N4. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all $n \geqslant k$, where k is some positive integer. Prove that there exists a positive integer m such that $a_n = a_{n+1}$ for all $n \geqslant m$.

Solution 1. The argument hinges on the following two facts: Let a, b, c be positive integers such that $N = b/c + (c-b)/a$ is an integer.

(1) If $\gcd(a, c) = 1$, then c divides b; and
(2) If $\gcd(a, b, c) = 1$, then $\gcd(a, b) = 1$.

To prove **(1)**, write $ab = c(aN + b - c)$. Since $\gcd(a, c) = 1$, it follows that c divides b. To prove **(2)**, write $c^2 - bc = a(cN - b)$ to infer that a divides $c^2 - bc$. Letting $d = \gcd(a, b)$, it follows that d divides c^2, and since the two are relatively prime by hypothesis, $d = 1$.

Now, let $s_n = a_1/a_2 + a_2/a_3 + \cdots + a_{n-1}/a_n + a_n/a_1$, let $\delta_n = \gcd(a_1, a_n, a_{n+1})$ and write

$$s_{n+1} - s_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1} - a_n}{a_1} = \frac{a_n/\delta_n}{a_{n+1}/\delta_n} + \frac{a_{n+1}/\delta_n - a_n/\delta_n}{a_1/\delta_n}.$$

Let $n \geqslant k$. Since $\gcd(a_1/\delta_n, a_n/\delta_n, a_{n+1}/\delta_n) = 1$, it follows by **(2)** that $\gcd(a_1/\delta_n, a_n/\delta_n) = 1$. Let $d_n = \gcd(a_1, a_n)$. Then $d_n = \delta_n \cdot \gcd(a_1/\delta_n, a_n/\delta_n) = \delta_n$, so d_n divides a_{n+1}, and therefore d_n divides d_{n+1}.

Consequently, from some rank on, the d_n form a nondecreasing sequence of integers not exceeding a_1, so $d_n = d$ for all $n \geqslant \ell$, where ℓ is some positive integer.

Finally, since $\gcd(a_1/d, a_{n+1}/d) = 1$, it follows by **(1)** that a_{n+1}/d divides a_n/d, so $a_n \geqslant a_{n+1}$ for all $n \geqslant \ell$. The conclusion follows.

Solution 2. We use the same notation s_n. This time, we explore the exponents of primes in the prime factorizations of the a_n for $n \geqslant k$.

To start, for every $n \geqslant k$, we know that the number

$$s_{n+1} - s_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} \qquad (*)$$

is integer. Multiplying it by a_1 we obtain that $a_1 a_n / a_{n+1}$ is integer as well, so that $a_{n+1} \mid a_1 a_n$. This means that $a_n \mid a_1^{n-k} a_k$, so all prime divisors of a_n are among those of $a_1 a_k$. There are finitely many such primes; therefore, it suffices to prove that the exponent of each of them in the prime factorization of a_n is eventually constant.

Choose any prime $p \mid a_1 a_k$. Recall that $v_p(q)$ is the standard notation for the exponent of p in the prime factorization of a nonzero rational number q. Say that an index $n \geqslant k$ is *large* if $v_p(a_n) \geqslant v_p(a_1)$. We separate two cases.

Case 1: There exists a large index n.

If $v_p(a_{n+1}) < v_p(a_1)$, then $v_p(a_n/a_{n+1})$ and $v_p(a_n/a_1)$ are nonnegative, while $v_p(a_{n+1}/a_1) < 0$; hence $(*)$ cannot be an integer. This contradiction shows that index $n+1$ is also large.

On the other hand, if $v_p(a_{n+1}) > v_p(a_n)$, then $v_p(a_n/a_{n+1}) < 0$, while $v_p\big((a_{n+1} - a_n)/a_1\big) \geqslant 0$, so $(*)$ is not integer again. Thus, $v_p(a_1) \leqslant v_p(a_{n+1}) \leqslant v_p(a_n)$.

The above arguments can now be applied successively to indices $n+1, n+2, \ldots$, showing that all the indices greater than n are large, and the sequence $v_p(a_n), v_p(a_{n+1}), v_p(a_{n+2}), \ldots$ is nonincreasing — hence eventually constant.

Case 2: There is no large index.

We have $v_p(a_1) > v_p(a_n)$ for all $n \geqslant k$. If we had $v_p(a_{n+1}) < v_p(a_n)$ for some $n \geqslant k$, then $v_p(a_{n+1}/a_1) < v_p(a_n/a_1) < 0 < v_p(a_n/a_{n+1})$ which would also yield that $(*)$ is not integer. Therefore, in this case the sequence $v_p(a_k), v_p(a_{k+1}), v_p(a_{k+2}), \ldots$ is nondecreasing and bounded by $v_p(a_1)$ from above; hence it is also eventually constant.

Comment. Given any positive odd integer m, consider the m-tuple $(2, 2^2, \ldots, 2^{m-1}, 2^m)$. Appending an infinite string of 1's to this m-tuple yields an eventually constant sequence of integers satisfying the condition in the statement, and shows that the rank from which the sequence stabilises may be arbitrarily large.

There are more sophisticated examples. The solution to part **(b)** of **10532**, *Amer. Math. Monthly*, Vol. 105 No. 8 (Oct. 1998), 775–777 (available at `https://www.jstor.org/stable/2589009`), shows that, for every integer $m \geqslant 5$, there exists an m-tuple $(a_1, a_2, \ldots, a_m)$ of pairwise distinct positive integers such that $\gcd(a_1, a_2) = \gcd(a_2, a_3) = \cdots = \gcd(a_{m-1}, a_m) = \gcd(a_m, a_1) = 1$, and the sum $a_1/a_2 + a_2/a_3 + \cdots + a_{m-1}/a_m + a_m/a_1$ is an integer. Letting $a_{m+k} = a_1$, $k = 1, 2, \ldots$, extends such an m-tuple to an eventually constant sequence of positive integers satisfying the condition in the statement of the problem at hand.

Here is the example given by the proposers of **10532**. Let $b_1 = 2$, let $b_{k+1} = 1 + b_1 \cdots b_k = 1 + b_k(b_k - 1)$, $k \geqslant 1$, and set $B_m = b_1 \cdots b_{m-4} = b_{m-3} - 1$. The m-tuple $(a_1, a_2, \ldots, a_m)$ defined below satisfies the required conditions:

$$a_1 = 1, \quad a_2 = (8B_m + 1)B_m + 8, \quad a_3 = 8B_m + 1, \quad a_k = b_{m-k} \quad \text{for } 4 \leqslant k \leqslant m - 1,$$
$$a_m = \frac{a_2}{2} \cdot a_3 \cdot \frac{B_m}{2} = \left(\frac{1}{2}(8B_m + 1)B_m + 4\right) \cdot (8B_m + 1) \cdot \frac{B_m}{2}.$$

It is readily checked that $a_1 < a_{m-1} < a_{m-2} < \cdots < a_3 < a_2 < a_m$. For further details we refer to the solution mentioned above. Acquaintance with this example (or more elaborated examples derived from) offers no advantage in tackling the problem.