Quiz 11: Graphs

Question 1. Let $G = (V, E)$ be a graph, where V is the set of all positive divisors of 144 (including 1 and 144 itself). Two different vertices d_1, d_2 are connected by an edge iff one of the numbers divides another ($d_1 \mid d_2$ or $d_2 \mid d_1$). Find the number of vertices $|V|$ and the number of edges $|E|$ in this graph.
Write two comma-separated integers.

Question 2. How long is the longest simple circuit in W_{20}? (A simple circuit is a circular path that may visit vertices multiple times, but does not contain any edge more than once.)
Write a positive integer.

Question 3. Let G be a planar connected graph with 60 vertices, each vertex has degree 3. How many regions are there in G?
Write a positive integer.

Question 4. This is an adjacency matrix for some graph:

$$M_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

It is known that G is a planar graph. Find the number of vertices $|V|$, number of edges $|E|$ and the number of regions $|R|$ for this graph.
Write 3 comma-separated integers.

Question 5 (Dudeney2016, Prob.434), "536 Puzzles".

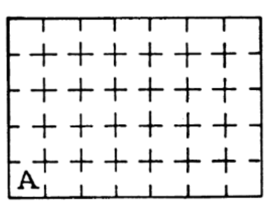

Figure 1. A weighted graph.

A prisoner currently is in the cell "A" (Figure 1). He has to visit each prison cell no more than once and return back to the cell "A". What is the largest number of prison cells that can be visited in this way?
(Visiting each cell once does not contradict with the requirement to return back to "A" – the prisoner uses a circular path between the rooms: every room on the path, including "A", is entered once and left once. We want to know the maximum length of this path.)
Write a positive integer.
Note. You may also want to prove to yourself that the number is the largest possible.

Question 6 There is a bipartite graph $G = (V, E)$ with exactly $|V| = 17$ vertices. (A graph is *bipartite*, if the set of vertices V can be split into two parts X, Y so that all edges are between a vertex in X and a vertex in Y.) Find the largest possible number of edges in such a graph.
Write a positive integer.

Question 7 Verify, if these statements are true. A simple undirected graph is called a *cubic* graph, if every vertex has degree 3.
(A) There exists a cubic graph with 7 vertices.
(B) There exists a cubic graph with 6 vertices that is not isomorphic to $K_{3,3}$.
(C) There exists a cubic graph with 8 edges.
Write a sequence of 3 comma-separated letters (e.g. `T,T,T` or `F,F,F`).

Question 8. Verify, if these statements are true:
(A) There exists a simple directed graph with indegrees $0, 1, 2, 4, 5$ and outdegrees $0, 3, 3, 3, 3$. (A graph is *simple*, if it is not a *multigraph* – there is no more than one edge (u, v) for any vertices u, v.)
(B) There exists a connected undirected simple planar graph with 5 regions and 8 vertices, each vertex has degree 3.
(C) There exists a connected undirected simple planar graph with 8 regions and 6 vertices, each region is surrounded with 3 edges.
Write a sequence of 3 comma-separated letters (e.g. `T,T,T` or `F,F,F`).

Question 9. Use Dijkstras Algorithm to find the shortest paths from the source vertex s to all other vertices t, x, y, z (Figure 2). The length of a path is obtained by adding the weights of the directed edges.

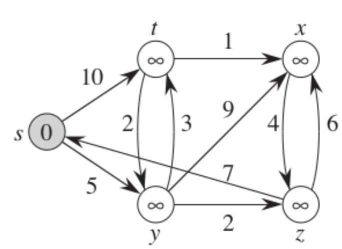

Figure 2. A weighted graph.

Write 4 comma-separated numbers – the shortest paths to the vertices t, x, y, z respectively.
Note. Dijkstra's algorithm (Rosen2019, p.747) initializes the set of vertices S that we know the shortest

paths to (initially it only contains the source vertex $S = \{s\}$; the distance from s to itself is 0; initialize the distances to all the other vertices to ∞). At every step consider all the edges that go from the set S to $\overline{S}$, i.e. to the vertices where we still do not know the shortest paths. Update all the shortest paths (if crossing from the set S to $\overline{S}$ finds a shorter path than ∞ or the currently known minimum length, then decrease the estimate for this vertex). Finally, add the minimum vertex from $\overline{S}$ to S. Repeat the steps until all vertices are added to S and all the shortest path estimates have reached their smallest values.

Question 10 (Dudeney2016, Prob.423), "536 Puzzles". A man starting from the town A, has to inspect all the roads shown from town to town (Figure 3). Their respective lengths, 13, 12, and 5 miles are all shown. What is the shortest possible route he can adopt, ending his journey wherever he likes?

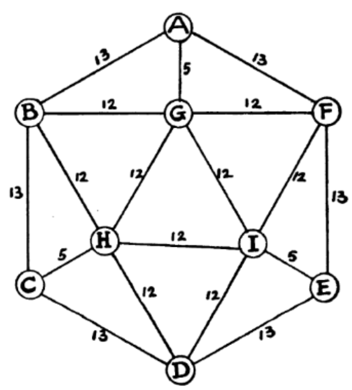

Figure 3. Path with repetitions

Write an integer – the length of the shortest route.
Note. This graph obviously has no Euler path (since there are more than 2 vertices with odd degrees). The problem is to find a path that is likely **not** simple (uses the same edge several times), but that includes every edge shown and the total of weights is minimal.

Question 11
Somebody placed 24 chess rooks on a 8×8 chessboard as shown in Figure 4 (each horizontal and each vertical has exactly 3 rooks).
We imagine that this chess-board defines a bipartite graph between the set of all verticals $X = \{A, B, C, D, E, F, G, H\}$ and the set of all horizontals $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Any rook defines an edge between these two sets. For example, the rook $C8$ defines an edge $(C, 8)$.
Find a subset of verticals $V \subseteq X$ such that $|V| = 3$, but the neighbor set has size $|N(V)| = 5$.
Write 3 comma-separated letters in your answer (the vertices from V). It is sufficient to write just one possible answer, if there are many.

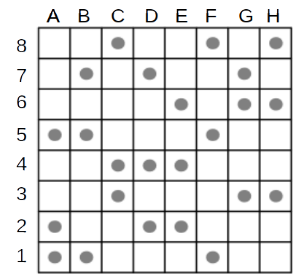

Figure 4. Path with repetitions

Note 1. For example, the answer $\mathtt{F}, \mathtt{G}, \mathtt{H}$ does not work, since the set of vertices $\{\mathtt{F}, \mathtt{G}, \mathtt{H}\} \subseteq X$ is neighboring with a set of six vertices $\{1, 3, 5, 6, 7, 8\} \subseteq Y$, i.e. the rooks on these three verticals attack six horizontals, but not five.
Note 2. For the condition of the Hall's marriage theorem we need the inequality $|V| \leq |N(V)|$ for **every** $V \subseteq X$. You could prove to yourself that it is always satisfied (also for all the other placements of 24 rooks where each horizontal and each vertical has 3 rooks).
Note 3. Interpret for yourself what does a "perfect matching" between the sets X and Y mean in this subject-area with a chessboard and rooks.

Answers

Question 1 Answer: $15, 75$
Since $144 = 2^4 \cdot 3^2$, number 144 has $(4+1)(2+1) = 15$ divisors (the number of ways to pick powers $2^a \cdot 3^b$). Their Hasse diagram is shown in Figure 5 (transitive closure has many more arrows that are not shown).

For each vertex d_1 we calculate the number of other vertices that are divisible by d_1 (i.e. can be reached by following one or more arrows in the Hasse diagram). Adding all those numbers gives the number of edges.

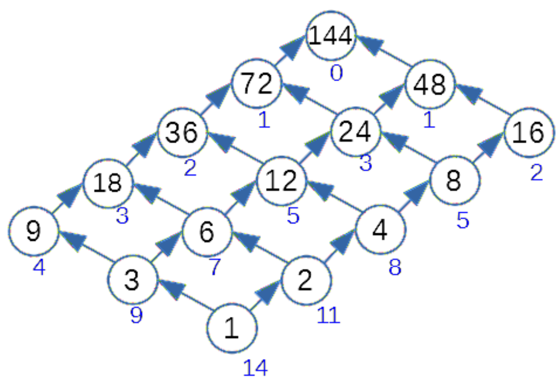

Figure 5. Divisibility Hasse diagram.

Question 2 Answer 30
All the vertices on the regular 20-gon have degree equal to 3. This means that we have to drop at least 10 edges before we get a simple path (because any simple path adds only even number to the degree of any vertex in a graph). Initially W_{20} has $20+20 = 40$ edges. After deleting 10 edges (every other edge on the perimeter of the 20-gon), we are left with 30 edges.

Question 3 Answer 32
60 vertices (having degree 3 each would create the sum of all degrees equal to $60 \cdot 3 = 180$. The number of edges equals one half of that; so $|E| = 90$. The number of regions can be computed using Euler's formula: $|V| - |E| + |R| = 2$ (in our case $60 - 90 + |R| = 2$; therefore $|R| = 32$.

One example of such graph is *truncated icosahedron*, see https://bit.ly/2WaSW6I, but there may be many others that are not isomorphic to it. Still, all of them would have the same number of regions due to Euler's formula.

Question 4 Answer: `9,17,10`
Number of vertices equals the size of the matrix 9×9, so $|V| = 9$. The number of edges is one half of all the 1s written in the adjacency matrix; therefore $|E| = 17$. Since we can assume that the graph G is planar, it satisfies Euler's formula:

$$|V| - |E| + |R| = 2.$$

Therefore the number of regions $|R| = 10$.
Graph (shown without edge intersections as a planar graph) is visible on Figure 6. In this picture we can simply count vertices, edges and regions. But it is usually time-consuming to create such pictures (and to verify that they match the adjacency matrix).

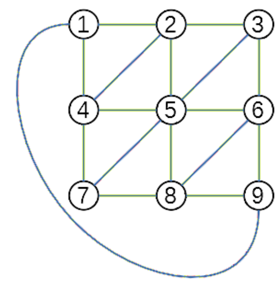

Figure 6. A planar graph

Question 5 Answer 34
It is easy to build a path that visits all rooms except one. There cannot be a circular path with exactly 35 steps – one can use checkerboard pattern (color all cells in black and white). Every step switches the color to the opposite; after exactly 35 color switches the color would be opposite – the path cannot return back to cell A.

Question 6 Answer: 72
We know that the sum of two sizes $|X| + |Y| = 17$, and the maximum number of edges is $|X| \cdot |Y|$. The greatest possible product of two numbers is when they are closest to each other: $8 \cdot 9 = 72$. (We can try out all combinations of two numbers that add up to 17 to see that this is the largest one.)
We can write the following algebraic inequalities:

$$|X| \cdot |Y| \leq \left(\frac{|X| + |Y|}{2} \right)^2,$$

$$4|X| \cdot |Y| \leq (|X| + |Y|)^2,$$

$$4|X| \cdot |Y| \leq |X|^2 + |Y|^2 + 2|X| \cdot |Y|,$$

$$0 \leq |X|^2 + |Y|^2 = 2|X| \cdot |Y| = (|X| + |Y|)^2.$$

From the first inequality we imply that $|X| \cdot |Y| \leq (17/2)^2 = 72.25$. Since the number of edges cannot be fractional, 72 is indeed the largest number.

Question 7 Answer: `FTT`
(A) False. No cubic graph can have odd number of vertices (the sum of all degrees of all vertices should be even – twice the number of edges).

(B) True. $K_{3,3}$ is bipartite graph (it does not contain any "triangles": three vertices that are all mutually connected). But the graph on Figure 7 is not bipartite (so it is not isomorphic to $K_{3,3}$.

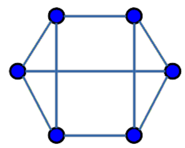

Figure 7. Cubic graph with 6 vertices.

(C) True. You can draw a regular octagon and add all the long diagonals. Now every vertex is adjacent with three vertices (both neighbors and the opposite one).

Question 8 Answer: `FFT`
(A) False. In a simple directed graph with 5 vertices, an indegree 5 means that all vertices should point arrows to the given vertex (including the vertex itself). But in this case there cannot be any vertices with outdegree 0.
(B) False. A graph with 8 vertices of degree 3 means that it has $\frac{8 \cdot 3}{2} = 12$ edges. According to Euler's formula, the number of regions should be $R = 2 + E - V = 6$. Therefore such a graph should have 6 (not 5) regions.
(C) True. Such graph exists. For example Octahedron - see https://bit.ly/3f1gnrG.

Question 9 Answer: `8,9,5,7`

Set S	Weights of $\overline{S}$	Added to S
$\{s\}$	$w(t,x,y,z) = (10,\infty,5,\infty)$	y (min path 5)
$\{s,y\}$	$w(t,x,z) = (8,\infty,7)$	z (min path 7)
$\{s,y,z\}$	$w(t,x) = (8,9)$	t (min path 8)
$\{s,t,y,z\}$	$w(x) = 9$	x (min path 9)

Question 10 Answer: `211`
There are altogether 6 vertices with odd degrees (one of them is A). If we start our travel in A and end it in any other vertex with odd degree (say, in G), then there are four more vertices with odd degrees. By adding edges (C,H) and (I,E) two times, we can build the required path (each of these edges has weight 5). Therefore the full length of the path is the sum of all weights:

$$3 \cdot (12 + 12 + 12) + 3 \cdot (13 + 13 + 5) + (5 + 5).$$

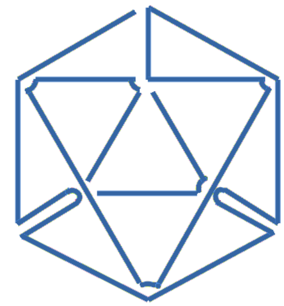

Figure 8. Path with Repetitions (Solved).

Question 11 Answer: `"A,B,D"`, `"A,B,F"`, `"C,E,H"`, `"C,G,H"`, `"D,E,G"`
There are five ways to select the verticals (any one of them is correct). Since the rooks (as edges linking horizontals with verticals) satisfy the Hall's Marriage theorem, there exists a perfect matching: One can select 8 rooks (out of the 24) so that each rook has its own horizontal and its own vertical. In other words, they do not attack each other.

We can start searching this perfect matching using "backtracking" – first try to pick the minimum possible horizontal in each vertical (avoiding any attacking position). If this leads in a dead end, then start moving the rooks that have been placed last. This very quickly leads to a solution (Figure 9). There are also many other perfect matchings.

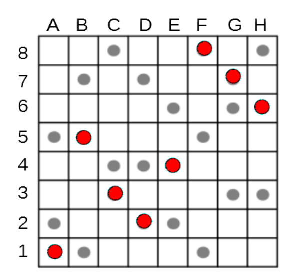

Figure 9. 8 selected rooks shown red