Discrete Structures – Homework 1

RBS
Affiliation

Due: January 20, 2020. Submit PDF file to the "Homework 1" folder in ORTUS.

Problem 1 [53, p.17] The n-th statement in a list of 100 statements is "Exactly n of the statements in this list are false."

(a) What conclusions can you draw from these statements?

(b) Answer part (a), if the n-th statement is "At least n of the statements in this list are false."

(c) Answer part (b) assuming that the list contains 99 statements.

Problem 2 [19, p.24] Each inhabitant of a remote village always tells the truth or always lies. A villager will give only a "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

Problem 3 [55, p.39] Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.
Note. Operator $\downarrow$ is named **Peirce arrow** (or NOR). Propositon $p \downarrow q$ is true when both p and q are false, and it is false otherwise. It is a shorthand: $p \downarrow q := \neg(p \vee q)$.

Problem 4 [39, p.114] Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

(a) Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

(b) Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

Problem 5 [39, p.119] Prove or disprove that if x^2 is irrational, then x^3 is irrational.

Problem 6 [43, p.133] Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Problem 7 [78, p.164] Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Problem 8 [28, p.179] Let a_n be the n-th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots$ constructed by including the integer k exactly k times. Show that $\left\lfloor \sqrt{2n} + \dfrac{1}{2} \right\rfloor$.

Problem 9 [31, p.187] Show that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by showing that the polynomial function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ with $f(m, n) = \dfrac{(m+n-2)(m+n+2)}{2} + m$ is one-to-one and onto.

Problem 10 [28, p.198] We define the **Ulam numbers** by setting $u_1 = 1$ and $u_2 = 2$. Furthermore, after dtermining whether the integers less than n are Ulam numbers, we set n equal to the next Ulam number, if it can be written uniquely as the sum of two different Ulam numbers. Note that $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, and $u_6 = 8$.

(a) Find these five consecutive Ulam numbers: $u_{2020}, u_{2021}, u_{2022}, u_{2023}, u_{2024}$.

(b) Prove that there are infinitely many Ulam numbers.