HOMEWORK 02, DUE BY 2022-01-20

Question 1: Let the domain $\mathbb{Z}_{>0}$ be the set of positive integers. Define these two predicates on this domain:

- $D(a, b)$ which is true iff a divides b (namely, b is divisible by a with remainder 0). For example $D(1, 6) = D(2, 6) = D(3, 6) = D(6, 6) =$ `True`, but $D(2, 1) = D(2, 5) =$ `False`.
- $O(a)$ which is true iff $a = 1$.

(A) Write these predicate sentences in English.

i. $\exists a \in S\ (\forall b \in S\ (D(a, b)) \wedge O(a))$.

ii. $\forall a \in S\ \forall b \in S\ \forall c \in S\ (D(a, b) \wedge D(b, c) \rightarrow D(a, c))$.

iii. $\forall a \in S\ \exists b \in S\ (D(a, b) \wedge D(b, a))$.

(B) Write these English sentences with predicates and quantifiers (use only the two given predicates, Boolean operations, parentheses and quantifiers).

i. There exist two numbers a and b such that neither divides another one.

ii. There exists a number a in the domain that is a prime number (only two numbers divide it: 1 and a itself).

iii. For any positive integer a there exist a mutual prime b (namely, some number that has no common divisors with a except 1).

Question 2: Consider the following predicate expression which uses the same "divides" predicate as the previous problem:

$$\forall x \in \mathbb{Z}_{>0}\ \forall y \in \mathbb{Z}_{>0}\ \exists z \in \mathbb{Z}_{>0}\ \Big(D(z, x) \wedge D(z, y) \wedge \forall t \in \mathbb{Z}_{>0}(D(t, x) \wedge D(t, y) \rightarrow D(t, z))\Big).$$

(A) Draw the abstract syntax tree for this predicate expression. (Assume that all quantifiers (such as $(\forall x \in \mathbb{Z}_{>0})$ and also $(\exists z \in \mathbb{Z}_{>0})$) are unary operators with the same precedence as negation.

(B) Let $x = 54$ and $y = 24$. Find a value $z \in \mathbb{Z}_{>0}$ such that the statement in the big parentheses is true.

Question 3: Introduce the following sets:

- Let P be the set of all points in a two-dimensional plane.
- Let L be the set of all lines in the same plane.

Points will be denoted by upper-case variables $A, B, C, \ldots \in P$, but lines will be denoted by the lower-case variables $a, b, c, \ldots \in L$. Define the following predicates:

- $S(A, a)$ is true iff the line a goes through the point A.
- $I(A, B)$ is true iff points A and B are identical (i.e. both variables represent the same point).
- $I(a, b)$ is true iff lines a and b are identical.

Using only sets P, L and predicates $S(A,a), I(A,B), I(a,b)$, express the following new predicates and Boolean propositions.

(A) Predicate $U(A,B,a)$ that is true iff the line a goes through both points A and B (we do not require for A, B to be different).

(B) Predicate $V(a,b,A)$ that is true iff two lines a and b have their only intersection point in A.

(C) Predicate $W(a,b)$ that is true iff two lines a and b are parallel (two lines are parallel, if they do not share any points; coinciding lines are not considered parallel).

(D) Predicate $X(A,B,C)$ that is true iff all three points are located on the same line (we do not require A, B, C to be mutually different points).

(E) Predicate $Y(A,B,C,D)$ that is true iff all four points A, B, C, D form a parallelogram (a parallelogram is a quadrangle where the opposite sides are parallel).

(F) Proposition K: For every two different points A, B there exist a third point C which is not on the same line as A and B.

(G) Proposition L: For every three different points A, B, C there exist three parallel lines a, b, c such that a goes through A, b goes through B, c goes through C.

(H) Proposition M: For any three points A, B, C that are not all on the same line there exists point D such that $ABCD$ is a parallelogram.

Note: You may use the predicates defined above and also the predicates already defined in earlier steps of your solution. You can also use Boolean connectors $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ and quantifiers. Always specify the domain for each quantifier and also enclose quantifier scopes in parentheses.

When defining a predicate such as $Y(A,B,C,D)$ you can use additional variables for other points or lines, but they should be all bound by quantifiers. The only free variables in your formulas should be A, B, C, D. When defining a proposition, it should not contain any free variables.

Question 4:

In a game the initial position is a pile with N stones (N is a nonnegative integer). Two players A and B make moves alternately (player A moves first). In a single step a player can remove either 1 or 4 stones from the pile. Whoever takes the last stone wins. Prove that a position of N stones is *cold* iff either N is divisible by 5 or N gives remainder 2 when divided by 5.

Note: A game position is called *hot*, if the player making the first move can win. A game position is called *cold*, if the player making the first move from this position loses. (In both cases assume that both players make optimal moves). See https://bit.ly/3noX5lQ.

Question 5: A larger-than-usual chessboard is subdivided into 10×10 smaller squares.

(A) Is it possible to remove one of the little squares so that the remaining chessboard can be cut into 33 rectangles of size 1×3? (The rectangles can be either horizontal or vertical).

(B) Is it possible to remove one of the little squares so that the remaining chessboard cannot be cut into 33 rectangles of size 1×3?

Note: The theory of chessboard cutting is in (Rosen2019, p.108); see subchapter *1.8.8 Tilings*.

Question 6 (Supplementary Task):

Introduction: Coq is a proof assistant that can verify the correctness of formal proofs. The simplest results to prove in Coq are Boolean tautologies – expressions built from propositional variables that are always true. You can run Coq in a browser (see https://coq.vercel.app/scratchpad.html) or install a standalone Coq IDE on your computer.

Problem: Use Coq proof assistant to prove the following tautologies (Rosen2019, p.38, Problem 12):

(A) $(\neg p \wedge (p \vee q)) \rightarrow q$

(B) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

(C) $(p \wedge (p \rightarrow q)) \rightarrow q$

(D) $((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$

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Lemma L12A: forall p q: Prop, (~p /\ (p \/ q)) -> q.
Proof.
  Admitted.

Lemma L12B: forall p q r: Prop, ((p -> q) /\ (q -> r)) -> (p -> r).
Proof.
  Admitted.

Lemma L12C: forall p q r: Prop, (p /\ (p -> q)) -> q.
Proof.
  Admitted.

Lemma L12D: forall p q r: Prop, ((p \/ q) /\ (p -> r) /\ (q -> r)) -> r.
Proof.
  Admitted.
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Note: Your proofs with Coq should not contain commands like `tauto` or `Admitted`. Proofs of many tautologies can be found here: https://bit.ly/321Osqc. Another good source to learn about Coq is Buffalo CSE191 (an equivalent of the RBS course *Discrete Structures*) – https://bit.ly/3rfIACe.