medium/easy?

– not nice because bounds are not tight

C3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

Solution. The stones are indistinguishable, and all have the same origin and the same final position. So, at any turn we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stones from 1 to n. At any turn, after choosing a square, Sisyphus moves the stone with the largest number from this square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least $\lceil n/k \rceil$ moves of stone k should have been made, for every $k = 1, 2, \ldots, n$.

By summing up over all $k = 1, 2, \ldots, n$, we get the required estimate.

Comment. The original submission contained the second part, asking for which values of n the equality can be achieved. The answer is $n = 1, 2, 3, 4, 5, 7$. The Problem Selection Committee considered this part to be less suitable for the competition, due to technicalities.