Discrete Sample Quiz 6

Question 1: Modifying Sudoku Rules; see Chapter 1.3.6 (Rosen2019, p.36).
We have a 9×9 table; each cell contains one number from $A_9 = \{1, 2, \ldots, 9\}$. We define predicate $p(i, j, n)$ which is true iff the cell in row i and column j has the given value n. All three arguments are integers from 1 to 9; this predicate is a function

$$p : (A_9)^3 \rightarrow \{\mathtt{true}, \mathtt{false}\}.$$

Formula to describe that every row i contains every number:

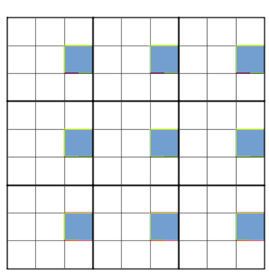
$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, k) = \forall i \in A_9,\ \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, k).$$

This formula describes that each 3×3 block contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n). \qquad (1)$$

(A) Count the number of conjunctions ($\wedge$), disjunctions ($\vee$) and predicates ($p(\ldots)$) in the formula (**??**).

(B) Similarly to the formula (**??**), write a Boolean formula to describe the following "rule" for sudoku tables: If there are any cells in the sudoku cells that have their row number difference AND column number difference divisible by 3, then they contain different numbers. For example, in the figure below, all the shaded cells have same relative position within their respective 3×3 blocks – therefore their row/column numbers differ by $0, 3, 6$, and accordingly to our new "rule" they should contain all nine different numbers.



Question 2: Long set operations. Denote $A_1 = \{1\}$, $A_2 = \{1, 2\}$, etc. In general, $A_k = \{1, 2, \ldots, k\}$.
By $A \oplus B = (A - B) \cup (B - A)$ we denote the symmetric difference: All elements that belong to just one of the sets A, B (but not the other one). Consider this set:

$$S = \bigoplus_{j=1}^{100} A_j = A_1 \oplus A_2 \oplus \ldots \oplus A_{100}.$$

Write a comma-separated list of the 10 smallest elements of S in increasing order.

Question 3: Using recurrent formula. Find the first 5 members of this sequence:

$$\begin{cases} f(0) = 1, \\ f(1) = 4, \\ f(n) = f(n-1) \cdot f(n-2) + 1, \ \forall n \geq 2. \end{cases}$$

Write comma-separated values $f(0), f(1), f(2), f(3), f(4)$.

Question 4: Reccurent sequence. A sequence of real numbers $f : \mathbb{N} \rightarrow \mathbb{R}$ satisfies the following properties:
(A) $f(k + 2) = f(k) + f(k + 1)$ for all integers $k \geq 2$.
(B) $f(n)$ is a growing geometric progression: Namely $f(1) = f(0) \cdot q$, $f(2) = f(0) \cdot q^2$ and so on.
Find the quotient of this geometric progression. Round it to the nearest thousandth (i.e. specify the first three digits after the decimal point).

Question 5: Finding a limit. Define the following sequence:

$$\begin{cases} x_0 = 1, \\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \text{ if } n \geq 0 \end{cases}$$

Assume that there exists limit $L = \lim_{n\rightarrow\infty} x_n$.
Find that limit L and round it to the nearest thousandth.

Question 6: Find recurrent formulas. We define three sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ explicitly. Find their recurrent formulas that allow to find next members of the sequence in terms of the previous ones.

- $a_n = 2^{\frac{1}{2^n}}$, where $n \geq 0$.
- $b_0 = 1$, $b_1 = 111$, $b_2 = 11111$, $b_3 = 1111111$, etc. (In general, the kth member b_k has $2k + 1$ digits "1" in its decimal notation).
- $c_n = n^2 + n$, where $n \geq 0$.

Initial member	**Recurrent expression**
$a_0 = \ldots$	$a_{n+1} = \ldots$ (express via a_n etc.)
$b_0 = \ldots$	$b_{n+1} = \ldots$
$c_0 = \ldots$	$c_{n+1} = \ldots$

Question 7: Taylor series There is a formula known from calculus (practically used to compute $y = \sin x$) for each $x \in \mathbb{R}$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Use Python or Scala to add the first 20 terms of this infinite sum to compute $\sin 30°$. Round the answer to the nearest thousandth. (Taylor series expects to have argument x in radians, so you have to convert degrees to radians before using the formula.)

Answers

Question 1. Answers:
(A1) $80 = 3 \cdot 3 \cdot 9 - 1$ conjunctions.
(A2) $648 = 81 \cdot 8$ disjunctions,
(A3) $3 \cdot 3 \cdot 9 \cdot 3 \cdot 3 = 729$ instances of predicate p.
(B) $\bigwedge_{i=1}^{3} \bigwedge_{j=1}^{3} \bigwedge_{n=1}^{9} \bigvee_{r=0}^{2} \bigvee_{s=0}^{2} p(3r+i, 3s+j, n)$.

(A1) To count conjunctions, expand the outer three big conjunctions in this expression:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n).$$

At the first step we expand the outermost $\bigwedge_{r=0}^{2}$ would get three similar subexpressions:

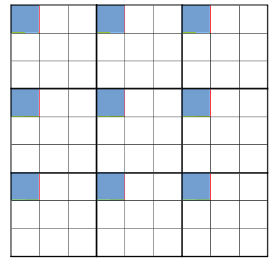
$$\left(\text{let } r=0 \text{ in } \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)\right) \bigwedge$$

$$\left(\text{let } r=1 \text{ in } \ldots\right) \bigwedge \left(\text{let } r=2 \text{ in } \ldots\right).$$

Then we expand the next large conjunction for $s = 0, 1, 2$. We would get 3 smaller terms inside every big parentheses:

$$\big((\ldots)\wedge(\ldots)\wedge(\ldots)\big)\wedge\big((\ldots)\wedge(\ldots)\wedge(\ldots)\big)\wedge\big((\ldots)\wedge(\ldots)\wedge(\ldots)\big).$$

So far we have 8 conjunctions: 2 of them are between the big parentheses, and $3 \cdot 2 = 6$ more conjunctions separate the smaller expressions $(\ldots)$. After that we are ready to expand each smaller $(\ldots)$ into a conjunction of 9 terms for $n = 1, 2, \ldots, 9$.
In all we have $3 \cdot 3 \cdot 9 = 81$ expressions to separate with conjunction symbols ($\wedge$); we need $81 - 1 = 80$ such separators.
(A2) The number of disjunctions can be calculated in a similar manner: we have 81 subexpressions $\bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$ for all different combinations of (r, s, n). Each subexpression has 8 disjunctions.
(A3) The number of predicate instances can be counted by multiplying the number of elements in all five loops: $3 \cdot 3 \cdot 9 \cdot 3 \cdot 3$.
(B) Let us express that all the cells in the upper right corners of all nine 3×3 blocks contain different numbers. One of such cells is $(i, j) = (1, 1)$ (all the others can be obtained by adding 3 or 6 to i and/or j - thus we get nine values $(1,1)$,$(1,4)$,$(1,7)$, $(4,1)$,$(4,4)$,$(4,7)$,$(7,1)$,$(7,4)$,$(7,7)$). Let us express the statement that all the numbers $n = 1, \ldots, 9$ should appear in the shaded cells (see figure):

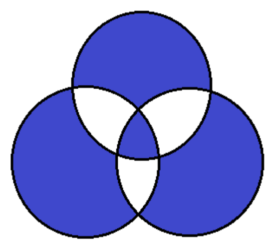


let $i = 1, j = 1$ in $\forall n \in \{1, \ldots, 9\} : \bigvee_{r=0}^{2} \bigvee_{s=0}^{2} p(3r+i, 3s+j, n)$.

If we want this to be true for any pair (i, j), where both i and j take all values $1, 2, 3$:

$$\bigwedge_{i=1}^{3} \bigwedge_{j=1}^{3} \bigwedge_{n=1}^{9} \bigvee_{r=0}^{2} \bigvee_{s=0}^{2} p(3r+i, 3s+j, n).$$

Question 2.
Answer: `2,4,6,8,10,12,14,16,18,20`



Notice that any symmetric difference of two or more sets includes only those elements that belong to an odd number of sets. For example, $A_1 \oplus A_2 \oplus A_3$ consists of the elements that belong to exactly one set (A_1, A_2 or A_3), or all three sets: $A_1 \cap A_2 \cap A_3$. If you do not believe this, it can be proven by induction.
Therefore the set $S = \bigoplus_{j=1}^{100} A_j$ contains just those elements that belong to an odd number of A_j. Number 1 belongs to all 100 sets (therefore $1 \notin S$). But number 2 belongs to 99 sets A_j (and therefore $2 \in S$). Similarly $4 \in S$, $6 \in S$, and so on.

Question 3: Answer: `1,4,5,21,106`
We can verify that $f(2) = f(0) \cdot f(1) + 1 = 5$ and so on.

Question 4: Answer: `1.618`
On one hand $f(k+2) = f(k) + f(k+1)$. On the other hand, $f(k) = f(0) \cdot q^k$ etc. If we plug into the above formula, we should get:

$$f(0) \cdot q^{k+2} = f(0) \cdot q^k + f(0) \cdot q^{k+1}. \qquad (2)$$

Since $f(k)$ is strictly increasing, we must have $f(0) \neq 0$ and $q > 0$. Namely, the first member in the geometric progression is nonzero, and the quotient is positive.

Therefore the equation (**??**) can be simplified by dividing both sides with $f(0) \cdot q^k$:

$$q^2 = 1 + q.$$

This quadratic equation has two roots:

$$q_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Only one of these roots is positive. It is approximately equal to 1.618.

Note. This quotient $q \approx 1.618$ is called *golden ratio*. For the regular Fibonacci numbers the ratio F_{n+1}/F_n approaches this number in the limit when $n \to \infty$.

Question 5: Answer: `1.414`

Since we know that the sequence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ has a limit, then both x_{n+1} and x_n go to the same limit as n grows. We have this identity:

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

and therefore $L^2 = 2$ and $L = \pm\sqrt{2}$. Since we start with a positive member, all subsequent members will be positive and $L = \sqrt{2}$.

Note. This recurrence is used in practice to calculate square-roots. It converges very fast. Here are the first few members with 15 decimal digits. After six steps we have square root with more than 15-digit precision.

```
x(0) = 1/1 = 1.000000000000000
x(1) = 3/2 = 1.500000000000000
x(2) = 17/12 = 1.416666666666667
x(3) = 577/408 = 1.414215686274510
x(4) = 1.414213562374690
x(5) = 1.414213562373095
x(6) = 1.414213562373095
```

Question 6: Answers:

Initial member	**Recurrent expression**
$a_0 = 2$	$a_{n+1} = \sqrt{a_n}$
$b_0 = 1$	$b_{n+1} = 100 \cdot b_n + 11$
$c_0 = 0$	$c_{n+1} = c_n + 2 \cdot (n+1)$

(A) Notice that the power in a_n decreases twice with each step. So it is equivalent to taking a square-root.

(B) Decimal representation of b_{n+1} is easy to obtain from b_n: We first shift all the digits two places to the left (multiply by 100), then add 11 to get the last two digits equal to "11".

(C) Compute the first few members of the sequence c_n: 0, 2, 6, 12, 20. Their differences make an arithmetic progression: $2 - 0 = 2$, $6 - 2 = 4$, $12 - 6 = 6$, etc. So it is possible to compute the next member by adding $2(n+1)$ (the nth even number).

Question 7: