medium, number theor/cal

A3. Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

Solution 1. Argue indirectly. Agree, as usual, that the empty sum is 0 to consider rationals in $[0, 1)$; adjoining 0 causes no harm, since $\sum_{x\in F} 1/x = 0$ for no nonempty finite subset F of S. For every rational r in $[0, 1)$, let F_r be the unique finite subset of S such that $\sum_{x\in F_r} 1/x = r$. The argument hinges on the lemma below.

Lemma. If x is a member of S and q and r are rationals in $[0, 1)$ such that $q - r = 1/x$, then x is a member of F_q if and only if it is not one of F_r.

Proof. If x is a member of F_q, then

$$\sum_{y\in F_q\smallsetminus\{x\}} \frac{1}{y} = \sum_{y\in F_q} \frac{1}{y} - \frac{1}{x} = q - \frac{1}{x} = r = \sum_{y\in F_r} \frac{1}{y},$$

so $F_r = F_q \smallsetminus \{x\}$, and x is not a member of F_r. Conversely, if x is not a member of F_r, then

$$\sum_{y\in F_r\cup\{x\}} \frac{1}{y} = \sum_{y\in F_r} \frac{1}{y} + \frac{1}{x} = r + \frac{1}{x} = q = \sum_{y\in F_q} \frac{1}{y},$$

so $F_q = F_r \cup \{x\}$, and x is a member of F_q. □

Consider now an element x of S and a positive rational $r < 1$. Let $n = \lfloor rx \rfloor$ and consider the sets $F_{r-k/x}$, $k = 0, \ldots, n$. Since $0 \leqslant r - n/x < 1/x$, the set $F_{r-n/x}$ does not contain x, and a repeated application of the lemma shows that the $F_{r-(n-2k)/x}$ do not contain x, whereas the $F_{r-(n-2k-1)/x}$ do. Consequently, x is a member of F_r if and only if n is odd.

Finally, consider $F_{2/3}$. By the preceding, $\lfloor 2x/3 \rfloor$ is odd for each x in $F_{2/3}$, so $2x/3$ is not integral. Since $F_{2/3}$ is finite, there exists a positive rational ε such that $\lfloor (2/3 - \varepsilon)x \rfloor = \lfloor 2x/3 \rfloor$ for all x in $F_{2/3}$. This implies that $F_{2/3}$ is a subset of $F_{2/3-\varepsilon}$ which is impossible.

Comment. The solution above can be adapted to show that the problem statement still holds, if the condition $r < 1$ in **(2)** is replaced with $r < \delta$, for an arbitrary positive δ. This yields that, if S does not satisfy **(1)**, then there exist *infinitely many* positive rational numbers $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

Solution 2. A finite S clearly satisfies **(2)**, so let S be infinite. If S fails both conditions, so does $S \smallsetminus \{1\}$. We may and will therefore assume that S consists of integers greater than 1. Label the elements of S increasingly $x_1 < x_2 < \cdots$, where $x_1 \geqslant 2$.

We first show that S satisfies **(2)** if $x_{n+1} \geqslant 2x_n$ for all n. In this case, $x_n \geqslant 2^{n-1}x_1$ for all n, so

$$s = \sum_{n\geqslant 1} \frac{1}{x_n} \leqslant \sum_{n\geqslant 1} \frac{1}{2^{n-1}x_1} = \frac{2}{x_1}.$$

If $x_1 \geqslant 3$, or $x_1 = 2$ and $x_{n+1} > 2x_n$ for some n, then $\sum_{x\in F} 1/x < s < 1$ for every finite subset F of S, so S satisfies **(2)**; and if $x_1 = 2$ and $x_{n+1} = 2x_n$ for all n, that is, $x_n = 2^n$ for all n, then every finite subset F of S consists of powers of 2, so $\sum_{x\in F} 1/x \neq 1/3$ and again S satisfies **(2)**.

Finally, we deal with the case where $x_{n+1} < 2x_n$ for some n. Consider the positive rational $r = 1/x_n - 1/x_{n+1} < 1/x_{n+1}$. If $r = \sum_{x\in F} 1/x$ for no finite subset F of S, then S satisfies **(2)**.

We now assume that $r = \sum_{x \in F_0} 1/x$ for some finite subset F_0 of S, and show that S satisfies **(1)**. Since $\sum_{x \in F_0} 1/x = r < 1/x_{n+1}$, it follows that x_{n+1} is not a member of F_0, so

$$\sum_{x \in F_0 \cup \{x_{n+1}\}} \frac{1}{x} = \sum_{x \in F_0} \frac{1}{x} + \frac{1}{x_{n+1}} = r + \frac{1}{x_{n+1}} = \frac{1}{x_n}.$$

Consequently, $F = F_0 \cup \{x_{n+1}\}$ and $G = \{x_n\}$ are distinct finite subsets of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$, and S satisfies **(1)**.