A4. Let $a_0, a_1, a_2, \dots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geqslant 2$ there exists $1 \leqslant k \leqslant n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal possible value of $a_{2018} - a_{2017}$.

Answer: The maximal value is $\frac{2016}{2017^2}$.

Solution 1. The claimed maximal value is achieved at

$$a_1 = a_2 = \cdots = a_{2016} = 1, \quad a_{2017} = \frac{a_{2016} + \cdots + a_0}{2017} = 1 - \frac{1}{2017},$$
$$a_{2018} = \frac{a_{2017} + \cdots + a_1}{2017} = 1 - \frac{1}{2017^2}.$$

Now we need to show that this value is optimal. For brevity, we use the notation

$$S(n, k) = a_{n-1} + a_{n-2} + \cdots + a_{n-k} \quad \text{for nonnegative integers } k \leqslant n.$$

In particular, $S(n, 0) = 0$ and $S(n, 1) = a_{n-1}$. In these terms, for every integer $n \geqslant 2$ there exists a positive integer $k \leqslant n$ such that $a_n = S(n, k)/k$.

For every integer $n \geqslant 1$ we define

$$M_n = \max_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \qquad m_n = \min_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \quad \text{and} \quad \Delta_n = M_n - m_n \geqslant 0.$$

By definition, $a_n \in [m_n, M_n]$ for all $n \geqslant 2$; on the other hand, $a_{n-1} = S(n, 1)/1 \in [m_n, M_n]$. Therefore,

$$a_{2018} - a_{2017} \leqslant M_{2018} - m_{2018} = \Delta_{2018},$$

and we are interested in an upper bound for Δ_{2018}.

Also by definition, for any $0 < k \leqslant n$ we have $km_n \leqslant S(n, k) \leqslant kM_n$; notice that these inequalities are also valid for $k = 0$.

Claim 1. For every $n > 2$, we have $\Delta_n \leqslant \frac{n-1}{n} \Delta_{n-1}$.

Proof. Choose positive integers $k, \ell \leqslant n$ such that $M_n = S(n, k)/k$ and $m_n = S(n, \ell)/\ell$. We have $S(n, k) = a_{n-1} + S(n-1, k-1)$, so

$$k(M_n - a_{n-1}) = S(n, k) - ka_{n-1} = S(n-1, k-1) - (k-1)a_{n-1} \leqslant (k-1)(M_{n-1} - a_{n-1}),$$

since $S(n-1, k-1) \leqslant (k-1)M_{n-1}$. Similarly, we get

$$\ell(a_{n-1} - m_n) = (\ell - 1)a_{n-1} - S(n-1, \ell-1) \leqslant (\ell-1)(a_{n-1} - m_{n-1}).$$

Since $m_{n-1} \leqslant a_{n-1} \leqslant M_{n-1}$ and $k, \ell \leqslant n$, the obtained inequalities yield

$$M_n - a_{n-1} \leqslant \frac{k-1}{k}(M_{n-1} - a_{n-1}) \leqslant \frac{n-1}{n}(M_{n-1} - a_{n-1}) \quad \text{and}$$
$$a_{n-1} - m_n \leqslant \frac{\ell-1}{\ell}(a_{n-1} - m_{n-1}) \leqslant \frac{n-1}{n}(a_{n-1} - m_{n-1}).$$

Therefore,

$$\Delta_n = (M_n - a_{n-1}) + (a_{n-1} - m_n) \leqslant \frac{n-1}{n}\big((M_{n-1} - a_{n-1}) + (a_{n-1} - m_{n-1})\big) = \frac{n-1}{n}\Delta_{n-1}. \ \square$$

Back to the problem, if $a_n = 1$ for all $n \leqslant 2017$, then $a_{2018} \leqslant 1$ and hence $a_{2018} - a_{2017} \leqslant 0$. Otherwise, let $2 \leqslant q \leqslant 2017$ be the minimal index with $a_q < 1$. We have $S(q, i) = i$ for all $i = 1, 2, \ldots, q-1$, while $S(q, q) = q - 1$. Therefore, $a_q < 1$ yields $a_q = S(q, q)/q = 1 - \frac{1}{q}$.

Now we have $S(q+1, i) = i - \frac{1}{q}$ for $i = 1, 2, \ldots, q$, and $S(q+1, q+1) = q - \frac{1}{q}$. This gives us

$$m_{q+1} = \frac{S(q+1,1)}{1} = \frac{S(q+1,q+1)}{q+1} = \frac{q-1}{q} \quad \text{and} \quad M_{q+1} = \frac{S(q+1,q)}{q} = \frac{q^2-1}{q^2},$$

so $\Delta_{q+1} = M_{q+1} - m_{q+1} = (q-1)/q^2$. Denoting $N = 2017 \geqslant q$ and using Claim 1 for $n = q+2, q+3, \ldots, N+1$ we finally obtain

$$\Delta_{N+1} \leqslant \frac{q-1}{q^2} \cdot \frac{q+1}{q+2} \cdot \frac{q+2}{q+3} \cdots \frac{N}{N+1} = \frac{1}{N+1}\left(1 - \frac{1}{q^2}\right) \leqslant \frac{1}{N+1}\left(1 - \frac{1}{N^2}\right) = \frac{N-1}{N^2},$$

as required.

Comment 1. One may check that the maximal value of $a_{2018} - a_{2017}$ is attained at the unique sequence, which is presented in the solution above.

Comment 2. An easier question would be to determine the maximal value of $|a_{2018} - a_{2017}|$. In this version, the answer $\frac{1}{2018}$ is achieved at

$$a_1 = a_2 = \cdots = a_{2017} = 1, \quad a_{2018} = \frac{a_{2017} + \cdots + a_0}{2018} = 1 - \frac{1}{2018}.$$

To prove that this value is optimal, it suffices to notice that $\Delta_2 = \frac{1}{2}$ and to apply Claim 1 obtaining

$$|a_{2018} - a_{2017}| \leqslant \Delta_{2018} \leqslant \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{2017}{2018} = \frac{1}{2018}.$$

Solution 2. We present a different proof of the estimate $a_{2018} - a_{2017} \leqslant \frac{2016}{2017^2}$. We keep the same notations of $S(n, k)$, m_n and M_n from the previous solution.

Notice that $S(n, n) = S(n, n-1)$, as $a_0 = 0$. Also notice that for $0 \leqslant k \leqslant \ell \leqslant n$ we have $S(n, \ell) = S(n, k) + S(n-k, \ell-k)$.

Claim 2. For every positive integer n, we have $m_n \leqslant m_{n+1}$ and $M_{n+1} \leqslant M_n$, so the segment $[m_{n+1}, M_{n+1}]$ is contained in $[m_n, M_n]$.

Proof. Choose a positive integer $k \leqslant n+1$ such that $m_{n+1} = S(n+1, k)/k$. Then we have

$$km_{n+1} = S(n+1, k) = a_n + S(n, k-1) \geqslant m_n + (k-1)m_n = km_n,$$

which establishes the first inequality in the Claim. The proof of the second inequality is similar. □

Claim 3. For every positive integers $k \geqslant n$, we have $m_n \leqslant a_k \leqslant M_n$.

Proof. By Claim 2, we have $[m_k, M_k] \subseteq [m_{k-1}, M_{k-1}] \subseteq \cdots \subseteq [m_n, M_n]$. Since $a_k \in [m_k, M_k]$, the claim follows. □

Claim 4. For every integer $n \geqslant 2$, we have $M_n = S(n, n-1)/(n-1)$ and $m_n = S(n,n)/n$.

Proof. We use induction on n. The base case $n = 2$ is routine. To perform the induction step, we need to prove the inequalities

$$\frac{S(n,n)}{n} \leqslant \frac{S(n,k)}{k} \quad \text{and} \quad \frac{S(n,k)}{k} \leqslant \frac{S(n,n-1)}{n-1} \tag{1}$$

for every positive integer $k \leqslant n$. Clearly, these inequalities hold for $k = n$ and $k = n-1$, as $S(n,n) = S(n,n-1) > 0$. In the sequel, we assume that $k < n-1$.

Now the first inequality in (1) rewrites as $nS(n,k) \geqslant kS(n,n) = k\big(S(n,k) + S(n-k, n-k)\big)$, or, cancelling the terms occurring on both parts, as

$$(n-k)S(n,k) \geqslant kS(n-k,n-k) \iff S(n,k) \geqslant k \cdot \frac{S(n-k,n-k)}{n-k}.$$

By the induction hypothesis, we have $S(n-k,n-k)/(n-k) = m_{n-k}$. By Claim 3, we get $a_{n-i} \geqslant m_{n-k}$ for all $i = 1, 2, \ldots, k$. Summing these k inequalities we obtain

$$S(n,k) \geqslant km_{n-k} = k \cdot \frac{S(n-k,n-k)}{n-k},$$

as required.

The second inequality in (1) is proved similarly. Indeed, this inequality is equivalent to

$$\begin{aligned}(n-1)S(n,k) \leqslant kS(n,n-1) &\iff (n-k-1)S(n,k) \leqslant kS(n-k,n-k-1)\\ &\iff S(n,k) \leqslant k \cdot \frac{S(n-k,n-k-1)}{n-k-1} = kM_{n-k};\end{aligned}$$

the last inequality follows again from Claim 3, as each term in $S(n,k)$ is at most M_{n-k}. □

Now we can prove the required estimate for $a_{2018} - a_{2017}$. Set $N = 2017$. By Claim 4,

$$\begin{aligned}a_{N+1} - a_N \leqslant M_{N+1} - a_N = \frac{S(N+1,N)}{N} - a_N &= \frac{a_N + S(N,N-1)}{N} - a_N\\ &= \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot a_N.\end{aligned}$$

On the other hand, the same Claim yields

$$a_N \geqslant m_N = \frac{S(N,N)}{N} = \frac{S(N,N-1)}{N}.$$

Noticing that each term in $S(N,N-1)$ is at most 1, so $S(N,N-1) \leqslant N-1$, we finally obtain

$$a_{N+1} - a_N \leqslant \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot \frac{S(N,N-1)}{N} = \frac{S(N,N-1)}{N^2} \leqslant \frac{N-1}{N^2}.$$

Comment 1. Claim 1 in Solution 1 can be deduced from Claims 2 and 4 in Solution 2.

By Claim 4 we have $M_n = \frac{S(n,n-1)}{n-1}$ and $m_n = \frac{S(n,n)}{n} = \frac{S(n,n-1)}{n}$. It follows that $\Delta_n = M_n - m_n = \frac{S(n,n-1)}{(n-1)n}$ and so $M_n = n\Delta_n$ and $m_n = (n-1)\Delta_n$

Similarly, $M_{n-1} = (n-1)\Delta_{n-1}$ and $m_{n-1} = (n-2)\Delta_{n-1}$. Then the inequalities $m_{n-1} \leqslant m_n$ and $M_n \leqslant M_{n-1}$ from Claim 2 write as $(n-2)\Delta_{n-1} \leqslant (n-1)\Delta_n$ and $n\Delta_n \leqslant (n-1)\Delta_{n-1}$. Hence we have the double inequality

$$\frac{n-2}{n-1}\Delta_{n-1} \leqslant \Delta_n \leqslant \frac{n-1}{n}\Delta_{n-1}.$$

Comment 2. Both solutions above discuss the properties of an arbitrary sequence satisfying the problem conditions. Instead, one may investigate only an *optimal* sequence which maximises the value of $a_{2018} - a_{2017}$. Here we present an observation which allows to simplify such investigation — for instance, the proofs of Claim 1 in Solution 1 and Claim 4 in Solution 2.

The sequence (a_n) is uniquely determined by choosing, for every $n \geqslant 2$, a positive integer $k(n) \leqslant n$ such that $a_n = S(n, k(n))/k(n)$. Take an arbitrary $2 \leqslant n_0 \leqslant 2018$, and assume that all such integers $k(n)$, for $n \neq n_0$, are fixed. Then, for every n, the value of a_n is a linear function in a_{n_0} (whose possible values constitute some discrete subset of $[m_{n_0}, M_{n_0}]$ containing both endpoints). Hence, $a_{2018} - a_{2017}$ is also a linear function in a_{n_0}, so it attains its maximal value at one of the endpoints of the segment $[m_{n_0}, M_{n_0}]$.

This shows that, while dealing with an optimal sequence, we may assume $a_n \in \{m_n, M_n\}$ for all $2 \leqslant n \leqslant 2018$. Now one can easily see that, if $a_n = m_n$, then $m_{n+1} = m_n$ and $M_{n+1} \leqslant \frac{m_n + nM_n}{n+1}$; similar estimates hold in the case $a_n = M_n$. This already establishes Claim 1, and simplifies the inductive proof of Claim 4, both applied to an optimal sequence.