Combinatorics

not KNOWN - RUSSIA? & AOPS

C1.

Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

Solution. We show that one of possible examples is the set

$$S = \{1 \cdot 3^k,\ 2 \cdot 3^k \colon k = 1, 2, \ldots, n-1\} \cup \left\{1,\ \frac{3^n + 9}{2} - 1\right\}.$$

It is readily verified that all the numbers listed above are distinct (notice that the last two are not divisible by 3).

The sum of elements in S is

$$\Sigma = 1 + \left(\frac{3^n + 9}{2} - 1\right) + \sum_{k=1}^{n-1} (1 \cdot 3^k + 2 \cdot 3^k) = \frac{3^n + 9}{2} + \sum_{k=1}^{n-1} 3^{k+1} = \frac{3^n + 9}{2} + \frac{3^{n+1} - 9}{2} = 2 \cdot 3^n.$$

Hence, in order to show that this set satisfies the problem requirements, it suffices to present, for every $m = 2, 3, \ldots, n$, an m-element subset $A_m \subset S$ whose sum of elements equals 3^n.

Such a subset is

$$A_m = \{2 \cdot 3^k \colon k = n - m + 1, n - m + 2, \ldots, n - 1\} \cup \{1 \cdot 3^{n-m+1}\}.$$

Clearly, $|A_m| = m$. The sum of elements in A_m is

$$3^{n-m+1} + \sum_{k=n-m+1}^{n-1} 2 \cdot 3^k = 3^{n-m+1} + \frac{2 \cdot 3^n - 2 \cdot 3^{n-m+1}}{2} = 3^n,$$

as required.

Comment. Let us present a more general construction. Let $s_1, s_2, \ldots, s_{2n-1}$ be a sequence of pairwise distinct positive integers satisfying $s_{2i+1} = s_{2i} + s_{2i-1}$ for all $i = 2, 3, \ldots, n-1$. Set $s_{2n} = s_1 + s_2 + \cdots + s_{2n-4}$.

Assume that s_{2n} is distinct from the other terms of the sequence. Then the set $S = \{s_1, s_2, \ldots, s_{2n}\}$ satisfies the problem requirements. Indeed, the sum of its elements is

$$\Sigma = \sum_{i=1}^{2n-4} s_i + (s_{2n-3} + s_{2n-2}) + s_{2n-1} + s_{2n} = s_{2n} + s_{2n-1} + s_{2n-1} + s_{2n} = 2s_{2n} + 2s_{2n-1}.$$

Therefore, we have

$$\frac{\Sigma}{2} = s_{2n} + s_{2n-1} = s_{2n} + s_{2n-2} + s_{2n-3} = s_{2n} + s_{2n-2} + s_{2n-4} + s_{2n-5} = \ldots,$$

which shows that the required sets A_m can be chosen as

$$A_m = \{s_{2n}, s_{2n-2}, \ldots, s_{2n-2m+4}, s_{2n-2m+3}\}.$$

So, the only condition to be satisfied is $s_{2n} \notin \{s_1, s_2, \ldots, s_{2n-1}\}$, which can be achieved in many different ways (e.g., by choosing properly the number s_1 after specifying $s_2, s_3, \ldots, s_{2n-1}$).

The solution above is an instance of this general construction. Another instance, for $n > 3$, is the set

$$\{F_1, F_2, \ldots, F_{2n-1}, F_1 + \cdots + F_{2n-4}\},$$

where $F_1 = 1$, $F_2 = 2$, $F_{n+1} = F_n + F_{n-1}$ is the usual Fibonacci sequence.