, ot – Accidentally made available.

A7. Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

Answer: $\dfrac{8}{\sqrt[3]{7}}$, reached when (a, b, c, d) is a cyclic permutation of $(1, 49, 1, 49)$.

Solution 1. Since the value $8/\sqrt[3]{7}$ is reached, it suffices to prove that $S \leqslant 8/\sqrt[3]{7}$.

Assume that x, y, z, t is a permutation of the variables, with $x \leqslant y \leqslant z \leqslant t$. Then, by the rearrangement inequality,

$$S \leqslant \left(\sqrt[3]{\frac{x}{t+7}} + \sqrt[3]{\frac{t}{x+7}}\right) + \left(\sqrt[3]{\frac{y}{z+7}} + \sqrt[3]{\frac{z}{y+7}}\right).$$

Claim. The first bracket above does not exceed $\sqrt[3]{\dfrac{x+t+14}{7}}$.

Proof. Since

$$X^3 + Y^3 + 3XYZ - Z^3 = \frac{1}{2}(X + Y - Z)\left((X - Y)^2 + (X + Z)^2 + (Y + Z)^2\right),$$

the inequality $X + Y \leqslant Z$ is equivalent (when $X, Y, Z \geqslant 0$) to $X^3 + Y^3 + 3XYZ \leqslant Z^3$. Therefore, the claim is equivalent to

$$\frac{x}{t+7} + \frac{t}{x+7} + 3\sqrt[3]{\frac{xt(x+t+14)}{7(x+7)(t+7)}} \leqslant \frac{x+t+14}{7}.$$

Notice that

$$\begin{aligned}
3\sqrt[3]{\frac{xt(x+t+14)}{7(x+7)(t+7)}} &= 3\sqrt[3]{\frac{t(x+7)}{7(t+7)} \cdot \frac{x(t+7)}{7(x+7)} \cdot \frac{7(x+t+14)}{(t+7)(x+7)}} \\
&\leqslant \frac{t(x+7)}{7(t+7)} + \frac{x(t+7)}{7(x+7)} + \frac{7(x+t+14)}{(t+7)(x+7)}
\end{aligned}$$

by the AM–GM inequality, so it suffices to prove

$$\frac{x}{t+7} + \frac{t}{x+7} + \frac{t(x+7)}{7(t+7)} + \frac{x(t+7)}{7(x+7)} + \frac{7(x+t+14)}{(t+7)(x+7)} \leqslant \frac{x+t+14}{7}.$$

A straightforward check verifies that the last inequality is in fact an equality. □

The claim leads now to

$$S \leqslant \sqrt[3]{\frac{x+t+14}{7}} + \sqrt[3]{\frac{y+z+14}{7}} \leqslant 2\sqrt[3]{\frac{x+y+z+t+28}{14}} = \frac{8}{\sqrt[3]{7}},$$

the last inequality being due to the AM–CM inequality (or to the fact that $\sqrt[3]{\ }$ is concave on $[0, \infty)$).

Solution 2. We present a different proof for the estimate $S \leqslant 8/\sqrt[3]{7}$.

Start by using Hölder's inequality:

$$S^3 = \left(\sum_{\text{cyc}} \frac{\sqrt[6]{a} \cdot \sqrt[6]{a}}{\sqrt[3]{b+7}}\right)^3 \leqslant \sum_{\text{cyc}} \left(\sqrt[6]{a}\right)^3 \cdot \sum_{\text{cyc}} \left(\sqrt[6]{a}\right)^3 \cdot \sum_{\text{cyc}} \left(\frac{1}{\sqrt[3]{b+7}}\right)^3 = \left(\sum_{\text{cyc}} \sqrt{a}\right)^2 \sum_{\text{cyc}} \frac{1}{b+7}.$$

Notice that

$$\frac{(x-1)^2(x-7)^2}{x^2+7} \geqslant 0 \iff x^2 - 16x + 71 \geqslant \frac{448}{x^2+7}$$

yields

$$\sum \frac{1}{b+7} \leqslant \frac{1}{448} \sum \left(b - 16\sqrt{b} + 71\right) = \frac{1}{448}\left(384 - 16 \sum \sqrt{b}\right) = \frac{48 - 2\sum \sqrt{b}}{56}.$$

Finally,

$$S^3 \leqslant \frac{1}{56}\left(\sum \sqrt{a}\right)^2 \left(48 - 2\sum \sqrt{a}\right) \leqslant \frac{1}{56}\left(\frac{\sum \sqrt{a} + \sum \sqrt{a} + \left(48 - 2\sum \sqrt{a}\right)}{3}\right)^3 = \frac{512}{7}$$

by the AM–GM inequality. The conclusion follows.

Comment. All the above works if we replace 7 and 100 with $k > 0$ and $2(k^2+1)$, respectively; in this case, the answer becomes

$$2\sqrt[3]{\frac{(k+1)^2}{k}}.$$

Even further, a linear substitution allows to extend the solutions to a version with 7 and 100 being replaced with arbitrary positive real numbers p and q satisfying $q \geqslant 4p$.