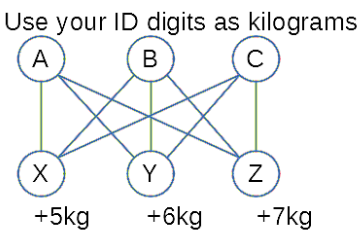
Discrete Quiz 10

Question 1
Some people participate in an Einkaufshelden program – they go to one of the shops A, B or C and deliver the products to one of the endpoints X, Y or Z. Each of the 9 edges in this full bipartite graph $K_{3,3}$ are selected by the same probability.



The sum of the numbers on both ends of an edge shows how many kilograms of stuff were delivered. (For example, if your Student ID has first digit $A = 0$, then edge AZ has $0 + 7 = 7$ kilograms. Let X denote the random variable: the kilograms of stuff delivered by an Einkaufsheld on a single edge.
The variance $V(X)$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Question 2.
Given a set of the first positive integers $S = \{1, 2, \ldots, A + B + (10 - C)\}$ (where $\mathtt{A}, \mathtt{B}, \mathtt{C}$ are the digits from your student ID: we define a relation: aRb iff $|a - b| \leq 2$.
Let R^2 be the second power of that relation R and let M_{R^2} be its matrix. Find the number of 1s in this matrix (In other words: how many pairs belong to this relation?)

Question 3.
Define a set S of these six positive integers:

$$S = \{1 + \mathtt{A},\ 2 + \mathtt{A} + \mathtt{B},\ 3 + \mathtt{A} + \mathtt{B} + \mathtt{C},\ 4 + 2\mathtt{A} + \mathtt{B} + \mathtt{C},\ 5 + 2\mathtt{A} + 2\mathtt{B} + \mathtt{C},\ 6 + 2\mathtt{A} + 2\mathtt{B} + 2\mathtt{C}\}.$$

Now compute the remainders of the elements of S when divided by 16. You should get another set S' where each element is between 0 and 15. (S' may contain fewer elements than S, if some remainders are identical.) Let b_i be a sequence of bits ($i = 0, \ldots, 15$):

$$b_i = 1 \text{ iff } i \in S'.$$

We define a matrix for relation R as follows:

$$M_R = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 & b_7 \\ b_8 & b_9 & b_{10} & b_{11} \\ b_{12} & b_{13} & b_{14} & b_{15} \end{pmatrix}$$

Let M^* be the matrix of the transitive closure of R. Find the number of 1s in the matrix M^*.

Question 4.
Let S be a set and its size is computed from the digits in your ID:

$$|S| = \mathtt{A} + \mathtt{B} + (10 - \mathtt{C}).$$

Let N be the number of binary relations on S that are reflexive and symmetric at the same time. Write the last 3 digits of N in your answer.
Hint If you need to find the last 3 digits of some large number, you can use periodicity (similar to this: https://bit.ly/33NrJKI). Euler's theorem about the period of remainders modulo 1000 being periodic with period $\varphi(1000)$ (see https://bit.ly/33PyI5Q) is not directly applicable in this situation, since your exponent a is not mutually prime with 1000. But with some additional reasoning you can use Euler's theorem as well.

Question 5.
Find the join of the 3-ary relation:

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{ (Wages,MS410,N507),
  (Rosen,CS540,N525),
  (Michaels,CS518,N504),
  (Michaels,MS410,N510) }
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and the 4-ary relation:

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{ (MS410,N507,Monday,6:00),
  (MS410,N507,Wednesday,6:00),
  (CS540,N525,Monday,7:30),
  (CS518,N504,Tuesday,6:00),
  (CS518,N504,Thursday,6:00) }
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with respect to the last two fields of the first relation and the first two fields of the second relation.
Write the number records in the join.

Question 6.
Let R be a relation on $\{a, b, c\}$ that is reflexive and transitive, but not antisymmetric. Denote its matrix by

$$M_R = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_3 & b_4 & b_5 \\ b_6 & b_7 & b_8 \end{pmatrix}$$

Write all the 9 bits (as a sequence of 0s and 1s) in your answer: $b_0b_1b_2b_3b_4b_5b_6b_7b_8$.

Question 7.
Let R be a relation on $\{a, b, c\}$ that is reflexive and transitive, but not symmetric. Denote its matrix by

$$M_R = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_3 & b_4 & b_5 \\ b_6 & b_7 & b_8 \end{pmatrix}$$

Write all the 9 bits (as a sequence of 0s and 1s) in your answer.