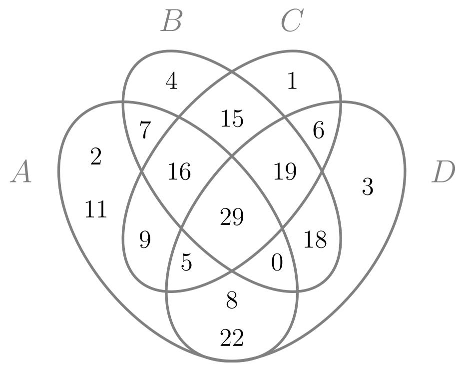
Homework 3

Discrete Structures
Due Tuesday, January 26, 2021

Submit each question separately in .pdf format only

---

1. (a) Given the Venn diagram on the left, write all the elements of the sets on the right.



$(A \cap C) \setminus (D \cap B)$

$\overline{(\overline{A} \cup D)} \cap B$

$(C \oplus D) \cap \overline{(D \oplus B)}$

$\overline{(D \cap B \cap C)} \oplus (D \cup B \cup C)$

(b) Describe the following sets using A, B, C, D from above and set operations on them.

$$E = \{16, 29, 0\} \qquad F = \{6, 7, 16, 19\} \qquad G = \{3, 18, 0, 4, 7\}$$

(c) Simplify the following sets as much as possible. That is, rewrite them without using the union $\cup$ or intersection $\cap$ symbols.

$$X = \bigcup_{i=0}^{\infty} [i, i+1] \qquad Y = \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] \qquad Z = \bigcap_{n=1}^{\infty} \left\{ \frac{n}{x} \ : \ x \in \mathbf{Z}_{\geqslant n} \right\}$$

2. Let A, B, C be sets, and let $f\colon A \to B$ and $g\colon B \to C$ be functions.

(a) Using logical symbols, express the following statements.

i. g is injective when restriced to the range of f

ii. there exists an element in C whose preimage in g is not $f(a)$ for any a in A

(b) If f and g are injective, prove that $g \circ f$ is injective.

(c) If $g \circ f$ is surjective, prove that g must be surjective.

3. Let A, B, C be arbitrary sets in the same universe U. Prove or disprove the following statements:

(a) $(B \cup C) - A = (B - C) \cup (C - A)$.

(b) $(B \oplus C) - A = (B - A) \oplus (C - A)$.

(c) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

4. Prove or disprove the following statements about power sets.

(a) There is a set X such that its powerset $\mathcal{P}(X)$ equals

$$\{\varnothing, \{a\}, \{\varnothing\}, \{a, \{\varnothing\}\}\}.$$

(b) There is a set X such that its powerset $\mathcal{P}(X)$ equals

$$\{\varnothing, \{\varnothing\}, \{\{a\}\}, \{\{a,b\}\}, \{\varnothing, \{a\}\}, \{\varnothing, \{a,b\}\}, \{\{a\}, \{a,b\}\}, \{\varnothing, \{a\}, \{a,b\}\}\}.$$

(c) For any two sets A and B, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ iff $A \subseteq B$.

5. Prove the following three tautologies using Coq. Submit your file `tautology.v` as the solution for your Problem 5.

```
Lemma Sample5A: forall P Q: ~(~P /\ ~Q) -> P \/ Q.
Proof.
  (* Place your proof here *)
Qed.

Lemma Sample5B: forall P Q: (P -> Q) -> (~P \/ Q).
Proof.
  (* Place your proof here *)
Qed.

Lemma Sample5C: forall P Q: (P -> Q) <-> (~Q -> ~P).
Proof.
  (* Place your proof here *)
Qed.
```

Note. Most lemmas in the non-constructive mathematics are proven using some tautology as an axiom. Either the "NNPP axiom" ($\neg\neg A \rightarrow A$, double negation elimination) or the "classic axiom" ($A \vee \neg A$, the law of the Excluded Middle). See the link *Week3 > Two Nonconstructive Proofs of the Same Lemma* in ORTUS. You can try out whichever method you want. For these axioms to work the first line in your proof should be:
`Require Import Classical_Prop.`