## Quiz 8: Probabilities

**Question 1** Two chess queens are placed on two different places in a $4 \times 4$ chess-board. Assume that all the $\binom{16}{2}$ possibilities how they are placed have equal probabilities. Find a probability that one queen *attacks* the other. (Two queens attack each other, if they are located on the same horizontal, vertical or diagonal).
Write your answer as a rational fraction: `P/Q`

**Question 2** Assume that you are generating 10-bit sequences (a string of 0's and 1's). All the $2^{10}$ sequences have equal probabilities. Find the probability of an event that the 10-bit sequence does NOT contain two consecutive 0's anywhere.
Write your answer as a rational fraction: `P/Q`

**Question 3** Two players have 4 cubic dices. Instead of the usual numbers, their faces have the following numbers on their faces:
**Dice A:** 4, 4, 4, 4, 0, 0;
**Dice B:** 3, 3, 3, 3, 3, 3;
**Dice C:** 6, 6, 2, 2, 2, 2;
**Dice D:** 5, 5, 5, 1, 1, 1.
In a single round Players Alice and Bob randomly select two of the dices (with equal probabilities they can select any of the six pairs - $(A, B)$, or $(A, C)$, or $(A, D)$, or $(B, C)$, or $(B, D)$, or $(C, D)$). There can be three outcomes:
**Outcome 1:** They have selected "opposite dices" - pairs $(A, C)$ or $(B, D)$. In this case the payoff is zero (nobody pays anything to the other).
**Outcome 2:** The dices win in the "clockwise manner" ($A > B$ or $B > C$ or $C > D$ or $D > A$) - then Alice wins 1 euro.
**Outcome 3:** The dices win in the "counter-clockwise manner" ($B > A$ or $C > B$ or $D > C$ or $A > D$) - then Bob wins 1 euro.
(*Note.* The expression $A > B$ means that the number that rolled out on the the dice $A$ was larger than the number on dice $B$; but $B > A$ denotes the opposite event.)
Find the expected value - how much money Alice is expected to win in a single round of such a game.
Write your answer as a rational fraction: `P/Q`
For example, if the expected win for Alice is 0.10 EUR, then write `1/10`. If Alice is expected on average to lose 0.10 EUR per one round of this game, then write `-1/10`.

**Question 4** For every year we count the number of Friday's that fall on the 13th date of some month (such as Friday, March 13, 2020). Denote this count by $X$ – it is your random variable. Find the expected value and the variance of $X$. Round them to the nearest thousandth.
Write your answer as two comma-separated numbers: `D.DDD,D.DDD`.

**Question 5** What is the probability that a randomly chosen positive integer between 1 and 600 is not divisible by either 6 or 10?
Write your answer as a rational fraction: `P/Q`

**Question 6** A chip factory *Intel* adds one toy animal to every bag of chips. There are three sorts of animals - Aligators, Bears or Cats (each one appears with probability $p = 1/3$). Find the expected number of the chip bags one needs to purchase to collect all three animals.
Write your answer as a rational fraction: `P/Q`

**Question 7** You create a random bit string of length five (all 32 bit strings are equally probable). Consider these events:
$E_1$: the bit string chosen begins with 1;
$E_3$: the bit string chosen has exactly three 1s.
**(A)** Find $p(E_1 \mid E_3)$.
**(B)** Find $p(E_3 \mid E_1)$.
Write your answer as a comma-separated rational fractions `P1/Q1,P2/Q2`

**Answers**

**Question 1.** Answer: `19/30`
To make the counting easier, let us assume that both queens are distinguishable (we can have $Q_1$ and $Q_2$ as white and black queen). Then the ways to place them are $16\cdot 15$ – which is twice the number of combinations $\binom{16}{2} = \frac{15\cdot 16}{1\cdot 2}$.

```
Q x x x    x Q x x    x x x .
x x . .    x x x .    x Q x x
x . x .    . x . x    x x x .
x . . x    . x . .    . x . x
```

Queen $Q_1$ can be placed in a corner of the $4 \times 4$ square (4 chances out of 16), on an edge (8 chances out of 16) or near the center (4 chances out of 16) – see the pictures above. In the first two cases queen $Q_1$ attacks 9 places (out of 15). In the last case it attacks 11 places (out of 15). The ultimate probability that $Q_1$ attacks $Q_2$ is

$$\frac{4}{16}\cdot\frac{9}{15}+\frac{8}{16}\cdot\frac{9}{15}+\frac{4}{16}\cdot\frac{11}{15}=\frac{152}{16\cdot 15}=\frac{19}{30}.$$

We could also count the mutual attack positions in the original problem (when $Q_1$ and $Q_2$ are indistinguishable). But we would get the same result, since attacking positions are symmetric (if $Q_1$ attacks $Q_2$, then $Q_2$ attacks $Q_1$).

**Question 2.** Answer: `9/64`
We can count all the sequences that do not contain two consecutive 0's: https://bit.ly/2TPwMaj. If we denote by $f_0(n)$ the count of all $n$-bit sequences that do not contain two consecutive 0s, we can prove that

$$f_0(0) = 1,\ f_0(1) = 2,\ f_0(2) = 3,\ f_0(3) = 5, \ldots$$

For arbitrary $n$ we get $f_0(n) = F_{n+2}$, where $F_n$ $(0, 1, 1, 2, 3, \ldots)$ is the Fibonacci sequence.
The 12th member of the Fibonacci sequence is $F_{12} = 144$. Therefore the proportion of 10-bit sequences equals $\frac{144}{1024} = \frac{9}{64}$.

**Question 3.** Answer: `2/9`
There is $1/3$ probability to pick $(A, C)$ or $(B, D)$ (payoff is 0 in this case).
Regarding the other four pairs, the probabilities are the following:

- $A$ wins $B$ with probability $2/3$.
- $B$ wins $C$ with probability $2/3$.
- $C$ wins $D$ with probability $\frac{1}{3}\cdot\frac{1}{1}+\frac{2}{3}\cdot\frac{1}{2}=\frac{2}{3}$: **Either** we have $C = 6$ (probability $1/3$); and it wins with probability 1. **Or** we have $C = 2$ (probability $2/3$) and it wins with probability $1/2$ (whenever $D = 1$).
- $D$ wins $A$ with probability $\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{3}=\frac{2}{3}$: **Either** we have $D = 5$ (probability $1/2$); and it wins with probability 1. **Or** we have $D = 1$ (probability $1/2$) and it wins with probability $1/3$ (whenever $A = 0$).

For each pair $(A, B)$, $(B, C)$, $(C, D)$, or $(D, A)$ the expected payoff for Alice is

$$\frac{2}{3}(+1\,\mathrm{EUR})+\frac{1}{3}(-1\,\mathrm{EUR})=\frac{1}{3}\,\mathrm{EUR}.$$

The expected probability for Alice to win in a single round can be obtained as a sum, where we multiply probabilities for each pair ($(A, C)$, $(B, D)$, $(A, B)$, $(B, C)$, $(C, D)$, or $(D, A)$) with their respective payoffs (either 0 or 1/3 euro):

$$\frac{1}{6}\cdot 0+\frac{1}{6}\cdot 0+\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3}=\frac{2}{9}.$$

**Question 4.** Answer: `1.714,0.490`
Assume that each year (and therefore each month) can start on each weekday with exactly the same probability (that is $1/7$). Every month has date 13, so there is a $1/7$ probability that this date will happen on Friday.
**(A)** If $X$ is the number of months having 13th date on Friday, then $E(X)$ must be $12/7 = 1.714$. It is not because the weekdays that start months in a single year are independent (which they are not!), but rather because every 28-year cycle should have the same number of each type of month (January, February, etc.) starting on each weekday: exactly 4 Januaries starting on Monday, exactly 4 Januaries starting on Tuesday, etc. $(4 + 4 + \ldots + 4 = 28)$. Only those Januaries that start on Sundays will have 13th date on Friday.
**(B)** Let us create a table reflecting the number of Fridays on 13th depending on whether the year is the leap year and the weekday it starts.

| Year | Non-leap | Leap |
|---|---|---|
| Sun | 2 (Jan,Oct) | 3 (Jan,Apr,Jul) |
| Mon | 2 (Apr,Jul) | 2 (Sep,Dec) |
| Tue | 2 (Sep,Dec) | 1 (Jun) |
| Wed | 1 (Jun) | 2 (Mar,Nov) |
| Thu | 3 (Feb,Mar,Nov) | 2 (Feb,Aug) |
| Fri | 1 (Aug) | 1 (May) |
| Sat | 1 (May) | 1 (Oct) |

The probability to get non-leap year starting on a certain weekday is $\frac{3}{4}\cdot\frac{1}{7}=\frac{3}{28}$. The probability to get leap year starting on a certain weekday is $\frac{1}{28}$.
We find the variance $V(X)$ using its definition:

$$V(X)=\sum(x_i-E(X))^2 p(x_i)=$$

$$=\frac{3}{28}\left(\left(3-\frac{12}{7}\right)^2+3\left(2-\frac{12}{7}\right)^2+3\left(1-\frac{12}{7}\right)^2\right)+$$

$$+\frac{1}{28}\left(\left(3-\frac{12}{7}\right)^2+3\left(2-\frac{12}{7}\right)^2+3\left(1-\frac{12}{7}\right)^2\right)=$$

$$=\frac{1}{7}\cdot\frac{9^2}{7^2}+\frac{3}{7}\cdot\frac{2^2}{7^2}+\frac{3}{7}\cdot\frac{(-5)^2}{7^2}=\frac{168}{7^3}=\frac{24}{49}\approx 0.490.$$

*Note.* In fact, the assumption that a year starts with each weekday with a probability $1/7$ is (slightly) false. https://bit.ly/2vo0YQt explains that each 400 year cycle in Gregorian calendar repeats the same weekdays. Namely, the calendar for year 1600 is identical to the calendar of year 2000; year 1620 starts on the same weekday as year 2020, and so on.

To verify this, notice that exactly 97 of all 400 years are leap years: (**either** years divisible by 4, but not with 100, **or** years divisible by 400. During this 400 year cycle the number of days:

$$303\cdot 365+97\cdot 366\equiv 303\cdot 1+97\cdot 2\equiv 497\equiv 0 \pmod 7).$$

Since this number is divisible by 7, we should count the proportions of the weekdays within one 400 year cycle.

**Question 5.** Answer: `23/30`

Let $U$ be the (universe) set of all numbers between 1 and 600. Define three more sets:

$$A=\{i\in U \mid i \text{ is divisible by } 2\},$$
$$B=\{i\in U \mid i \text{ is divisible by } 3\},$$
$$C=\{i\in U \mid i \text{ is divisible by } 5\},$$

Numbers $x\in\overline{A\cap B}=\overline{A}\cup\overline{B}$ are not divisible by 6. Numbers $x\in\overline{A\cap C}=\overline{A}\cup\overline{C}$ are not divisible by 10. Numbers in the intersection are not divisible either by 6 or by 10:

$$(\overline{A}\cup\overline{B})\cap(\overline{A}\cup\overline{C})=\overline{A}\cup(\overline{B}\cap\overline{C})=\overline{A}\cup\overline{B\cup C}.$$

Inclusion-exclusion principle tells that the number of elements in this union:

$$|\overline{A}\cup\overline{B\cup C}|=|\overline{A}|+|\overline{B\cup C}|-|\overline{A}\cap\overline{B\cup C}|.$$

We can compute:

- $|\overline{A}|=(1/2)\cdot 600=300$ ... not divisible by 2.
- $|\overline{B\cup C}|=600-|B\cup C|=600-(|B|+|C|-|B\cap C|)=$ $=600-(200+120-40)=320$ ... not divisible either by 3 or 5.
- $|\overline{A}\cap\overline{B\cup C}|=160.$

Let us return to the original question: counting the elements in $\overline{A}\cup\overline{B\cup C}$:

$$|\overline{A}\cup\overline{B\cup C}|=300+320-160=460.$$

All these numbers are not divisible either by 10 or by 6, so their proportion is $\frac{460}{600}=\frac{23}{30}$.

**Question 6.** Answer: `11/2`

When you buy the first bag of chips, you necessarily get a toy animal you did not have before. Waiting time for this is always 1.

Assume that you already have one toy; then it might take $x_1=1$, $x_2=2$, $x_3=3$, etc. more bags to find a different toy animal. The respective probabilities of these events are $p_1=\frac{2}{3}$, $p_2=\frac{1}{3}\cdot\frac{2}{3}$, $p_3=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}$, etc. These probabilities make a *geometric distribution*. The expected waiting time is the sum $x_1p_1+x_2p_2+x_3p_3+\ldots$:

$$S=1\cdot\frac{2}{3}+2\cdot\frac{1}{3}\cdot\frac{2}{3}+3\cdot\frac{1^2}{3^2}\cdot\frac{2}{3}+\ldots$$

To find the value of $S$, we multiply it by 3:

$$3S=3\cdot\frac{2}{3}+2\cdot\frac{2}{3}+3\cdot\frac{1}{3}\cdot\frac{2}{3}+4\cdot\frac{1^2}{3^2}\cdot\frac{2}{3}\ldots$$

If we subtract 2nd from the 1st and bring the factor $2/3$ to the front:

$$3S-S=\frac{2}{3}\left(4+\left(3\cdot\frac{1}{3}-2\cdot\frac{1}{3}\right)+\left(4\cdot\frac{1^2}{3^2}-3\cdot\frac{1}{3^2}\right)+\ldots\right).$$

$$2S=\frac{2}{3}\left(4+\frac{1}{3}+\frac{1}{3^2}+\ldots\right)$$

We get that $3S=4\frac{1}{2}$ and $S=\frac{3}{2}$. I.e. we expect to get the second toy after 1.5 bags of chips.

Once we have 2 toys, the final toy can also be found by summation and infinite geometrical progression. The expected waiting time is 3 (bags of chips).

The total waiting time for all 3 toys is $\frac{3}{3}+\frac{3}{2}+\frac{3}{1}=\frac{11}{2}$.

**Question 7.** Answer: `3/5,3/8`

The conditional probability of $E_1$ given $E_3$ is defined like this:

$$p(E_1 \mid E_3)=\frac{p(E_1\cap E_3)}{p(E_3)}.$$

The conditional probability of $E_3$ given $E_1$ is defined similarly:

$$p(E_3 \mid E_1)=\frac{p(E_1\cap E_3)}{p(E_1)}.$$

- Out of 32 bit sequences there are 6 sequences that are in $E_1\cap E_3$: They start with 1 and also contain exactly three 1's. Indeed, the first bit is 1, and there are $\binom{4}{2}$ ways to select the remaining two bits that equal 1.
- Out of 32 bit sequences there are 16 sequences that are in $E_1$. They start with 1.
- Out of 32 bit sequences there are 10 sequences that are in $E_3$. There are $\binom{5}{3}$ ways to select three bits that equal 1.

Compute both conditional probabilities:

$$p(E_1 \mid E_3) = \frac{p(E_1 \cap E_3)}{p(E_3)} = \frac{6/32}{10/32} = \frac{3}{5}.$$

$$p(E_3 \mid E_1) = \frac{p(E_1 \cap E_3)}{p(E_1)} = \frac{6/32}{16/32} = \frac{3}{8}.$$