N5. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t. \tag{$*$}$$

Is it possible that both xy and zt are perfect squares?

Answer: No.

Solution 1. Arguing indirectly, assume that $xy = a^2$ and $zt = c^2$ with $a, c > 0$.

Suppose that the number $x + y = z + t$ is odd. Then x and y have opposite parity, as well as z and t. This means that both xy and zt are even, as well as $xy - zt = x + y$; a contradiction. Thus, $x + y$ is even, so the number $s = \frac{x+y}{2} = \frac{z+t}{2}$ is a positive integer.

Next, we set $b = \frac{|x-y|}{2}$, $d = \frac{|z-t|}{2}$. Now the problem conditions yield

$$s^2 = a^2 + b^2 = c^2 + d^2 \tag{1}$$

and

$$2s = a^2 - c^2 = d^2 - b^2 \tag{2}$$

(the last equality in (2) follows from (1)). We readily get from (2) that $a, d > 0$.

In the sequel we will use only the relations (1) and (2), along with the fact that a, d, s are positive integers, while b and c are nonnegative integers, at most one of which may be zero. Since both relations are symmetric with respect to the simultaneous swappings $a \leftrightarrow d$ and $b \leftrightarrow c$, we assume, without loss of generality, that $b \geqslant c$ (and hence $b > 0$). Therefore, $d^2 = 2s + b^2 > c^2$, whence

$$d^2 > \frac{c^2 + d^2}{2} = \frac{s^2}{2}. \tag{3}$$

On the other hand, since $d^2 - b^2$ is even by (2), the numbers b and d have the same parity, so $0 < b \leqslant d - 2$. Therefore,

$$2s = d^2 - b^2 \geqslant d^2 - (d-2)^2 = 4(d-1), \qquad \text{i.e.,} \qquad d \leqslant \frac{s}{2} + 1. \tag{4}$$

Combining (3) and (4) we obtain

$$2s^2 < 4d^2 \leqslant 4\left(\frac{s}{2} + 1\right)^2, \qquad \text{or} \qquad (s-2)^2 < 8,$$

which yields $s \leqslant 4$.

Finally, an easy check shows that each number of the form s^2 with $1 \leqslant s \leqslant 4$ has a unique representation as a sum of two squares, namely $s^2 = s^2 + 0^2$. Thus, (1) along with $a, d > 0$ imply $b = c = 0$, which is impossible.

Solution 2. We start with a complete description of all 4-tuples (x, y, z, t) of positive integers satisfying $(*)$. As in the solution above, we notice that the numbers

$$s = \frac{x+y}{2} = \frac{z+t}{2}, \quad p = \frac{x-y}{2}, \quad \text{and} \quad q = \frac{z-t}{2}$$

are integers (we may, and will, assume that $p, q \geqslant 0$). We have

$$2s = xy - zt = (s+p)(s-p) - (s+q)(s-q) = q^2 - p^2,$$

so p and q have the same parity, and $q > p$.

Set now $k = \frac{q-p}{2}$, $\ell = \frac{q+p}{2}$. Then we have $s = \frac{q^2-p^2}{2} = 2k\ell$ and hence

$$\begin{aligned} x &= s+p = 2k\ell - k + \ell, \qquad & y &= s-p = 2k\ell + k - \ell, \\ z &= s+q = 2k\ell + k + \ell, \qquad & t &= s-q = 2k\ell - k - \ell. \end{aligned} \tag{5}$$

Recall here that $\ell \geqslant k > 0$ and, moreover, $(k, \ell) \neq (1,1)$, since otherwise $t = 0$.

Assume now that both xy and zt are squares. Then $xyzt$ is also a square. On the other hand, we have

$$\begin{aligned} xyzt &= (2k\ell - k + \ell)(2k\ell + k - \ell)(2k\ell + k + \ell)(2k\ell - k - \ell) \\ &= \left(4k^2\ell^2 - (k-\ell)^2\right)\left(4k^2\ell^2 - (k+\ell)^2\right) = (4k^2\ell^2 - k^2 - \ell^2)^2 - 4k^2\ell^2. \end{aligned} \tag{6}$$

Denote $D = 4k^2\ell^2 - k^2 - \ell^2 > 0$. From (6) we get $D^2 > xyzt$. On the other hand,

$$\begin{aligned} (D-1)^2 = D^2 - 2(4k^2\ell^2 - k^2 - \ell^2) + 1 &= (D^2 - 4k^2\ell^2) - (2k^2-1)(2\ell^2-1) + 2 \\ &= xyzt - (2k^2-1)(2\ell^2-1) + 2 < xyzt, \end{aligned}$$

since $\ell \geqslant 2$ and $k \geqslant 1$. Thus $(D-1)^2 < xyzt < D^2$, and $xyzt$ cannot be a perfect square; a contradiction.

Comment. The first part of Solution 2 shows that all 4-tuples of positive integers $x \geqslant y$, $z \geqslant t$ satisfying $(*)$ have the form (5), where $\ell \geqslant k > 0$ and $\ell \geqslant 2$. The converse is also true: every pair of positive integers $\ell \geqslant k > 0$, except for the pair $k = \ell = 1$, generates via (5) a 4-tuple of positive integers satisfying $(*)$.