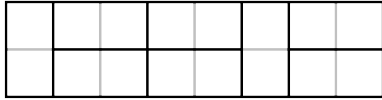


## Discrete Quiz 9

**Question 1.** You have a rectangular chocolate bar of  $2 \times n$  squares. Denote by  $a(n)$  the ways how you can divide the chocolate bar into little dominoes (rectangles  $1 \times 2$ ). For example,  $a(1) = 1$  and  $a(2) = 2$ .



In your answer write the recurrent expression of  $a(n)$  (express it through the previous members of the sequence:  $a(n-1), \dots$ ).

*Note.* Assume that all the little chocolate squares are distinguishable; the cuts that differ by some symmetry (rotation or flip of the chocolate bar) are considered to be different.

**Question 2.** Somebody wants to find out, in how many ways it is possible to pay \$15 in a vending machine, using \$1 coins, \$1 bills, \$2 bills and \$5 bills, where the order, how you insert them into the machine matters. In your answer write an integer number.

*Note.* In order to solve this, you probably want to define a recurrent sequence for  $b_n$  and find  $b_{15}$ . People sometimes use characteristic functions for such problems as well: see <https://bit.ly/2Qu4Re4>. Then they can solve some variations of this problem (what happens, if you have limited number of \$5 bills, etc.), but they need to use infinite power series and other calculus techniques.

**Question 3.** Consider a recurrent sequence:

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_n = 2a_{n-1} + 2a_{n-2}, \quad n \geq 2. \end{cases}$$

Find both roots of the characteristic equation  $r_1, r_2$ .

In your answer write two real numbers, round them to the nearest thousandth.

**Question 4** There is a sequence  $a(n)$  such that  $a(0) = 0$ ,  $a(1) = 0$ ,  $a(2) = 1$ , but its characteristic equation is  $(r-1)^2(r-2) = 0$ . (See <https://bit.ly/3a72Ps0> where the characteristic equations with repeated roots are explained.) Find the value  $a(8)$ .

In your answer write a number.

**Question 5** Write the following expression:

$((A / (B - (C + D))) * E) - (F + (G + H))$  in prefix (Polish) notation and also in the postfix (reverse-Polish) notation.

In your answer separate both expressions with a comma.

**Question 6** Find the number of ways to parenthesize the following expression:

$$A / B - C + D * E - F + G + H$$

You should not assume any associativity for the operations. For example  $(F + G) + H$  and  $F + (G + H)$  are two different ways to insert parentheses. On the other hand  $(F + G) + H$  and  $((F + (G)) + H)$  is the same way, since the order of execution in both is the same.

In your answer write an integer number.

**Question 7.** Suppose  $f(n) = 3f(n/3) + 2n$ ,  $f(1) = 1$ . Find  $f(3^8)$ .

In your answer write an integer number.

## Answers

**Question 1.** Answer:  $a(n) = a(n-1) + a(n-2)$

Notice that the leftmost two squares in the  $2 \times n$  rectangle can be filled in two different ways:

**Alternative 1:** They can be filled by a single vertical domino. In this case the remaining rectangle  $2 \times (n-1)$  can be filled in  $a(n-1)$  ways.

**Alternative 2:** They can be filled by two horizontal dominoes. In this case the remaining rectangle  $2 \times (n-2)$  can be filled in  $a(n-2)$  ways.

The total number of ways is obtained by adding  $a(n-1)$  and  $a(n-2)$ .

*Note.* The sequence  $a(n)$  is a shifted Fibonacci sequence. We can easily verify that  $a(n) = F_{n+1}$  for  $n \geq 0$ . (We define  $F_0 = 0$ ,  $F_1 = 1$  and  $F_n = F_{n-1} + F_{n-2}$  is the regular Fibonacci sequence.)

**Question 2.** Answer: 527383

Let us define the sequence  $b_n$  (in how many ways you can insert \$1 coins, \$1 bills, \$2 bills and \$5 bills into a vending machine so that it adds up to  $n$  dollars).

But before that we consider a simpler sequence  $a_n$  (in how many ways you can insert \$1 coins, \$1 bills, \$2 bills into a vending machine to pay  $n$  dollars – i.e. you do not use \$5 bills at all).

Sequence  $a_n$  was already discussed in the *Sample Quiz 9*, Problem 2. It is defined as a recurrent sequence:

$$\begin{cases} a_1 = 2, \\ a_2 = 5, \\ a_n = 2a_{n-1} + a_{n-2}, \quad n \geq 3. \end{cases}$$

If you wish, you can also define  $a_0 = 1$  (there is exactly one way how to pay \$0 – use zero instances of every coin and bill). Compute the initial members of this sequence:

$$(a_1, a_2, a_3, a_4) = (2, 5, 12, 29).$$

Notice that the sequence  $b_n$  (where we also allow \$5 bills) would have exactly same members up to  $b_4$  (because sums up to 4 dollars cannot use any \$5 bills anyway). Further members of  $b_n$  can be defined recursively:

$$\begin{cases} b_0 = 1, \\ b_1 = 2, \\ b_2 = 5, \\ b_3 = 12, \\ b_4 = 29, \\ b_n = 2b_{n-1} + b_{n-2} + b_{n-5}, \quad n \geq 5. \end{cases}$$

The last (recurrent) line means that  $b_n$  ( $n \geq 5$ ) is a total of four different kinds of sequences:

- Some sequences start from a \$1 coin; the rest is paid in  $b_{n-1}$  different ways.

- Some sequences start from a \$1 bill; the rest is paid in  $b_{n-1}$  different ways.
- Some sequences start from a \$2 bill; the rest is paid in  $b_{n-2}$  different ways.
- Some sequences start from a \$5 bill; the rest is paid in  $b_{n-5}$  different ways.

Here are the first members  $b_0, \dots, b_{15}$  of this sequence: 1, 2, 5, 12, 29, 71, 173, 422, 1029, 2509, 6118, 14918, 36376, 88699, 216283, 527383.

**Question 3.** Answer:

$$2.732, -0.732 \text{ or } -0.732, 2.732$$

The characteristic equation is

$$r^2 - 2r - 2 = 0.$$

And the roots of this square equation are  $r_{1,2} = 1 \pm \sqrt{3}$ . By rounding them to the nearest thousandth, we get the answer.

*Note.* By the way, we can also find the formula for  $a_n$ . (See *Sample Quiz 9*, Problem 3, (B).)

**Question 4** Answer: 247

Rewrite the characteristic equation:

$$\begin{aligned} (r-1)^2(r-2) &= (r^2 - 2r + 1)(r-2) = \\ &= r^3 - 2r^2 + r - 2r^2 + 4r - 2 = \\ &= r^3 - 4r^2 + 5r - 2 = 0. \end{aligned}$$

We can restore the recurrent relationship from here:

$$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}.$$

$n$	0	1	2	3	4	5	6	7	8
$a_n$	0	0	1	4	11	26	57	120	247

**Question 5** Answer:

$$-*/A-B+CDE+F+GH, \quad ABCD+-/E*FGH++-$$

In order to transform infix notation into postfix notation we can do this step by step. We start with the last/outermost operation (minus in our case). And leave both subexpressions in the infix form. Then we find the last/outermost operation the subexpressions and so on.

$$\begin{aligned} &((A/(B-(C+D))) * E) - (F+(G+H)) \\ &- ((A/(B-(C+D))) * E) (F+(G+H)) \\ &- * (A/(B-(C+D))) E (F+(G+H)) \\ &- * / A (B-(C+D)) E (F+(G+H)) \\ &- * / A - B (C+D) E (F+(G+H)) \\ &- * / A - B + C D E (F+(G+H)) \\ &- * / A - B + C D E + F (G+H) \\ &- * / A - B + C D E + F + G H \end{aligned}$$

In these expressions the regular (infix) arithmetic operations are shown in black, but prefix arithmetic operations are shown in red. Postfix transformation is very similar.

**Question 6** Answer: 429

The expression  $A/B-C+D \cdot E-F+G+H$  contains 7 operations and 8 operands/letters. It can be parenthesized in  $C_7 = 429$  ways, where  $C_n$  is the sequence of *Catalan numbers* defined recursively:

$$\begin{cases} C_0 = 1, \\ C_{n+1} = \sum_{i=0}^n C_i \cdot C_{n-i} \end{cases}$$

Here are the first few members:

$C_0 = 1$ ,  $C_1 = 1$ ,  $C_2 = 2$ ,  $C_3 = 5$ ,  $C_4 = 14$ ,  $C_5 = 42$ ,  
 $C_6 = 132$ ,  $C_7 = 429$ ,  $C_8 = 1430$ ,  $C_9 = 4862$ ,  
 $C_{10} = 16796$ .

**Question 7** Answer: 111537

We can compute the subsequent values, using the recursive formula.

$n$	$f(n)$
$3^0$	1
$3^1$	$9 = 3 \cdot 3$
$3^2$	$45 = 5 \cdot 9$
$3^3$	$189 = 7 \cdot 27$
$3^4$	$729 = 9 \cdot 81$
$3^5$	$2673 = 11 \cdot 243$
$3^6$	$9477 = 13 \cdot 729$
$3^7$	$32805 = 15 \cdot 2187$
$3^8$	$111537 = 17 \cdot 6561$

It is possible to prove by induction that

$$f(n) = (2n + 1) \cdot 3^n.$$

We can also apply this theorem:

**Master Theorem:** Let  $f$  be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever  $n = b^k$ , where  $k$  is a positive integer greater than 1, and  $c$  and  $d$  are real numbers with  $c$  positive and  $d$  nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

In our case  $a = 3$ ,  $b = 3$ ,  $c = 2$  and  $d = 1$ . Observe that  $a = b^d$ , therefore we get that  $f(n)$  is in  $O(n \log n)$ .

For example, if  $n = 3^8 = 6561$ , then  $f(n) = 6561 \cdot 17$ . We clearly see both factors: 6561 grows as  $O(n)$ , but  $17 = 2 \cdot (\log_3 6561) + 1$  grows as  $O(\log n)$  (we do not care about the base of a logarithm in the Big-O notation). So their product  $f(n)$  grows as  $O(n \log n)$ .