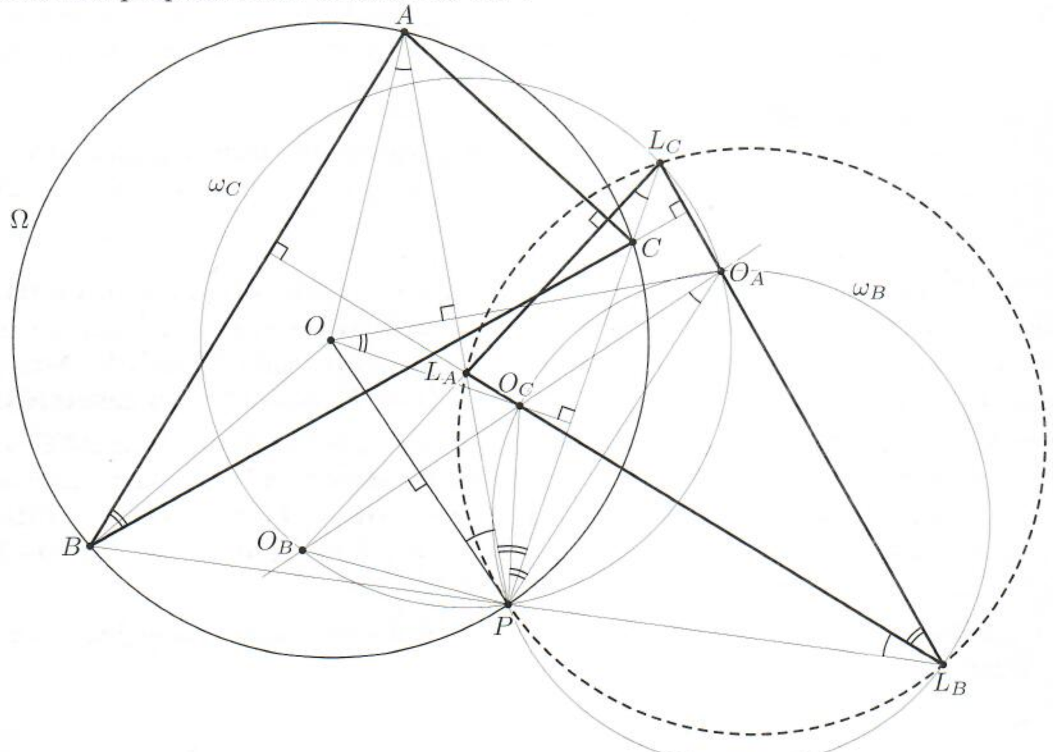
— Bash solution

G7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, and ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

Solution. As usual, we denote the directed angle between the lines a and b by $\measuredangle(a, b)$. We frequently use the fact that $a_1 \perp a_2$ and $b_1 \perp b_2$ yield $\measuredangle(a_1, b_1) = \measuredangle(a_2, b_2)$.

Let the lines ℓ_B and ℓ_C meet at L_A; define the points L_B and L_C similarly. Note that the sidelines of the triangle $L_A L_B L_C$ are perpendicular to the corresponding sidelines of ABC. Points O_A, O_B, O_C are located on the corresponding sidelines of $L_A L_B L_C$; moreover, O_A, O_B, O_C all lie on the perpendicular bisector of OP.



Claim 1. The points L_B, P, O_A, and O_C are concyclic.

Proof. Since O is symmetric to P in $O_A O_C$, we have

$$\measuredangle(O_A P, O_C P) = \measuredangle(O_C O, O_A O) = \measuredangle(CP, AP) = \measuredangle(CB, AB) = \measuredangle(O_A L_B, O_C L_B). \qquad \square$$

Denote the circle through L_B, P, O_A, and O_C by ω_B. Define the circles ω_A and ω_C similarly.

Claim 2. The circumcircle of the triangle $L_A L_B L_C$ passes through P.

Proof. From cyclic quadruples of points in the circles ω_B and ω_C, we have

$$\begin{aligned}\measuredangle(L_C L_A, L_C P) &= \measuredangle(L_C O_B, L_C P) = \measuredangle(O_A O_B, O_A P)\\ &= \measuredangle(O_A O_C, O_A P) = \measuredangle(L_B O_C, L_B P) = \measuredangle(L_B L_A, L_B P).\end{aligned} \qquad \square$$

Claim 3. The points P, L_C, and C are collinear.

Proof. We have $\measuredangle(PL_C, L_C L_A) = \measuredangle(PL_C, L_C O_B) = \measuredangle(PO_A, O_A O_B)$. Further, since O_A is the centre of the circle AOP, $\measuredangle(PO_A, O_A O_B) = \measuredangle(PA, AO)$. As O is the circumcentre of the triangle PCA, $\measuredangle(PA, AO) = \pi/2 - \measuredangle(CA, CP) = \measuredangle(CP, L_C L_A)$. We obtain $\measuredangle(PL_C, L_C L_A) = \measuredangle(CP, L_C L_A)$, which shows that $P \in CL_C$. $\square$

Similarly, the points P, L_A, A are collinear, and the points P, L_B, B are also collinear. Finally, the computation above also shows that

$$\measuredangle(OP, PL_A) = \measuredangle(PA, AO) = \measuredangle(PL_C, L_C L_A),$$

which means that OP is tangent to the circle $PL_A L_B L_C$.

Comment 1. The proof of Claim 2 may be replaced by the following remark: since P belongs to the circles ω_A and ω_C, P is the Miquel point of the four lines ℓ_A, ℓ_B, ℓ_C, and $O_A O_B O_C$.

Comment 2. Claims 2 and 3 can be proved in several different ways and, in particular, in the reverse order.

Claim 3 implies that the triangles ABC and $L_A L_B L_C$ are perspective with perspector P. Claim 2 can be derived from this observation using spiral similarity. Consider the centre Q of the spiral similarity that maps ABC to $L_A L_B L_C$. From known spiral similarity properties, the points L_A, L_B, P, Q are concyclic, and so are L_A, L_C, P, Q.

Comment 3. The final conclusion can also be proved it terms of spiral similarity: the spiral similarity with centre Q located on the circle ABC maps the circle ABC to the circle $PL_A L_B L_C$. Thus these circles are orthogonal.

Comment 4. Notice that the homothety with centre O and ratio 2 takes O_A to A' that is the common point of tangents to Ω at A and P. Similarly, let this homothety take O_B to B' and O_C to C'. Let the tangents to Ω at B and C meet at A'', and define the points B'' and C'' similarly. Now, replacing labels O with I, Ω with ω, and swapping labels $A \leftrightarrow A''$, $B \leftrightarrow B''$, $C \leftrightarrow C''$ we obtain the following

Reformulation. Let ω be the incircle, and let I be the incentre of a triangle ABC. Let P be a point of ω (other than the points of contact of ω with the sides of ABC). The tangent to ω at P meets the lines AB, BC, and CA at A', B', and C', respectively. Line ℓ_A parallel to the internal angle bisector of $\angle BAC$ passes through A'; define lines ℓ_B and ℓ_C similarly. Prove that the line IP is tangent to the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C.

Though this formulation is equivalent to the original one, it seems more challenging, since the point of contact is now "hidden".