Quiz 2: Predicates

Question 1. Let $p \in \mathbb{Z}^+$ be a positive integer. Translate into predicate logic: "p is a prime number." (Prime numbers have exactly two positive divisors: 1 and the number itself).

Note. Use "infix" notation in your expressions: write $a \mid b$ whenever a divides b; write $x < y$, if x is less than y.

(Write predicate expression; specify domains for quantifiers.)

Question 2. We define $P(n)$ to be true iff n is a prime. For example, $P(2)$, $P(3)$, $P(5)$ etc. are true, but $P(1)$, $P(4)$ etc. are all false.

Translate into predicate logic: "There are arbitrarily large primes", i.e. there is no such thing as the largest prime. (Use just the $P(n)$ and inequaly symbols as predicates.)

(Write predicate expression; specify domains for quantifiers.)

Question 3. You can express the exclusive OR as a *disjunction of conjunctions*:

$$a \oplus b \;\equiv\; (a \land \neg b) \lor (\neg a \land b).$$

Indeed, for $a \oplus b$ to be true, you should either have a true and b false: $(a \land \neg b)$ or a false and b true: $(\neg a \land b)$. Express this truth table as a *disjunction of conjunctions* as well – list all cases when it takes value T:

p	q	r	$E(p,q,r)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

(Write Boolean expression as disjunction of conjunctions)

Question 4. There are altogether 10 children: $\{c_1, \ldots, c_{10}\}$ and 10 hats: $\{h_1, \ldots, h_{10}\}$. Initially, every child c_i has his own hat h_i. When they were about to leave a party, there was an electricity blackout, and they grabbed hats at random (not necessarily their own). Predicate $G(i,j)$ is true iff child c_i grabbed hat h_j.

(a) Write the domain set and the range set of the predicate function G.

(b) Translate this statement into predicate logic: "Nobody grabbed his/her own hat."

Question 5. Translate these two sequences into predicate logic:

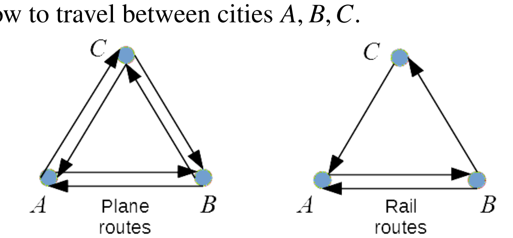
(a) In the interval of real numbers $(0;1)$ there is no smallest number.

(Write predicate expression; specify domains for quantifiers)

(b) For the function $f(x) = x^2 - x$ defined on $(0;1)$ there exists the smallest value.

(Write predicate expression; specify domains for quantifiers)

Question 6. Predicates Plane(x,y), Rail(x,y) show how to travel between cities A, B, C.



They have these following truth tables:

Plane	A	B	C
A	F	T	T
B	T	F	T
C	T	T	F

Rail	A	B	C
A	F	T	F
B	T	F	T
C	T	F	F

Find the truth values of these statements:

(a) $\forall x \exists y, \neg \text{Plane}(x,y)$

(b) $\forall x \forall y \exists z, \text{Plane}(x,z) \land \text{Plane}(z,y)$

(c) $\exists x \exists y \exists z, \text{Rail}(x,y) \land \text{Rail}(y,z) \land \text{Rail}(z,x)$

(d) $\forall x \forall y \exists z, \text{Rail}(x,z) \land \text{Rail}(z,y)$

Note. In the truth tables the first argument is represented by row, the second is represented by column. For example, $\text{Rail}(\mathtt{C},\mathtt{B}) = \mathtt{F}$ (3rd row, 2nd column).

Question 7. Two positive real numbers $x, y \in \mathbb{R}^+$ are given. Translate into predicate logic this statement: "Values x and y are the same, if we round them to two decimal places." Use variable names, arithmetic operations and the floor function ($\lfloor x \rfloor$ is the largest integer not exceeding x).

Note. Rounding x to two decimal places finds the number $\frac{p}{100}$ closest to x. If two are equally close, then round up. (For example, 3.14159 rounds to 3.14; 3.144999 rounds to 3.14, but 3.145 rounds to 3.15.)

(Write predicate expression; specify domains for quantifiers)

Answers

Question 1: Answer: Number $n \in \mathbb{Z}^+$ is a *prime* iff

$$P(n) := \forall d \in \mathbb{Z}^+, \quad (n > 1 \land (d \mid n \rightarrow (d = 1 \lor d = n)))\,.$$

It is essential to require that d is positive: $d \in \mathbb{Z}^+$, since prime numbers may have negative divisors, say $n = 7$ is divisible by four numbers: $d = -7, -1, 1, 7$. It is simpler to deal with only positive d such that $d \mid n$. Also it is essential that $n > 1$; otherwise $n = 1$ would also turn out to be a prime. Which is wrong, since it only has one divisor, not **exactly** two.

Question 2: Answer:

$$\boxed{\forall N \in \mathbb{Z}^+\ \exists p \in \mathbb{Z}^+,\ (p > N \ \land\ P(p))\,.}$$

In the predicate language this is a statement that for any (arbitrarily large) positive integer N there exists an even larger positive integer p such that p is a prime.

Question 3: Answer:

$$\boxed{(p \land q \land r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land \neg r).}$$

For each row, where the truth table has value `T`, we write one more subexpression. Since there are just 3 such rows in the table, there should be a disjunction $(\ldots) \lor (\ldots) \lor (\ldots)$. For example, the 2nd term $(\ldots)$ corresponds to the case, where $p = \mathtt{F}$, $q = \mathtt{T}$, $r = \mathtt{F}$, so we write $(\neg p \land q \land \neg r)$ (and other cases are treated similarly).

Question 4: **(a)** Answer:

$$\boxed{G\ :\ \{1, \ldots, 10\} \times \{1, \ldots, 10\}\ \rightarrow\ \{\mathtt{T}, \mathtt{F}\}.}$$

Note that the predicate $G(i, j)$ acts not on the sets of children and hats, but on their *numbers*. Both i and j change over the same set $\{1, \ldots, 10\}$.
(b) Answer:

$$\boxed{\forall i \in \{1, \ldots, 10\}\ \forall j \in \{1, \ldots, 10\},\ (G(i, j) \rightarrow i \neq j)\,.}$$

Note. From the problem statement, some other predicate logic statements follow. For example, we could also write another predicate logic statement saying that no child grabbed more than 1 hat and every hat was taken. It is not wrong to add such clauses, but there were not part of the requirement of the problem, so the above answer is the simplest possible.

Question 5: **(a)** In the interval of real numbers $(0; 1)$ there is no smallest number:

$$\boxed{\forall x_1 \in (0; 1)\, \exists x_2 \in (0; 1),\ (x_2 < x_1).}$$

In this example the domain is $(0; 1)$. It has no smallest number, since for every positive $x_1 > 0$ there exists an even smaller positive number. For example, we can take $x_2 = x_1/2$.
(b) For the function $f(x) = x^2 - x$ defined on $(0; 1)$ there exists the smallest value.

$$\boxed{\exists x_0 \in (0; 1)\, \forall x \in (0; 1),\ (f(x) \geq f(x_0)).}$$

In this case $f(x_0)$ is the smallest value, which is reached for the argument x_0. By the way, in our specific case this predicate logic statement is true. We can take $x_0 = \frac{1}{2} \in (0; 1)$. This value gives the smallest value to the quadratic function $f(x) = x^2 - x$. But predicate logic statement can be written for any function $f(x) : (0; 1) \rightarrow \mathbb{R}$: If it does not have the minimum on $(0; 1)$, the statement would be false.

Question 6: Answers:
(a) $\forall x \exists y, \neg\text{Plane}(x, y) = \mathtt{True}$.
Indeed, for each city x there is a city y such that there is no plane connection. You can take $x = y$ (no plane goes from $x = A$ $x = B$, or $x = C$ to itself).
(b) $\forall x \forall y \exists z, \text{Plane}(x, z) \land \text{Plane}(z, y) = \mathtt{True}$.
Reasoning with 3 variables and truth tables is not too convenient, but we can reason in plain English: Is it true that for every two cities x and y (possibly $x = y$), there exists a city z, so that one can fly from x to y via z? This is true and visible from the triangle graph.
(c) $\exists x \exists y \exists z, \text{Rail}(x, y) \land \text{Rail}(y, z) \land \text{Rail}(z, x) = \mathtt{True}$.
There exists a circular trip with rail. For example, we can take $(x, y, z) = (A, B, C)$. Then we can go from A to B, B to C and C to A.
(d) $\forall x \forall y \exists z, \text{Rail}(x, z) \land \text{Rail}(z, y) = \mathtt{False}$.
If we take cities $x = A$ and $x = B$, then there exists no two-leg trip from A to B going through some city z.

Question 7: Answer:

$$\boxed{\left\lfloor 100x + \frac{1}{2} \right\rfloor = \left\lfloor 100y + \frac{1}{2} \right\rfloor.}$$

The fact that two numbers (for example, expressing money) round to exactly the same number of euros and eurocents, can be written without any quantifiers. It is a predicate: $P(x, y)$, with $P\ :\ \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \{\mathtt{T}, \mathtt{F}\}$.
Note. We could also write this with quantifiers. For example: for the given x, y there exists a non-negative integer number $z \in \mathbb{Z}^{0+}$ (z expresses the number of eurocents) such that both $|100x - z| \leq 0.5$ and $|100y - z| \leq 0.5$. But writing such an expression would be longer, since we need to sort out the special case, when exactly one half of a eurocent ($x = 0.005$) is rounded up.