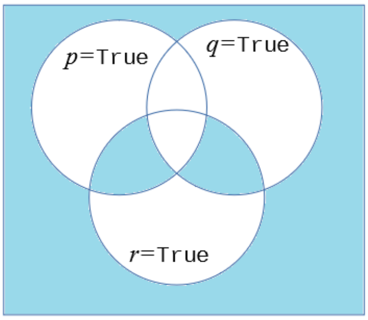
Discrete Structures

# Exam, Part 1

Spring 2021
May 24

1. In the Venn diagram each oval represents the area where statements $p$, $q$, and $r$ are true (outside their respective ovals they are false). Regions where $f(p,q,r) = \texttt{True}$ are shaded.




(a) Create a truth table for this Boolean function $f(p,q,r)$.

| $p$ | $q$ | $r$ | $f(p,q,r)$ |
|---|---|---|---|
| False | False | False | True |
| False | False | True | False |
| False | True | False | False |
| False | True | True | True |
| True | False | False | False |
| True | False | True | True |
| True | True | False | False |
| True | True | True | False |

Region outside of all ovals is shaded, so $f(\texttt{False}, \texttt{False}, \texttt{False}) = \texttt{True}$ (1st line in the truth table); also two more regions are shaded, where just two of the variables are True. □

(b) Create either a Disjunctive Normal Form (DNF) or a Conjunctive Normal Form (CNF) for this function $f(p,q,r)$.
*Note.* You can pick either a DNF or a CNF – whatever you like; there is no need to construct both.

In our situation the DNF would be shorter – it is a disjunction of just 3 conjunctions (each conjunction describes one row where the Boolean function is True).

$$f(p,q,r) = (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r).$$

(CNF would be a conjunction of 5 expressions – one per every row in the table where the function is False.) □

2. Let $H$ be the set of all humans. We define the following predicates:
$K(a,b)$ is true iff when the human $a \in H$ knows a human $b \in H$.
$R(a)$ is true iff the human $a$ has travelled from Riga to Frankfurt by train.

Express the following English sentences using the predicates and quantifiers:

(a) For any two humans $a$ and $b$, person $a$ knows person $b$ iff person $b$ knows person $a$.

$$\forall a \in H\ \forall b \in H\ (K(a,b) \leftrightarrow K(b,a)).$$

This condition tells that the predicate $K(a,b)$ (which is also a relation of "knowing" between two humans) is *symmetric*. It is also possible to express this as one-way implication.

$$\forall a \in H\ \forall b \in H\ (K(a,b) \rightarrow K(b,a)).$$

The 2nd expression tells the same thing (you can get the reverse implication, if you switch variables $a$ and $b$). So both answers are fine; the 1st answer reflects the English sentence better. □

(b) Not every human knows someone who has travelled from Riga to Frankfurt by train.

$$\neg \forall a \in H\ \exists b \in H\ (K(a,b) \wedge R(b)).$$

Literally: It is not the case that for any human $a \in H$ there exist a human $b \in H$ such that $a$ knows $b$ and $b$ has traveled to Frankfurt.

If you wish, you can also rewrite this using De Morgan's laws:

$$\exists a \in H\ \forall b \in H\ (\neg K(a,b) \vee \neg R(b)).$$

Literally: There is a human $a \in H$ such that for any human $b \in H$ – either $a$ does not know $b$ or $b$ did not travel from Riga to Frankfurt by train. □

3. Let $H$ be the set of all humans and the predicates $K(a,b)$ and $R(a)$ are the same as in the previous exercise. Rewrite the statement so that negations are applied only to the predicates directly.

$$\neg\left(\forall a \in H\ \exists b \in H\ \exists c \in H\ \left(R(a)\ \vee\ \left(K(a,b) \wedge K(a,c) \wedge b \neq c \wedge R(b) \wedge R(c)\right)\right)\right).$$

Here we negate the sentence that every human $a \in H$ either himself/herself traveled to Frankfurt or he knows two other people $b$ and $c$ (not equal between themselves: $b \neq c$) such that both of them traveled to Frankfurt.

$$\left(\exists a \in H\ \forall b \in H\ \forall c \in H\ \left(\neg R(a)\ \wedge\ \left(\neg K(a,b) \vee \neg K(a,c) \vee b = c \vee \neg R(b) \vee \neg R(c)\right)\right)\right).$$

Literally: Some human $a$ is such that s/he did not travel to Frankfurt and also for any other two humans $b, c$, either $a$ does not know at least one of them, or $b$ is equal to $c$ or at least one among $b$ and $c$ did not travel to Frankfurt. □

4. Let $f(x) = x^3 + 1$ be a function $f : \mathbf{R} \rightarrow \mathbf{R}$ with real arguments and real values.

(a) Is $f$ an injective function?

Yes, $f$ is an injective function. The easiest way to see it is to notice that $f(x) = x^3+1$ is strictly monotone (growing) function as a cubic parabola: whenever $x_1 < x_2$ we should also have $f(x_1) < f(x_2)$.
So it is impossible that some different values $x_1 \neq x_2$ would have equal values $f(x_1) = f(x_2)$, since one of the $x_1, x_2$ would be smaller than the other (and we apply the monotonicity). $\square$

(b) Is $f$ a surjective function?

Yes, $f$ is a surjection, since for every $a \in \mathbf{R}$ we can solve the equation:

$$x^3 + 1 = a.$$

The root is $x = \sqrt[3]{a-1}$. If we compute $f(x)$ for this argument we will get the value $a$ we need. $\square$

5. Let $S = \{1, 2, 4, 5, 7, 8\}$ be the set of all digits not divisible by 3. Define a relation $R \subseteq S^2$:

$$R = \left\{(a, b) \in S^2 \quad | \quad |a-b| \leq 1\right\}.$$

(a) Is the relation $R$ reflexive? Is it symmetric? Is it transitive?

The condition tells that the distance between $a$ and $b$ is no more than 1. So the only pairs in the relation are $R(1,1)$, $R(1,2)$, $R(2,1)$, $R(2,2)$ (and similarly for 4 and 5 and also for 7 and 8): Two digits are in this relation if they are the same ($|a-b| = 0$) or they are next to each other.
Clearly, this relation is reflexive (as $|a-a| = 0$); it is symmetric (since $|a-b| = |b-a|$). It is also transitive (because the digits fall into three groups of neighbors – if $a$ and $b$ are in the same group and also $b$ and $c$ are in the same group, then also $a$, $c$ are in the same group. $\square$

(b) Let $R^t$ be the transitive closure of the relation $R$. What are the pairs that belong to $R^t$, but not to $R$ (or vice versa)?

As we saw in the previous item – the relation $R$ is itself transitive. Therefore its transitive closure $R^t$ does not add any new pairs to the relation. We have $R^t = R$: both relations contain exactly the same pairs. $\square$

6. Let $\alpha$ be an irrational number.

(a) Is $\dfrac{2\alpha}{3}$ a rational or an irrational number? (Or can it be either depending on $\alpha$.)

The number $\dfrac{2\alpha}{3}$ must be irrational.
**Prove this by contradiction.**
Assume that $\dfrac{2\alpha}{3} = \frac{p}{q}$ for some rational fraction $p/q$. in this case $\alpha = \frac{3p}{2q}$, so it is also a rational fraction. But this contradicts the known fact that $\alpha$ must be irrational.

Generally, if you multiply irrational numbers by rational ones (such as $2/3$), you always get irrational results. $\square$

(b) Is $\alpha^2 + 3$ a rational or an irrational number? (Or can it be either depending on $\alpha$.)

The number $\alpha^2 + 3$ can be either rational or irrational.

**Prove this by counterexamples.**

Case 1. Let $\alpha = \sqrt{2}$. In this case $\alpha^2 + 3 = (\sqrt{2})^2 + 3 = 2 + 3 = 5$ which is rational.

Case 2. Let $\alpha = \sqrt[4]{2}$. Notice that the 4th degree root of 2 must be irrational; so $\alpha \notin \mathbf{Q}$. In this case $\alpha^2 + 3 = (\sqrt[4]{2})^2 + 3 = \sqrt{2} + 3$. This number is irrational, since $\sqrt{2}$ is also irrational.

*Note.* You can safely assume that all roots of integer numbers are either integers themselves (or they are irrational). Therefore $\sqrt[4]{2}$ must be irrational, since it is not an integer number: it is between 1 and 2.

You can also prove the fact that $\sqrt[4]{2}$ is irrational:

Assume that $\sqrt[4]{2} = p/q$ where $p/q$ is an irreducible rational fraction with $q > 1$. Then $2 = \frac{p^4}{q^4}$ is also an irreducible fraction with $q^4 > 1$, which is a contradiction, since the number $2 = \frac{2}{1}$ has number 1 in the denominator. $\square$