Baltic Way shortlist

St.Petersburg

2018

Solutions

1 Algebra

A-1. Consider $m \geq 3$ positive real numbers $g_1, \ldots, g_m$, each number being less than the sum of the others. For any subset $M \subseteq \{1, \ldots, m\}$, denote

$$S_M = \sum_{k \in M} g_k.$$

Find all m for which it is always possible to partition the indices $1, \ldots, m$ into three sets A, B, C, with the property that

$$S_A < S_B + S_C, \qquad S_B < S_A + S_C \qquad \text{and} \qquad S_C < S_A + S_B.$$

Solution.

Answer: The partition is always possible precisely when $m \neq 4$.

For $m = 3$ it is trivially possible, and for $m = 4$ the four equal numbers g, g, g, g provide a counter-example. Henceforth, we assume $m \geq 5$.

Among all possible partitions $A \sqcup B \sqcup C = \{1, \ldots, m\}$ such that

$$S_A \leq S_B \leq S_C,$$

select one for which the difference $S_C - S_A$ is minimal. If there are several such, select one so as to maximise the number of elements in C. We will show that $S_C < S_A + S_B$, which is clearly sufficient.

If C consists of a single element, this number is by assumption less than the sum of the remaining ones, hence $S_C < S_A + S_B$ holds true.

Suppose now C contains at least two elements, and let g_c be a minimal number indexed by a $c \in C$. We have the inequality

$$S_C - S_A \leq g_c \leq \frac{1}{2} S_C.$$

The first is by the minimality of $S_C - S_A$, the second by the minimality of g_c. These two inequalities together yield

$$S_A + S_B \geq 2S_A \geq 2(S_C - g_c) \geq S_C.$$

If either of these inequalities is strict, we are finished.

Hence suppose all inequalities are in fact equalities, so that

$$S_A = S_B = \frac{1}{2} S_C = g_c.$$

It follows that $C = \{c, d\}$, where $g_d = g_c$. If A contained more than one element, we could increase the number of elements in C by creating instead a partition

$$\{1, \ldots, m\} = \{c\} \sqcup B \sqcup (A \cup \{d\}),$$

resulting in the same sums. A similar procedure applies to B. Consequently, A and B must be singleton sets, whence

$$m = |A| + |B| + |C| = 4.$$

A-2. Let a_1, a_2, $\ldots$, a_{100} be a permutation of numbers 1, 2, $\ldots$, 100. Denote by N the number of different values of the sums

$$\sum_{i=u}^{v} a_i, \qquad \text{where } 1 \leq u \leq v \leq 100.$$

Is it possible that $N \geq 2500$?

Solution.

Answer: yes.

For example consider a permutation 1, 100, 2, 99, 3, 98, $\ldots$ For odd i we have $a_i + a_{i+1} = 101$. It is not difficult to check that if u and v have the same parity (and therefore the number of summands is odd) then for all choices of u and $v = u + 2\ell$ all the sums

$$\sum_{i=2k-1}^{2k-1+2\ell} a_i = 101\ell + a_{2k-1+2\ell}, \qquad \sum_{i=2k}^{2k+2\ell} a_i = 101\ell + a_{2k}$$

are different! Therefore the total number of different values is at least $51 \cdot 50 = 2550 > 2500$.

A-3. Non negative integers are written in some cells of 100×100 table. For each k, $1 \leq k \leq 100$, the k-th row of the table contains numbers from 1 to k written in increasing order (from left to right) but not necessarily in consecutive cells. The empty cells are filled with zeroes. Prove that there exist two columns such that the sum of numbers in one of them is at least 19 times greater than the sum in the second column.

Solution.

Observe that the sum of numbers in the first column is at most $1 \cdot 100 = 100$, the sum in the first and second columns is at most $1 \cdot 100 + 2 \cdot 99$, the sum in the first, second and third columns is at most $1 \cdot 100 + 2 \cdot 99 + 3 \cdot 98$, etc. But the sum of all nonzero numbers equals $\sum_{i=1}^{100} i(101-i)$, therefore the sum in the columns from 31^{th} to 100^{th} is at least

$$\sum_{i=31}^{100} i(101-i) = \sum_{i=1}^{70} i(101-i) = 101\sum_{i=1}^{70} i - \sum_{i=1}^{70} i^2 = 35 \cdot 71(101 - 141/3) = 70 \cdot 27 \cdot 71.$$

Therefore one of these columns has a sum at least $27 \cdot 71 = 1917$. Therefore the ratio of sums in this column and in the first one is more that 19.

A-4. Let a, b, c be positive real numbers with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that

$$3(ab+bc+ca)+\frac{9}{a+b+c}\leq\frac{9abc}{a+b+c}+2(a^2+b^2+c^2)+1$$

and find all numbers a, b, c for which equality holds.

Solution.

By symmetry, we can assume $a\geq b\geq c$. And the claim is equivalent to

$$a^2+b^2+c^2+\frac{9}{a+b+c}\leq\frac{9abc}{a+b+c}+3(a^2+b^2+c^2-ab-bc-ca)+1,$$

$$\begin{aligned}(a+b+c)(a^2+b^2+c^2)+9\leq\;&9abc+3(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+(a+b+c)\\=\;&3(a^3+b^3+c^3)+(a+b+c),\end{aligned}$$

$$(a+b+c)(a^2+b^2+c^2-1)\leq 3(a^3+b^3+c^3-3),$$

$$\frac{a+b+c}{3}\cdot\frac{(a^2-\frac{1}{a})+(b^2-\frac{1}{b})+(c^2-\frac{1}{c})}{3}\leq\frac{(a^3-1)+(b^3-1)+(c^3-1)}{3}.$$

Note that we also have the order $a^2-\frac{1}{a}\geq b^2-\frac{1}{b}\geq c^2-\frac{1}{c}$, and now it suffices to recall Chebyshev's inequality, and we see that the last line is true.

We have equality only if $a=b=c$, hence each of them equals 3.

A-5. Let a, b, c, d be positive numbers such that $abcd=1$. Prove the inequality

$$\frac{1}{\sqrt{a+2b+3c+10}}+\frac{1}{\sqrt{b+2c+3d+10}}+\frac{1}{\sqrt{c+2d+3a+10}}+\frac{1}{\sqrt{d+2a+3b+10}}\leq 1.$$

Solution.

Let x, y, z, t be positive numbers such that $a=x^4$, $b=y^4$, $c=z^4$, $d=t^4$.

By AM-GM ineguality $x^4+y^4+z^4+1\geq 4xyz$, $y^4+z^4+1+1\geq 4yz$ and $z^4+1+1+1\geq 4z$. Therefore we have the following estimation for the first fraction

$$\frac{1}{\sqrt{x^4+2y^4+3z^4+10}}\leq\frac{1}{\sqrt{4xyz+4yz+4z+4}}=\frac{1}{2\sqrt{xyz+yz+z+1}}.$$

Transform analogous estimations for the other fractions:

$$\frac{1}{\sqrt{b+2c+3d+10}}\leq\frac{1}{2\sqrt{yzt+zt+t+1}}=\frac{1}{2\sqrt{t}\sqrt{yz+z+1+xyz}}=\frac{\sqrt{xyz}}{2\sqrt{xyz+yz+z+1}};$$

$$\frac{1}{\sqrt{c+2d+3a+10}}\leq\frac{1}{2\sqrt{ztx+tx+x+1}}=\frac{1}{2\sqrt{tx}\sqrt{z+1+xyz+yz}}=\frac{\sqrt{yz}}{2\sqrt{xyz+yz+z+1}};$$

$$\frac{1}{\sqrt{d+2a+3b+10}}\leq\frac{1}{2\sqrt{txy+xy+y+1}}=\frac{1}{2\sqrt{txy}\sqrt{1+xyz+yz+z}}=\frac{\sqrt{z}}{2\sqrt{xyz+yz+z+1}}.$$

Thus, the sum does not exceed

$$\frac{1+\sqrt{xyz}+\sqrt{yz}+\sqrt{z}}{2\sqrt{xyz+yz+z+1}}.$$

It remains to apply inequality $\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}+\sqrt{\delta}\leq 2\sqrt{\alpha+\beta+\gamma+\delta}$, which can be easily proven by taking squares.

A-6. Required are all functions f mapping non-negative reals to non-negative reals, fulfilling the identity

$$f(x_1^2 + \cdots + x_n^2) = f(x_1)^2 + \cdots + f(x_n)^2$$

for any choice of numbers $x_1, \ldots, x_n$.

Solution.

Answer: the functions $f(x) = 0$ and $f(x) = x$.

A first observation is that

$$f(1) = f(1^2) = f(1)^2,$$

so that $f(1)$ is either 0 or 1.

Assume first that $f(1) = 0$. For each positive integer n, we find

$$f(n) = f(n \cdot 1^2) = nf(1)^2 = 0.$$

Given an arbitrary x, find y so that $x^2 + y^2$ becomes a positive integer n. Then

$$f(x)^2 + f(y)^2 = f(x^2 + y^2) = f(n) = 0.$$

Consequently, $f(x) = 0$ for all x.

Now assume $f(0) = 1$. We shall prove that $f(x) = x$ for all x. For each positive integer n, we find

$$f(n) = f(n \cdot 1^2) = nf(1)^2 = n.$$

For a non-negative rational number $\frac{p}{q}$, we find

$$p^2 = f(p^2) = f\left(q^2 \cdot \left(\frac{p}{q}\right)^2\right) = q^2 f\left(\frac{p}{q}\right)^2,$$

hence $f(x) = x$ also for rational numbers.

Finally, let x be an irrational number. Select a rational number $\frac{p}{q} > x$. Choosing y so that $x^2 + y^2 = \frac{p^2}{q^2}$, we deduce

$$\frac{p^2}{q^2} = f\left(\frac{p^2}{q^2}\right) = f(x^2 + y^2) = f(x)^2 + f(y)^2 \geq f(x)^2,$$

hence $f(x) \leq \frac{p}{q}$. Next, select a (positive) rational number $\frac{r}{s} < \sqrt{x}$, i.e. $\frac{r^2}{s^2} < x$. Choosing z so that $\frac{r^2}{s^2} + z^2 = x$, we deduce

$$f(x) = f\left(\frac{r^2}{s^2} + z^2\right) = f\left(\frac{r}{s}\right)^2 + f(z)^2 = \frac{r^2}{s^2} + f(z)^2 \geq \frac{r^2}{s^2},$$

hence $f(x) \geq \frac{r^2}{s^2}$. Together, these two bounds for $f(x)$ imply $f(x) = x$, and we are finished.

Remark of the Problem committee. The main part of this solution is the proof of well known fact that if an additive function is non negative (for non negative arguments) then it is linear.

A-7. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that for all real x and y satisfy the equation

$$f\big(y^2 - f(x)\big) = yf(x)^2 + f(x^2y + y).$$

Solution.

Answer: The only such a function is $f(x) = 0$.

At first, assume that $f(x) > 0$ for some $x \in \mathbb{R}$. It means that we can choose y such that

$$y^2 - f(x) = x^2y + y$$

(because for $f(x) > 0$ this equation has two solutions with respect to y), and if we insert it into the given equation we obtain an equality $yf(x)^2 = 0$. As $f(x) > 0$ then $y = 0$. But $y = 0$ is not a solution of $y^2 - f(x) = x^2y + y$ — contradiction. Thus $f(x) \le 0$ for all $x \in \mathbb{R}$.

Note that $f(x) = 0$ is a solution. So assume that $f(x_0) < 0$ for some $x_0 \in \mathbb{R}$. At first we show that f is unbounded. Assume the contrary and put x_0 into the equation. We get that

$$f(y^2 - f(x_0)) - f(x_0^2y + y) = yf(x_0)^2,$$

and see that if $f(x)$ is bounded then the left hand side of this equality also is bounded, but the right hand side is unbounded, that is impossible.

If we put $y = 0$ in the original equation we get that $f(-f(x)) = f(0)$. As $f(x)$ is unbounded and nonpositive we conclude that we can find arbitrarily large y such that $f(y) = f(0)$. Now put $x = x_0$ and choose y_0 such that $y_0 > \frac{-f(0)}{f(x_0)^2}$ and $f(y_0(x_0^2 + 1)) = f(0)$. We get that

$$f(y_0^2 - f(x_0)) = y_0f(x_0)^2 + f(y_0(x_0^2 + 1)) > -f(0) + f(0) = 0,$$

what contradicts the fact, that $f(x) \le 0$ for all real x.

A-8. Solve the equation

$$2x^3 + 3yx^2 + 2y^2x + y^3 = 0$$

in real numbers.

Solution.

Answer: all pairs $(a, -a)$ are solutions.

We write the equation as follows:

$$(x + y)^3 = x(x + y)^2 - 2(x + y)x^2.$$

This shows that all points $(a, -a)$ are solutions. Then dividing out $x + y$ and writing $t = (x + y)$ we get $t^2 - xt + 2x^2 = 0$. The discriminant of this quadratic equation is $x^2 - 8x^2 = -7x^2 < 0$. Thus there are no further solutions.

A-9. Compute the following product:

$$\prod_{m=1}^{2018} \frac{(2m-1)^4+\frac{1}{4}}{(2m)^4+\frac{1}{4}}$$

Solution.

By applying Sophie-Germain identity we can obtain following equality:

$$\frac{(2m-1)^4+\frac{1}{4}}{(2m)^4+\frac{1}{4}} = \frac{((2m-\frac{1}{2})^2+\frac{1}{4})((2m-\frac{3}{2})^2+\frac{1}{4})}{((2m+\frac{1}{2})^2+\frac{1}{4})((2m-\frac{1}{2})^2+\frac{1}{4})} = \frac{(2m-\frac{3}{2})^2+\frac{1}{4}}{(2m+\frac{1}{2})^2+\frac{1}{4}}$$

It is now easy to see that the product, that we want to compute, can be shortened:

$$\prod_{m=1}^{2018} \frac{(2m-1)^4+\frac{1}{4}}{(2m)^4+\frac{1}{4}} = \prod_{m=1}^{2018} \frac{(2m-\frac{3}{2})^2+\frac{1}{4}}{(2m+\frac{1}{2})^2+\frac{1}{4}} = \frac{(2-\frac{3}{2})^2+\frac{1}{4}}{(2\cdot 2018+\frac{1}{2})^2+\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{8073^2+1}{4}} = \frac{2}{8073^2+1}.$$

A-10. A polynomial $f(x)$ with real coefficients is called generating, if for each polynomial $\varphi(x)$ with real coefficients there exists positive integer k and polynomials $g_1(x), \ldots, g_k(x)$ such that

$$\varphi(x) = f(g_1(x)) + \cdots + f(g_k(x)).$$

Find all generating polynomials.

Solution.

Answer: the generating polynomials are exactly the polynomials of odd degree.

Take an arbitrary polynomial f. We call a polynomial *good* if it can pe represented as $\sum f(g_i(x))$ for some polynomials g_i. It is clear that the sum of good polynomials is good, and if ϕ is a good polynomial then each polynomial of the form $\phi(g(x))$ is good also. Therefore for the proof that f is generating it is sufficient to show that x is good polynomial. Consider two cases.

1) Let the degree n of f is odd. Check that x is good polynomial. Observe that by substitutions of the form $f(ux)$ we can obtain a good polynomial ϕ_n of degree n with leading coefficient 1, and a good polynomial ψ_n of degree n with leading coefficient -1 (because n is odd). Then for each a a polynomial $\phi_n(x+a)+\psi_n(x)$ is good. It is clear that its coefficient of x^n equals 0; moreover, by choosing appropriate a we can obtain a good polynomial ϕ_{n-1} of degree $n-1$ with leading coefficient 1, and a good polynomial ψ_{n-1} with leading coefficient -1. Continuing in this way we will obtain a good poynomial $\phi_1(x) = x + c$. Then $\phi_1(x-c) = x$ is also good.

2) Let the degree n of f is even. Prove that $f(x)$ is not generating. It follows from the observation that the degree of every good polynomial is even in this case. Indeed, the degree of each polynomial $f(g_i)$ is even and the leading coefficient has the same sign as the leading coefficient of f. Therefore the degree of polynomial $\sum f(g_i(x))$ is even.

A-11. A grasshopper is jumping along the set $\mathbb{Z}$ of integers. He starts at the origin; and for each jump, he may decide whether to jump to the left or to the right. For each $n \in \mathbb{N}_0$, the n-th jump has length n^2.

Prove or disprove that for each $k \in \mathbb{Z}$ the grasshopper can arrive at k starting from origin.

Solution.

Clearly, the grasshopper can arrive at the integers $+1$ and -1, in one jump. And also, he can arrive at the integer 14 using three jumps to the right $(1 + 4 + 9 = 14)$.

Note the following: if the grasshopper can arrive at a number $a \in \mathbb{Z}$ using n jumps, then he can also arrive at the numbers $a - 4$ and $a + 4$ using $n + 4$ jumps, by jumping left, right, right, left (for $a - 4$), and right, left, left, right (for $a + 4$), because of

$$a - (n+1)^2 + (n+2)^2 + (n+3)^2 - (n+4)^2 = a - 4,$$
$$a + (n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2 = a + 4.$$

Since the numbers 0, $+1$, -1 and 14 all have distinct remainders modulo four, the grasshopper can arrive at each integer in a finite number of jumps.

Remark of the Problem committee. The problem statement is very similar to problem C-3. But the question here is rather algebraic: it is about existence of some identities, while the question of C-3 is more combinatorial.

A-12. Let there be an operator $*$. Given an expression that includes this operator, one can make the following transformations:

1. An expression of the form $x * (y * z)$ can be rewritten as $((1 * x) * y) * z$;

2. An expression of the form $x * 1$ can be rewritten as x.

The transformations may be performed only on the entire expression and not on the subexpressions. For example, $(1 * 1) * (1 * 1)$ may only be rewritten using the first kind of transformation as $((1 * (1 * 1)) * 1) * 1$, but it cannot be transformed into $1 * (1 * 1)$ or $(1 * 1) * 1$ using a single step – in the latter two cases the second kind of transformation would have been applied just to the left or right subexpression of the form $1 * 1$.

For which natural numbers n can the expression $\underbrace{1 * (1 * (1 * (\cdots * (1 * 1))))}_{n \text{ ones}}$ be rewritten to an expression that does not include a single occurrence of the $*$ operator?

Solution.

Answer: 1, 2, 3, 4.

Let's look at the transformation operation in reverse. The end result can only be 1. This can only come from the expression $1 * 1$ using the second transformation. Any intermediate expression of the form $1 * x$ can also only be obtained with the help of the second transformation from the expression $(1 * x) * 1$ and whichever intermediate result of the from $(1 * x) * y$ also only from $((1 * x) * y) * 1$ using the second transformation. Any intermediate result of the form $((1 * x) * y) * z$ can only be obtained from the expression $x * (y * z)$ using the first transformation, because if it was obtained from the expression $(((1 * x) * y) * z) * 1$ using the second transformation, then this in turn could have only been obtained from a longer expression using the second transformation, this in turn from a longer expression and so on, none of which can be represented in the form $1 * (1 * (1 * (\cdots * (1 * 1))))$. Therefore the end result uniquely determines all the previous expressions:

$$
\begin{aligned}
1 &\stackrel{(2)}{\Longleftarrow} 1 * 1 \\
&\stackrel{(2)}{\Longleftarrow} (1 * 1) * 1 \\
&\stackrel{(2)}{\Longleftarrow} ((1 * 1) * 1) * 1 \\
&\stackrel{(1)}{\Longleftarrow} 1 * (1 * 1) \\
&\stackrel{(2)}{\Longleftarrow} (1 * (1 * 1)) * 1 \\
&\stackrel{(2)}{\Longleftarrow} ((1 * (1 * 1)) * 1) * 1 \\
&\stackrel{(1)}{\Longleftarrow} (1 * 1) * (1 * 1) \\
&\stackrel{(2)}{\Longleftarrow} ((1 * 1) * (1 * 1)) * 1 \\
&\stackrel{(1)}{\Longleftarrow} 1 * ((1 * 1) * 1) \\
&\stackrel{(2)}{\Longleftarrow} (1 * ((1 * 1) * 1)) * 1 \\
&\stackrel{(2)}{\Longleftarrow} ((1 * ((1 * 1) * 1)) * 1) * 1 \\
&\stackrel{(1)}{\Longleftarrow} ((1 * 1) * 1) * (1 * 1) \\
&\stackrel{(1)}{\Longleftarrow} 1 * (1 * (1 * 1)) \\
&\stackrel{(2)}{\Longleftarrow} (1 * (1 * (1 * 1))) * 1 \\
&\stackrel{(2)}{\Longleftarrow} ((1 * (1 * (1 * 1))) * 1) * 1 \\
&\stackrel{(1)}{\Longleftarrow} (1 * (1 * 1)) * (1 * 1) \\
&\stackrel{(2)}{\Longleftarrow} ((1 * (1 * 1)) * (1 * 1)) * 1 \\
&\stackrel{(1)}{\Longleftarrow} (1 * 1) * ((1 * 1) * 1) \\
&\stackrel{(2)}{\Longleftarrow} ((1 * 1) * ((1 * 1) * 1)) * 1 \\
&\stackrel{(1)}{\Longleftarrow} 1 * (((1 * 1) * 1) * 1) \\
&\stackrel{(2)}{\Longleftarrow} (1 * (((1 * 1) * 1) * 1)) * 1 \\
&\stackrel{(2)}{\Longleftarrow} ((1 * (((1 * 1) * 1) * 1)) * 1) * 1 \\
&\stackrel{(1)}{\Longleftarrow} (((1 * 1) * 1) * 1) * (1 * 1).
\end{aligned}
$$

However, the expression $(((1 * 1) * 1) * 1) * (1 * 1)$ cannot be an intermediate result based on what has been showed earlier. Therefore all expressions that can be transformed into 1 are shown in the chain above. Only four of them are in the required form – for $n = 1, 2, 3, 4$.

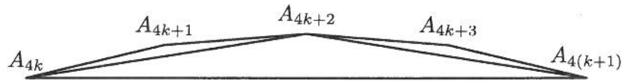
2 Combinatorics

C-1. Baron Münchhausen says that his new stained-glass window looks as inscribed 400-gon which is split by non-intersecting diagonals onto triangular pieces of glass. He claims that it is possible to construct at least $(3/2)^{400}$ different convex polygons by taking unions of these triangles. Can it be a truth?

Solution.

Answer: yes, it is possible.

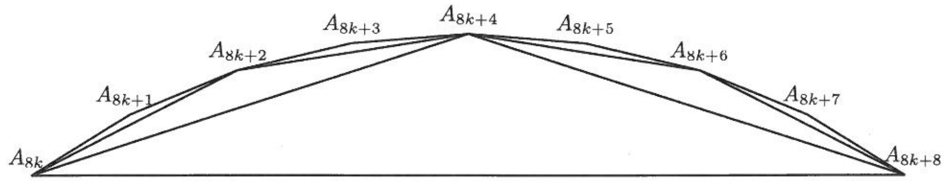
Let A_{4k}, A_{4k+1}, A_{4k+2}, A_{4k+3}, A_{4k+4} be five consecutive points of 400-gon. For each k triangulate pentagon $S_k = A_{4k}A_{4k+1}A_{4k+2}A_{4k+3}A_{4k+4}$ as follows.



It is clear that for each k we can choose 5 different convex polygons inside pentagon S_k containing segment $A_{4k}A_{4k+4}$:

$$A_{4k}A_{4k+1}A_{4k+2}A_{4k+3}A_{4k+4}, \quad A_{4k}A_{4k+1}A_{4k+2}A_{4k+4}, \quad A_{4k}A_{4k+2}A_{4k+3}A_{4k+4},$$
$$A_{4k}A_{4k+2}A_{4k+4} \quad \text{and (degenerate case)} \quad A_{4k}A_{4k+4}.$$

Now let each fragment of 400-gon containing two consecutive S_{2k} and S_{2k+1} looks as follows, denote this polygon as D_k.



For this picture we can choose $5^2+1=26$ different convex polygons, containing segment $A_{8k}A_{8k+8}$: first, we can take the degenerate case, i.e. a segment $A_{8k}A_{8k+8}$ itself; second, we can take triangle $A_{8k}A_{8k+4}A_{8k+8}$ and append one of 5 possible convex polygons inside S_{2k} and independently append one of 5 possible convex polygons inside S_{2k+1}.

After that triangulate polygon $S = A_8A_{16}A_{24}\dots A_{400}$ in an arbitrary way.

Obviously, baron Münchhausen just counted convex polygons that contain the whole polygon S and differ by parts that he chooses inside polygons D_k. All these choices are compatible and can be made independently. Thus we obtain $26^{50} = (\sqrt[8]{26})^{400} \approx (1.502)^{400} > (3/2)^{400}$ convex polygons.

C-2. Let M be a subset of a plane sufficing following properties:

1) There is no sinlge line k, such that $M \subset k$.
2) For any parallelogram $ABCD$ if $A, B, C \in M$, then $D \in M$.
3) If $A, B \in M$, then $|AB| > 1$.

Prove, that there are two families of parallel lines, such that M is a set consisting of all intersection points of lines from the first family with lines from the second family.

Solution.

At the beginning we can see that property 3) implies that

$$\text{in any bonded subset of a plain there is only finite number of points from } M. \qquad (\star)$$

Next, we can see, that if for some points $A, B \in M$ we will define by ϕ a translation by vector $\overrightarrow{AB}$, then

$$\text{for any point } C \in M \text{ we also have } \phi(C) \in M. \qquad (\star\star)$$

Indeed: thanks to property 1) we know, that there is a point $P \in M$, that does not belong to line AB. Therefore, from 2) we can imply, that there is a point $R \in M$, such that $ABRP$ is a parallelogram, and therefore $\overrightarrow{PR} = \overrightarrow{AB}$. Now it is sufficient to see, that point C does not belong to line AB or does not belong to line PR, so $\phi(C) \in M$ by property 2), as the fourth vertex of parallelogram $BAC\phi(C)$ or $RPC\phi(C)$, which concludes the proof of $(\star\star)$. It is worth mentioning, that we can say the same about ϕ^{-1} (translation by vector $\overrightarrow{BA}$). It shows, that ϕ is an one to one mapping of set M on itself.

We will show now, that we can choose such parallelogram (we will call it "basic") with vertices in M, which does not contain any other points form M (except vertices). Indeed: thanks to $(\star)$ we can pick line segment AB with ends in M, which won't contain any other points from M. Thanks to properties 1) and 2) we know, that we can find parallelogram $ABCD$ with vertices in M. If $ABCD$ is not basic, by $(\star)$, from the finitely many points from M contained inside $ABCD$ we can pick point E, that lies closest to the line AB. Then, as we know from 2) we can get parallelogram $ABEF$ with vertices in M, which either is basic or it contains point $Q \in M$ belonging $ABEF$ but not on segment EF (then translation of Q by vector $\overrightarrow{AB}$ belongs to $ABCD$ and lies closer to line AB than E, a contradiction) or it contains point $Q \in M$ on segment EF (in this case the translation of A by vector $\pm\overrightarrow{EQ}$ is a point of M laying inside segment AB, a contradiction).

To sum things up, we have to see, that if we get basic parallelogram $ABCD$ with vertices in M, and define ϕ as a translation by vector $\overrightarrow{AB}$, and define ψ as a translation by vector $\overrightarrow{AD}$, then we can define a set $Z = \{\phi^k \circ \psi^l(A) : k, l \in \mathbb{Z}\}$, which now is easy to see, is equal to M. It is obvious, that $Z = M$ fulfills the thesis.

C-3. Let n be a positive integer. Elfie the Elf lives in a three dimensional space $\mathbb{Z}^3$. She starts at the origin: $(0,0,0)$. In each turn she can teleport into any point in $\mathbb{Z}^3$ which lies at the distance $\sqrt{n}$ from her current location. However, teleportation is a complicated procedure. Elfie starts off *normal* but she turns *strange* with her first teleportation. Next time she teleports she becomes *normal* again, then *strange* again... etc.

For which n can Elfie travel to any given point in $\mathbb{Z}^3$ and be *normal* when she gets there?

Solution.

Answer: there are no such n.

We colour all the points in $\mathbb{Z}^3$ white and black: The point (x,y,z) is colored white if $x+y+z \equiv_2 0$ and black if if $x+y+z \equiv_2 1$.

After the first move Elfie is at a point (a,b,c) where $a^2+b^2+c^2 = n$. Thus, $a+b+c \equiv_2 a^2+b^2+c^2 = n$.

Now, if n is even then (a,b,c) is white. Thus, in that case Elfie only jumps between white points.

On the other hand, if n is odd, then (a,b,c) is certainly black. And one can easily see that Elfie alternates between black and white squares after each move. But since Elfie is normal after even number of moves, and is then on a white point, she can never reach any black point being normal. Thus, there no n such that Elfie can travel to any given point and be normal when she gets there.

C-4. Let b_i, c_i, $0 \le i \le 100$ be two sequences of positive integers with two exceptions: $c_0 = 0$, $b_{100} = 0$. Several villages are connected by roads, each road connects two villages which are called *neighbours* and has length 1 km. Roads do not intersect each other, but can pass over/under each other. The *distance* between two villages X and Y is the length of the shortest path between them. In this country the maximal distance between two villages equals 100 km and for every pair of villages (X,Y) (the case $X=Y$ is allowed) the following condition holds: if distance between X and Y is k km, then there are exactly b_k (c_k, respectively) neighbours of Y that are 1 km further from (closer to, respectively) X than Y. Show that the number

$$\frac{b_0 b_1 \dots b_{99}}{c_1 c_2 \dots c_{100}}$$

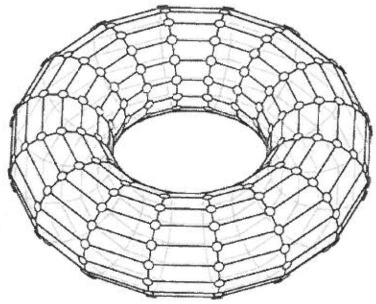
is an integer.

Solution.

Let z be an arbitrary village, $S_i(z)$ be the set of villages at distance i from z, and k_i be the number of elements in $S_i(z)$. Then the sequence k_i does not depend on z!

We prove this statement by induction on i. We will show also that $k_{i+1} = k_i \cdot \frac{b_i}{c_i}$. Clearly, $k_0 = 1$. This is the base of induction. To prove the step of induction we count roads between $S_i(z)$ and $S_{i+1}(z)$ in two ways: we can choose the village x in $S_i(z)$ and the road to $S_{i+1}(z)$ by $k_i b_i$ ways; form the other hand we can choose the village x in $S_{i+1}(z)$ and the road to $S_i(z)$ by $k_{i+1}c_{i+1}$ ways. Therefore $k_{i+1}c_i = k_i b_i$.

The problem statement follows immediately from this formula.



C-5. On a 16 x 16 torus as shown the edges are all colored red and blue in such a way that every vertex is an endpoint of an even number of red edges. A move consists of changing the color of all edges on a unit square. If one coloring can be converted to another by a sequence of moves we put the two colorings into the same box. After all the colorings are partitioned into boxes how many boxes do we have?

Solution.

Answer: 4. Representatives of the equivalence classes are: all blue, all blue with one longitudinal red ring, all blue with one transversal red ring, all blue with one longitudinal and one transversal red ring.

First, show that these four classes are non equivalent. Consider any ring transversal or longitudinal and count the number of red edges going out from vertices of this ring in the same halftorus. This number can not be changed mod 2.

Now we show that each configuration can be transformed to one of these four classes. We suggest two independent reasoning.

Scanning of the square.

Cut the torus up in a square 16×16. In order to restore the initial torus we will identify the opposite sides of the square, but we will do it in the end of solution. Now we will work with the square. It is clear that during all recolorings each vertex of torus has even red degree. The same is true for the degrees of the inner vertices of the 16×16 square when we deal with it instead of the torus.

Scan all cells of this square one by one from left to right and from bottom to top. For convenience we may think that in each moment the scanned area is colored grey. First we take bottom left corner cell ($a1$ in chess notations) and color it grey. Then we consider the next cell ($b1$ in chess notations) color it grey and if the edge between the cells $a1$ and $b1$ is red, change the colors of the cell $b1$ edges. We obtain a grey area with no red edges in its interior. After that when we scan each new cell we append this cell to the grey figure and if it is necessary change the colors of edges of the new cell to make the color of all new edges in the grey area blue.

The latter is always possible because the new cell have either one common edge with the grey figure (as in the case "$a1$–$b1$" above) or two common edges. For example let grey figure consist of the first row of the square and $a2$ cell. When we append the cell $b2$ to the grey figure two edges of its lower left corner vertex already belong to the grey figure, they are blue. Therefore the other two edges $a2$–$b2$ and $b1$–$b2$ have the same color and we can make them both blue (if they are not) by recoloring the edges of cell $b2$.

So by doing that with all cells of the square we obtain 16×16 square with blue edges inside it. Now its time to recall that the sides of the square should be identified, and the red degree of each vertex of torus is even. It follows that the whole (identified) vertical sides of the square are either red or blue, and the same for horizontal sides.

Deformations of red loops (sketch).

To see that any configuration can be made into one of the above four configurations it is most clear to cut the torus up in a square with opposite edges identified.

Since the red degree of each vertex is even we can always find a loop consisting of red edges only. Now, suppose that one can make a (simple) red loop that does not cross the boundary of the square. We can change the color of this loop by changing one by one the colors of unit squares inside it. In the remaining configuration every vertex is still an endpoint of an even number of red edges and we can repeat the operation. So by doing that to every red loop we are left with a configuration where one can not make red loops that do not intersect the boundary. Second, any red loop left that passes through more than one boundary vertex can be deformed into a loop containing only one boundary vertex. Finally, any two loops crossing the same side of the square can be removed by changing colors of all unit squares between these loops. Thus, we are left with only the four possibilities mentioned.

C-6. One of the cells of 20×20 torus contains a buried treasure. Today, in order to find the treasure we select several rectangles 1×4 or 4×1 on this torus and ask the sapper to investigate them by a mine detector. The results of all investigations well be known tomorrow, for each rectangle the sapper will tell us if the treasure is in this rectangle. What is the minimal number of rectangles we should select in order to find the cell that contains the treasure?

Solution.

Answer: 160.

In our torus each cell is determined by coordinates (i, j), $1 \le i, j \le 20$, the two cells being neighbours if one of their coordinates is the same, and the others differ by ± 1 mod 20.

Example. Select the following 160 rectangles

$$\begin{array}{llll} (a-1,b);(a-2,b);(a-3,b);(a-4,b) \pmod{20}, & \text{where } 5 \mid (a+b), & \text{and} \\ (a,b-1);(a,b-2);(a,b-3);(a,b-4) \pmod{20}, & \text{where } 5 \mid (a+b-1). & \end{array}$$

If the sapper says that the treasure belongs to only one of the rectangles, then the cell is uniquely determined, because each rectangle contains the unique cell not covered by the other rectangles. If the sapper says that the treasure belongs to two of rectangles then the treasure is in their intersection cell.

Estimation. Suppose that we select 159 rectangles only. It is clear that the torus is fully covered by the rectangles except at most one cell. Therefore at least $399\cdot 2 - 159\cdot 4 = 162$ is covered by only one rectangle, and hence two of these cells belong to the same rectangle. If the rectangle contains the treasure we can not distinguish on which of these cells it is hidden.

C-7. An invisible hare sits one of N vertices of a graph G. Several hunters try to kill the hare. Each minute all of them simultaneously shoot: each hunter shoots to a single vertex, they choose the target vertices cooperatively. If the hare was in the target vertex during a shoot, the hunting is finished. Otherwise the hare can jump to one of the neighbouring vertices or stay in its vertex.

The hunters know an algorithm that allows to kill the hare by at most $N!$ shoots. Prove that then there exists an algorithm that allows to kill the hare by at most 2^N shoots.

Solution.

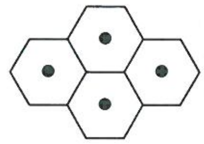
Let hunters apply optimal (fastest) algorithm. Let say that *a vertex has a smell of a hare*, if there exists an initial vertex and a sequence of moves of the hare for which the hare is still alive and now occupies this vertex. After every shoot mark the set of all the vertices that have a smell of a hare. In the beginning all the vertices of the graph have a smell of hare, and after finish of hunting this set is empty. The idea is that in optimal strategy these sets can not repeat!

Indeed, the hunting does not imply feedback, the hunters' shoots do not depend on hare's moves because the hunters try to foresee all possible moves of hare. So if a set of vertices A appears after the k-th shoot and once again after the m-th shoot, then then the strategy is not optimal because all shoots form k-th to $(m-1)$-th can be omitted with the same result of hunting.

Since it is possible to mark at most 2^N sets the hunting will finish in at most $2^N - 1$ shoots.

C-8. Olga and Sasha play a game on an infinite hexagonal grid. They take alternating turns in placing a counter on a free hexagon of their choice, with Olga opening the game. Beginning from the 2018th move, a new rule will come into play. A counter may now be placed only on those free hexagons having at least two occupied neighbours.

A player loses when she or he either is unable to make a turn, or has filled a pattern of the rhomboid shape below with counters (rotated in any possible way). Determine which player, if any, possesses a winning strategy.



Solution.

Answer: Olga has a winning strategy.

The game cannot go on forever. Draw a large hexagon enclosing all 2017 counters in play after the 2017th move, as in Figure 1. While it will be possible to place future counters in the hexagonal frame at distance 1 from the shaded part (i.e. immediately surrounding it), where D and E are located, it will be impossible to reach cells at distance 2 from the shaded part, where F is located. Indeed, in order to place a counter at F, first counters must be placed on cells D and E.

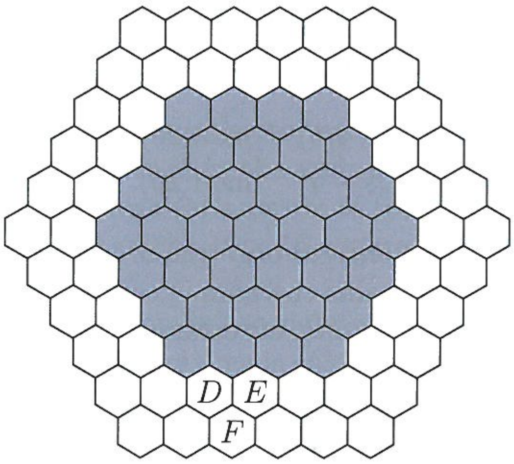


Figure 1: A large shaded hexagon enclosing all 2017 counters in play after the 2017th move.

Assume that the cells E_1, E_2, …, E_n to the right of E contain counters, but the next cell to the right is E_{n+1} and it is empty. Observe that the counter on E_{n-1} has been placed before the counter on E_n, because otherwise the forbidden rhombus is formed by the cells E_{n-1}, E_n and two ancestors of E_n in the previous row. By analogous reasoning considering the moment of placing the counter on E_{n-1} one can prove that the counter on E_{n-2} has been placed before the counter on E_{n-1}, etc. Thus we conclude that the counter on D has been placed before the counter on E. But changing the direction of our reasoning to the left we similarly conclude that counter on E has been placed before the counter on D. A contradiction.

Now, let Olga place her first counter in any hexagon H, and then respond to each of Sasha's successive moves by symmetry, choosing to place her counter on the reflexion in H of his chosen hexagon (in other words, diametrically opposite to his with respect to H). It is clear that the gameplay will be completely symmetrical after each of Olga's moves. Hence she may respond, even under the additional rule, to any move Sasha might make. It is also evident that she will never complete a forbidden rhombus if Sasha did not already do so before. Hence Olga is always certain to have a legal move at her disposal, and so will eventually win.

C-9. There are 2019 plates placed around a round table and on each of them there is one coin. Alice and Bob are playing a game that proceeds in rounds indefinitely as follows. In each round, Alice first chooses a plate on which there is at least one coin. Then Bob moves one coin from this plate to one of the two adjacent plates, chosen by him. Determine whether it is possible for Bob to select his moves so that, no matter how Alice selects her moves, there are never more than two coins on any plate.

Solution.

Answer: Yes, it is possible.

We provide a suitable strategy for Bob. Given a configuration of coins on the plates, let a block be any inclusion-wise maximal contiguous interval consisting of non-empty plates. The idea of Bob's strategy is to maintain the following invariant throughout the game: in every block, all plates except at most one contain exactly one coin, while the remaining one contains two coins. Since the total number of coins is always equal to the total number of plates, this is equivalent to the following condition: either every plate contains exactly one coin (as in the initial configuration), or in every block there is exactly one plate with two coins and all the other plates of the block contain one coin each, and moreover the blocks are delimited by single plates containing zero coins. It now suffices to show that in any configuration C satisfying the invariant, regardless of which plate Alice picks, Bob can always select his move so that the invariant is maintained after the move. We consider two cases: either Alice picks a plate with two coins, or with one coin. Suppose first that Alice picks a plate with two coins. If any of the adjacent plates contains one coin, then Bob moves a coin to this plate and the invariant is maintained - the set of blocks remains unchanged and only within one block the plate with two coins has moved. If both of the adjacent plates contain zero coins, then Bob moves a coin to any of them. Thus, one single-plate block disappears and some other block gets extended with two plates with one coin each; hence, the invariant is maintained.

Suppose now that Alice picks a plate with one coin. If the configuration is as the initial one - every plate contains one coin - then any move of Bob maintains the invariant. Otherwise, within the block B containing the plate P chosen by Alice there is another plate P' containing two coins, and B does not contain all the plates. Without loss of generality, suppose that in order to get from P' to P within B one needs to go in the clockwise direction. Then the move of Bob is to move the coin from P also in the clockwise direction. Then either P is the clockwise endpoint of B, and we just move one plate with one coin from B to the next block in the clockwise direction, or P is not the clockwise endpoint of B, and the move results in dividing B into two blocks, each containing exactly one plate with two coins. In both cases, the invariant is maintained.

C-10. Positive integers from 1 to n are written on the blackboard. The first player chooses a number and erases it. Then the second player chooses two consecutive numbers and erases them. After that the first player chooses three consecutive numbers and erases them. And finally the second player chooses four consecutive numbers and erases them. What is the smallest value of n for which the second player can ensure that he completes both his moves?

Solution.

Answer: $n = 14$.

At first, let's show that for $n = 13$ the first player can ensure that after his second move no 4 consecutive numbers are left. In the first move he can erase number 4 and in the second move he can ensure that numbers 8, 9 and 10 are erased. No interval of length 4 is left.

If $n = 14$ the second player can use the following strategy. Let the first player erase number k in his first move, because of symmetry assume that that $k \leq 7$. If $k \geq 5$ then the second player can erase $k+1$ and $k+2$ and there are two intervals left of length at least 4: $1..(k-1)$ and $(k+3)..14$, but the first player can destroy at most one of them. But if $k \leq 4$, then the second player can erase numbers 9 and 10 in his first move and again there are two intervals left of length at least 4: $(k+1)..8$ and $11..14$.

C-11. Grandfather has a finite number of empty dustbins in his attic. Each dustbin is a rectangular parallelepiped with integral side lengths. A dustbin can be thrown away into another iff the side lengths of these dustbins can be set to one-to-one correspondence in such a way that the side lengths of the first dustbin are less than the corresponding side lengths of the other dustbin. No dustbin can contain two other dustbins unless the latters have been placed one into another. Grandfather wants to throw away as many dustbins as possible for saving space. He developed the following algorithm for it: find the longest chain of dustbins that can be thrown away into each other, then repeat the same with remaining dustbins, etc., until no more dustbins can be thrown away. When following this algorithm, the longest chain of dustbins to be chosen turned out to be unique at each step. Is it necessarily true that, as the result of the process, the maximal possible number of dustbins have been thrown away?

Solution.

Answer: No.

Suppose grandfather has 6 dustbins with sizes $20 \times 20 \times 20$, $19 \times 19 \times 19$, $16 \times 16 \times 16$, $21 \times 18 \times 15$, $18 \times 15 \times 12$ and $17 \times 14 \times 11$. The first dustbin can contain the second one, the second can contain the third or the fifth, the fifth can contain the sixth. The fourth also can contain the fifth. It is impossible to throw the first and the fourth into each other, the second and the fourth into each other, the third and the fourth into each other, the third and the fifth into each other, or the third and the sixth into each other. In the longest chain of dustbins that can be thrown into each other is 4 dustbins: the sixth can be thrown into the fifth, which can be thrown into the second, which can be thrown into the first. As the remaining two dustbins cannot be thrown into each other, 3 dustbins in total are not thrown away. However, by throwing the third dustbin into the second, the second into the first, the sixth into the fifth and the fifth into the fourth, only 2 dustbins are not thrown away. Hence grandfather's algorithm does not provide an optimal solution.

3 Geometry

G-1. Points A, B, C, D lie, in this order, on a circle ω, where AD is a diameter of ω. Furthermore, $AB = BC = a$ and $CD = c$ for some relatively prime positive integers a and c. Show that if the diameter d of ω is also an integer, then d is a perfect square or $2d$ is a perfect square.

Solution.

By Pythagoras, the lengths of the diagonals of quadrangle $ABCD$ are $\sqrt{d^2 - a^2}$ and $\sqrt{d^2 - c^2}$. Applying Ptolemaios' Theorem to the quadrilateral $ABCD$ gives

$$\sqrt{d^2 - a^2} \cdot \sqrt{d^2 - c^2} = ab + ac,$$

which after squaring and simplifying becomes

$$d^3 - (2a^2 + c^2)d - 2a^2c = 0.$$

Then $d = -c$ is a root of this equation, hence, $c + d$ is a positive factor of the left-hand side. Hence, the remaining factor (which is quadratic in d) must vanish, and we obtain $d^2 = cd + 2a^2$. Let $e = 2d - c$. The number $c^2 + 8a^2 = (2d - c)^2 = e^2$ is a square, and it follows that $8a^2 = e^2 - c^2$. If e and c both were even, then by $8 \mid (e^2 - c^2)$ we also have $16 \mid (e^2 - c^2) = 8a^2$ which implies $2 \mid a$, a contradiction to the fact that a and c are relatively prime. Hence, e and c both must be odd. Moreover, e and c are obviously relatively prime. Consequently, the factors on the right-hand side of $2a^2 = \frac{e-c}{2} \cdot \frac{e+c}{2}$ are relatively prime. It follows that $d = \frac{e+c}{2}$ is a perfect square or twice a perfect square.

G-2. Given a triangle Δ with circumradius R and inradius r, prove that the area of the circle with radius $R + r$ is at least 5 times greater than the area of the triangle Δ.

Solution.

Let the area of the triangle Δ be S. Among the triangles with fixed circumradius, the one with largest perimeter is equilateral (as can be easily seen from Jensen's inequality). Hence

$$S = \frac{a+b+c}{2} \cdot r \leq \frac{3\sqrt{3}}{2} Rr.$$

By Euler's inequality, $R \geq 2r$. Thus

$$R^2 + r^2 = \frac{3}{4}R^2 + \left(\frac{1}{4}R^2 + r^2\right) \geq \frac{3}{2}Rr + Rr = \frac{5}{2}Rr.$$

Hence

$$\pi(R+r)^2 \geq \pi \cdot \frac{9}{2}Rr > 5 \cdot \frac{3\sqrt{3}}{2}Rr \geq 5S.$$

Remark. It is possible to make the problem easier by replacing 5 with 4, 5. Then one can use easier inequality $a, b, c \leq 2R$ in the first part (yielding $S \leq 3Rr$), or instead of using Euler's inequality in the second part, one can use AM-GM ($(R+r)^2 \geq 4Rr$).

G-3. A convex quadrilateral $ABCD$ is right-angled at A and fulfils $BC + CD = 1$. Determine its greatest possible area.

Solution 1.

Answer: $\frac{1}{8}(1+\sqrt{2})$.

Reflect the quadrilateral in line AB, and reflect the resulting pentagon again in line AD; see Figure 1.This produces an octagon of fixed perimeter

$$4(BC + CD) = 4$$

and area four times that of $ABCD$. The maximal area is obtained for a regular octagon of edge length $\frac{1}{2}$. Since the area of a regular octagon of edge a is known (or easily verified) to be $2(1+\sqrt{2})a^2$, the maximal area of the original quadrilateral is $\frac{1}{4}\cdot 2(1+\sqrt{2})\left(\frac{1}{2}\right)^2 = \frac{1}{8}(1+\sqrt{2})$.

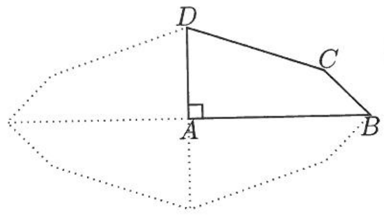


Fig. 1. Double reflexion into an octagon.

Solution 2.

Let us fix the segment BD, and consider the points A and C as variable.

The locus of points A fulfilling $\angle A = 90°$ is a (semi-)circle with diameter BD. In order to maximise the area of triangle BAD, the length of the altitude from A must be maximised, which occurs when A lies on the perpendicular bisector of BD, so that $AB = AD$.

The locus of points C fulfilling $BC+CD = 1$ is (an arc of) an ellipse with foci B and D. Again, in order to maximise the area of triangle BCD, the length of the altitude from C must be maximised, which again occurs for C on the perpendicular bisector of BD, so that $BC = CD = \frac{1}{2}$.

The maximal quadrilateral satisfying the conditions will thus be mirror-symmetric with

$$AB = AD \qquad \text{and} \qquad BC = CD = \frac{1}{2},$$

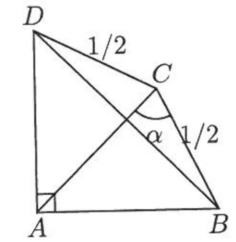


Fig. 2. Symmetric quadrilateral.

as in Figure 2.

Put $\alpha = \angle BCA = \angle DCA$. Using some trigonometry, we find $BD = \sin\alpha$ and $AB = \frac{1}{\sqrt{2}}\sin\alpha$, so that the area of $ABCD$ can be expressed as

$$\begin{aligned}
|ABCD| &= |BCD| + |BAD| \\
&= \frac{1}{8}\sin 2\alpha + \frac{1}{4}\sin^2\alpha \\
&= \frac{1}{8}(\sin 2\alpha + 1 - \cos 2\alpha) \\
&= \frac{1}{8}\left(1+\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin 2\alpha - \frac{1}{\sqrt{2}}\cos 2\alpha\right)\right) \\
&= \frac{1}{8}\left(1+\sqrt{2}\sin(2\alpha - 45°)\right).
\end{aligned}$$

Since $0° \le \alpha \le 90°$, this function obtains its unique maximum $\frac{1}{8}(1+\sqrt{2})$ for $\alpha = 62.5°$.

G-4. Let R be the radius of the circumcircle of a regular 2018-gon $A_1A_2\dots A_{2018}$. Prove that

$$A_1A_{1008} - A_1A_{1006} + A_1A_{1004} - A_1A_{1002} + \dots + A_1A_4 - A_1A_2 = R.$$

Solution.

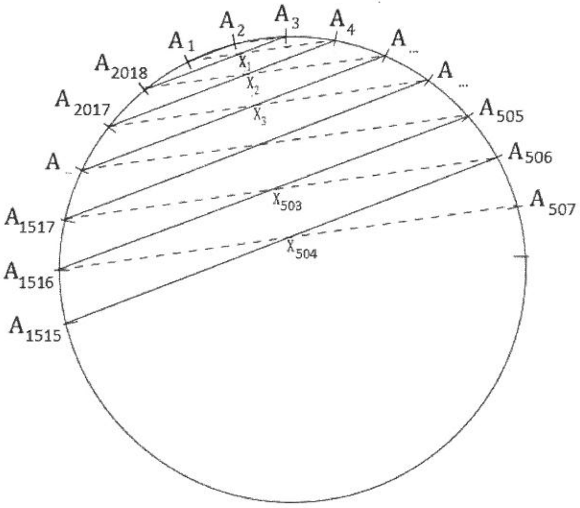
Let X_1 be the intersection of A_1A_4 and $A_{2018}A_3$, X_2 be the intersection of $A_{2018}A_5$ and $A_{2017}A_4$, $\dots$, X_{504} be the intersection of $A_{1516}A_{507}$ and $A_{1515}A_{506}$. All quadrilaterals $A_1A_2A_3X_1$, $A_{2018}X_1A_4X_2$, $A_{2017}X_2A_5X_3$, $\dots$, $A_{1516}X_{503}A_{506}X_{504}$ are parallelograms since the following lines are parallel:

$$A_1A_2 \parallel A_{2018}A_3 \parallel A_{2017}A_4 \parallel \dots \parallel A_{1515}A_{506} \quad \text{and} \quad A_2A_3 \parallel A_1A_4 \parallel A_{2018}A_5 \parallel \dots \parallel A_{1516}A_{507}.$$

It means that

$$\begin{aligned}
A_{2018}X_1 &= A_{2018}A_3 - X_1A_3 = A_{2018}A_3 - A_1A_2\\
A_{2017}X_2 &= A_{2017}A_4 - X_2A_4 = A_{2017}A_4 - A_{2018}X_1 = A_{2017}A_4 - A_{2018}A_3 + A_1A_2\\
A_{2016}X_3 &= A_{2016}A_5 - X_3A_5 = A_{2016}A_5 - A_{2017}X_2 = A_{2016}A_5 - A_{2017}A_4 + A_{2018}A_3 - A_1A_2\\
&\dots\\
A_{1516}X_{503} &= A_{1516}A_{505} - X_{503}A_{505} = A_{1516}A_{505} - A_{1517}X_{502} =\\
&= A_{1516}A_{505} - A_{1517}A_{504} + A_{1518}A_{503} - A_{1519}A_{502} + \dots + A_{2018}A_2 - A_1A_2 \qquad (*)
\end{aligned}$$

Note that X_{504} is the circumcenter of the given 2018-gon as it is the intersection of two diameters $A_{1516}A_{507}$ and $A_{1515}A_{506}$. Therefore from the parallelogram $A_{1516}X_{503}A_{506}X_{504}$ we can conclude that $A_{1516}X_{503} = X_{504}A_{506} = R$. To complete the proof one has to substitute in the right hand side of $(*)$ $A_{1516}A_{505} = A_1A_{1008}$, $A_{1517}A_{504} = A_1A_{1006}$, $\dots$, $A_{2018}A_2 = A_1A_4$.



G-5. Let ABC be an acute triangle, H its orthocentre, and M the midpoint of BC. Furthermore, let k_1 and k_2 be the circle with diameter AH and the circle with center M that touches the circumcircle of triangle ABC interiorly, respectively. Prove that k_1 and k_2 are touching circles.

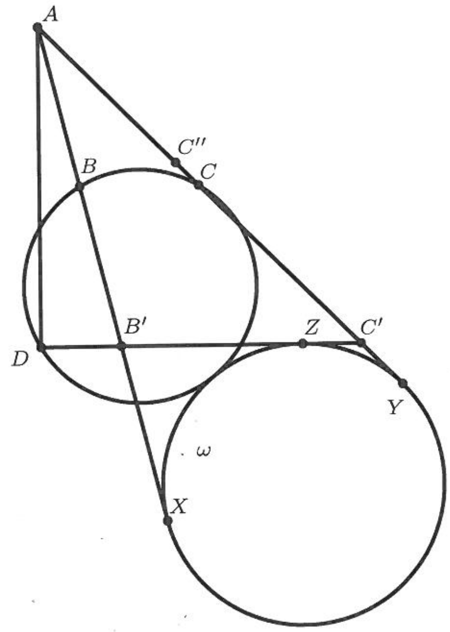
Solution.

Let N be the midpoint of AH (and of k_1), and let X be the image of H with respect to reflection about M. Then X lies on the circumcircle of ABC, opposite to A. As OM and AH are parallel, by the Intercept Theorem, we have $AH = 2OM$. Hence, $AN = OM$, i.e., $ANMO$ is a parallelogram. Let r_1 and r_2 be the radii of k_1 and k_2, respectively, and let R be the radius of ABC's circumcircle. Then $R - r_2 = OM = AN = r_1$ and, hence, $r_1 + r_2 = R = AO = NM$. This means that the distance between the midpoints of k_1 and k_2 is the sum of their radii. Consequently, k_1 and k_2 touch each other.

Remark. It is easy to show that the touching point of k_1 and k_2 lies on the bisector of $\angle BAC$.

G-6. Let ω be a circle and A a point outside of ω. Draw the tangents from A to ω and call the points of tangency X and Y. Let B and C be points on the segments AX and AY, respectively, such that the perimeter of $\triangle ABC$ is equal to the length of the segment AX. Let D be the reflection of A in the line BC. Show that the circumcircle BDC touches ω.

Solution.



Let B' be the reflection of A through B. Since the perimeter of $\triangle ABC$ equals the length of the segment AX, AB is less than half of AX and, therefore, B' lies on the segment AX.

Let the point C' lie on AY such that $B'C'$ touches ω in the point Z. Let C'' be the midpoint of AC'. Since B and C'' are midpoints of the sides AB' and AC', respectively, we have that the perimeter of $\triangle AB'C'$ is double that of $\triangle ABC''$.

We also have that the perimeter of $\triangle AB'C'$ equals

$$\begin{aligned}|AB'| + |B'C'| + |C'A| &= |AB'| + |B'Z| + |ZC'| + |C'A| \\ &= |AB'| + |B'X| + |YC'| + |C'A| \\ &= |AX| + |AY| \\ &= 2|AX|.\end{aligned}$$

Therefore, the perimeter of triangles $\triangle ABC$ and $\triangle ABC''$ is the same, namely $|AX|$.

If we assume that C'' lies between A and C we have that $|BC''| + |AC''| = |BC| + |AC|$, so

$$|BC''| = |BC| + |CC''|.$$

Which contradicts the triangle inequality, so C'' does not lie between A and C. Similarly, C'' cannot lie between C and Y, and must therefore lie on C. Hence, C and C'' are the same point so C is the midpoint of AC'.

Now, ω is tangent to the extensions of the sides AB' and AC' of $\triangle AB'C'$ as well as being tangent to the side $B'C'$ and is therefore an excircle of the triangle.

Also, B and C are the midpoints of sides AB' and AC', respectively.

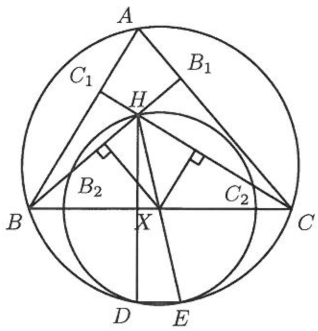
We have that the point D lies on the line $B'C'$, for D, B' and C' are reflections of A through points on BC. Also, since $BC || B'C'$, and $AD \perp BC$, we have that $AD \perp B'C'$. So D is the foot of the altitude from A in $\triangle AB'C'$. Thus, the circle through B, D, C is the nine-point circle of $\triangle AB'C'$. According to Feuerbach's theorem, it touches the excircle ω, as required.

G-7. Heights BB_1 and CC_1 of acute triangle ABC intersect in point H. B_2 and C_2 are points on segments BH and CH respectively such that $BB_2 = B_1H$ and $CC_2 = C_1H$. Circumcircle of the triangle B_2HC_2 intersects circumcircle of triangle ABC in points D and E. Prove that triangle DEH is right.

Solution.

Despite of the logical symmetry of the picture the right angle in triangle $\triangle DEH$ is not H but either D or E.

Denote by w the circumcircle of the triangle B_2HC_2. Midperpendicular to the segment C_2H is also the midperpendicular to CC_1 therefore it passes through the midpoint X of side BC. By the similar reasoning the midperpendicular to B_2H passes through X. Therefore X is the center of the circle w.

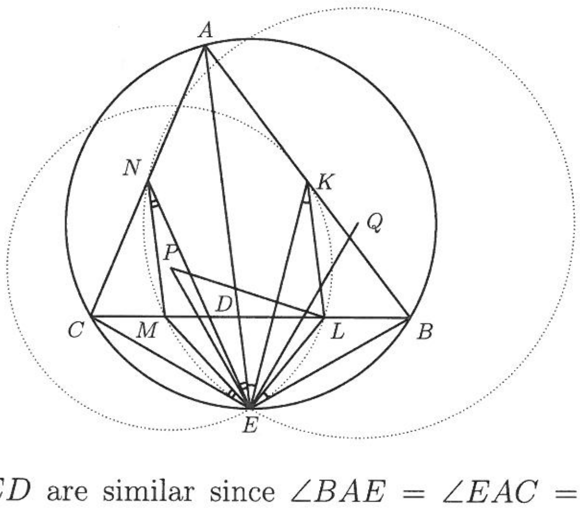


It is well known that the point which is symmetrical to the orthocenter H with respect to the side BC belongs to the circumcircle of the triangle ABC. The distance from this point to X equals XH due to symmetry, hence this point belongs w, therefore it coincides with D or E, without loss of generality with D. Thus $DH \perp BC$.

Finally, the centers of w and circumcircle (ABC) belong to the midperpendicular of BC, therefore their common chord DE is parallel BC. Thus $\angle HDE = 90°$.

G-8. AD is a bisector if the triangle ABC. Line AD intersects second time the cirsumcircle of $\triangle ABC$ in point E. Let K, L, M and N be the midpoints of the segments AB, BD, CD and AC respectively, P be the circumcenter of the triangle EKL, Q be the circumcenter of the triangle EMN. Prove that $\angle PEQ = \angle BAC$.

Solution.



Triangles AEB and BED are similar since $\angle BAE = \angle EAC = \angle DBE$. Hence $\angle AEK = \angle BEL$ as the angles between a median and a side in similar triangles. Denote these angles by φ. Then $\angle EKL = \varphi$ since KL is a midline of $\triangle ABD$. Analogously, let $\psi = \angle AEN = \angle CEM = \angle ENM$. And let $\beta = \angle ABC$, $\gamma = \angle ACB$.

The triangle PEL is isosceles, therefore $\angle PEL = 90° - \frac{1}{2}\angle EPL = 90° - \angle EKL = 90° - \varphi$ and

$$\angle PEA = \angle PEL - \angle AEL = \angle PEL - (\angle AEB - \angle BEL) = 90° - \varphi - (\gamma - \varphi) = 90° - \gamma.$$

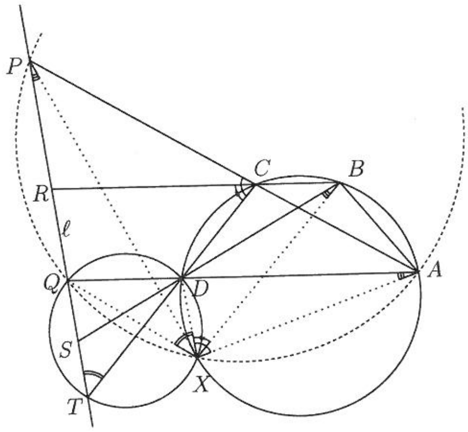
Analogously $\angle QEA = 90° - \beta$.

Thus $\angle PEQ = \angle PEA + \angle QEA = 180° - \beta - \gamma = \angle BAC$.

G-9. Let $ABCD$ be a trapezoid with $AD \parallel BC$ and $\angle ADC = \angle BAD$, and let ℓ be a line not intersecting the line segments AC or BD. Assume that ℓ intersects the lines AC, AD, BC, BD and CD in the points P, Q, R, S and T respectively. Show that the three circles $\odot(DQT)$, $\odot(BRQ)$ and $\odot(BPS)$ intersect in a common point, where $\odot(XYZ)$ denotes the circumcircle of XYZ.

Remark. The condition that ℓ does not intersect the linesegments AC or BD is not strictly needed, but it reduces casework.

Solution.



Note first that $ABCD$ is concyclic since it is an isosceles trapezoid. We define X to be the intersection of $\odot(DQT)$ and $\odot(ABCD)$, and it now suffices to prove that $BRQX$ and $BPSX$ are cyclic quadrilaterals. We have that

$$\begin{aligned}\angle BXQ &= \angle BXD + \angle DXQ = (\pi - \angle BCD) + \angle DTQ \\ &= \angle RCD + \angle CDR = \angle RCT + \angle CTR = \pi - \angle CRT\end{aligned}$$

and hence $BRQX$ is a cyclic quadrilateral. Now observe that

$$\begin{aligned}\angle AXQ = \angle AXD + \angle DXQ &= (\pi - \angle ACD) + \angle DTQ \\ &= \angle PCD + \angle CTP = \angle PCT + \angle CTP \\ &= \pi - \angle CPT = \pi - \angle APQ\end{aligned}$$

so $APQX$ is concyclic, and it now follows that:

$$\angle SBX = \angle DBX = \angle DAX = \angle QAX = \angle QPX = \angle SPX$$

Hence, $BPSX$ is a cyclic quadrilateral.

Remark 1. Note that we only use that $ABCD$ is cyclic, and one can in fact replace $ABCD$ an isosceles trapezoid wtih the weaker condition $ABCD$ cyclic. The stronger condition only serves the purpose as a red herring.

Remark 2. One can entirely avoid angles and instead apply the extended Miquel's theorem three times:
Letting $X = \odot(DQT) \cap \odot(ABCD)$, we get that $BRQX$ is cyclic by application of the extended Miquel's theorem on $\triangle CRT$ and circles $\odot(CDB), \odot(TDQ)$ and $\odot(RQB)$.
Applying the theorem again on $\triangle CPT$ and circles $\odot(CDA), \odot(TDQ)$ and $\odot(PQA)$ we get that $APQX$ is cyclic.
Having established that $APQX$ is cyclic, it is also follows that $BPSX$ is cyclic by applying the extended Miquel's theorem on $\triangle DQS$ and the circles $\odot(DAB), \odot(QAP)$ and $\odot(SBQ)$.

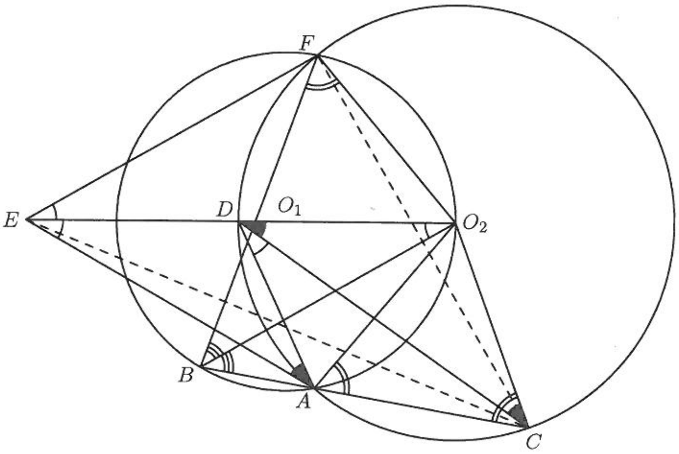
G-10. Let ω_1 and ω_2 be two circles with centers O_1 and O_2, respectively, with O_2 lying on ω_1. Let A be a common point of ω_1 and ω_2. A line through A intersects ω_1 in $B \neq A$ and ω_2 in $C \neq A$ such that A lies between B and C. The ray O_2O_1 intersects ω_2 in D and contains a point E such that $\angle EAD = \angle DCO_2$ and D lies between O_2 and E. Show that BO_2 bisects CE.

Solution 1.

Let F be the second intersection of ω_1 and ω_2. Notice that

$$\angle BFO_2 = 180^\circ - \angle BAO_2 = \angle CAO_2 = \angle O_2CA$$

and since $AO_2 = EO_2$ that $\angle FBO_2 = \angle O_2BA$. It follows that F is the reflection of C over BO_2, thus it suffices to prove that EF is parallel to BO_2, as then BO_2 is a midline of triangle CEF.



Notice that $\angle O_2DC = \angle DCO_2 = \angle EAD$. Hence by symmetry about O_1O_2, we have

$$\begin{aligned}\angle FEO_2 = \angle AED &= 180^\circ - \angle EAD - \angle ADE \\ &= \angle ADO_2 - \angle CDO_2 = \angle ADC = \frac{1}{2}\angle AO_2C\end{aligned}$$

Moreover

$$\begin{aligned}90^\circ - \angle O_1O_2B &= \frac{1}{2}\angle BO_1O_2 = 180^\circ - \angle BAO_2 \\ &= \angle CAO_2 = 90^\circ - \frac{1}{2}\angle AO_2C\end{aligned}$$

so we conclude that $\angle FEO_2 = \angle EO_2B$ which proves that EF and BO_2 are parallel.

Solution 2.

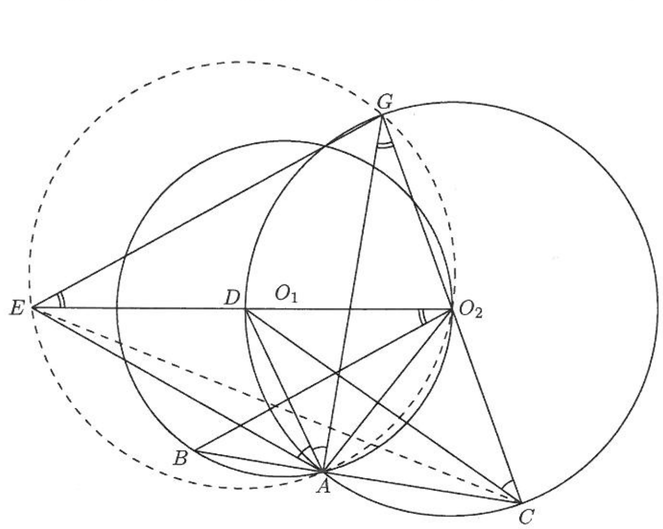
Let G be the point on ω_2 diametrically opposite C. As in the first solution, it suffices to prove that EG is parallel to BO_2, as then BO_2 is a midline of triangle CEG. Note first that

$$\angle GO_2E = 2\angle ECD = \angle DAE + \angle GAD = \angle GAE$$

so the points A, E, G, and O_2 lie on a circle. Since $GO_2 = AO_2$, we see that O_2 is the midpoint of the arc AG of this circle. It follows then that

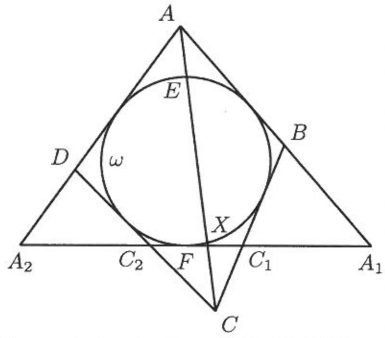
$$\angle GEO_2 = AGO_2 = \frac{1}{2}\angle AO_2C$$

and as we showed in the first solution this implies that $\angle GEO_2 = \angle EO_2B$, so lines EG and BO_2 are parallel.



G-11. Quadrilateral $ABCD$ is circumscribed about a circle ω. E is the intersection point of ω and the diagonal AC, which is nearest to A. Point F is diametrically opposite to point E in the circle ω. The line which is tangent to ω in the point F intersects lines AB and BC in points A_1 and C_1, and lines AD and CD in points A_2 and C_2 respectively. Prove that $A_1C_1 = A_2C_2$.

Solution.



Denote by X the intersection point of the lines A_1A_2 and AC. Prove that X is a contact point of escribed circle of $\triangle AA_1A_2$ with side A_1A_2. Indeed, consider a homothety with center A which maps incircle ω of $\triangle AA_1A_2$ to its escribed circle. This homothety maps the line that is tangent to ω in point E to the parallel line which is tangent to the escribed circle, i.e. to the line A_1A_2. Therefore the point E maps to the point X, hence A_1A_2 is tangent to the escribed circle of $\triangle AA_1A_2$ in the point X.

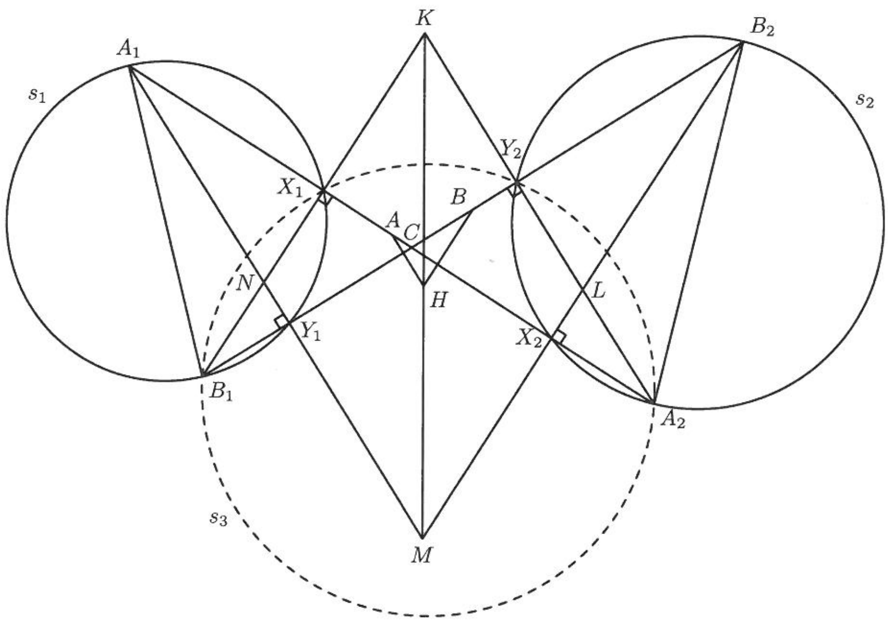
One can similarly prove that X is a tangent point of the line c C_1C_2 and incircle of $\triangle C_1CC_2$.

From the first statement we conclude that $A_1X = FA_2$, and from the second one that $C_1X = FC_2$. It remains to subtract the second equality from the first one.

G-12. Given two circles on the plane do not intersect. We choose diameters A_1B_1 and A_2B_2 of these circles such that the segments A_1A_2 and B_1B_2 intersects. Let A and B be the midpoints of segments A_1A_2 and B_1B_2, C be its intersection point. Prove that the orthocenter of the triangle ABC belongs to the fixed line that does not depend on the choice of the diameters.

Comment. The statement is true if the lines (not necessarily segments) A_1A_2 and B_1B_2 intersect. We formulate the statement for the segments in order to reduce the number of cases in solutions.

Solution.



Prove that the orthocenter H of $\triangle ABC$ belongs to their radical axe.

Denote the circles by s_1 и s_2. Let the line A_1A_2 intrersect circles s_1 and s_2 second time in points X_1 and X_2 respectively, and the line B_1B_2 intrersect the circles second time in points Y_1 and Y_2.

The lines A_1Y_1 and A_2Y_2 are parallel (because both of them are orthogonal to B_1B_2), analogously B_1X_1 and B_2X_2 are parallel. Hence these four lines form a parallelogram $KLMN$ (see fig.). It is clear that perpendiculars from the point A to the line BC and from the point B to the line AC lay on the midlines of this parallelogram. Therefore H is the center of parallelogram $KLMN$ and coincide with the midpoint of segment KM.

In order to prove that H lays on the radical axe of s_1 and s_2 it is sufficient to show that both points K and M belong to that radical axe.

The points X_1 and Y_2 lay on the circle s_3 with diameter B_1A_2. The line B_1X_1 is radical axe of s_1 and s_3, and the line A_2Y_2 is radical axe of s_2 and s_3. Therefore k is radical center of these three circles and hence K lays on the radical axe of s_1 and s_2. Analogously M lays on the radical axe of s_1 and s_2.

4 Number Theory

NT-1. Prove that there are infinitely many positive integers n, which are not divisible by 10 and such that $s(n^2) < s(n) - 5$ where $s(n)$ is the sum of digits of n.

Solution.

All integers of the form $499\ldots99$ satisfy the condition. Indeed, if $n = 4\underbrace{99\ldots99}_{k} = 5 \cdot 10^k - 1$ then

$$n^2 = 25 \cdot 10^{2k} - 10^{k+1} + 1 = 24\underbrace{99\ldots9}_{k-1}\underbrace{00\ldots00}_{k}1.$$

In such a case $s(n) = 4 + 9k$, but $s(n^2) = 7 + 9(k-1) = 9k - 2$.

Remark of the Problem committee. We suggest more elegant and sharp question.

Prove that for each N there exists a positive integer n such that $s(n) - s(n^2) > N$.

Solution.

Consider a sequence $10^{3m} - 10^{2m} - 1$. Similarly to the original solution it is easy to check that if m increases by 1 then $s(n)$ increases by 27, but $s(n^2)$ increases by 18 only.

NT-2. Let's say that a digit is eternal for a positive integer n, if it is contained in every multiple of n. Find all digits which are eternal for at least one positive integer.

Solution.

The only such a digit is 0, it is contained in every multiple of 10. Let's show that no other digit is eternal for any positive integer.

Assume that some digit is eternal for integer n. Consider remainders of numbers

$$1, \quad 11, \quad 111, \quad \ldots, \quad \underbrace{11..11}_{n+1}$$

modulo n. By the pigeonhole principle two of these remainders are equal, therefore their difference which has the form 11..100..0, is a multiple of n. If we multiply this number by 2 then we get a multiple of n of the form 22..200..0. But the only common digit for these two multiples is 0.

NT-3. Let p be an odd prime. Find all positive integers n for which $\sqrt{n^2 - np}$ is a positive integer?

Solution.

Answer: $n = \left(\frac{p+1}{2}\right)^2$.

Assume that $\sqrt{n^2 - pn} = m$ is a positive integer. Then $n^2 - pn - m^2 = 0$, and hence

$$n = \frac{p \pm \sqrt{p^2 + 4m^2}}{2}.$$

Now $p^2+4m^2 = k^2$ for some positive integer k, and $n = \frac{p+k}{2}$ since $k > p$. Thus $p^2 = (k+2m)(k-2m)$, and since p is prime we get $p^2 = k + 2m$ and $k - 2m = 1$. Hence $k = \frac{p^2+1}{2}$ and

$$n = \frac{p + \frac{p^2+1}{2}}{2} = \left(\frac{p+1}{2}\right)^2$$

is the only possible value of n. In this case we have

$$\sqrt{n^2 - pn} = \sqrt{\left(\frac{p+1}{2}\right)^4 - p\left(\frac{p+1}{2}\right)^2} = \frac{p+1}{2}\sqrt{\left(\frac{p^2+1}{2}\right)^2 - p} = \frac{p+1}{2} \cdot \frac{p-1}{2}.$$

NT-4. Prove, that for any $p, q \in \mathbb{N}$, such that $\sqrt{11} > \frac{p}{q}$, following inequality holds:

$$\sqrt{11} - \frac{p}{q} > \frac{1}{2pq}.$$

Solution.

We can assume that p and q are coprime, and since both sides of first inequality are positive, we can change it to $11q^2 > p^2$. The same way we can change second inequality:

$$11p^2q^2 > p^4 + p^2 + \frac{1}{4}.$$

To see this one holds, we will prove stronger one:

$$11p^2q^2 \geq p^4 + 2p^2.$$

Indeed, dividing this inequality by p^2 we get $11q^2 \geq p^2+2$, and since we already know that $11q^2 > p^2$ we only have to see, that $11q^2$ can't be equal to $p^2 + 1$. Since we know that the only reminders of squares (mod 11) are 0, 1, 3, 4, 5 and 9, p^2+1 can't be divisible by 11, and therefore $11q^2 \neq p^2+1$.

NT-5. Let $n \geq 3$ be an integer, such that $4n + 1$ is a prime number. Prove that $4n + 1$ divides $n^{2n} - 1$.

Solution.

Since $p := 4n + 1$ is a prime number, each non-zero remainder modulo p possesses a unique multiplicative inverse. Since $-4 \cdot n \equiv 1 \bmod p$, we have $n \equiv (-4)^{-1} \bmod p$, from which we deduce that $n \equiv -(2^{-1})^2$. Consequently,

$$n^{2n} - 1 \equiv \left(-(2^{-1})^2\right)^{2n} - 1 \equiv (2^{-1})^{4n} - 1 \equiv 0 \bmod p,$$

by Fermat's Little Theorem.

NT-6. Fix a prime p, and let m be a number for which $p \mid m^2 - 2$. Suppose there exists a number a for which $p \mid a^2 + m - 2$. Prove there exists a number b for which

$$p \mid b^2 - m - 2.$$

Solution.

Computing modulo p, our assumptions are $m^2 \equiv 2$ and $2 - m \equiv a^2$.

Suppose first that $m \equiv 2 \pmod p$. In that case $2 \equiv m^2 \equiv 4$, so we conclude

$$p = 2, \quad m \equiv 0 \pmod 2 \quad \text{and} \quad a \equiv 0 \pmod 2.$$

We may choose $b = 0$.

Assuming now $m \not\equiv 2 \pmod p$, the number $2 - m \equiv a^2$ is inversible modulo p. From

$$(2+m)(2-m) = 4 - m^2 \equiv 2 \equiv m^2 \pmod p,$$

we deduce

$$2 + m \equiv m^2(2-m)^{-1} = m^2 a^{-2} \pmod p,$$

hence we may choose $b = ma^{-1}$.

NT-7. Let $\mathbb{T} = \{1, 3, 6, 10, 15 \ldots\}$ be the set of triangular numbers, i.e. numbers of the form $T_n = \frac{n(n+1)}{2}$. Let f be a function defined on the set of positive integers such that

1) $f(n)$ is a positive integer for each n;
2) $f(uv) = f(u)f(v)$ for any pair (u, v) of coprime numbers;
3) $f(a + b + c) = f(a) + f(b) + f(c)$ for $a, b, c \in \mathbb{T}$.

Prove that $f(n) = n$ for all n.

Solution.

It is not difficult to find $f(n)$ for small n:

$f(1 \cdot 1) = f(1)f(1)$ therefore $f(1) = 1$;

$f(3) = f(1) + f(1) + f(1) = 3$;

$f(5) = f(1 + 1 + 3) = 5$;

$f(10) = f(1 + 3 + 6) = 4 + 3f(2)$ and $f(10) = f(2 \cdot 5) = f(2)f(5) = 5f(2)$ therefore $f(2) = 2$.

Now we use induction. Suppose that $f(n) = n$ for all $n < N$. Let us show that $f(N) = N$. Since f is multiplicative we may assume that $N = p^r$ for some prime p. Consider several similar cases.

1) $N = 3^r$. Then

$$f(3T_{3^r-1}) = 3f(T_{3^r-1}) = 3f\Big(\frac{3^{r-1}(3^{r-1}+1)}{2}\Big) = 3f(3^{r-1})f\Big(\frac{3^{r-1}+1}{2}\Big)$$

And from the other hand

$$f(3T_{3^r-1}) = f\Big(\frac{3^r(3^{r-1}+1)}{2}\Big) = f(3^r)f\Big(\frac{3^{r-1}+1}{2}\Big)$$

So we conclude that $f(3^r) = 3^r$ since $f(3^{r-1}) = 3^{r-1}$ by induction hypothesis.

2) $N = p^r$, where p is an odd prime and $p^r = 3s - 1$. Note that $f(T_{s-1}) = T_{s-1}$ and $f(T_s) = T_s$ by induction hypothesis since T_s can be factored into integers smaller than N. Once again write the two equalities:

$$f(T_{s-1} + T_{s-1} + T_s) = \frac{s(s-1)}{2} + \frac{s(s-1)}{2} + \frac{s(s+1)}{2} = \frac{s(3s-1)}{2} = \frac{sp^r}{2}$$

and

$$f(T_{s-1}+T_{s-1}+T_s)=f\Big(\frac{s(3s-1)}{2}\Big)=f\Big(\frac{s}{2}\Big)f(3s-1)=\frac{s}{2}f(p^r).$$

Hence $f(p^r)=p^r$.

3) $N=p^r$, where p is an odd prime and $p^r=3s+1$. Similarly we have

$$\begin{aligned}f(T_{s-1}+T_s+T_s)&=\frac{s(s-1)}{2}+\frac{s(s+1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s+1)}{2}=\frac{sp^r}{2}\\&=f\Big(\frac{s(3s+1)}{2}\Big)=f\Big(\frac{s}{2}\Big)f(3s+1)=\frac{s}{2}f(p^r).\end{aligned}$$

Hence $f(p^r)=p^r$.

4) $N=2^r$. Let $2^{r+1}=3s\pm 1$ then we use the following pairs of equalities: either

$$\begin{aligned}f(T_{s-1}+T_{s-1}+T_s)&=\frac{s(s-1)}{2}+\frac{s(s-1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s-1)}{2}=s2^r\\&=f\Big(\frac{s(3s-1)}{2}\Big)=f(s)f\Big(\frac{3s-1}{2}\Big)=sf(2^r)\end{aligned}$$

or

$$\begin{aligned}f(T_{s-1}+T_s+T_s)&=\frac{s(s-1)}{2}+\frac{s(s+1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s+1)}{2}=s2^r\\&=f\Big(\frac{s(3s+1)}{2}\Big)=f(s)f\Big(\frac{3s+1}{2}\Big)=sf(2^r).\end{aligned}$$

Thus $f(2^r)=2^r$.

NT-8. An infinite set B consisting of non negative integers has the following property. For each $a,b\in B$ $(a>b)$ the number $\frac{a-b}{(a,b)}\in B$. Prove that B contains all non negative integers. Here (a,b) is the greatest common divisor of numbers a and b.

<u>Solution.</u>

If d is g.c.d. of all the numbers in set B, let $A=\{b/d:\ b\in B\}$. Then for each $a,b\in A$ $(a>b)$ we have

$$\frac{a-b}{d(a,b)}\in A. \tag{$*$}$$

Observe that g.c.d of the set A equals 1, therefore we can find a finite subset $A_1\in A$ for which the $\gcd A_1=1$. We may think that the sum of elements of A_1 is minimal possible. Choose numbers $a,b\in A_1$ $(a>b)$ and replace a in the set A_1 with $\frac{a-b}{d(a,b)}$. The g.c.d. of the obtained set equals 1. But the sum of numbers decreases by this operations that contradicts minimality of A_1.

Thus, $A_1=\{1\}$. Therefore all the numbers in the set A have residue 1 modulo d. Take an arbitrary $a=kd+1\in A$ and $b=1$. Then $k\in A$ by $(*)$ and hence $k=ds+1$. But $(k,kd+1)=1$, therefore $\frac{kd+1-ds-1}{d}=k-s=(d-1)s+1\in A$, so s is divisible by d. But $s\in A$, therefore $s-1$ is also divisible by d, hence $d=1$ (that means that $B=A$). Thus we have checked that if $a=kd+1=k+1\in A$ then $a-1=k\in A$. Then all non-negative integers belong to A because it is infinite.

NT-9. Prove that there exist only finitely many triples of positive integers (n, a, b) such that:

$$n! = 2^a - 2^b.$$

Solution.

Since $n!$ is divisible by $3^{[n/3]}$, $(2^{a-b}-1)2^b = 2^a - 2^b$ is divisible by $3^{[n/3]}$. From the Lifting the Exponent Lemma we obtain that $a-b$ is divisible by $3^{[n/3]-1}$. So $a - b \geq 3^{[n/3]-1} \geq 3^{n/3-2}$. Hence, the right hand side of our prior equality is greater than

$$2^{a-b} - 1 \geq 2^{a-b-1} \geq 2^{3^{n/3-2}-1}$$

which for sufficiently large n is obviously greater than $2^{n^2} > n!$.

NT-10. Find all triples (a, b, c) of integers that satisfy

$$(a-b)^3(a+b)^2 = c^2 + 2(a-b) + 1.$$

Solution.

Answer: $(0, 1, 0)$ and $(-1, 0, 0)$.

Substituting $x = a - b$, $y = a + b$ we obtain the equation

$$x^3y^2 = c^2 + 2x + 1,$$

which suffices to be solved in integers such that x and y have equal parity. We shall consider the equivalent equation

$$x(x^2y^2 - 2) = c^2 + 1.$$

If both x and y are even then $x^2y^2 - 2$ is even, whence the l.h.s. is divisible by 4. Thus $c^2 \equiv -1 \pmod 4$, but this is impossible since -1 is not a quadratic residue modulo 4.

Let now both x and y be odd; then x^2y^2 is a positive odd number. Consider two cases:

- If $x^2y^2 = 1$ then $x^2 = 1$ and $y^2 = 1$. Since $x^2y^2 - 2 = -1 < 0$ whereas the product $x(x^2y^2 - 2)$ equals the positive number $c^2 + 1$, we must have $x = -1$. The possibilities $x = -1, y = 1$ and $x = -1, y = -1$ give $a = 0, b = 1$ and $a = -1, b = 0$, respectively. In both cases $c = 0$.
- If $x^2y^2 > 1$ then $x^2y^2 - 2 > 0$. Squares of odd numbers give remainder 1 upon division by 4, whence $x^2y^2 - 2 \equiv 1 \cdot 1 - 2 = -1 \pmod 4$. Consequently there exists a prime divisor p of $x^2y^2 - 2$ such that $p \equiv -1 \pmod 4$. But then $c^2 \equiv -1 \pmod p$, which is impossible since -1 is not a quadratic residue modulo p.

NT-11. Find all the triples of non negative integers (a, b, c) for which the number

$$\frac{(a+b)^4}{c} + \frac{(b+c)^4}{a} + \frac{(c+a)^4}{b}$$

is integer and $a + b + c$ is prime.

Solution.

Answer $(1, 1, 1)$, $(1, 2, 2)$, $(2, 3, 6)$.

Let $p = a + b + c$, then $a + b = p - c$, $b + c = p - a$, $c + a = p - b$ and

$$\frac{(p-c)^4}{c} + \frac{(p-a)^4}{a} + \frac{(p-b)^4}{b}$$

is a non-negative integer. By expanding brackets we obtain that the number $p^4(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$ is integer, too. But the numbers a, b, c are not divisible by p, therefore the number $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is (non negative) integer. That is possible for the triples $(1, 1, 1)$, $(1, 2, 2)$, $(2, 3, 6)$ only.

NT-12. Find all quadruples (x, y, z, t) of positive integers that satisfy the system of equations

$$\begin{cases} xyz = t! \\ (x+1)(y+1)(z+1) = (t+1)! \end{cases}.$$

Solution.

Answer: $t = 3$ and (x, y, z) is any permutation of $(1, 2, 3)$.

Since the equations are symmetrical with respect to variables x, y and z, we can assume that $x \le y \le z$. Dividing the second equation by the first one we obtain the equality

$$t + 1 = \frac{(t+1)!}{t!} = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right). \tag{1}$$

Assume that $y \ge 3$, then also $z \ge 3$. Now from $x \ge 1$ and (1) we get that $t + 1 \le 2 \cdot \frac{4}{3} \cdot \frac{4}{3} < 4$, therefore $t \le 2$, but that contradicts the inequality $y \ge 3$.

Thus $y \le 2$. It leaves us with three possibilities.

- $x = y = 1$. Writing (1) in the form $t + 1 = 4 + \frac{4}{z}$ we conclude that $z \in \{1, 2, 4\}$, but none of these values leads to a solution.

- $x = 1$ and $y = 2$. From (1) we get that

  $$t + 1 = 2 \cdot \frac{3}{2}\left(1 + \frac{1}{z}\right) = 3 + \frac{3}{z},$$

  what means that $z \in \{1, 3\}$ and as $z \ge y \ge 2$ then $z = 3$ and $t = 3$. One can check, that this is a solution, therefore we get 6 solutions where $t = 3$ and (x, y, z) is any permutation of $(1, 2, 3)$.

- $x = y = 2$. From (1) we get that $t + 1 = \frac{9}{4} + \frac{9}{4z}$. The expression of the right hand side is larger than 2 and less than 3 if $z \ge 4$, what is impossible. Therefore $z \le 3$ and it remains to check that $(x, y, z) = (2, 2, 2)$ and $(x, y, z) = (2, 2, 3)$ are not solutions.