Preparation for Midterm - 1

Part A. Computational Problems

In this part we apply known algorithms to particular situations. The goal is to obtain correct result and and add a short explanation, what you did and why.

A1. Propositional Logic

Question 1. Consider this Boolean expression: $E = p \rightarrow q \rightarrow r$. Implication ($\rightarrow$) is right-associative.

(a) Write an equivalent Boolean expression using only conjunctions ($\wedge$) and negations ($\neg$).

(b) Fill in the missing parts in the truth table:

p	q	r	E
T	T	T	...
T	T	F	...
T	F	T	...
T	F	F	...
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Question 2. Assume the following precedence order and associativity of the 5 Boolean operations:

`~a`	1st precedence	right-associative
`a/\b`	2nd precedence	right-associative
`a\/b`	3rd precedence	right-associative
`a<->b`	4th precedence	left-associative
`a->b`	5th precedence	right-associative

Show step-by-step how would you restore parentheses in the following Boolean expression (each step adds one pair of parentheses – in the order that follows from the rules of precedence and associativity):

```
a -> b /\ ~ c \/ ~ ~ d <-> e -> f <-> g
```

A2. Sets and Quantifiers

Question 3. The universe U is all integers between 1 and 600. $K_2 \subseteq U$ denotes all even numbers from U. Similarly, $K_3 \subseteq U$ denotes all numbers divisible by 3; $K_5 \subseteq U$ denotes all numbers divisible by 5.

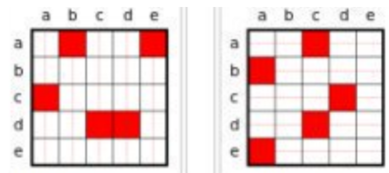
(A) Draw Euler-Venn diagram with a big rectangle (the set U), the subsets K_2, K_3, K_5 as three intersecting circles.

(B) Shade the region that corresponds to the set $(K_2 \cap \overline{K}_3) \cup K_5$.

(C) Find the cardinality of the set $\left|(K_2 \cap \overline{K}_3) \cup K_5\right|$, justify your answer.

(D) Describe, which elements are in the set $(K_2 \cap \overline{K}_3) \cup K_5$ in English. For example: *"All ... such that ... is divisible ... and/or is not divisible..."*.

Question 4. In these pictures a red square on the intersection of row i and column j means that the predicate $L(i, j)$ is true (person i loves person j). White square means that the predicate $L(i, j)$ is false. Here $i, j \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$.



Which statement is shown in the left picture?

(A) Everyone is loved by someone.

(B) Eveyone loves someone.

(C) Someone loves everyone.

(D) Someone is loved by everyone.

Which statement is shown in the right picture?

(A) $\forall x \exists y,\ L(y, x)$.

(B) $\forall x \exists y,\ L(x, y)$.

(C) $\exists x \forall y,\ L(x, y)$.

(D) $\exists x \forall y,\ L(y, x)$.

A3. Algorithms and Big-O Notation

In these problems you can verify the definition of the Big-O Notation or find the limit $f(n)/g(n)$ as $n \rightarrow \infty$.

Question 5. For each function find the smallest k such that $f(n)$ is in $O(n^k)$. Justify your answer.

(A) $f(n) = \sum_{j=1}^{n} (j^3 + j \log_2 j)$.

(B) $f(n) = n^3 + \sin n^7$.

(C) $f(n) = 1^2 + 2^2 + \ldots + n^2$.

Question 6. We say that the functions $f(n)$ and $g(n)$ *are of the same order*, if $f(n)$ is in $O(g(n))$ and $g(n)$ is in $O(f(n))$. Find all pairs of functions in this list that are of the same order:

$$n^2 + \log_2 n,\ 2^n + 3^n,\ 100n^3 + n^2,\ n^2 + 2^n,$$

$$n^2 + n^3,\ 3n^3 + 2^n.$$

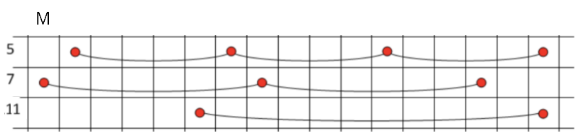
A4. Number Theory

Question 7.

(A) Somebody has written a long hexadecimal number `F0F0F0...F0F` – it uses 28 digits "F" separated by 27 digits "0". Express the value of this number as a short expression (without ...), using the formula of a geometric progression: $\frac{b_1(q^{n+1}-1)}{q-1}$.

(B) Somebody has written an infinite hexadecimal fraction `0.F0F0F0...`. Express it as a rational number in decimal notation.

Question 8



(A) Find some number $M \in \mathbb{Z}^+$ that gives the following remainders when divided by 5, 7 and 11:

$$\begin{cases} x \equiv 4 \pmod{5} \\ x \equiv 0 \pmod{7} \\ x \equiv 6 \pmod{11} \end{cases}$$

(B) Find arithmetic progression containing all such numbers.

Question 9.

(A) Alice has 13-cent coins; Bob has 21-cent coins. Alice wants to pay Bob exactly 1 cent. How to do this?

(B) Alice has 21-cent coins; Bob has 13-cent coins. Alice wants to pay Bob exactly 1 cent. How to do this?

(C) Solve the congruence equation – find x such that $13x \equiv 1$ (mod 21).

(D) Solve the congruence equation – find x such that $13x \equiv 4$ (mod 21).

Part B. Analysis Tasks

In these problems you have to apply concepts (such as predicates or quantifiers) to new situations, describe algorithms or procedures for new tasks or sort cases in adaptive ways.

B1. Propositional Logic

Question 2.1. Among the people A, B, C one is a truth-teller, the other two are liars. Every person (A, B, and C) has a closed box in front of himself/herself. Exactly one of the boxes has a candy inside. A, B, C know everything about each other and the location of candy.
Find out, which YES/NO questions you need to ask to find out, which box contains candy. Ask as few questions as possible ("brute-force" strategies that ask clearly redundant questions may only get partial credit).

Question 2.2.

(a) Find out, if this formula is a tautology:

$$((\mathtt{A} \vee \mathtt{B}) \wedge (\mathtt{A} \rightarrow \mathtt{C}) \wedge (\mathtt{B} \rightarrow \mathtt{C})) \rightarrow \mathtt{C}.$$

(b) Find out, if this formula is satisfiable:

$$\neg(((\mathtt{A} \vee \mathtt{B}) \wedge (\mathtt{A} \rightarrow \mathtt{C}) \wedge (\mathtt{B} \rightarrow \mathtt{C})) \rightarrow \mathtt{C}).$$

Note. A Boolean expression is called *tautology*, if it is true for all possible truth values of its variables. An expression is called *satisfiable*, if there is a way to assign variables so that it becomes true.

B2. Sets and Quantifiers

Part C. Proofs

In proof problems the goal is to prove some general statement by showing every essential step of your reasoning. Below we describe various proof strategies and give some sample statements that can be proven by that strategy.

Translate the problem into algebra
In K.Rosen's textbook these are called "direct proofs" of IF-THEN statements. You assume that the condition is true, introduce some

- If n is odd, then n^2 is also odd.
- If the decimal notation of a number n ends with the digit "5", then n^2 ends with digits "25".

Reason by cases

- $n^2 \equiv 5 \pmod{11}$ if and only if
  $n \equiv 4 \pmod{11}$ or $n \equiv -4 \pmod{11}$.
- Assume that person A says "B always lies." and person B says "We both always tell the truth.".
  It can happen only if A tells the truth and B lies (you can analyze all other cases to see that only this works).

Proofs by Contradiction
In these examples you assume that the statement is false and get a contradiction.

- There are infinitely many primes.
- There are infinitely many primes of the form $4n + 3$.
- Assume that there are infinitely many primes that divide some value of the polynomial $P(n) = n^2 + n + 1$.
- $\sqrt{2}$ is irrational.
- $\log_2 10$ is irrational.

Building Bijective Functions
Any combinations of these tactics can be used to prove that two sets have the same cardinality (and that there is a bijective function from one set to another).

- There is a bijection from $\mathbb{Z}^+ \cup \{1, 2, \ldots, n\}$ to $\mathbb{Z}^+$ (one can add n new guests to the Hilbert's hotel).
- There is a bijection from $\mathbb{Z}^+ \times \{1, 2\}$ to $\mathbb{Z}^+$ (if there are two infinite buses with guests: $(1, 1), (2, 1), (3, 1), \ldots$ and $(1, 2), (2, 2), (2, 3), \ldots$, then they can be hosted in a single Hilbert's hotel).
- There is a bijection from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$ (infinitely many infinite buses can be hosted in a single Hilbert's hotel).
- There is a bijection from $\mathbb{Q}$ to $\mathbb{Z}^+$.

For subsets of real numbers you might need different proof methods:

- There is a bijective function from any closed segment $[a; b]$ to any other closed segment $[c; d]$ (regardless of their lengths). Can be achieved by a linear function.
- There is a bijective function from any open segment $(a; b)$ to any other open segment $(c; d)$.
- There is a bijective function from any open segment $(a; b)$ to the set of all real numbers $\mathbb{R}$ or to the half-line of all positive reals $(0; +\infty)$.
- There is a bijective function from a semi-open segment $[0; 1)$ to an open segment $(0; 1)$.

Proving that there is no Bijection

- Numbers on $(0; 1)$ are uncountable. (Cantor's diagonalization).
- All real numbers $\mathbb{R}$ are uncountable - there is no bijection from $\mathbb{R}$ to $\mathbb{Z}^+$ or to $\mathbb{Q}$.
- All infinite sequences of integers are uncountable (Proof by Cantor's diagonalization).
- All non-decreasing sequences of integers are uncountable.
- The Power-set of any countable set is uncountable. For example, there is no bijection from $\mathcal{P}(\mathbb{Z}^+)$ to $\mathbb{Z}^+$ (i.e. no mapping from the set of all subsets of positive integers to the set of integers themselves).