Worksheet 11: Graphs

Question 1.
(A) K_n (the complete graph on n vertices) has ...... edges and ...... vertices.
(B) $K_{m,n}$ (the complete bipartite graph on sets of sizes m, n) has ...... edges and ...... vertices.
(C) W_n (wheel graph on n vertices – n-gonal pyramid viewed from above) has ...... edges and ...... vertices.
(D) Q_n (n-dimensional cube) has ...... edges and ...... vertices.

Question 2 (Rosen7e, Ch.10, Q15-Q17).
(A) The length of the longest simple circuit in K_5 is ....
(B) The length of the longest simple circuit in W_{10} is ....
(C) The length of the longest simple circuit in $K_{4,10}$ is ....
Note. A simple circuit in (Rosen2019) is defined as a circular sequence of vertices $v_0, v_1, \ldots, v_n = v_0$, where each two neighboring vertices are connected by an edge (and it does not contain any edge more than once). It can return to the same vertex multiple times.

Question 3 (Rosen7e, Ch.10, Q19-Q24). In each example find the dimensions of a matrix; and number of 0s and 1s in it: Find X, Y, Z, T.
(A) The adjacency matrix for $K_{m,n}$ has size (rows times columns) $X \times Y$; it has Z 0's and T 1's.
(B) The adjacency matrix for K_n has size $X \times Y$; it has Z 0's and T 1's.
(C) The adjacency matrix for C_n has size $X \times Y$; it has Z 0's and T 1's.
(D) The adjacency matrix for Q_4 has size $X \times Y$; it has Z 0's and T 1's.
(E) The incidence matrix for W_n has size $X \times Y$; it has Z 0's and T 1's.
(F) The incidence matrix for Q_5 has size $X \times Y$; it has Z 0's and T 1's.
Note. Adjacency matrix is a square matrix of size $|V| \times |V|$, but incidence matrix is a rectangular matrix of size $|V| \times |E|$.

Question 4 (Rosen7e, Ch.10, Q28-Q31).
(A) List all positive integers n such that K_n has an Euler circuit; what is its length in terms of n?
(B) List all positive integers n such that Q_n has an Euler circuit.

Question 5 (Rosen7e, Ch.10, Q43).
If G is a planar connected graph with 12 regions and 20 edges, then G has ...... vertices.

Question 6 (Rosen7e, Ch.10, Q44).
If G is a planar connected graph with 20 vertices, each of degree 3, then G has ...... regions.

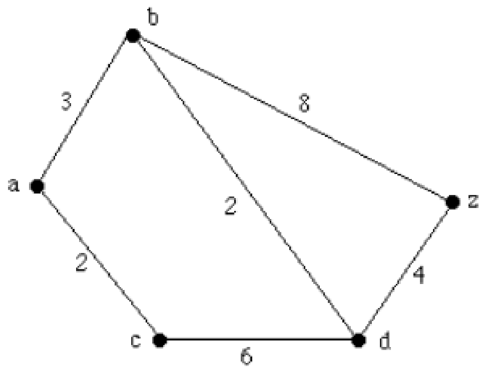
Question 7 (Rosen7e, Ch.10, Q45).
If a regular graph G has 10 vertices and 45 edges, then each vertex of G has degree ....

Note. A *regular graph* is a graph where all vertices have the same degree.

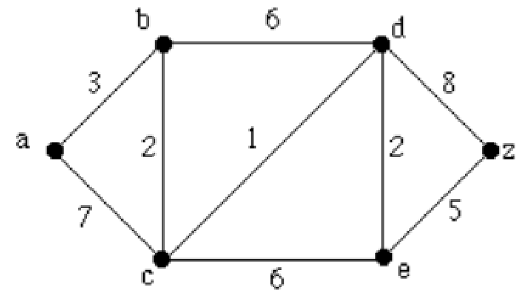
Question 8 (Rosen7e, Ch.10, Q59-Q82).
(A) A simple graph with 6 vertices, whose degrees are $2, 2, 2, 3, 4, 4$.
(B) A simple graph with 8 vertices, whose degrees are $0, 1, 2, 3, 4, 5, 6, 7$.
(C) A simple graph with degrees $1, 2, 2, 3$.
(D) A simple graph with degrees $2, 3, 4, 4, 4$.
(E) A simple graph with degrees $1, 1, 2, 4$.
(F) A simple digraph with indegrees $0, 1, 2$ and outdegrees $0, 1, 2$.
(G) A simple digraph with indegrees $1, 1, 1$ and outdegrees $1, 1, 1$.
(H) A simple digraph with indegrees $0, 1, 2, 2$ and outdegrees $0, 1, 1, 3$.
(I) A simple digraph with indegrees $0, 1, 2, 4, 5$ and outdegrees $0, 3, 3, 3, 3$.
(J) A simple digraph with indegrees $0, 1, 1, 2$ and outdegrees $0, 1, 1, 1$.
(K) A simple digraph with indegrees: $0, 1, 2, 2, 3, 4$ and outdegrees: $1, 1, 2, 2, 3, 4$.
(L) A simple graph with 6 vertices and 16 edges.
(M) A connected simple planar graph with 5 regions and 8 vertices, each of degree 3.
(N) A graph with 4 vertices that is not planar.
(O) A planar graph with 10 vertices.
(P) A planar graph with 8 vertices, 12 edges, and 6 regions.
(Q) A planar graph with 7 vertices, 9 edges, and 5 regions.

Question 9 (Rosen7e, Ch.10, Q108).
Use Dijkstras Algorithm to find the shortest path length between the vertices a and z in these weighted graphs.

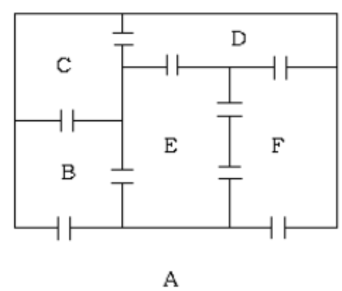
(A)



(B)



Question 10 (Rosen7e, Ch.10, Q113).
The picture at the right shows the floor plan of an office. Show that it is impossible to plan a walk that passes through each doorway exactly once, starting and ending at A.



Hall's Marriage Theorem (Rosen2019, p.772)
The bipartite graph $G = (V, E)$ with partition of vertices into 2 disjoint sets $V = X \cup Y$ has a *maximum matching* that saturates X iff for all $A \subseteq X$ we have $|X| \leq |N(X)|$.
Note. A *matching* in a graph is a set of of edges such that no two edges share a common endpoint. A *maximum matching* is matching containing the greatest number of edges. And a matching *saturates* a set X, if each vertex $v \in X$ belongs to some matching edge.

Question 11 (Königs Marriage Theorem)
Prove that if all the vertices of a bipartite graph have the same degree, then it has a perfect matching.
(Quines2017, p.11); https://cjquines.com/files/halls.pdf
Note. A matching is *perfect*, if it saturates all vertices (every vertex has a pair).

Question 12
We have a regular deck of 52 playing cards, with exactly 4 cards of each of the 13 ranks. The cards have been randomly dealt into 13 piles, each with 4 cards in it. Prove that there is a way to take a card from each pile so that after we take a card from every pile, we

have exactly a card of every rank.
(Quines2017, p.11).

Answers

Question 1. Answer:

(A) K_n has $\frac{n(n-1)}{2}$ edges and n vertices.
(B) $K_{m,n}$ has $m \cdot n$ edges and $m + n$ vertices.
(C) W_n has $2n$ edges and $n + 1$ vertices.
(D) Q_n has $n \cdot 2^{n-1}$ edges and 2^n vertices.
The number of edges in Q_n could be first expressed by a recurrent formula (and then proven by mathematical induction).

Question 2. Answer: TBD

Question 3. Answer: TBD

Question 4. Answer: TBD

Question 5. Answer: TBD

Question 6. Answer: TBD

Question 7. Answer: TBD

Question 8. Answer: TBD

Question 9. Answer: TBD

Question 10. Answer: TBD

Question 11. Answer: TBD

Question 12. Answer: TBD