Final Exam

RBS, Discrete Structures

2020-04-23

Problem 1

By U we denote the set of all positive integers between 1 and 120. This is our *universe* in which we define several subsets:

$$\begin{cases} A = \{x \in U \;\mid\; 2 \mid x\}, \\ B = \{x \in U \;\mid\; 3 \mid x\}, \\ C = \{x \in U \;\mid\; 5 \mid x\}, \\ X = \{x \in U \;\mid\; 2 \mid x \;\vee\; 3 \mid x\}, \\ Y = \{x \in U \;\mid\; (3 \mid x \;\wedge\; 5 \mid x) \;\vee \neg(2 \mid x)\}. \end{cases}$$

(A) Express X using the sets A, B, C (using set union $V \cup W$, set intersection $V \cap W$, set complement $\overline{V}$ operations).
(B) Express Y using the sets A, B, C in a similar way.
(C) Find $|X|$ - the size of the set X.
(D) Find $|Y|$ - the size of the set Y.

Problem 2

Let A and B be sets with sizes $|A| = 8$ and $|B| = 5$ and $|A \cap B| = 3$.

Calculate the largest and the smallest possible values for each of the following set sizes:

(A) $|A \cup B|$.
(B) $|A \times (B \times B)|$.
(C) $|\mathcal{P}(\mathcal{P}(A \cap B))|$ - the powerset of a powerset of $A \cap B$.
(D) $|A \oplus B|$ - the symmetric difference of the sets A and B.

Problem 3

Consider the following recurrent sequence:

$$\begin{cases} a_0 = 3 \\ a_1 = 4 \\ a_{n+2} = 5a_{n+1} - 6a_n, \;\text{ if } n \geq 0 \end{cases}$$

Assume that b_n is another sequence satisfying the recurrence rule

$$b_{n+2} = 5b_{n+1} - 6b_n, \;\text{ if } n \geq 0$$

(The first two members b_0, b_1 are not known.)

(A) Write the first 6 members of this sequence $(a_1, \ldots, a_6)$.
(B) Write the characteristic equation for this sequence.
(C) Write the general expression for an arbirary sequence b_n satisfying the recurrent expression as a sum of two geometric progressions (you can leave unknown coefficients in your answer; just explain which ones they are).
(D) Write the formula to compute a_n (that would satisfy the initial conditions $a_1 = 3$ and $a_2 = 4$).

Problem 4

Consider this code snippet in Python:

```
n = 1000
sum = 0
for i in range(1, n*n+1):
    for j in range(1,i+1):
        sum += i % j
```

And a similar one in R:

```
n <- 1000
sum <- 0
for (i in 1:(n*n)) {
    for (j in 1:i) {
        sum <- sum + i %% j
    }
}
```

(A) Explain in human language what this algorithm does.
(B) Denote by $f(n)$ the number of times the variable `sum` is incremented. Write the Big-O-Notation for $f(n)$. Find a function $g(n)$ such that $f(n)$ is in $O(g(n))$. (If there are multiple functions, pick the one with the slowest growth.)
(C) Express the function $f(n)$ precisely - how many times `sum` is incremented in terms of variable n.

Problem 5

Let A be the set of all positive divisors of the number 120 (including 1 and 120 itself).
(A) What is the multiplication of all numbers in the set A?
(B) Express this number as the product of prime powers.

Problem 6

Define the following binary relationship on the set of integer numbers $\mathbb{Z}$:
We say that aRb (numbers $a, b \in \mathbb{Z}$ are in the relation R) iff

$$\begin{cases} a - b \equiv 0 \pmod{11} \\ a - b \equiv 0 \pmod{12} \\ a - b \equiv 0 \pmod{13} \end{cases}$$

Item	Statement	True or False?
(A)	R is reflexive	
(B)	R is symmetric	
(C)	R is antisymmetric	
(D)	R is transitive	
(E)	aRb iff $a = b$	

For all items where you answered `FALSE`, specify a counterexample (values for some numbers that would make the condition true, but the conclusion false). If the statement was true, write “none”.

(A) counterexample ___

(B) counterexample ___

(C) counterexample ___

(D) counterexample ___

(E) counterexample ___

Problem 7

Four people A, B, C, D each has his own hat. After the meeting they leave their building in a hurry, everyone grabs some hat at random so that all 4! permutations of the hats have equal probabilities.

Let the random variable X denote the number of hats that were picked up correctly. (For example, if the hat assignment is this: $(A \to A, B \to B, C \to D, D \to C)$, then $X = 2$, because two people got their own hats.)

(A) Find $E(X)$ - the expected value of X.
(B) Find $V(X)$ - the variance of X.

Problem 8

There was a crooked man who had a crooked 1 euro coin. On lucky days it would flip the *heads* with probability $p = \frac{2}{3}$, and the *tails* with probability $p = \frac{1}{3}$, but on unlucky days it was the opposite ($p(\texttt{heads}) = \frac{1}{3}$, but $p(\texttt{tails}) = \frac{2}{3}$). There were equal probabilities of $\frac{1}{2}$ for lucky and unlucky days.

One morning he flipped the coin 5 times and altogether got three *heads* and two *tails*.

Let us introduce the following events:

- E (evidence): Five coin tosses result in three *heads* and two *tails*.
- H (hypothesis): The current day is lucky.

(A) Find $P(E|H)$ - the conditional probability of E given that the day is lucky.
(B) Find $P(E|H) \cdot P(H)$ - the probability that the day is lucky and E happens.
(C) Find $P(E|\overline{H})$ - the conditional probability of E given that the day is not lucky.
(D) Find $P(E|\overline{H}) \cdot P(\overline{H})$ - the probability that the day is unlucky and E happens.
(E) Find $P(E)$ - as the sum of two probabilities (E happened on a lucky day and also E happened on unlucky day).
(F) Find the conditional probability $P(H|E)$ - the likelyhood that the croocked man has a lucky day, given that the event E has happened.

Problem 9

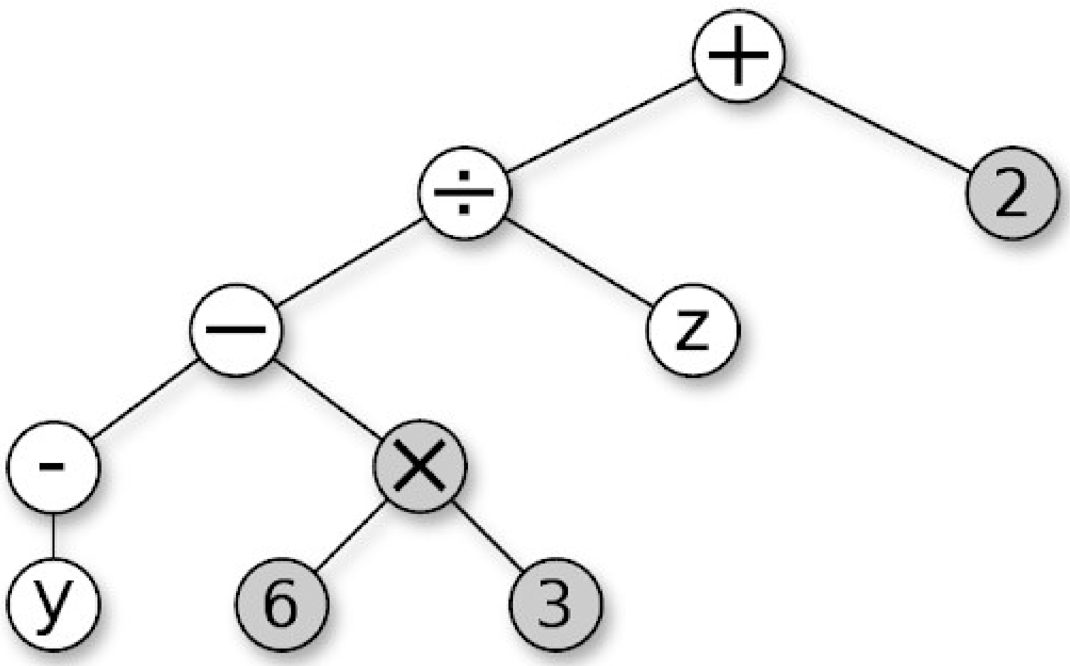

Figure: A syntax tree for an expression

The syntax tree describes an algebraic expression (please note the difference between the unary minus that flips the value of the variable y and the binary minus that subtracts the two subexpressions: $-y$ and 6×3).

(A) Write the preorder DFS traversal of this tree.
(B) Write the inorder DFS traversal of this tree.
(C) Write the postorder DFS traversal of this tree.

Note. In all 3 answers denote the unary minus with the tilde sign ~, but the regular/binary minus with –.

Problem 10

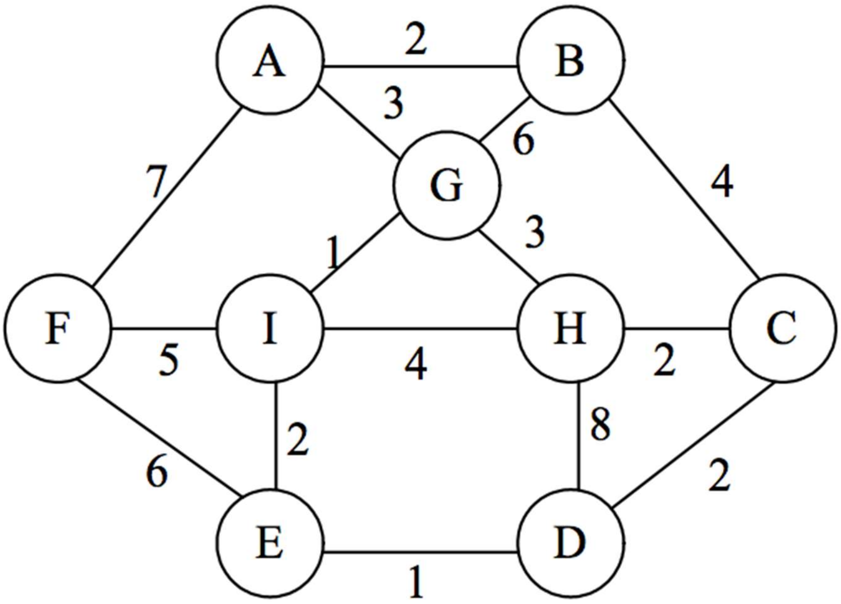

Figure: Graph with 9 vertices

Run the Prim's algorithm on the following weighted graph, start growing the tree from the vertex *I*.

Step	Newly Added Edge
Step 1	
Step 2	
Step 3	
Step 4	
Step 5	
Step 6	
Step 7	
Step 8	

What is the total weight of the obtained Minimum Spanning Tree? ______