Similar to Romanian.

N6. Let $f\colon \{1,2,3,\dots\} \to \{2,3,\dots\}$ be a function such that $f(m+n) \mid f(m)+f(n)$ for all pairs m,n of positive integers. Prove that there exists a positive integer $c>1$ which divides all values of f.

Solution 1. For every positive integer m, define $S_m = \{n\colon m \mid f(n)\}$.

Lemma. If the set S_m is infinite, then $S_m = \{d, 2d, 3d, \dots\} = d\cdot\mathbb{Z}_{>0}$ for some positive integer d.

Proof. Let $d = \min S_m$; the definition of S_m yields $m \mid f(d)$.

Whenever $n \in S_m$ and $n > d$, we have $m \mid f(n) \mid f(n-d)+f(d)$, so $m \mid f(n-d)$ and therefore $n-d \in S_m$. Let $r \leqslant d$ be the least positive integer with $n \equiv r \pmod{d}$; repeating the same step, we can see that $n-d, n-2d, \dots, r \in S_m$. By the minimality of d, this shows $r = d$ and therefore $d \mid n$.

Starting from an arbitrarily large element of S_m, the process above reaches all multiples of d; so they all are elements of S_m. $\square$

The solution for the problem will be split into two cases.

Case 1: The function f is bounded.

Call a prime p *frequent* if the set S_p is infinite, i.e., if p divides $f(n)$ for infinitely many positive integers n; otherwise call p *sporadic*. Since the function f is bounded, there are only a finite number of primes that divide at least one $f(n)$; so altogether there are finitely many numbers n such that $f(n)$ has a sporadic prime divisor. Let N be a positive integer, greater than all those numbers n.

Let $p_1, \dots, p_k$ be the frequent primes. By the lemma we have $S_{p_i} = d_i \cdot \mathbb{Z}_{>0}$ for some d_i. Consider the number

$$n = Nd_1d_2\cdots d_k + 1.$$

Due to $n > N$, all prime divisors of $f(n)$ are frequent primes. Let p_i be any frequent prime divisor of $f(n)$. Then $n \in S_{p_i}$, and therefore $d_i \mid n$. But $n \equiv 1 \pmod{d_i}$, which means $d_i = 1$. Hence $S_{p_i} = 1\cdot\mathbb{Z}_{>0} = \mathbb{Z}_{>0}$ and therefore p_i is a common divisor of all values $f(n)$.

Case 2: f is unbounded.

We prove that $f(1)$ divides all $f(n)$.

Let $a = f(1)$. Since $1 \in S_a$, by the lemma it suffices to prove that S_a is an infinite set.

Call a positive integer p a *peak* if $f(p) > \max\bigl(f(1), \dots, f(p-1)\bigr)$. Since f is not bounded, there are infinitely many peaks. Let $1 = p_1 < p_2 < \dots$ be the sequence of all peaks, and let $h_k = f(p_k)$. Notice that for any peak p_i and for any $k < p_i$, we have $f(p_i) \mid f(k) + f(p_i - k) < 2f(p_i)$, hence

$$f(k) + f(p_i - k) = f(p_i) = h_i. \tag{1}$$

By the pigeonhole principle, among the numbers $h_1, h_2, \dots$ there are infinitely many that are congruent modulo a. Let $k_0 < k_1 < k_2 < \dots$ be an infinite sequence of positive integers such that $h_{k_0} \equiv h_{k_1} \equiv \dots \pmod{a}$. Notice that

$$f(p_{k_i} - p_{k_0}) = f(p_{k_i}) - f(p_{k_0}) = h_{k_i} - h_{k_0} \equiv 0 \pmod{a},$$

so $p_{k_i} - p_{k_0} \in S_a$ for all $i = 1, 2, \dots$. This provides infinitely many elements in S_a.

Hence, S_a is an infinite set, and therefore $f(1) = a$ divides $f(n)$ for every n.

Comment. As an extension of the solution above, it can be proven that if f is not bounded then $f(n) = an$ with $a = f(1)$.

Take an arbitrary positive integer n; we will show that $f(n+1) = f(n) + a$. Then it follows by induction that $f(n) = an$.

Take a peak p such that $p > n+2$ and $h = f(p) > f(n) + 2a$. By (1) we have $f(p-1) = f(p) - f(1) = h - a$ and $f(n+1) = f(p) - f(p-n-1) = h - f(p-n-1)$. From $h - a = f(p-1) \mid f(n) + f(p-n-1) < f(n) + h < 2(h-a)$ we get $f(n) + f(p-n-1) = h - a$. Then

$$f(n+1) - f(n) = \big(h - f(p-n-1)\big) - \big(h - a - f(p-n-1)\big) = a.$$

On the other hand, there exists a wide family of bounded functions satisfying the required properties. Here we present a few examples:

$$f(n) = c; \quad f(n) = \begin{cases} 2c & \text{if } n \text{ is even} \\ c & \text{if } n \text{ is odd;} \end{cases} \quad f(n) = \begin{cases} 2018c & \text{if } n \leqslant 2018 \\ c & \text{if } n > 2018. \end{cases}$$

Solution 2. Let $d_n = \gcd\big(f(n), f(1)\big)$. From $d_{n+1} \mid f(1)$ and $d_{n+1} \mid f(n+1) \mid f(n) + f(1)$, we can see that $d_{n+1} \mid f(n)$; then $d_{n+1} \mid \gcd\big(f(n), f(1)\big) = d_n$. So the sequence $d_1, d_2, \ldots$ is nonincreasing in the sense that every element is a divisor of the previous elements. Let $d = \min(d_1, d_2, \ldots) = \gcd(d_1.d_2, \ldots) = \gcd\big(f(1), f(2), \ldots\big)$; we have to prove $d \geqslant 2$.

For the sake of contradiction, suppose that the statement is wrong, so $d = 1$; that means there is some index n_0 such that $d_n = 1$ for every $n \geqslant n_0$, i.e., $f(n)$ is coprime with $f(1)$.

Claim 1. If $2^k \geqslant n_0$ then $f(2^k) \leqslant 2^k$.

Proof. By the condition, $f(2n) \mid 2f(n)$; a trivial induction yields $f(2^k) \mid 2^k f(1)$. If $2^k \geqslant n_0$ then $f(2^k)$ is coprime with $f(1)$, so $f(2^k)$ is a divisor of 2^k. □

Claim 2. There is a constant C such that $f(n) < n + C$ for every n.

Proof. Take the first power of 2 which is greater than or equal to n_0: let $K = 2^k \geqslant n_0$. By Claim 1, we have $f(K) \leqslant K$. Notice that $f(n+K) \mid f(n) + f(K)$ implies $f(n+K) \leqslant f(n) + f(K) \leqslant f(n) + K$. If $n = tK + r$ for some $t \geqslant 0$ and $1 \leqslant r \leqslant K$, then we conclude

$$f(n) \leqslant K + f(n-K) \leqslant 2K + f(n-2K) \leqslant \ldots \leqslant tK + f(r) < n + \max\big(f(1), f(2), \ldots, f(K)\big),$$

so the claim is true with $C = \max\big(f(1), \ldots, f(K)\big)$. □

Claim 3. If $a, b \in \mathbb{Z}_{>0}$ are coprime then $\gcd\big(f(a), f(b)\big) \mid f(1)$. In particular, if $a, b \geqslant n_0$ are coprime then $f(a)$ and $f(b)$ are coprime.

Proof. Let $d = \gcd\big(f(a), f(b)\big)$. We can replicate Euclid's algorithm. Formally, apply induction on $a + b$. If $a = 1$ or $b = 1$ then we already have $d \mid f(1)$.

Without loss of generality, suppose $1 < a < b$. Then $d \mid f(a)$ and $d \mid f(b) \mid f(a) + f(b-a)$, so $d \mid f(b-a)$. Therefore d divides $\gcd\big(f(a), f(b-a)\big)$ which is a divisor of $f(1)$ by the induction hypothesis. □

Let $p_1 < p_2 < \ldots$ be the sequence of all prime numbers; for every k, let q_k be the lowest power of p_k with $q_k \geqslant n_0$. (Notice that there are only finitely many positive integers with $q_k \neq p_k$.)

Take a positive integer N, and consider the numbers

$$f(1), f(q_1), f(q_2), \ldots, f(q_N).$$

Here we have $N + 1$ numbers, each being greater than 1, and they are pairwise coprime by Claim 3. Therefore, they have at least $N + 1$ different prime divisors in total, and their greatest prime divisor is at least p_{N+1}. Hence, $\max(f(1), f(q_1), \ldots, f(q_N)) \geqslant p_{N+1}$.

Choose N such that $\max(q_1, \ldots, q_N) = p_N$ (this is achieved if N is sufficiently large), and $p_{N+1} - p_N > C$ (that is possible, because there are arbitrarily long gaps between the primes). Then we establish a contradiction

$$p_{N+1} \leqslant \max(f(1), f(q_1), \ldots, f(q_N)) < \max(1 + C, q_1 + C, \ldots, q_N + C) = p_N + C < p_{N+1}$$

which proves the statement.