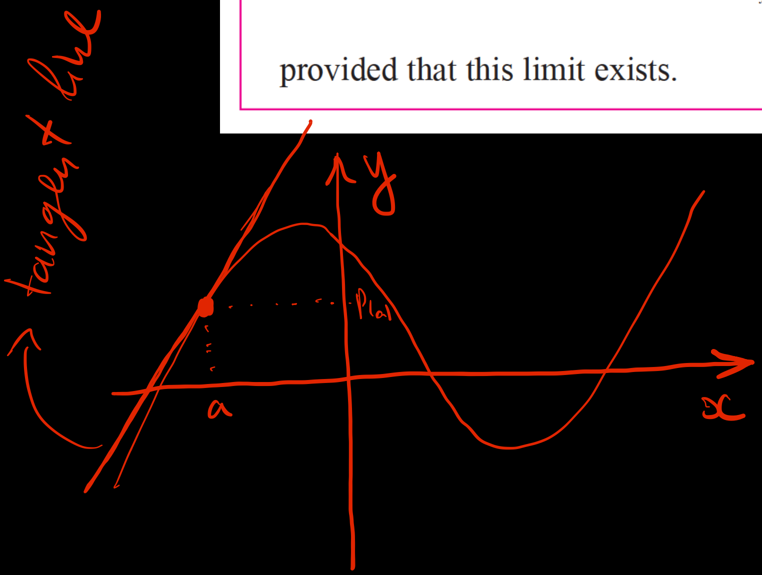
$\boxed{1}$ **Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

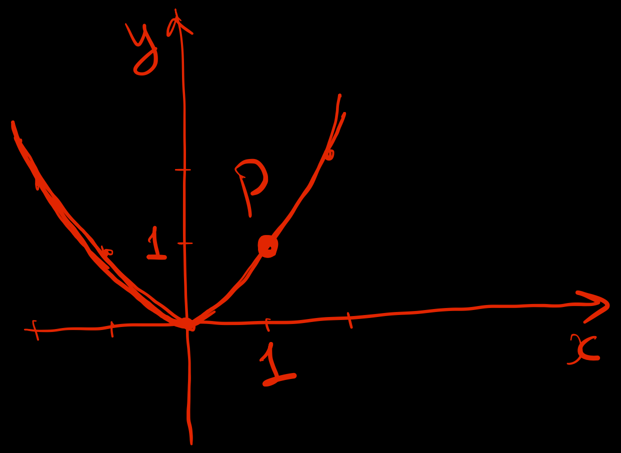



$$M = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The linear function $h(x) = mx + b$ is called tangent line of the function $f(x)$ at point $x = a$.

**EXAMPLE 1** Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Sol: $f(x) = x^2$ is quadratic function $a=1$, $f(a)=1$




$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} =$$

$$= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1}(x+1) = 2$$

Since the tangent line $h(x) = mx + b$ passes through the point $P(1,1)$ then $h(1) = 1$

Hence $1 = 2 \cdot 1 + b \Rightarrow b = 1 - 2 = -1$

Ans: Analytic form of tangent line is

$h(x) = 2x - 1$

**EXAMPLE 2** Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Sol: $f(x) = \frac{3}{x}$ hyperbolic function $a=3$, $f(a)=1$

Hence, $m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3} = \frac{0}{0}$

$$\lim_{x\to 3} \frac{\frac{3}{x}-1}{x-3} = \lim_{x\to 3} \frac{\frac{3-x}{x}}{x-3} = \lim_{x\to 3} \frac{3-x}{x(x-3)} =$$

$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{x\cancel{(x-3)}} = \lim_{x\to 3} \frac{-1}{x} = -\frac{1}{3}.$$

If $h(x) = mx + b$ is tangent line then in order to find $b$ we use the point $P(3,1)$. Since the graph passes throug the point $P(3,1)$ then $h(3) = 1$. Hence $1 = \left(-\frac{1}{3}\right)\cdot 3 + b \Rightarrow$

$$1 = -1 + b \Rightarrow b = 1 + 1 = 2.$$

Ans: The analytic form of the tangent line is $h(x) = -\frac{1}{3}x + 2$.

**4 Definition** The **derivative of a function $f$ at a number $a$**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{0}{0}$$

if this limit exists.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

**EXAMPLE** Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number $a$.

Sol: According to the definition above

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^2 - 8(a+h) + 9 - (a^2 - 8a + 9)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{9} - \cancel{a^2} + \cancel{8a} - \cancel{9}}{h} =$$

$$= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}} =$$

$$= \lim_{h \to 0} (2a + h - 8) = 2a + 0 - 8 = 2a - 8.$$ Hence

$$f'(a) = 2a - 8, \quad \text{i.e.} \quad (x^2 - 8x + 9)'\Big|_{x=a} = 2a - 8$$

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of $f$ at $a$.

Indeed, according to the definition of tangent line $y = mx + b$, where $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

If $x - a = h$ then $h \to 0$ when $x \to a$.

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Therefore, $m = f'(a)$

**EXAMPLE 5** Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

Sol: Due to the previous example $f'(a) = 2a - 8$. Hence, $f'(3) = 2 \cdot 3 - 8 = -2$

Hence the tangent line to the given function $f(x)$ at point $x = 3$ has form $y = -2x + b$. Taking into account that the tangent line passes throug the point $(3, -6)$ we have:

$$-6 = -2 \cdot 3 + b \Rightarrow b = 0$$

Ans: The equation of the tangent line to the function $f(x) = x^2 - 8x + 9$ at the point $(3, -6)$ is $y = -2x$.

In the preceding section we considered the derivative of a function $f$ at a fixed number $a$:

**1**
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number $a$ vary. If we replace $a$ in Equation 1 by a variable $x$, we obtain

**2**
$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

**EXAMPLE 2**

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate by comparing the graphs of $f$ and $f'$.

Sol: $f(x) = x^3 - x$. Hence

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$$

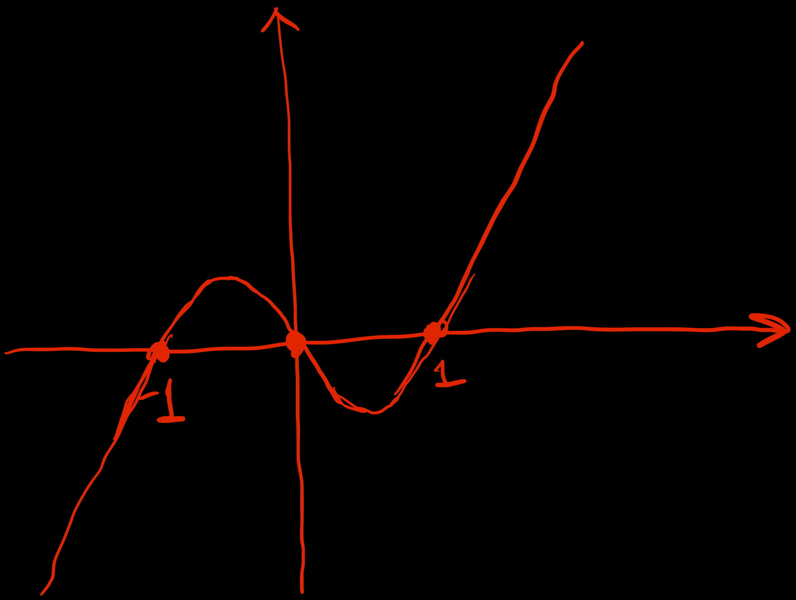
$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) =$$

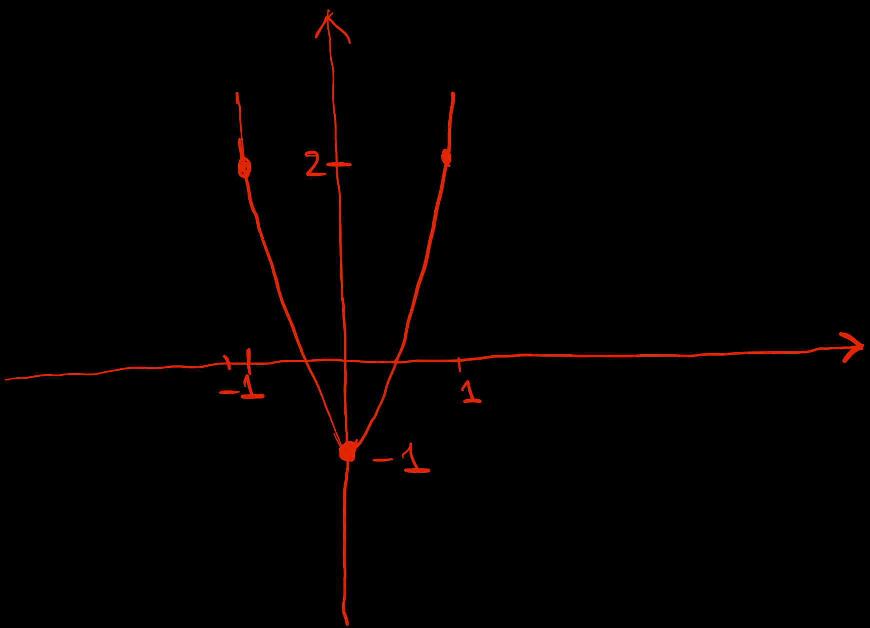
$$= 3x^2 + 3x \cdot 0 + 0^2 - 1 = 3x^2 - 1. \text{ Hence}$$

$$f'(x) = 3x^2 - 1, \quad \text{i.e. } (x^3 - x)' = 3x^2 - 1$$

$f(x) = x^3 - x$




$f'(x) = 3x^2 - 1$



**EXAMPLE 3** If $f(x) = \sqrt{x}$, find the derivative of $f$. State the domain of $f'$.

Sol: $f(x) = \sqrt{x}$. Hence

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$ Therefore

$$f'(x) = \frac{1}{2\sqrt{x}},$$ i.e. $$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$\underline{\mathrm{Dom}(f') = (0, +\infty)}$$

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is $x$ and the dependent variable is $y$, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols $D$ and $d/dx$ are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

**3 Definition** A function $f$ is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

**EXAMPLE 5** Where is the function $f(x) = |x|$ differentiable?

Sol: We show that the given function $f(x)$ is differentiable for any $x \in \mathbb{R} \setminus \{0\}$.

Take any $a > 0$. Hence

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{|a+h| - |a|}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{a} + h - \cancel{a}}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Now take any $a < 0$. Hence

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{|a+h| - |a|}{h} =$$

$$= \lim_{h\to 0} \frac{-(a+h)-(-a)}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0}(-1) = -1$$

Suppose now $a=0$. And consider one-sided limit

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{|a+h|-|a|}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} =$$

$$= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1.$$

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{|a+h|-|a|}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1.$$ So,

$$-1 = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \neq \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = 1$$

Hence, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ doesn't exist

So, $f'(a)$ doesn't exist when $a=0$.

**4** **Theorem** If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

Prove: The function $f$ is differentiable at point $x=a$ means that the following limit exists:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

If $f'(a) = b \in \mathbb{R}$ then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = b$.

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |x-a| < \delta \Rightarrow \left| \frac{f(x)-f(a)}{x-a} - b \right| < \varepsilon$$

We have to show that $\lim_{x\to a} f(x) = f(a)$.

For $\varepsilon > 0$ consider $\left| \frac{f(x)-f(a)}{x-a} - b \right| < \varepsilon \Leftrightarrow$

$$-\varepsilon < \frac{f(x)-f(a)}{x-a} - b < \varepsilon \Rightarrow b-\varepsilon < \frac{f(x)-f(a)}{x-a} < b+\varepsilon$$

$$\Rightarrow |f(x)-f(a)| < |x-a| \cdot (|b|+\varepsilon) \ \forall x : |x-a| < \delta$$

$$\text{Since } 0 \le |f(x)-f(a)| < \underbrace{|x-a| \cdot (|b|+\varepsilon)}^{\forall x:\ |x-a|<\delta}$$

Taking into account $|x-a| \cdot (|b|+\varepsilon) \xrightarrow[x\to a]{} 0$

then due to the squeeze theorem

$$|f(x) - f(a)| \longrightarrow 0 \text{ when } x \to a, \text{ i.e.}$$

$$\lim_{x \to a} f(x) = f(a).$$

Hence the function is continuous at point $x = a$. $\blacksquare$

## Higher Derivatives

If $f$ is a differentiable function, then its derivative $f'$ is also a function, so $f'$ may have a derivative of its own, denoted by $(f')' = f''$. This new function $f''$ is called the **second derivative** of $f$ because it is the derivative of the derivative of $f$. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

**EXAMPLE** If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Sol: $f(x) = x^3 - x$. First we need to find $f'(x)$. Hence

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - h - \cancel{x^3}}{h} =$$

$$\lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}} = \lim_{h\to 0}(3x^2 + 3xh + h - 1) = 3x^2 - 1$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 - 1 - (3x^2 - 1)}{h} =$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} = \lim_{h\to 0}(6x + 3h) = 6x$$

Ans: $f''(x) = (x^3 - x)'' = 6x$

The **third derivative** $f'''$ is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

The process can be continued. The fourth derivative $f''''$ is usually denoted by $f^{(4)}$. In general, the $n$th derivative of $f$ is denoted by $f^{(n)}$ and is obtained from $f$ by differentiating $n$ times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

**EXAMPLE** If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of $x$. So $f'''$ is a constant function and its graph is a horizontal line. Therefore, for all values of $x$,

$$f^{(4)}(x) = 0$$

Hence $f^{(5)}(x) = f^{(6)}(x) = f^{(7)}(x) = \ldots = 0$

where $f(x) = x^3 - x$