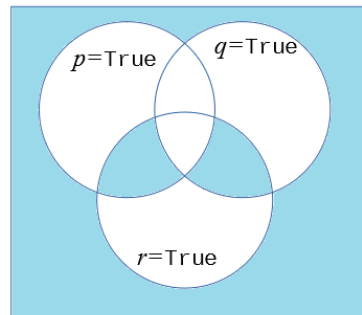


1. In the Venn diagram each oval represents the area where statements p , q , and r are true (outside their respective ovals they are false). Regions where $f(p, q, r) = \text{True}$ are shaded.



- (a) Create a truth table for this Boolean function $f(p, q, r)$.

p	q	r	$f(p, q, r)$
False	False	False	True
False	False	True	False
False	True	False	False
False	True	True	True
True	False	False	False
True	False	True	True
True	True	False	False
True	True	True	False

Region outside of all ovals is shaded, so $f(\text{False}, \text{False}, \text{False}) = \text{True}$ (1st line in the truth table); also two more regions are shaded, where just two of the variables are **True**. \square

- (b) Create either a Disjunctive Normal Form (DNF) or a Conjunctive Normal Form (CNF) for this function $f(p, q, r)$.

Note. You can pick either a DNF or a CNF – whatever you like; there is no need to construct both.

In our situation the DNF would be shorter – it is a disjunction of just 3 conjunctions (each conjunction describes one row where the Boolean function is **True**).

$$f(p, q, r) = (\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r).$$

(CNF would be a conjunction of 5 expressions – one per every row in the table where the function is **False**.) \square

2. Let H be the set of all humans. We define the following predicates:

$K(a, b)$ is true iff when the human $a \in H$ knows a human $b \in H$.

$R(a)$ is true iff the human a has travelled from Riga to Frankfurt by train.

Express the following English sentences using the predicates and quantifiers:

- (a) For any two humans a and b , person a knows person b iff person b knows person a .

$$\forall a \in H \forall b \in H (K(a, b) \leftrightarrow K(b, a)).$$

This condition tells that the predicate $K(a, b)$ (which is also a relation of “knowing” between two humans) is *symmetric*. It is also possible to express this as one-way implication.

$$\forall a \in H \forall b \in H (K(a, b) \rightarrow K(b, a)).$$

The 2nd expression tells the same thing (you can get the reverse implication, if you switch variables a and b). So both answers are fine; the 1st answer reflects the English sentence better. \square

- (b) Not every human knows someone who has travelled from Riga to Frankfurt by train.

$$\neg \forall a \in H \exists b \in H (K(a, b) \wedge R(b)).$$

Literally: It is not the case that for any human $a \in H$ there exist a human $b \in H$ such that a knows b and b has traveled to Frankfurt.

If you wish, you can also rewrite this using De Morgan’s laws:

$$\exists a \in H \forall b \in H (\neg K(a, b) \vee \neg R(b)).$$

Literally: There is a human $a \in H$ such that for any human $b \in H$ – either a does not know b or b did not travel from Riga to Frankfurt by train. \square

3. Let H be the set of all humans and the predicates $K(a, b)$ and $R(a)$ are the same as in the previous exercise. Rewrite the statement so that negations are applied only to the predicates directly.

$$\neg \left(\forall a \in H \exists b \in H \exists c \in H \left(R(a) \vee (K(a, b) \wedge K(a, c) \wedge b \neq c \wedge R(b) \wedge R(c)) \right) \right).$$

Here we negate the sentence that every human $a \in H$ either himself/herself traveled to Frankfurt or he knows two other people b and c (not equal between themselves: $b \neq c$) such that both of them traveled to Frankfurt.

$$\left(\exists a \in H \forall b \in H \forall c \in H \left(\neg R(a) \wedge (\neg K(a, b) \vee \neg K(a, c) \vee b = c \vee \neg R(b) \vee \neg R(c)) \right) \right).$$

Literally: Some human a is such that s/he did not travel to Frankfurt and also for any other two humans b, c , either a does not know at least one of them, or b is equal to c or at least one among b and c did not travel to Frankfurt. \square

4. Let $f(x) = x^3 + 1$ be a function $f : \mathbf{R} \rightarrow \mathbf{R}$ with real arguments and real values.

(a) Is f an injective function?

Yes, f is an injective function. The easiest way to see it is to notice that $f(x) = x^3 + 1$ is strictly monotone (growing) function as a cubic parabola: whenever $x_1 < x_2$ we should also have $f(x_1) < f(x_2)$.

So it is impossible that some different values $x_1 \neq x_2$ would have equal values $f(x_1) = f(x_2)$, since one of the x_1, x_2 would be smaller than the other (and we apply the monotonicity). \square

(b) Is f a surjective function?

Yes, f is a surjection, since for every $a \in \mathbf{R}$ we can solve the equation:

$$x^3 + 1 = a.$$

The root is $x = \sqrt[3]{a-1}$. If we compute $f(x)$ for this argument we will get the value a we need. \square

5. Let $S = \{1, 2, 4, 5, 7, 8\}$ be the set of all digits not divisible by 3. Define a relation $R \subseteq S^2$:

$$R = \{(a, b) \in S^2 \mid |a - b| \leq 1\}.$$

(a) Is the relation R reflexive? Is it symmetric? Is it transitive?

The condition tells that the distance between a and b is no more than 1. So the only pairs in the relation are $R(1, 1)$, $R(1, 2)$, $R(2, 1)$, $R(2, 2)$ (and similarly for 4 and 5 and also for 7 and 8): Two digits are in this relation if they are the same ($|a - b| = 0$) or they are next to each other.

Clearly, this relation is reflexive (as $|a - a| = 0$); it is symmetric (since $|a - b| = |b - a|$). It is also transitive (because the digits fall into three groups of neighbors – if a and b are in the same group and also b and c are in the same group, then also a, c are in the same group). \square

(b) Let R^t be the transitive closure of the relation R . What are the pairs that belong to R^t , but not to R (or vice versa)?

As we saw in the previous item – the relation R is itself transitive. Therefore its transitive closure R^t does not add any new pairs to the relation. We have $R^t = R$: both relations contain exactly the same pairs. \square

6. Let α be an irrational number.

(a) Is $\frac{2\alpha}{3}$ a rational or an irrational number? (Or can it be either depending on α .)

The number $\frac{2\alpha}{3}$ must be irrational.

Prove this by contradiction.

Assume that $\frac{2\alpha}{3} = \frac{p}{q}$ for some rational fraction p/q . In this case $\alpha = \frac{3p}{2q}$, so it is also a rational fraction. But this contradicts the known fact that α must be irrational.

Generally, if you multiply irrational numbers by rational ones (such as $2/3$), you always get irrational results. \square

- (b) Is $\alpha^2 + 3$ a rational or an irrational number? (Or can it be either depending on α .)

The number $\alpha^2 + 3$ can be either rational or irrational.

Prove this by counterexamples.

Case 1. Let $\alpha = \sqrt{2}$. In this case $\alpha^2 + 3 = (\sqrt{2})^2 + 3 = 2 + 3 = 5$ which is rational.

Case 2. Let $\alpha = \sqrt[4]{2}$. Notice that the 4th degree root of 2 must be irrational; so $\alpha \notin \mathbf{Q}$. In this case $\alpha^2 + 3 = (\sqrt[4]{2})^2 + 3 = \sqrt{2} + 3$. This number is irrational, since $\sqrt{2}$ is also irrational.

Note. You can safely assume that all roots of integer numbers are either integers themselves (or they are irrational). Therefore $\sqrt[4]{2}$ must be irrational, since it is not an integer number: it is between 1 and 2.

You can also prove the fact that $\sqrt[4]{2}$ is irrational:

Assume that $\sqrt[4]{2} = p/q$ where p/q is an irreducible rational fraction with $q > 1$. Then $2 = \frac{p^4}{q^4}$ is also an irreducible fraction with $q^4 > 1$, which is a contradiction, since the number $2 = \frac{2}{1}$ has number 1 in the denominator. \square