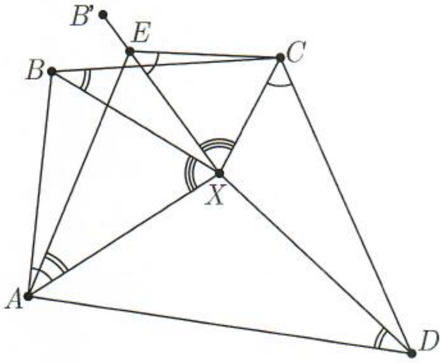
– IMO 6 – a geom!

G6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. A point X is chosen inside the quadrilateral so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180°$.

Solution 1. Let B' be the reflection of B in the internal angle bisector of $\angle AXC$, so that $\angle AXB' = \angle CXB$ and $\angle CXB' = \angle AXB$. If X, D, and B' are collinear, then we are done. Now assume the contrary.

On the ray XB' take a point E such that $XE \cdot XB = XA \cdot XC$, so that $\triangle AXE \sim \triangle BXC$ and $\triangle CXE \sim \triangle BXA$. We have $\angle XCE + \angle XCD = \angle XBA + \angle XAB < 180°$ and $\angle XAE + \angle XAD = \angle XDA + \angle XAD < 180°$, which proves that X lies inside the angles $\angle ECD$ and $\angle EAD$ of the quadrilateral $EADC$. Moreover, X lies in the interior of exactly one of the two triangles EAD, ECD (and in the exterior of the other).



The similarities mentioned above imply $XA \cdot BC = XB \cdot AE$ and $XB \cdot CE = XC \cdot AB$. Multiplying these equalities with the given equality $AB \cdot CD = BC \cdot DA$, we obtain $XA \cdot CD \cdot CE = XC \cdot AD \cdot AE$, or, equivalently,

$$\frac{XA \cdot DE}{AD \cdot AE} = \frac{XC \cdot DE}{CD \cdot CE}. \qquad (*)$$

Lemma. Let PQR be a triangle, and let X be a point in the interior of the angle QPR such that $\angle QPX = \angle PRX$. Then $\dfrac{PX \cdot QR}{PQ \cdot PR} < 1$ if and only if X lies in the interior of the triangle PQR.

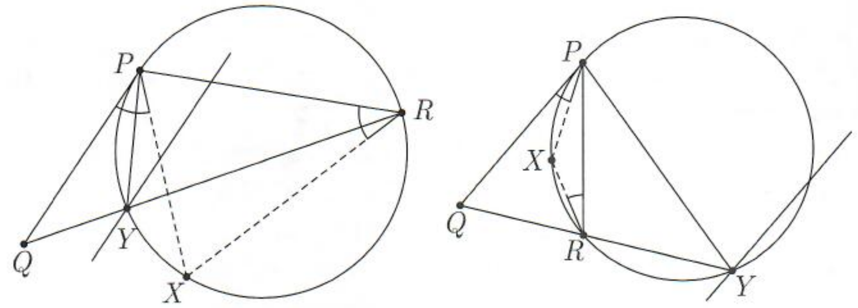
Proof. The locus of points X with $\angle QPX = \angle PRX$ lying inside the angle QPR is an arc α of the circle γ through R tangent to PQ at P. Let γ intersect the line QR again at Y (if γ is tangent to QR, then set $Y = R$). The similarity $\triangle QPY \sim \triangle QRP$ yields $PY = \dfrac{PQ \cdot PR}{QR}$. Now it suffices to show that $PX < PY$ if and only if X lies in the interior of the triangle PQR. Let m be a line through Y parallel to PQ. Notice that the points Z of γ satisfying $PZ < PY$ are exactly those between the lines m and PQ.

Case 1: Y lies in the segment QR (see the left figure below).

In this case Y splits α into two arcs $\widehat{PY}$ and $\widehat{YR}$. The arc $\widehat{PY}$ lies inside the triangle PQR, and $\widehat{PY}$ lies between m and PQ, hence $PX < PY$ for points $X \in \widehat{PY}$. The other arc $\widehat{YR}$ lies outside triangle PQR, and $\widehat{YR}$ is on the opposite side of m than P, hence $PX > PY$ for $X \in \widehat{YR}$.

Case 2: Y lies on the ray QR beyond R (see the right figure below).

In this case the whole arc α lies inside triangle PQR, and between m and PQ, thus $PX < PY$ for all $X \in \alpha$. □



Applying the Lemma (to $\triangle EAD$ with the point X, and to $\triangle ECD$ with the point X), we obtain that exactly one of two expressions $\frac{XA \cdot DE}{AD \cdot AE}$ and $\frac{XC \cdot DE}{CD \cdot CE}$ is less than 1, which contradicts $(*)$.

Comment 1. One may show that $AB \cdot CD = XA \cdot XC + XB \cdot XD$. We know that D, X, E are collinear and $\angle DCE = \angle CXD = 180° - \angle AXB$. Therefore,

$$AB \cdot CD = XB \cdot \frac{\sin \angle AXB}{\sin \angle BAX} \cdot DE \cdot \frac{\sin \angle CED}{\sin \angle DCE} = XB \cdot DE.$$

Furthermore, $XB \cdot DE = XB \cdot (XD + XE) = XB \cdot XD + XB \cdot XE = XB \cdot XD + XA \cdot XC$.

Comment 2. For a convex quadrilateral $ABCD$ with $AB \cdot CD = BC \cdot DA$, it is known that $\angle DAC + \angle ABD + \angle BCA + \angle CDB = 180°$ (among other, it was used as a problem on the Regional round of All-Russian olympiad in 2012), but it seems that there is no essential connection between this fact and the original problem.

Solution 2. The solution consists of two parts. In Part 1 we show that it suffices to prove that

$$\frac{XB}{XD} = \frac{AB}{CD} \tag{1}$$

and

$$\frac{XA}{XC} = \frac{DA}{BC}. \tag{2}$$

In Part 2 we establish these equalities.

Part 1. Using the sine law and applying (1) we obtain

$$\frac{\sin \angle AXB}{\sin \angle XAB} = \frac{AB}{XB} = \frac{CD}{XD} = \frac{\sin \angle CXD}{\sin \angle XCD},$$

so $\sin \angle AXB = \sin \angle CXD$ by the problem conditions. Similarly, (2) yields $\sin \angle DXA = \sin \angle BXC$. If at least one of the pairs $(\angle AXB, \angle CXD)$ and $(\angle BXC, \angle DXA)$ consists of supplementary angles, then we are done. Otherwise, $\angle AXB = \angle CXD$ and $\angle DXA = \angle BXC$. In this case $X = AC \cap BD$, and the problem conditions yield that $ABCD$ is a parallelogram and hence a rhombus. In this last case the claim also holds.

Part 2. To prove the desired equality (1), invert $ABCD$ at centre X with unit radius; the images of points are denoted by primes.

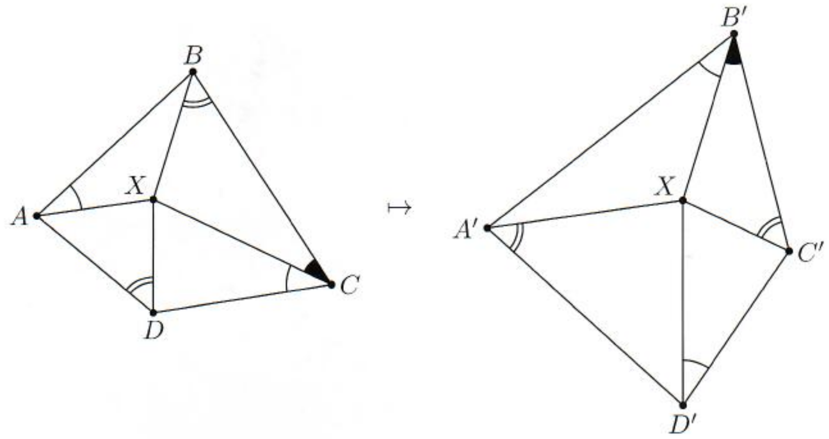
We have

$$\angle A'B'C' = \angle XB'A' + \angle XB'C' = \angle XAB + \angle XCB = \angle XCD + \angle XCB = \angle BCD.$$

Similarly, the corresponding angles of quadrilaterals $ABCD$ and $D'A'B'C'$ are equal.

Moreover, we have

$$A'B' \cdot C'D' = \frac{AB}{XA \cdot XB} \cdot \frac{CD}{XC \cdot XD} = \frac{BC}{XB \cdot XC} \cdot \frac{DA}{XD \cdot DA} = B'C' \cdot D'A'.$$



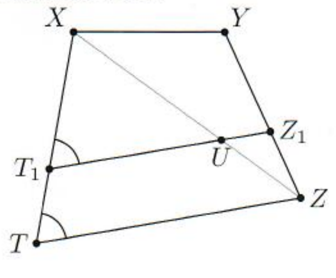
Now we need the following Lemma.

Lemma. Assume that the corresponding angles of convex quadrilaterals $XYZT$ and $X'Y'Z'T'$ are equal, and that $XY \cdot ZT = YZ \cdot TX$ and $X'Y' \cdot Z'T' = Y'Z' \cdot T'X'$. Then the two quadrilaterals are similar.

Proof. Take the quadrilateral XYZ_1T_1 similar to $X'Y'Z'T'$ and sharing the side XY with $XYZT$, such that Z_1 and T_1 lie on the rays YZ and XT, respectively, and $Z_1T_1 \parallel ZT$. We need to prove that $Z_1 = Z$ and $T_1 = T$. Assume the contrary. Without loss of generality, $TX > XT_1$. Let segments XZ and Z_1T_1 intersect at U. We have

$$\frac{T_1X}{T_1Z_1} < \frac{T_1X}{T_1U} = \frac{TX}{ZT} = \frac{XY}{YZ} < \frac{XY}{YZ_1},$$

thus $T_1X \cdot YZ_1 < T_1Z_1 \cdot XY$. A contradiction. □



It follows from the Lemma that the quadrilaterals $ABCD$ and $D'A'B'C'$ are similar, hence

$$\frac{BC}{AB} = \frac{A'B'}{D'A'} = \frac{AB}{XA \cdot XB} \cdot \frac{XD \cdot XA}{DA} = \frac{AB}{AD} \cdot \frac{XD}{XB},$$

and therefore

$$\frac{XB}{XD} = \frac{AB^2}{BC \cdot AD} = \frac{AB^2}{AB \cdot CD} = \frac{AB}{CD}.$$

We obtain (1), as desired; (2) is proved similarly.

Comment. Part 1 is an easy one, while part 2 seems to be crucial. On the other hand, after the proof of the similarity $D'A'B'C' \sim ABCD$ one may finish the solution in different ways, e.g., as follows. The similarity taking $D'A'B'C'$ to $ABCD$ maps X to the point X' *isogonally conjugate* of X with respect to $ABCD$ (i.e. to the point X' inside $ABCD$ such that $\angle BAX = \angle DAX'$, $\angle CBX = \angle ABX'$, $\angle DCX = \angle BCX'$, $\angle ADX = \angle CDX'$). It is known that the required equality $\angle AXB + \angle CXD = 180°$ is one of known conditions on a point X inside $ABCD$ equivalent to the existence of its isogonal conjugate.