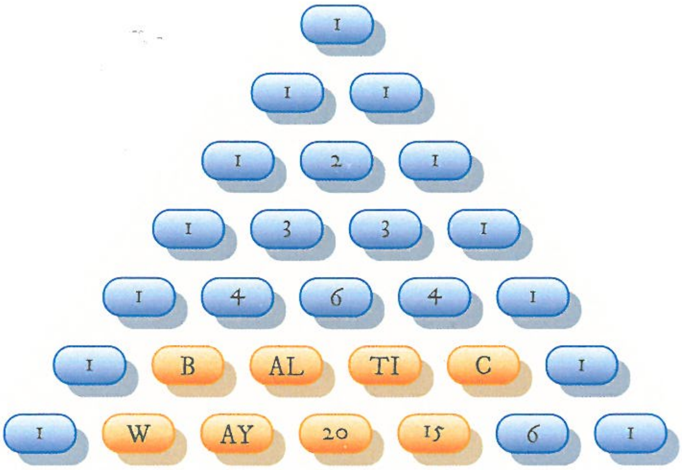


Baltic Way 2015

Shortlisted Problems

A1 A5 A7 A9 A12

Algebra

Problems

Problem A.1 *Denmark*

For $n \geqslant 2$, an equilateral triangle is divided into n^2 congruent smaller equilateral triangles. Determine all ways in which real numbers can be assigned to the $\frac{n(n+1)}{2}$ vertices so that three such numbers sum to zero whenever the three vertices form an equilateral triangle with edges parallel to the sides of the big triangle.

Problem A.2 *Latvia*

The numbers

$$\frac{1}{2016}, \frac{2}{2016}, \frac{3}{2016}, \ldots, \frac{2015}{2016}$$

are written on a blackboard. With each move, one may erase any two numbers a and b and replace them with

$$3ab - 2a - 2b + 2.$$

What will be the single remaining number after 2014 moves?

Problem A.3 *Denmark*

A function $f : \mathbf{R} \to \mathbf{R}$ satisfies

$$f(f(a)) = f(a) \quad \text{and} \quad f(a+b) = f(a) + f(b)$$

for all real numbers a, b. Prove that, for all real x, there exists a unique y such that $f(y) = 0$ and $x = y + f(z)$ for some real z.

Problem A.4 *Saint Petersburg*

Let m and n be positive integers and let the integer $X \geqslant \max(m,n)$. Show that there exist integers u and v, not both equal to 0, such that

$$\max(|u|,|v|) \leqslant \sqrt{X} \qquad \text{and} \qquad 0 \leqslant mu + nv \leqslant 2\sqrt{X}.$$

Problem A.5 *Denmark*

Let $a_1, \ldots, a_n$ be real numbers, fulfilling $0 \leqslant a_i \leqslant 1$ for $i = 1, \ldots, n$. Prove the inequality

$$(1-a_1^n)(1-a_2^n)\cdots(1-a_n^n) \leqslant (1-a_1a_2\cdots a_n)^n.$$

Problem A.6 *Sweden*

Let B, A, L, T, I, C be positive numbers. Find all possible values of the expression

$$\frac{BA}{(C+B)(A+L)} + \frac{LT}{(A+L)(T+I)} + \frac{IC}{(T+I)(C+B)}.$$

Problem A.7 *Poland*

Let $n > 1$. Find all non-constant real polynomials $P(x)$ fulfilling, for any real x, the identity

$$P(x)P(x^2)P(x^3)\cdots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right).$$

Problem A.8 *Finland*

Let P be a real polynomial of degree 2015 and Q a real quadratic polynomial. Could it be that the polynomial $P(Q(x))$ has precisely the roots

$$-2014,\ -2013,\ \ldots,\ -2,\ -1,\ 1,\ 2,\ \ldots,\ 2014,\ 2015,\ 2016?$$

Problem A.9 *Norway*

A family wears three colours of clothing: red, blue and green, with a separate laundry bin for each colour. Each week, the family generates a total of K kilogrammes of laundry (the proportion of each colour is subject to variation). The laundry is first sorted by colour and disposed of in the bins. Next, the heaviest bin is emptied and its contents washed. What is the storing capacity required of the laundry bins if they must never overflow?

Problem A.10 *Germany*

Consider four positive real numbers a, b, c and d, satisfying

$$a^2 + ab + b^2 = 3c^2 \quad \text{and} \quad a^3 + a^2b + ab^2 + b^3 = 4d^3.$$

Prove that

$$a + b + d \leqslant 3c.$$

Problem A.11 *Estonia*

Which number is greater,

$$\sin 1 - \cos 1 \quad \text{or} \quad \frac{1}{4}?$$

Problem A.12 *Estonia*

Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all real numbers x and y, the equation

$$|x|f(y) + yf(x) = f(xy) + f(x^2) + f(f(y)).$$

Problem A.13 *Estonia*

Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all real x and y, the equality

$$f(2^x + 2y) = 2^y f(f(x))f(y).$$

Problem A.14 *Norway*

Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all $x \neq 0$ and all y,

$$f(x + y^2) = f(x) + f(y)^2 + \frac{2f(xy)}{x}.$$

Problem A.15 *Sweden*

Let a and b be positive numbers. Find all pairs of functions $f, g\colon \mathbf{R} \to \mathbf{R}$, each assuming the value 1 and fulfilling, for any $y \neq 0$ and any x, the equations

$$f\left(\frac{1}{y^2}g(xy) - ax^2\right) = 0 = g\left(\frac{1}{y}f(xy) - bx\right).$$

Problem A.16 *Saint Petersburg*

Prove that, for positive x, y, z, the following inequality holds:

$$(x + y + z)(4x + y + 2z)(2x + y + 8z) \geqslant \frac{375}{2}xyz.$$

Problem A.17 *Latvia*

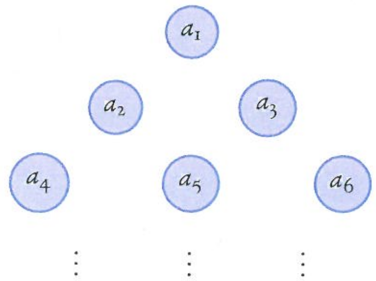
For $x \geqslant \frac{1}{2}$, what is the largest possible value of the expression

$$\frac{x^4 - x^2}{x^6 + 16x^3 - 1}?$$

Solutions

Problem A.1
For $n \geqslant 2$, an equilateral triangle is divided into n^2 congruent smaller equilateral triangles. Determine all ways in which real numbers can be assigned to the $\frac{n(n+1)}{2}$ vertices so that three such numbers sum to zero whenever the three vertices form an equilateral triangle with edges parallel to the sides of the big triangle.

Solution. We label the vertices (and the corresponding real numbers) as follows.

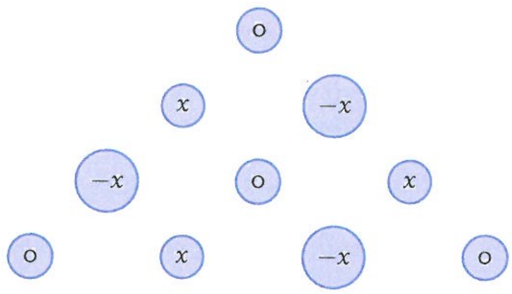


For $n = 2$, the only requirement is obviously $a_1 = -a_2 - a_3$.
For $n = 3$, we see that

$$a_2 + a_4 + a_5 = 0 = a_2 + a_3 + a_5,$$

which shows that $a_3 = a_4$ and similarly $a_1 = a_5$ and $a_2 = a_6$. Now the only requirement is the stated equalities and $a_1 = -a_2 - a_3$.

For $n = 4$, observe that $a_1 = a_7 = a_{10}$ since they all equal a_5. Since also $a_1 + a_7 + a_{10} = 0$, they all equal zero. By considering the top triangle, we get $x = a_2 = -a_3$ and this uniquely determines the rest. It is easily checked that, for any real x, this is actually a solution:



For $n > 4$ we can apply the same argument as above for any collection of 10 vertices. Any vertex not on the sides of the big triangle has to equal zero, since it is the centre of such a collection of 10 vertices. Any vertex a on the sides of the big triangle forms some parallelogram similar to a_4, a_2, a_5, a_8, where the point opposite a is in the interior of the big triangle. Since such opposite numbers are equal, all a_i have to be zero in this case. □

Problem A.2
The numbers
$$\frac{1}{2016}, \frac{2}{2016}, \frac{3}{2016}, \ldots, \frac{2015}{2016}$$
are written on a blackboard. With each move, one may erase any two numbers a and b and replace them with
$$3ab - 2a - 2b + 2.$$
What will be the single remaining number after 2014 moves?

Solution. Note that if $a = \frac{1344}{2016} = \frac{2}{3}$, then
$$3ab - 2a - 2b + 2 = \frac{2}{3},$$
irrespective of the value of b. Hence, $\frac{2}{3}$ will always remain on the blackboard. □

Problem A.3
A function $f : \mathbf{R} \to \mathbf{R}$ satisfies
$$f(f(a)) = f(a) \quad \text{and} \quad f(a+b) = f(a) + f(b)$$
for all real numbers a, b. Prove that, for all real x, there exists a unique y such that $f(y) = 0$ and $x = y + f(z)$ for some real z.

Solution. Let x be given. First the uniqueness is proved. Assume that $x = y + f(z)$ with $f(y) = 0$. If f is applied on both sides, then
$$f(x) = f(y + f(z)) = f(y) + f(f(z)) = 0 + f(z) = f(z),$$
because $f(y) = 0$ and $f(f(a)) = f(a)$. Hence $y = x - f(z) = x - f(x)$ is the only possible candidate for y.

Now we prove that $y = x - f(x)$ has the assumed property. Observe that $f(a-b) = f(a) - f(b)$, and hence

$$f(y) = f(x - f(x)) = f(x) - f(f(x)) = f(x) - f(x) = 0.$$

Thus $x = y + f(x)$ as was to be proved. $\square$

Problem A.4

Let m and n be positive integers and let the integer $X \geqslant \max(m,n)$. Show that there exist integers u and v, not both equal to 0, such that

$$\max(|u|,|v|) \leqslant \sqrt{X} \qquad \text{and} \qquad 0 \leqslant mu + nv \leqslant 2\sqrt{X}.$$

Solution. There are $\lfloor \sqrt{X} + 1 \rfloor^2 \geqslant X + 1$ pairs (a,b) such that $0 \leqslant a,b \leqslant \sqrt{X}$, and for these

$$0 \leqslant ma + nb \leqslant 2X\sqrt{X}.$$

Two linear combinations $ma + nb \geqslant ma' + nb'$ differ by at most $2\sqrt{X}$, and so

$$\max(|a-a'|,|b-b'|) \leqslant \sqrt{X} \quad \text{and} \quad 0 \leqslant m(a-a') + n(b-b') \leqslant 2\sqrt{X}. \ \square$$

Problem A.5

Let $a_1, \dots, a_n$ be real numbers, fulfilling $0 \leqslant a_i \leqslant 1$ for $i = 1, \dots, n$. Prove the inequality

$$(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n) \leqslant (1 - a_1 a_2 \cdots a_n)^n.$$

Solution. AM–GM gives

$$\begin{aligned}(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n) &\leqslant \left(\frac{(1 - a_1^n) + (1 - a_2^n) + \cdots + (1 - a_n^n)}{n} \right)^n \\ &= \left(1 - \frac{a_1^n + \cdots + a_n^n}{n} \right)^n.\end{aligned}$$

By applying AM–GM again we obtain

$$a_1 a_2 \cdots a_n \leqslant \frac{a_1^n + \cdots + a_n^n}{n} \quad \Rightarrow \quad \left(1 - \frac{a_1^n + \cdots + a_n^n}{n} \right)^n \leqslant (1 - a_1 a_2 \cdots a_n)^n,$$

and hence the desired inequality. $\square$

Remark. It is possible to use Jensen's inequality applied to $f(x) = \log(1 - e^x)$.

Problem A.6
Let B, A, L, T, I, C be positive numbers. Find all possible values of the expression

$$\frac{BA}{(C+B)(A+L)} + \frac{LT}{(A+L)(T+I)} + \frac{IC}{(T+I)(C+B)}.$$

Solution. The range of the expression is the interval $(0, 1)$. Writing $x = \frac{A}{A+L}$, $y = \frac{T}{T+I}$, $z = \frac{C}{C+B}$, and observing that $\frac{L}{A+L} = 1 - x$ &c., the expression transforms into

$$x(1-z) + y(1-x) + z(1-y) = 1 - xyz - (1-x)(1-y)(1-z),$$

where $0 < x, y, z < 1$. Clearly the expression may assume arbitrarily small positive values, by letting x, y, z approach 0, and also values arbitrarily close to 1, by letting x, y approach 0 and z approach 1. $\square$

Problem A.7
Let $n > 1$. Find all non-constant real polynomials $P(x)$ fulfilling, for any real x, the identity

$$P(x)P(x^2)P(x^3)\cdots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right).$$

Solution. Answer: $P(x) = x^n$ if n is even; $P(x) = \pm x^n$ if n is odd.

Consider first the case of a monomial $P(x) = ax^m$ with $a \neq 0$. Then

$$ax^{\frac{mn(n+1)}{2}} = P\left(x^{\frac{n(n+1)}{2}}\right) = P(x)P(x^2)P(x^3)\cdots P(x^n) = ax^m \cdot ax^{2m} \cdot \cdots \cdot ax^{nm} = a^n x^{\frac{mn(n+1)}{2}}$$

implies $a^n = a$. Thus, $a = 1$ when n is even and $a = \pm 1$ when n is odd. Obviously these polynomials satisfy desired equality.

Suppose now that P is not a monomial. Write $P(x) = ax^m + Q(x)$, where Q is non-zero polynomial with $\deg Q = k < m$. We have

$$ax^{\frac{mn(n+1)}{2}} + Q\left(x^{\frac{n(n+1)}{2}}\right) = P\left(x^{\frac{n(n+1)}{2}}\right) =$$
$$P(x)P(x^2)P(x^3)\cdots P(x^n) = (ax^m + Q(x))(ax^{2m} + Q(x^2))\cdots(ax^{nm} + Q(x^n)).$$

The highest degree of a monomial, on both sides of the equality, is $\frac{mn(n+1)}{2}$. The second highest degree in the right-hand side is

$$2m + 3m + \cdots + nm + k = \frac{m(n+2)(n-1)}{2} + k,$$

while in the left-hand side it is $\frac{kn(n+1)}{2}$. Thus

$$\frac{m(n+2)(n-1)}{2} + k = \frac{kn(n+1)}{2},$$

which leads to

$$(m-k)(n+2)(n-1) = 0,$$

and so $m = k$, contradicting the assumption that $m > k$. Consequently, no polynomial of the form $ax^m + Q(x)$ fulfils the given condition. □

Problem A.8

Let P be a real polynomial of degree 2015 and Q a real quadratic polynomial. Could it be that the polynomial $P(Q(x))$ has precisely the roots

$$-2014,\ -2013,\ \ldots,\ -2,\ -1,\ 1,\ 2,\ \ldots,\ 2014,\ 2015,\ 2016?$$

Solution. The values of Q at the 4030 points indicated in the problem need be a subset of the zeroes of P. But these are at most 2015 in number, and Q, being quadratic, assumes any given value at most twice. Therefore, the 4030 numbers can be split into 2015 pairs (p_i, q_i), for which $Q(p_i) = Q(q_i)$ runs through all the 2015 zeroes of P as $i = 1, \ldots, 2015$.

Put $Q(x) = ax^2 + bx + c$. By Vieta's formulæ, $p_i + q_i = -\frac{b}{a}$ for such a pair, so the 2015 pairs need have equal sums. Since the numbers are integers, this is possible only if their sum is a multiple of 2015. But it is not, in fact the sum is

$$-1007 \cdot 2015 + 1008 \cdot 2017,$$

which is not even a multiple of 5. □

Problem A.9

A family wears three colours of clothing: red, blue and green, with a separate laundry bin for each colour. Each week, the family generates a total of K kilogrammes of laundry (the proportion of each colour is subject to variation). The laundry is first sorted by colour and disposed of in the bins. Next, the heaviest bin is emptied and its contents washed. What is the storing capacity required of the laundry bins if they must never overflow?

Solution. Answer: $\frac{5}{2}K$.

Each week, the accumulation of laundry increases the total amount by K, after which the washing decreases it by at least one third, because, by the pigeon-hole principle, the bin with the most laundry must contain at least a third of the total. Hence the amount of laundry post-wash after the nth week is bounded above by the sequence $a_{n+1} = \frac{2}{3}(a_n + K)$ with $a_0 = 0$, which is clearly bounded above by $2K$. The total amount of laundry is less than $2K$ post-wash and $3K$ pre-wash.

Now suppose pre-wash state (a, b, c) precedes post-wash state $(a, b, 0)$, which precedes pre-wash state (a', b', c'). The relations $a \leqslant c$ and $a' \leqslant a + K$ lead to

$$3K > a + b + c \geqslant 2a \geqslant 2(a' - K),$$

and similarly for b', whence $a', b' < \frac{5}{2}K$. Since also $c' \leqslant K$, a pre-wash bin, and *a fortiori* a post-wash bin, always contains less than $\frac{5}{2}K$.

Consider now the following scenario. For a start, we keep packing the three bins equally full before washing. Initialising at $(0, 0, 0)$, the first week will end at $(\frac{1}{3}K, \frac{1}{3}K, \frac{1}{3}K)$ pre-wash and $(\frac{1}{3}K, \frac{1}{3}K, 0)$ post-wash, the second week at $(\frac{5}{9}K, \frac{5}{9}K, \frac{5}{9}K)$ pre-wash and $(\frac{5}{9}K, \frac{5}{9}K, 0)$ post-wash, &c. Following this scheme, we can get arbitrarily close to the state $(K, K, 0)$ after washing. Supposing this accomplished, placing $\frac{1}{2}K$ kg of laundry in each of the non-empty bins leaves us in a state close to $(\frac{3}{2}K, \frac{3}{2}K, 0)$ pre-wash and $(\frac{3}{2}K, 0, 0)$ post-wash. Finally, the next week's worth of laundry is directed solely to the single non-empty bin. It may thus contain any amount of laundry below $\frac{5}{2}K$ kg. □

Problem A.10

Consider four positive real numbers a, b, c and d, satisfying

$$a^2 + ab + b^2 = 3c^2 \qquad \text{and} \qquad a^3 + a^2b + ab^2 + b^3 = 4d^3.$$

Prove that

$$a + b + d \leqslant 3c.$$

Solution. Setting $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$, we have $a = x + y$ and $b = x - y$. The given equations transform into

$$c^2 = x^2 + \frac{y^2}{3} \tag{1}$$

$$d^3 = x(x^2 + y^2), \tag{2}$$

and the inequality to be proved into $2x + d \leqslant 3c$.

Solution 2. As before, we first deduce that $X = p^k$, $Y = p^l$ and assume $k \geqslant l$.

Suppose $k = l$. Then

$$0 = X - Y = -6a^6 - 14a^4b^2 + 14a^2b^4 + 6b^6 = 6(-a^6 + b^6) + 14a^2b^2(-a^2 + b^2).$$

If $a > b$, then the right-hand side is negative; if $a < b$, it is positive. So we must have $a = b$. Then $X = Y = 64a^6$, so a must be a power of 2. We find the solutions $(a, b) = (2^i, 2^i)$ with $i \geqslant 0$.

Now suppose $k > l$. Then

$$X - 7Y = -48a^6 - 224a^4b^2 - 112a^2b^4 = -16(3a^6 + 14a^4b^2 + 7a^2b^2) < 0.$$

Hence $X < 7Y$. Furthermore, we have $X = p^k \geqslant p^{l+1} = pY$, from which $p < 7$. This leaves the possibilities $p = 2$, $p = 3$ and $p = 5$. Modulo 8 we have $X - 7Y \equiv 0$, so $p^k + p^l \equiv 0$. Suppose $p \neq 2$. Then $p^k \equiv p$ or $p^k \equiv 1$, depending on whether k is even or odd, and also $p^l \equiv p$ or $p^l \equiv 1$. But then we must have $1 + 1 \equiv 0$ or $p + 1 \equiv 0$ or $p + p \equiv 0$. For $p = 3$ or $p = 5$ this is impossible.

Consequently, $p = 2$. If a and b are both even, then we can divide out by a factor 2^{12}, so without loss of generality we assume at least one of them is odd. If this is b, then

$$p^k \cdot p^l = XY \equiv 7b^6 \cdot b^6 \equiv 1 \bmod 2,$$

contradicting $p = 2$. Similarly, assuming a is odd will give a contradiction.

So a and b are both odd. As $X < 7Y$, we have $2^k < 7 \cdot 2^l$, so either $k = l + 1$ or $k = l + 2$. Now, a^2 is a divisor of

$$-16(3a^6 + 14a^4b^2 + 7a^2b^2) = X - 7Y = \begin{cases} 2^{l+1} - 7 \cdot 2^l = -5 \cdot 2^l & \text{if } k = l + 1; \\ 2^{l+2} - 7 \cdot 2^l = -3 \cdot 2^l & \text{if } k = l + 2. \end{cases}$$

Since a is odd, it follows that $a = 1$.

Similarly, by considering $Y - 7X$, we derive

$$b^2 \mid (1 - 7 \cdot 2) = -13 \qquad \text{or} \qquad b^2 \mid (1 - 7 \cdot 4) = -27,$$

from which $b = 1$ or $b = 3$. We assumed that $k > l$, so $a \neq b$, and it must be that $a = 1$ and $b = 3$. However, then $X = 8128$ and this is not a power of 2.

We conclude that the solutions are the pairs $(a, b) = (2^i, 2^i)$ with $i \geqslant 0$. □

Problem N.11

For any positive integer $n \geqslant 2$, we define A_n to be the number of positive integers m with the following property: the distance from n to the nearest integral multiple of m is equal to the distance from n^3 to the nearest multiple of m. Find all positive integers $n \geqslant 2$ for which A_n is odd.

(The distance between two integers a and b is defined as $|a - b|$.)

Solution. There are infinitely many solutions, since for each pair (a, b) of positive integers, there holds

$$\left(a(a^{2015} + b^{2015})\right)^{2015} + \left(b(a^{2015} + b^{2015})\right)^{2015} = (a^{2015} + b^{2015})^{2016}. \quad \square$$

Problem N.10

Determine all pairs (a, b) of positive integers for which the number

$$(a^6 + 21a^4b^2 + 35a^2b^4 + 7b^6)(b^6 + 21b^4a^2 + 35b^2a^4 + 7a^6)$$

is a prime power.

Solution 1. Let

$$X = a^6 + 21a^4b^2 + 35a^2b^4 + 7b^6 \qquad \text{and} \qquad Y = b^6 + 21b^4a^2 + 35b^2a^4 + 7a^6.$$

If XY is a prime power, then X and Y are both powers of the same prime. Hence there exists a prime p and positive integers k and l such that $X = p^k$, $Y = p^l$ and without loss of generality $k \geqslant l$.

First we determine all solutions with $\gcd(a, b) = 1$. Suppose $p > 2$. As $p \mid X$ and $p \mid Y$, we have

$$p \mid aX \pm bY = (a \pm b)^7.$$

So $p \mid a+b$ and $p \mid a-b$, hence $p \mid (a+b)+(a-b) = 2a$ and $p \mid a$. Since $p \mid a+b$ we also have $p \mid b$, contradicting $\gcd(a, b) = 1$. Now suppose $p = 2$. If $l \geqslant 8$ we find $2^8 \mid 2^l = Y$ and $2^8 \mid X$, so

$$2^8 \mid aX \pm bY = (a \pm b)^7.$$

This means $4 \mid a+b$ and $4 \mid a-b$, so a and b are even, contradicting $\gcd(a, b) = 1$. Hence we have $Y \leqslant 2^7$. We have $Y \geqslant a^6 + b^6 \geqslant 3^6 + 1^6 > 2^7$ if $(a, b) \neq (1, 1)$ (since a and b are odd as $2 = p \mid a-b$), so $a = b = 1$. Now XY is equal to 2^{12} and that is indeed a prime power. So there is one solution satisfying $\gcd(a, b) = 1$, and that is $a = b = 1$.

Now suppose that $\gcd(a, b) = d > 1$. Then d must also be a power of p, so we may put $d = p^i$ with $i > 0$. Write $a = dr$ and $b = ds$. Then the number

$$(r^6 + 21r^4s^2 + 35r^2s^4 + 7s^6)(s^6 + 21s^4r^2 + 35s^2r^4 + 7r^6) = \frac{XY}{p^{12i}} = p^{k+l-12i}$$

is a prime power as well, so (r, s) is a solution with $\gcd(r, s) = 1$. It follows that $r = s = 1$ and $p = 2$, so $(a, b) = (2^i, 2^i)$. This is indeed a solution, since then we have $XY = 2^{12(i+1)}$.

We conclude that the solutions are the pairs $(a, b) = (2^i, 2^i)$ with $i \geqslant 0$. $\square$

Remark. Enquiring for natural, rather than integral, solutions attenuates the problem.

Problem N.8

Given are three pairwise distinct positive integers a, b, c. They have no common divisor, and satisfy

$$a \mid (b-c)^2, \qquad b \mid (c-a)^2 \qquad \text{and} \qquad c \mid (a-b)^2.$$

Prove that there does not exist a triangle with side lengths a, b, c.

Solution. First observe that these numbers are pairwise coprime. Indeed, if, say, a and b are divisible by a prime p, then p divides b, which divides $(a-c)^2$; hence p divides $a-c$, and therefore p divides c. Thus, p is a common divisor of these three numbers, a contradiction.

Now consider the number

$$M = 2ab + 2bc + 2ac - a^2 - b^2 - c^2.$$

It is clear from the problem condition that M is divisible by a, b, c, and therefore M is divisible by abc.

Assume that a triangle with sides a, b, c exists. Then $a < b + c$, and so $a^2 < ab + ac$. Analogously, we have $b^2 < bc + ba$ and $c^2 < ca + cb$. Summing these three inequalities leads to $M > 0$, and hence $M \geqslant abc$.

On the other hand,

$$a^2 + b^2 + c^2 > ab + bc + ac,$$

and therefore $M < ab+bc+ac$. Supposing, with no loss of generality, $a > b > c$, we must have $M < 3ab$. Taking into account the inequality $M \geqslant abc$, we conclude that $c = 1$ or $c = 2$ are the only possibilities.

For $c = 1$ we have $b < a < b + 1$ (the first inequality is our assumption, the second is the triangle inequality), a contradiction.

For $c = 2$ we have $b < a < b + 2$, i.e. $a = b + 1$. But then $1 = (a-b)^2$ is not divisible by $c = 2$. □

Problem N.9

How many solutions does the equation

$$x^{2015} + y^{2015} = z^{2016}$$

have in distinct positive integers: none, finitely many or infinitely many?

Summing the equations gives

$$a^3 + b^3 + c^3 = 3abc + a + b + c.$$

Using the identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\big((a - b)^2 + (a - c)^2 + (b - c)^2\big),$$

we get

$$a + b + c = 0 \qquad \text{or} \qquad (a - b)^2 + (a - c)^2 + (b - c)^2 = 2$$

The latter case implies that a, b, c are some numbers k, k, $k \pm 1$ in some order. If $a = b$, then equations (14) and (15) imply

$$2a + 2c = -c \qquad \Rightarrow \qquad 0 = 2a + 3c = 5k \pm 3,$$

which is impossible. If $a = c$, then equations (14) and (16) imply

$$2a + 2c = -a + b \qquad \Rightarrow \qquad 0 = 3a - b + 2c = 4k \pm 1,$$

which is impossible. If $b = c$, then equations (15) and (16) imply

$$-c = -a + b \qquad \Rightarrow \qquad 0 = a - b - c = -k \pm 1,$$

whence $k = \pm 1$. These values lead up to the solutions $(a, b, c) = (2, 1, 1)$ or $(a, b, c) = (-2, -1, -1)$.

There remains to investigate the case when $a + b + c = 0$. We may assume that exactly one of the unknowns is negative since the set of equations is invariant under $(a, b, c) \mapsto (-a, -b, -c)$. If $a < 0$ (and hence $b, c > 0$), substitution of $a = -b - c$ into equation (15) gives

$$b^3 = -(b + c)bc - c,$$

which is impossible. If $b < 0$, substitution of $b = -a - c$ into (16) gives

$$c^3 = -ac(a + c) - 2a - c,$$

which is impossible. If $c < 0$, substitution of $c = -a - b$ into (14) gives

$$a^3 = -ab(a + b) - 2b,$$

which is again impossible.

(Alternatively, when $a + b + c = 0$, it is quite easy to prove that $\gcd(a, b) = 1$ and $(b - a) \mid 4$ and start analysing cases.) $\square$

Problem N.6

Let $h(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree $n \geqslant 1$ with n integer roots. It is known that there exist distinct primes $p_0, p_1, \ldots, p_{n-1}$ such that a_i is a power of p_i, for all $i = 0, \ldots, n-1$. Find all possible values of n.

Solution. Obviously all the roots have to be negative by the positivity of the coefficients. If at least two of the roots are unequal to -1, then both of them have to be powers of p_0. Now Vieta's formulæ yield $p_0 \mid a_1$, which is a contradiction. Thus we can factor h as

$$h(x) = (x + a_0)(x + 1)^{n-1}.$$

Expanding yields

$$a_2 = \binom{n-1}{1} + a_0\binom{n-1}{2} \qquad \text{and} \qquad a_{n-2} = a_0\binom{n-1}{n-2} + \binom{n-1}{n-3}.$$

If $n \geqslant 5$, we see that $2 \neq n-2$ and so the two coefficients above are relatively prime, being powers of two distinct primes. However, depending on the parity of n, we have that a_2 and a_{n-2} are both divisible by $n-1$ or $\frac{n-1}{2}$, which is a contradiction.

For $n = 1, 2, 3, 4$, the following polynomials meet the requirements:

$$\begin{aligned} h_1(x) &= x + 2 \\ h_2(x) &= (x+2)(x+1) = x^2 + 3x + 2 \\ h_3(x) &= (x+3)(x+1)^2 = x^3 + 5x^2 + 7x + 3 \\ h_4(x) &= (x+2)(x+1)^3 = x^4 + 5x^3 + 9x^2 + 7x + 2 \end{aligned} \qquad \square$$

Problem N.7

Find all triples of integers (a, b, c) satisfying the system of equations

$$a^3 = abc + 2a + 2c \tag{14}$$
$$b^3 = abc - c \tag{15}$$
$$c^3 = abc - a + b. \tag{16}$$

Solution. Answer: There are three solutions: $(0,0,0)$, $(2,1,1)$ and $(-2,-1,-1)$.

Note that $abc = 0$ implies that $(a,b,c) = (0,0,0)$, which is a solution. Assume from now on that a, b and c are non-zero.

Solution. Answer: The equality holds only for $n = 3$.

It is easy to see that $P(n) \neq P(n+1)$. Therefore we need also that $\lfloor\sqrt{n}\rfloor \neq \lfloor\sqrt{n+1}\rfloor$ in order for equality to hold. This is only possible if $n+1$ is a perfect square. In this case,

$$\lfloor\sqrt{n}\rfloor + 1 = \lfloor\sqrt{n+1}\rfloor,$$

and hence $P(n) = P(n+1) + 1$. As both $P(n)$ and $P(n+1)$ are primes, it must be that $P(n) = 3$ and $P(n+1) = 2$.

It follows that $n = 3^a$ and $n+1 = 2^b$, and we are required to solve the equation $3^a = 2^b - 1$. Calculating modulo 3, we find that b is even. Put $b = 2c$:

$$3^a = (2^c - 1)(2^c + 1).$$

As both factors cannot be divisible by 3 (their difference is 2), $2^c - 1 = 1$. From this we get $c = 1$, which leads to $n = 3$. □

Problem N.4

Find all positive integers a for which $a^{a-1} - 1$ is divisible by 2^{2015}, but not by 2^{2016}.

Solution. Since a must be odd, write $a = 2^d u + 1$, where $u, d \in \mathbf{N}$ and u is odd. Now

$$a^{a-1} - 1 = (a^{2^d} - 1)(\underbrace{a^{2^d \cdot (u-1)} + \cdots + a^{2^d \cdot 1} + 1}_{u}),$$

and hence $2^{2015} \parallel (a^{a-1} - 1)$ iff $2^{2015} \parallel (a^{2^d} - 1)$. (The notation $p^n \parallel m$ denotes that $p^n \mid m$ and $p^{n+1} \nmid m$.)

We factorise once more:

$$\begin{aligned} a^{2^d} - 1 &= (a-1)(a+1)\underbrace{(a^2+1)\cdots(a^{2^{d-1}}+1)}_{d-1} \\ &= 2^d u \cdot 2(2^{d-1}u + 1)\underbrace{(a^2+1)\cdots(a^{2^{d-1}}+1)}_{d-1}. \end{aligned}$$

If $k \geqslant 1$, then $2 \parallel a^{2^k} + 1$, and so from the above

$$2^{2d} \parallel 2^d u \cdot 2 \cdot \underbrace{(a^2+1)\cdots(a^{2^{d-1}}+1)}_{d-1} \qquad \text{and} \qquad 2^{2015-2d} \parallel (2^{d-1}u + 1).$$

It is easy to see that this is the case exactly when $d = 1$ and $u = 2^{2013}v - 1$ where v is odd.

Solutions

Problem N.1
For n a natural number, what is the greatest possible value of

$$\text{GCD}(n^2 + 3, (n+1)^2 + 3)?$$

Solution. If k divides both $n^2 + 3$ and $n^2 + 2n + 4$, it must also divide the difference $2n + 1$. Then also

$$k \mid n(2n+1) - 2(n^2+3) = n - 6,$$

and finally

$$k \mid (2n+1) - 2(n-6) = 13.$$

The value 13 is attained for $n = 6$:

$$\text{GCD}(6^2 + 3, 7^2 + 3) = \text{GCD}(39, 52) = 13. \quad \square$$

Problem N.2
Prove that the equation

$$a^2 + 2015a + b^2 + 2015b = c^2 + 2015c$$

has infinitely many solutions in positive integers.

Solution. Taking $c = b + 1$ transforms the equation to

$$a^2 + 2015a = 2b + 2016.$$

As $a^2 + 2015a$ is even and greater than 2016 for any $a \geqslant 2$, the equations

$$\begin{cases} b = \frac{a^2+2015a-2016}{2} \\ c = b + 1 \end{cases}$$

give a solution for any $a \geqslant 2$. $\square$

Problem N.3
Denote by $P(n)$ the greatest prime divisor of n. Find all $n \geqslant 2$ for which

$$P(n) + \lfloor \sqrt{n} \rfloor = P(n+1) + \lfloor \sqrt{n+1} \rfloor.$$

integral multiple of m is equal to the distance from n^3 to the nearest multiple of m. Find all positive integers $n \geqslant 2$ for which A_n is odd.

(The distance between two integers a and b is defined as $|a - b|$.)

Problem N.12 *Latvia*

Does there exist an $n \geqslant 2$, not divisible by 10, for which reversing the order of the decimal digits of n^k produces a prime number for all positive integers k?

Problem N.13 *Finland*

Find all positive integers n such that the congruence

$$x^{60} \equiv 1 \bmod n$$

is satisfied for all x with $\gcd(x, n) = 1$.

Problem N.6 ***Denmark***

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree $n \geqslant 1$ with n integer roots. It is known that there exist distinct primes $p_0, p_1, \dots, p_{n-1}$ such that a_i is a power of p_i, for all $i = 0, \dots, n-1$. Find all possible values of n.

Problem N.7 ***Norway***

Find all triples of integers (a, b, c) satisfying the system of equations

$$a^3 = abc + 2a + 2c \tag{9}$$
$$b^3 = abc - c \tag{10}$$
$$c^3 = abc - a + b. \tag{11}$$

Problem N.8 ***Saint Petersburg***

Given are three pairwise distinct positive integers a, b, c. They have no common divisor, and satisfy

$$a \mid (b-c)^2, \qquad b \mid (c-a)^2 \quad \text{and} \quad c \mid (a-b)^2.$$

Prove that there does not exist a triangle with side lengths a, b, c.

Problem N.9 ***Sweden***

How many solutions does the equation

$$x^{2015} + y^{2015} = z^{2016}$$

have in distinct positive integers: none, finitely many or infinitely many?

Problem N.10 ***The Netherlands***

Determine all pairs (a, b) of positive integers for which the number

$$(a^6 + 21a^4b^2 + 35a^2b^4 + 7b^6)(b^6 + 21b^4a^2 + 35b^2a^4 + 7a^6)$$

is a prime power.

Problem N.11 ***The Netherlands***

For any positive integer $n \geqslant 2$, we define A_n to be the number of positive integers m with the following property: the distance from n to the nearest

N3 N4 N6 N8 N11

Number Theory

Problems

Problem N.1 *Latvia*

For n a natural number, what is the greatest possible value of

$$\gcd(n^2+3,(n+1)^2+3)?$$

Problem N.2 *Latvia*

Prove that the equation

$$a^2+2015a+b^2+2015b=c^2+2015c$$

has infinitely many solutions in positive integers.

Problem N.3 *Latvia*

Denote by $P(n)$ the greatest prime divisor of n. Find all $n \geqslant 2$ for which

$$P(n)+\lfloor\sqrt{n}\rfloor=P(n+1)+\lfloor\sqrt{n+1}\rfloor.$$

Problem N.4 *Denmark*

Find all positive integers a for which $a^{a-1}-1$ is divisible by 2^{2015}, but not by 2^{2016}.

Problem N.5 *Germany*

Do there exist five prime numbers p, q, r, s and t such that

$$p^3+q^3+r^3+s^3=t^3\ ?$$

By the power-of-a-point theorem, we have $AM \cdot AF = AT \cdot AD$. Substituting the previous equalities, we get $VY \cdot VZ = VU \cdot VX$. Thus, the points X, Y, Z, U are concyclic.

We finish the solution by showing that U, T, Y, Z are concyclic. Note that $FLUT$ and $FLYZ$ are isosceles trapezia, so $UTZY$ is an isosceles trapezium as well. Thus U, T, Z, Y are concyclic. $\square$

Remark. It is also possible to create the diagram starting with the points B, C, Q and P. Explicitly, an alternative formulation is:

> Let $ABCD$ be a convex quadrilateral and let K and L be the midpoints of AB and CD, respectively. Suppose there exists a point P inside the quadrilateral such that $\angle PAB = \angle DBL$ and $\angle PBA = \angle CAL$. If CD is tangent to the circumcircle of $\triangle ABL$, show that P lies on KL.

The points have been renamed here, so that the points A, B, C, D, L and P correspond to the points B, C, Q, P, A and S, respectively, in the first formulation.

Problem G.13

Let ABC be a triangle. Let its altitudes AD, BE and CF concur at H. Let K, L and M be the midpoints of BC, CA and AB, respectively. Prove that, if $\angle BAC = 60^\circ$, then the midpoints of the segments AH, DK, EL, FM are concyclic.

Solution. Denote the midpoints of AH, DK, EL and FM by T, X, Y and Z, respectively. The points D, E, F, K, L, M, T lie on the Feuerbach circle of triangle ABC. Let U be midpoint of arc LM of this circle.

Observe that the triangle AME is equilateral. Indeed, triangle ABE is right-angled and since M is the midpoint of its hypotenuse we have $AM = ME$. Since $\angle AME = 60^\circ$, the conclusion follows. Analogously, the triangle AFL is equilateral.

Let V be the midpoint of segment ML. Then $VY \parallel ME \parallel FL \parallel ZV$, so $V \in YZ$.

It is easy to see that

$$2 \cdot VX = AD, \qquad 2 \cdot VY = ME = AM \qquad \text{and} \qquad 2 \cdot VZ = FL = AF.$$

We shall show that $2 \cdot VU = AT$. We have

$$\angle MUL = 180^\circ - \angle LDM = 180^\circ - \angle MAL = 120^\circ.$$

Moreover, $\angle FTE = 2\angle FAE = 120^\circ$, because T is the circumcentre of triangle AEF. Thus, the isosceles triangles FTE and MUL are similar. In fact, they are congruent, because they are inscribed in the same circle. Thus

$$AT = FT = MU = 2 \cdot UV.$$

Symmetrically we obtain

$$\frac{|BF|}{|FA|} = \frac{|AP| \cdot |BC|}{|AC| \cdot |AB|}.$$

From the definition of P and Q we know $|AP| = |AQ|$, and from the definition of D we know $|BD| = |CD|$. Thus we have

$$\frac{|AE|}{|EC|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|BF|}{|FA|} = \frac{|AB| \cdot |AC|}{|AQ| \cdot |BC|} \cdot 1 \cdot \frac{|AP| \cdot |BC|}{|AC| \cdot |AB|} = \frac{|AP|}{|AQ|} = 1.$$

Since D, E and F lie on the interior sides of the triangle, we obtain by Ceva's theorem that BE, CF and AD intersect in a point. As S lies on both BE and CF, this intersection must be S. Therefore the median of A runs through S. □

Solution 3. By the tangency of line PQ to the circumcircle of triangle ABC, we obtain

$$\angle APB = \angle QAB - \angle ABP = \angle ACB - \angle SCB = \angle ACS,$$

and, similarly,

$$\angle AQC = \angle PAC - \angle ACQ = \angle ABC - \angle SBC = \angle ABS.$$

By the sine law in triangle APB and AQC, we obtain

$$\frac{AB}{AP} = \frac{\sin \angle APB}{\sin \angle ABP} = \frac{\sin \angle ACS}{\sin \angle SCB}$$

and

$$\frac{AC}{AQ} = \frac{\sin \angle AQC}{\sin \angle ACQ} = \frac{\sin \angle ABS}{\sin \angle SBC}.$$

Hence, we conclude that

$$\frac{AB}{AC} = \frac{\sin \angle ACS}{\sin \angle ABS} \cdot \frac{\sin \angle SBC}{\sin \angle SCB}.$$

By the trigonometric Ceva theorem in triangle ABC with respect to point S, we obtain

$$\frac{\sin \angle ACS}{\sin \angle ABS} \cdot \frac{\sin \angle SBC}{\sin \angle SCB} = \frac{\sin \angle CAS}{\sin \angle BAS}.$$

Therefore,

$$AB \cdot \sin \angle BAS = AC \cdot \sin \angle CAS.$$

Let M be the intersection of AS and BC. We have

$$AM \cdot AB \cdot \sin \angle BAS = AM \cdot AC \cdot \sin \angle CAS,$$

so the areas of triangles ABM and ACM are equal, which means that AM is the median of triangle ABC. □

This implies that the triangles $\triangle QAC$ and $\triangle BAT$ are similar. Thus we have

$$|AT| = \frac{|AB||AC|}{|AQ|}.$$

Analogously one can show that

$$|AU| = \frac{|AB||AC|}{|AP|}.$$

Since $|AP| = |AQ|$, it follows that $|AT| = |AU|$, hence A is the midpoint of TU, as desired. $\square$

Remark. A similar proof may start by considering the line parallel to BC through S, and defining the intersections of that line with AB and AC.

Solution 2. Let D be the midpoint of BC, E the intersection of AC and BS, and F the intersection of AB and CS. The proof will proceed by using Ceva's theorem, that is, we will show

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1.$$

Let CQ meet the circle again in Q', and let BS meet the circle again in E'. Then observe that

$$\angle ACQ' = \angle ACQ = \angle CBS = \angle CBE' = \angle CAE'.$$

Hence $AE' \parallel Q'C$. Also,

$$\angle QAQ' = \angle ACQ' = \angle CAE'$$

and

$$\angle AQ'Q = 180^\circ - \angle AQ'C = \angle AE'C.$$

Hence $\triangle QAQ' \sim \triangle CAE'$, thus

$$\frac{|AQ'|}{|AQ|} = \frac{|AE'|}{|AC|}.$$

From $\triangle CEE' \sim \triangle BEA$ and $\triangle BEC \sim \triangle AEE'$ we find

$$\frac{|CE|}{|BE|} = \frac{|CE'|}{|AB|}, \qquad \frac{|AE|}{|BE|} = \frac{|AE'|}{|BC|}.$$

Since $AQ'CE'$ is a cyclic trapezium we have $|AQ'| = |CE'|$. Combining these data we obtain

$$\frac{|CE|}{|EA|} = \frac{|CE'| \cdot |BE|}{|AB|} \cdot \frac{|BC|}{|BE| \cdot |AE'|} = \frac{|AQ'| \cdot |BC|}{|AB| \cdot |AE'|} = \frac{|AQ| \cdot |BC|}{|AB| \cdot |AC|}$$

$$= (4OA^2 - AF^2 + AD^2) + (AE^2 + AF^2) - (4OB^2 - BC^2 + (EB + CD)^2)$$
$$= AE^2 + AD^2 + BC^2 - (EB + CD)^2.$$

By Pythagoras's theorem again, we obtain $AE^2 + AD^2 = ED^2$, and hence

$$D'F^2 + EF^2 - D'E^2 = ED^2 + BC^2 - (EB + CD)^2.$$

We have

$$\begin{aligned} ED^2 + BC^2 &= (EB + BD)^2 + (BD + CD)^2 \\ &= EB^2 + BD^2 + 2 \cdot EB \cdot BD + BD^2 + CD^2 + 2 \cdot BD \cdot CD \\ &= EB^2 + CD^2 + 2 \cdot BD \cdot (EB + BD + CD) \\ &= EB^2 + CD^2 + 2 \cdot BD \cdot EC. \end{aligned}$$

By the internal and external bisector theorems, we have

$$\frac{BD}{CD} = \frac{BA}{CA} = \frac{BE}{CE},$$

hence

$$ED^2 + BC^2 = EB^2 + CD^2 + 2 \cdot CD \cdot BE = (EB + CD)^2.$$

So

$$D'F^2 + EF^2 - D'E^2 = 0,$$

which implies that $\angle D'FE = 90°$. □

Problem G.12

Let $\triangle ABC$ be a triangle, and let P and Q be two distinct points on the tangent line at A to the circumscribed circle. They are such that $|AP| = |AQ|$ and that BP and CQ meet inside the triangle. Let S be a point inside triangle $\triangle ABC$ such that $\angle ABP = \angle BCS$ and $\angle ACQ = \angle CBS$. Prove that AS is the median of $\triangle ABC$ through A.

Solution 1. Let T and U be the intersections of the line through A parallel to BC with BS and CS, respectively. We will show that A is the midpoint of TU. Since there exists an homothety centreed at S that sends T to B, U to C and A to the intersection point of AS and BC, this will imply that AS is a median in $\triangle ABC$.

Notice that $\angle ATB = \angle SBC = \angle ACQ$ and

$$\angle BAT = \angle BAC + \angle CAT = \angle BAC + \angle ACB = \angle BAC + \angle QAB = \angle QAC.$$

Now assume that $\angle BAC \neq 90°$. We consider the configuration where $\angle BAC < 90°$. Let M, N and L be the midpoints of the line segments BC, DE and DF, respectively. Note that N is the circumcentre of $\triangle ADE$, so we find

$$\begin{aligned}\angle NAF = \angle NAD = \angle NDA &= \angle DAC + \angle ACD \\ &= \frac{1}{2}\angle A + \angle ACD = \angle BAF + \angle ACD = \angle BCF + \angle ACD = \angle ACF.\end{aligned}$$

Hence NA is tangent to the circumcircle of $\triangle ABC$, thus $NA \perp OA$. Furthermore, we have $NM \perp OM$, so $AOMN$ is cyclic with ON as diameter. Now, since L is the circumcentre of $\triangle DMF$, we find

$$\angle LMN = \angle LMD = \angle LDM = \angle ADN = \angle DAN = \angle LAN,$$

so $ANLM$ is cyclic. Combining this with what we found before, we now conclude that $AOMLN$ is cyclic with ON as diameter, thus $\angle OLN = 90°$. Using a dilation with centre D and factor 2 we now can conclude $\angle D'FE = \angle OLN = 90°$.

In case $\angle BAC > 90°$, the proof is similar (the cyclic quadrilateral will this time be $AMON$). □

Solution 4. We consider the configuration where C, D, B and E are on the line BC in that order. The other configuration can be proved analogously. Let P and R be the feet of the perpendiculars from D' and O to the line AD, respectively, and let Q and S be the feet of the perpendiculars from D' and O to the line CD, respectively. Since D' is the reflection of D with respect to O, we have $PR = RD$. Since we also have $OA = OF$ and therefore $RA = RF$, we obtain $AD = PF$. Similarly, $CD = BQ$. By Pythagoras's theorem,

$$D'F^2 = D'P^2 + PF^2 = 4OR^2 + AD^2 = 4OA^2 - 4AR^2 + AD^2 = 4OA^2 - AF^2 + AD^2$$

and

$$\begin{aligned}D'E^2 = D'Q^2 + EQ^2 = 4OS^2 + EQ^2 &= 4OB^2 - 4BS^2 + EQ^2 \\ &= 4OB^2 - BC^2 + (EB + CD)^2.\end{aligned}$$

And since $\angle EAF = 90°$ we have

$$EF^2 = AE^2 + AF^2.$$

As $OA = OB$, we conclude that

$$D'F^2 + EF^2 - D'E^2$$

which is also the second intersection of the line AE with the circumcircle of $\triangle ABC$. We now have

$$\angle D'FO = \angle OF'D.$$

Since $\angle DMF' = 90^\circ = \angle DAF'$, the quadrilateral $MDAF'$ is cyclic, thus

$$\angle OF'D = \angle MF'D = \angle MAD.$$

Furthermore, $\angle FME = 90^\circ = \angle FAE$, so $FMAE$ is cyclic as well. This implies that

$$\angle MAD = \angle MAF = \angle MEF.$$

Combining these three equalities we find that $\angle D'FO = \angle MEF$, thus

$$\angle D'FE = \angle D'FO + \angle OFE = \angle MEF + \angle MFE = 180^\circ - \angle EMF = 90^\circ. \quad \square$$

Solution 2. Again, assume $AB < AC$ and define F' as in the previous solution. Let G be the intersection of the lines DF' and EF.

We can easily see that FA is perpendicular to EF', and BC to FF'. Now, in triangle $\triangle EFF'$, we have that FD and ED are altitudes, so D is the orthocentre of this triangle. Now, $F'D$ is an altitude as well and we find that $F'G$ is perpendicular to EF. Since FF' is a diameter of the circumcircle of $\triangle ABC$, G must lie on this circle as well.

We now find

$$\angle EFA = \angle GFA = \angle GF'A = \angle DF'A.$$

Also, $\angle DF'F = \angle D'FF'$. This implies that

$$\begin{aligned}\angle D'FE &= \angle D'FF' + \angle F'FA + \angle AFE \\ &= \angle DF'F + \angle F'FA + \angle AF'D = \angle F'FA + \angle AF'F.\end{aligned}$$

Now, in triangle $\triangle AFF'$ we have

$$\angle F'FA + \angle AF'F = 180^\circ - \angle FAF' = 90^\circ,$$

as required. $\square$

Solution 3. Again, assume $AB < AC$. We first consider the case $\angle BAC = 90^\circ$. Define F' as in the previous solution. Now O and D' lie on BC, so $\triangle D'FO$ and $\triangle DFO$ are mirror images with respect to FF', while $\triangle OFE$ and $\triangle OF'E$ are mirror images with respect to BC. We find that

$$\begin{aligned}\angle D'FE &= \angle D'FO + \angle OFE = \angle DFO + \angle OF'E \\ &= \angle AFF' + \angle FF'A = 180^\circ - \angle FAF' = 90^\circ.\end{aligned}$$

Solution 2. We consider the same configuration as in Solution 1. Let K be the second intersection of AH with the circumcircle of $\triangle AMN$. Since H is the midpoint of MN and AH is perpendicular to MN, the line AH is the perpendicular bisector of MN, which contains the circumcentre of $\triangle AMN$. So AK is a diameter of this circumcircle. Now we have $\angle BPK = 90^\circ = \angle BHK$, so $BPHK$ is a cyclic quadrilateral. Also, A, M, N, K, P and Q are concyclic. Using both circles, we find

$$180^\circ - \angle CQP = \angle AQP = \angle AKP = \angle HKP = \angle HBP = \angle CBP.$$

This implies that $BPQC$ is cyclic as well. Using the power theorem we find $AP \cdot AB = AQ \cdot AC$, with directed lengths. Also, $BN \cdot BM = BP \cdot BA$ and $CN \cdot CM = CQ \cdot CA$. Hence

$$AP \cdot AB \cdot BN \cdot BM \cdot CQ \cdot CA = AQ \cdot AC \cdot BP \cdot BA \cdot CN \cdot CM.$$

Changing the signs of all six lengths on the right-hand side and replacing MC by BM, we find

$$AP \cdot AB \cdot BN \cdot BM \cdot CQ \cdot CA = QA \cdot CA \cdot PB \cdot AB \cdot NC \cdot BM.$$

Cleaning this up, we have

$$AP \cdot BN \cdot CQ = QA \cdot PB \cdot NC,$$

implying

$$\frac{BN}{NC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

With Ceva's theorem we can conclude that AN, BQ and CP are concurrent. □

Problem G.11

In triangle $\triangle ABC$, the interior and exterior angle bisectors of $\angle BAC$ cut the line BC in D and E, respectively. Let $F \neq A$ be the second intersection of the line AD with the circumcircle of $\triangle ABC$. Let O be the circumcentre of $\triangle ABC$ and let D' be the reflection of D in O. Prove that $\angle D'FE = 90^\circ$.

Solution 1. Note that $AB \neq AC$, since otherwise the exterior angle bisector of $\angle BAC$ would be parallel to BC. So assume without loss of generality that $AB < AC$. Let M be the midpoint of BC and let F' the reflection of F in O, Since H is the midpoint of MN and AH is perpendicular to MN, the line AH is the perpendicular bisector of MN, which contains the circumcentre of $\triangle AMN$. As A is on the perpendicular bisector of MN, we have $|AM| = |AN|$. In the chosen configuration we now have $\angle APM = \angle AQN$. Therefore

$$\angle CQN = 180^\circ - \angle AQN = 180^\circ - \angle APM = \angle BPM.$$

Furthermore, as $NMPQ$ is cyclic, we have

$$\angle NQP = 180^\circ - \angle NMP = \angle BMP.$$

Hence

$$\angle AQP = 180^\circ - \angle CQN - \angle NQP = 180^\circ - \angle BPM - \angle BMP = \angle PBM = \angle ABC.$$

Similarly,

$$\angle APQ = \angle BCA.$$

Now we have $\triangle APQ \sim \triangle ACB$. So

$$\frac{|AP|}{|AQ|} = \frac{|AC|}{|AB|}.$$

Furthermore, $\angle MAB = \angle MAP = \angle MNP = \angle BNP$, so $\triangle BMA \sim \triangle BPN$, and hence

$$\frac{|BN|}{|BP|} = \frac{|BA|}{|BM|}.$$

We also have $\angle CAM = \angle QAM = 180^\circ - \angle QNM = \angle QNC$. This implies $\triangle CMA \sim \triangle CQN$, so

$$\frac{|CQ|}{|CN|} = \frac{|CM|}{|CA|}.$$

Putting everything together, we find

$$\frac{|BN|}{|BP|} \cdot \frac{|CQ|}{|CN|} \cdot \frac{|AP|}{|AQ|} = \frac{|BA|}{|BM|} \cdot \frac{|CM|}{|CA|} \cdot \frac{|AC|}{|AB|}.$$

As $|BM| = |CM|$, the right-hand side is equal to 1. This means that

$$\frac{|BN|}{|NC|} \cdot \frac{|CQ|}{|QA|} \cdot \frac{|AP|}{|PB|} = 1.$$

In our configuration N is on segment BC, P is on segment AB and Q is on segment AC. Hence with directed lengths we have

$$\frac{BN}{NC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

With Ceva's theorem we can conclude that AN, BQ and CP are concurrent. □

Observe that

$$\angle PXY = \angle PQY = \angle PD'A' \quad \text{and} \quad \angle YPX = \angle YQX = \angle A'PD'.$$

It follows that $\triangle PXY \sim \triangle PD'A'$, and thus

$$\frac{XY}{AD} = \frac{XY}{A'D'} = \frac{XP}{PD'}.$$

Let now Γ' be a circle passing through A and D', and tangent to BC at X'. Denote $\{A, P'\} = AX' \cap \Gamma$ and $\{Y'\} = P'D \cap BC$. By the previous paragraph,

$$\frac{X'Y'}{AD} = \frac{X'P'}{P'D'}.$$

We have $\angle D'PA = \angle D'P'A$. Moreover, $\angle D'XA \leqslant D'X'A$, because X lies outside circle Γ'. Take the point $X'' \in P'X'$ so that $\angle D'XA = \angle D'X''A$. Then X'' lies between X' and P according to the previous inequality, so $P'X'' \leqslant P'X'$. Note that $\triangle PXD' \sim \triangle P'X''D'$. Thus

$$\frac{XP}{PD'} = \frac{X''P'}{P'D'}.$$

Combining all the results above, we arrive at

$$\frac{XY}{AD} = \frac{XP}{PD'} = \frac{X''P'}{P'D'} \leqslant \frac{X'P'}{P'D'} = \frac{X'Y'}{AD},$$

whence $XY \leqslant X'Y'$. It is easy to see that, if $P \neq P'$, the inequality is strict.

We need to show that $BX' = CY'$. We have

$$\angle Y'X'D' = \angle X'AD' = \angle P'AC + \angle CAD' = \angle P'DC + \angle DCB = \angle DY'X'.$$

Obviously, $\angle X'BD' = \angle DCY'$ and $BD' = CD$, so $\triangle X'BD' \cong \triangle Y'CD$. In particular, $BX' = CY'$. □

Problem G.10

In the non-isosceles triangle $\triangle ABC$ the altitude from A meets side BC in H. Let M be the midpoint of BC and let N be the reflection of M in H. The circumcircle of $\triangle AMN$ intersects the line AB in $P \neq A$ and the line AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.

Solution 1. Consider the configuration in which the order of the points on the line BC is B, M, H, N, C. This implies that P is on the segment AB and Q is on the segment AC. In any other configuration a similar proof can be used.

is perpendicular to AK.

Solution. Let $\angle BAC = \alpha$, $\angle CBA = \beta$ and $\angle ACB = \gamma$. Let I be the incentre, let F be the point where the incircle touches AB, and let DF intersect AK in X.

Points B, D, I, F are concyclic as $\angle IDB = \angle BFI = 90°$. Thus $\angle DFI = \angle DBI = \frac{\beta}{2}$. So

$$\angle IXD = \angle AIF - \angle DFI = 90° - \frac{\alpha}{2} - \frac{\beta}{2} = \frac{\gamma}{2} = \angle ICD,$$

which implies that points C, I, D, X are concyclic. Thus $\angle CXI = \angle CDI = 90°$.

Let CX intersect AD in Q. It is enough to show that $P = Q$. Let D' and Q' be the reflections of D and Q, respectively, in AK. Note that $D' \in XE$ because E and F are symmetric with respect to AK. By Menelaus's theorem, applied to triangle ACQ' and the line $ED'X$, we obtain

$$1 = \frac{AD'}{D'Q'} \cdot \frac{Q'X}{XC} \cdot \frac{CE}{EA} = \frac{AD}{DQ} \cdot \frac{QX}{XC} \cdot \frac{CE}{EA}.$$

By Ceva's theorem, the lines CD, QE, AX are concurrent. But this means that P and Q coincide. This finishes the proof. □

Problem G.9

Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Let P be a variable point on that arc BC not containing the points A and D. Suppose BC intersects the lines AP and DP in X and Y, respectively. Show that, if we choose P in such a way as to maximise the length of the segment XY, then $BX = CY$.

Solution. Let ω be the circumcircle of triangle PXY. Denote $\{P, Q\} = \Gamma \cap \omega$, and further $\{Q, A'\} = QY \cap \Gamma$ and $\{Q, D'\} = QX \cap \Gamma$.

First we shall show that the points A' and D' do not depend on the choice of P. To avoid cases we make use of directed angles. We have

$$\angle PQY = \angle PXY = \angle PBX + \angle XPB.$$

Subtracting $\angle PQC = \angle PBC$, we obtain $\angle CQA' = \angle APB$, so (directed) arcs AB and CA' are equal, which means $BC \parallel AA'$. Analogously, $DD' \parallel BC$. Consequently, A' and D' are independent of the choice of P. In particular, $A'D' = AD$.

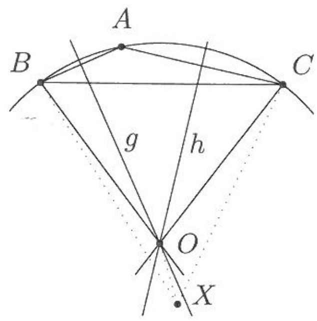


FIGURE 5: Problem G.7.

Prove that $\angle BAC \leqslant 150°$.

Solution. Arguing indirectly, we suppose that $\angle BAC > 150°$ Let g and h denote the perpendicular bisectors of the segments AB and AC.

Due to $AX \geqslant BX$, the points B and X belong to a common closed half-plane determined by g. Similarly, there is a closed half-plane with boundary h to which C and X belong. This shows, in particular, that A cannot be an interior point of the segment BC. Because of $\angle BAC > 150°$, the triangle ABC is non-degenerate, and g and h meet at the circumcentre O of triangle ABC.

The inscribed angle theorem and $\angle BAC > 150°$ yield $\angle COB < 60°$, which in turn reveals that $BO, CO > BC$. Thus we have $BX < BO$ and $X \neq O$. As a greater side of a triangle subtends a greater angle, the angle $\angle XOB$ is acute. For the same reason the angle $\angle COX$ is acute as well, but both statements together contradict the position required of X from the previous paragraph. □

Remark. Actually, the same statement holds in higher dimensions as well, i.e., when A, B, C and X are any four distinct points from some Euclidean vector space. To show this we may just project X onto a plane passing through A, B, and C and argue as above. It seems that the planar version is more suitable for the competition, but we would like to remark that the general version is relevant to recent joint work of Frankl, Pach, Reiher and Rödl.

Problem G.8

Let ABC be a scalene triangle. Let D and E be the points where the incircle touches sides BC and CA, respectively. Let K be the common point of line BC and the bisector of the angle $\angle BAC$. Let AD intersect EK in P. Prove that PC

$$
\begin{aligned}
|PA_1|^2 + \cdots + |PA_n|^2 &= \sum_{i=1}^{n} \left((x-a_i)^2 + (y-c_i)^2 \right) \\
&= n(x^2+y^2) - 2x\sum_{i=1}^{n} a_i - 2y\sum_{i=1}^{n} c_i + \sum_{i=1}^{n}(a_i^2+c_i^2) \\
&= n(x^2+y^2+1) = |PB_1|^2 + \cdots + |PB_n|^2. \quad \square
\end{aligned}
$$

Problem G.6
Let BD be an altitude of triangle ABC. The incentre of triangle BCD coincides with the centroid of triangle ABC. If $AB = c$, find the lengths of AC and BC.

Solution. Answer: $AC = BC = \sqrt{\frac{5}{2}}c$.

The centroid of ABC lies on the median CC'. It will also, by the assumption, lie on the angle bisector through C. Since the median and the angle bisector coincide, ABC is isosceles with $AC = BC = a$.

Furthermore, the centroid lies on the median BB' and the bisector of $\angle DBC$, again by hypothesis. By the Angle Bisector Theorem,

$$\frac{B'D}{BD} = \frac{B'C}{BC} = \frac{a/2}{a} = \frac{1}{2}.$$

The triangles $ABD \sim ACC'$ since they have equal angles, whence

$$\frac{c}{a} = \frac{AB}{AC} = \frac{AD}{AC'} = \frac{a/2 - B'D}{c/2} = \frac{a - BD}{c}.$$

Using the fact that the length of the altitude CC' is $\sqrt{a^2 - \frac{c^2}{4}}$, this leads to

$$a^2 - c^2 = aBD = 2|ABC| = c\sqrt{a^2 - \frac{c^2}{4}},$$

or, equivalently,

$$\frac{a^2}{c^2} - 1 = \sqrt{\frac{a^2}{c^2} - \frac{1}{4}}.$$

Clearly, $\frac{a}{c} > 1$, and the only solution is $\frac{a}{c} = \sqrt{\frac{5}{2}}$. $\square$

Problem G.7
Suppose that A, B, C, and X are any four distinct points in the plane with

$$\max(BX, CX) \leqslant AX \leqslant BC.$$

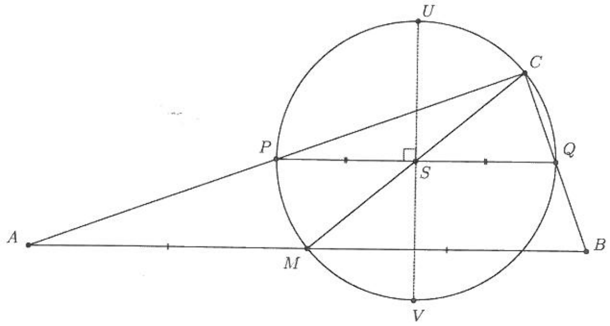


FIGURE 4: Problem G.4.

of Γ. Since S belongs to PQ, the latter is also a diameter of Γ, and hence $\angle PCQ = 90^\circ$. In this case $\angle ACB = 90^\circ$.

Conversely, it is easy to check that any $\triangle ABC$ with $\angle ABC = \angle BAC$ or $\angle ACB = 90^\circ$ fulfils the requirements. □

Problem G.5

For what positive numbers m and n do there exist points $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ in the plane such that, for any point P, the equation

$$|PA_1|^2 + \cdots + |PA_m|^2 = |PB_1|^2 + \cdots + |PB_n|^2$$

holds true?

Solution. Answer: for $m = n$.

Denote $A_i = (a_i, c_i)$ and $B_j = (b_j, d_j)$. Further writing $P = (sx, sy)$, the equation becomes

$$\sum_{i=1}^{m}\left((sx - a_i)^2 + (sy - c_i)^2\right) = \sum_{j=1}^{n}\left((sx - b_j)^2 + (sy - d_j)^2\right).$$

Assuming $(x, y) \neq (0, 0)$, both sides are quadratic polynomials in s, in which s^2 carries the co-efficient $m(x^2 + y^2)$ and $n(x^2 + y^2)$, respectively. Necessarily $m = n$.

Conversely, for $m = n$, let $A_1B_1A_2B_2 \ldots A_nB_n$ be a regular $2n$-gon of circumradius 1, centred on the origin. Then $a_i^2 + c_i^2 = 1$ and $\sum a_i = \sum c_i = 0$. Writing $P = (x, y)$, we thus have

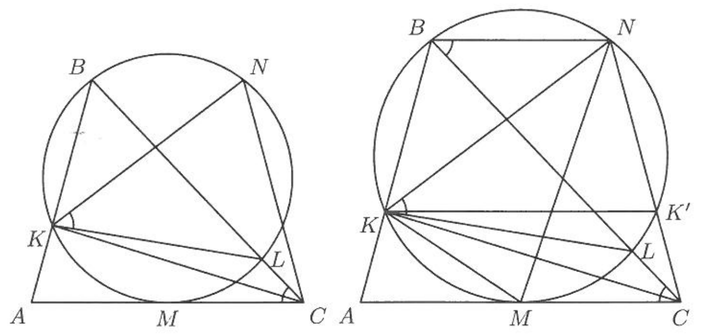


FIGURE 3: Problem G.3.

Denote by K' the intersection point of s and CN. Then the line KK' is parallel to the bases of the trapezium. Hence M is the midpoint of arc KK' and the line NM is an angle bisector of the equilateral triangle KNC.

Thus we obtain that $MC = MK$. Therefore the length of median KM of the triangle AKC equals $\frac{1}{2}AC$; hence $\angle AKC = 90^\circ$. We have

$$2\angle A = \angle KAC + \angle ACN = \angle KAC + \angle ACK + \angle KCN = 90^\circ + 60^\circ = 150^\circ,$$

and so $\angle A = 75^\circ$. □

Problem G.4

Let CM be a median of $\triangle ABC$. The circle with diameter CM intersects segments AC and BC in points P and Q, respectively. Given that $AB \parallel PQ$ and $\angle BAC = \alpha$, what are the possible values of $\angle ACB$?

Solution. Answer: $180^\circ - 2\alpha$ and 90°.

Let S be the intersection point of CM and PQ. By Thales's theorem, $\frac{PS}{SQ} = \frac{AM}{MB}$; hence $PS = SQ$.

Denote the circle with diameter CM by Γ. Let the perpendicular bisector of the segment PQ intersect Γ in points U and V. Then UV is a diameter of Γ. Examine two possible cases:

- UV coincides with CM. Then $PQ \perp CM$, and also $AB \perp CM$. In this case CM is both a median and an altitude in $\triangle ABC$. Hence $\angle ABC = \angle BAC$ and $\angle ACB = 180^\circ - 2\alpha$.

- UV and CM are distinct diameters of Γ. It follows that S is the centre

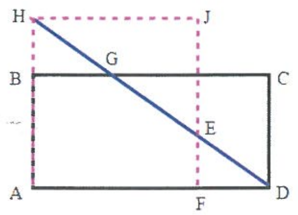


FIGURE 2: Problem G.2.

BC and FJ be G and E respectively. Denote

$$a = AH = \sqrt{2015}, \qquad b = AB = 65, \qquad c = AD = 31,$$

so that $a^2 = bc = 2015$.

Triangles BHG and FED are congruent. Indeed, both of them are similar to the triangle AHD (their angles are equal), and the coefficients of similarity are $\frac{BH}{AH} = \frac{a-b}{a}$ and $\frac{FD}{AD} = \frac{c-a}{c}$, respectively, and those are equal because $\frac{a-b}{a} = \frac{c-a}{c}$ if (and only if) $a^2 = bc$.

Also, triangles GCD and HJE are congruent, because

$$HJ = a = c - (c - a) = BC - BG = GC$$
$$CD = b = a - (a - b) = JF - FE = JE.$$

Therefore, one can cut the square $AHJF$ into a pentagon $ABGEF$ and two triangles BHG and HJE, from which it is possible to form the rectangle $ABCD$, as composed of the pieces $ABGEF$, GCD and FED. □

Problem G.3

A circle s passes through vertex B of the triangle ABC, intersects its sides AB and BC at points K and L, and touches the side AC in its midpoint M. The point N on the arc BL is such that $\angle LKN = \angle ACB$. It so happens that triangle CKN is equilateral. Find $\angle BAC$.

Solution. Answer: $\angle BAC = 75^\circ$.

Since $\angle ACB = \angle LKN = \angle LBN$, the lines AC and BN are parallel. Hence $ACNB$ is a trapezium. Moreover, $ACNB$ is an isosceles trapezium, because the segment AC touches the circle s in the midpoint (and so the trapezium is symmetrical with respect to the perpendicular bisectors of BN).

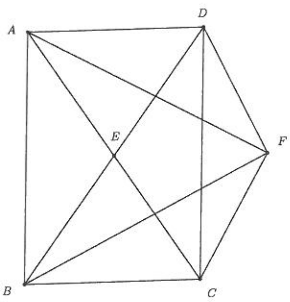


FIGURE 1: Problem G.1.

Solutions

Problem G.1
The diagonals of the parallelogram $ABCD$ intersect at E. The bisectors of $\angle DAE$ and $\angle EBC$ intersect at F. It is known that $ECFD$ is a parallelogram. Determine the ratio $AB : AD$.

Solution. Since $ECFD$ is a parallelogram, we have $ED \parallel CF$ and $\angle CFB = \angle EBF = \angle FBC$ (BF bisects $\angle DBC$). So CFB is an isosceles triangle and $BC = CF = ED$ ($ECFD$ is a parallelogram). In a similar manner, $EC = AD$. But since $ABCD$ is a parallelogram, $AD = BC$, whence $EC = ED$. So the diagonals of $ABCD$ are equal, which means that $ABCD$ is in fact a rectangle. Also, the triangles EDA and EBC are equilateral, and so AB is twice the altitude of EDA, or $AB = \sqrt{3} \cdot AD$. □

Problem G.2
Is it possible to cut a square with side $\sqrt{2015}$ into no more than five pieces so that these pieces can be rearranged into a rectangle with sides of integer length? (The cuts should be made using straight lines, and flipping of the pieces is disallowed.)

Solution. Yes, it is possible, and 3 pieces is enough.

Take a rectangle $ABCD$ of size 65×31, and draw a square of the same area $AHJF$ as shown in Figure 2. Draw the line DH, and let its intersections with

Problem G.11 *The Netherlands*

In triangle $\triangle ABC$, the interior and exterior angle bisectors of $\angle BAC$ cut the line BC in D and E, respectively. Let $F \neq A$ be the second intersection of the line AD with the circumcircle of $\triangle ABC$. Let O be the circumcentre of $\triangle ABC$ and let D' be the reflection of D in O. Prove that $\angle D'FE = 90^\circ$.

Problem G.12 *The Netherlands*

Let $\triangle ABC$ be a triangle, and let P and Q be two distinct points on the tangent line at A to the circumscribed circle. They are such that $|AP| = |AQ|$ and that BP and CQ meet inside the triangle. Let S be a point inside triangle $\triangle ABC$ such that $\angle ABP = \angle BCS$ and $\angle ACQ = \angle CBS$. Prove that AS is the median of $\triangle ABC$ through A.

Problem G.13 *Poland*

Let ABC be a triangle. Let its altitudes AD, BE and CF concur at H. Let K, L and M be the midpoints of BC, CA and AB, respectively. Prove that, if $\angle BAC = 60^\circ$, then the midpoints of the segments AH, DK, EL, FM are concyclic.

Problem G.5 *Sweden*

For what positive numbers m and n do there exist points $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ in the plane such that, for any point P, the equation

$$|PA_1|^2 + \cdots + |PA_m|^2 = |PB_1|^2 + \cdots + |PB_n|^2$$

holds true?

Problem G.6 *Sweden*

Let BD be an altitude of triangle ABC. The incentre of triangle BCD coincides with the centroid of triangle ABC. If $AB = c$, find the lengths of AC and BC.

Problem G.7 *Germany*

Suppose that A, B, C, and X are any four distinct points in the plane with

$$\max(BX, CX) \leqslant AX \leqslant BC.$$

Prove that $\angle BAC \leqslant 150^\circ$.

Problem G.8 *Poland*

Let ABC be a scalene triangle. Let D and E be the points where the incircle touches sides BC and CA, respectively. Let K be the common point of line BC and the bisector of the angle $\angle BAC$. Let AD intersect EK in P. Prove that PC is perpendicular to AK.

Problem G.9 *Poland*

Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Let P be a variable point on that arc BC not containing the points A and D. Suppose BC intersects the lines AP and DP in X and Y, respectively. Show that, if we choose P in such a way as to maximise the length of the segment XY, then $BX = CY$.

Problem G.10 *The Netherlands*

In the non-isosceles triangle $\triangle ABC$ the altitude from A meets side BC in H. Let M be the midpoint of BC and let N be the reflection of M in H. The circumcircle of $\triangle AMN$ intersects the line AB in $P \neq A$ and the line AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.

G1 G3 G6 G10 G11

Geometry

Problems

Problem G.1 *Finland*

The diagonals of the parallelogram $ABCD$ intersect at E. The bisectors of $\angle DAE$ and $\angle EBC$ intersect at F. It is known that $ECFD$ is a parallelogram. Determine the ratio $AB : AD$.

Problem G.2 *Latvia*

Is it possible to cut a square with side $\sqrt{2015}$ into no more than five pieces so that these pieces can be rearranged into a rectangle with sides of integer length? (The cuts should be made using straight lines, and flipping of the pieces is disallowed.)

Problem G.3 *Saint Petersburg*

A circle s passes through vertex B of the triangle ABC, intersects its sides AB and BC at points K and L, and touches the side AC in its midpoint M. The point N on the arc BL is such that $\angle LKN = \angle ACB$. It so happens that triangle CKN is equilateral. Find $\angle BAC$.

Problem G.4 *Latvia*

Let CM be a median of $\triangle ABC$. The circle with diameter CM intersects segments AC and BC in points P and Q, respectively. Given that $AB \parallel PQ$ and $\angle BAC = \alpha$, what are the possible values of $\angle ACB$?

We see that the actions of first two policemen force the coordinate $y \to +\infty$ Similar actions of policemen P_3 and P_4 will force $y \to -\infty$. This contradiction shows that the criminal cannot escape indefinitely. $\square$

Roughly speaking, the first pair of policemen should drive the criminal around in the clockwise direction, and the second pair in the anti-clockwise direction.

If police P_k (or criminal) occupies location A_i or B_i we say that it has coordinate $x_k = i$ (resp. $y = i$), taken modulo 2015.

The strategy of policemen P_1 and P_2 consists of two parts.

First part: Police P_2 stays in some fixed city, while P_1 moves along the cities in incrementing direction. His aim is to make his coordinate x_1 equal y modulo 17. Consider the remainder of $y - x_1$ modulo 17. If the criminal moves along a village road or a road A_iB_i, the remainder does not change, but P_1, in his next move, will decrease this reminder by 1. If the criminal moves from A_i to A_{i+1}, then the next move of P_1 will again decrease the remainder to the value it had. But he cannot continue this indefinitely due to policeman P_2. Finally, if the criminal moves from A_{i+1} to A_i, then P_1 remains where he is.

Thus, we eventually reach a situation where $x_1 \equiv y \bmod 17$. Moreover, P_1 can arrange the next move so that he is in a city if the criminal is, and in a village if the criminal is, all the while upholding the relation $x_1 \equiv y \bmod 17$.

Second part: Now we opt to keep more careful track of people's locations by no longer considering the coordinates modulo 2015. By allowing x_k and y to assume any integer values, we keep track of when the criminal skips over a policeman. Choose a value of x_1 such that $x_1 \equiv y \bmod 17$ and $x_1 < y$. Let $k = y - x_1$ be the difference at this moment.

Police P_1 should now keep the congruence $x_1 \equiv y \bmod 17$, and keep to the same urban type (city or village) as the criminal. For this, he may just copy the criminal's moves, the single exception being if the criminal moves from village B_{i+17} to B_i, in which case P_1 simply remains where he is. This last type of move we call *critical*. During a critical move, the difference $y - x_1$ decreases by 17. Clearly, the criminal can make only k critical moves.

As for P_2, we choose his coordinate at the beginning of this part to be $x_2 < y$. His strategy is to successively move from city A_i to A_{i+1}.

If the criminal does not wish to be caught under this stratagem, his coordinate y must tend to $+\infty$. Indeed, y avoids increment in only three cases:

(1) The criminal moves between a city and a village or remains in place; then the difference $y - x_2$ decreases after the next move of P_2.

(2) The criminal moves from city A_{i+1} to A_i; then the difference $y - x_2$ decreases.

(3) The criminal makes a critical move from B_{i+17} to B_i, skipping over P_2. At this instant, we change the value of x_2 to $x_2 - 2015$, and therefore the difference $y - x_2$ increases by almost 2000.

Since the third case can occur at most k times, our claim follows.

divide n. Thus the claim of the problem holds for all natural numbers n except for some finite set. By case study, it is easy to see that the only numbers for which the claim of the problem fails are 1, 2, 4, 6, 12.

- The auxiliary claim, the inequality $kl \geqslant n$, directly implies the Erdős–Szekeres theorem. Our proof of this inequality is also a direct generalisation of the proof of the Erdős–Szekeres theorem presented on Wikipedia.
- There is a great temptation to prove the inequality $kl \geqslant n$ by induction, but it comes to a standstill at the problem of finding a longest increasing subsequence such that its removal would eliminate an element from every longest decreasing subsequence *or* a longest decreasing subsequence whose removal would eliminate an element from every longest increasing subsequence. If this were always possible, the induction step would go through. In the first case, for example, removing such an increasing subsequence would produce a sequence of length $n-k$, with increasing subsequences of length at most k and decreasing subsequences of length at most $l-1$. Thus $n-k \leqslant k(l-1)$ by the induction hypothesis, implying also $n \leqslant kl$.

  In reality, this kind of choice is not always possible, as shown by the counterexample
$$6,\ 5,\ 1,\ 2,\ 7,\ 8,\ 4,\ 3.$$
  Both the longest increasing and the longest decreasing subsequence have length 4, both are unique, and they do not contain common elements.

Problem C.10

A country consists of 2015 cities $A_1, A_2, \ldots, A_{2015}$, and 2015 villages $B_1, B_2, \ldots, B_{2015}$. There is a road between city A_i and village B_i, between cities A_i and A_{i+1}, and between villages B_i and B_{i+17} (in cyclic numbering).

Four policemen and one criminal are roaming the country. One day all the policemen move, the next day the robber moves, then the policemen again, and so on. A move consists of remaining in place, or travelling to a neighbouring city or village. Everybody is completely informed of everybody else's actions.

Can the policemen always catch the criminal, independently of their initial positions?

Solution. Yes, they can.

Remark.

- For part (a) we do not use that n is a power of 2, indeed all we use is that n is a multiple of 4. One could generalise the statement to all $n \geqslant 4$ by changing $\frac{3n}{4}$ to $\frac{3n+3}{4}$.
- Part (b) is still true for general $n \geqslant 4$, which can be shown by a variation on the third proof. In the other two proofs the calculation of $|S|$ becomes messier and we might get a little (order $\log n$) below $\frac{2n}{3}$, so those proofs will not be valid.

Problem C.9
Let n be a positive integer such that there exists a prime number that is less than $\sqrt{n}$ and does not divide n. Let $a_1, \ldots, a_n$ denote the numbers $1, \ldots, n$, rearranged in any order. Let $a_{i_1} < \cdots < a_{i_k}$ be an increasing subsequence of maximal length ($i_1 < \cdots < i_k$) and $a_{j_1} > \cdots > a_{j_l}$ a decreasing subsequence of maximal length ($j_1 < \cdots < j_l$). Prove that one of the sequences $a_{i_1}, \ldots, a_{i_k}$ and $a_{j_1}, \ldots, a_{j_l}$ contains a number not dividing n.

Solution. The first phase of the solution consists in showing that $kl \geqslant n$. For every $i = 1, \ldots, n$, let $f(i)$ denote the length of the longest increasing subsequence ending with a_i, and let $g(i)$ be the length of the longest decreasing subsequence ending with a_i.

Consider two indices $i < j$. If $a_i < a_j$, then $f(i) < f(j)$, and if $a_i > a_j$, then $g(i) < g(j)$. Hence the n pairs $(f(i), g(i))$ are all distinct. By assumption, $\max f(i) = k$ and $\max g(i) = l$, and there can be at most distinct kl pairs $(f(i), g(i))$. Consequently, $n \leqslant kl$.

At most one number can belong to an increasing and a decreasing subsequence simultaneously. Using AM–GM, the subsequences $(a_{i_1}, \ldots, a_{i_k})$ and $(a_{j_1}, \ldots, a_{j_l})$ together contain at least

$$k + l - 1 \geqslant 2\sqrt{kl} - 1 \geqslant 2\sqrt{n} - 1$$

different numbers in total. By assumption, n has at most $\lfloor \sqrt{n} \rfloor - 1$ divisors not exceeding $\sqrt{n}$, whence its total number of divisors is at most $2\lfloor \sqrt{n} \rfloor - 2$. Consequently, the given subsequences must contain a non-divisor of n. □

Remark.

- By Chebyshev's theorem, one can show that, for each natural number $n \geqslant 25$, there exists a prime number that is less than $\sqrt{n}$ and does not and set

$$S = A_k \cup A_{k-2} \cup \cdots \cup A_l \cup \{1\}, \qquad \text{where } l = \begin{cases} 2 & \text{if } k \text{ even,} \\ 1 & \text{if } k \text{ odd.} \end{cases}$$

Note that $A_j \subseteq \{1, 2, \ldots, n\}$ whenever $j \leqslant k$, and that $|A_j| = 2^{j-1}$. We find

$$|S| = 2^{k-1} + 2^{k-3} + \cdots + 2^{l-1} + 1 = \frac{2^{l-1} - 2^{k+1}}{1-4} + 1 = -\frac{2^{l-1}}{3} + \frac{2n}{3} + 1 > \frac{2n}{3}.$$

We show that S is not balanced. Take $a = 1 \in S$, and consider a $1 \neq b \in S$. Then $b \in A_j$ for some j. If b is even, then $\frac{1+b}{2}$ is not integral. If b is odd, then also $1 + b \in A_j$, so $\frac{1+b}{2} \in A_{j-1}$ and does not lie in S. Thus S is not balanced. □

Solution 3 of part (b). Let us introduce the concept of *lonely element* as an $a \in S$ for which there does not exist a $b \in S$, distinct from a, such that $\frac{a+b}{2} \in S$.

We will construct an unbalanced set S with $|S| > \frac{2n}{3}$ for all k. For $n = 4$ we can use $S = \{1, 2, 4\}$ (all elements are lonely), and for $n = 8$ we can use $S = \{1, 2, 3, 5, 6, 7\}$ (2 and 6 are lonely).

Now we will construct an unbalanced set $S \subseteq \{1, 2, \ldots, 4n\}$, given an unbalanced set $T \subseteq \{1, 2, \ldots, n\}$ with $|T| > \frac{2n}{3}$. Take

$$S = \{\, i \in \{1, 2, \ldots, 4n\} \mid i \equiv 1 \bmod 2 \,\} \cup \{\, 4t - 2 \mid t \in T \,\}.$$

Then

$$|S| = 2n + |T| > 2n + \frac{2n}{3} = \frac{8n}{3} = \frac{2 \cdot 4n}{3}.$$

Supposing $a \in T$ is lonely, we will show that $4a - 2 \in S$ is lonely. Indeed, suppose $4a - 2 \neq b \in S$ with

$$\frac{4a - 2 + b}{2} = 2a - 1 + \frac{b}{2} \in S.$$

Then b must be even, so $b = 4t - 2$ for some $a \neq t \in T$. But then

$$\frac{4a - 2 + 4t - 2}{2} = 4\frac{a+t}{2} - 2,$$

again an even element. However, as a is lonely we know that $\frac{a+t}{2} \notin T$, and hence $4\frac{a+t}{2} - 2 \notin S$. We conclude that $4a - 2$ is lonely in S.

Thus S is an unbalanced set, and by induction we can find an unbalanced set of size exceeding $\frac{2n}{3}$ for all $k > 1$. □

Problem C.8

A subset S of $\{1, 2, \dots, n\}$ is called *balanced* if for every $a \in S$ there exists a $b \in S$, distinct from a, such that $\frac{a+b}{2} \in S$ as well.

(a) Let $k > 1$ be an integer and let $n = 2^k$. Show that every subset S of $\{1, 2, \dots, n\}$ with $|S| > \frac{3n}{4}$ is balanced.

(b) Does there exist an $n = 2^k$, with $k > 1$ integral, for which every subset S of $\{1, 2, \dots, n\}$ with $|S| > \frac{2n}{3}$ is balanced?

Solution of part (a). Let $m = n - |S|$, thus $m < \frac{n}{4}$ and (as n is a multiple of 4) $m \leqslant \frac{n}{4} - 1$. Let $a \in S$. There are $\frac{n}{2} - 1$ elements in $\{1, 2, \dots, n\}$ distinct from a and with the same parity as a. At most m of those elements are not in S, hence at least $\frac{n}{2} - 1 - m \geqslant \frac{n}{4}$ of them are in S. For each such b, the number $\frac{a+b}{2}$ is an integer, and all of these at least $\frac{n}{4}$ numbers are distinct. But at most $m < \frac{n}{4}$ of them are not in S, so at least one is a member of S. Hence S is balanced. □

Solution 1 of part (b). For convenience we work with $\{0, 1, \dots, n-1\}$ rather than $\{1, 2, \dots, n\}$; this does not change the problem. We show that one can always find an unbalanced subset containing more than $\frac{2n}{3}$ elements.

Let $\operatorname{ord}_2(i)$ denote the number of factors 2 occurring in the prime factorisation of i. We set

$$T_j = \{\, i \in \{1, 2, \dots, n-1\} \mid \operatorname{ord}_2(i) = j \,\}.$$

Then we choose

$$S = \{0, 1, 2, \dots, n-1\} \setminus (T_1 \cup T_3 \cup \dots \cup T_l), \qquad \text{where } l = \begin{cases} k-1 & \text{if } k \text{ even,} \\ k-2 & \text{if } k \text{ odd.} \end{cases}$$

Observe that $|T_j| = \frac{n}{2^{j+1}}$, so

$$|S| = n - \left(\frac{n}{4} + \frac{n}{16} + \dots + \frac{n}{2^{l+1}}\right) = n - n \cdot \frac{\frac{1}{4} - \frac{1}{2^{l+3}}}{1 - \frac{1}{4}} > n - \frac{n}{3} = \frac{2n}{3}.$$

We show that S is not balanced. Take $a = 0 \in S$, and consider a $0 \neq b \in S$. If b is odd, then $\frac{0+b}{2}$ is not integral. If b is even, then $b \in T_2 \cup T_4 \cup \cdots$, so $\frac{b}{2} \in T_1 \cup T_3 \cup \cdots$, hence $\frac{b}{2} \notin S$. Thus S is not balanced. □

Solution 2 of part (b). We define the sets

$$A_j = \{2^{j-1} + 1, 2^{j-1} + 2, \dots, 2^j\},$$

Now look at two different stacks a_v and a_u. Then, without loss of generality, we may assume that $u = v + i$ for some $i = 1, 2, \ldots, m$ (again we consider the index modulo $n = 2m + 1$). Since there is a card in stack a_v with number $km + i$, the card $km + i + 1$ is in stack $a_{v+i} = a_u$. Hence among the cards in any two stacks there is at least one magic pair. Since there is the same number of pairs of stacks as of magic pairs, there must be exactly one magic pair among the cards of any two stacks. □

Problem C.7

In a school for girls, all the girls either intensely love or hate each other, and these feelings are always mutual.

(a) Prove that, among any $3n - 2$ girls, for $n \geqslant 2$, one of the following situations must occur.

- There exist n girls $A_1, \ldots, A_n$ such that girls A_i and A_{i+1} love each other, for $i = 1, \ldots, n-1$, or
- there exist seven girls $B_1, \ldots, B_7$ that all hate each other, except possibly the three pairs (B_1, B_2), (B_3, B_4) and (B_5, B_6), which may either love or hate each other.

(b) Prove that the same conclusion does not necessarily hold in a collection of $3n - 3$ girls.

Solution. (a) Suppose there are no n girls fulfilling the first condition. Choose the longest sequence B_1, $C_1, \ldots, C_i$, B_2 of girls, in which each girls loves the next. By assumption, there are at most $n-1$ girls in the sequence. Now remove these girls from consideration. Among the remaining girls, again choose the longest sequence B_3, $D_1, \ldots, D_j$, B_4 of girls, in which each girls loves the next. Again this sequence contains at most $n - 1$ girls. Removing these also, we repeat the procedure a third time for a sequence $B_5, \ldots, B_6$. At most $3n - 3$ girls have now been removed, so there is at least one girl B_7 remaining. The girls $B_1, \ldots, B_7$ now fulfil the second condition.

(b) Consider three classes, each consisting of $n - 1$ girls. Girls from separate classes hate each other, but they love their classmates. Then the longest sequence of girls that succesively love each other has length $n - 1$, and among any seven girls, three must attend the same class, and hence love each other, which violates the second condition. □

Problem C.6

Let $n > 2$ be a positive integer. A deck contains $\frac{n(n-1)}{2}$ cards, numbered

$$1,\ 2,\ 3,\ \ldots,\ \frac{n(n-1)}{2}.$$

Two cards form a *magic pair* if their numbers are consecutive, or if their numbers are 1 and $\frac{n(n-1)}{2}$.

For which n is it possible to distribute the cards into n stacks in such a manner that, among the cards in any two stacks, there is exactly one magic pair?

Solution. Answer: for all odd n.

First assume a stack contains two cards that form a magic pair; say cards number i and $i+1$. Among the cards in this stack and the stack with card number $i+2$ (they might be identical), there are two magic pairs — a contradiction. Hence no stack contains a magic pair.

Each card forms a magic pair with exactly two other cards. Hence if n is even, each stack must contain at least $\lceil\frac{n-1}{2}\rceil = \frac{n}{2}$ cards, since there are $n-1$ other stacks. But then we need at least $n\frac{n}{2} > \frac{n(n-1)}{2}$ cards — a contradiction.

In the odd case we distribute the cards like this: Let $a_1, a_2, \ldots, a_n$ be the n stacks and let $n = 2m+1$. Card number 1 is put into stack a_1. If card number $km+i$, for $i = 1, 2, \ldots, m$, is put into stack a_j, then card number $km+i+1$ is put into stack number a_{j+i}, where the indices are calculated modulo n.

There are

$$\frac{n(n-1)}{2} = \frac{(2m+1)(2m)}{2} = m(2m+1)$$

cards. If we look at all the card numbers of the form $km+1$, there are exactly $n = 2m+1$ of these, and we claim that there is exactly one in each stack. Card number 1 is in stack a_1, and card number $km+1$ is in stack

$$a_{1+k(1+2+3+\cdots+m)}.$$

Since

$$1+2+3+\cdots+m = \frac{m(m+1)}{2}$$

and $\gcd(2m+1, \frac{m(m+1)}{2}) = 1$, all the indices

$$1+k(1+2+3+\cdots+m), \qquad k = 0, 1, 2, \ldots, 2m$$

are different modulo $n = 2m+1$. In the same way we see that each stack contains exactly one of the $2m+1$ cards with the numbers $km+i$ for a given $i = 2, 3, \ldots, m$.

At any moment, let k denote the number of positive numbers on the blackboard and let l denote the number of substantial moves that have already taken place. A move that is not substantial will affect neither k nor l, and, in particular, the sum $k + l$ will not change. A substantial move replaces the pair (k, l) by $(k - 1, l + 1)$, and does not change the value of $k + l$ either.

This proves that the sum $k+l$ remains invariant. Initially, $k+l = n+0 = n$. At the end we have $k = 1$ and consequently $l = n - 1$. This means that there are indeed always $n - 1$ substantial moves. □

Problem C.5

With inspiration drawn from the rectilinear network of streets in New York, the *Manhattan distance* between two points (a, b) and (c, d) is defined to be

$$|a - c| + |b - d|.$$

Suppose only two distinct Manhattan distances occur between all the points of some point set. What is the maximal number of points in such a set?

Solution. Answer: nine.

Let

$$\{(x_1, y_1), \ldots, (x_m, y_m)\}, \qquad \text{where} \qquad x_1 \leqslant \cdots \leqslant x_m,$$

be the set, and suppose $m \geqslant 10$.

A special case of the Erdős–Szekeres Theorem asserts that a real sequence of length $n^2 + 1$ contains a monotonic subsequence of length $n + 1$. (Proof: Given a sequence $a_1 \ldots, a_{n^2+1}$, let p_i denote the length of the longest increasing subsequence ending with a_i, and q_i the length of the longest decreasing subsequence ending with a_i. If $i < j$ and $a_i \leqslant a_j$, then $p_i < p_j$. If $a_i \geqslant a_j$, then $q_i < q_j$. Hence all $n^2 + 1$ pairs (p_i, q_i) are distinct. If all of them were to satisfy $1 \leqslant p_i, q_i \leqslant n$, it would violate the Pigeon-Hole Principle.)

Applied to the sequence $y_1, \ldots, y_m$, this will produce a subsequence

$$y_i \leqslant y_j \leqslant y_k \leqslant y_l \qquad \text{or} \qquad y_i \geqslant y_j \geqslant y_k \geqslant y_l.$$

One of the shortest paths from (x_i, y_i) to (x_l, y_l) will pass through first (x_j, y_j) and then (x_k, y_k). At least three distinct Manhattan distances will occur.

Conversely, among the nine points

$$(0, 0), \qquad (\pm 1, \pm 1), \qquad (\pm 2, 0), \qquad (0, \pm 2)$$

only the Manhattan distances 2 and 4 occur. □

Solution. The first player wins.

He should present his opponent with one of the following positions:

$$(0,0), \quad (1,1), \quad (2,2), \quad \ldots, \quad (2014,2014).$$

All these positions have different total numbers of tokens modulo 2015. Therefore, if the game starts from two piles of arbitrary sizes, it is possible to obtain one of these positions just by the first move. In our case

$$10,000 + 20,000 \equiv 1790 \bmod 2015,$$

and the first player can leave to his opponent the position $(895, 895)$. The remaining part of the game is trivial. □

Problem C.3
There are 100 members of a ladies' club. Each lady has had tea (in private) with exactly 56 of her lady friends. The Board, consisting of the 50 most distinguished ladies, have all had tea with one another. Prove that the entire ladies' club may be split into two groups in such a way that, within each group, any lady has had tea with any other.

Solution. Each lady in the Board has had tea with 49 ladies within the Board, and 7 ladies without. Each lady not in the Board has had tea with at most 49 ladies not in the Board, and at least 7 ladies in the Board. Comparing these two observations, we conclude that each lady not in the Board has had tea with exactly 49 ladies not in the Board and exactly 7 ladies in the Board. Hence the club may be split into Board members and non-members. □

Problem C.4
Let $m \geqslant 0$ and $n \geqslant 1$. Suppose that, initially, there are m zeros and n ones written on a blackboard. Now we are allowed to erase two arbitrary numbers form the blackboard, replacing them by their sum. Such a move is said to be *substantial* if both of the erased numbers were positive. The process is repeated until a single number remains on the blackboard.

Determine all possibilities for the number of substantial moves.

Solution. Answer: There are always $n - 1$ substantial moves.

Solutions

Problem C.1

In the parliament of Neverland, all legislative work is carried out in committees of three people. The constitution dictates that any four people can be in at most two committees. We call a collection of committees a *clique* if any two of them have exactly two people in common, and any manner of including another committee in the collection would break this condition. Prove that two different cliques cannot have two committees in common.

Solution 1. It is easy to see that, given three different committees, each pair of them can have two people in common only if all three committees share the same two people, for otherwise the people in the committees would contain four people from whom three committees have been formed. As a corollary, for any clique, there are some two people who belong to all the committees in the clique, and these people are unique.

To derive a contradiction, let us consider two cliques C_1 and C_2 with two committees in common. There are some two people A and B who belong to all the committees in C_1. These two people must also belong to the two committees shared by C_1 and C_2. But then all the committees in C_2 must also include A and B. Now, we can extend the clique C_1 into $C_1 \cup C_2$, which violates the definition of a clique. □

Solution 2. Consider three committees in a clique. As above, each pair of them can have two people in common only if all three of them share the same two people. Therefore, all committees in a clique share the same two people, and the clique with intersection $\{A, B\}$ consists, by maximality, of all possible committees

$$\{A, B, P_1\}, \ \ldots, \ \{A, B, P_n\}.$$

The clique is thus uniquely determined by the intersection of any two of its elements. □

Problem C.2

Two players play the following game. At the outset there are two piles, containing $10,000$ and $20,000$ tokens, respectively. A move consists of removing any positive number of tokens from a single pile *or* removing a total of $2015k$ tokens from both piles, where k is a positive integer. The player who cannot make a move loses. Which player has a winning strategy?

(b) Prove that the same conclusion does not necessarily hold in a collection of $3n - 3$ girls.

Problem C.8 *The Netherlands*

A subset S of $\{1, 2, \ldots, n\}$ is called *balanced* if for every $a \in S$ there exists a $b \in S$, distinct from a, such that $\frac{a+b}{2} \in S$ as well.

(a) Let $k > 1$ be an integer and let $n = 2^k$. Show that every subset S of $\{1, 2, \ldots, n\}$ with $|S| > \frac{3n}{4}$ is balanced.

(b) Does there exist an $n = 2^k$, with $k > 1$ integral, for which every subset S of $\{1, 2, \ldots, n\}$ with $|S| > \frac{2n}{3}$ is balanced?

Problem C.9 *Estonia*

Let n be a positive integer such that there exists a prime number that is less than $\sqrt{n}$ and does not divide n. Let $a_1, \ldots, a_n$ denote the numbers $1, \ldots, n$, rearranged in any order. Let $a_{i_1} < \cdots < a_{i_k}$ be an increasing subsequence of maximal length ($i_1 < \cdots < i_k$) and $a_{j_1} > \cdots > a_{j_l}$ a decreasing subsequence of maximal length ($j_1 < \cdots < j_l$). Prove that one of the sequences $a_{i_1}, \ldots, a_{i_k}$ and $a_{j_1}, \ldots, a_{j_l}$ contains a number not dividing n.

Problem C.10 *Saint Petersburg*

A country consists of 2015 cities $A_1, A_2, \ldots, A_{2015}$, and 2015 villages $B_1, B_2, \ldots, B_{2015}$. There is a road between city A_i and village B_i, between cities A_i and A_{i+1}, and between villages B_i and B_{i+17} (in cyclic numbering).

Four policemen and one criminal are roaming the country. One day all the policemen move, the next day the robber moves, then the policemen again, and so on. A move consists of remaining in place, or travelling to a neighbouring city or village. Everybody is completely informed of everybody else's actions.

Can the policemen always catch the criminal, independently of their initial positions?

written on a blackboard. Now we are allowed to erase two arbitrary numbers form the blackboard, replacing them by their sum. Such a move is said to be *substantial* if both of the erased numbers were positive. The process is repeated until a single number remains on the blackboard.

Determine all possibilities for the number of substantial moves.

Problem C.5 ***Sweden***

With inspiration drawn from the rectilinear network of streets in New York, the *Manhattan distance* between two points (a, b) and (c, d) is defined to be

$$|a - c| + |b - d|.$$

Suppose only two distinct Manhattan distances occur between all the points of some point set. What is the maximal number of points in such a set?

Problem C.6 ***Denmark***

Let $n > 2$ be a positive integer. A deck contains $\frac{n(n-1)}{2}$ cards, numbered

$$1, 2, 3, \ldots, \frac{n(n-1)}{2}.$$

Two cards form a *magic pair* if their numbers are consecutive, or if their numbers are 1 and $\frac{n(n-1)}{2}$.

For which n is it possible to distribute the cards into n stacks in such a manner that, among the cards in any two stacks, there is exactly one magic pair?

Problem C.7 ***Saint Petersburg***

In a school for girls, all the girls either intensely love or hate each other, and these feelings are always mutual.

(a) Prove that, among any $3n - 2$ girls, for $n \geqslant 2$, one of the following situations must occur.

- There exist n girls $A_1, \ldots, A_n$ such that girls A_i and A_{i+1} love each other, for $i = 1, \ldots, n - 1$, or
- there exist seven girls $B_1, \ldots, B_7$ that all hate each other, except possibly the three pairs (B_1, B_2), (B_3, B_4) and (B_5, B_6), which may either love or hate each other.

C2 C3 C5 C6 C8

Combinatorics

Problems

Problem C.1 *Finland*

In the parliament of Neverland, all legislative work is carried out in committees of three people. The constitution dictates that any four people can be in at most two committees. We call a collection of committees a *clique* if any two of them have exactly two people in common, and any manner of including another committee in the collection would break this condition. Prove that two different cliques cannot have two committees in common.

Problem C.2 *Saint Petersburg*

Two players play the following game. At the outset there are two piles, containing $10,000$ and $20,000$ tokens, respectively. A move consists of removing any positive number of tokens from a single pile *or* removing a total of $2015k$ tokens from both piles, where k is a positive integer. The player who cannot make a move loses. Which player has a winning strategy?

Problem C.3 *Saint Petersburg*

There are 100 members of a ladies' club. Each lady has had tea (in private) with exactly 56 of her lady friends. The Board, consisting of the 50 most distinguished ladies, have all had tea with one another. Prove that the entire ladies' club may be split into two groups in such a way that, within each group, any lady has had tea with any other.

Problem C.4 *Germany*

Let $m \geqslant 0$ and $n \geqslant 1$. Suppose that, initially, there are m zeros and n ones which can be easily checked by means of derivatives. One finds the minimum 0 to be attained for $t = 4/3$. (So the minimum in the initial inequality holds for $(x, y, z) = (4, 8, 3)$.) □

Problem A.17

For $x \geqslant \frac{1}{2}$, what is the largest possible value of the expression

$$\frac{x^4 - x^2}{x^6 + 16x^3 - 1}?$$

Solution. Answer: $\frac{1}{15}$.

Note that, if $\frac{1}{2} \leqslant x < 1$, then $x^4 - x^2$ is negative, while $x^6 + 16x^3 - 1$ is positive. Therefore, in this interval, the expression takes on only negative values. We can then consider only $x \geqslant 1$.

Denoting $t = x - \frac{1}{x}$, hence $x^3 - \frac{1}{x^3} = t^3 + 3t$, the expression can be rewritten as

$$\frac{x - \frac{1}{x}}{x^3 - \frac{1}{x^3} + 16} = \frac{t}{t^3 + 3t + 16} = \frac{1}{t^2 + 3 + \frac{16}{t}}.$$

The minimum of the denominator $t^2 + 3 + \frac{16}{t}$ for $t \geqslant 0$ (as $x \geqslant 1$) can be done with the help of some simple calculus or by noting that

$$t^2 + \frac{8}{t} + \frac{8}{t} + 3 \geqslant \sqrt[3]{t^2 \cdot \frac{8}{t} \cdot \frac{8}{t}} + 3 = 15.$$

This value is reached when $t = 2$, which, in turn, is obtained when $x - \frac{1}{x} = 2$ or $x = 1 + \sqrt{2}$. □

Solution. Answer: Either $f(z) = g(z) = \delta_{z,0}$ or $f(z) = bz$ and $g(z) = az^2$.

Putting $xy = w$ in the second equation gives

$$0 = g\left(\frac{1}{y}f(w) - b\frac{w}{y}\right) = g\left(\frac{1}{y}(f(w) - bw)\right)$$

for all $y \neq 0$. Hence, if $f(w) \neq bw$ for some w, it must be that $g(z) = 0$ for $z \neq 0$. Since g must assume the value 1 somewhere, $g(0) = 1$.

The function g now being known, the first equation transforms, for $x = 0$ and $x \neq 0$, respectively, into

$$f\left(\frac{1}{y^2}\right) = 0 \qquad \text{and} \qquad f(-ax^2) = 0.$$

Consequently, $f(z) = 0$ for $z > 0$ or $z < 0$. Again, f must assume the value 1, and so $f = g$.

There remains the case when $f(z) = bz$ for all z. Substitute $y = 1$ into the first equation to find $b(g(x) - ax^2) = 0$, so that $g(z) = az^2$ for all z.

One easily verifies that these two possibilities satisfy the requirements. □

Problem A.16

Prove that, for positive x, y, z, the following inequality holds:

$$(x + y + z)(4x + y + 2z)(2x + y + 8z) \geqslant \frac{375}{2}xyz.$$

Solution. Consider the first two brackets and observe that

$$(x + y + z)(4x + y + 2z) = (2x + y)^2 + 3z(2x + y) + 2z^2 + xy.$$

Therefore, we can write the inequality in the form

$$\left(\frac{(2x + y)^2 + 3z(2x + y) + 2z^2}{xy} + 1\right) \cdot \frac{2x + y + 8z}{z} \geqslant \frac{375}{2}.$$

Now fix z and $2x + y$ and move $2x$ and y closer to each other. Then we see that xy increases during this movement and attains its maximum when $2x = y$.

Therefore the inequality follows from the inequality obtained by the substitution of $y = 2x$ into initial inequality, i.e.

$$(3x + z)(6x + 2z)(4x + 8z) \geqslant 375x^2z.$$

Letting $t = x/z$, we can rewrite the inequality in the form

$$8(3t + 1)^2(t + 2) - 375t^2 \geqslant 0,$$

Let $y = 1$ (or swap x and y) to deduce

$$f(x) = (1 + f(1))x - 1 = ax - 1, \qquad x \neq 0,$$

where $a = 1 + f(1)$. Insert this expression into the original equation:

$$(a^2 - a)y^2 + 1 = \frac{2}{x}, \qquad x, y, x + y^2 \neq 0.$$

This is clearly impossible (fix one $y \neq 0$ and let $x = 1$ and $x = 2$).

The contradiction shows that $f(-y) = f(y)$ for some $y \neq 0$. Then $f(-xy) = f(xy)$ for all $x \neq 0$ by (7), so that $f(-z) = f(z)$ for all z.

Returning to the original equation, let $y = 1$:

$$f(x + 1) = \left(1 + \frac{2}{x}\right)f(x) + f(1)^2, \qquad x \neq 0.$$

Exchange x for $-x - 1$ and use $f(-x) = f(x)$:

$$f(x) = \left(1 - \frac{2}{x + 1}\right)f(x + 1) + f(1)^2, \qquad x \neq -1.$$

Eliminate $f(x + 1)$ from these two equations and simplify:

$$f(x) = x^2 f(1)^2, \qquad x \neq 0, -1. \tag{8}$$

This formula is, in fact, accurate even for $x = 0$ and $x = -1$. Indeed, letting $y = 0$ in the original equation leads to

$$f(0) = 0 = 0^2 f(1)^2.$$

Moreover, letting $x = 1$ in (8) yields $f(1) = f(1)^2$, so that

$$f(-1) = f(1) = (-1)^2 f(1)^2.$$

Finally, $f(1) = f(1)^2$ determines $f(1) = 0$ or $f(1) = 1$. Therefore, (8) provides two easily verified solutions $f(z) = 0$ and $f(z) = z^2$. □

Problem A.15

Let a and b be positive numbers. Find all pairs of functions $f, g\colon \mathbf{R} \to \mathbf{R}$, each assuming the value 1 and fulfilling, for any $y \neq 0$ and any x, the equations

$$f\left(\frac{1}{y^2}g(xy) - ax^2\right) = 0 = g\left(\frac{1}{y}f(xy) - bx\right).$$

for all positive z. On the other hand, applying (5) to (4) gives

$$f(-2^{x-1}) = \frac{2^{2^{x-1}}}{2^{2^x}} \cdot f(0) = 2^{-2^{x-1}} f(0)$$

for all x, which implies (6) also for all negative z.

We have shown above that $f(0) = 0$ implies $f(x) = 0$ for all x. By taking $x = 0$ in (5) and applying (6), we obtain

$$2 = 2^{2^0} = f(f(0)) = 2^{f(0)} \cdot f(0),$$

which can only happen for $f(0) = 1$. (Clearly $f(0) > 0$, and $z \mapsto 2^z z$ is strictly increasing for positive z.) The only non-zero solution is thus $f(x) = 2^x$. □

Problem A.14

Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all $x \neq 0$ and all y,

$$f(x + y^2) = f(x) + f(y)^2 + \frac{2f(xy)}{x}.$$

Solution. Answer: $f(z) = 0$ or $f(z) = z^2$.

Replacing y by $-y$ gives us

$$f(x) + f(y)^2 + \frac{2f(xy)}{x} = f(x + y^2) = f(x) + f(-y)^2 + \frac{2f(-xy)}{x},$$

which implies that

$$f(y)^2 + \frac{2f(xy)}{x} = f(-y)^2 + \frac{2f(-xy)}{x} \tag{7}$$

for all $x \neq 0$ and all y. Let $x = 1$ and complete the squares:

$$\big(f(y) + 1\big)^2 = f(y)^2 + 2f(y) + 1 = f(-y)^2 + 2f(-y) + 1 = \big(f(-y) + 1\big)^2.$$

Hence

$$f(y) + 1 = \pm\big(f(-y) + 1\big),$$

so that, for any y,

$$\text{either} \quad f(y) = f(-y) \quad \text{or} \quad f(y) + f(-y) = -2.$$

Suppose $f(y) + f(-y) = -2$ for all $y \neq 0$. Equation (7) then simplifies to

$$f(xy) + 1 = x\big(f(y) + 1\big), \qquad x, y \neq 0.$$

for all y. Simplifying the original equation and swapping x and y leads to

$$|x|f(y) + yf(x) = f(xy) = |y|f(x) + xf(y).$$

Choose $y = -1$ and put $c = \frac{f(-1)}{2}$:

$$|x|f(-1) - f(x) = f(x) + xf(-1) \quad \Rightarrow \quad f(x) = \frac{f(-1)}{2}(|x| - x) = c(|x| - x).$$

One easily verifies that these functions satisfy the functional equation for any parameter c. □

Remark. The problem can be made simpler by removing the term $f(f(y))$ from the functional equation. In this case, we solve by first choosing $x = y = 0$, next $y = 0$ to find $f(x^2) = 0$, and then solve as above after equation (3).

Problem A.13
Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all real x and y, the equality

$$f(2^x + 2y) = 2^y f(f(x))f(y).$$

Solution. Answer: $f(x) = 0$ and $f(x) = 2^x$.
Substitute $y = -2^x$:

$$f(-2^x) = \frac{1}{2^{2^x}} f(f(x))f(-2^x).$$

Hence, for each x, either $f(-2^x) = 0$ or $f(f(x)) = 2^{2^x}$.
Substitute $y = -2^{x-1}$:

$$f(0) = \frac{1}{2^{2^{x-1}}} f(f(x))f(-2^{x-1}). \tag{4}$$

So, if $f(-2^z) = 0$ for at least one z, then also $f(0) = 0$. Taking $x = 0$ in the original equation gives $f(1 + 2y) = 0$, i.e., $f \equiv 0$.
Assume from now on that $f(-2^z)$ is never 0. By the first paragraph,

$$f(f(x)) = 2^{2^x} \tag{5}$$

for all x. By taking $y = 0$ in the original equation and applying (5), we obtain

$$f(2^x) = f(f(x))f(0) = 2^{2^x} f(0)$$

for all x, which implies that

$$f(z) = 2^z f(0) \tag{6}$$

Solution 2. Using $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, observe that

$$\sin 1 - \cos 1 = \sqrt{2}\left(\sin 1 \cos\frac{\pi}{4} - \cos 1 \sin\frac{\pi}{4}\right) = \sqrt{2}\,\sin\left(1 - \frac{\pi}{4}\right).$$

Hence, the numbers $\sin 1 - \cos 1$ and $\frac{1}{4}$ are ordered in the same way as the numbers $\sin\left(1 - \frac{\pi}{4}\right)$ and $\frac{1}{4\sqrt{2}}$. We show that the first number is greater.

Since $0 < \frac{\pi}{16} < 1 - \frac{\pi}{4} < \frac{\pi}{2}$ (the middle inequality is equivalent to $\pi < \frac{16}{5}$) and the sine is concave in the first quadrant,

$$\sin\left(1 - \frac{\pi}{4}\right) > \sin\frac{\pi}{16} > \frac{1}{4}\sin\frac{\pi}{4} = \frac{1}{4\sqrt{2}}. \quad \square$$

Remark. The exact value is

$$\sin 1 - \cos 1 = 0.30116867893975\ldots.$$

Problem A.12

Find all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying, for all real numbers x and y, the equation

$$|x|f(y) + yf(x) = f(xy) + f(x^2) + f(f(y)).$$

Solution. Answer: all functions $f(x) = c(|x| - x)$, where c is a real number.

Choosing $x = y = 0$, we find

$$f(f(0)) = -2f(0).$$

Denote $a = f(0)$, so that $f(a) = -2a$, and choose $y = 0$ in the initial equation:

$$a|x| = a + f(x^2) + f(a) = a + f(x^2) - 2a \quad \Rightarrow \quad f(x^2) = a(|x| + 1).$$

In particular, $f(1) = 2a$. Choose $(x, y) = (z^2, 1)$ in the initial equation:

$$\begin{aligned} & z^2 f(1) + f(z^2) = f(z^2) + f(z^4) + f(f(1)) \\ \Rightarrow \quad & 2az^2 = z^2 f(1) = f(z^4) + f(f(1)) = a(z^2 + 1) + f(2a) \\ \Rightarrow \quad & az^2 = a + f(2a). \end{aligned}$$

The right-hand side is constant, while the left-hand side is a quadratic function in z, which can only happen if $a = 0$. (Choose $z = 1$ and then $z = 0$.)

We now conclude that $f(x^2) = 0$, and so $f(x) = 0$ for all non-negative x. In particular, $f(0) = 0$. Choosing $x = 0$ in the initial equation, we find

$$f(f(y)) = 0 \qquad (3)$$

By (1), we have $c \geqslant x > 0$. Moreover, a simple calculation shows

$$(3c - 2x)^2 = 9c^2 - 12cx + 4x^2 = 3c^2 + 6(c - x)^2 - 2x^2 \geqslant 3c^2 - 2x^2 = x^2 + y^2,$$

using (1) in the last step.

Next, we observe that (2) implies $d \geqslant x$ and hence also $x^2 + y^2 \geqslant d^2$. In combination with $3c - 2x \geqslant c > 0$, this leads to

$$3c - 2x \geqslant \sqrt{x^2 + y^2} \geqslant d,$$

whereby the problem is solved. □

Problem A.11
Which number is greater,

$$\sin 1 - \cos 1 \qquad \text{or} \qquad \frac{1}{4}\,?$$

Solution 1. Answer: The greater number is $\sin 1 - \cos 1$.

The sine is increasing and the cosine decreasing in the first quadrant. Since $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, we have

$$\sin 1 - \cos 1 > \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0.$$

Hence, the numbers $\sin 1 - \cos 1$ and $\frac{1}{4}$ are ordered in the same way as their squares $(\sin 1 - \cos 1)^2$ and $\frac{1}{16}$. Since

$$(\sin 1 - \cos 1)^2 = \sin^2 1 + \cos^2 1 - 2\sin 1 \cos 1 = 1 - \sin 2,$$

it suffices to compare the numbers $1 - \sin 2$ and $\frac{1}{16}$. We show that the former number is greater by showing that $\sin 2 < \frac{15}{16}$.

Since $\pi < 3.2 = \frac{16}{5}$, we have $\frac{\pi}{2} < \frac{5}{8}\pi < 2 < \pi$, which implies

$$\sin 2 < \sin\frac{5}{8}\pi = \sqrt{\frac{1 - \sin\frac{5}{4}\pi}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 + \sqrt{2}}}{2} < \frac{15}{16},$$

using the half-angle sine formula. The last inequality is clear from

$$\sqrt{2} < \frac{3}{2} < \frac{97}{64} \quad\Rightarrow\quad 2 + \sqrt{2} < \frac{225}{64} \quad\Rightarrow\quad \sqrt{2 + \sqrt{2}} < \frac{15}{8}. \quad \square$$