Number Theory

Erasers, nice, time consuming? / medium

N1. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

Answer: All pairs (n, k) such that $n \nmid k$ and $k \nmid n$.

Solution. As usual, the number of divisors of a positive integer n is denoted by $d(n)$. If $n = \prod_i p_i^{\alpha_i}$ is the prime factorisation of n, then $d(n) = \prod_i (\alpha_i + 1)$.

We start by showing that one cannot find any suitable number s if $k \mid n$ or $n \mid k$ (and $k \neq n$). Suppose that $n \mid k$, and choose any positive integer s. Then the set of divisors of sn is a proper subset of that of sk, hence $d(sn) < d(sk)$. Therefore, the pair (n, k) does not satisfy the problem requirements. The case $k \mid n$ is similar.

Now assume that $n \nmid k$ and $k \nmid n$. Let $p_1, \ldots, p_t$ be all primes dividing nk, and consider the prime factorisations

$$n = \prod_{i=1}^{t} p_i^{\alpha_i} \quad \text{and} \quad k = \prod_{i=1}^{t} p_i^{\beta_i}.$$

It is reasonable to search for the number s having the form

$$s = \prod_{i=1}^{t} p_i^{\gamma_i}.$$

The (nonnegative integer) exponents γ_i should be chosen so as to satisfy

$$\frac{d(sn)}{d(sk)} = \prod_{i=1}^{t} \frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = 1. \tag{1}$$

First of all, if $\alpha_i = \beta_i$ for some i, then, regardless of the value of γ_i, the corresponding factor in (1) equals 1 and does not affect the product. So we may assume that there is no such index i. For the other factors in (1), the following lemma is useful.

Lemma. Let $\alpha > \beta$ be nonnegative integers. Then, for every integer $M \geqslant \beta + 1$, there exists a nonnegative integer γ such that

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} = \frac{M+1}{M}.$$

Proof.

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} \iff \frac{\alpha - \beta}{\beta + \gamma + 1} = \frac{1}{M} \iff \gamma = M(\alpha - \beta) - (\beta + 1) \geqslant 0. \qquad \square$$

Now we can finish the solution. Without loss of generality, there exists an index u such that $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, u$, and $\alpha_i < \beta_i$ for $i = u + 1, \ldots, t$. The conditions $n \nmid k$ and $k \nmid n$ mean that $1 \leqslant u \leqslant t - 1$.

Choose an integer X greater than all the α_i and β_i. By the lemma, we can define the numbers γ_i so as to satisfy

$$\frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = \frac{uX + i}{uX + i - 1} \qquad \text{for } i = 1, 2, \ldots, u, \text{ and}$$

$$\frac{\beta_{u+i} + \gamma_{u+i} + 1}{\alpha_{u+i} + \gamma_{u+i} + 1} = \frac{(t-u)X + i}{(t-u)X + i - 1} \qquad \text{for } i = 1, 2, \ldots, t - u.$$

Then we will have

$$\frac{d(sn)}{d(sk)}=\prod_{i=1}^{u}\frac{uX+i}{uX+i-1}\cdot\prod_{i=1}^{t-u}\frac{(t-u)X+i-1}{(t-u)X+i}=\frac{u(X+1)}{uX}\cdot\frac{(t-u)X}{(t-u)(X+1)}=1,$$

as required.

Comment. The lemma can be used in various ways, in order to provide a suitable value of s. In particular, one may apply induction on the number t of prime factors, using identities like

$$\frac{n}{n-1}=\frac{n^2}{n^2-1}\cdot\frac{n+1}{n}.$$