AIME Problems

1. Logarithms (also powers and roots) – interplay between irrational and rational numbers.
2. Complex numbers – how they relate to problems with rational/integer numbers.
3. Trigonometric functions and their properties.
4. Algebraic manipulations with polynomials in number theory.
5. Geometry interpretations of the algebra of rational/irrational numbers.

## 1983.1

Let $x$, $y$ and $z$ all exceed $1$ and let $w$ be a positive number such that $\log_xw=24$, $\log_y w = 40$ and $\log_{xyz}w=12$. Find $\log_zw$.

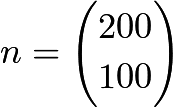
## 1983.5

Suppose that the sum of the squares of two complex numbers $x$ and $y$ is $7$ and the sum of the cubes is $10$. What is the largest real value that $x + y$ can have?

## 1983.6

Let $a_n=6^{n}+8^{n}$. Determine the remainder on dividing $a_{83}$ by $49$.

## 1983.8

What is the largest $2$-digit prime factor of the integer ?

## 1984.1

Find the value of $a_2+a_4+a_6+a_8+\ldots+a_{98}$ if $a_1$, $a_2$, $a_3\ldots$ is an arithmetic progression with common difference 1, and $a_1+a_2+a_3+\ldots+a_{98}=137$.

## 1984.2

The integer $n$ is the smallest positive multiple of $15$ such that every digit of $n$ is either $8$ or $0$. Compute $\frac{n}{15}$.

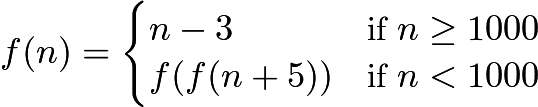
## 1984.4

Let $S$ be a list of positive integers - not necessarily distinct - in which the number $68$ appears. The arithmetic mean of the numbers in $S$ is $56$. However, if $68$ is removed, the arithmetic mean of the numbers is $55$. What's the largest number that can appear in $S$?

## 1984.5

Determine the value of $ab$ if $\log_8a+\log_4b^2=5$ and $\log_8b+\log_4a^2=7$.

## 1984.7

The function f is defined on the set of integers and satisfies 

## 1984.8

The equation $z^6+z^3+1=0$ has complex roots with argument $\theta$ between $90^\circ$ and $180^\circ$ in the complex plane. Determine the degree measure of $\theta$.

## 1985.7

Assume that $a$, $b$, $c$ and $d$ are positive integers such that $a^5 = b^4$, $c^3 = d^2$ and $c - a = 19$. Determine $d - b$.

## 1985.10

How many of the first $1000$ positive integers can be expressed in the form

$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$,

where $x$ is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to $z$?

## 1986.2

Evaluate the product

$(\sqrt 5+\sqrt6+\sqrt7)(-\sqrt 5+\sqrt6+\sqrt7)(\sqrt 5-\sqrt6+\sqrt7)(\sqrt 5+\sqrt6-\sqrt7)$.

## 1986.5

What is that largest [positive integer](https://artofproblemsolving.com/wiki/index.php/Positive_integer) $n$ for which $n^3+100$ is [divisible](https://artofproblemsolving.com/wiki/index.php/Divisible) by $n+10$?

## 1986.7

The increasing sequence $1,3,4,9,10,12,13\cdots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the $100^{\mbox{th}}$ term of this sequence.

## 1986.8

Let $S$ be the sum of the base $10$ logarithms of all the proper divisors of $1000000$. What is the integer nearest to $S$?

## 1987.1

An ordered pair $(m,n)$ of non-negative integers is called "simple" if the addition $m+n$ in base $10$ requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to $1492$.

## 1987.3

By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

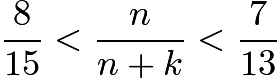
## 1987.5

Find $3x^2 y^2$ if $x$ and $y$ are integers such that $y^2 + 3x^2 y^2 = 30x^2 + 517$.

## 1987.7

Let $[r,s]$ denote the least common multiple of positive integers $r$ and $s$. Find the number of ordered triples $(a,b,c)$ of positive integers for which $[a,b] = 1000$, $[b,c] = 2000$, and $[c,a] = 2000$.

## 1987.8

What is the largest positive integer $n$ for which there is a unique integer $k$ such that ?

## 1988.2

For any positive integer $k$, let $f_1(k)$ denote the square of the sum of the digits of $k$. For $n \ge 2$, let $f_n(k) = f_1(f_{n - 1}(k))$. Find $f_{1988}(11)$.

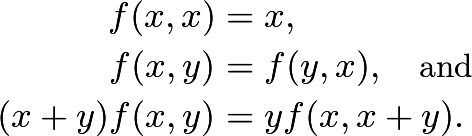
## 1988.3

Find $(\log_2 x)^2$ if $\log_2 (\log_8 x) = \log_8 (\log_2 x)$.

## 1988.5

Let $\frac{m}{n}$, in lowest terms, be the probability that a randomly chosen positive divisor of $10^{99}$ is an integer multiple of $10^{88}$. Find $m + n$.

## 1988.8

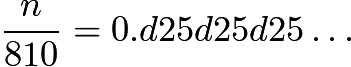
The function $f$, defined on the set of ordered pairs of positive integers, satisfies the following properties:Calculate $f(14,52)$.

## 1988.9

Find the smallest positive integer whose [cube](https://artofproblemsolving.com/wiki/index.php/Perfect_cube) ends in $888$.

## 1989.3

Suppose $n_{}^{}$ is a positive integer and $d_{}^{}$ is a single digit in base 10. Find $n_{}^{}$ if



## 1989.4

If $a<b<c<d<e^{}_{}$ are consecutive positive integers such that $b+c+d^{}_{}$ is a perfect square and $a+b+c+d+e^{}_{}$ is a perfect cube, what is the smallest possible value of $c^{}_{}$?

## 1989.5

When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to $0^{}_{}$ and is the same as that of getting heads exactly twice. Let $\frac ij^{}_{}$, in lowest terms, be the probability that the coin comes up heads in exactly $3_{}^{}$ out of $5^{}_{}$ flips. Find $i+j^{}_{}$.

## 1989.7

If the integer $k^{}_{}$ is added to each of the numbers $36^{}_{}$, $300^{}_{}$, and $596^{}_{}$, one obtains the squares of three consecutive terms of an arithmetic series. Find $k^{}_{}$.

## 1989.8

Assume that $x_1,x_2,\ldots,x_7$ are real numbers such that

$x_1+4x_2+9x_3+16x_4+25x_5+36x_6+49x_7=1^{}_{}$

$4x_1+9x_2+16x_3+25x_4+36x_5+49x_6+64x_7=12^{}_{}$

$9x_1+16x_2+25x_3+36x_4+49x_5+64x_6+81x_7=123^{}_{}$

Find the value of $16x_1+25x_2+36x_3+49x_4+64x_5+81x_6+100x_7^{}$.

## 1989.9

One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer $n$ such that $133^5+110^5+84^5+27^5=n^{5}_{}$. Find the value of $n^{}_{}$.

## 1990.5

Let $n^{}_{}$ be the smallest positive integer that is a multiple of $75_{}^{}$ and has exactly $75_{}^{}$ positive integral divisors, including $1_{}^{}$ and itself. Find $\frac{n}{75}$.

## 1990.9

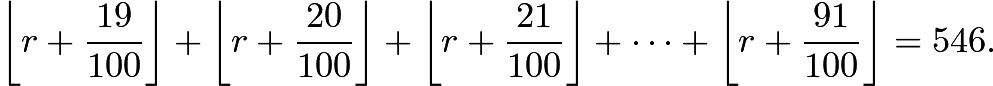
A fair coin is to be tossed $10_{}^{}$ times. Let $i/j^{}_{}$, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i+j_{}^{}$.

## 1991.5

Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will $20_{}^{}!$ be the resulting product?

## 1991.6

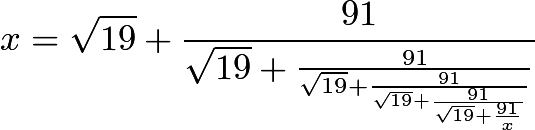
Suppose $r^{}_{}$ is a real number for which



Find $\lfloor 100r \rfloor$. (For real $x^{}_{}$, $\lfloor x \rfloor$ is the greatest integer less than or equal to $x^{}_{}$.)

## 1991.7

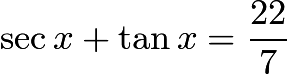
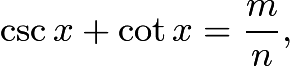
Find $A^2_{}$, where $A^{}_{}$ is the sum of the absolute values of all roots of the following equation:



## 1991.8

For how many real numbers $a^{}_{}$ does the quadratic equation $x^2 + ax^{}_{} + 6a=0$ have only integer roots for $x^{}_{}$?

## 1991.9

Suppose that  and that  where $\frac mn$ is in lowest terms. Find $m+n^{}_{}.$

## 1991.10

Two three-letter strings, $aaa^{}_{}$ and $bbb^{}_{}$, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a 1/3 chance of being received incorrectly, as an $a^{}_{}$ when it should have been a $b^{}_{}$, or as a $b^{}_{}$ when it should be an $a^{}_{}$. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. Let $S_a^{}$ be the three-letter string received when $aaa^{}_{}$ is transmitted and let $S_b^{}$ be the three-letter string received when $bbb^{}_{}$ is transmitted. Let $p$ be the probability that $S_a^{}$ comes before $S_b^{}$ in alphabetical order. When $p$ is written as a fraction in lowest terms, what is its numerator?

## 1992.1

Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

## 1992.4

In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3: 4: 5$?

## 1992.5

Let $S^{}_{}$ be the set of all rational numbers $r^{}_{}$, $0^{}_{}<r<1$, that have a repeating decimal expansion in the form $0.abcabcabc\ldots=0.\overline{abc}$, where the digits $a^{}_{}$, $b^{}_{}$, and $c^{}_{}$ are not necessarily distinct. To write the elements of $S^{}_{}$ as fractions in lowest terms, how many different numerators are required?

## 1992.8

For any sequence of real numbers $A=(a_1,a_2,a_3,\ldots)$, define $\Delta A^{}_{}$ to be the sequence $(a_2-a_1,a_3-a_2,a_4-a_3,\ldots)$, whose $n^{th}$ term is $a_{n+1}-a_n^{}$. Suppose that all of the terms of the sequence $\Delta(\Delta A^{}_{})$ are $1^{}_{}$, and that $a_{19}=a_{92}^{}=0$. Find $a_1^{}$.

## 1992.10

Consider the region $A^{}_{}$ in the complex plane that consists of all points $z^{}_{}$ such that both $\frac{z^{}_{}}{40}$ and $\frac{40^{}_{}}{\overline{z}}$ have real and imaginary parts between $0^{}_{}$ and $1^{}_{}$, inclusive. What is the integer that is nearest the area of $A^{}_{}$?

## 1993.6

What is the smallest positive integer than can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

## 1993.7

Three numbers, $a_1\,$, $a_2\,$, $a_3\,$, are drawn randomly and without replacement from the set $\{1, 2, 3, \dots, 1000\}\,$. Three other numbers, $b_1\,$, $b_2\,$, $b_3\,$, are then drawn randomly and without replacement from the remaining set of 997 numbers. Let $p\,$ be the probability that, after a suitable rotation, a brick of dimensions $a_1 \times a_2 \times a_3\,$ can be enclosed in a box of dimensions $b_1 \times b_2 \times b_3\,$, with the sides of the brick parallel to the sides of the box. If $p\,$ is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

## 1994.1

The increasing sequence $3, 15, 24, 48, \ldots\,$ consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994th term of the sequence is divided by 1000?

## 1994.3

The function $f_{}^{}$ has the property that, for each real number $x,\,$

$f(x)+f(x-1) = x^2\,$.

If $f(19)=94,\,$ what is the remainder when $f(94)\,$ is divided by 1000?

## 1994.4

Find the positive integer $n\,$ for which

$\lfloor \log_2{1}\rfloor+\lfloor\log_2{2}\rfloor+\lfloor\log_2{3}\rfloor+\cdots+\lfloor\log_2{n}\rfloor=1994$.

(For real $x\,$, $\lfloor x\rfloor\,$ is the greatest integer $\le x.\,$)

## 1994.5

Given a positive integer $n\,$, let $p(n)\,$ be the product of the non-zero digits of $n\,$. (If $n\,$ has only one digit, then $p(n)\,$ is equal to that digit.) Let

$S=p(1)+p(2)+p(3)+\cdots+p(999)$.

What is the largest prime factor of $S\,$?

## 1994.7

For certain ordered pairs $(a,b)\,$ of real numbers, the system of equations

$ax+by=1\,$

$x^2+y^2=50\,$

has at least one solution, and each solution is an ordered pair $(x,y)\,$ of integers. How many such ordered pairs $(a,b)\,$ are there?

## 1994.8

The points $(0,0)\,$, $(a,11)\,$, and $(b,37)\,$ are the vertices of an equilateral triangle. Find the value of $ab\,$.

## 1994.9

A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. The probability that the bag will be emptied is $p/q,\,$ where $p\,$ and $q\,$ are relatively prime positive integers. Find $p+q.\,$

## 1995.2

Find the last three digits of the product of the positive roots of $\sqrt{1995}x^{\log_{1995}x}=x^2.$

## 1995.3

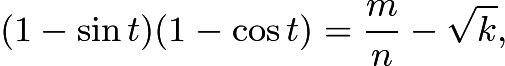
Starting at $(0,0),$ an object moves in the coordinate plane via a sequence of steps, each of length one. Each step is left, right, up, or down, all four equally likely. Let $p$ be the probability that the object reaches $(2,2)$ in six or fewer steps. Given that $p$ can be written in the form $m/n,$ where $m$ and $n$ are relatively prime positive integers, find $m+n.$

## 1995.6

Let $n=2^{31}3^{19}.$ How many positive integer divisors of $n^2$ are less than $n_{}$ but do not divide $n_{}$?

## 1995.7

Given that $(1+\sin t)(1+\cos t)=5/4$ and



where $k, m,$ and $n_{}$ are positive integers with $m_{}$ and $n_{}$ relatively prime, find $k+m+n.$

## 1995.8

For how many ordered pairs of positive integers $(x,y),$ with $y<x\le 100,$ are both $\frac xy$ and  integers?

## 1995.10

What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a positive composite integer?

## 1996.2

For each real number $x$, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed $x$. For how many positive integers $n$ is it true that $n<1000$ and that $\lfloor \log_{2} n \rfloor$ is a positive even integer?

## 1996.6

In a five-team tournament, each team plays one game with every other team. Each team has a $50\%$ chance of winning any game it plays. (There are no ties.) Let $\dfrac{m}{n}$ be the probability that the tournament will produce neither an undefeated team nor a winless team, where $m$ and $n$ are relatively prime integers. Find $m+n$.

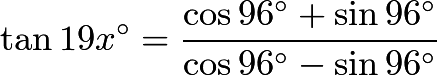
## 1996.8

The harmonic mean of two positive integers is the reciprocal of the arithmetic mean of their reciprocals. For how many ordered pairs of positive integers $(x,y)$ with $x<y$ is the harmonic mean of $x$ and $y$ equal to $6^{20}$?

## 1996.9

A bored student walks down a hall that contains a row of closed lockers, numbered 1 to 1024. He opens the locker numbered 1, and then alternates between skipping and opening each locker thereafter. When he reaches the end of the hall, the student turns around and starts back. He opens the first closed locker he encounters, and then alternates between skipping and opening each closed locker thereafter. The student continues wandering back and forth in this manner until every locker is open. What is the number of the last locker he opens?

## 1996.10

Find the smallest positive integer solution to .

## 1997.1

How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

## 1997.2

The nine horizontal and nine vertical lines on an $8\times8$ checkerboard form $r$ rectangles, of which $s$ are squares. The number $s/r$ can be written in the form $m/n,$ where $m$ and $n$ are relatively prime positive integers. Find $m + n.$

## 1997.3

Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

## 1997.5

The number $r$ can be expressed as a four-place decimal $0.abcd,$ where $a, b, c,$ and $d$ represent digits, any of which could be zero. It is desired to approximate $r$ by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to $r$ is $\frac 27.$ What is the number of possible values for $r$?

## 1997.9

Given a nonnegative real number $x$, let $\langle x\rangle$ denote the fractional part of $x$; that is, $\langle x\rangle=x-\lfloor x\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to $x$. Suppose that $a$ is positive, $\langle a^{-1}\rangle=\langle a^2\rangle$, and $2<a^2<3$. Find the value of $a^{12}-144a^{-1}$.

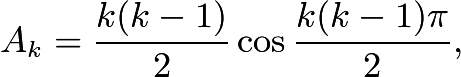
## 1998.1

For how many values of $k$ is $12^{12}$ the [least common multiple](https://artofproblemsolving.com/wiki/index.php/Least_common_multiple) of the positive integers $6^6$ and $8^8$, and $k$?

## 1998.4

Nine tiles are numbered $1, 2, 3, \cdots, 9,$ respectively. Each of three players randomly selects and keeps three of the tiles, and sums those three values. The [probability](https://artofproblemsolving.com/wiki/index.php/Probability) that all three players obtain an [odd](https://artofproblemsolving.com/wiki/index.php/Odd) sum is $m/n,$ where $m$ and $n$ are [relatively prime](https://artofproblemsolving.com/wiki/index.php/Relatively_prime) [positive integers](https://artofproblemsolving.com/wiki/index.php/Positive_integer). Find $m+n.$

## 1998.5

Given that  find $|A_{19} + A_{20} + \cdots + A_{98}|.$

## 1998.8

Except for the first two terms, each term of the sequence $1000, x, 1000 - x,\ldots$ is obtained by subtracting the preceding term from the one before that. The last term of the sequence is the first [negative](https://artofproblemsolving.com/wiki/index.php/Negative) term encountered. What positive integer $x$ produces a sequence of maximum length?

## 1998.9

Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly $m$ minutes. The [probability](https://artofproblemsolving.com/wiki/index.php/Probability) that either one arrives while the other is in the cafeteria is $40 \%,$ and $m = a - b\sqrt {c},$ where $a, b,$ and $c$ are [positive](https://artofproblemsolving.com/wiki/index.php/Positive) [integers](https://artofproblemsolving.com/wiki/index.php/Integer), and $c$ is not divisible by the square of any [prime](https://artofproblemsolving.com/wiki/index.php/Prime). Find $a + b + c.$

## 1999.1

Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms also being prime.

## 1999.3

Find the sum of all positive integers $n$ for which $n^2-19n+99$ is a perfect square.

## 1999.5

For any positive integer $x_{}$, let $S(x)$ be the sum of the digits of $x_{}$, and let $T(x)$ be $|S(x+2)-S(x)|.$ For example, $T(199)=|S(201)-S(199)|=|3-19|=16.$ How many values of $T(x)$ do not exceed 1999?

## 1999.7

There is a set of 1000 switches, each of which has four positions, called $A, B, C$, and $D$. When the position of any switch changes, it is only from $A$ to $B$, from $B$ to $C$, from $C$ to $D$, or from $D$ to $A$. Initially each switch is in position $A$. The switches are labeled with the 1000 different integers $(2^{x})(3^{y})(5^{z})$, where $x, y$, and $z$ take on the values $0, 1, \ldots, 9$. At step $i$ of a 1000-step process, the $i$-th switch is advanced one step, and so are all the other switches whose labels divide the label on the $i$-th switch. After step 1000 has been completed, how many switches will be in position $A$?

## 1999.9

A function $f$ is defined on the complex numbers by $f(z)=(a+bi)z,$ where $a_{}$ and $b_{}$ are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi|=8$ and that $b^2=m/n,$ where $m_{}$ and $n_{}$ are relatively prime positive integers. Find $m+n.$

## 1999.10

Ten points in the plane are given, with no three collinear. Four distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. The probability that some three of the segments form a triangle whose vertices are among the ten given points is $m/n,$ where $m_{}$ and $n_{}$ are relatively prime positive integers. Find $m+n.$