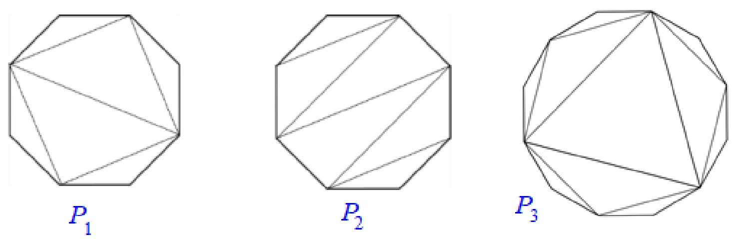
HOMEWORK 06, DUE BY 2022-03-14

Question 1:



The above pictures show three polygons that are cut into triangles by their diagonals. For every picture P_1, P_2, P_3 create the following graph $G_i = (V_i, E_i)$:

Each triangle in the polygon picture P_i is represented by a vertex in V_i. Two vertices are connected by an edge iff their triangles in the polygon picture have a common side.

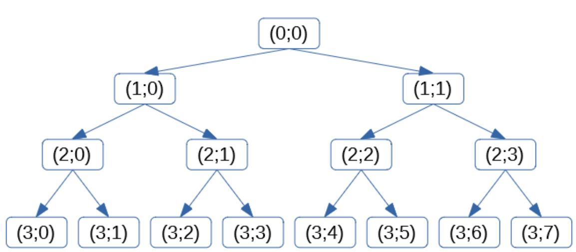
(A) Draw all three graphs G_1, G_2, G_3 created in this way.

(B) For every polygon picture P_i we need to know, how many quadrangles and how many pentagons were created by the diagonals. Describe, how you would count the number of quandrangles and pentagons in picture P_i using only the graph G_i (i.e. without looking at the polygon picture P_i itself). Fill in the table:

Example	Vertice count $\|V_i\|$	Total quadrangles	Total pentagons
P_1			
P_2			
P_3			

Question 2: Let $G = (V, E)$ be a graph with $n = |V|$ vertices. Assume that every vertex $v \in V$ has degree $\deg(v) \geq \frac{n}{2}$. Prove that the graph G is connected.

Question 3: Consider a perfect binary tree of some height h. (The tree in image has height $h = 3$.)



Nodes are identified as pairs (ℓ, k), where ℓ is the level number in the tree, and $k \in \{0, 1, \ldots, 2^\ell - 1\}$ is the position of this node on the given level.

(A) Assume that the nodes in this perfect tree of some height $h > 0$ are being visited in the in-order DFS sequence. (For the tree of height $h = 3$ this sequence is $(3;0), (2;0), (3;1), (1;0), (3;2), (2;1), (3;3), (0;0), (3;4), (2;2), \ldots, (3;7)$.)

Given the node (ℓ, k) define $\text{inOrderNext}(\ell, k)$ to find the next node in the in-order sequence. For example, $\text{inOrderNext}(3;0) = (2;0)$ and $\text{inOrderNext}(3;3) = (0;0)$.

(B) Define a function $\text{postOrderNext}(\ell, k)$ that computes the next node in the post-order sequence. For example, $\text{postOrderNext}(3;0) = (3;1)$ and $\text{postOrderNext}(3;3) = (2;1)$.

Note: Your formulas should work for any tree height $h > 0$. Both functions may use all four arithmetic operations, modular arithmetic (integer division “div” and remainder “mod”), roots, logarithms, floor and ceiling functions. It can also use definition by cases (assignments with a curly bracket). The inputs are arguments ℓ, k – the location of the current node and also the height of the tree h.

Question 4: Assume there is a bipartite graph with two sets of vertices: V_1 and V_2 with sizes $|V_1| = n_1$ and $|V_2| = n_2$ respectively. Every edge in this graph is between some vertex in V_1 and some vertex in V_2.

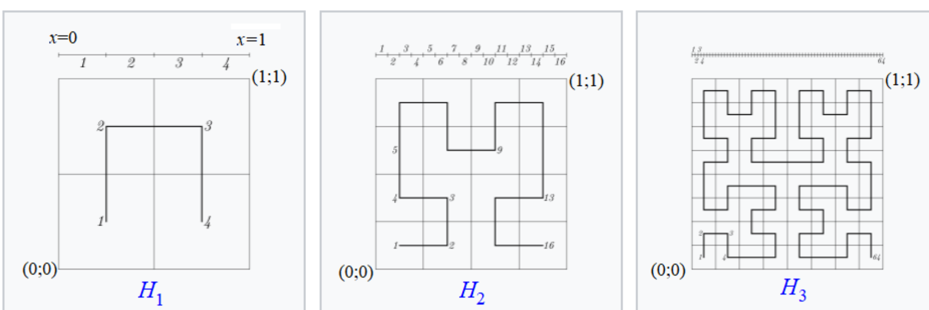
It is known that every vertex $v \in V_1$ is connected with exactly k vertices in V_2 (and also every vertex $v \in V_2$ is connected with exactly k vertices in V_1).

(A) Prove that $n_1 = n_2$.

(B) Prove that it is possible to find a *complete matching* from V_1 to V_2 – find n_1 edges in this graph so that no two edges are connected to the same vertex in either V_1 or in V_2.

Question 5: In a full m-ary tree every node has either no children (is a leaf) or it has exactly m children. A perfect m-ary tree of height h is a full m-ary tree in which every node on levels $0, 1, 2, \ldots, h-1$ is an internal node, but each node on level h is a leaf. Prove by induction that a perfect m-ary tree of height h has $\dfrac{m(m^h - 1)}{m-1}$ edges.

Question 6: Hilbert’s curve is defined as the limit of a sequence of recursively defined curves:



The image shows iterations H_1, H_2, H_3. Each iteration maps the interval $[0;1]$ to the square $[0;1] \times [0;1]$.

To define mapping H_1, split the interval $[0;1]$ as $[0;1/4) \cup [1/4;2/4) \cup [2/4;3/4) \cup [3/4;1]$. Map each of these intervals to $[0;1] \times [0;1]$:

$$H_1(z) = \begin{cases} (x,y) = (\frac{1}{4}, \frac{1}{4}), & \text{if } z \in [0/4;1/4) \\ (x,y) = (\frac{1}{4}, \frac{3}{4}), & \text{if } z \in [1/4;2/4) \\ (x,y) = (\frac{3}{4}, \frac{3}{4}), & \text{if } z \in [2/4;3/4) \\ (x,y) = (\frac{3}{4}, \frac{1}{4}), & \text{if } z \in [3/4;4/4] \end{cases}$$

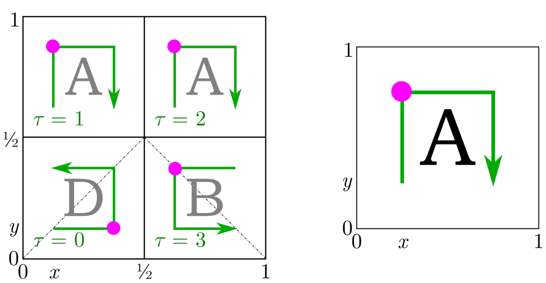
Regarding the curve H_2 – split the interval $[0;1]$ into sixteen parts: $[0;1/16) \cup [1/16;2/16) \cup \ldots \cup [14/16;15/16) \cup [15/16;16/16]$. Each point $z \in [0;1]$ belongs to one of these sixteen parts and $H_2(z)$ maps this point to the center of one of the 16 squares shown in the above picture:

$$H_2(z) = \begin{cases} (x,y) = (\frac{1}{8}, \frac{1}{8}), & \text{if } z \in [0/16; 1/16) \\ (x,y) = (\frac{3}{8}, \frac{1}{8}), & \text{if } z \in [1/16; 2/16) \\ (x,y) = (\frac{3}{8}, \frac{3}{8}), & \text{if } z \in [2/16; 3/16) \\ \ldots & \\ (x,y) = (\frac{7}{8}, \frac{7}{8}), & \text{if } z \in [15/16; 16/16] \end{cases}$$

For curve H_k – split the interval $[0;1]$ into 2^{2k} equal intervals $\left[0; \frac{1}{2^{2k}}\right), \left[\frac{1}{2^{2k}}; \frac{2}{2^{2k}}\right), \ldots, \left[\frac{2^{2k}-1}{2^{2k}}; 1\right]$. And also split the square $[0;1] \times [0;1]$ into $2^k \times 2^k$ smaller squares. As before, each interval $I_k \subset [0;1]$ maps to a center of some little square inside $[0;1] \times [0;1]$.

The order how these little squares are visited is constructed iteratively: the square order for H_{k+1} is obtained by using four copies of H_k order – two copies of H_k are scaled by factor 0.5 and translated into the upper-left and upper-right squares $[0;0.5] \times [0.5;1]$ and $[0.5;1] \times [0.5;1]$.

Building H_{k+1} in the two bottom squares ($[0;0.5] \times [0;0.5]$ and $[0.5;1] \times [0.5;1]$ also involves copying the previous iteration H_k, but they are rotated and flipped around the diagonal as shown in the image below:



Once all the functions $H_n(z)$ are defined, proceed by definining the Hilbert's curve itself.

$$H(z) = \lim_{n \to \infty} H_n(z), \quad \text{where } z \in [0;1] \text{ is some fixed number.}$$

So the Hilbert's curve is the limit of iterations $H_n(z)$. You can assume basic facts about Hilbert's curve: it is a mapping between a one-dimensional $[0;1]$ (a line segment) and a two-dimensional $[0;1] \times [0;1]$ (a unit square). It is a surjective mapping – every point in the unit square (x,y) has a primitive image $z \in [0;1]$ that satisfies $H(z) = (x,y)$.

(A) Solve the equations – find all values $z \in [0;1]$ for which the following equalities hold:

- Equation1: $H(z) = \left(\frac{1}{2}; \frac{1}{2}\right)$.
- Equation2: $H(z) = \left(\frac{1}{2}; \frac{1}{4}\right)$.
- Equation3: $H(z) = \left(\frac{1}{2}; \frac{1}{5}\right)$.

Try to find all the solutions for every equation.

(B) Verify that the Hilbert curve $H\colon [0;1] \to [0;1] \times [0;1]$ is a continuous mapping. Namely, for every $z_0 \in [0;1]$ and every $(x_0, y_0) \in [0;1] \times [0;1]$ such that $H(z_0) = (x_0, y_0)$ this definition is satisfied:

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall z, x, y \in [0;1] \left(|z - z_0| < \delta \land (x,y) = H(z) \to \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon \right).$$

(C) Consider the function $x = h_x(z)$ which finds the x-coordinate of the image $H(z)$. For example, $h_x(0) = 0$, $h_x(3/16) = 1/4$ and $h_x(1) = 1$. Plot the graph of the function $x = h_x(z)$ (preferably with Python's Matplotlib or a similar tool). Is $z \mapsto h_x(z)$ a continuous function?