

Note *Nice, easy/med*
A2. Find all positive integers $n \geq 3$ for which there exist real numbers a_1, a_2, \dots, a_n , $a_{n+1} = a_1$, $a_{n+2} = a_2$ such that

$$a_i a_{i+1} + 1 = a_{i+2}$$

for all $i = 1, 2, \dots, n$.

Answer: n can be any multiple of 3.

Solution 1. For the sake of convenience, extend the sequence a_1, \dots, a_{n+2} to an infinite periodic sequence with period n . (n is not necessarily the shortest period.)

If n is divisible by 3, then $(a_1, a_2, \dots) = (-1, -1, 2, -1, -1, 2, \dots)$ is an obvious solution.

We will show that in every periodic sequence satisfying the recurrence, each positive term is followed by two negative values, and after them the next number is positive again. From this, it follows that n is divisible by 3.

If the sequence contains two consecutive positive numbers a_i, a_{i+1} , then $a_{i+2} = a_i a_{i+1} + 1 > 1$, so the next value is positive as well; by induction, all numbers are positive and greater than 1. But then $a_{i+2} = a_i a_{i+1} + 1 \geq 1 \cdot a_{i+1} + 1 > a_{i+1}$ for every index i , which is impossible: our sequence is periodic, so it cannot increase everywhere.

If the number 0 occurs in the sequence, $a_i = 0$ for some index i , then it follows that $a_{i+1} = a_{i-1} a_i + 1$ and $a_{i+2} = a_i a_{i+1} + 1$ are two consecutive positive elements in the sequences and we get the same contradiction again.

Notice that after any two consecutive negative numbers the next one must be positive: if $a_i < 0$ and $a_{i+1} < 0$, then $a_{i+2} = a_i a_{i+1} + 1 > 1 > 0$. Hence, the positive and negative numbers follow each other in such a way that each positive term is followed by one or two negative values and then comes the next positive term.

Consider the case when the positive and negative values alternate. So, if a_i is a negative value then a_{i+1} is positive, a_{i+2} is negative and a_{i+3} is positive again.

Notice that $a_i a_{i+1} + 1 = a_{i+2} < 0 < a_{i+3} = a_{i+1} a_{i+2} + 1$; by $a_{i+1} > 0$ we conclude $a_i < a_{i+2}$. Hence, the negative values form an infinite increasing subsequence, $a_i < a_{i+2} < a_{i+4} < \dots$, which is not possible, because the sequence is periodic.

The only case left is when there are consecutive negative numbers in the sequence. Suppose that a_i and a_{i+1} are negative; then $a_{i+2} = a_i a_{i+1} + 1 > 1$. The number a_{i+3} must be negative. We show that a_{i+4} also must be negative.

Notice that a_{i+3} is negative and $a_{i+4} = a_{i+2} a_{i+3} + 1 < 1 < a_i a_{i+1} + 1 = a_{i+2}$, so

$$a_{i+5} - a_{i+4} = (a_{i+3} a_{i+4} + 1) - (a_{i+2} a_{i+3} + 1) = a_{i+3} (a_{i+4} - a_{i+2}) > 0,$$

therefore $a_{i+5} > a_{i+4}$. Since at most one of a_{i+4} and a_{i+5} can be positive, that means that a_{i+4} must be negative.

Now a_{i+3} and a_{i+4} are negative and a_{i+5} is positive; so after two negative and a positive terms, the next three terms repeat the same pattern. That completes the solution.

Solution 2. We prove that the shortest period of the sequence must be 3. Then it follows that n must be divisible by 3.

Notice that the equation $x^2 + 1 = x$ has no real root, so the numbers a_1, \dots, a_n cannot be all equal, hence the shortest period of the sequence cannot be 1.

By applying the recurrence relation for i and $i + 1$,

$$\begin{aligned} (a_{i+2} - 1)a_{i+2} &= a_i a_{i+1} a_{i+2} = a_i (a_{i+3} - 1), \quad \text{so} \\ a_{i+2}^2 - a_i a_{i+3} &= a_{i+2} - a_i. \end{aligned}$$

By summing over $i = 1, 2, \dots, n$, we get

$$\sum_{i=1}^n (a_i - a_{i+3})^2 = 0.$$

That proves that $a_i = a_{i+3}$ for every index i , so the sequence a_1, a_2, \dots is indeed periodic with period 3. The shortest period cannot be 1, so it must be 3; therefore, n is divisible by 3.

Comment. By solving the system of equations $ab + 1 = c$, $bc + 1 = a$, $ca + 1 = b$, it can be seen that the pattern $(-1, -1, 2)$ is repeated in all sequences satisfying the problem conditions.