MIDTERM, FRIDAY, 2022-02-25

You must justify all your answers to recieve full credit.

Question 1: Consider the following Boolean expression:

$$E(A,B,C) \equiv (A \rightarrow B) \wedge (B \rightarrow C)$$

(A) Find the negation $\neg E(A,B,C)$ – in your answer all three Boolean variables (A,B,C) and any of the operator(s) $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus)$ can appear, but all the negations should be only applied to variables (rather than to expressions in parentheses).

(B) Assume that Boolean variable is assigned $B = \texttt{True}$. Express $E(A, \texttt{True}, C)$ and simplify this expression. After simplification it only contains the remaining variables, but no references to the Boolean constants `True` or `False`.

Answer:

(A) Apply the negation to $E(A,B,C)$ and then transform, using Boolean identities:

$$\begin{aligned} \neg E(A,B,C) \equiv \neg((A \rightarrow B) \wedge (B \rightarrow C)) \equiv \\ \equiv \neg(A \rightarrow B) \vee \neg(B \rightarrow C) \equiv \\ \equiv \neg(\neg A \vee B) \vee \neg(\neg B \vee C) \equiv \\ \equiv (\neg\neg A \wedge \neg B) \vee (\neg\neg B \wedge \neg C) \equiv \\ \equiv (A \wedge \neg B) \vee (B \wedge \neg C). \end{aligned}$$

The first step uses De Morgans law, the next step rewrites implication $(\rightarrow)$ as a disjunction $(\vee)$, the third step uses De Morgans law again (to the inner parentheses), the final step applies the double negation identity (cancels out negation of negation).

(B) Plug in the value for B:

$$E(A, \texttt{True}, C) \equiv (A \rightarrow \texttt{True}) \wedge (\texttt{True} \rightarrow C) \equiv \texttt{True} \wedge C \equiv C.$$

Question 2: Describe the given sets using the set-builder notation (in the following form $\{x \in \ldots \mid \ldots\}$), where one can specify the universe, which encloses all the elements x in this set, and some predicate – a property which must be satisfied by elements in this universe x in order to belong to this set.

(A) $S_1 = \{-100, -98, -96, -94, \ldots, 94, 96, 98, 100\}$

(B) $S_2 = \{1, 81, 121, 361, 441, \ldots, 1681\}$ (full squares up to 2000 ending with digit 1).

Answer:

(A) Even number is a number congruent to 0 (modulo 2). (Or, you can also use a different formalization: Even numbers are those n for which there is some k such that $n = 2k$):

$$S_1 = \{n \in \mathbb{Z} \ \mid \ \exists k \in \mathbb{Z}\ (2k = n) \ \wedge \ n \geq -100 \ \wedge \ n \leq 100\}$$

or

$$S_1 = \{n \in \mathbb{Z} \mid n \equiv 0 \bmod 2 \ \wedge \ n \geq -100 \ \wedge \ n \leq 100\}$$

(B) A full square that ends with digit 1 can be formalized as a number n for which there exists a number k such that $k^2 = n$ and also congruent to 1 modulo 10:

$$S_2 = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N}\,(k^2 = n) \ \wedge \ n \leq 2000 \ \wedge \ n \equiv 1 \bmod 10\}.$$

Note: Instead of the set universe $\mathbb{N}$ one can use universe $\mathbb{Z}$, since full squares are non-negative (i.e. natural) numbers anyway.

Question 3:

(A) Find a rational number $a \in \mathbb{Q}$ for which the equation $x + \dfrac{1}{x} = a$ has a rational root x. Express this root as $x = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$.

(B) Find a rational number $b \in \mathbb{Q}$ for which the equation $x + \dfrac{1}{x} = b$ has an irational root x. Prove that the root is irrational. (You can use the fact that the square root of an integer which is not a full square is an irrational number.)

Answer:

(A) Select $a = \frac{2}{1} = 2$. It is a rational number and the equation $x + \dfrac{1}{x} = 2$ has root $x = \frac{1}{1} = 1$ which is a rational number.

(B) Select $b = \frac{3}{1} = 3$. It is a rational number and let us solve the equation $x + \dfrac{1}{x} = 3$:

$$\begin{aligned} x + \frac{1}{x} &= 3, \\ x^2 + 1 &= 3x, \\ x^2 - 3x + 1 &= 0, \\ x_{1,2} = \frac{3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} &= \frac{3 \pm \sqrt{5}}{2}. \end{aligned}$$

Let us prove that one of the roots $x_1 = \frac{3+\sqrt{5}}{2}$ is irrational number.

Proof: By contradiction, assume that $x_1 = \frac{3+\sqrt{5}}{2} = \frac{p}{q}$ is a rational fraction. In this case the number $3 + \sqrt{5} = \frac{2p}{q}$ is also a rational fraction. And finally, $\sqrt{5} = \frac{2p}{q} - 3 = \frac{2p-3q}{q}$ must also be a rational fraction.

We conclude that $\sqrt{5}$ is a rational number (a contradiction – since 5 is not a full square, its square root must be irational number. (The proof for this is similar to the proof that $\sqrt{2}$ is irational.) ■

Question 4: Find a closed interval of real numbers $[a, b] \subseteq \mathbb{R}$, which satisfies the following properties:

- $10 \leq a < b \leq 15$.
- Function $f(x) = \sin x$ is a bijection from $[a; b]$ to the interval $[-1; 1]$.

Also prove that for your values a and b the function $f(x) = \sin x$ is bijective mapping from $[a; b]$ to $[-1; 1]$ using the definitions of surjective and injective functions.

Answer:

We can solve two equations:

$$\sin x = -1;\quad x = \frac{3\pi}{2} + 2\pi k_1,\ \text{where } k_1 \in \mathbb{Z}.$$

$$\sin x = 1;\quad x = \frac{\pi}{2} + 2\pi k_2,\ \text{where } k_2 \in \mathbb{Z}.$$

If we pick $k_1 = 1$, but $k_2 = 2$, then we get $\sin x$ function arguments to have values -1 and 1:

$$\left\{ \begin{array}{l} \sin\left(\frac{3\pi}{2} + (2\pi)\cdot 1\right) = -1 \\ \sin\left(\frac{\pi}{2} + (2\pi)\cdot 2\right) = 1 \end{array} \right.$$

We can select $a = 3\pi/2 + 2\pi = \frac{7\pi}{2} \approx 10.99557$ and $b = \pi/2 + 4\pi = \frac{9\pi}{2} \approx 14.13717$. In this case $10 \leq a < b \leq 15$ is satisfied.

Function is surjective: We have $\sin a = -1$ and $\sin b = 1$; also the function $f(x) = \sin x$ is continuous; therefore it must take all the intermediate values between $f(a)$ and $f(b)$. This is the consequence of *Intermediate value theorem*, see https://bit.ly/3vnAzif.

Function is injective: Function $f(x) = \sin x$ is growing in the whole interval $[a; b]$, since its first derivative $(f(x))' = (\sin x)' = \cos x > 0$. Growing function means $x_1, x_2 \in [a; b]$, where $x_1 < x_2$ implies $\sin x_1 < \sin x_2$. ■

Question 5: Define a binary relation on the set of plane points with integer coordinates: $(x, y) \in \mathbb{Z}^2$.

$$R = \left\{ \left((x_1, y_1), (x_2, y_2)\right) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ \mid\ \left(2 \mid (x_1 - x_2) \wedge 3 \mid (y_1 - y_2)\right) \vee \left(3 \mid (x_1 - x_2) \wedge 2 \mid (y_1 - y_2)\right) \right\}$$

In other words, either the difference of x-coordinates of both points in this relation is divisible by 2 and the difference of y-coordinates is divisible by 3, or the other way round (the difference of x-coordinates is divisible by 3 and the difference of y-coordinates is divisible by 2)

(A) Is the relation R reflexive?

(B) Is the relation R symmetric?

(C) Is the relation R transitive?

(D) Is the relation R equivalence?

Answer:

(A) Yes, the relation is reflexive, since any point (any pair of integers) $(x, y) \in \mathbb{Z}^2$ is in relation to itself: $(x, y)R(x, y)$ means that 2 divides $(x - x) = 0$ and also 2 divides $(y - y) = 0$.

(B) Yes, the relation is symmetric, since switching the order of two points (x_1, y_1) and (x_2, y_2), we would replace $(x_1 - x_2)$ by $(x_2 - x_1)$ and also $(y_1 - y_2)$ by $(y_2 - y_1)$. That means simply switching the sign, since $(x_1 - x_2) = -(x_2 - x_1)$. If some number is divisible by 2 or by 3, then the same number with minus sign is also divisible by 2 or by 3.

(C) No, the relation is not transitive.

For example, $(0, 0)$ is in relation R with $(2, 3)$, and also $(2, 3)$ is in relation R with $(5, 5)$, but $(0, 0)$ is not in relation with $(5, 5)$.

(D) Since the relation is not transitive, it cannot be an equivalence relation.

Question 6: Consider the following binary relation $R \subseteq \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ which is defined as a set of pairs (a, b):

$$R = \{(1, 1), (1, 2), (2, 1), (2, 5), (3, 4), (4, 3)\}.$$

(A) Create the matrix M_R for this binary relation.

(B) Draw the matrix M_{R^t} for the transitive closure of this binary relation R^t.

Answer:

(A) The matrix M_R should have entry $m_{ij} = 1$ iff the pair (i, j) is in relation R. Here is the matrix:

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(B) Transitive closure would also contain additional pairs (that are implied by existing pairs):

- $(2,1) \in R \wedge (1,2) \in R \rightarrow (2,2) \in R^t$.
- $(1,2) \in R \wedge (2,5) \in R \rightarrow (1,5) \in R^t$.
- $(3,4) \in R \wedge (4,3) \in R \rightarrow (3,3) \in R^t$.
- $(4,3) \in R \wedge (3,4) \in R \rightarrow (4,4) \in R^t$.

$$M_{R^t} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Question 7:

(A) Run Euclidean algorithm to find $\gcd(1000, 711)$.

(B) Find an integer solution $x, y \in \mathbb{Z}$ for the following Bézouts identity:

$$1000x + 711y = 0.$$

(C) Find an integer number that is divisible by 711 and its decimal notation ends with these three digits: `001`.

Answer:

(A)

$$\begin{array}{l} \gcd(1000, 711) = \gcd(711, 1000 \bmod 711) = \\ \gcd(711, 289) = \gcd(289, 711 \bmod 289) = \\ \gcd(289, 133) = \gcd(289, 289 \bmod 133) = \\ \gcd(133, 23) = \gcd(133, 133 \bmod 23) = \\ \gcd(23, 18) = \gcd(23, 23 \bmod 18) = \\ \gcd(18, 5) = \gcd(18, 18 \bmod 5) = \\ \gcd(5, 3) = \gcd(5, 5 \bmod 3) = \\ \gcd(3, 2) = \gcd(3, 3 \bmod 2) = \\ \gcd(2, 1) = \gcd(2, 2 \bmod 1) = \\ \gcd(1, 0) = 1. \end{array}$$

Note: Even though we could have seen immediately that $1000 = 2^3 \cdot 5^3$ does not have common divisors with $711 = 3^2 \cdot 79$, in this exercise we need to do step-by-step computation (for larger numbers factorization into primes is very slow, in contrast – Euclidean algorithm is efficient).

(B) Such linear integer equations can be solved by https://www.wolframalpha.com/ (see image):

So, the WolframAlphas solution is $x = 433$ and $y = -609$ – if we plug in the parameter $n = 0$ (the parameter n can take any integer value). One can indeed compute and verify that $1000 \cdot 433 - 609 \cdot 711 = 1$.

If WolframAlpha is not available, one can solve this using Blankinships algorithm (also known as extended Euclidean algorithm). Here is the computation – at every step one line of a matrix is multiplied by something and subtracted from another line:

$$\left(\begin{array}{r|rr} 1000 & 1 & 0 \\ 711 & 0 & 1 \end{array}\right) \rightsquigarrow \text{ Subtract Line2 from Line1}$$

$$\left(\begin{array}{r|rr} 289 & 1 & -1 \\ 711 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 289 & 1 & -1 \\ 133 & -2 & 3 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 23 & 5 & -7 \\ 133 & -2 & 3 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 23 & 5 & -7 \\ 18 & -27 & 38 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 5 & 32 & -45 \\ 18 & -27 & 38 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 5 & 32 & -45 \\ 3 & -123 & 173 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 2 & 155 & -218 \\ 3 & -123 & 173 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{r|rr} 2 & 155 & -218 \\ 1 & -278 & 391 \end{array}\right) \rightsquigarrow$$

The line highlighted in red shows that $1 = 1000 \cdot (-278) + 711 \cdot 391$. (This solution was also found by WolframAlpha, if we plug in $n = -1$.)

(C) As computed in (B), we have the following equality: $1000 \cdot (-278) + 711 \cdot 391 = 1$. If this is rewritten as a congruence modulo 1000, then the first term turns to zero:

$$711 \cdot 371 \equiv 1 \pmod{1000}.$$

From here $711 \cdot 371 = 278001$ ends with the digits 001.

Question 8:

(A) A sequence of natural numbers $a_0, a_1, a_2, \ldots$ is defined by this recurrence:

$$\begin{cases} a_0 = 1, \\ a_{k+1} = (10 \cdot a_k) \bmod 2624, \ \text{for every } k \in \mathbb{N}. \end{cases}$$

Compute the first 12 members of this sequence.

(B) Consider the definition of an eventually periodic sequence:

$$\exists M \in \mathbb{N}\, \exists T \in \mathbb{N}\, \forall n \in \mathbb{N}\, \big(T > 0 \wedge (n \geq M \rightarrow a_n = a_{n+T})\big)$$

Show that the given sequence a_n matches the definition of an eventually periodic sequence. Find the smallest natural numbers M and also $T > 0$ in this definition that would make it true for the sequence a_n.

(C) Consider the fraction $\dfrac{1}{2624}$. It is a rational number – therefore it can be expressed as a repeating/periodic decimal. Identify the *repetend* (also called the *period*) – the sequence of digits that repeats itself infinitely. Identify the *prefix* (the digits preceding the repetend).

Note: For example, the fraction $1/44 = 0.02272727272727272\ldots = 0.02\overline{27}$ has two-digit repetend $\overline{27}$ and the prefix: 02.

Answer:

(A) The first 12 terms of sequence: $a_0, a_1, a_2, \ldots, a_{11}$.

k	0	1	2	3	4	5	6	7	8	9	10	11
a_k	1	10	100	1000	2128	288	256	2560	1984	1472	1600	256

The mapping $x \mapsto (10 \cdot x) \bmod 2624$ is not an injection (since 10 and 2624 are not mutually prime). Therefore we have $288 \mapsto 256$ and also $1600 \mapsto 256$ (two values $288 \neq 1600$ collide – see image).

On the other hand, in this sequence the next term fully depends on the previous term; therefore, 256 will always map to 2560, 2560 will always map to 1984 and so on. After every 5 steps the term 256 (along with other four terms) will repeat.

(B) To make the definition of eventually periodic sequence true, we pick specific values for parameters: $M = 6$ (namely, there are six terms in the sequence $a_0, \ldots, a_6$) and $T = 5$. In this case for each $n \geq 6$ we have $n \geq M \to a_n = a_{n+T}$.

Proof for this just depends on the observation that $a_6 = a_{11}$ (and all subsequent terms in the sequence are determined by the previous terms, so they must repeat as well – by induction we can show that $a_7 = a_{12}$, $a_8 = a_{13}$ and so on).

(C) Can run computation in Python console:

```
>>> 1/2624
0.00038109756097560977
```

We can observe that the first 6 digits after the comma (`000381`) appear only once – it is the *prefix*. After that the group of digits `09756` appears many times. It is thus the *repetend* or the *period*. We have this representation (an overline is drawn over the repeating period):

$$\frac{1}{2624} = 0.000381\overline{09756}.$$

To justify this answer, we should explain, why the period is actually the right one. (It is known from the theory that $1/2624$ is periodic, but from the calculator we do not see, if the period 09756 is the right one – maybe the real period is longer or appears later.)

Simple calculation to verify that 09756 works. Express the fraction $\frac{1}{2624}$ like this:

$$\frac{1}{2624} = \frac{000381}{1000000} + \frac{1}{1000000} \cdot \frac{09756}{99999}.$$

(Ignore leading zeroes – they are left just to show how this is obtained from the decimal notation.) If we do arithmetic with the rational fractions, we find that the equality is true, so the *actual* period must bet 09756.

Note: Part (C) (prefix length 6 and period length 5) is exactly equal to the answer in part (B) – about the constants M and T for the sequence a_n. This is not a coincidence, as the sequence a_n is formed from the remainders as we divide 1 by 2624 according to the school algorithm.