G5. Let ABC be a triangle with circumcircle ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to ω.

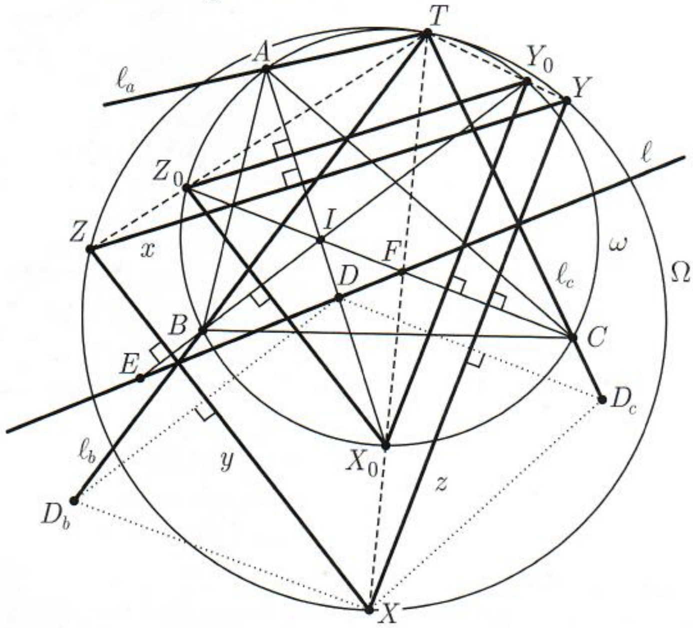
Preamble. Let $X = y \cap z$, $Y = x \cap z$, $Z = x \cap y$ and let Ω denote the circumcircle of the triangle XYZ. Denote by X_0, Y_0, and Z_0 the second intersection points of AI, BI and CI, respectively, with ω. It is known that Y_0Z_0 is the perpendicular bisector of AI, Z_0X_0 is the perpendicular bisector of BI, and X_0Y_0 is the perpendicular bisector of CI. In particular, the triangles XYZ and $X_0Y_0Z_0$ are homothetic, because their corresponding sides are parallel.

The solutions below mostly exploit the following approach. Consider the triangles XYZ and $X_0Y_0Z_0$, or some other pair of homothetic triangles Δ and δ inscribed into Ω and ω, respectively. In order to prove that Ω and ω are tangent, it suffices to show that the centre T of the homothety taking Δ to δ lies on ω (or Ω), or, in other words, to show that Δ and δ are perspective (i.e., the lines joining corresponding vertices are concurrent), with their perspector lying on ω (or Ω).

We use directed angles throughout all the solutions.

Solution 1.

Claim 1. The reflections ℓ_a, ℓ_b and ℓ_c of the line ℓ in the lines x, y, and z, respectively, are concurrent at a point T which belongs to ω.



Proof. Notice that $\measuredangle(\ell_b, \ell_c) = \measuredangle(\ell_b, \ell) + \measuredangle(\ell, \ell_c) = 2\measuredangle(y, \ell) + 2\measuredangle(\ell, z) = 2\measuredangle(y, z)$. But $y \perp BI$ and $z \perp CI$ implies $\measuredangle(y, z) = \measuredangle(BI, IC)$, so, since $2\measuredangle(BI, IC) = \measuredangle(BA, AC)$, we obtain

$$\measuredangle(\ell_b, \ell_c) = \measuredangle(BA, AC). \tag{1}$$

Since A is the reflection of D in x, A belongs to ℓ_a; similarly, B belongs to ℓ_b. Then (1) shows that the common point T' of ℓ_a and ℓ_b lies on ω; similarly, the common point T'' of ℓ_c and ℓ_b lies on ω.

If $B \notin \ell_a$ and $B \notin \ell_c$, then T' and T'' are the second point of intersection of ℓ_b and ω, hence they coincide. Otherwise, if, say, $B \in \ell_c$, then $\ell_c = BC$, so $\measuredangle(BA, AC) = \measuredangle(\ell_b, \ell_c) = \measuredangle(\ell_b, BC)$, which shows that ℓ_b is tangent at B to ω and $T' = T'' = B$. So T' and T'' coincide in all the cases, and the conclusion of the claim follows. $\square$

Now we prove that X, X_0, T are collinear. Denote by D_b and D_c the reflections of the point D in the lines y and z, respectively. Then D_b lies on ℓ_b, D_c lies on ℓ_c, and

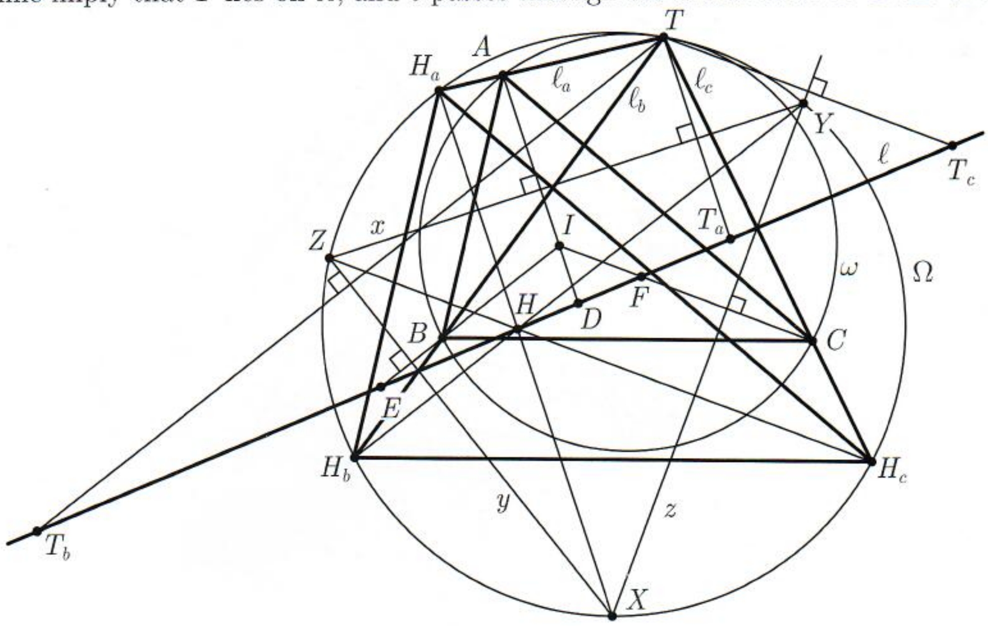
$$\begin{aligned}\measuredangle(D_bX, XD_c) &= \measuredangle(D_bX, DX) + \measuredangle(DX, XD_c) = 2\measuredangle(y, DX) + 2\measuredangle(DX, z) = 2\measuredangle(y, z)\\ &= \measuredangle(BA, AC) = \measuredangle(BT, TC),\end{aligned}$$

hence the quadrilateral XD_bTD_c is cyclic. Notice also that since $XD_b = XD = XD_c$, the points D, D_b, D_c lie on a circle with centre X. Using in this circle the diameter $D_cD'_c$ yields $\measuredangle(D_bD_c, D_cX) = 90^\circ + \measuredangle(D_bD'_c, D'_cX) = 90^\circ + \measuredangle(D_bD, DD_c)$. Therefore,

$$\begin{aligned}\measuredangle(\ell_b, XT) = \measuredangle(D_bT, XT) = \measuredangle(D_bD_c, D_cX) = 90^\circ + \measuredangle(D_bD, DD_c)\\ = 90^\circ + \measuredangle(BI, IC) = \measuredangle(BA, AI) = \measuredangle(BA, AX_0) = \measuredangle(BT, TX_0) = \measuredangle(\ell_b, X_0T),\end{aligned}$$

so the points X, X_0, T are collinear. By a similar argument, Y, Y_0, T and Z, Z_0, T are collinear. As mentioned in the preamble, the statement of the problem follows.

Comment 1. After proving Claim 1 one may proceed in another way. As it was shown, the reflections of ℓ in the sidelines of XYZ are concurrent at T. Thus ℓ is the Steiner line of T with respect to ΔXYZ (that is the line containing the reflections T_a, T_b, T_c of T in the sidelines of XYZ). The properties of the Steiner line imply that T lies on Ω, and ℓ passes through the orthocentre H of the triangle XYZ.



Let H_a, H_b, and H_c be the reflections of the point H in the lines x, y, and z, respectively. Then the triangle $H_aH_bH_c$ is inscribed in Ω and homothetic to ABC (by an easy angle chasing). Since $H_a \in \ell_a$, $H_b \in \ell_b$, and $H_c \in \ell_c$, the triangles $H_aH_bH_c$ and ABC form a required pair of triangles Δ and δ mentioned in the preamble.

Comment 2. The following observation shows how one may guess the description of the tangency point T from Solution 1.

Let us fix a direction and move the line ℓ parallel to this direction with constant speed.

Then the points D, E, and F are moving with constant speeds along the lines AI, BI, and CI, respectively. In this case x, y, and z are moving with constant speeds, defining a family of homothetic triangles XYZ with a common centre of homothety T. Notice that the triangle $X_0Y_0Z_0$ belongs to this family (for ℓ passing through I). We may specify the location of T considering the degenerate case when x, y, and z are concurrent. In this degenerate case all the lines x, y, z, ℓ, ℓ_a, ℓ_b, ℓ_c have a common point. Note that the lines ℓ_a, ℓ_b, ℓ_c remain constant as ℓ is moving (keeping its direction). Thus T should be the common point of ℓ_a, ℓ_b, and ℓ_c, lying on ω.

Solution 2. As mentioned in the preamble, it is sufficient to prove that the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to ω. Thus, it suffices to prove that $\sphericalangle(TX_0, TY_0) = \sphericalangle(Z_0X_0, Z_0Y_0)$, or, equivalently, $\sphericalangle(XX_0, YY_0) = \sphericalangle(Z_0X_0, Z_0Y_0)$.

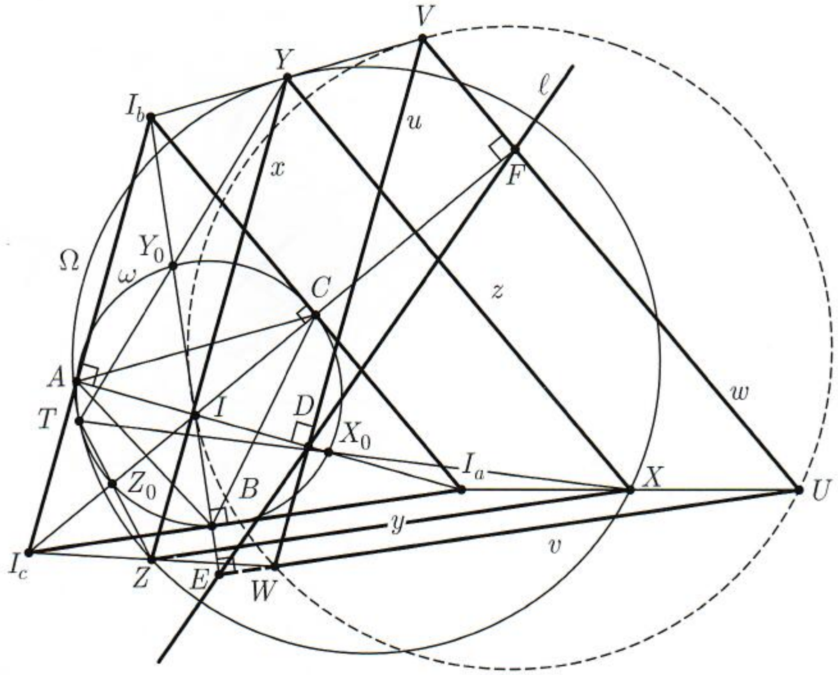
Recall that YZ and Y_0Z_0 are the perpendicular bisectors of AD and AI, respectively. Then, the vector $\vec{x}$ perpendicular to YZ and shifting the line Y_0Z_0 to YZ is equal to $\frac{1}{2}\overrightarrow{ID}$. Define the shifting vectors $\vec{y} = \frac{1}{2}\overrightarrow{IE}$, $\vec{z} = \frac{1}{2}\overrightarrow{IF}$ similarly. Consider now the triangle UVW formed by the perpendiculars to AI, BI, and CI through D, E, and F, respectively (see figure below). This is another triangle whose sides are parallel to the corresponding sides of XYZ.

Claim 2. $\overrightarrow{IU} = 2\overrightarrow{X_0X}$, $\overrightarrow{IV} = 2\overrightarrow{Y_0Y}$, $\overrightarrow{IW} = 2\overrightarrow{Z_0Z}$.

Proof. We prove one of the relations, the other proofs being similar. To prove the equality of two vectors it suffices to project them onto two non-parallel axes and check that their projections are equal.

The projection of $\overrightarrow{X_0X}$ onto IB equals $\vec{y}$, while the projection of $\overrightarrow{IU}$ onto IB is $\overrightarrow{IE} = 2\vec{y}$. The projections onto the other axis IC are $\vec{z}$ and $\overrightarrow{IF} = 2\vec{z}$. Then $\overrightarrow{IU} = 2\overrightarrow{X_0X}$ follows. □

Notice that the line ℓ is the Simson line of the point I with respect to the triangle UVW; thus U, V, W, and I are concyclic. It follows from Claim 2 that $\sphericalangle(XX_0, YY_0) = \sphericalangle(IU, IV) = \sphericalangle(WU, WV) = \sphericalangle(Z_0X_0, Z_0Y_0)$, and we are done.



Solution 3. Let I_a, I_b, and I_c be the excentres of triangle ABC corresponding to A, B, and C, respectively. Also, let u, v, and w be the lines through D, E, and F which are perpendicular to AI, BI, and CI, respectively, and let UVW be the triangle determined by these lines, where $u = VW$, $v = UW$ and $w = UV$ (see figure above).

Notice that the line u is the reflection of I_bI_c in the line x, because u, x, and I_bI_c are perpendicular to AD and x is the perpendicular bisector of AD. Likewise, v and I_aI_c are reflections of each other in y, while w and I_aI_b are reflections of each other in z. It follows that X, Y, and Z are the midpoints of UI_a, VI_b and WI_c, respectively, and that the triangles UVW, XYZ and $I_aI_bI_c$ are either translates of each other or homothetic with a common homothety centre.

Construct the points T and S such that the quadrilaterals $UVIW$, $XYTZ$ and $I_aI_bSI_c$ are homothetic. Then T is the midpoint of IS. Moreover, note that ℓ is the Simson line of the point I with respect to the triangle UVW, hence I belongs to the circumcircle of the triangle UVW, therefore T belongs to Ω.

Consider now the homothety or translation h_1 that maps $XYZT$ to $I_aI_bI_cS$ and the homothety h_2 with centre I and factor $\frac{1}{2}$. Furthermore, let $h = h_2 \circ h_1$. The transform h can be a homothety or a translation, and

$$h(T) = h_2(h_1(T)) = h_2(S) = T,$$

hence T is a fixed point of h. So, h is a homothety with centre T. Note that h_2 maps the excentres I_a, I_b, I_c to X_0, Y_0, Z_0 defined in the preamble. Thus the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to Ω, and this completes the proof.