

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r -*permuation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S .
- The ordered arrangement 3,2 is a 2-permutation of S .
- The number of r -permuatations of a set with n elements is denoted by $P(n,r)$.
 - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$
 r -permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

Note that $P(n, 0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$$

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: Consider all words of all 8 letters using every {A,B,C,D,E,F,G,H} just once. How many of them contain letters "ABC" in this order.

Solution: We solve this problem by counting the permutations of six objects, *ABC, D, E, F, G, and H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This is considerably less than all the permutations out of 8:

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320.$$

Combinations

Definition: An *r-combination* is an unordered selection of *r* elements from a set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with *n* distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section 6.4.*)

Example: Let *S* be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from *S*. It is the same as $\{d, c, a\}$ since the order listed does not matter.

$C(4, 2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r -combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52, 5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n, n - r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

Definition 1: A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

- *Bijective Proof:* Suppose that S is a set with n elements. The function that maps a subset A of S to \bar{A} is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements.
- *Double Counting Proof:* By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A , i.e., those which are in \bar{A} . Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.



Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$