A5. Determine all functions $f:(0,\infty)\to\mathbb{R}$ satisfying

$$\left(x+\frac{1}{x}\right)f(y)=f(xy)+f\left(\frac{y}{x}\right) \tag{1}$$

for all $x,y>0$.

Answer: $f(x)=C_1x+\dfrac{C_2}{x}$ with arbitrary constants C_1 and C_2.

Solution 1. Fix a real number $a>1$, and take a new variable t. For the values $f(t)$, $f(t^2)$, $f(at)$ and $f(a^2t^2)$, the relation (1) provides a system of linear equations:

$$\begin{aligned}
&x=y=t: && \left(t+\frac{1}{t}\right)f(t) &&= f(t^2)+f(1) && \text{(2a)}\\
&x=\frac{t}{a},\ y=at: && \left(\frac{t}{a}+\frac{a}{t}\right)f(at) &&= f(t^2)+f(a^2) && \text{(2b)}\\
&x=a^2t,\ y=t: && \left(a^2t+\frac{1}{a^2t}\right)f(t) &&= f(a^2t^2)+f\left(\frac{1}{a^2}\right) && \text{(2c)}\\
&x=y=at: && \left(at+\frac{1}{at}\right)f(at) &&= f(a^2t^2)+f(1) && \text{(2d)}
\end{aligned}$$

In order to eliminate $f(t^2)$, take the difference of (2a) and (2b); from (2c) and (2d) eliminate $f(a^2t^2)$; then by taking a linear combination, eliminate $f(at)$ as well:

$$\left(t+\frac{1}{t}\right)f(t)-\left(\frac{t}{a}+\frac{a}{t}\right)f(at)=f(1)-f(a^2)\quad\text{and}$$

$$\left(a^2t+\frac{1}{a^2t}\right)f(t)-\left(at+\frac{1}{at}\right)f(at)=f(1/a^2)-f(1),\quad\text{so}$$

$$\left(\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)\right)f(t)$$
$$=\left(at+\frac{1}{at}\right)\big(f(1)-f(a^2)\big)-\left(\frac{t}{a}+\frac{a}{t}\right)\big(f(1/a^2)-f(1)\big).$$

Notice that on the left-hand side, the coefficient of $f(t)$ is nonzero and does not depend on t:

$$\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)=a+\frac{1}{a}-\left(a^3+\frac{1}{a^3}\right)<0.$$

After dividing by this fixed number, we get

$$f(t)=C_1t+\frac{C_2}{t} \tag{3}$$

where the numbers C_1 and C_2 are expressed in terms of a, $f(1)$, $f(a^2)$ and $f(1/a^2)$, and they do not depend on t.

The functions of the form (3) satisfy the equation:

$$\left(x+\frac{1}{x}\right)f(y)=\left(x+\frac{1}{x}\right)\left(C_1y+\frac{C_2}{y}\right)=\left(C_1xy+\frac{C_2}{xy}\right)+\left(C_1\frac{y}{x}+C_2\frac{x}{y}\right)=f(xy)+f\left(\frac{y}{x}\right).$$

Solution 2. We start with an observation. If we substitute $x = a \neq 1$ and $y = a^n$ in (1), we obtain

$$f(a^{n+1}) - \left(a + \frac{1}{a}\right) f(a^n) + f(a^{n-1}) = 0.$$

For the sequence $z_n = a^n$, this is a homogeneous linear recurrence of the second order, and its characteristic polynomial is $t^2 - \left(a + \frac{1}{a}\right)t + 1 = (t-a)(t-\frac{1}{a})$ with two distinct nonzero roots, namely a and $1/a$. As is well-known, the general solution is $z_n = C_1 a^n + C_2(1/a)^n$ where the index n can be as well positive as negative. Of course, the numbers C_1 and C_2 may depend of the choice of a, so in fact we have two functions, C_1 and C_2, such that

$$f(a^n) = C_1(a) \cdot a^n + \frac{C_2(a)}{a^n} \quad \text{for every } a \neq 1 \text{ and every integer } n. \tag{4}$$

The relation (4) can be easily extended to rational values of n, so we may conjecture that C_1 and C_2 are constants, and whence $f(t) = C_1 t + \frac{C_2}{t}$. As it was seen in the previous solution, such functions indeed satisfy (1).

The equation (1) is linear in f; so if some functions f_1 and f_2 satisfy (1) and c_1, c_2 are real numbers, then $c_1 f_1(x) + c_2 f_2(x)$ is also a solution of (1). In order to make our formulas simpler, define

$$f_0(x) = f(x) - f(1) \cdot x.$$

This function is another one satisfying (1) and the extra constraint $f_0(1) = 0$. Repeating the same argument on linear recurrences, we can write $f_0(a) = K(a)a^n + \frac{L(a)}{a^n}$ with some functions K and L. By substituting $n = 0$, we can see that $K(a) + L(a) = f_0(1) = 0$ for every a. Hence,

$$f_0(a^n) = K(a)\left(a^n - \frac{1}{a^n}\right).$$

Now take two numbers $a > b > 1$ arbitrarily and substitute $x = (a/b)^n$ and $y = (ab)^n$ in (1):

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) f_0\big((ab)^n\big) = f_0\big(a^{2n}\big) + f_0\big(b^{2n}\big), \quad \text{so}$$

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) K(ab)\left((ab)^n - \frac{1}{(ab)^n}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right), \quad \text{or equivalently}$$

$$K(ab)\left(a^{2n} - \frac{1}{a^{2n}} + b^{2n} - \frac{1}{b^{2n}}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right). \tag{5}$$

By dividing (5) by a^{2n} and then taking limit with $n \to +\infty$ we get $K(ab) = K(a)$. Then (5) reduces to $K(a) = K(b)$. Hence, $K(a) = K(b)$ for all $a > b > 1$.

Fix $a > 1$. For every $x > 0$ there is some b and an integer n such that $1 < b < a$ and $x = b^n$. Then

$$f_0(x) = f_0(b^n) = K(b)\left(b^n - \frac{1}{b^n}\right) = K(a)\left(x - \frac{1}{x}\right).$$

Hence, we have $f(x) = f_0(x) + f(1)x = C_1 x + \frac{C_2}{x}$ with $C_1 = K(a) + f(1)$ and $C_2 = -K(a)$.

Comment. After establishing (5), there are several variants of finishing the solution. For example, instead of taking a limit, we can obtain a system of linear equations for $K(a)$, $K(b)$ and $K(ab)$ by substituting two positive integers n in (5), say $n = 1$ and $n = 2$. This approach leads to a similar ending as in the first solution.

Optionally, we define another function $f_1(x) = f_0(x) - C\left(x - \frac{1}{x}\right)$ and prescribe $K(c) = 0$ for another fixed c. Then we can choose $ab = c$ and decrease the number of terms in (5).