hard. – for 6

C7. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice — once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

Solution 1. Letting $n = 2018$, we will show that, if every region has at least one non-yellow vertex, then every circle contains at most $n + \lfloor\sqrt{n-2}\rfloor - 2$ yellow points. In the case at hand, the latter equals $2018 + 44 - 2 = 2060$, contradicting the hypothesis.

Consider the natural geometric graph G associated with the configuration of n circles. Fix any circle C in the configuration, let k be the number of yellow points on C, and find a suitable lower bound for the total number of yellow vertices of G in terms of k and n. It turns out that k is even, and G has at least

$$k + 2\binom{k/2}{2} + 2\binom{n-k/2-1}{2} = \frac{k^2}{2} - (n-2)k + (n-2)(n-1) \qquad (*)$$

yellow vertices. The proof hinges on the two lemmata below.

Lemma 1. Let two circles in the configuration cross at x and y. Then x and y are either both yellow or both non-yellow.

Proof. This is because the numbers of interior vertices on the four arcs x and y determine on the two circles have like parities. $\square$

In particular, each circle in the configuration contains an even number of yellow vertices.

Lemma 2. If $\widehat{xy}$, $\widehat{yz}$, and $\widehat{zx}$ are circular arcs of three pairwise distinct circles in the configuration, then the number of yellow vertices in the set $\{x, y, z\}$ is odd.

Proof. Let C_1, C_2, C_3 be the three circles under consideration. Assume, without loss of generality, that C_2 and C_3 cross at x, C_3 and C_1 cross at y, and C_1 and C_2 cross at z. Let k_1, k_2, k_3 be the numbers of interior vertices on the three circular arcs under consideration. Since each circle in the configuration, different from the C_i, crosses the cycle $\widehat{xy} \cup \widehat{yz} \cup \widehat{zx}$ at an even number of points (recall that no three circles are concurrent), and self-crossings are counted twice, the sum $k_1 + k_2 + k_3$ is even.

Let Z_1 be the colour z gets from C_1 and define the other colours similarly. By the preceding, the number of bichromatic pairs in the list (Z_1, Y_1), (X_2, Z_2), (Y_3, X_3) is odd. Since the total number of colour changes in a cycle Z_1–Y_1–Y_3–X_3–X_2–Z_2–Z_1 is even, the number of bichromatic pairs in the list (X_2, X_3), (Y_1, Y_3), (Z_1, Z_2) is odd, and the lemma follows. $\square$

We are now in a position to prove that $(*)$ bounds the total number of yellow vertices from below. Refer to Lemma 1 to infer that the k yellow vertices on C pair off to form the pairs of points where C is crossed by $k/2$ circles in the configuration. By Lemma 2, these circles cross pairwise to account for another $2\binom{k/2}{2}$ yellow vertices. Finally, the remaining $n - k/2 - 1$ circles in the configuration cross C at non-yellow vertices, by Lemma 1, and Lemma 2 applies again to show that these circles cross pairwise to account for yet another $2\binom{n-k/2-1}{2}$ yellow vertices. Consequently, there are at least $(*)$ yellow vertices.

Next, notice that G is a plane graph on $n(n-1)$ degree 4 vertices, having exactly $2n(n-1)$ edges and exactly $n(n-1) + 2$ faces (regions), the outer face inclusive (by Euler's formula for planar graphs).

Lemma 3. Each face of G has equally many red and blue vertices. In particular, each face has an even number of non-yellow vertices.

Proof. Trace the boundary of a face once in circular order, and consider the colours each vertex is assigned in the colouring of the two circles that cross at that vertex, to infer that colours of non-yellow vertices alternate. □

Consequently, if each region has at least one non-yellow vertex, then it has at least two such. Since each vertex of G has degree 4, consideration of vertex-face incidences shows that G has at least $n(n-1)/2+1$ non-yellow vertices, and hence at most $n(n-1)/2-1$ yellow vertices. (In fact, Lemma 3 shows that there are at least $n(n-1)/4+1/2$ red, respectively blue, vertices.)

Finally, recall the lower bound $(*)$ for the total number of yellow vertices in G, to write $n(n-1)/2-1 \geqslant k^2/2-(n-2)k+(n-2)(n-1)$, and conclude that $k \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$, as claimed in the first paragraph.

Solution 2. The first two lemmata in Solution 1 show that the circles in the configuration split into two classes: Consider any circle C along with all circles that cross C at yellow points to form one class; the remaining circles then form the other class. Lemma 2 shows that any pair of circles in the same class cross at yellow points; otherwise, they cross at non-yellow points.

Call the circles from the two classes *white* and *black*, respectively. Call a region *yellow* if its vertices are all yellow. Let w and b be the numbers of white and black circles, respectively; clearly, $w+b=n$. Assume that $w \geqslant b$, and that there is no yellow region. Clearly, $b \geqslant 1$, otherwise each region is yellow. The white circles subdivide the plane into $w(w-1)+2$ larger regions — call them *white*. The white regions (or rather their boundaries) subdivide each black circle into *black arcs*. Since there are no yellow regions, each white region contains at least one black arc.

Consider any white region; let it contain $t \geqslant 1$ black arcs. We claim that the number of points at which these t arcs cross does not exceed $t-1$. To prove this, consider a multigraph whose vertices are these black arcs, two vertices being joined by an edge for each point at which the corresponding arcs cross. If this graph had more than $t-1$ edges, it would contain a cycle, since it has t vertices; this cycle would correspond to a closed contour formed by black sub-arcs, lying inside the region under consideration. This contour would, in turn, define at least one yellow region, which is impossible.

Let t_i be the number of black arcs inside the i^{th} white region. The total number of black arcs is $\sum_i t_i = 2wb$, and they cross at $2\binom{b}{2}=b(b-1)$ points. By the preceding,

$$b(b-1) \leqslant \sum_{i=1}^{w^2-w+2}(t_i-1) = \sum_{i=1}^{w^2-w+2} t_i-(w^2-w+2) = 2wb-(w^2-w+2),$$

or, equivalently, $(w-b)^2 \leqslant w+b-2=n-2$, which is the case if and only if $w-b \leqslant \lfloor\sqrt{n-2}\rfloor$. Consequently, $b \leqslant w \leqslant \left(n+\lfloor\sqrt{n-2}\rfloor\right)/2$, so there are at most $2(w-1) \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$ yellow vertices on each circle — a contradiction.