A6. Let $m, n \geqslant 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \ldots + x_n}{m} \right\rfloor \quad \text{for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

Solution. We transform the problem to a single variable question by the following

Lemma. Let $a_1, \ldots, a_n$ be nonnegative integers and let $G(x)$ be a nonzero polynomial with $\deg G \leqslant a_1 + \ldots + a_n$. Suppose that some polynomial $F(x_1, \ldots, x_n)$ satisfies

$$F(x_1, \ldots, x_n) = G(x_1 + \ldots + x_n) \quad \text{for } (x_1, \ldots, x_n) \in \{0, 1, \ldots, a_1\} \times \ldots \times \{0, 1, \ldots, a_n\}.$$

Then F cannot be the zero polynomial, and $\deg F \geqslant \deg G$.

For proving the lemma, we will use *forward differences* of polynomials. If $p(x)$ is a polynomial with a single variable, then define $(\Delta p)(x) = p(x+1) - p(x)$. It is well-known that if p is a nonconstant polynomial then $\deg \Delta p = \deg p - 1$.

If $p(x_1, \ldots, x_n)$ is a polynomial with n variables and $1 \leqslant k \leqslant n$ then let

$$(\Delta_k p)(x_1, \ldots, x_n) = p(x_1, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_n) - p(x_1, \ldots, x_n).$$

It is also well-known that either $\Delta_k p$ is the zero polynomial or $\deg(\Delta_k p) \leqslant \deg p - 1$.

Proof of the lemma. We apply induction on the degree of G. If G is a constant polynomial then we have $F(0, \ldots, 0) = G(0) \neq 0$, so F cannot be the zero polynomial.

Suppose that $\deg G \geqslant 1$ and the lemma holds true for lower degrees. Since $a_1 + \ldots + a_n \geqslant \deg G > 0$, at least one of $a_1, \ldots, a_n$ is positive; without loss of generality suppose $a_1 \geqslant 1$.

Consider the polynomials $F_1 = \Delta_1 F$ and $G_1 = \Delta G$. On the grid $\{0, \ldots, a_1 - 1\} \times \{0, \ldots, a_2\} \times \ldots \times \{0, \ldots, a_n\}$ we have

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= F(x_1 + 1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) = \\ &= G(x_1 + \ldots + x_n + 1) - G(x_1 + \ldots + x_n) = G_1(x_1 + \ldots + x_n). \end{aligned}$$

Since G is nonconstant, we have $\deg G_1 = \deg G - 1 \leqslant (a_1 - 1) + a_2 + \ldots + a_n$. Therefore we can apply the induction hypothesis to F_1 and G_1 and conclude that F_1 is not the zero polynomial and $\deg F_1 \geqslant \deg G_1$. Hence, $\deg F \geqslant \deg F_1 + 1 \geqslant \deg G_1 + 1 = \deg G$. That finishes the proof. $\square$

To prove the problem statement, take the unique polynomial $g(x)$ so that $g(x) = \left\lfloor \frac{x}{m} \right\rfloor$ for $x \in \{0, 1, \ldots, n(m-1)\}$ and $\deg g \leqslant n(m-1)$. Notice that precisely $n(m-1)+1$ values of g are prescribed, so $g(x)$ indeed exists and is unique. Notice further that the constraints $g(0) = g(1) = 0$ and $g(m) = 1$ together enforce $\deg g \geqslant 2$.

By applying the lemma to $a_1 = \ldots = a_n = m-1$ and the polynomials f and g, we achieve $\deg f \geqslant \deg g$. Hence we just need a suitable lower bound on $\deg g$.

Consider the polynomial $h(x) = g(x+m) - g(x) - 1$. The degree of $g(x+m) - g(x)$ is $\deg g - 1 \geqslant 1$, so $\deg h = \deg g - 1 \geqslant 1$, and therefore h cannot be the zero polynomial. On the other hand, h vanishes at the points $0, 1, \ldots, n(m-1) - m$, so h has at least $(n-1)(m-1)$ roots. Hence,

$$\deg f \geqslant \deg g = \deg h + 1 \geqslant (n-1)(m-1) + 1 \geqslant n.$$

Comment 1. In the lemma we have equality for the choice $F(x_1, \ldots, x_n) = G(x_1 + \ldots + x_n)$, so it indeed transforms the problem to an equivalent single-variable question.

Comment 2. If $m \geqslant 3$, the polynomial $h(x)$ can be replaced by Δg. Notice that

$$(\Delta g)(x) = \begin{cases} 1 & \text{if } x \equiv -1 \pmod{m} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x = 0, 1, \ldots, n(m-1) - 1.$$

Hence, Δg vanishes at all integers x with $0 \leqslant x < n(m-1)$ and $x \not\equiv -1 \pmod{m}$. This leads to $\deg g \geqslant \frac{(m-1)^2 n}{m} + 1$.

If m is even then this lower bound can be improved to $n(m-1)$. For $0 \leqslant N < n(m-1)$, the $(N+1)^{\text{st}}$ forward difference at $x = 0$ is

$$(\Delta^{N+1}) g(0) = \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} (\Delta g)(k) = \sum_{\substack{0 \leqslant k \leqslant N \\ k \equiv -1 \pmod{m}}} (-1)^{N-k} \binom{N}{k}. \qquad (*)$$

Since m is even, all signs in the last sum are equal; with $N = n(m-1) - 1$ this proves $\Delta^{n(m-1)} g(0) \neq 0$, indicating that $\deg g \geqslant n(m-1)$.

However, there are infinitely many cases when all terms in $(*)$ cancel out, for example if m is an odd divisor of $n+1$. In such cases, $\deg f$ can be less than $n(m-1)$.

Comment 3. The lemma is closely related to the so-called

> **Alon–Füredi bound.** *Let $S_1, \ldots, S_n$ be nonempty finite sets in a field and suppose that the polynomial $P(x_1, \ldots, x_n)$ vanishes at the points of the grid $S_1 \times \ldots \times S_n$, except for a single point. Then* $\deg P \geqslant \sum_{i=1}^{n} \big(|S_i| - 1\big)$.
>
> (A well-known application of the Alon–Füredi bound was the former IMO problem 2007/6. Since then, this result became popular among the students and is part of the IMO training for many IMO teams.)

The proof of the lemma can be replaced by an application of the Alon–Füredi bound as follows. Let $d = \deg G$, and let G_0 be the unique polynomial such that $G_0(x) = G(x)$ for $x \in \{0, 1, \ldots, d-1\}$ but $\deg G_0 < d$. The polynomials G_0 and G are different because they have different degrees, and they attain the same values at $0, 1, \ldots, d-1$; that enforces $G_0(d) \neq G(d)$.

Choose some nonnegative integers $b_1, \ldots, b_n$ so that $b_1 \leqslant a_1, \ldots, b_n \leqslant a_n$, and $b_1 + \ldots + b_n = d$, and consider the polynomial

$$H(x_1, \ldots, x_n) = F(x_1, \ldots, x_n) - G_0(x_1 + \ldots + x_n)$$

on the grid $\{0, 1, \ldots, b_1\} \times \ldots \times \{0, 1, \ldots, b_n\}$.

At the point $(b_1, \ldots, b_n)$ we have $H(b_1, \ldots, b_n) = G(d) - G_0(d) \neq 0$. At all other points of the grid we have $F = G$ and therefore $H = G - G_0 = 0$. So, by the Alon–Füredi bound, $\deg H \geqslant b_1 + \ldots + b_n = d$. Since $\deg G_0 < d$, this implies $\deg F = \deg(H + G_0) = \deg H \geqslant d = \deg G$. $\square$