Geometry

Easy

G1. Let ABC be an acute-angled triangle with circumcircle Γ. Let D and E be points on the segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of the segments BD and CE intersect the small arcs $\widehat{AB}$ and $\widehat{AC}$ at points F and G respectively. Prove that $DE \parallel FG$.

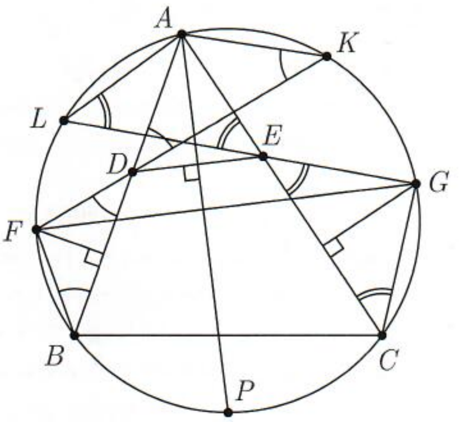
Solution 1. In the sequel, all the considered arcs are small arcs.

Let P be the midpoint of the arc $\widehat{BC}$. Then AP is the bisector of $\angle BAC$, hence, in the isosceles triangle ADE, $AP \perp DE$. So, the statement of the problem is equivalent to $AP \perp FG$.

In order to prove this, let K be the second intersection of Γ with FD. Then the triangle FBD is isosceles, therefore

$$\angle AKF = \angle ABF = \angle FDB = \angle ADK,$$

yielding $AK = AD$. In the same way, denoting by L the second intersection of Γ with GE, we get $AL = AE$. This shows that $AK = AL$.



Now $\angle FBD = \angle FDB$ gives $\widehat{AF} = \widehat{BF} + \widehat{AK} = \widehat{BF} + \widehat{AL}$, hence $\widehat{BF} = \widehat{LF}$. In a similar way, we get $\widehat{CG} = \widehat{GK}$. This yields

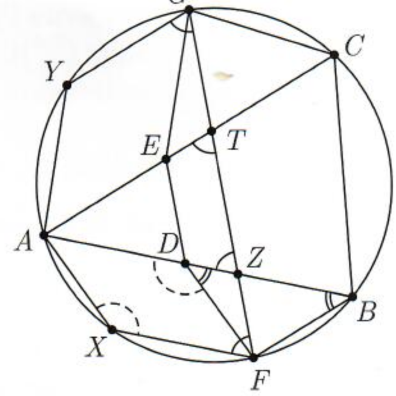
$$\angle(AP, FG) = \frac{\widehat{AF} + \widehat{PG}}{2} = \frac{\widehat{AL} + \widehat{LF} + \widehat{PC} + \widehat{CG}}{2} = \frac{\widehat{KL} + \widehat{LB} + \widehat{BC} + \widehat{CK}}{4} = 90^\circ.$$

Solution 2. Let $Z = AB \cap FG$, $T = AC \cap FG$. It suffices to prove that $\angle ATZ = \angle AZT$.

Let X be the point for which $FXAD$ is a parallelogram. Then

$$\angle FXA = \angle FDA = 180^\circ - \angle FDB = 180^\circ - \angle FBD,$$

where in the last equality we used that $FD = FB$. It follows that the quadrilateral $BFXA$ is cyclic, so X lies on Γ.

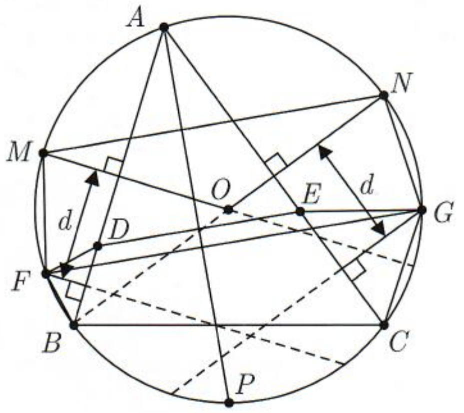


Analogously, if Y is the point for which $GYAE$ is a parallelogram, then Y lies on Γ. So the quadrilateral $XFGY$ is cyclic and $FX = AD = AE = GY$, hence $XFGY$ is an isosceles trapezoid.

Now, by $XF \parallel AZ$ and $YG \parallel AT$, it follows that $\angle ATZ = \angle YGF = \angle XFG = \angle AZT$.

Solution 3. As in the first solution, we prove that $FG \perp AP$, where P is the midpoint of the small arc $\widehat{BC}$.

Let O be the circumcentre of the triangle ABC, and let M and N be the midpoints of the small arcs $\widehat{AB}$ and $\widehat{AC}$, respectively. Then OM and ON are the perpendicular bisectors of AB and AC, respectively.



The distance d between OM and the perpendicular bisector of BD is $\frac{1}{2}AB - \frac{1}{2}BD = \frac{1}{2}AD$, hence it is equal to the distance between ON and the perpendicular bisector of CE.

This shows that the isosceles trapezoid determined by the diameter δ of Γ through M and the chord parallel to δ through F is congruent to the isosceles trapezoid determined by the diameter δ' of Γ through N and the chord parallel to δ' through G. Therefore $MF = NG$, yielding $MN \parallel FG$.

Now

$$\angle(MN, AP) = \frac{1}{2}\left(\widehat{AM} + \widehat{PC} + \widehat{CN}\right) = \frac{1}{4}\left(\widehat{AB} + \widehat{BC} + \widehat{CA}\right) = 90^\circ,$$

hence $MN \perp AP$, and the conclusion follows.