Pigeonhole principle

2022-04-22

# Part 1: Questions

**Question 1:**

In a very dark room there is a dark wardrobe – it contains socks of different colors; there are socks of every color. What is the smallest number of socks that should be randomly retrieved from this wardrobe to be guaranteed to get socks of the same color.

**Question 2:**

Some factory produces bricks – each brick has weight between 2.7 kilograms and 3.0 kilograms. Each brick has its weight written on it (rounded to the nearest gram).

What is the smallest number of bricks that we need to purchase to ensure that there are at least two bricks with the same weight in grams written on them.

**Question 3:**

What is the smallest possible count of positive integers you need to remove from the sequence so that among the remaining numbers one cannot find two numbers such that ?

**Question 4:**

rabbits are somehow placed into cages. Which statement is always correct (regardless of how they are placed):

1. There must be a cage containing exactly two rabbits.
2. There must be a cage that is empty.
3. There must be a cage containing exactly one rabbit.
4. There must be at least two cages containing at least one rabbit each.

**Question 5:**

We randomly draw play cards (out of a pack of playing cards). Playing cards are in two colors - black or red.

What number of the playing cards (out of the ) is guaranteed to be in the same color?

**Question 6:**

In a very dark wardrobe there are black, blue and green socks. What is the smallest number of socks to be drawn randomly from this wardrobe in order to get either two black or two blue socks? The answer should reflect the worst-case scenario – what number of socks always guarantees that two black or two blue ones will be selected.

**Question 7:**

Language contains all -letter words made from two letters “A” and “B” (words containing several equal letters next to each other are also allowed - for example “AAAAA” or “BBBBA”).

How many words in this language should be written down so that we necessarily have two equal words among them?

**Question 8:**

What is the smallest number of numbers to be selected out of the numbers so that there must be two such numbers (among the selected ones) which have sum ?

**Question 9:**

In a open-air concert there are very long benches. There are girls already sitting on these benches (we do not know how many of the girls sit on every bench).

What is the largest number of boys that can be seated in this concert, if no two boys can sit next to each other on the same bench?

**Question 10:**

What is the smallest people that should be riding on the same bus so that at least of them have their birthdays on the same month?

**Question 11:**

A dealer of used cars has Audi cars, BMW cars, VW cars and Volvo cars. For what smallest number of cars sold it is possible to say that among the sold cars there must be at least cars of the same type?

**Question 12:**

There are people in a room; and the following statement is true: “At least three of them are born on the same weekday (Monday through Sunday). But it might happen that it is impossible to find four people who are all born on the same weekday.” Find the largest value of for which this statement is true.

**Question 13:**

Decorating a cake requires either two oranges, or three apples, or five apricots or seven cherries. Little Mia bought fruit (every fruit is either an orange, an apple, an apricot or a cherry). What is the smallest value of to guarantee that there will be enough fruit to decorate the cake?

**Question 14:**

On a remote island there is an elementary school with students. Each student has exactly two grandfathers. Every two students have at least one grandfather in common. It is also known that there does not exist a grandfather who is common to all students.

What is the largest possible number of the grandfathers (who are grandfathers for at least one student in this school)?

**Question 15:**

What is the largest possible amount of numbers that can be chosen out of so that there do **not** exist two selected numbers having their sum divisible by ?

**Question 16:**

Consider a quadratic grid of points ( points ordered in horizontal lines and vertical columns). What is the smallest number of points out of the that should be colored black so that there exists a horizontal or a vertical line containing at least black points?

**Question 17:**

There are students in some class. Each student somehow chooses other students and sends an Easter greeting to each of the chosen students. What is the smallest value of which guarantees that there are two such students who have sent Easter greetings to each other?

**Question 18:**

Candies are put into bags – each bag may contain any number of candy between and (including both endpoints). It is known that whenever candies are put in the bags in this way – either there are two bags with the same number of candy (or two bags where the counts of candy add up to ).

Find the smallest possible value of .

**Question 20:**

Fröken Bock (Nanny Bock/Bokas jaunkundze) every day goes up the stair exactly steps (Level 0 to Level 49). Each of her steps goes up either , or levels. She always steps up so that the total number of steps (to overcome all levels) is . (She never needs to step down, since Karlsson-on-the-roof always magically brings her down again.)

Assume that Fröken Bock has already stepped up that staircase five times. Let denote the largest number of times she has stepped on some level (not counting Level 0 and Level 49 - her starting point and destination).

What is the smallest possible value of ?

**Question 21:**

There are identically looking coins; their masses are all different. We need to find the heaviest coin. We also have scales (which can compare masses of two coins at a time). We know that uses of scales is insufficient to find the heaviest coin.

Find the smallest for which this is true.

**Question 22:**

Michael tossed a coin times; and wrote either or (depending on whether it was heads or tails). What is the smallest value such that there is necessarily two identical substrings of length . (Substrings may also overlap; for example, “HTHTH” contains two identical fragments “HTH”.)

**Question 23:**

What is the largest number of chess bishops to be placed on a chessboard of size so that these bishops do not attack each other. (Bishop attacks all the cells that are placed on the same diagonal as the current position of the bishop.)

**Question 24:**

What is the smallest number of rooks to place on a chessboard of size cells so that there are necessarily three rooks who do not attack each other (i.e. are not located on the same horizontal or the same vertical)?

In this problem you choose the number of rooks, but not the way how they are placed on the board. Namely, what is the number of rooks that guarantees that some rooks can be chosen so that neither of them attacks the other two.

**Question 25:**

On a horizontal line we paint a sequence of points (the distance of any two neighboring points is exactly centimenter) Each point is either blue, green or red.

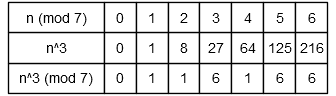
What is the smallest number of points you should select (without looking at their colors and exact locations) so that there will always exist some three points such that they are all of the same color; and all their pairwise distances are divisible by ?

**Question 26:**

On the plane there are points; no three points are located on the same line. Every two of these points are connected with a line in one of four colors (blue, green, orange or red). What is the **largest** such that you can always find some line segments that are in the same color?

**Question 27:**

The table summarizes positive integers, their cubes, and the remainders of their cubes when divided by . How many positive integers (none of them divisible by ) should you choose so that you can always find three integers such that their cubes are all congruent modulo .



**Question 28:**

Michael chose some integers from the interval ; their squares give different remainders, if divided by . (For example, Michael cannot choose and at the same time, since and are both congruent to modulo .) What is the largest number of numbers that Michael can choose in this way?

**Question 29:**

All the numbers from to are partitioned into multiple geometric series with common ratio :

Each progression starts with an odd number, every next number in a series is twice the previous one. Some series (if they start with or something larger) may contain just one term. This is how some series look like:

What is the smallest number of integers that one should select from to guarantee that two selected numbers are in the same progression?

**Question 30:**

What is the largest number of positive integers from to that one can select so that no two selected numbers have their difference , or , or . (All the other differences are allowed.)

**Question 31:**

There is a set of positive integers that contains some numbers from . It satisfies the following property: If contains numbers and , then is not divisible by . What is the largest possible value of (the number of elements in )?

**Question 32:**

On a remote island there is an elementary school with students. Each student has exactly two grandfathers. Every two students have at least one grandfather in common.

Somebody claims that there exists a grandfather with at least grandchildren among the students in this school (without even knowing anything about their relationships). What is the largest value of for which this claim can be reliably made?

**Question 33:**

rabbits have been somehow distributed among cages. Let denote the number of rabbits in the cage, which has the largest rabbit population.

Identify the inequality that is necessarily satisfied by .  
(Here denotes the *floor function* - the largest integer not exceeding . For example, and .)

**Question 33:**

In a full set of playing cards there are different cards in four suits (diamonds, clubs, hearts and spades). In every suit there are ranks (“2” (the Two), “3”, “4”, “5”, “6”, “7”, “8”, “9”, “10” (the Ten), “J” (Jack), “Q” (Queen), “K” (King), and “A” (ace)).

What is the smallest number of playing cards that one must select to guarantee that among the selected cards there will be cards of the same suit and neighboring ranks? (For example “2 and 3 of diamonds” or “10 and J of hearts” or “J and Q of clubs” or “K and A of spades”.)

**Question 34:**

The set contains some positive integers from . It is known that for any two elements , . What is the largest possible value of ?

# Part 2: Answers

**Question 1:**

Answer: 13

The retrieved socks are like “pigeons” and their possible colors are like “pigeonholes”. Since there are colors/pigeonholes, we need at least socks to guarantee that two will belong to the same category.

socks clearly does not guarantee this, since every sock can be in a different color.

**Question 2:**

Answer: 302

In the interval there are exactly integer numbers. Once you choose bricks, at least two of them will round to the same number of grams.

**Question 3:**

Answer: 4

There are "pigeonholes - number pairs that add up to : , , , . If we remove one number from each pair, then we destroy all the possibilities to get the sum equal to .

If we would remove fewer numbers (three or less), then at least one pair stays intact and their sum will be .

**Question 4:**

Answer: b

If there are more cages than the rabbits, then one cage must be empty. (This is a variant of pigeonhole principle - not every cage receives a rabbit.) Other statements can be refuted by counterexamples.

**Question 5:**

Answer: 8

If there would be only 7 cards from each color (black and red), then their total cannot be more than .

**Question 6:**

Answer: 103

It is not enough to retrieve socks, since the “worst-case” scenario can give you green socks, black and blue. Retrieving socks is always enough, since at least of them cannot be green (so two will be of the same color).

**Question 7:**

Answer: 9

In language there different words. Once you write words, there must be a word that is written twice.

**Question 8:**

Answer: 5

One can choose the number and then one number from every pair , , , .

If one chooses more than five numbers, then at least two numbers would fall into the same pair; so there will be a sum equal to .

**Question 9:**

Answer: 110

Before any girls had arrived, there were “cages” (every bench is like a “cage”). Every seated girl increases the number of “cages” by , as it divides some bench or its part into two. When all girls are seated, there will be “cages”  
(bench segments which are appropriate to seat one boy each).  
Therefore the count of boys cannot exceed .

**Question 10:**

Answer: 61

There are months. If there just people, it is possible to have people born every month. As soon as you have people, at least one month should have at least people.

**Question 11:**

Answer: 17

If there are just cars, it is possible that only cars of every category are sold. Once there are cars it is impossible, since .

**Question 12:**

Answer: 21

If the number of people is between and (including both endpoints) we would get (by the generalized Pigeonhole principle) at least people born on the “most popular” weekday (in the meantime, we do not necessarily need any weekday with people as )

As soon as , there will be a weekday with people, using the same generalized Pigeonhole principle.

**Question 13:**

Answer: 14

The number is the worst-case way to bring the fruit ( orange, apples, appricots and cherries). As soon as one more fruit is added (the total count of fruit is ) then at least one category of fruit will reach its threshold value.

**Question 14:**

Answer: 3

If there are three grandfathers , then the conditions are possible to satisfy: If every student has a pair of grandfathers (, , or ), then every two students share some grandfather – since every two pairs intersect.

What happens, if we have grandfathers? Let us consider the students one at a time. Let the grandfathers of be named and . Denote by some student who does not have the same pair of grandfathers; let be his grandfathers. Such must exist, since no grandfather (say, ) can be a grandfather of everyone.

For a similar reason there must be with grandfather pair – since there cannot be everyone’s grandfather and grandfathers must intersect with both and also .

As we consider students – they must have some existing pair , , or . Introducing a new grandfather is impossible – otherwise it won’t intersect with some of the existing pairs , , or .

**Question 15:**

Answer: 23

Kādu lielāko skaitu skaitļu var izvēlēties no kopas tā, lai starp izvēlētajiem neatrastos divi dažādi skaitļi, kuru summa dalās ar ?

Classify the numbers into congruence classes:

* There are numbers () with remainder when dividing with .
* There are numbers () with remainder when dividing with .
* There are numbers () with remainder . And so on.
* There are numbers () with remainder .

To avoid any sum divisible by , we must not select two numbers with remainders . We also must not pick remainders and (or and , or and ).

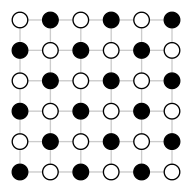
The best option is to pick one number with remainder , all possible numbers with remainders , , and . The total count is .

**Question 16:**

Answer: 19

If you color points black, there will be at least black points on the same horizontal line.

If you choose only black points, then it is possible to color the grid to avoid that, for example, in the checkerboard pattern.



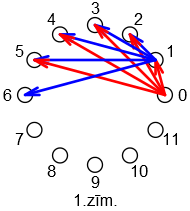
**Question 17:**

Answer: 6

Represent the students as points on a circle. They are connected by edges (which are “channels” to exchange the letters - each channel can be used in one or in both directions).

If eveybody sends greetings, there is the total of = letters, and some letter will not get a separate “channel” - so two letters will be exchanged on the same channel (these two students will send letters to each other).

To create a counterexample with just greetings, denote the students with numbers to . A student with number sends five letters to the students (modulo ). For this construction it is impossible to have two students who send letters to each other, since (where ) cannot be congruent to modulo .



**Question 18:**

Answer: 106

To build the counterexample for , compute the arithmetic expression:

As soon as the count of candy exceeds , it is impossible to have bags of candy. We need at least bags, so at least two of the bags will contain identical number of candy (or we will need to choose both numbers from some pair , , , , – so we will end up having two bags that add up to ).

**Question 20:**

Answer: 4

The total number of levels (excepting levels and ) is . After making steps upward Fröken Bock is on level , but after making steps upward she is on level ).

Therefore she spends exactly steps somewhere in the middle. The product is . If we try to distribute “evenly” all her steps into “buckets” (i.e. levels on the staircase), we will get that at least one level gets of her steps.

(**Note:** Fröken Bock’s ability to step up exactly , , or levels is not very essential – one could make it possible for her to step up any larger amount of steps as well, we would still get the same answer.)

**Question 21:**

Answer: 5

If there are coins, it is possible to make exactly comparisons so that at every stage a coin (the current maximum) is compared with all the remaining ones. And whichever coin turns out to be heavier, is promoted to the next round.

If there are coins, imagine them all as “isolated islands” Every time you compare any two coins, you build a bridge between the islands (reducing the number of islands by ).

After you build three bridges, there are still isolated pieces that have never been compared between themeselves. Announcing that some coin is the heaviest at this point may lead to an error (since the actual heaviest coin may happen to be on another isolated piece).

**Question 22:**

Answer: 11

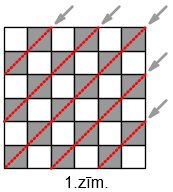
If the length of the coin tosses is , then it must have fragments of length (the first fragment starts from the 1st letter, the last one starts from the 9th letter). From these fragments at least two will be identical, since the two letters () can form only fragments of length .

The answer is too small, since you can have the sequence , where every 3-letter fragment occurs exactly once.

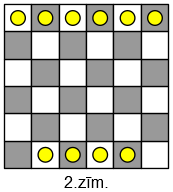
**Question 23:**

Answer: 10

There are two kinds of bishops – on black and on white cells. Let us consider black diagonals only. There is one diagonal direction which has only black diagonals (the other has ). Clearly, each of these black diagonals can contain no more than one bishop. So, there can be no more than “black” bishops. Same thing is true about the “white” bishops. So the total number of bishops cannot exceed .



Here is a way to place bishops:



**Question 24:**

Answer: 13

If there are or fewer rooks, you can place them on two horizontal lines (). Clearly, you cannot select rooks that mutually do not attack each other (as two of them will be on the same horizontal).

As soon as you have rooks, find some horizontal containing at least three rooks. (Such a horizontal must exist by the generalized Pigeonhole principle). Mark all the three rooks on this horizontal, and after that - strike out this horizontal.

We how have an estimate from above (no horizontal can contain more than rooks); so there must be rooks on the horizontals that have not been striked. From the remaining horizontals find some horizontal containing at least rooks - and strike out that as well. Finally there is at least rook on some other horizontal .

Now we pick the only remaining rook from , then some non-conflicting rook from (it has two rooks), and, finally, some non-conflicting rook from (which does not attack anything we have selected before). After this whole process we have constructed three rooks that do not attack each other.

**Question 25:**

Answer: 19

Introduce some “origin” point on this coordinate line – declare that its coordinate is (and all the other colored points get other integer coordinates).

For each colored point find the remainder of its coordinate when divided by (you can get three remainders , , or ). Now categorize all points into categories – depending on their color and remainder (you could have, e.g. “red points congruent to modulo ” or “blue points congruent to modulo ” and so on).

As soon as the number of points reaches , you would get at least points in some category – in this case they are all in the same color, and all their mutual distances are divisible by .

**Question 26:**

Answer: 14

Out of points you can pick the two endpoints of some line segment in different ways. One can color these line segments in different colors. So there must be at least segments that have the same color.

**Question 27:**

Answer: 4

There are altogether remainders, if you divide cubes of integer numbers by . So, if you choose at least numbers, there will be two equal remainders.

**Question 28:**

Answer: 6

We will always have . Therefore we must have:

* ,
* ,
* ,
* ,
* .

So, we can have only different remainders (also including ), but cannot have remainders.

**Question 29:**

Answer: 51

In every progression there is exactly one number that is not divisible by (and there are exactly odd numbers – ). For each odd number we have its own progression.

As soon as you select one more number (i.e.  numbers out of ), there will be two numbers in the same progression, and their ratio will be some power of two: .

**Question 30:**

Answer: 240

Out of every numbers it is possible to choose no more than numbers. Let us consider the sequence (or any other consecutive numbers shifted forward). Obviously, one cannot choose more than one number out of the triplets or , since all the differences between the numbers in these triplets are “forbidden” (equal to , vai ).

So, the only way to choose three numbers out of these seven would be - to choose the number and also one number out of each triplet ; . The number conflicts with numbers and (so we must choose number from the triplet ).

On the other hand, coflicts with both and (so we must also choose from the triplet ). But and at the same time are impossible, since their difference is . So it is impossible to pick three numbers out of any seven consecutive numbers; we can pick at most two non-conflicting numbers.

Next, subdivide all the numbers from to into groups (seven consecutive numbers in each group). The maximum that can be selected in those groups is .

**Question 31:**

Answer: 34

You can choose the set with all those numbers that are congruent to modulo . Namely,

Then the sum of any two numbers will be congruent to modulo . Meanwhile, their difference will be divisible by . Thus, the sum is never divisible by (since it is not divisible by ).

As soon as you pick more than numbers, there will be at least two numbers with their distance equal to or (and in this case is always divisible by ).