Solutions

Algebra

Nice & easy

A1. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y) \tag{$*$}$$

for all $x, y \in \mathbb{Q}_{>0}$.

Answer: $f(x) = 1$ for all $x \in \mathbb{Q}_{>0}$.

Solution. Take any $a, b \in \mathbb{Q}_{>0}$. By substituting $x = f(a)$, $y = b$ and $x = f(b)$, $y = a$ into $(*)$ we get

$$f\big(f(a)\big)^2 f(b) = f\big(f(a)^2 f(b)^2\big) = f\big(f(b)\big)^2 f(a),$$

which yields

$$\frac{f\big(f(a)\big)^2}{f(a)} = \frac{f\big(f(b)\big)^2}{f(b)} \qquad \text{for all } a, b \in \mathbb{Q}_{>0}.$$

In other words, this shows that there exists a constant $C \in \mathbb{Q}_{>0}$ such that $f\big(f(a)\big)^2 = Cf(a)$, or

$$\left(\frac{f\big(f(a)\big)}{C}\right)^2 = \frac{f(a)}{C} \qquad \text{for all } a \in \mathbb{Q}_{>0}. \tag{1}$$

Denote by $f^n(x) = \underbrace{f(f(\ldots(f}_{n}(x))\ldots))$ the n^{th} iteration of f. Equality (1) yields

$$\frac{f(a)}{C} = \left(\frac{f^2(a)}{C}\right)^2 = \left(\frac{f^3(a)}{C}\right)^4 = \cdots = \left(\frac{f^{n+1}(a)}{C}\right)^{2^n}$$

for all positive integer n. So, $f(a)/C$ is the 2^n-th power of a rational number for all positive integer n. This is impossible unless $f(a)/C = 1$, since otherwise the exponent of some prime in the prime decomposition of $f(a)/C$ is not divisible by sufficiently large powers of 2. Therefore, $f(a) = C$ for all $a \in \mathbb{Q}_{>0}$.

Finally, after substituting $f \equiv C$ into $(*)$ we get $C = C^3$, whence $C = 1$. So $f(x) \equiv 1$ is the unique function satisfying $(*)$.

Comment 1. There are several variations of the solution above. For instance, one may start with finding $f(1) = 1$. To do this, let $d = f(1)$. By substituting $x = y = 1$ and $x = d^2$, $y = 1$ into $(*)$ we get $f(d^2) = d^3$ and $f(d^6) = f(d^2)^2 \cdot d = d^7$. By substituting now $x = 1$, $y = d^2$ we obtain $f(d^6) = d^2 \cdot d^3 = d^5$. Therefore, $d^7 = f(d^6) = d^5$, whence $d = 1$.

After that, the rest of the solution simplifies a bit, since we already know that $C = \frac{f(f(1))^2}{f(1)} = 1$. Hence equation (1) becomes merely $f(f(a))^2 = f(a)$, which yields $f(a) = 1$ in a similar manner.

Comment 2. There exist nonconstant functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(*)$ for all real $x, y > 0$ — e.g., $f(x) = \sqrt{x}$.