

Advanced Combinatorics: Solving Linear Recurrences

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$, where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

- it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n .
- it is *homogeneous* because no terms occur that are not multiples of the a_j s. Each coefficient is a constant.
- the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1, a_1 = C_2, \dots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrences

- Look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:
$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 . Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \beta r_2^n$$

for $n = 0, 1, 2, \dots$, where α and β are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution: The characteristic equation is

$$r^2 - r - 2 = 0.$$

Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$, for some constants α_1 and α_2 .

To find the constants α_1 and α_2 , note that

$$\begin{aligned}a_0 &= 2 = \alpha_1 + \alpha_2 \text{ and} \\a_1 &= 7 = \alpha_1 2 + \alpha_2 (-1).\end{aligned}$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2 .

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1.$$

Solving, we obtain

$$\alpha_1 = \frac{1}{\sqrt{5}} \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

Hence,

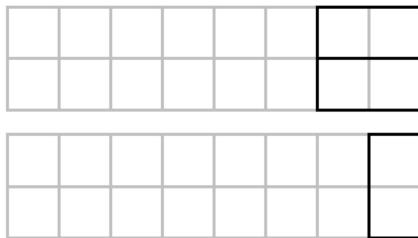
$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Illustration with Domino Tilings

- How many ways to fill rectangle 2×8 with dominoes 1×2 ?



- Can reduce to 2×6 and 2×7 problems.



Linear Recurrences with Complex Roots

Problem: A sequence x_n is defined by $x_0 = 0$, $x_1 = 1$, and the recurrence relation

$$x_n = x_{n-1} - 4x_{n-2}$$

Find the closed-form of x_n .

Example is analyzed here - <https://brilliant.org/wiki/linear-recurrence-relations/#solving-linear-recurrence-with-complex-roots>

Solving the Recurrence

Solution: The characteristic polynomial of the recurrence is $r^2 = r - 4$

Since the solutions of this equation are

$$r = \frac{1 \pm i\sqrt{15}}{2},$$

Can still use the formula $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are found from the two initial conditions. (And hope that i always cancels).

Can also rewrite $r = 2e^{\pm i\theta}$, where $\theta = \arctan \sqrt{15}$.

Then $x_n = c_1 2^n \cos n\theta + c_2 2^n \sin n\theta$.

From $x_0 = 0, x_1 = 1$ we can find $c_1 = 0, 2c_1 \cos \theta + 2c_2 \sin \theta = 1$. Therefore $c_2 = \frac{2}{\sqrt{15}}$. Ultimate solution: $x_n = \frac{2^{n+1}}{\sqrt{15}} \sin n\theta$.

Finding the Recurrence in Python

```
x=[0,1]
for i in range(2,21):
    x.append(x[i-1] - 4*x[i-2])
[0, 1, 1, -3, -7, 5, 33, 13, -119, -171, 305,
989, -231, -4187, -3263, 13485, 26537, -27403, -133551, -23939, 510265]
```

More complicated pattern how positive and negative values mix.

In fact, it is a sum of two (complex) geometric progressions, where imaginary parts cancel out.

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has one repeated root r_0 . Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation
 $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is
 $r^2 - 6r + 9 = 0$.

The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2 , note that

$$a_0 = 1 = \alpha_1$$

$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.

$$\text{Hence, } a_n = 3^n + n3^n.$$

Linear Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has k distinct roots r_1, r_2, \dots, r_k . Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has t distinct roots r_1, r_2, \dots, r_t with multiplicities m_1, m_2, \dots, m_t , respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad \text{if and only if}$$

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \dots + (\alpha_{t,0} + \alpha_{t,1} n + \dots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \dots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Linear Nonhomogeneous Recurrences – 1

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

where c_1, c_2, \dots, c_k are real numbers, and $F(n)$ is a function not identically zero depending only on n .

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrences – 2

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$$a_n = a_{n-1} + 2^n,$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1,$$

$$a_n = 3a_{n-1} + n3^n,$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

where the following are the associated linear homogeneous recurrence relations, respectively:

$$a_n = a_{n-1},$$

$$a_n = a_{n-1} + a_{n-2},$$

$$a_n = 3a_{n-1},$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

Linear Nonhomogeneous Recurrences – 3

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Linear Nonhomogeneous Recurrences – 4

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.

What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n) = 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n , say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.

Then $a_n = 3a_{n-1} + 2n$ becomes
 $cn + d = 3(c(n-1) + d) + 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. We get that $cn + d$ is a solution if and only if

$2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution iff $c = -1$ and $d = -3/2$.

Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.

Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$.
Hence solution $a_n = -n - 3/2 + (11/6)3^n$.