Homework 1: Grading Criteria

As a general rule, the statements in your homework have to be justified. Rigorous proofs can only happen in Coq, so in the homework you need to write **why** do you think your solution is true.

If you quote an external source, you can use the results found on that page, but you still need to restate the statement (plus a short justification, why do you think the statement is credible).

You are encouraged to write answers concisely – the best solution would contain the minimal set of sentences that allow the reader to check, what you have done. On the other hand, grading does not penalize you for writing extra stuff. The important thing is to avoid writing nonsense that casts doubt on your understanding, and also to add all the essential parts for your solution.

Problem 1 The question in this problem from your textbook was this: *"What conclusions can you draw from these statements?"* One can certainly draw various conclusions (verify, which statements are mutually exclusive or imply each other; find how many are true or false; are they consistent; or even analyze all possible ways how they can be true or false and prove that there are no other possibilities).

5 points It is clearly said, if the statements are consistent at all (is it possible to assign any truth values to them without getting a contradiction). And if so, then

4 points 1 point subtracted, if there is an incorrect conclusion in **(c)**. For example, a statement that 49 (or 50) statements should be true.

2 points Deficiencies in reasoning for (a), (b). Unjustified claims about (c).

Problem 2 Answer (in form of the question to ask the villager) has to be given; it should be analyzed for correctness.

1 point Only the answer is given; no analysis.

2 points Some attempt to write analysis, but it has serious deficiences; most cases are not analyzed.

Problem 3 Answer is not difficult to get in this case; justification of it might be somewhat longer.

- **1 point** Just an answer such as $p \vee q = (p \downarrow q) \downarrow (p \downarrow q)$, but no justification. Writing mere answers does not get much credit, since the homeworks are meant for technical communication. Answer (without any procedure, proof, algorithm to obtain similar answers in the future) is not sufficient communication in computer science.
- **2 points** An answer plus a link, but still no justification.
- **5 points** An answer plus a link, plus a short explanation (that the formula you quoted from an external source can be obtained via truth tables or similar).
- **5 points** Expression is derived by equivalences.
- **4 points** $\neg p \vee q \equiv (p \downarrow p) \vee q$ directly goes to answer, without explaining, how the $\vee$ was transformed into $\downarrow$.
- **4 points** Formulas for $p \wedge q$, $p \vee q$, $\neg p$ are all given, but $p \to q$ only mentions the answer, without any relation to the previous formulas.
- **3 points** Same as above, but with some typos in the formulas; still no explanation why pq should be equal the expression.

Problem 4 [39, p.114] Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

(a) Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

(b) Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

Problem 5 Fermat's last theorem not really needed here; there are much easier ways to prove that $\sqrt[3]{4}$ cannot be expressed as a rational number p/q.

- Just the counterexample - 1 point.
- A counterexample (and one proof that the cube is rational) - 2 points.

Problem 6

- **4 points** If there is a counterexample, but cases with very small sizes for A and B (one or two elements) are not properly explained.

- **Problem 7 [78, p.164]** Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.
  Note. By $\lfloor x \rfloor$ we denote the largest integer number that does not exceed x. For example $\lfloor 3.14 \rfloor = 3$, $\lfloor 17 \rfloor = 17$, $\lfloor -4.5 \rfloor = -5$.

Problem 8 [28, p.179] Let a_n be the n-th term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots$ constructed by including the integer k exactly k times. Show that $\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$.

Problem 9 [31, p.187] Show that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by showing that the polynomial function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ with $f(m, n) = \dfrac{(m + n - 2)(m + n - 1)}{2} + m$ is one-to-one and onto.

Problem 10 Subitem **(a)** has 1 point max credit.

- **1 point.** Everybody who correctly quotes the source of the code and computes the numbers, gets max points for this.

Subitem **(b)** has 4 points max credit.

- **4 points.** A correct explanation, why there should be infinitely many Ulam numbers. (The easiest proof is from the contradiction.)

We define the **Ulam numbers** by setting $u_1 = 1$ and $u_2 = 2$. Furthermore, after determining whether the integers less than n are Ulam numbers, we set n equal to the next Ulam number, if it can be written uniquely as the sum of two different Ulam numbers. Note that $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, and $u_6 = 8$.

(a) Find these five consecutive Ulam numbers: $u_{2020}, u_{2021}, u_{2022}, u_{2023}, u_{2024}$.

(b) Prove that there are infinitely many Ulam numbers.