Comprehensive Exam Topics

Discrete Structures

You must justify all your answers to recieve full credit

Comprehensive exam in Discrete Structures revisits the essential theory material. The exam is split into four parts – every part has mandatory office hours followed by the exam on that part (typically on a subsequent day). Each part lasts 60 minutes, it has 6 questions (up to 10 points each). You will need to pass all four parts; all four results should be at least 30 points – half of the maximum.

Part 1: Logic and Proofs

1. **Boolean expressions.** Truth tables, logical equivalences, Venn diagrams.
   (a) Express an English sentence as a Boolean expression.
   (b) Build a truth table for a Boolean Expression.
   (c) Build a DNF or a CNF for a given truth table.
   (d) Simplify a Boolean Expression using identities; prove, disprove tautologies.
   (e) Shade regions in a Venn diagram corresponding to a Boolean Expression.
2. **Quantifiers.** Predicates, quantifiers.
   (a) Express math and programming concepts with predicates.
   (b) Express an English sentence as a predicate logic expression.
   (c) Restore parentheses, identify free/bound variables in a predicate expression.
   (d) Write the negation for a predicate expression; simplify using De Morgan's laws.
   (e) Read and write set-builder notation.
3. **Functions and Relations** Injections, surjections, bijections, relations and their properties.
   (a) Given a function, prove/disprove that it is injective, surjective or bijective.
   (b) Given function definitions, evaluate their compositions and inverses.
   (c) Given a sequence, identify its properties, is it (eventually) constant/periodic, etc.
   (d) Convert a desription of a binary relation into another form.
   (e) Given a binary relation determine if it is a (injective, surjective, bijective) function.
   (f) Given a binary relation determine if it is reflexive, symmetric, antisymmetric or transitive.
   (g) Compute compositions and powers for relations, find transitive closures.
4. **Proofs.** Simple statements grouped by the method of proof.
   (a) Prove an implication directly.
   (b) Prove an implication by contradiction.
   (c) Prove a logical equivalence (if and only if).
   (d) Prove by counterexample.
   (e) Prove by mathematical induction.
   (f) Prove or disprove equality of two numbers or sets.

Part 2: Structures

1. **Number Theory.** Congruences. Bezout identity. Inverses. Chinese remainder theorem.
   (a) Factorize a number into a product of prime powers.
   (b) Divide numbers with remainders as in $n = qd + r$, express decimal digits.
   (c) Find members of arithmetic progressions, also by modulo m.
   (d) Given two integers, find their GCD (also LCM) by Euclid algorithm.
   (e) Given integers, solve Bezout identity with Blankenship algorithm.
   (f) Given m and x, compute multiplicative inverse $\overline{x}$ modulo m; solve linear congruences.
   (g) Convert periodic decimal numbers to rational fractions.
   (h) Given a decimal integer, convert it to binary, hexadecimal (and vice versa).
2. **Graphs.** Graph concepts, subgraphs, graph families and isomorphisms.
   (a) Count the number of graphs with a given property or parameter.
   (b) Check a propert for a special graph (complete, cycle, wheel, n-cube, complete bipartite).
   (c) Check if a given graph is bipartite, complete, or connected.
   (d) Justify whether or not a graph has a particular subgraph, cycle or path.
   (e) Convert between different representations of graphs.
   (f) Given a tree, check the condition for a Euler circuit (or path) and find it.
   (g) Given two graphs prove or disprove they are isomorphic.
3. **Trees.** Tree concepts, traversing trees with BFS, DFS.
   (a) Given the count of vertices, edges, height or other parameter, estimate other parameters.
   (b) Given an n-ary tree and some parameters, estimate other parameters.
   (c) Convert between representations: tree diagrams, lists of edges, or traversals.
   (d) Given a prefix, infix or postfix notation, convert it into the syntax tree or other notation(s).
   (e) Given an undirected graph, do a DFS and BFS traversal, indicating all steps.

Part 3: Counting and Estimation

1. **Combinatorics.** Permutations, combinations, binomial coefficients, pigeonhole principle.
   (a) Given a word problem, count variants using the product, sum, difference rules.
   (b) Given a set of restrictions and symmetries, count variants using the division rule.
   (c) Count variants using combinations and permutation formulas with or without repetition.
   (d) Given a polynomial, find coefficients using binomial and multinomial rules.
2. **Recurrent Sequences.** Periodicity, 1st and 2nd order recurrences, Master theorem.
   (a) Evaluate $\sum\limits_{i=0}^{n} \ldots$ and similar constructs.
   (b) Prove a property of a recurrent sequence by induction or using invariants.
   (c) Prove that a recurrent sequence has a closed formula using induction.
   (d) Given a 1st order non-homogeneous recurrence, solve it.
   (e) Given a 2nd order homogeneous recurrence, solve it.

(f) Given a word problem (sets of strings, Tower of Hanoi, tilings, etc.) build recurrences.

(g) Given a divide-and-conquer type recurrence, solve it with Master theorem.

3. **Big-O notation.**

    (a) Given functions f, g, check by definition that $f(n)$ is in $O(g(n))$, $\Omega(g(n))$, $\Theta(g(n))$.

    (b) Given a function $f(x)$, simplify it to get its "optimal" $O(g(x))$ or $\Theta(g(x))$ class.

    (c) Given a collection of functions, arrange them by growth.

    (d) Given a pseudocode, basic operations and input length, estimate its time as $O(g(n))$.

Part 4: Probabilities

1. **Events.** Events, complements, independence, conditional probability, Bernoulli trials.

    (a) Describe events in a sample space and compute probabilities using Laplace's definition.

    (b) Compute probabilities of derived events (complementary, intersection, union, etc.).

    (c) Find conditional probabilities for events and their combinations.

    (d) Prove or disprove pairwise and mutual independence of events.

    (e) Analyze the probabilities of the outcomes of a probabilistic 2-player game.

    (f) Express conditional probabilities using Bayes' theorem.

2. **Random Variables.** Expected value, variance, distributions, Chebyshev's inequality.

    (a) Identify the geometric, binomial, and Bernoulli distributions.

    (b) Given a definition of a random variable, find its probability mass function.

    (c) Given a distribution for a discrete random variable X, compute $E(X)$ and $V(X)$.

    (d) Estimate the probability of X being in an interval by Chebyshev's inequality.