Number Theory Problems by Grade

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Abstract: In this article we study olympiad problems in number theory (NT) offered in Latvia and Estonia. We compare the olympiad papers by grade and sort them by these attributes:

- **skills** - ability to perform certain groups of tasks,
- **question type** - what is the required result. Is it a proof for a general case, a proof by providing a counterexample, finding all numbers satisfying some statement, etc.),
- **genre** a narrower category that often implies the question type such as integer equations, analyzing adversarial games, filling a table,
- **concepts** appearing in the problem (for example, problems mentioning prime numbers, greatest common divisors, floor function).

Introduction

Secondary schools and gymnasiums in Latvia and Estonia share many similarities. * Primary education (Grades 1-6, typically for 7-13 year olds) is a prerequisite for secondary education. Some olympiads and competitions target this age group (for example, Latvian Open Mathematical Olympiad includes grades 5-6). This study only includes problems starting from Grade 7. * Secondary education has 2 levels - lower secondary is Grades 7-9 (ages 14-16), upper secondary is Grades 10-12 (ages 17-19). * Various NT concepts (divisibility, remainders, prime numbers, lcd and gcm, decimal notation) are already covered in Grades 5-6, so the "theory" part in the number theory problems is almost the same for Grades 7-12. * Olympiads in Latvia and in Estonia follow a similar pattern. It starts with the regional tour followed by the national tour (both events limit the number of participants; in particular, only the highest scoring students from the regional tours may enter the national tour). In addition, there is the open tour (in Estonia there are even two open tours per year), where everyone can participate.

Existing guidelines:

Most olympiad papers in Latvian and Estonian olympiads (regional, national and open tours) have 5 problems. On average, each olympiad paper has 1 NT problem (others are on geometry, algebera, combinatorics and/or "school problems").

1. Junior grades do not get NT problems that use some advanced concepts that are mostly covered after Grade 9 (geometrical progressions, exponential functions or logarithms, floor function $\lfloor x \rfloor$).
2. Problems involving ellipsis or sum notation such as $F(1) + F(2) + \ldots + F(n)$ or $\sum_{k=1}^{n} F(k)$.
3. Steps that introduce a higher level of abstraction. For example, problem using the "regular" rule of divisibility by 9 (9 divides a number if and only if 9 divides the sum of its digits) is fine for all grades 7-12, but a generalization (each number n is congruent to the sum of its digits modulo 9) is only used for grades 10-12.

The goal of this paper is to find empirical evidence of other differences caused by conscious or unconscious choices of the problem selection committees in both countries. This would be necessary to plan curricula for additional training in the number theory for different age groups.

Metadata for Olympiad Problems

Skills for "Junior" and "Senior" NT Problems

Differences by Question Type

- **Find.All** - find all objects satisfying the condition and prove that there are no others. In some problems the question implicitly assumes that the number was unique (for example, "Which was the number guessed by Joe?", EE.LVT.2016.noorem.2). Still, a full solution would need to prove the uniqueness of the answer. The objects to find may also differ. They may be integer numbers, pairs or n-tuples of integer numbers, series of moves to achieve some result, sequences or solutions to functional equations.
- `Prove.Exists` (typically this requires a constructive proof, but the question is about the fact of existence rather than finding an example.) Sometimes the object that should exist is a natural or integer number. A pair of numbers or even a path.
- `ProveDisprove.Exists`

Differences by Genre

Differences by Concepts

5 categories - only junior concepts; predominantly junior, mixed, predominantly senior, only senior.

Conclusions

1. NT problems offered to senior students (grades 10-12) need more skills compared to the NT problems offered to junior students (grades 7-9); in fact the former skills are a proper superset.
2. Question types and genres differ more strongly by country (Latvia vs Estonia) than by age. There are certain "national traditions" how the olympiad problems in NT are formulated.
3. The concepts used in junior grades are partly replaced

Bibliography