**N7.** Let $n \geqslant 2018$ be an integer, and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n} \tag{1}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

**Solution.** Suppose that (1) is an arithmetic progression with nonzero difference. Let the difference be $\Delta = \frac{c}{d}$, where $d > 0$ and $c, d$ are coprime.

We will show that too many denominators $b_i$ should be divisible by $d$. To this end, for any $1 \leqslant i \leqslant n$ and any prime divisor $p$ of $d$, say that *the index $i$ is $p$-wrong*, if $v_p(b_i) < v_p(d)$. ($v_p(x)$ stands for the exponent of $p$ in the prime factorisation of $x$.)

*Claim 1.* For any prime $p$, all $p$-wrong indices are congruent modulo $p$. In other words, the $p$-wrong indices (if they exist) are included in an arithmetic progression with difference $p$.

*Proof.* Let $\alpha = v_p(d)$. For the sake of contradiction, suppose that $i$ and $j$ are $p$-wrong indices (i.e., none of $b_i$ and $b_j$ is divisible by $p^\alpha$) such that $i \not\equiv j \pmod p$. Then the least common denominator of $\frac{a_i}{b_i}$ and $\frac{a_j}{b_j}$ is not divisible by $p^\alpha$. But this is impossible because in their difference, $(i-j)\Delta = \frac{(i-j)c}{d}$, the numerator is coprime to $p$, but $p^\alpha$ divides the denominator $d$. □

*Claim 2.* $d$ has no prime divisors greater than 5.

*Proof.* Suppose that $p \geqslant 7$ is a prime divisor of $d$. Among the indices $1, 2, \ldots, n$, at most $\left\lceil \frac{n}{p} \right\rceil < \frac{n}{p} + 1$ are $p$-wrong, so $p$ divides at least $\frac{p-1}{p}n - 1$ of $b_1, \ldots, b_n$. Since these denominators are distinct,

$$5n \geqslant \max\{b_i : \; p \mid b_i\} \geqslant \left(\frac{p-1}{p}n - 1\right)p = (p-1)(n-1) - 1 \geqslant 6(n-1) - 1 > 5n,$$

a contradiction. □

*Claim 3.* For every $0 \leqslant k \leqslant n - 30$, among the denominators $b_{k+1}, b_{k+2}, \ldots, b_{k+30}$, at least $\varphi(30) = 8$ are divisible by $d$.

*Proof.* By Claim 1, the 2-wrong, 3-wrong and 5-wrong indices can be covered by three arithmetic progressions with differences 2, 3 and 5. By a simple inclusion-exclusion, $(2-1)\cdot(3-1)\cdot(5-1) = 8$ indices are not covered; by Claim 2, we have $d \mid b_i$ for every uncovered index $i$. □

*Claim 4.* $|\Delta| < \frac{20}{n-2}$ and $d > \frac{n-2}{20}$.

*Proof.* From the sequence (1), remove all fractions with $b_n < \frac{n}{2}$, There remain at least $\frac{n}{2}$ fractions, and they cannot exceed $\frac{5n}{n/2} = 10$. So we have at least $\frac{n}{2}$ elements of the arithmetic progression (1) in the interval $(0, 10]$, hence the difference must be below $\frac{10}{n/2-1} = \frac{20}{n-2}$.

The second inequality follows from $\frac{1}{d} \leqslant \frac{|c|}{d} = |\Delta|$. □

Now we have everything to get the final contradiction. By Claim 3, we have $d \mid b_i$ for at least $\left\lfloor \frac{n}{30} \right\rfloor \cdot 8$ indices $i$. By Claim 4, we have $d \geqslant \frac{n-2}{20}$. Therefore,

$$5n \geqslant \max\{b_i : \; d \mid b_i\} \geqslant \left(\left\lfloor \frac{n}{30} \right\rfloor \cdot 8\right) \cdot d > \left(\frac{n}{30} - 1\right) \cdot 8 \cdot \frac{n-2}{20} > 5n.$$

**Comment 1.** It is possible that all terms in (1) are equal, for example with $a_i = 2i - 1$ and $b_i = 4i - 2$ we have $\frac{a_i}{b_i} = \frac{1}{2}$.

**Comment 2.** The bound $5n$ in the statement is far from sharp; the solution above can be modified to work for $9n$. For large $n$, the bound $5n$ can be replaced by $n^{\frac{3}{2}-\varepsilon}$.