1/3$ ). Find the expected number of the chip bags one needs to purchase to collect all three animals. Write your answer as a rational fraction: P/Q

**Question 7** You create a random bit string of length five (all 32 bit strings are equally probable). Consider these events:

$E_1$ : the bit string chosen begins with 1;

$E_3$ : the bit string chosen has exactly three 1s.

(A) Find  $p(E_1 | E_3)$ .

(B) Find  $p(E_3 | E_1)$ .

Write your answer as a comma-separated rational fractions P1/Q1, P2/Q2

## Answers

### Question 1. Answer: 19/30

To make the counting easier, let us assume that both queens are distinguishable (we can have  $Q_1$  and  $Q_2$  as white and black queen). Then the ways to place them are  $16 \cdot 15$  – which is twice the number of combinations  $\binom{16}{2} = \frac{15 \cdot 16}{1 \cdot 2}$ .

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Q x x x   x Q x x   x x x .
x x . .   x x x .   x Q x x
x . x .   . x . x   x x x .
x . . x   . x . .   . x . x

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Queen  $Q_1$  can be placed in a corner of the  $4 \times 4$  square (4 chances out of 16), on an edge (8 chances out of 16) or near the center (4 chances out of 16) – see the pictures above. In the first two cases queen  $Q_1$  attacks 9 places (out of 15). In the last case it attacks 11 places (out of 15). The ultimate probability that  $Q_1$  attacks  $Q_2$  is

$$\frac{4}{16} \cdot \frac{9}{15} + \frac{8}{16} \cdot \frac{9}{15} + \frac{4}{16} \cdot \frac{11}{15} = \frac{152}{16 \cdot 15} = \frac{19}{30}.$$

We could also count the mutual attack positions in the original problem (when  $Q_1$  and  $Q_2$  are indistinguishable). But we would get the same result, since attacking positions are symmetric (if  $Q_1$  attacks  $Q_2$ , then  $Q_2$  attacks  $Q_1$ ).

### Question 2. Answer: 9/64

We can count all the sequences that do not contain two consecutive 0's: <https://bit.ly/2TPwMaj>. If we denote by  $f_0(n)$  the count of all  $n$ -bit sequences that do not contain two consecutive 0s, we can prove that

$$f_0(0) = 1, f_0(1) = 2, f_0(2) = 3, f_0(3) = 5, \dots$$

For arbitrary  $n$  we get  $f_0(n) = F_{n+2}$ , where  $F_n$  (0, 1, 1, 2, 3, ...) is the Fibonacci sequence.

The 12th member of the Fibonacci sequence is  $F_{12} = 144$ . Therefore the proportion of 10-bit sequences equals  $\frac{144}{1024} = \frac{9}{64}$ .

### Question 3. Answer: 4/9

There is  $1/3$  probability to pick  $(A, C)$  or  $(B, D)$  (payoff is 0 in this case).

Regarding the other four pairs, the probabilities are the following:

- $A$  wins  $B$  with probability  $2/3$ .
- $B$  wins  $C$  with probability  $2/3$ .
- $C$  wins  $D$  with probability  $\frac{1}{3} \cdot \frac{1}{1} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$ : **Either** we have  $C = 6$  (probability  $1/3$ ); and it wins with probability 1. **Or** we have  $C = 2$  (probability  $2/3$ ) and it wins with probability  $1/2$  (when-ever  $D = 1$ ).

- $D$  wins  $A$  with probability  $\frac{1}{2} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$ : **Either** we have  $D = 5$  (probability  $1/2$ ); and it wins with probability 1. **Or** we have  $D = 1$  (probability  $1/2$ ) and it wins with probability  $1/3$  (when-ever  $A = 0$ ).

The expected probability for Alice to win in a single round can be obtained as a sum, where we multiply probabilities for each pair  $((A, C), (B, D), (A, B), (B, C), (C, D),$  or  $(D, A))$  with their respective payoffs:

$$\frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{2}{3} = \frac{4}{9}.$$

### Question 4. Answer: 1.714, 0.490

Assume that each year (and therefore each month) can start on each weekday with exactly the same probability (that is  $1/7$ ). Every month has date 13, so there is a  $1/7$  probability that this date will happen on Friday.

(A) If  $X$  is the number of months having 13th date on Friday, then  $E(X)$  must be  $12/7 = 1.714$ . It is not because the weekdays that start months in a single year are independent (which they are not!), but rather because every 28-year cycle should have the same number of each type of month (January, February, etc.) starting on each weekday: exactly 4 Januaries starting on Monday, exactly 4 Januaries starting on Tuesday, etc. ( $4 + 4 + \dots + 4 = 28$ ). Only those Januaries that start on Sundays will have 13th date on Friday.

(B) Let us create a table reflecting the number of Fridays on 13th depending on whether the year is the leap year and the weekday it starts.

Year	Non-leap	Leap
Sun	2 (Jan, Oct)	3 (Jan, Apr, Jul)
Mon	2 (Apr, Jul)	2 (Sep, Dec)
Tue	2 (Sep, Dec)	1 (Jun)
Wed	1 (Jun)	2 (Mar, Nov)
Thu	3 (Feb, Mar, Nov)	2 (Feb, Aug)
Fri	1 (Aug)	1 (May)
Sat	1 (May)	1 (Oct)

The probability to get non-leap year starting on a certain weekday is  $\frac{3}{4} \cdot \frac{1}{7} = \frac{3}{28}$ . The probability to get leap year starting on a certain weekday is  $\frac{1}{28}$ .

We find the variance  $V(X)$  using its definition:

$$\begin{aligned}
 V(X) &= \sum (x_i - E(X))^2 p(x_i) = \\
 &= \frac{3}{28} \left( \left(3 - \frac{12}{7}\right)^2 + 3 \left(2 - \frac{12}{7}\right)^2 + 3 \left(1 - \frac{12}{7}\right)^2 \right) + \\
 &\quad + \frac{1}{28} \left( \left(3 - \frac{12}{7}\right)^2 + 3 \left(2 - \frac{12}{7}\right)^2 + 3 \left(1 - \frac{12}{7}\right)^2 \right) = \\
 &= \frac{1}{7} \cdot \frac{9^2}{7^2} + \frac{3}{7} \cdot \frac{2^2}{7^2} + \frac{3}{7} \cdot \frac{(-5)^2}{7^2} = \frac{168}{7^3} = \frac{24}{49} \approx 0.490.
 \end{aligned}$$

*Note.* In fact, the assumption that a year starts with each weekday with a probability  $1/7$  is (slightly) false. <https://bit.ly/2vo0YQt> explains that each 400 year cycle in Gregorian calendar repeats the same weekdays. Namely, the calendar for year 1600 is identical to the calendar of year 2000; year 1620 starts on the same weekday as year 2020, and so on.

To verify this, notice that exactly 97 of all 400 years are leap years: (**either** years divisible by 4, but not with 100, **or** years divisible by 400. During this 400 year cycle the number of days:

$$303 \cdot 365 + 97 \cdot 366 \equiv 303 \cdot 1 + 97 \cdot 2 \equiv 497 \equiv 0 \pmod{7}.$$

Since this number is divisible by 7, we should count the proportions of the weekdays within one 400 year cycle.

**Question 5.** Answer: 23/30

Let  $U$  be the (universe) set of all numbers between 1 and 600. Define three more sets:

$$A = \{i \in U \mid i \text{ is divisible by } 2\},$$

$$B = \{i \in U \mid i \text{ is divisible by } 3\},$$

$$C = \{i \in U \mid i \text{ is divisible by } 5\},$$

Numbers  $x \in \overline{A \cap B} = \overline{A} \cup \overline{B}$  are not divisible by 6. Numbers  $x \in \overline{A \cap C} = \overline{A} \cup \overline{C}$  are not divisible by 10. Numbers in the intersection are not divisible either by 6 or by 10:

$$(\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C}) = \overline{A} \cup (\overline{B} \cap \overline{C}) = \overline{A} \cup \overline{B \cap C}.$$

Inclusion-exclusion principle tells that the number of elements in this union:

$$|\overline{A} \cup \overline{B \cap C}| = |\overline{A}| + |\overline{B \cap C}| - |\overline{A} \cap \overline{B \cap C}|.$$

We can compute:

- $|\overline{A}| = (1/2) \cdot 600 = 300 \dots$  not divisible by 2.
- $|\overline{B \cap C}| = 600 - |B \cap C| = 600 - (|B| + |C| - |B \cap C|) = 600 - (200 + 120 - 40) = 320 \dots$  not divisible either by 3 or 5.
- $|\overline{A} \cap \overline{B \cap C}| = 160.$

Let us return to the original question: counting the elements in  $\overline{A} \cup \overline{B \cap C}$ :

$$|\overline{A} \cup \overline{B \cap C}| = 300 + 320 - 160 = 460.$$

All these numbers are not divisible either by 10 or by 6, so their proportion is  $\frac{460}{600} = \frac{23}{30}$ .

**Question 6.** Answer: 11/2

When you buy the first bag of chips, you necessarily get a toy animal you did not have before. Waiting time for this is always 1.

Assume that you already have one toy; then it might take  $x_1 = 1, x_2 = 2, x_3 = 3$ , etc. more bags to find a different toy animal. The respective probabilities of these events are  $p_1 = \frac{2}{3}, p_2 = \frac{1}{3} \cdot \frac{2}{3}, p_3 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$ , etc. These probabilities make a *geometric distribution*. The expected waiting time is the sum  $x_1 p_1 + x_2 p_2 + x_3 p_3 + \dots$ :

$$S = 1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} + 3 \cdot \frac{1^2}{3^2} \cdot \frac{2}{3} + \dots$$

To find the value of  $S$ , we multiply it by 3:

$$3S = 3 \cdot \frac{2}{3} + 2 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 \cdot \frac{1^2}{3^2} \cdot \frac{2}{3} \dots$$

If we subtract 2nd from the 1st and bring the factor  $2/3$  to the front:

$$3S - S = \frac{2}{3} \left( 4 + \left( 3 \cdot \frac{1}{3} - 2 \cdot \frac{1}{3} \right) + \left( 4 \cdot \frac{1^2}{3^2} - 3 \cdot \frac{1}{3^2} \right) + \dots \right).$$

$$2S = \frac{2}{3} \left( 4 + \frac{1}{3} + \frac{1}{3^2} + \dots \right)$$

We get that  $3S = 4\frac{1}{2}$  and  $S = \frac{3}{2}$ . I.e. we expect to get the second toy after 1.5 bags of chips.

Once we have 2 toys, the final toy can also be found by summation and infinite geometrical progression. The expected waiting time is 3 (bags of chips).

The total waiting time for all 3 toys is  $\frac{3}{3} + \frac{3}{2} + \frac{3}{1} = \frac{11}{2}$ .

**Question 7.** Answer:  $3/5, 3/8$

The conditional probability of  $E_1$  given  $E_3$  is defined like this:

$$p(E_1 \mid E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)}.$$

The conditional probability of  $E_3$  given  $E_1$  is defined similarly:

$$p(E_3 \mid E_1) = \frac{p(E_1 \cap E_3)}{p(E_1)}.$$

- Out of 32 bit sequences there are 6 sequences that are in  $E_1 \cap E_3$ : They start with 1 and also contain exactly three 1's. Indeed, the first bit is 1, and there are  $\binom{4}{2}$  ways to select the remaining two bits that equal 1.
- Out of 32 bit sequences there are 16 sequences that are in  $E_1$ . They start with 1.
- Out of 32 bit sequences there are 10 sequences that are in  $E_3$ . There are  $\binom{5}{3}$  ways to select three bits that equal 1.

Compute both conditional probabilities:

$$p(E_1 \mid E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)} = \frac{6/32}{10/32} = \frac{3}{5}.$$

$$p(E_3 \mid E_1) = \frac{p(E_1 \cap E_3)}{p(E_1)} = \frac{6/32}{16/32} = \frac{3}{8}.$$