easy? hard to coordinate

**G3.** A circle $\omega$ of radius 1 is given. A collection $T$ of triangles is called *good*, if the following conditions hold:

($i$) each triangle from $T$ is inscribed in $\omega$;

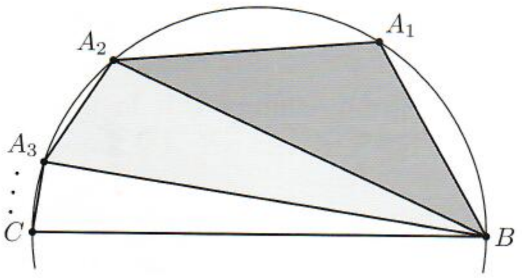
($ii$) no two triangles from $T$ have a common interior point.

Determine all positive real numbers $t$ such that, for each positive integer $n$, there exists a good collection of $n$ triangles, each of perimeter greater than $t$.

**Answer:** $t \in (0, 4]$.

**Solution.** First, we show how to construct a good collection of $n$ triangles, each of perimeter greater than 4. This will show that all $t \leqslant 4$ satisfy the required conditions.

Construct inductively an $(n+2)$-gon $BA_1A_2 \ldots A_nC$ inscribed in $\omega$ such that $BC$ is a diameter, and $BA_1A_2$, $BA_2A_3$, $\ldots$, $BA_{n-1}A_n$, $BA_nC$ is a good collection of $n$ triangles. For $n = 1$, take any triangle $BA_1C$ inscribed in $\omega$ such that $BC$ is a diameter; its perimeter is greater than $2BC = 4$. To perform the inductive step, assume that the $(n+2)$-gon $BA_1A_2 \ldots A_nC$ is already constructed. Since $A_nB + A_nC + BC > 4$, one can choose a point $A_{n+1}$ on the small arc $\widehat{CA_n}$, close enough to $C$, so that $A_nB + A_nA_{n+1} + BA_{n+1}$ is still greater than 4. Thus each of these new triangles $BA_nA_{n+1}$ and $BA_{n+1}C$ has perimeter greater than 4, which completes the induction step.




We proceed by showing that no $t > 4$ satisfies the conditions of the problem. To this end, we assume that there exists a good collection $T$ of $n$ triangles, each of perimeter greater than $t$, and then bound $n$ from above.

Take $\varepsilon > 0$ such that $t = 4 + 2\varepsilon$.

*Claim.* There exists a positive constant $\sigma = \sigma(\varepsilon)$ such that any triangle $\Delta$ with perimeter $2s \geqslant 4 + 2\varepsilon$, inscribed in $\omega$, has area $S(\Delta)$ at least $\sigma$.

*Proof.* Let $a$, $b$, $c$ be the side lengths of $\Delta$. Since $\Delta$ is inscribed in $\omega$, each side has length at most 2. Therefore, $s - a \geqslant (2+\varepsilon) - 2 = \varepsilon$. Similarly, $s - b \geqslant \varepsilon$ and $s - c \geqslant \varepsilon$. By Heron's formula, $S(\Delta) = \sqrt{s(s-a)(s-b)(s-c)} \geqslant \sqrt{(2+\varepsilon)\varepsilon^3}$. Thus we can set $\sigma(\varepsilon) = \sqrt{(2+\varepsilon)\varepsilon^3}$. □

Now we see that the total area $S$ of all triangles from $T$ is at least $n\sigma(\varepsilon)$. On the other hand, $S$ does not exceed the area of the disk bounded by $\omega$. Thus $n\sigma(\varepsilon) \leqslant \pi$, which means that $n$ is bounded from above.

**Comment 1.** One may prove the Claim using the formula $S = \dfrac{abc}{4R}$ instead of Heron's formula.

**Comment 2.** In the statement of the problem condition ($i$) could be replaced by a weaker one: each triangle from $T$ lies within $\omega$. This does not affect the solution above, but reduces the number of ways to prove the Claim.