medium/hard

C4.   An *anti-Pascal pyramid* is a finite set of numbers, placed in a triangle-shaped array so that the first row of the array contains one number, the second row contains two numbers, the third row contains three numbers and so on; and, except for the numbers in the bottom row, each number equals the absolute value of the difference of the two numbers below it. For instance, the triangle below is an anti-Pascal pyramid with four rows, in which every integer from 1 to $1 + 2 + 3 + 4 = 10$ occurs exactly once:

$$4$$
$$2 \quad 6$$
$$5 \quad 7 \quad 1$$
$$8 \quad 3 \quad 10 \quad 9 \ .$$

Is it possible to form an anti-Pascal pyramid with 2018 rows, using every integer from 1 to $1 + 2 + \cdots + 2018$ exactly once?

Answer: No, it is not possible.

Solution. Let T be an anti-Pascal pyramid with n rows, containing every integer from 1 to $1+2+\cdots+n$, and let a_1 be the topmost number in T (Figure 1). The two numbers below a_1 are some a_2 and $b_2 = a_1 + a_2$, the two numbers below b_2 are some a_3 and $b_3 = a_1 + a_2 + a_3$, and so on and so forth all the way down to the bottom row, where some a_n and $b_n = a_1 + a_2 + \cdots + a_n$ are the two neighbours below $b_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$. Since the a_k are n pairwise distinct positive integers whose sum does not exceed the largest number in T, which is $1 + 2 + \cdots + n$, it follows that they form a permutation of $1, 2, \ldots, n$.
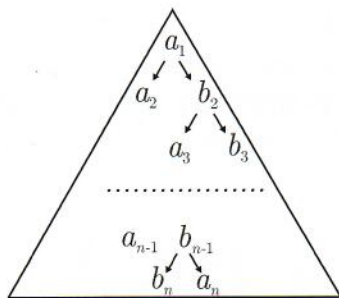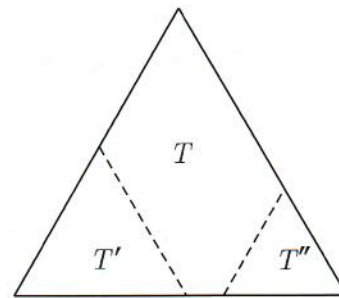

Figure 1   Figure 2

Consider now (Figure 2) the two 'equilateral' subtriangles of T whose bottom rows contain the numbers to the left, respectively right, of the pair a_n, b_n. (One of these subtriangles may very well be empty.) At least one of these subtriangles, say T', has side length $\ell \geqslant \lceil (n-2)/2 \rceil$. Since T' obeys the anti-Pascal rule, it contains ℓ pairwise distinct positive integers $a_1', a_2', \ldots, a_\ell'$, where a_1' is at the apex, and a_k' and $b_k' = a_1' + a_2' + \cdots + a_k'$ are the two neighbours below b_{k-1}' for each $k = 2, 3 \ldots, \ell$. Since the a_k all lie outside T', and they form a permutation of $1, 2, \ldots, n$, the a_k' are all greater than n. Consequently,

$$b_\ell' \geqslant (n+1) + (n+2) + \cdots + (n+\ell) = \frac{\ell(2n+\ell+1)}{2}$$

$$\geqslant \frac{1}{2} \cdot \frac{n-2}{2}\left(2n + \frac{n-2}{2} + 1\right) = \frac{5n(n-2)}{8},$$

which is greater than $1 + 2 + \cdots + n = n(n+1)/2$ for $n = 2018$. A contradiction.

Comment. The above estimate may be slightly improved by noticing that $b_\ell' \neq b_n$. This implies $n(n+1)/2 = b_n > b_\ell' \geqslant \lceil (n-2)/2 \rceil (2n + \lceil (n-2)/2 \rceil + 1)/2$, so $n \leqslant 7$ if n is odd, and $n \leqslant 12$ if n is even. It seems that the largest anti-Pascal pyramid whose entries are a permutation of the integers from 1 to $1 + 2 + \cdots + n$ has 5 rows.