N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = 1. \tag{1}$$

(Hungary)

Solution. Notice first that $\left\lfloor x + \frac{1}{2} \right\rfloor$ is an integer nearest to x, so $\|x\| = \left|\left\lfloor x + \frac{1}{2} \right\rfloor - x\right|$. Thus we have

$$\left\lfloor x + \frac{1}{2} \right\rfloor = x \pm \|x\|. \tag{2}$$

For every rational number r and every prime number p, denote by $v_p(r)$ the exponent of p in the prime factorisation of r. Recall the notation $(2n-1)!!$ for the product of all odd positive integers not exceeding $2n-1$, i.e., $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$.

Lemma. For every positive integer n and every odd prime p, we have

$$v_p\big((2n-1)!!\big) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Proof. For every positive integer k, let us count the multiples of p^k among the factors $1, 3, \ldots, 2n-1$. If ℓ is an arbitrary integer, the number $(2\ell - 1)p^k$ is listed above if and only if

$$0 < (2\ell - 1)p^k \leqslant 2n \quad \Longleftrightarrow \quad \frac{1}{2} < \ell \leqslant \frac{n}{p^k} + \frac{1}{2} \quad \Longleftrightarrow \quad 1 \leqslant \ell \leqslant \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Hence, the number of multiples of p^k among the factors is precisely $m_k = \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor$. Thus we obtain

$$v_p\big((2n-1)!!\big) = \sum_{i=1}^{n} v_p(2i-1) = \sum_{i=1}^{n} \sum_{k=1}^{v_p(2i-1)} 1 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} 1 = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor. \qquad \square$$

In order to prove the problem statement, consider the rational number

$$N = \frac{(2a+2b-1)!!}{(2a-1)!! \cdot (2b-1)!!} = \frac{(2a+1)(2a+3)\cdots(2a+2b-1)}{1 \cdot 3 \cdots (2b-1)}.$$

Obviously, $N > 1$, so there exists a prime p with $v_p(N) > 0$. Since N is a fraction of two odd numbers, p is odd.

By our lemma,

$$0 < v_p(N) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor \right).$$

Therefore, there exists some positive integer k such that the integer number

$$d_k = \left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor$$

is positive, so $d_k \geqslant 1$. By (2) we have

$$1 \leqslant d_k = \frac{a+b}{p^k} - \frac{a}{p^k} - \frac{b}{p^k} \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\| = \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\|. \tag{3}$$

Since $\|x\| < \frac{1}{2}$ for every rational x with odd denominator, the relation (3) can only be satisfied if all three signs on the right-hand side are positive and $d_k = 1$. Thus we get

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = d_k = 1,$$

as required.

Comment 1. There are various choices for the number N in the solution. Here we sketch such a version.

Let x and y be two rational numbers with odd denominators. It is easy to see that the condition $\|x\| + \|y\| + \|x+y\| = 1$ is satisfied if and only if

$$\text{either}\quad \{x\} < \tfrac{1}{2}, \quad \{y\} < \tfrac{1}{2}, \quad \{x+y\} > \tfrac{1}{2}, \qquad \text{or}\quad \{x\} > \tfrac{1}{2}, \quad \{y\} > \tfrac{1}{2}, \quad \{x+y\} < \tfrac{1}{2},$$

where $\{x\}$ denotes the fractional part of x.

In the context of our problem, the first condition seems easier to deal with. Also, one may notice that

$$\{x\} < \tfrac{1}{2} \iff \varkappa(x) = 0 \qquad \text{and} \qquad \{x\} \geqslant \tfrac{1}{2} \iff \varkappa(x) = 1, \tag{4}$$

where

$$\varkappa(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor.$$

Now it is natural to consider the number

$$M = \frac{\binom{2a+2b}{a+b}}{\binom{2a}{a}\binom{2b}{b}},$$

since

$$v_p(M) = \sum_{k=1}^{\infty}\left(\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right)\right).$$

One may see that $M > 1$, and that $v_2(M) \leqslant 0$. Thus, there exist an odd prime p and a positive integer k with

$$\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) > 0.$$

In view of (4), the last inequality yields

$$\left\{\frac{a}{p^k}\right\} < \frac{1}{2}, \quad \left\{\frac{b}{p^k}\right\} < \frac{1}{2}, \quad \text{and} \quad \left\{\frac{a+b}{p^k}\right\} > \frac{1}{2}, \tag{5}$$

which is what we wanted to obtain.

Comment 2. Once one tries to prove the existence of suitable p and k satisfying (5), it seems somehow natural to suppose that $a \leqslant b$ and to add the restriction $p^k > a$. In this case the inequalities (5) can be rewritten as

$$2a < p^k, \quad 2mp^k < 2b < (2m+1)p^k, \quad \text{and} \quad (2m+1)p^k < 2(a+b) < (2m+2)p^k$$

for some positive integer m. This means exactly that one of the numbers $2a+1, 2a+3, \ldots, 2a+2b-1$ is divisible by some number of the form p^k which is greater than $2a$.

Using more advanced techniques, one can show that such a number p^k exists even with $k = 1$. This was shown in 2004 by Laishram and Shorey; the methods used for this proof are elementary but still quite involved. In fact, their result generalises a theorem by Sylvester which states that for every pair of integers (n, k) with $n \geqslant k \geqslant 1$, the product $(n+1)(n+2)\cdots(n+k)$ is divisible by some prime $p > k$. We would like to mention here that Sylvester's theorem itself does not seem to suffice for solving the problem.