Discrete Sample Quiz 6

Question 1: Modifying Sudoku Rules; see Chapter 1.3.6 (Rosen2019, p.36).
We have a 9×9 table; each cell contains one number from $A_9 = \{1, 2, \ldots, 9\}$. We define predicate $p(i, j, n)$ which is true iff the cell in row i and column j has the given value n. All three arguments are integers from 1 to 9; this predicate is a function

$$p : (A_9)^3 \rightarrow \{\texttt{true}, \texttt{false}\}.$$

Formula to describe that every row i contains every number:

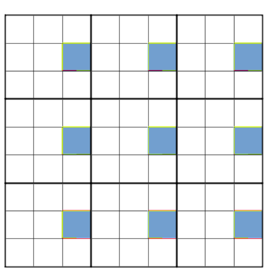
$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, k) = \forall i \in A_9,\ \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, k).$$

This formula describes that each 3×3 block contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n). \qquad (1)$$

(A) Count the number of conjunctions ($\wedge$), disjunctions ($\vee$) and predicates ($p(\ldots)$) in the formula (1).

(B) Similarly to the formula (1), write a Boolean formula to describe the following "rule" for sudoku tables: If there are any cells in the sudoku cells that have their row number difference AND column number difference divisible by 3, then they contain different numbers. For example, in the figure below, all the shaded cells have same relative position within their respective 3×3 blocks – therefore their row/column numbers differ by $0, 3, 6$, and accordingly to our new "rule" they should contain all nine different numbers.



Question 2: Long set operations. Denote $A_1 = \{1\}$, $A_2 = \{1, 2\}$, etc. In general, $A_k = \{1, 2, \ldots, k\}$.
By $A \oplus B = (A - B) \cup (B - A)$ we denote the symmetric difference: All elements that belong to just one of the sets A, B (but not the other one). Consider this set:

$$S = \bigoplus_{j=1}^{100} A_j = A_1 \oplus A_2 \oplus \ldots \oplus A_{100}.$$

Write a comma-separated list of the 10 smallest elements of S in increasing order.

Question 3: Using recurrent formula. Find the first 5 members of this sequence:

$$\begin{cases} f(0) = 1, \\ f(1) = 4, \\ f(n) = f(n-1) \cdot f(n-2) + 1,\ \forall n \geq 2. \end{cases}$$

Write comma-separated values $f(0), f(1), f(2), f(3), f(4)$.

Question 4: Reccurent sequence. A sequence of real numbers $f : \mathbb{N} \rightarrow \mathbb{R}$ satisfies the following properties:
(A) $f(k+2) = f(k) + f(k+1)$ for all integers $k \geq 2$.
(B) $f(n)$ is a growing geometric progression: Namely $f(1) = f(0) \cdot q$, $f(2) = f(0) \cdot q^2$ and so on.

Find the quotient of this geometric progression. Round it to the nearest thousandth (i.e. specify the first three digits after the decimal point).

Question 5: Finding a limit. Define the following sequence:

$$\begin{cases} x_0 = 1, \\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \text{ if } n \geq 0 \end{cases}$$

Assume that there exists limit $L = \lim_{n\to\infty} x_n$.
Find that limit L and round it to the nearest thousandth.

Question 6: Find recurrent formulas. We define three sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ explicitly. Find their recurrent formulas that allow to find next members of the sequence in terms of the previous ones.

- $a_n = 2^{\frac{1}{2^n}}$, where $n \geq 0$.
- $b_0 = 1$, $b_1 = 111$, $b_2 = 11111$, $b_3 = 1111111$, etc. (In general, the kth member b_k has $2k + 1$ digits "1" in its decimal notation).
- $c_n = n^2 + n$.

Initial member	Recurrent expression
$a_0 = \ldots$	$a_{n+1} = \ldots$ (express via a_n etc.)
$b_0 = \ldots$	$b_{n+1} = \ldots$
$c_0 = \ldots$	$c_{n+1} = \ldots$

Question 7: Taylor series There is a formula known from calculus (practically used to compute $y = \sin x$) for each $x \in \mathbb{R}$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Use Python or Scala to add the first 20 terms of this infinite sum to compute $\sin 30°$. Round the answer to the nearest thousandth. (Taylor series expects to have argument x in radians, so you have to convert degrees to radians before using the formula.)