# Homework 1

Discrete Structures
Due Tuesday, January 12, 2021

**Submit each question separately in .pdf format only**

1. Complete the following truth table.

| $P$ | $Q$ | $R$ | $P \wedge Q \wedge R$ | $(P \vee Q) \wedge R$ | $P \vee (Q \wedge R)$ | $P \rightarrow (Q \vee R)$ | $(P \wedge Q) \leftrightarrow (Q \vee R)$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | | | | | |
| $T$ | $T$ | $F$ | | | | | |
| $T$ | $F$ | $T$ | | | | | |
| $T$ | $F$ | $F$ | | | | | |
| $F$ | $T$ | $T$ | | | | | |
| $F$ | $T$ | $F$ | | | | | |
| $F$ | $F$ | $T$ | | | | | |
| $F$ | $F$ | $F$ | | | | | |

2. Rewrite the following statements using only symbols (logical quantifiers, connectives, and the symbols $\mathbf{N}, \mathbf{Q}, \mathbf{R}$).

   (a) There is a largest natural number.

   (b) The sum of two rational numbers is a rational number.

   (c) The sum of a rational and an irrational number is a rational number.

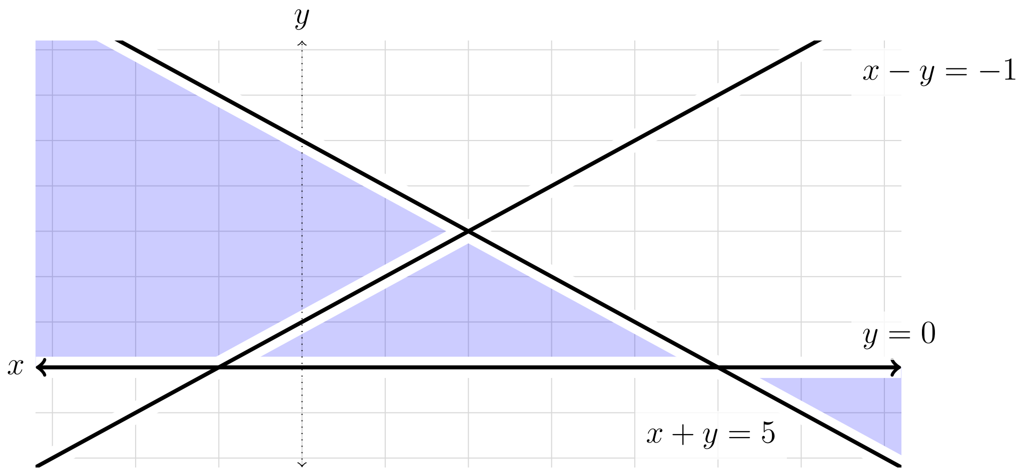
   (d) Between any two rational numbers there is another rational number.

3. Let $P, Q, T, F$ be logical propositions, where $T$ is always true, but $F$ is always false. Prove that the following propositions are tautologies. Indicate every logical equivalence or definition you are using.

   (a) $T \wedge (((P \vee (P \wedge Q)) \vee \neg P) \vee (\neg(\neg Q \vee Q)))$

   (b) $\neg(P \leftrightarrow \neg P)$

4. Consider three lines in the plane, which decompose it into 7 parts.




Let $(a, b)$ be a point in the plane. You are given three propositions about $a$ and $b$:

- $P$ asserts that $a + b \geqslant 5$
- $Q$ asserts that $a - b \leqslant -1$
- $R$ asserts that $b \geqslant 0$

Using these propositions, answer the questions below.

(a) Write a compound proposition that is true whenever $(a, b)$ belongs to one of the three shaded areas of the diagram. The shaded areas **do not** contain their bounding lines.

(b) Write a compound proposition that is logically satisfiable, but no value of $(a, b) \in \mathbf{R}^2$ makes it true.

*Note 1.* By $\mathbf{R}^2$ we denote the set of pairs of real numbers; namely, $a$ and $b$ are both real.
*Note 2.* On a border the truth values of $P, Q, R$ may be indistinguishable from truth values in an adjacent region to the "wrong" side of the line. Everywhere in this problem you can assume that all the points $(a, b)$ only belong to the internal regions, they never fall on any of the three lines.

5. Consider the following proposition, which joins 10 atomic propositions $P_i$ with exclusive or.

$$P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8 \oplus P_9 \oplus P_{10} \tag{1}$$

(a) How many terms does its disjunctive normal form (DNF) have? Justify that your DNF contains the smallest possible number of terms.

(b) How many terms does its conjunctive normal form (CNF) have?

(c) Write the equation (1) using as few logical connectors ($\vee$, $\wedge$, $\neg$) as possible. You may use the connectors $\vee$, $\wedge$, $\neg$, and parentheses in any way, but your statement does not need to be in CNF or in DNF. Explain, why your expression is logically equivalent to (1).

*Note.* A "term" in a DNF is any conjunction of $P_i$ or their negations participating in the long disjunction. For example, the following DNF:

$$(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E) \vee C \vee (A \wedge \neg E)$$

has 4 terms: $(A \wedge \neg B \wedge \neg C)$, $(\neg D \wedge E)$, $(C)$ and $(A \wedge \neg E)$.