# Discrete Sample Quiz 4

**Question 1.** We define three sets in the universe $U$ of integer numbers between 1 and 70 (inclusive):

$$\begin{cases} K_2 &= \{x \in U \mid 2 \mid x\}, \\ K_5 &= \{x \in U \mid 5 \mid x\}, \\ K_7 &= \{x \in U \mid 7 \mid x\}, \end{cases}$$

Find the size of the following sets. Here $|X|$ denotes the number of elements in a finite set (BTW, $|X|$ is also the notation for the cardinality of an infinite set).
*Note.* It was not intentional, but vertical bar: $|$ in this exercise happens to be used in three different ways: It is a separator when defining sets $K_2$; it is used to denote divisibility; it also denotes set cardinality.

$|K_2 \cup K_5|$ ...
$|K_2 \cap K_7|$ ...
$\left|\overline{K_7}\right|$ ...
$\left|\overline{K_2 \cap K_5}\right|$ ...
$|K_2 \cap K_5 \cap K_7|$ ...
$\left|K_2 \cap K_5 \cap \overline{K_7}\right|$ ...
$|K_2 \cup K_5 \cup K_7|$ ...
$\left|\overline{K_2 \cup K_5 \cup K_7}\right|$ ...

**Question 2.** Find a counterexample to refute the following predicate expression:
$(\exists x \in U,\ P(x)) \wedge (\exists x \in U,\ Q(x)) \to$
$\to \exists x \in U,\ (P(x) \wedge Q(x))$.
Here $P(x)$ is true iff $P$ is a full square (a square of some integer number), $Q(x)$ is true iff $x$ is divisible by 5, and $U$ is the set of all integers from the interval $[120; 130]$.
*Note.* The three $x$'s in this formula refer to three unrelated (local) variables. If it looks confusing, you can rewrite it like this:
$(\exists x_1 \in U,\ P(x_1)) \wedge (\exists x_2 \in U,\ Q(x_2)) \to$
$\to \exists x_3 \in U,\ (P(x_3) \wedge Q(x_3))$.
**(A)** Identify the variables which you need to pick for your counter-example.
**(B)** Pick the values for these variables to make the above statement false.

**Question 3.** Determine the cardinality of the following sets (some finite number? equal to $|\mathbb{N}|$? equal to $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$? equal to $|\mathcal{P}(\mathbb{R})|$?)

(A) The set of positive real numbers from $(0; 1)$ with decimal representation containing only digits 0 and 1?

(B) The set of positive real numbers from $(0; 1)$ with decimal representation containing only digits 0 and 1 (and it is known that the number of 0s is finite)?

(C) The set of positive real numbers from $(0; 1)$ that are fully periodic decimal fractions with a period of 2020 digits?

(D) The set of positive real numbers from $(0; 1)$ that have decimal representation without any digits "9"?

(E) The set of all irrational $x \in (0; 1)$ such that $x^3$ is rational?

(F) Ordered pairs of real numbers $x_1, x_2$ such that $x_1, x_2 \in (0; 1)$.

(G) Finite sequences of real numbers: $x_1, \ldots, x_n$, and all $x_i \in (0; 1)$? (Here $n$ can be any positive integer)?

(H) Infinite sequences of real numbers from $(0; 1)$: $\{x_n\}$: $x_1, x_2, x_3, \ldots$.

**Question 4.** Let $f(x) = \left\lfloor \frac{x^3}{3} \right\rfloor$. Find $f(S)$ if $S$ is:

(A) $S = \{2, 1, 0, 1, 2, 3\}$.

(B) $S = \{0, 1, 2, 3, 4, 5\}$.

(C) $S = \{1, 5, 7, 11\}$.

Is function $f : \mathbb{Z} \to \mathbb{Z}$ injective? Is it surjective? If it is not, mention counterexamples to show this.

**Question 5.** Determine, if the given set is a powerset of some other set. If yes, which one?

(A) $\{\emptyset,\ \{\emptyset\},\ \{a\},\ \{\{a\}\},\ \{\{\{a\}\}\},\ \{\emptyset, a\},\ \{\emptyset, \{a\}\},\ \{\emptyset, \{\{a\}\}\},$
$\{a, \{a\}\},\ \{a, \{\{a\}\}\},\ \{\{a\}, \{\{a\}\}\},\ \{\emptyset, a, \{a\}\},\ \{\emptyset, a, \{\{a\}\}\},$
$\{\emptyset, \{a\}, \{\{a\}\}\},\ \{a, \{a\}, \{\{a\}\}\},\ \{\emptyset, a, \{a\}, \{\{a\}\}\}\}$.

(B) $\{\emptyset, \{a\}\}$.

(C) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$.

(D) $\{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}$.

(E) $\{\emptyset, \{a, \emptyset\}\}$.

**Question 6.** Given two sets $A = \{x, y\}$ and $B = \{x, \{x\}\}$, check, if statements are true or false:

(A) $x \subseteq B$.

(B) $\emptyset \in \mathcal{P}(B)$.

(C) $\{x\} \subseteq A - B$.

(D) $|\mathcal{P}(A)| = 4$.

**Question 7.** We define functions $g : A \to A$ and $f : A \to A$, where $A\{1, 2, 3, 4\}$ by listing all argument-value pairs:

$$g = \{(1, 4), (2, 1), (3, 1), (4, 2)\}, \quad f = \{(1, 3), (2, 2), (3, 4), (4, 1)\}.$$

Find these functions by listing their argument/value pairs (or establish that they do not exist).

(A) Find $f \circ g$.

(B) Find $g \circ f$.

(C) Find $g \circ g$.

(D) Find $g \circ (g \circ g)$.

(E) Find $f^{-1}$.

(F) Find $g^{-1}$.

**Question 8.** Find these sums:

(A) $1/4 + 1/8 + 1/16 + 1/32 + \ldots$.

(B) $2 + 4 + 8 + 16 + 32 + \ldots + 2^{28}$.

(C) $2 - 4 + 8 - 16 + 32 - \ldots - 2^{28}$.

(D) $1 - 1/2 + 1/4 - 1/8 + 1/16 - \ldots$.

**Question 9.** Find an appropriate $O(g(n))$ for each function $f(n)$ defined below (pick your $g(n)$ to be the slowest growing among the functions such that $f(n)$ is in $O(g(n))$).

(A) $f(n) = 1^2 + 2^2 + \ldots + n^2$.

(B) $f(n) = \dfrac{3n - 8 - 4n^3}{2n - 1}$.

(C) $f(n) = \sum_{k=1}^{n} k^3$.

(D) $f(n) = \dfrac{6n + 4n^5 - 4}{7n^2 - 3}$.

(E) $f(n) = \sum_{k=2}^{n} k \cdot (k - 1)$.

(F) $f(n) = 3n^2 + 8n + 7$

**Question 10.** For the given functions, find an optimal $O(g(n))$; find $C$ and $n_0$ (from the definition $|f(n)| < C \cdot |g(n)|$ as long as $n > n_0$).

(A) $f(n) = 3n^4 + \log_2 n^8$.

(B) $f(n) = \sum_{k=1}^{n} (k^3 + k)$.

(C) $f(n) = (n + 2)\log_2(n^2 + 1) + \log_2(n^3 + 1)$.

(D) $f(n) = n^3 + \sin n^7$.

**Question 11.** This is a Python fragment; variable `n` can become very large; `t` is some fixed parameter. Denote by $f(n)$ the number of operations depending on the variable `n`, where an operation is an addition or a multiplication, or raising to the power 2. Find the slowest growing $g(n)$ so that $f(n)$ is in $O(g(n))$.

```
sum = 0
for i in range(1,n+1):
    for j in range(1,n+1):
        sum += (i*t + j*t + 1)**2
```

**Question 12.** There are two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined for all real numbers and taking real values. Find, which predicate logic expressions describe a statement that is logically equivalent to the English sentence "The function $f(n)$ is in $O(g(n))$".
*Note.* There may be multiple correct answers.

(A) $\forall n \in \mathbb{R}\ \exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}$, $(n > n_0 \to |f(n)| \le C \cdot |g(n)|)$.

(B) $\exists n_0 \in \mathbb{R}\ \forall n \in \mathbb{R}\ \exists C \in \mathbb{R}$, $(n > n_0 \to |f(n)| \le C \cdot |g(n)|)$.

(C) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$, $(n > n_0 \to |f(n)| \le C \cdot |g(n)|)$.

(D) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$, $(n > n_0 \to f(n) \le C \cdot |g(n)|)$.

(E) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$, $(n > n_0 \to |f(n)| \le C \cdot g(n))$.

(F) $\exists n_0 \in \mathbb{R}\ \exists C \in \mathbb{R}\ \forall n \in \mathbb{R}$, $(n \ge n_0 \to |f(n)| < C \cdot |g(n)|)$.

(G) $\exists n_0 \in \mathbb{Z}^+\ \exists C \in \mathbb{Z}^+\ \forall n \in \mathbb{R}$, $(n > n_0 \to |f(n)| \le C \cdot |g(n)|)$.