Quiz 10: Binary Relations

Question 1
Some people participate in an Einkaufshelden program – they go to one of the shops A, B or C and deliver the products to the endpoints X, Y or Z. Each of the 9 edges is selected by the same probability (Figure 1)

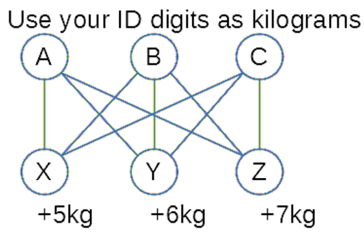

Figure 1. The Paths of a Delivery Service.

The sum of the numbers on both ends of an edge shows how many kilograms of stuff were delivered. (For example, if your Student ID has $A = 0$, then on AZ there are $0 + 7 = 7$ kilograms. Let X denote the random variable: the kilograms of stuff delivered on a single edge.
Write the variance $V(X)$ as an irreducible fraction $\mathtt{P/Q}$.

Question 2.
Given a set of the first positive integers $S = \{1, 2, \ldots, A + B + (10 - C)\}$ (where $\mathtt{A,B,C}$ are the digits from your student ID). We define a relation on S: aRb is true iff $|a - b| \leq 2$.
Let R^2 be the second power of that relation R and let M_{R^2} be its matrix. Find the number of 1s in this matrix (In other words: how many pairs belong to this relation?)

Question 3.
Define a set S of these six positive integers:

$$S = \{1 + \mathtt{A},\ 2 + \mathtt{A} + \mathtt{B},\ 3 + \mathtt{A} + \mathtt{B} + \mathtt{C},\ 4 + 2\mathtt{A} + \mathtt{B} + \mathtt{C},\ 5 + 2\mathtt{A} + 2\mathtt{B} + \mathtt{C},\ 6 + 2\mathtt{A} + 2\mathtt{B} + 2\mathtt{C}\}.$$

Now compute the remainders of the elements of S when divided by 16. You should get another set S' where each element is between 0 and 15. (S' may contain fewer elements than S, if some remainders are identical.) Let b_i be a sequence of bits ($i = 0, \ldots, 15$):

$$b_i = 1 \text{ iff } i \in S'.$$

We define a matrix for relation R as follows:

$$M_R = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 & b_7 \\ b_8 & b_9 & b_{10} & b_{11} \\ b_{12} & b_{13} & b_{14} & b_{15} \end{pmatrix}$$

Let M^* be the matrix of the transitive closure of R. Find the number of 1s in the matrix M^*.

Question 4.
Let S be a set and its size is computed from the digits in your ID:

$$|S| = \mathtt{A} + \mathtt{B} + (10 - \mathtt{C}).$$

Let N be the number of binary relations on S that are reflexive and symmetric at the same time. Write the last 3 digits of N in your answer.
Hint If you need to find the last 3 digits of some large number, you can use periodicity (similar to this: https://bit.ly/33NrJKI). Euler's theorem about the period of remainders modulo 1000 being periodic with period $\varphi(1000)$ (see https://bit.ly/33PyI5Q) is not directly applicable in this situation, since your exponent a is not mutually prime with 1000. But with some additional reasoning you can use Euler's theorem as well.

Question 5.
Find the join of the 3-ary relation:

```
{ (Wages,MS410,N507),
  (Rosen,CS540,N525),
  (Michaels,CS518,N504),
  (Michaels,MS410,N510) }
```

and the 4-ary relation:

```
{ (MS410,N507,Monday,6:00),
  (MS410,N507,Wednesday,6:00),
  (CS540,N525,Monday,7:30),
  (CS518,N504,Tuesday,6:00),
  (CS518,N504,Thursday,6:00) }
```

with respect to the last two fields of the first relation and the first two fields of the second relation.
Write the number records in the join.

Question 6.
Let R be a relation on the set $\{x_1, x_2, x_3\}$ that is reflexive and transitive, but not antisymmetric. Denote its matrix by

$$M_R = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Write all the 9 bits (as a sequence of 0s and 1s) in your answer: $b_{11}b_{12}b_{13}b_{21}b_{22}b_{23}b_{31}b_{32}b_{33}$. If there are multiple answers, write the lexicographically first one.

Question 7.
Let R be a relation on $\{a, b, c\}$ that is reflexive and transitive, but not symmetric. Denote its matrix by

$$M_R = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Write all the 9 bits (as a sequence of 0s and 1s) in your answer: $b_{11}b_{12}b_{13}b_{21}b_{22}b_{23}b_{31}b_{32}b_{33}$. If there are multiple answers, write the lexicographically first one.

Answers

In solutions we assume that $A = 4$, $B = 5$, $C = 6$.

Question 1. Answer: 7 (if $A = 4$, $B = 5$, $C = 6$)

Figure 2 shows all the weight combinations in this case. The list of total weights on all 9 edges is the following:

$$x_1 = 9,\ 10,\ 11,\ 10,\ 11,\ 12,\ 11,\ 12,\ x_9 = 13.$$

The mean value is 11. Variance $V(X)$ is the arithmetic mean of all squared deviations from that mean:

$$V(X) = \frac{\sum_{i=1}^{9}(x_i - 11)^2}{9} = \frac{12}{9} = \frac{4}{3}.$$

Therefore $m + n = 4 + 3 = 7$.

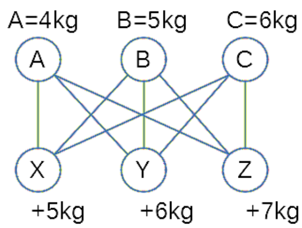

Figure 2. Delivery Service (specific values).

Question 2. Answer: 97 (if $A = 4$, $B = 5$, $C = 6$)
$S = \{1, 2, \ldots, A + B + (10 - C)\} = \{1, 2, \ldots, 13\}$,
The relation R^2 on any two $a, b \in S$ holds, iff the distance between a, b is no more than 4. This follows from the "triangle inequality".
Namely, $a(R^2)b$ holds, if there exists c that aRc and cRb. Therefore $|a - c| \leq 2$ and $|c - b| \leq 2$ and we should have $|a - b| \leq 4$. The matrix of R^2 has 13×13 entries. Out of these entries the main diagonal (where $a = b$) and also other diagonal lines that run parallel to that contain 1s. The total number of 1's is

$$9 + 10 + 11 + 12 + 13 + 12 + 11 + 10 + 9 = 97.$$

Question 3. Answer: 16 (if $A = 4$, $B = 5$, $C = 6$)
Plug in the numbers A, B, C:

$$S = \{5, 11, 18, 23, 29, 36\},$$

$$S' = \{5, 11, 2, 7, 13, 4\},$$

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This matrix only contains 6 elements equal to 1, but its transitive closure is all 16 elements equal to 1. Indeed, if we consider the relation R on a set of four elements: $A = \{a_1, a_2, a_3, a_4\}$, then there is a cyclical path:

$$a_1Ra_3,\ a_3Ra_4,\ a_4Ra_2,\ a_2Ra_1.$$

Therefore the transitive closure is a relation consisting of all pairs of elements $A \times A$.

Question 4. Answer: 544 (if $A = 4$, $B = 5$, $C = 6$)
We get $|S| = A + B + (10 - C) = 13$. The matrix of any binary relation R defined on S has $13 \times 13 = 169$ elements. Since R is reflexive, all the elements on the main diagonal of that matrix are equal to 1. Moreover, any elements above the diagonal are symmetric to some element below the main diagonal. Ultimately, we can only choose $(169 - 13)/2 = 78$ elements in the matrix. Each element can have two values, therefore $N = 2^{78}$.
The last three digits in that number are 544. Indeed,

$$\begin{cases} 2^{78} \equiv 0 \pmod{8}, \\ 2^{78} \equiv 44 \pmod{125}. \end{cases}$$

The first congruence is obvious (any large power of 2 is divisible by 8). The second one follows from

$$\begin{aligned} 2^{78} &= 2^{64} \cdot 2^8 \cdot 2^4 \cdot 2^2 \equiv \\ &\equiv 116 \cdot 6 \cdot 16 \cdot 4 = \\ &= 44544 \equiv 44 \pmod{125} \end{aligned}$$

Question 5. Answer: 5

Table1	Joined		Table2	
(T1.L1) Wages	MS410	N507	(T2.L1) Mon	6:00
(T1.L1) Wages	MS410	N507	(T2.L2) Wed	6:00
(T1.L2) Rosen	CS540	N525	(T2.L3) Mon	7:30
(T1.L3) Michaels	CS518	N504	(T2.L4) Tue	6:00
(T1.L3) Michaels	CS518	N504	(T2.L5) Thu	6:00

In this table the middle section shows the two joined columns. The left section shows the lines from Table 1 (with respective line numbers in that table). The right section shows the lines from Table 2 (also with line numbers).

Question 6. Answer: 100011011
We know that the relation is reflexive, so all the elements on its diagonal are 1 ($b_{11} = b_{22} = b_{33} = 1$). It must not be antisymmetric, so there must be $x \neq y$ such that xRy and yRx. Since we want to have the lexicographically first matrix, we set the entries b_{23} and $b_{32} = 1$. The matrix is now as follows:

$$M_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

We can easily verify that the relationship R is also transitive (in fact, it is an equivalence relationship with two equivalence classes $\{x_1\}$ and $\{x_2, x_3\}$).

Question 7. Answer: `100010011`
We know that the relation is reflexive, so all the elements on its diagonal are 1 ($b_{11} = b_{22} = b_{33} = 1$). It must not be symmetric, so we can set $b_{23} \neq b_{32}$. (In order to be the lexicographically first, we do not want to touch earlier entries in that matrix.)

$$M_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$