Dasgy partial progress

**N2.** Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

($i$) Each number in the table is congruent to 1 modulo $n$;

($ii$) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to $n$ modulo $n^2$.

Let $R_i$ be the product of the numbers in the $i^{\text{th}}$ row, and $C_j$ be the product of the numbers in the $j^{\text{th}}$ column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo $n^4$.

**Solution 1.** Let $A_{i,j}$ be the entry in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column; let $P$ be the product of all $n^2$ entries. For convenience, denote $a_{i,j} = A_{i,j} - 1$ and $r_i = R_i - 1$. We show that

$$\sum_{i=1}^{n} R_i \equiv (n-1) + P \pmod{n^4}. \tag{1}$$

Due to symmetry of the problem conditions, the sum of all the $C_j$ is also congruent to $(n-1)+P$ modulo $n^4$, whence the conclusion.

By condition ($i$), the number $n$ divides $a_{i,j}$ for all $i$ and $j$. So, every product of at least two of the $a_{i,j}$ is divisible by $n^2$, hence

$$R_i = \prod_{j=1}^{n}(1+a_{i,j}) = 1+\sum_{j=1}^{n} a_{i,j} + \sum_{1 \leqslant j_1 < j_2 \leqslant n} a_{i,j_1}a_{i,j_2} + \cdots \equiv 1+\sum_{j=1}^{n} a_{i,j} \equiv 1-n+\sum_{j=1}^{n} A_{i,j} \pmod{n^2}$$

for every index $i$. Using condition ($ii$), we obtain $R_i \equiv 1 \pmod{n^2}$, and so $n^2 \mid r_i$.

Therefore, every product of at least two of the $r_i$ is divisible by $n^4$. Repeating the same argument, we obtain

$$P = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n}(1+r_i) \equiv 1 + \sum_{i=1}^{n} r_i \pmod{n^4},$$

whence

$$\sum_{i=1}^{n} R_i = n + \sum_{i=1}^{n} r_i \equiv n + (P-1) \pmod{n^4},$$

as desired.

**Comment.** The original version of the problem statement contained also the condition

($iii$) The product of all the numbers in the table is congruent to 1 modulo $n^4$.

This condition appears to be superfluous, so it was omitted.

**Solution 2.** We present a more straightforward (though lengthier) way to establish (1). We also use the notation of $a_{i,j}$.

By condition ($i$), all the $a_{i,j}$ are divisible by $n$. Therefore, we have

$$P = \prod_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j}) \equiv 1 + \sum_{(i,j)} a_{i,j} + \sum_{(i_1,j_1),\,(i_2,j_2)} a_{i_1,j_1}a_{i_2,j_2} + \sum_{(i_1,j_1),\,(i_2,j_2),\,(i_3,j_3)} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \pmod{n^4},$$

where the last two sums are taken over all unordered pairs/triples of *pairwise different* pairs $(i,j)$; such conventions are applied throughout the solution.

Similarly,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \prod_{j=1}^{n} (1 + a_{i,j}) \equiv n + \sum_i \sum_j a_{i,j} + \sum_i \sum_{j_1, j_2} a_{i,j_1} a_{i,j_2} + \sum_i \sum_{j_1, j_2, j_3} a_{i,j_1} a_{i,j_2} a_{i,j_3} \pmod{n^4}.$$

Therefore,

$$\begin{aligned} P + (n-1) - \sum_i R_i \equiv \sum_{\substack{(i_1,j_1),(i_2,j_2)\\ i_1 \neq i_2}} a_{i_1,j_1} a_{i_2,j_2} &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1 \neq i_2 \neq i_3 \neq i_1}} a_{i_1,j_1} a_{i_2,j_2} a_{i_3,j_3} \\ &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1 \neq i_2 = i_3}} a_{i_1,j_1} a_{i_2,j_2} a_{i_3,j_3} \pmod{n^4}. \end{aligned}$$

We show that in fact each of the three sums appearing in the right-hand part of this congruence is divisible by $n^4$; this yields (1). Denote those three sums by $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ in order of appearance. Recall that by condition *(ii)* we have

$$\sum_j a_{i,j} \equiv 0 \pmod{n^2} \qquad \text{for all indices } i.$$

For every two indices $i_1 < i_2$ we have

$$\sum_{j_1} \sum_{j_2} a_{i_1,j_1} a_{i_2,j_2} = \Big(\sum_{j_1} a_{i_1,j_1}\Big) \cdot \Big(\sum_{j_2} a_{i_2,j_2}\Big) \equiv 0 \pmod{n^4},$$

since each of the two factors is divisible by $n^2$. Summing over all pairs $(i_1, i_2)$ we obtain $n^4 \mid \Sigma_1$.

Similarly, for every three indices $i_1 < i_2 < i_3$ we have

$$\sum_{j_1} \sum_{j_2} \sum_{j_3} a_{i_1,j_1} a_{i_2,j_2} a_{i_3,j_3} = \Big(\sum_{j_1} a_{i_1,j_1}\Big) \cdot \Big(\sum_{j_2} a_{i_2,j_2}\Big) \cdot \Big(\sum_{j_3} a_{i_3,j_3}\Big)$$

which is divisible even by $n^6$. Hence $n^4 \mid \Sigma_2$.

Finally, for every indices $i_1 \neq i_2 = i_3$ and $j_2 < j_3$ we have

$$a_{i_2,j_2} \cdot a_{i_2,j_3} \cdot \sum_{j_1} a_{i_1,j_1} \equiv 0 \pmod{n^4},$$

since the three factors are divisible by $n$, $n$, and $n^2$, respectively. Summing over all 4-tuples of indices $(i_1, i_2, j_2, j_3)$ we get $n^4 \mid \Sigma_3$.