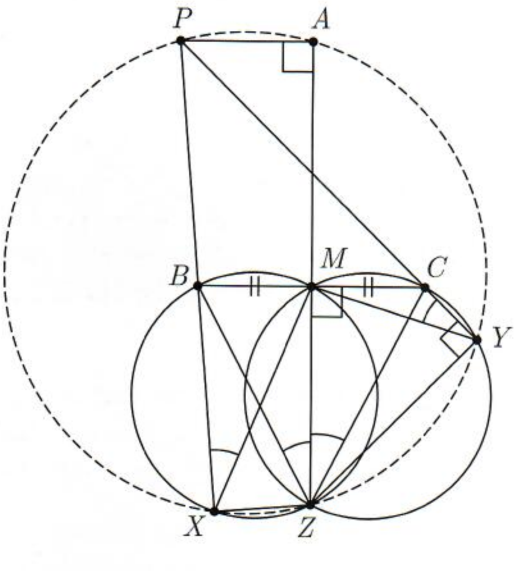
medium, has high powered solution, easy if you have tools
/easy

G2. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

Solution. Since $AB = AC$, AM is the perpendicular bisector of BC, hence $\angle PAM = \angle AMC = 90°$.



Now let Z be the common point of AM and the perpendicular through Y to PC (notice that Z lies on to the ray AM beyond M). We have $\angle PAZ = \angle PYZ = 90°$. Thus the points P, A, Y, and Z are concyclic.

Since $\angle CMZ = \angle CYZ = 90°$, the quadrilateral $CYZM$ is cyclic, hence $\angle CZM = \angle CYM$. By the condition in the statement, $\angle CYM = \angle BXM$, and, by symmetry in ZM, $\angle CZM = \angle BZM$. Therefore, $\angle BXM = \angle BZM$. It follows that the points B, X, Z, and M are concyclic, hence $\angle BXZ = 180° - \angle BMZ = 90°$.

Finally, we have $\angle PXZ = \angle PYZ = \angle PAZ = 90°$, hence the five points P, A, X, Y, Z are concyclic. In particular, the quadrilateral $APXY$ is cyclic, as required.

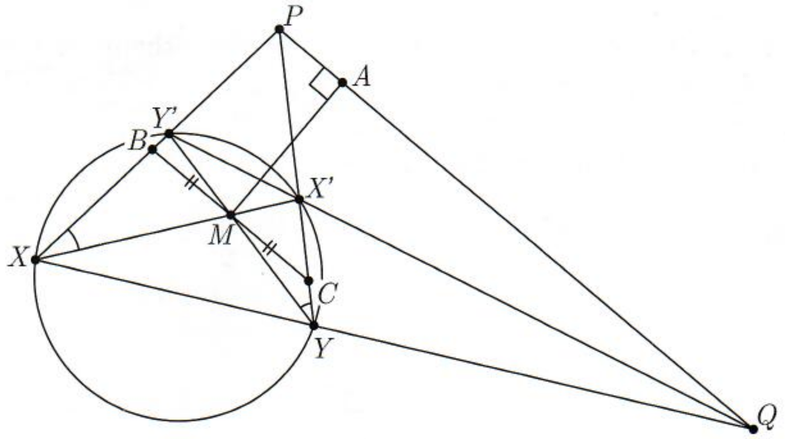
Comment 1. Clearly, the key point Z from the solution above can be introduced in several different ways, e.g., as the second meeting point of the circle CMY and the line AM, or as the second meeting point of the circles CMY and BMX, etc.

For some of definitions of Z its location is not obvious. For instance, if Z is defined as a common point of AM and the perpendicular through X to PX, it is not clear that Z lies on the ray AM beyond M. To avoid such slippery details some more restrictions on the construction may be required.

Comment 2. Let us discuss a connection to the Miquel point of a cyclic quadrilateral. Set $X' = MX \cap PC$, $Y' = MY \cap PB$, and $Q = XY \cap X'Y'$ (see the figure below).

We claim that $BC \parallel PQ$. (One way of proving this is the following. Notice that the quadruple of lines PX, PM, PY, PQ is harmonic, hence the quadruple B, M, C, $PQ \cap BC$ of their intersection points with BC is harmonic. Since M is the midpoint of BC, $PQ \cap BC$ is an ideal point, i.e., $PQ \parallel BC$.)

It follows from the given equality $\angle PXM = \angle PYM$ that the quadrilateral $XYX'Y'$ is cyclic. Note that A is the projection of M onto PQ. By a known description, A is the Miquel point for the sidelines $XY, XY', X'Y, X'Y'$. In particular, the circle PXY passes through A.



Comment 3. An alternative approach is the following. One can note that the (oriented) lengths of the segments CY and BX are both linear functions of a parameter $t = \cot \angle PXM$. As t varies, the intersection point S of the perpendicular bisectors of PX and PY traces a fixed line, thus the family of circles PXY has a fixed common point (other than P). By checking particular cases, one can show that this fixed point is A.

Comment 4. The problem states that $\angle PXM = \angle PYM$ implies that $APXY$ is cyclic. The original submission claims that these two conditions are in fact equivalent. The Problem Selection Committee omitted the converse part, since it follows easily from the direct one, by reversing arguments.