Exam, Part 2

1. Let a_n be an arithmetic series defined recursively:

$$\begin{cases} a_0 = 0, \\ a_{n+1} = a_n + 19, \text{ for } n \geq 0. \end{cases}$$

(a) How many members of this series a_n are 3-digit numbers (between 100 and 999)?

Answer: 47.

All members of this series are divisible by 19. The smallest member a_n that is at least 100 (and is divisible by 19) is $a_6 = 6 \cdot 19 = 114$.

The largest member a_n that does not exceed 999 is $a_{52} = 52 \cdot 19 = 988$.

The total number of such members is $52 - 6 + 1 = 47$. (We add 1, because both members a_{52} and a_6 should be included in the count.)

Note. We can do common-sense check for this answer. In general, there are $999-100+1 = 900$ numbers having exactly three digits, and $900/19 = 47.36842$. So, among arithmetic series with common difference $d = 19$ some will have 47 members in the interval $[100; 999]$, and some will have 48 members in this interval. In our case the sequence has 47 members in the interval $[100; 999]$ □

(b) What is the smallest member a_k that satisfies the congruence $a_k \equiv 1 \pmod{16}$.

Answer: That member is $a_{11} = 11 \cdot 19 = 208$.

There are different ways to get this answer.

Method 1. Observe that 19 and 16 are relative primes ($\gcd(19, 16) = 1$). We solve the Bezout identity: Find integers x, y satisfying the equation

$$19x + 16y = 1$$

(You can apply Blankenship's algorithm or type this equation into Wolfram Alpha or a similar tool. You will get $x = 11$ and $y = -13$, so $11 \cdot 19 + (-13) \cdot 16 = 1$. We conclude that $a_{11} \equiv 1$ (modulo 19).

Method 2. In this problem we are interested into remainders modulo 16. So, we just need to find the multiplicative inverse of $19 \equiv 3$ (modulo 16). Observe that $3 \cdot 11 \equiv 33 \equiv 1 \pmod{16}$. □

2. Consider the following two 5-digit numbers $a = 18696$ and $b = 99999$.

(a) Apply Euclidean algorithm to find the greatest common divisor $\gcd(18696, 99999)$; show the steps of this algorithm. Also find $\operatorname{lcm}(18696, 99999)$.

We start by dividing 99999 with 18696 (we get remainder 6519). Then we replace the largest number 99999 with this remainder, and so on. This is the Euclidean algorithm.

$$\gcd(99999, 18696) = \gcd(99999, 18696) = \gcd(18696, 6519) = \gcd(6519, 5658) =$$

$= \gcd(5658, 861) = \gcd(861, 492) = \gcd(492, 369) = \gcd(369, 123) = \gcd(123, 0) = 123.$

The least common multiple can be found by the formula $lcm(a, b) = ab/gcd(a, b)$:

$$\text{lcm}(99999, 18696) = \frac{99999 \cdot 18696}{\gcd(99999, 18696)} =$$

$$= \frac{99999 \cdot 18696}{123} = \frac{99999}{123} \cdot 18696 = 813 \cdot 18696 = 15199848.$$

$\square$

(b) Express $\frac{18696}{99999}$ as an irreducible fraction.

We can divide the numerator and denominator with their GCD:

$$\frac{18696}{99999} = \frac{18696/123}{99999/123} = \frac{152}{813}.$$

Note. If we divide both numbers as an infinite decimal fraction, we will get the same 5 digits (used in the problem statement) as the period:

$$\frac{152}{813} = 0.186961869618696\ldots = 0.(18696).$$

$\square$

3. (a) Let $\texttt{1101011110}_2$ be a positive integer written in binary notation. Convert this number into hexadecimal and also into decimal notation.

Hexadecimal notation is easy – just group the binary digits into groups of 4 (from the right side). You will get this:

$$\texttt{1101011110}_2 = \texttt{11.0101.1110}_2 = \texttt{35E}_{16}.$$

Decimal notation can be obtained either from the binary notation directly (or from our hexadecimal notation). In the latter case we get:

$$\texttt{35E}_{16} = 3 \cdot 16^2 + 5 \cdot 16 + 14 = 862.$$

$\square$

(b) Let $P(x)$ be polynomial with coefficients same as in this binary number (namely, $1, 1, 0, 1, 0, 1, 1, 1, 1, 0$):

$P(x) = 1 \cdot x^9 + 1 \cdot x^8 + 0 \cdot x^7 + 1 \cdot x^6 + 0 \cdot x^5 + 1 \cdot x^4 + 1 \cdot x^3 + 1 \cdot x^2 + 1 \cdot x^1 + 0 \cdot x^0.$

Evaluate this polynomial for $x = 2$, i.e. find $P(2)$.

You can simply note that the polynomial for $x = 2$ evaluates to the numeric value of the binary number $\texttt{1101011110}_2$. So it must be 862.
You can also compute this directly on a calculator. Figure 1 shows the screenshot from RStudio that computes $P(2)$: $\square$

```
> x = 2
> ((((((((1*x) + 1)*x + 0)*x + 1)*x + 0)*x + 1)*x + 1)*x + 1)*x + 0
[1] 862
```

Figure 1: Evaluating Polynomial $P(x)$.

4. Count the number of graphs having exactly 4 vertices and exactly 4 edges. All vertices in the graph are distinguishable, they are labeled with letters A, B, C, D, and isomorphic graphs are considered different, if their vertex labels differ.

Altogether there are 6 edges you can draw between 4 vertices in an undirected graph. It will be easier for us to count the 2 edges in the graph's complement (i.e. the two not-taken edges in the graph) rather than 4 edges that actually belong to the graph.

Case 1. We can count those cases when the 2 (non-existent) edges share one vertex. In this case we would get a 'snake' or a 'path' of 3 vertices connected with edges. For example, edges $\{AB, BC\}$ create a short path ABC with endpoints A and C and B in the middle. Altogether there are 12 ways to arrange three vertices on a path.

$$ABC, ABD, ACB, ACD, ADB, ADC, BAC, BAD, BCD, BDC, CAD, CBD.$$

In all these 12 paths the first vertex alphabetically precedes the last one. (There are 12 more paths where the first vertex is alphabetically after the last one, but they can be reversed so as to match one of the 12 cases above.)

Case 2. We can also count the cases where the 2 (non-existent) edges do not share any vertices. And there are just 3 ways to connect 4 vertices with disconnected edges: $\{AB, CD\}$, $\{AC, BD\}$, and $\{AD, BC\}$.

Altogether there are $12 + 3 = 15$ ways to **unselect** two vertices (and the same number of ways to **select** four vertices). □

5. An undirected graph $G = (V, E)$, where the vertices $V = \{0, 1, 2, 3, 4, 5\}$ is given by the following matrix:

$$M_G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) Draw the representation of this graph as adjacency lists.

(b) Find a Eulerian circuit in this graph: list its vertices or prove that it cannot exist.

Eulerian circuit (a path that uses every edge exactly once and returns to the starting point) in this graph exists. This follows from the graph being connected and every vertex there has even degree – every row in the matrix has even number of 1s). The order of visiting vertices is shown with the following path (it starts and ends in the vertex (0)):

$$(0) \to (1) \to (3) \to (4) \to (0) \to (2) \to (4) \to (1) \to (5) \to (3) \to (0).$$

Figure 2 shows this circuit in the graph itself. □

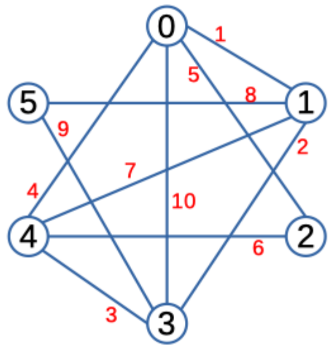

Figure 2: Graph with Eulerian circuit (edge order enumerated in red)

6. Let G be a bipartite graph; all edges in this graph connect some vertex in $V_1 = \{v_1, v_2, v_3, v_4\}$ with some vertex in $V_2 = \{v_5, v_6, v_7\}$. By $d(v_i)$ we denote the degree of some vertex.

   (a) Does there exist a bipartite graph G with these vertex sets, if the degrees of all 7 vertices are as follows:

   $$(d(v_1), d(v_2), d(v_3), d(v_4), d(v_5), d(v_6), d(v_7)) = (2, 2, 1, 1, 2, 2, 1).$$

   This graph is impossible, since $2+2+1+1$ (sum of degrees on the left side of the bipartite graph) is not equal to $2+2+1$ (the sum of degrees on the right side of the bipartite graph). □

   (b) Does there exist a bipartite graph G with these vertex sets, if the degrees of all 7 vertices are as follows:

   $$(d(v_1), d(v_2), d(v_3), d(v_4), d(v_5), d(v_6), d(v_7)) = (3, 3, 1, 2, 3, 3, 3).$$

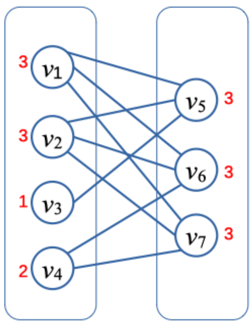

Figure 3: Example of a bipartite graph.)

   Yes, such a graph exists (the sum of degrees are equal on both sides: $3+3+1+2 = 3+3+3$, and no degree exceeds the number of vertices on the other side that it can connect with). Figure 3 □

Note. For each example draw some bipartite graph with these degrees (or explain why it cannot exist).