medium Best solution

G4. A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

Solution. By $\measuredangle(\ell, n)$ we always mean the directed angle of the lines ℓ and n, taken modulo $180°$.

Let CC_2 meet Ω again at K (as usual, if CC_2 is tangent to Ω, we set $T = C_2$). We show that the line BB_2 contains K; similarly, AA_2 will also pass through K. For this purpose, it suffices to prove that

$$\measuredangle(C_2C, C_2A_1) = \measuredangle(B_2B, B_2A_1). \tag{1}$$

By the problem condition, CB and CA are the perpendicular bisectors of TA_1 and TB_1, respectively. Hence, C is the circumcentre of the triangle A_1TB_1. Therefore,

$$\measuredangle(CA_1, CB) = \measuredangle(CB, CT) = \measuredangle(B_1A_1, B_1T) = \measuredangle(B_1A_1, B_1B_2).$$

In circle Ω we have $\measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2)$. Thus,

$$\measuredangle(CA_1, CB) = \measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2). \tag{2}$$

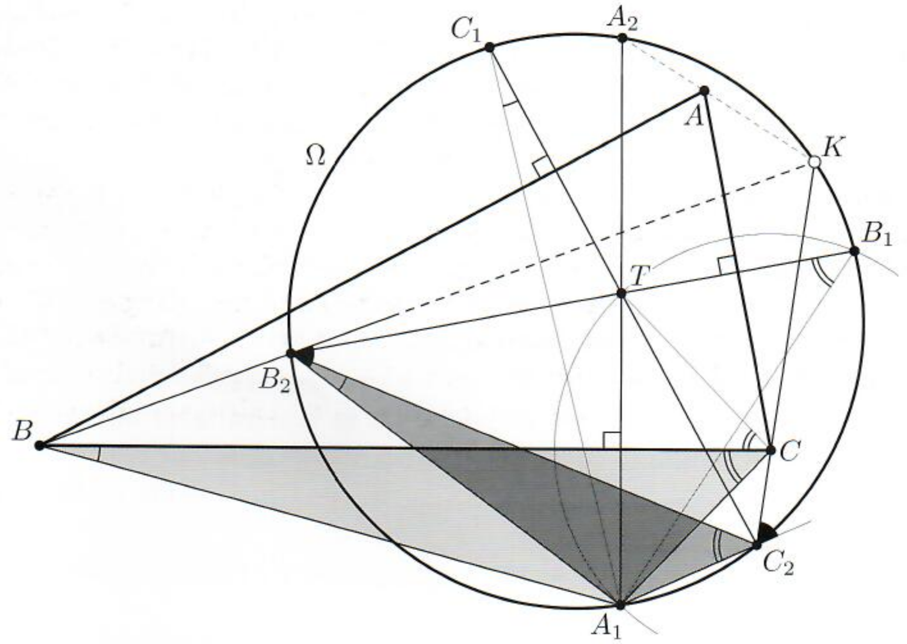
Similarly, we get

$$\measuredangle(BA_1, BC) = \measuredangle(C_1A_1, C_1C_2) = \measuredangle(B_2A_1, B_2C_2). \tag{3}$$

The two obtained relations yield that the triangles A_1BC and $A_1B_2C_2$ are similar and equioriented, hence

$$\frac{A_1B_2}{A_1B} = \frac{A_1C_2}{A_1C} \quad \text{and} \quad \measuredangle(A_1B, A_1C) = \measuredangle(A_1B_2, A_1C_2).$$

The second equality may be rewritten as $\measuredangle(A_1B, A_1B_2) = \measuredangle(A_1C, A_1C_2)$, so the triangles A_1BB_2 and A_1CC_2 are also similar and equioriented. This establishes (1).



Comment 1. In fact, the triangle A_1BC is an image of $A_1B_2C_2$ under a spiral similarity centred at A_1; in this case, the triangles ABB_2 and ACC_2 are also spirally similar with the same centre.

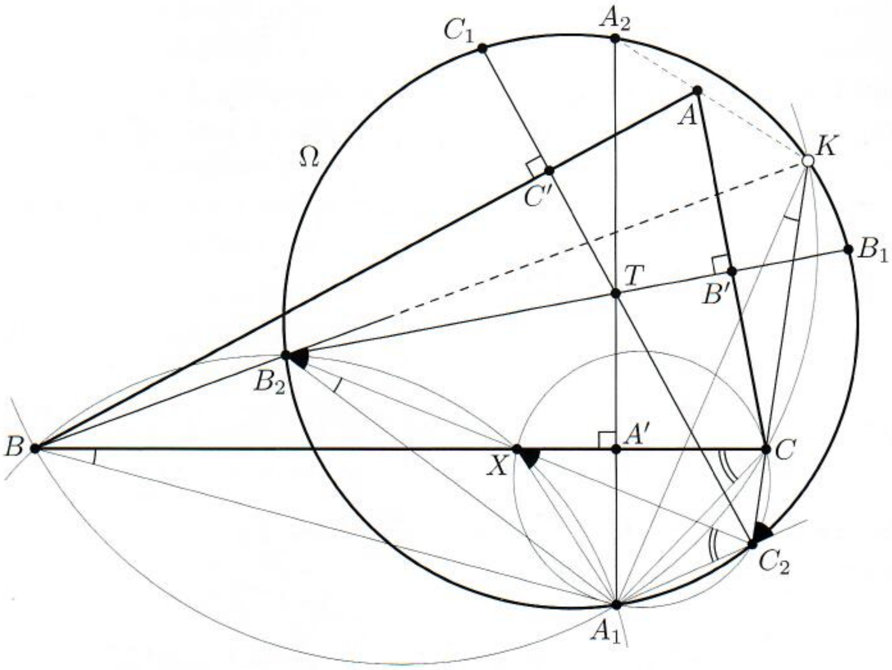
Comment 2. After obtaining (2) and (3), one can finish the solution in different ways.

For instance, introducing the point $X = BC \cap B_2C_2$, one gets from these relations that the 4-tuples (A_1, B, B_2, X) and (A_1, C, C_2, X) are both cyclic. Therefore, K is the Miquel point of the lines BB_2, CC_2, BC, and B_2C_2; this yields that the meeting point of BB_2 and CC_2 lies on Ω.

Yet another way is to show that the points A_1, B, C, and K are concyclic, as

$$\measuredangle(KC, KA_1) = \measuredangle(B_2C_2, B_2A_1) = \measuredangle(BC, BA_1).$$

By symmetry, the second point K' of intersection of BB_2 with Ω is also concyclic to A_1, B, and C, hence $K' = K$.



Comment 3. The requirement that the common point of the lines AA_2, BB_2, and CC_2 should lie on Ω may seem to make the problem easier, since it suggests some approaches. On the other hand, there are also different ways of showing that the lines AA_2, BB_2, and CC_2 are just concurrent.

In particular, the problem conditions yield that the lines A_2T, B_2T, and C_2T are perpendicular to the corresponding sides of the triangle ABC. One may show that the lines AT, BT, and CT are also perpendicular to the corresponding sides of the triangle $A_2B_2C_2$, i.e., the triangles ABC and $A_2B_2C_2$ are *orthologic*, and their orthology centres coincide. It is known that such triangles are also *perspective*, i.e. the lines AA_2, BB_2, and CC_2 are concurrent (in projective sense).

To show this mutual orthology, one may again apply angle chasing, but there are also other methods. Let A', B', and C' be the projections of T onto the sides of the triangle ABC. Then $A_2T \cdot TA' = B_2T \cdot TB' = C_2T \cdot TC'$, since all three products equal (minus) half the power of T with respect to Ω. This means that A_2, B_2, and C_2 are the poles of the sidelines of the triangle ABC with respect to some circle centred at T and having *pure imaginary* radius (in other words, the reflections of A_2, B_2, and C_2 in T are the poles of those sidelines with respect to some regular circle centred at T). Hence, dually, the vertices of the triangle ABC are also the poles of the sidelines of the triangle $A_2B_2C_2$.