

Final Exam, 2020-04-23

Question 1.

By U we denote the set of all positive integers between 1 and 120. This is the *universe* in which we define several subsets:

$$\begin{cases} A = \{x \in U \mid 2 \mid x\}, \\ B = \{x \in U \mid 3 \mid x\}, \\ C = \{x \in U \mid 5 \mid x\}, \\ X = \{x \in U \mid 2 \mid x \vee 3 \mid x\}, \\ Y = \{x \in U \mid (3 \mid x \wedge 5 \mid x) \vee \neg(2 \mid x)\}. \end{cases}$$

- (A) Express X using the sets A, B, C (using set union \cup , set intersection \cap , set complement $\overline{}$ operations).
 (B) Express Y using the sets A, B, C in a similar way.
 (C) Find $|X|$ - the size of the set X .
 (D) Find $|Y|$ - the size of the set Y .

Question 2.

Let A and B be sets with sizes $|A| = 8$ and $|B| = 5$ and $|A \cap B| = 3$.

Calculate the largest and the smallest possible values for each of the following set sizes:

- (A) $|A \cup B|$.
 (B) $|A \times (B \times B)|$.
 (C) $|\mathcal{P}(\mathcal{P}(A \cap B))|$ - the powerset of a powerset of $A \cap B$.
 (D) $|A \oplus B|$ - the symmetric difference of the sets A and B .

Question 3.

Consider the following recurrent sequence:

$$\begin{cases} a_0 = 3 \\ a_1 = 4 \\ a_{n+2} = 5a_{n+1} - 6a_n, \text{ if } n \geq 0 \end{cases}$$

Assume that b_n is another sequence satisfying the recurrence rule

$$b_{n+2} = 5b_{n+1} - 6b_n, \text{ if } n \geq 0$$

(The first two members b_0, b_1 are not known.)

- (A) Write the first 6 members of this sequence (a_0, \dots, a_5).
 (B) Write the characteristic equation for this sequence.
 (C) Write the general expression for an arbitrary sequence b_n satisfying the recurrent expression as a sum of two geometric progressions (you can leave unknown coefficients in your answer; just explain which ones they are).
 (D) Write the formula to compute a_n (that would satisfy the initial conditions $a_0 = 3$ and $a_1 = 4$).

Question 4.

Consider this code snippet in Python:

```
n = 1000
sum = 0
for i in range(1, n*n+1):
    for j in range(1, i+1):
        sum += i % j
```

And a similar one in R:

```
n <- 1000
sum <- 0
for (i in 1:(n*n)) {
    for (j in 1:i) {
        sum <- sum + i %% j
    }
}
```

- (A) Explain in human language what this algorithm does.
 (B) Denote by $f(n)$ the number of times the variable 'sum' is incremented. Write the Big-O-Notation for $f(n)$. Find a function $g(n)$ such that $f(n)$ is in $O(g(n))$. (If there are multiple functions, pick the one with the slowest growth.)
 (C) Express the function $f(n)$ precisely - how many times 'sum' is incremented in terms of variable n .

Question 5.

Let A be the set of all positive divisors of the number 120 (including 1 and 120 itself).

- (A) What is the multiplication of all numbers in the set A ?
 (B) Express this number as the product of prime powers.

Question 6.

Define the following binary relationship on the set of integer numbers \mathbb{Z} : We say that aRb (numbers $a, b \in \mathbb{Z}$ are in the relation R) iff

$$\begin{cases} a - b \equiv 0 \pmod{11} \\ a - b \equiv 0 \pmod{12} \\ a - b \equiv 0 \pmod{13} \end{cases}$$

Item	Statement	True or False?
(A)	R is reflexive	
(B)	R is symmetric	
(C)	R is antisymmetric	
(D)	R is transitive	
(E)	aRb iff $a = b$	

For all items where you answered 'FALSE', specify a counterexample (values for some numbers that would make the condition true, but the conclusion false). If the statement was true, write "none".

- (A) counterexample: ...
 (B) counterexample: ...
 (C) counterexample: ...
 (D) counterexample: ...
 (E) counterexample: ...

Question 7.

Four people A, B, C, D each has his own hat. After the meeting they leave their building in a hurry, everyone grabs some hat at random so that all $4!$ permutations of the hats have equal probabilities.

Let the random variable X denote the number of hats that were picked up correctly. (For example, if the hat assignment is this: $(A \rightarrow A, B \rightarrow B, C \rightarrow D, D \rightarrow C)$, then $X = 2$, because two people got their own hats.)

(A) Find $E(X)$ - the expected value of X .

(B) Find $V(X)$ - the variance of X .

Question 8.

There was a crooked man who had a crooked 1 euro coin. On lucky days it would flip the *heads* with probability $p = \frac{2}{3}$, and the *tails* with probability $p = \frac{1}{3}$, but on unlucky days it was the opposite ($p(\text{heads}) = \frac{1}{3}$, but $p(\text{tails}) = \frac{2}{3}$). There were equal probabilities of $\frac{1}{2}$ for lucky and unlucky days.

One morning he flipped the coin 5 times and altogether got three *heads* and two *tails*.

Let us introduce the following events:

- E (evidence): Five coin tosses result in three *heads* and two *tails*.
- H (hypothesis): The current day is lucky.

(A) Find $P(E|H)$ - the conditional probability of E given that the day is lucky.

(B) Find $P(E|H) \cdot P(H)$ - the probability that the day is lucky and E happens.

(C) Find $P(E|\bar{H})$ - the conditional probability of E given that the day is not lucky.

(D) Find $P(E|\bar{H}) \cdot P(\bar{H})$ - the probability that the day is unlucky and E happens.

(E) Find $P(E)$ - as the sum of two probabilities (E happened on a lucky day and also E happened on unlucky day).

(F) Find the conditional probability $P(H|E)$ - the likelihood that the crooked man has a lucky day, given that the event E has happened.

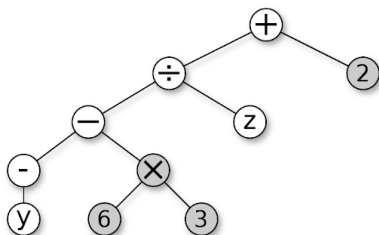
Question 9.

Figure 1. A tree for an expression.

The syntax tree describes an algebraic expression (please note the difference between the unary minus

that flips the value of the variable y and the binary minus that subtracts the two subexpressions: $-y$ and 6×3).

(A) Write the preorder DFS traversal of this tree.

(B) Write the inorder DFS traversal of this tree.

(C) Write the postorder DFS traversal of this tree.

Note. In all 3 answers denote the unary minus with the tilde sign \sim , but the regular/binary minus with $-$.

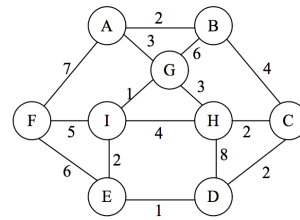
Question 10.

Figure 2. A graph with 9 vertices.

Run the Prim's algorithm on the weighted graph in Figure 2, start growing the tree from the vertex I .

Step	Newly Added Edge
Step 1	
Step 2	
Step 3	
Step 4	
Step 5	
Step 6	
Step 7	
Step 8	

What is the total weight of the obtained Minimum Spanning Tree?

Answers

Question 1.

- (A) $X = A \cup B$ (Boolean OR means set union)
 (B) $Y = (B \cap C) \cup \bar{A}$ (Boolean and means set intersection; negation means set complement)
 (C) $|X| = |A| + |B| - |A \cap B| = 120 + 60 - 20 = 160$ (principle of inclusion-exclusion).
 (D) $|Y|$ is all odd numbers and also four even numbers divisible by 15 (30, 60, 90, 120). The total is $60 + 4 = 64$.

Question 2.

In all the answers the largest and the smallest value are equal, because we know exactly how the two sets intersect; how many elements belong to just one of the sets A , B , and how many elements belong to the both sets.

- (A) $|A \cup B| = |A| + |B| - |A \cap B| = 8 + 5 - 3 = 10$ (the principle of inclusion-exclusion).
 (B) $|A \times (B \times B)| = 8 \cdot 5 \cdot 5 = 200$
 (Cartesian product has size that is the product of all participant sets: one can combine three elements from the sets A , B and B in this many ways).
 (C) $2^{2^3} = 2^8 = 256$ (the number of elements in the powerset of any set X can be obtained by raising 2 to the power $|X|$).
 (D) $|A \oplus B| = (8 - 3) + (5 - 3) = 7$ (we remove the common elements from both A and B).

Question 3.

- (A) $a_0 = 3$,
 $a_1 = 4$,
 $a_2 = 5 \cdot 4 - 6 \cdot 3 = 2$,
 $a_3 = 5 \cdot 2 - 6 \cdot 4 = -14$,
 $a_4 = 5 \cdot (-14) - 6 \cdot 2 = -82$,
 $a_5 = 5 \cdot (-82) - 6 \cdot (-14) = -326$,
 $a_6 = 5 \cdot (-326) - 6 \cdot (-82) = -1138$.
 (B) The characteristic equation is obtained, if we try to find a_n in the form of a geometric progression r^n : $r^{n+2} = 5r^{n+1} - 6r^n$, or $r^2 - 5r + 6 = 0$. It has two roots: $r_1 = 2$, $r_2 = 3$.
 (C) The general form of the expression for any iterative sequence b_n satisfying the relationship $b_{n+2} = 5b_{n+1} - 6b_n$ is as follows:

$$b_n = A \cdot 2^n + B \cdot 3^n,$$

where A, B are two constants that depend on the two initial values of the sequence b_n .

- (D) We need to solve a system of two equations, to ensure that the formula $a_n = A \cdot 2^n + B \cdot 3^n$ has correct values for $n = 0$ and $n = 1$. We get the following system:

$$\begin{cases} A + B = 3, \\ 2A + 3B = 4. \end{cases}$$

Substitute $B = 3 - A$ into the second equation. We get that $2A + 9 - 3A = 4$ and $A = 5$. We also get that $B = -2$. Therefore the exact formula to calculate the sequence a_n is this:

$$a_n = 5 \cdot 2^n - 2 \cdot 3^n, \text{ where } n \geq 0.$$

This actually works, if we plug in the values calculated in (A) for $n = 0, \dots, 6$.

Question 4.

- (A) The algorithm takes all numbers i from 1 to n^2 and divides them by all the smaller numbers $j < i$, and adds up all the obtained remainders.
 (C) The outer loop is repeated n^2 times. The inner loop is repeated $1 + 2 + 3 + \dots + n^2$ times. This is an arithmetic progression. The sum of an arithmetic progression is the arithmetic mean of the first and the last member multiplied by the number of members:

$$f(n) = \frac{1 + n^2}{2} \cdot n^2 = \frac{n^4 + n^2}{2}.$$

- (B) $f(n)$ is in $O(n^4)$. Therefore we can take $g(n) = n^4$. We can pick another $g(n)$ that is multiplied by some nonzero constant (such as $\frac{n^4}{2}$ or $17n^4$ or anything else - that also counts as a valid answer). Certainly, $f(n)$ is also in $O(n^k)$ for any $k > 4$, but the function $g(n) = n^4$ is the slowest growing.

Question 5.

- (A) If expressed as a product of two positive integers $120 = ab$, one of the divisors a or b would be smaller than $\sqrt{120} \approx 11$, and the other one would be bigger. We can easily list all the ways to express 120 as a product of two integers:

$$1 \cdot 120 = 2 \cdot 60 = 3 \cdot 40 = 4 \cdot 30 = 5 \cdot 24 = 6 \cdot 20 = 8 \cdot 15 = 10 \cdot 12,$$

and there are no other factorizations, since all the divisors less than 11 are already listed. Multiplying them all together would give

$$(120)^8 = 429981696000000000$$

- (B) As a product of prime factors:

$$(120)^8 = (2^3 \cdot 3 \cdot 5)^8 = 2^{24} \cdot 3^8 \cdot 5^8.$$

Question 6.

Item	Statement	True or False?
(A)	R is reflexive	TRUE
(B)	R is symmetric	TRUE
(C)	R is antisymmetric	FALSE
(D)	R is transitive	TRUE
(E)	aRb iff $a = b$	FALSE

- (A) Counterexample: None
 (B) Counterexample: None
 (C) Consider counterexample $a = 0, b = 11 \cdot 12 \cdot 13 = 1716$. While it is true that aRb and bRa , nevertheless $a \neq b$.
 (D) Counterexample: None
 (E) Counterexample is same as in (C): $a = 0, b = 1716$.

Question 7. Answer: 17

- For 1 of 24 permutations $X = 4$ (all hats stay in place),
- For 0 permutations $X = 3$ (it is not possible for exactly three hats to stay in place, because then the 4th hat also returns to its owner),
- For 6 of 24 permutations $X = 2$ (there are $\binom{4}{2} = 6$ ways how to pick 2 hats that stay in place; and the remaining two hats can switch places only in one way),
- For 8 of 24 permutations $X = 1$ (there are $\binom{4}{1} = 4$ ways how to pick 1 hat that stays in place; and the remaining three hats can rotate in two ways).
- For the remaining $24 - (1 + 6 + 8) = 9$ permutations $X = 0$ (no hats stay in place).

(A) $E(X) = \frac{1}{24} \cdot 4 + \frac{6}{24} \cdot 2 + \frac{8}{24} \cdot 1 = 1$. This means that the expected number of hats that stay in place is exactly 1.

(B) For all 24 permutations, subtract the value $E(X) = 1$ from every hat experiment outcome. Define x_1, \dots, x_{24} - all 24 values of the random variable X (exactly one value is 4, exactly six values are 2, exactly 8 values are 1, exactly 9 values are 0):

$$V(X) = \frac{\sum_{i=1}^{24} (x_i - 1)^2}{24} = \frac{24}{24} = 1.$$

Therefore, $V(X) = 1$ (variance also equals 1, but the unit of measurement is not hats but “hats squared”).

Question 8.

(A) $P(E|H)$ is the outcome of the Binomial distribution: There are $n = 5$ coin-toss experiments; the probability of success for any single experiment is $p = \frac{2}{3}$ (since we know that the day is lucky and hypothesis H holds). Therefore,

$$P(E|H) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{80}{243}$$

(B) $P(E|H) \cdot P(H) = \frac{80}{243} \cdot \frac{1}{2} = \frac{40}{243}$, since $P(H) = \frac{1}{2}$ (the *a priori* probability of a lucky day is exactly 1/2).

(C) $P(E|\bar{H})$ is the outcome of the Binomial distribution: Again, there are $n = 5$ coin-toss experiments,

but now the probability of a single experiment is just $p = \frac{1}{3}$. Therefore,

$$P(E|\bar{H}) = \binom{5}{3} p^3 (1-p)^2 = 10 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 = \frac{40}{243}$$

(D) $P(E|\bar{H}) \cdot P(\bar{H}) = \frac{40}{243} \cdot \frac{1}{2} = \frac{20}{243}$.

(E) We can compute $P(E)$ as the sum of two mutually incompatible events: event E can happen either on a lucky day or on an unlucky day:

$$P(E) = P(E|H) \cdot P(H) + P(E|\bar{H}) \cdot P(\bar{H}) = \frac{40}{243} + \frac{20}{243} = \frac{60}{243}.$$

(F) Use Bayes formula:

$$\begin{aligned} P(H|E) &= \frac{P(E|H) \cdot P(H)}{P(E|H) \cdot P(H) + P(E|\bar{H}) \cdot P(\bar{H})} = \\ &= \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{\frac{40}{243}}{\frac{60}{243}} = \frac{2}{3}. \end{aligned}$$

Bayes formula is intuitive: It shows the proportion of the subcase $P(E|H) \cdot P(H)$ (i.e. event E happens on a lucky day) out of the whole probability $P(E) = P(E|H) \cdot P(H) + P(E|\bar{H}) \cdot P(\bar{H})$ (i.e. event E happens either on a lucky or unlucky day).

Question 9.

(A) $+: \sim y \times 6 \ 3 \ z \ 2$,

(B) $y \sim - \ 6 \times 3 : z + 2$,

(C) $y \sim 6 \ 3 \times - z : 2 +$.

Note. In inorder traversal (B) we first visit the first subtree (e.g., y), and only then the parent node (e.g., unary minus \sim). See (Rosen2019, p.811).

Question 10.

We start from vertex I . At every step we grow the tree by a single edge (so that it stays connected and the newly added edge has the smallest possible weight).

Step	Newly Added Edge
Step 1	$IG, w = 1$
Step 2	$IE, w = 2$
Step 3	$ED, w = 1$
Step 4	$DC, w = 2$
Step 5	$CH, w = 2$
Step 6	$GA, w = 3$
Step 7	$AB, w = 2$
Step 8	$IF, w = 5$

The total weight of all added edges (same as the total weight of the MST) is 18.

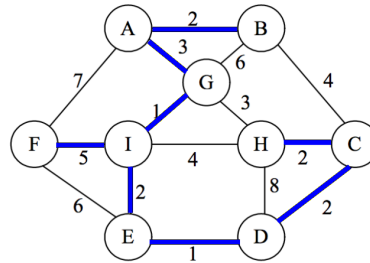


Figure 3. MST edges shown in blue.