Sample Quiz for Week02

Discrete Structures, Fall 2020

RBS

Question 1. Let $a, b \in \mathbb{Z}^+$ be two positive integers. Translate into predicate logic: "d is the greatest common divisor of a, b." (That is, the greatest number that divides both a and b).

Note 1. "Translate into predicate logic" means - use predicates and quantifiers to express that statement.

Note 2. Use "infix" notation for common predicates: write $a \mid b$ whenever a divides b; write $x < y$, if x is less than y. For example, $3 \mid 6$ (3 divides 6) is preferred compared to "$\texttt{divides}(3, 6)$". Also $10 < 17$ is more readable than "$\texttt{lessThan}(\texttt{10}, \texttt{17})$".

Question 2. Use quantifiers to write a statement to tell that a quadratic function $f(x) = ax^2 + bx + c$ has two different integer roots.

Question 3. Assume that for some argument `x`, Python functions `a(x)` and `b(x)` return value `True`, but other two functions `c(x)` and `d(x)` return value `False`. Which functions are called and evaluated, if you run the following conditional statement:

```
if (a(x) or b(x)) and (c(x) & d(x)):
    ## ... some Python code ...
```

Note. Note that Boolean operators `and`, `or` use *short-circuit evaluation*, but operators `&`, `|` do not.

Question 4. There is a set C of several children and a set H of several hats. There is a predicate $W(c, h)$ which is `True` iff the child $c \in C$ has ever worn the hat $h \in H$.

(a) Write the domain set and the range set of the predicate function W.

(b) Use quantifiers to write a statement: "Every two children have at least one hat in common (that is, both of them have worn it)."

Question 5. Let $\mathcal{H}$ be the set of all humans, and the predicate $F(x, y)$ is true iff x is the father of y. Express these sentences in plain English:

(a) $\forall y \in \mathcal{H}\ \exists x \in \mathcal{H},\ F(x, y)$.

(b) $\exists x \in \mathcal{H}\ \forall y \in \mathcal{H},\ F(x, y)$.

(c) Which of the two statements **(a)**, **(b)** (if any) is true, if we replace the non-empty domain of all humans $\mathcal{H}$ by an empty domain $\mathcal{Z} = \emptyset$ of all zombies?

Question 6. We define predicates $P(x, y)$, $Q(x, y)$ with two arguments, where x, y can be any of the three letters: $\mathtt{a}, \mathtt{b}, \mathtt{c}$. (The type for all these predicates is $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^2 \to \{\mathtt{T}, \mathtt{F}\}$.)

They have these following truth tables:

P	a	b	c
a	T	F	T
b	F	F	F
c	T	T	T

Q	a	b	c
a	F	T	F
b	F	F	T
c	T	F	F

Find the truth values of these statements:

(a) $\exists x \forall y, P(y, x)$

(b) $\exists x \forall y, P(x, y)$

(c) $\exists x \exists y, Q(x, y)$

(d) $\forall x \exists y, Q(x, y)$

Note. In all truth tables the first argument is represented by row, the second is represented by column. For example $P(\mathtt{b}, \mathtt{c}) = \mathtt{F}$ (2nd row and 3rd column). Meanwhile $P(\mathtt{c}, \mathtt{b}) = \mathtt{T}$ (3rd row and 2nd column).

Question 7. For a real number x we know that $\lfloor x \rfloor \neq \lfloor x + 0.5 \rfloor$. Write this statement in predicate logic without using any $\lfloor \ldots \rfloor$ notation. (Write instead that x and $x + 0.5$ have some integer number betwen x and $x + 0.5$.)

Note. By $\lfloor x \rfloor$ we denote the largest integer number that does not exceed x. For example $\lfloor 3.14 \rfloor = 3$, $\lfloor 17 \rfloor = 17$, $\lfloor -4.5 \rfloor = -5$.

Answers

Question 1. Answer:

$$\boxed{\forall k \in \mathbb{Z}^+,\ (k \mid a \wedge k \mid b) \rightarrow k \leq d.}$$

Recite: "For all positive integers k, if k divides both a and b, then k does not exceed $d = \gcd(a, b)$."
This also means that d is the largest number among all common divisors of a, b (*greatest common divisor*, GCD).
Note: In the above formula a, b, d are "free variables" (they need to be assigned independently; and if $d \neq \gcd(a, b)$, then the statement is false). On the other hand, k is a "bound variable". You can rename k into x – and nothing will change.

Question 2. Answer:

$$\begin{aligned} &\exists x_1 \in \mathbb{Z}^+ \exists x_2 \in \mathbb{Z}^+ \forall x_3 \in \mathbb{Z}^+, \\ &(x_1 \neq x_2 \wedge f(x_1) = 0 \wedge f(x_2) = 0) \wedge \\ \wedge\, &(f(x_3) = 0 \rightarrow x_3 = x_1 \vee x_3 = x_2)\,. \end{aligned}$$

Recite: "There exist two positive integers x_1, x_2 such that they are different and both of them are roots of $f(x) = 0$ and, furthermore, for any other root x_3, it equals either x_1 or x_2."
Note: The above statement says that there are *exactly* two roots x_1 and x_2. You can easily write a modified statement saying that the equation $f(x) = 0$ has *at least* two roots – in this case you can skip the x_3 part. For quadratic equations it is the same (since no equation has more than 2 roots), but the idea expressed here is slightly different:

$$\begin{aligned} &\exists x_1 \in \mathbb{Z}^+ \exists x_2 \in \mathbb{Z}^+ \\ &(x_1 \neq x_2 \wedge f(x_1) = 0 \wedge f(x_2) = 0)\,. \end{aligned}$$

Question 3. Answer:
The following functions are called: `a(x)`, `c(x)`, `d(x)`.
Function `b(x)` is not called, because short-circuit operator **or** skips the second argument in "`a(x) or b(x)`", if the first argument is `True`.

Question 4. Answer: **(a)**

$$\boxed{W\ :\ C \times H \rightarrow \{\mathtt{T}, \mathtt{F}\}.}$$

Note. In Coq the set of "True" and "False" is denoted by `Prop` or $\{\mathtt{T}, \mathtt{F}\}$.
(b)

$$\boxed{\forall c_1 \in C\ \forall c_2 \in C\ \exists h \in H, W(c_1, h) \wedge W(c_2, h).}$$

Every two children have at least one hat in common (that is, both of them have worn it).

Question 5. Answers:
(a) Every human has a father.
(b) There exists someone, who is the father of everyone.
(c) Both (a) and (b) are false for empty sets.
Every time we write $\exists x \in \mathcal{Z}$ regarding anything (where $\mathcal{Z} = \emptyset$ – an empty set) it is false.

On the other hand, this statement is `True`:

$$\forall x \in \mathcal{Z}\ \forall y \in \mathcal{Z},\ F(x, y) \equiv \mathtt{True}.$$

Recite: "In the (empty) set $\mathcal{Z}$ of all zombies, any zomby is a father of any other zomby."
Note. For empty domains we do not need to know anything about the predicate values, because their truth tables have zero rows and zero columns. You can also safely state the negation: "No zomby is a father of another zomby."

$$\forall x \in \mathcal{Z}\ \forall y \in \mathcal{Z},\ \neg F(x, y) \equiv \mathtt{True}.$$

Question 6. Answers:
(a) False. In the truth table of P, there is NO column (denoted by x) containing `T` in every row.
(b) True. In the truth table of P there is a row x (it is the last row $x = \mathtt{c}$), containing `T` in every column.
(c) True. We can indeed make $Q(x, y)$ true, if we pick row and column. For example $Q(\mathtt{c}, \mathtt{a}) = \mathtt{T}$.
(d) True. For any row in the truth table of Q, we can find at least one `T` in some column.

Question 7. Answer:

$$\exists k \in \mathbb{Z}, x < k \wedge x + 0.5 \geq k.$$

In this example $\lfloor x \rfloor = k - 1$, but $\lfloor x + 0.5 \rfloor = k$, so they are different.