Signale und Systeme Boxen

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1 Motivation, Wiederholung und Überblick

a

<div style="border:1px dashed">

1

$$\underline{U}_1 = U_1\angle\varphi_1 = \frac{30}{\sqrt{2}}\angle\frac{\pi}{3}$$

$$\underline{Z}_R = R = 1000$$

$$\underline{Z}_C = \frac{1}{j\omega C} = \frac{1}{sC} = \frac{1000}{s}$$

</div>

2

$$H(s) := \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_C}{\underline{Z}_R + \underline{Z}_C} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{1+s}$$

3

$$|H(j\omega)| = \left|\frac{1}{1 + j\omega RC}\right| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + \omega^2}}$$

$$\sphericalangle H(j\omega) = \sphericalangle\frac{1}{1 + j\omega RC} = \sphericalangle\frac{1 - j\omega RC}{1 + (\omega RC)^2} = \arctan\left(\frac{\frac{-\omega RC}{1+(\omega RC)^2}}{\frac{1}{1+(\omega RC)^2}}\right)$$

$$= \arctan(\omega RC) = -\arctan(\omega)$$

$$\underline{U}_2 = H(s)\cdot\underline{U}_1 = |H(j\omega)|\cdot|\underline{U}_1|\angle\varphi_1\sphericalangle H(j\omega)$$

$$= \frac{1}{\sqrt{1+\omega^2}}\cdot|\underline{U}_1|\angle(\frac{\pi}{3} - \arctan(\omega))$$

$\boxed{4}$

a) $\quad \underline{U}_2 = \dfrac{1}{\sqrt{1 + (2\pi \cdot 0,5)^2}} \cdot 21.2\angle\dfrac{\pi}{3} - \arctan(2\pi \cdot 0,5) \approx 6,43\angle - 0,21$

$\qquad \Rightarrow u_2(t) = 6,43 \cdot \sqrt{2} \cdot \sin(2\pi \cdot 0,5 \cdot t - 0,21) \approx 9,09\,\text{V} \cdot \sin(\pi t - 0,21)$

b) $\quad \underline{U}_2 \approx 0,67\angle - 0,49 \Rightarrow u_2(t) \approx 0,95\,\text{V} \cdot \sin(10\pi t - 0,49)$

c) $\quad \underline{U}_2 \approx 0,0067\angle - 0,523 \Rightarrow u_2(t) \approx 9,55\,\text{mV} \cdot \sin(100\pi t - 0,523)$

$\boxed{5}$

Für $\quad x(t) = 30\,\text{V}\sin(\pi t + \pi/3) \quad$ ist $\quad \mathcal{H}\{x(t)\} = 9,09\,\text{V}\sin(\pi t - 0,21)$

Für $\quad y(t) = 30\,\text{V}\sin(10\pi t + \pi/3) \quad$ ist $\quad \mathcal{H}\{y(t)\} = 0,95\,\text{V}\sin(10\pi t - 0,49)$

$\boxed{6}$

$$u_1(t) = 15\,\text{V}\sin(\pi t + \pi/3) + 60\,\text{V}\sin(10\pi t + \pi/3) = 0,5x(t) + 2y(t)$$

und damit $a = 0,5, b = 2$ und

$$u_2(t) := \mathcal{H}\{u_1(t)\} = \mathcal{H}\{0,5x(t) + 2y(t)\} \overset{??}{=}$$

$\boxed{7}$
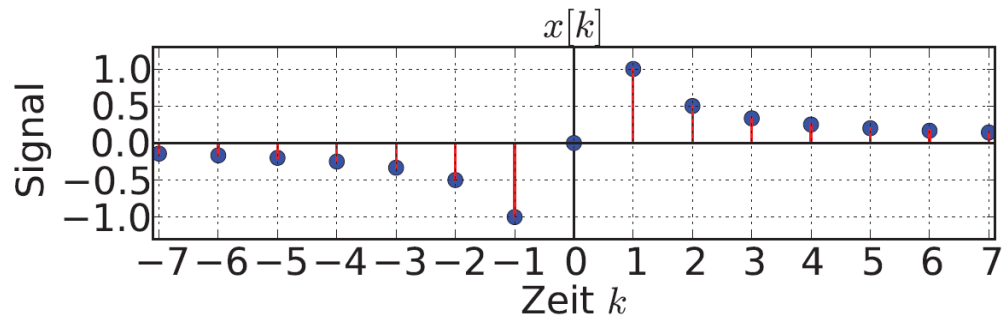
$$\mathcal{H}\{ax(t) + by(t) + cz(t)\} = \mathcal{H}\{ax(t) + 1 \cdot (by(t) + cz(t))\}$$

$$\overset{(??)}{=} a\mathcal{H}\{x(t)\} + 1 \cdot \mathcal{H}\{by(t) + cz(t)\} \overset{(??)}{=} a\mathcal{H}\{x(t)\} + b\mathcal{H}\{y(t)\} + c\mathcal{H}\{z(t)\}$$

$\boxed{8}$

$$x[-\infty], \ldots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \ldots, x[\infty]$$

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- $x[-k]$ die Spiegelung von $x[k]$ an der Signalpegel-Achse

- $x[k + k_0]$ die Verschiebung von $x[k]$ um k_0 nach links

- $x[k - k_0]$ die Verschiebung von $x[k]$ um k_0 nach rechts

2 Diskrete Signale

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$$(b) \quad x[-k] = \begin{cases} -\frac{1}{k}, k/ne0 \\ 0, k = 0 \end{cases}$$

$$(c) \quad x[k + k_0] = x[k + 3] = \begin{cases} \frac{1}{k+3}, k/ne - 3 \\ 0, k = -3 \end{cases}$$

$$(d) \quad x[k - k_0] = x[k - 3] = \begin{cases} \frac{1}{k-3}, k/ne3 \\ 0, k = 3 \end{cases}$$

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$$x[k_0 - k] = x[-(k - k_0)]$$
$$= x[(-k) + k_0]$$

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$$\text{mit} \quad x[k_0 - k] = x[3 - k] = \begin{cases} \frac{1}{3-k}, k \neq 3 \\ 0, k = 3 \end{cases}$$

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- $x[k]$ heißt <u>gerades Signal</u>, falls $x[k] = x[-k] \forall k \in \mathbb{Z}$ gilt.

- $x[k]$ heißt <u>ungerades Signal</u>, falls $x[k] = -x[-k] \forall k \in \mathbb{Z}$ gilt.

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$$x[-k] = \begin{cases} \frac{1}{-k}, k \neq 0 \\ 0, k = 0 \end{cases} = \begin{cases} -\frac{1}{k}, k/ne0 \\ 0, k = 0 \end{cases} = -x[k]$$

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$$y[-k] = \begin{cases} \frac{1}{(-k)^2}, k \neq 0 \\ 0, k = 0 \end{cases} = \begin{cases} -\frac{1}{k^2}, k \neq 0 \\ 0, k = 0 \end{cases} = y[k]$$

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- $x[k]$ heißt <u>kausales Signal</u>, falls gilt: $x[k] = 0 \forall k < 0$

- $x[k]$ heißt <u>nicht-kausales Signal</u>, falls gilt $\exists k < 0 : x[k] \neq 0$

- $x[k]$ heißt <u>anti-kausales Signal</u>, falls $x[-k-1]$ kausal ist, d.h. falls gilt: $x[k] = 0 \forall k \leqslant 0$

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- $x[k]$ ist nicht-kausal

- $u[k]$ ist kausal

- $v[k]$ ist anti-kausal

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$$\delta[k] := \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases}$$

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$$\epsilon[k] := \begin{cases} 1, k \geq 0 \\ 0, k < 0 \end{cases}$$

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$$\boxed{21}$$

$$\delta[k - k_0] = \begin{cases} 1, k = k_0 \\ 0, k \neq k_0 \end{cases}$$

$$\text{bzw.}$$

$$\delta[k + k_0] = \begin{cases} 1, k \neq -k_0 \\ 0, k = -k_0 \end{cases}$$

$$\boxed{22}$$

$$x[k] \cdot \delta[k - i] = \begin{cases} x[i], k = i \\ 0, k \neq i \end{cases}$$

$$= x[i] \cdot \delta[k - i] \qquad (2.1)$$

$$\text{Siebeigenschaft}$$

$$\boxed{23}$$

$$x[k] = \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k - i] \quad \text{für alle} \quad k \in \mathbb{Z}$$

$$\boxed{24}$$

$$x[k] = \sum_{i=-K}^{K} x[i] \cdot \delta[k - i]$$

$$\boxed{25}$$

$$u[k] = \delta[k + 2] + \delta[k + 1] + \delta[k] + \delta[k - 1]$$
$$v[k] = 2 \cdot \delta[k + 3] + \delta[k + 1] - \delta[k - 1] - 2 \cdot \delta[k - 3]$$

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$$sgn[k] := \epsilon[k] - \epsilon[-k] = \begin{cases} 1, k > 0 \\ 0, k = 0 \\ -1, k < 0 \end{cases}$$

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$$Ш[k] := \epsilon[k] + \epsilon[-k-1] = 1 \text{für alle} k \in \mathbb{Z}$$

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$$rect_{k_1,k_2}[k] :) \epsilon[k-k1] - \epsilon[k-k_2-1] = \left\{ 1, k_1 \leqslant k \leqslant k_2 \right.$$

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$$x[k] = q^k \cdot \epsilon[k]$$

30

$$x[k] : 0, ..., 0, x[0] = 1, x[1] = -0.7, x[2] = 0.49, x[3] = 0.343, ...$$

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$$x[k] : 0, ..., 0, x[0] = 1, x[1] = -0.8, x[2] = 0.64, x[3] = -0.512, ...$$

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$$x[k] + y[k] : x[-\infty] + y[-\infty]..., x[0] + y[0], x[1] + y[1], ..., x[\infty] + y[\infty]$$
$$x[k] \cdot y[k] : x[-\infty] \cdot y[-\infty]..., x[0] \cdot y[0], x[1] \cdot y[1], ..., x[\infty] \cdot y[\infty]$$
$$c \cdot x[k] : c \cdot x[-\infty]..., c \cdot x[0], c \cdot x[1], ..., c \cdot x[\infty]$$

$$S_{k_1,k_2} := \{\overrightarrow{x} \in S \mid x[k] = 0 \, \forall k < k_1 \, \text{oder} \, k > k_2\}$$

$$\overrightarrow{x} = (0 \quad 3 \quad 2 \quad 5 \quad 0 \quad 0)$$
$$\overrightarrow{y} = (0 \quad 0 \quad 2 \quad -3 \quad 0 \quad 2)$$
$$\overrightarrow{x} + \overrightarrow{y} = (0 \quad 3 \quad 4 \quad 2 \quad 0 \quad 2)$$
$$\overrightarrow{x} - \overrightarrow{y} = (0 \quad 3 \quad 0 \quad 8 \quad 0 \quad -2)$$
$$\overrightarrow{x} \cdot \overrightarrow{y} = (0 \quad 0 \quad 4 \quad -15 \quad 0 \quad 0)$$
$$c + \overrightarrow{x} = (0 \quad 15 \quad 10 \quad 25 \quad 0 \quad 0)$$

$$(x * y)[k] := \sum_{i=-\infty}^{\infty} x[i] \cdot y[k-i]$$

$$
\begin{array}{lll}
i = 0 & i = 0 & i = 0 \\
\downarrow & \downarrow & \downarrow \\
x[i] = (3 \quad 2 \quad 1), & y[i] = (1 \quad -1 \quad 2) \text{ bzw.} & z[0-i] = (2 \quad -1 \quad 1)
\end{array}
$$

	$x[i] =$			3	2	1		$\sum x[i]y[k-i] =$	$(x*y)[k]$
$k=0$	$y[k-i] =$	2	-1	1				$3 \cdot 1$	$= 3$
$k=1$			2	-1	1			$3 \cdot (-1) + 2 \cdot 1$	$= -1$
$k=2$				2	-1	1		$3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1$	$= 5$
$k=3$					2	-1	1	$2 \cdot 2 + 1 \cdot (-1)$	$= 3$
$k=4$						2	-1	1	$1 \cdot 2$	$= 2$

$$x[k] * y[k] = 3\delta[k] - \delta[k-1] + 5\delta[k-2] + 3\delta[k-3] + 2\delta[k-4]$$

$\boxed{39}$

$$i = -43$$
$$\downarrow$$
$$x[i] = (-1 \quad 3 \quad -2) \text{ und}$$
$$i = 19$$
$$\downarrow$$
$$y[i] = (1 \quad -2 \quad 4 \quad -1) \text{ bzw. } y[-i] = (-1 \quad 4 \quad -2 \quad 1)$$

$\boxed{40}$

$i = -43$

k	$x[i]$					-1	3	-2		$(x*y)[k]$
-24	$y[k-i] =$	-1	4	-2	1					-1
-23			-1	4	-2	1				$2 + 3 = 5$
-22				-1	4	-2	1			$-4 - 6 - 2 = -12$
-21					-1	4	-2	1		$1 + 12 + 4 = 17$
-20						-1	4	-2	1	$-3 - 8 = -11$
-19							-1	4	-2	1	2

$\boxed{41}$

$$(x*y)[k] = -\delta[k+24] + 5\delta[k+23] - 12\delta[k+22] + 17\delta[k+21]$$
$$- 11\delta[k+20] + 2\delta[k+19]$$

$\boxed{42}$

$$x[k] * y[k] \in \mathcal{S}_{a+c,b+d} \quad \text{und hat Länge} \quad n + m - 1.$$

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I) Kommutativität: $x * y = y * x$

II) Assoziativität: $w * (x * y) = (w * x) * y$ und $c \cdot (x * y) = (c \cdot) * y$

III) Distributivität: $w * (x + y) = w * x + w * y$

IV) Neutrales Element: $x * \delta = x$

V) Verschiebung: $x[k] * \delta[k_0 - k] = x[k - k_0]$

VI) Zeitinvarianz: $x[k] * y[k - k_0] = (x[k] * y[k])[k - k_0]$

VII) Linearität: $(c \cdot x + d \cdot y) * w = c \cdot (x * w) + d \cdot (y * w)$

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$$p(z) := a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n$$

45

$$x[k] = a_0 \delta[k] + a_1 \delta[k - 1] + a_2 \delta[k - 2] + ... + a_n \delta[k - n]$$

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$$p(z) \cdot q(z) = c_0 + c1_z + c_2 z^2 + ... + c_{2n} z^{2n} \quad \text{Mit Koeffizenten } c_k = (c * y)[k]$$

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$$p(z) = 3 + 2z + z^2 \text{ und } q(z) = 1 - z + 2z^2$$

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$$\begin{aligned}
p(z) \cdot q(z) &= (3 + 2z + z^2) \cdot (2z^2 - z + 1) \\
&= 3 \cdot 1 + z(3 \cdot (-1) + 2 \cdot 1) + z^2(3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1) \\
&\quad + z^3(2 \cdot 2 + 1 \cdot (-1)) + z^4(1 \cdot 2) \\
&= 3 - z + 5z^2 + 3z^3 + 2z^4
\end{aligned}$$

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$$E_x := \sum_{i=-\infty}^{\infty} |x[i]|^2$$

50

$$P_x := \lim_{K \to \infty} \frac{1}{2K+1} \sum_{i=-K}^{K} |x[i]|^2$$

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$$\langle x[k], y[k] \rangle_E := \sum_{k=-\infty}^{\infty} x^*[k] \cdot y[k]$$

52

$$\langle x[k], y[k] \rangle_P := \lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x^*[k] \cdot y[k]$$

53

$$||x[k]||_E := \sqrt{\langle x[k], x[k] \rangle_E} = \sqrt{E_x} \text{ bzw.}$$
$$||x[k]||_P := \sqrt{\langle x[k], x[k] \rangle_P} = \sqrt{P_x}$$

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$$\cos \Phi = \frac{\langle x[k], y[k] \rangle}{||x[k]|| \cdot ||y[k]||}$$

55

$$\varphi_{xy}[\kappa] := \langle x[k], y[k+\kappa] \rangle$$

56

$$\varphi_{xx}[\kappa] := \langle x[k], x[k + \kappa] \rangle$$

57

$$\varphi_{xy}^E[\kappa] = x^*[-\kappa] * y[\kappa] \text{ bzw. } \varphi_{xy}^P[\kappa] = \lim_{K \to \infty} \frac{1}{2K + 1} x_K^*[-\kappa] * y_K[\kappa]$$

3 Diskrete Systeme

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$$y[k] = \mathcal{H}\{x[k]\}$$

59

$$x[k] = x_0 \cdot \delta[k] = \begin{cases} x_0, k = 0 \\ 0, k \neq 0 \end{cases}$$

entwickelt sich nun das Guthaben des Sparbuchs wie folgt:

zu Beginn: $y[0] = x_0$

nach 1 Jahr: $y[1] = x_0 + p \cdot x_0 = (1 + p) \cdot x_0$ nach 2 Jahren: $y[2] = (1 + p)x_0 + p \cdot (1 + p) \cdot x_0 = (1 + p) \cdot (1 + p) \cdot x_0 = (1 + p)^2 \cdot x_0$

nach 3 Jahren: $y[3] = ... = (1 + p)^3 \cdot x_0$

nach i Jahren: $y[i] = (1 + p)^i \cdot x_0$

D.h. das Ausgangssignal ist die kausale Exponentialfolge $y[k] = x_0 \cdot (1 + p)^k \cdot \epsilon[k]$

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$$y[k + 1] = y[k] \cdot (1 + p) + x[k + 1] \tag{3.1}$$

Das heißt $y[k+1]$ ergibt sich aus dem verzinsten Guthaben $y[k]$ des vorigen Jahres und zusätzlich den neuen Einzahlungen $x[k + 1]$.

61

$$\mathcal{H}\{c \cdot x_1[k] + d \cdot x_2[k]\} = c \cdot \mathcal{H}\{x_1[k]\} + d \cdot \mathcal{H}\{x_2[k]\}$$

62

$$y[0] = x[0] = c \cdot x_1[0] + d \cdot x_2[0]$$

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$$y[k+1] \overset{(3.1)}{=} y[k] \cdot (1+p) + x[k+1]$$
$$\overset{(I.V)}{=} (cy_1[k] + d \cdot y_2[k]) \cdot (1+p) + c \cdot x_1[k+1] + d \cdot x_2[k+1]$$
$$= c \cdot (y_1[k] \cdot (1+p) + x_1[k+1]) + d \cdot (y_2[k] \cdot (1+p) + x_2[k+1])$$
$$\overset{(3.1)}{=} c \cdot y[k+1] + d \cdot y_2[k+1]$$

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$$\mathcal{H}\{x[k-k_0]\} = y[k-k_0]$$

65

$$z[k_0] = x[k_0 - k_0] = x[k_0] = y[0] = y[k_0 - k_0]$$
$$\text{und } z[k] = 0 = y[k - k_0] \text{ für } k < k_0$$

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$$z[k+1] \overset{(3.1)}{=} z[k] \cdot (1+p) + x[k+1-k_0]$$
$$\overset{(I.V.)}{=} y[k-k_0] \cdot (1+p) + x[k-k_0+1]$$
$$\overset{(3.1)}{=} y[k-k_0+1]$$

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... wenn der Ausgabewert $y[k_0]$ zur Zeit k_0 nur von früheren Eingabewerten $x[k], k \leq k_0$ abhängig ist.

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$$|x[k]| < C \forall k \Rightarrow |y[k]| < D \forall k$$

69

$$y[k] = x_0 \cdot (1+p)^k \cdot \epsilon[k] \to \infty \text{ für } k \to \infty$$

70

..., wenn der Ausgang $y[k]$ zur Zeit k nur vom Eingang $x[k]$ zur Zeit k abhängt.

71

..., falls $y[k]$ nur von $x[\kappa]$ für $|\kappa - k| \leq L$ abhängt.

72

$$h[k] := \mathcal{H}\{\delta[k]\}$$

73

$$y[k] = \mathcal{H}\{x[k]\} \overset{(2.6)}{=} \mathcal{H}\left\{\sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k-i]\right\}$$
$$= \sum_{i=-\infty}^{\infty} x[i]\mathcal{H}\{\delta[k-i]\}$$
$$= \sum_{i=-\infty}^{\infty} x[i] \cdot h[k-i]$$
$$= x[k] * h[k]$$

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$$y[k] = x[k] * h[k] \text{ für alle } x[k] \in \mathcal{S}$$

75

$$h[k] := \mathcal{H}\{\delta[k]\} = (1+p)^k \epsilon[k]$$

76

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} (1+p)^i \epsilon[i] \cdot x[k-i]$$

$$= \sum_{i=0}^{\infty} (1+p)^i \cdot x[k-i]$$

77

$$y[k] = \sum_{i=0}^{k} (1+p)^i \cdot x[k-i]$$

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$$\sum_{y=-\infty}^{\infty} |h[i]| < \infty$$

79

$$|y[k]| = |h[k] * x[k]| = |\sum_{i=-\infty}^{\infty} h[k]x[k-i]| \overset{DUG}{\leq} \sum_{i=-\infty}^{\infty} |h[i] \cdot x[k-i]|$$

$$= \sum_{i=-\infty}^{\infty} |h[i]| \cdot |x[k-i]| < M \sum_{i=-\infty}^{\infty} |h[i]| < M \cdot C < \infty$$

80

$$x[k] := sgn(h[-k]) = \begin{cases} 1, & h[-k] > 0 \\ 0, & h[-k] = 0 \\ -1, & h[-k] < 0 \end{cases}$$

81

$$x[k] \cdot h[-k] = sgn(h[-k]) \cdot h[-k] = |h[-k]| \geq 0$$

$$|x[0]| = |(x \cdot h)[0]| = \left| \sum_{i=-\infty}^{\infty} x[i] \cdot h[-i] \right| = \sum_{i=-\infty}^{\infty} |h[-i]| = \sum_{i=-\infty}^{\infty} |h[i]| = \infty$$

$$\sum_{i=-\infty}^{\infty} |h[i]| = \sum_{i=-\infty}^{\infty} (1+p)^i \cdot \epsilon[i] = \sum_{i=0}^{\infty} (1+p)^i$$

$$|1+p| < 1 \quad \text{bzw. äquivalent für} \quad -2 < p < 0$$

$$h[k] = 0, \quad \forall k < 0$$

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^{\infty} h[i] \cdot x[k-i]$$

<u>Lösung:</u> Folgendes Blockschaltbild realisiert die Rekursion $y[k+1] = y[k] \cdot (1+p) + x[k+1]$ von (3.1) auf Seite 40: $y[k] = y[k-1] \cdot (1+p) + x[k]$
Hier tikz einfügen!

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^{n} \beta i \cdot x[k-i]$$

$$\boxed{89} \qquad y[k] = \frac{1}{\alpha_0} \cdot \left(\sum_{i=0}^{N} \beta i \cdot x[k-i] - \sum_{i=0}^{N} \alpha i \cdot y[k-i] \right)$$

$$\boxed{90} \qquad \begin{aligned} \vec{v}[k+1] &= fv(\vec{v}[k], \vec{x}[k]) \\ \vec{y}[k] &= fy(\vec{v}[k], \vec{x}[k]) \end{aligned}$$

$$\boxed{91} \quad \begin{aligned} &\text{I)} \quad \vec{y}[k_0] = fx(\vec{v}[k_0], \vec{x}[k_0]) \quad \text{II)} \quad \vec{v}[k_0 + 1] = fv(\vec{v}[k_0], \vec{x}[k_0]) \\ &\text{III)} \ \vec{y}[k_0 + 1] = fy(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1]) \ \text{IV)} \ \vec{v}[k_0 + 2] = fv(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1]) \\ &\text{V)} \ \vec{y}[k_0 + 2] = \ldots \end{aligned}$$

$$\boxed{92} \qquad \vec{v}[k] : \begin{pmatrix} x[k-1] & x[k-2] & x[k-3] & \ldots & x[k-L] \end{pmatrix}$$

$$\boxed{93} \qquad \begin{aligned} \vec{v}[k+1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot v[k] + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot x[k] \\ y[k] &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \cdot \vec{v}[k] + (a_0) \cdot x[k] \end{aligned}$$

$$\boxed{94} \qquad X(z) := \mathcal{Z}\{x[k]\} := \sum x[k] \cdot z^{-k}$$

$$a := \limsup_{h \to \infty} \sqrt[k]{|x[k]|}$$

$$b := \frac{1}{\limsup\limits_{h \to \infty} \sqrt[k]{|x[-k]|}}$$

$$X^+(z) := \mathcal{Z}\{x[k]\} := \sum_{h=0}^{\infty} x[k] \cdot z^{-k}$$

a) $\mathcal{Z}\{\delta[k]\} = \displaystyle\sum_{k=-\infty}^{\infty} \delta[k] \cdot z^{-k} = z^{-0} = 1$ für $z \in \mathbb{C}$

b) $\mathcal{Z}\{\delta[k-i]\} = \displaystyle\sum_{k=-\infty}^{\infty} \delta[k-i] \cdot z^{-k} = z^{-i}$ für $0 < |z| < \infty$

c) $\mathcal{Z}\{\epsilon[k]\} = \displaystyle\sum_{k=-\infty}^{\infty} \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right)^k$

$\overset{geom.}{\underset{Reihe}{=}} \dfrac{1}{1-\frac{1}{z}} = \dfrac{z}{z-1}$ für $\left|\dfrac{1}{z}\right| < 1$ bzw. $|z| > 1$

d) $\mathcal{Z}\{a^k \cdot \epsilon[k]\} = \displaystyle\sum_{k=-\infty}^{\infty} a^k \cdot \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \dfrac{1}{1-\frac{a}{z}}$

$= \dfrac{z}{z-a}$ für $\left|\dfrac{a}{z}\right| < 1 \Leftrightarrow |z| > |a|$

e) $\mathcal{Z}\{a^k \cdot \epsilon[-k-1]\} = \displaystyle\sum_{k=-\infty}^{\infty} -a^k \cdot \epsilon[-k-1] \cdot z^{-k} = -\sum_{k=-\infty}^{-1} a^k \cdot z^{-k}$

$= -\displaystyle\sum_{k=1}^{\infty} a^{-k} \cdot z^k = -\sum_{k=1}^{\infty} \left(\frac{a}{z}\right)^k = -\frac{z}{a} \cdot \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^k = \frac{z}{a} \cdot \dfrac{1}{1-\frac{z}{a}}$

$= \dfrac{z}{z-a}$ für $\left|\dfrac{z}{a}\right| < 1 \Leftrightarrow |z| < |a|$

$$\boxed{98}\qquad \mathcal{Z}\{\alpha x[k] + \beta y[k]\} \overset{Def.}{=} \sum_{k=-\infty}^{\infty} (\alpha x[k] + \beta y[k]) \cdot z^{-k}$$

$$= \alpha \left(\sum_{k=-\infty}^{\infty} x[k]z^{-k} \right) + \beta \left(\sum_{k=-\infty}^{\infty} y[k]z^{-k} \right)$$

$$= \alpha X(z) + \beta Y(z)$$

$$\boxed{99}\qquad \mathcal{Z}\{x[k+k_0]\} \overset{Def.}{=} \sum_{k=-\infty}^{\infty} x[k+k_0]z^{-k} \overset{k'=k+k_0}{\underset{(k=k'-k_0)}{=}} \sum_{k'=-\infty}^{\infty} \underbrace{x[k']z^{-k'+k_0}}_{=z^{-k'} \cdot z^{k_0}}$$

$$= z^{k_0} \cdot \underbrace{\sum_{k'=-\infty}^{\infty} x[k']z^{-k'}}_{=X(z)} = z^{k_0} \cdot X(z)$$

$$\boxed{100}\qquad \mathcal{Z}\{a^k \cdot x[k]\} \overset{Def.}{=} \sum_{k=-\infty}^{\infty} \underbrace{a^k}_{=(\frac{1}{\alpha})^{-k}} \cdot x[k] \cdot z^{-k} = \sum_{k=-\infty}^{\infty} x[k] \cdot \left(\frac{z}{\alpha}\right)^{-k} = X\left(\frac{z}{\alpha}\right)$$

$$\boxed{101}\qquad \mathcal{Z}\{x[-k]\} = \sum_{k=-\infty}^{\infty} x[-k]z^{-k} \overset{k'=-k}{\underset{(k=-k')}{=}} \sum_{k'=-\infty}^{\infty} x[k']z^{k'} = \sum_{k'=-\infty}^{\infty} x[k'] \cdot \left(\frac{1}{z}\right)^{-k'} = X\left(\frac{1}{z}\right)$$

$$\boxed{102}\qquad x[0] = \lim_{z \to \infty} X(z)$$

103

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1}(z-1) \cdot X(z)$$

104

$$\mathcal{Z}\{\alpha^{k-1} \cdot \epsilon[k-1]\} = z^{-1} \cdot \mathcal{Z}\{\alpha^k \cdot \epsilon[k]\} = z^{-1} \cdot \frac{z}{z-\alpha} = \frac{1}{z-\alpha}$$

105

$$\mathcal{Z}\{A \cdot \alpha^{k-k_0} \cdot \epsilon[k-k_0]\} = A \cdot z^{-k} \cdot \mathcal{Z}\{\alpha^k \epsilon[k]\} = A \cdot z^{-k_0} \cdot \frac{z}{z-\alpha} = \frac{A \cdot z^{-(k_0-1)}}{z-\alpha}$$

106

$$\mathcal{Z}\{k \cdot \alpha^k \cdot \epsilon[k]\} = -z \cdot \frac{d}{dz} \mathcal{Z}\{\alpha^k \epsilon[k]\} = -z \cdot \left(\frac{z}{z-\alpha}\right)'$$

$$= -z \cdot \left(\frac{1 \cdot (z-\alpha) - z \cdot 1}{(z-\alpha)^2}\right) = \frac{\alpha \cdot z}{(z-\alpha)^2}$$

107

$$Y(z) = H(z) \cdot X(z)$$

108

$$H(z) = \frac{Y(z)}{X(z)}$$

109

$$h[k] = (1+p)^k \cdot \epsilon[k]$$

$$H(z) = \frac{z}{z-(1+p)}$$

110

$$y[k+1] = y[k] \cdot (1+p) + x[k+1]$$

$$Y(z) = Y(z) \cdot (1+p) + zX(z) \Leftrightarrow Y(z)(z-(1+p)) = zX(z)$$

111

$$H(z) := \frac{Y(z)}{X(z)} = \frac{z}{z-(1+p)}$$

112

$$x[k] = x_0 \cdot \epsilon[k]$$

$$X(z) = x_0 \cdot \frac{z}{z-1}$$

113

$$Y(z) = H(z) \cdot X(z) = \frac{z}{z-(a+p)} \cdot x_0 \cdot \frac{z}{z-q} = x_0 \cdot z^2 \cdot \frac{1}{(z-(a+p)) \cdot (z-1)}$$

114

$$\frac{1}{(z-(a+p)) \cdot (z-1)} = \frac{p^{-1}}{z-(a+p)} - \frac{p^{-1}}{z-1}$$

115

$$\frac{p^{-1}}{z-(a+p)} \bullet\!\!-\!\!\circ p^{-1}(a+p)^{K-1}\epsilon[K-1] \text{ und}$$

$$\frac{p^{-1}}{z-1} \bullet\!\!-\!\!\circ p^{-1}1^{K-1}\epsilon[K-1] = p^{-1}\epsilon[K-1]$$

$\boxed{117}$

$$\sum_{i=0}^{n} \alpha_i y[k-i] = \sum_{i=N-M}^{N} \beta_i x[k-i] \qquad (a_0 \neq 0, \beta_{N-M} \neq 0)$$

$$\sum_{i=0}^{N} \alpha_i \cdot z^{-i} \cdot Y(z) = \sum_{i=N-M}^{N} \beta_i \cdot z^{-i} \cdot X(z)$$

$$\Leftrightarrow Y(z) \cdot \sum_{i=0}^{N} \alpha_i \cdot z^{-i} = X(z) \cdot \sum_{i=N-M}^{N} \beta_i \cdot z^{-i}$$

$\boxed{118}$

$$H(z) := \frac{Y(z)}{X(z)} = \frac{\sum_{i=N-M}^{N} \beta_i \cdot z^{-i}}{\sum_{i=0}^{N} \alpha_i \cdot z^{-i}}$$

$$= \frac{\sum_{i=N-M}^{N} \beta_i \cdot z^{N-i}}{\sum_{i=0}^{N} \alpha_i \cdot z^{N-i}}$$

$$= \frac{\beta_{N-M} \cdot z^M + \beta_{N-M+1} \cdot z^{M-1} + ... + \beta_N}{\alpha_0 \cdot z^N + \alpha_1 \cdot z^{N-1} + ... + \alpha_N}$$

$\boxed{119}$

$$y(k) = \frac{1}{\alpha_0} \left(\sum_{i=N-M}^{N} \beta_i x[k-i] - \sum_{i=1}^{N} \alpha_i y[k-i] \right)$$

$\boxed{120}$

$$h[k] = 3 \cdot \left(\frac{1}{5}\right)^k \cdot epsilon[k] + 2 \cdot \left(\frac{1}{2}\right)^k \cdot \epsilon[k]$$

$$H(z) = 3 \cdot \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}} = \frac{15z}{5z - 1} + \frac{4z}{2z - 1}$$

$$= \frac{15z(2z - 1) + 4z(5z - 1)}{(5z - 1)(2z - 1)} = \frac{50z^2 - 19z}{10z^2 - 7z + 1}$$

$\boxed{121}$

$$\alpha_0 = a_2 = 10, \alpha_1 = -7, \alpha_1 = 1 \text{ und } \beta_0 = 50, \beta_1 = -19, \beta_2 = 0$$

$\boxed{122}$

$$10y[k] - 7y[k-1] + 1y[k-2] = 50x[k] - 19x[k-1] \text{ bzw. äquivalent}$$

$$y[k] = \frac{1}{10} \cdot (50x[k] - 19x[k-1] + 7y[k-1] - y[k-2])$$

$\boxed{123}$

$$h[k] = 3 \cdot 5^{-(k+2)} \epsilon[k+2] + 2 \cdot 2^{-k} \epsilon[k]$$

$$H(z) = 3 \cdot z^2 \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}}$$

$$= \frac{15z^3}{5z - 1} + \frac{4z}{2z - 1} = \frac{30z^4 - 15z^3 + 20z^2 - 4z}{10z^2 - 7z + 1}$$

$$(\Rightarrow M = 4, N = 2)$$

$\boxed{124}$

$$\alpha_0 = a_{2-0} = 10, \alpha_1 = -7, \alpha_2 = 1 \text{ und}$$

$$\beta_{-2} = b_4 = 30, \beta_{-1} = 15, \beta_0 = 20, \beta_1 = -4$$

$\boxed{125}$

$$10y[k] - 7y[k-1] + y[k-2] = 30x[k+2] - 15[k+1] + 20x[k] - 4x[k-1]$$

$$y[k] = \frac{1}{10}(30x[k+2] - 15[k+1] + 20x[k] - 4x[k-1] + 7y[k-1] - y[k-2])$$

$\boxed{126}$

$$y[k-2] = \frac{1}{10}(30x[k+2] - 15[k-1] + 20x[k-2] - 4x[k-3] + 7y[k-3] - y[k-4])$$

$$127 \quad x[k] = \mathcal{Z}^{-1}\left\{X(z)\right\} = \mathcal{Z}^{1}\left\{\sum_{k=-\infty}^{\infty} x[k]z^{-k}\right\}$$

$$128 \quad Y(z) = 1 \cdot z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{4}z^{-4} + ... + \frac{(-1)^{k+1}}{k}z^{-k} + ...$$

$$129 \quad y[k] = \begin{cases} 0 & , k \leq 0 \\ (-1)^{k+1} \cdot \frac{1}{k} & , k > 0 \end{cases}$$

$$130 \quad Y(z) \overset{!}{=} \frac{A_1}{z - \lambda_1} + \frac{A_2}{z - \lambda_2} + ... + \frac{A_n}{z - \lambda_n}$$

$$131 \quad Y(z) \overset{!}{=} \frac{A_1(\lambda_2)...(z - \lambda_N) + ... + A_N(z - \lambda_1)...(z - \lambda_{N-1})}{HN}$$

$$132 \quad Y(z) = \frac{1}{(z - (1 + P)) \cdot (z - 1)} = \frac{A(z - 1) + B(z - (1 + P))}{HN}$$
$$= \frac{(A + B) \cdot z - A - B(a + P)}{HN}$$

$$133 \qquad \begin{array}{lll} -A & -(a + p)B = 1 & (1) \\ A & +B = 0 & (2) \end{array}$$

134

$$(-1 - p + 1) \cdot B = 1 \text{ bzw. } B = -\frac{1}{p}$$

135

$$A = -B = \frac{1}{p}$$

136

$$Y(z) = \frac{p^{-1}}{z - (a + p)} - \frac{p^{-1}}{z - 1}$$

$$y[k] = p^{-1} \cdot ((1 + p)^{k-1} - 1) \cdot \epsilon[k - 1]$$

137

$$Y(z) = \frac{A}{z - 5} + \frac{B}{z - 3} + \frac{C}{(z - 3)^2} = \frac{A(z - 3)^2 + B(z - 5)(z - 3)C(z - 5)}{HN}$$

$$= \frac{A(z^2 - 6z + 9) + B(z^2 - 8z + 15) + C(z - 5)}{HN}$$

$$= \frac{z^2(A + B) + z(-6A - 8B + C) + 9A + 15B - 5C}{HN}$$

138

$A+$	B	$= 2(1)$
$-6A$	$-8B + 1C$	$= -9(2)$
$9A$	$+15B - 5C$	$= 3(3)$
$5 \cdot (2) + (3): -21A$	$-25B$	$= -42(4)$
$21 \cdot (1 + 4):$	$-4B$	$= 0(5)$

139

$$Y(z) = \frac{2}{z - 5} + \frac{3}{(z - 3)^2}$$

$$y[k] = 2 \cdot 5^{k-1}\epsilon[k - 1] + (k - 1)3^{k-1}\epsilon[k - 1]$$

140

$$Y(z) = s(z) + \frac{r(z)}{q(z)}$$

141

$$s(z) = s_0 + s_1 z + ... + s_{M-N}Z^{M-N}$$

$$s[k] = s_0\delta[k] + s_1\delta[k + 1] + ... + s_{M-N}\delta[k + M - N]$$

142

$$Y(z) = 3z^2 - 2z + 1\frac{2z^2 - 9z + 3}{(z - 5)(z - 3)^2}$$

143

$$Y(z) = 3z^2 - 2z + 1\frac{2z^2 - 9z + 3}{(z - 5)(z - 3)^2}$$

$$y[k] = 3\delta[k + 2] - 2\delta[k + 1] + \delta[k] + 2 \cdot 5^{k-1}\epsilon[k - 1] + (k - 1)3^{k-1}\epsilon[k - 1]$$

144

$$Y(z) = \sum_{i=0}^{M-N} s_i z^i + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} \frac{A_{i,v}}{(z-\lambda_i)^v}$$

$$y[k] = \sum_{i=0}^{M-N} s_i \delta[k+i] + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} A_{i,v} \cdot \binom{k-1}{v-1} \lambda_i^{k-v} \epsilon[k-1]$$

147

$$A = \frac{1}{0!} \left(\frac{d}{dz}\right)^0 [(z-1) \cdot \frac{z}{z-1}(z+1)]|_{z=1} = \frac{z}{z+1}|_{z=1} = \frac{1}{2}$$

$$B = \frac{1}{0!} \left(\frac{d}{dz}\right)^0 [(z+1) \cdot \frac{z}{z-1}(z+1)]|_{z=-1} = \frac{z}{z-1}|_{z=-1} = \frac{1}{2}$$

4 Kontinuierliche Signale

Inhalt...

171

$$\int_{-\infty}^{t} \epsilon(\tau)d\tau = \begin{cases} 0 & ,t \leq 0 \\ \int_0^1 1d\tau = [\tau]_0^t = t & ,t > 0 \end{cases} = t \cdot \epsilon(t) =: ramp(t)$$

172

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \epsilon(t)$$

173

$$\delta(t) := \frac{d}{dt}\epsilon(t) = \lim_{T \to 0} \frac{1}{T}\text{rect}(\frac{t}{T}) \qquad = \lim_{T \to 0} \frac{1}{T}\text{tri}(\frac{t}{T}) = \lim_{T \to 0} \frac{2}{T}\text{si}(s\pi\frac{t}{T})$$

174

I) $x(\tau) \cdot \delta(\tau - t) = x(t) \cdot \delta(\tau - t)$

II) $\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = 1$

III) $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$ für alle $t \in \mathbb{R}$

175

$$\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = \int_{-\infty}^{\infty} \delta(\lambda)d\lambda = \epsilon(\infty) = 1$$

$$\boxed{176}$$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(\tau-t)d\tau \stackrel{(I)}{=} \int_{-\infty}^{\infty} x(t)\delta(\tau-t)d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \delta(\tau-t)d\tau \stackrel{(II)}{=} x(t)\cdot 1 = x(t)$$

$$\boxed{177}$$

$$(x*y)(t) := \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

$$\boxed{178}$$

a)

$$\epsilon(t)*\epsilon(t) = \int_{-\infty}^{\infty} \epsilon(\tau)\epsilon(t-\tau)d\tau = \int_{0}^{\infty} \epsilon(t-\tau)d\tau = \int_{0}^{t} 1\cdot d\tau \stackrel{\lambda:=t-\tau}{=}$$

$$-\int_{t}^{-\infty} \epsilon(\lambda)d\lambda = \int_{-\infty}^{0} \epsilon(\lambda)d\lambda \stackrel{(42)}{=} t\epsilon(t) = \mathrm{ramp}(t)$$

b)

$$\mathrm{rect}(t)*\epsilon(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\epsilon(t-\tau)d\tau$$

$$= \int_{-0,5}^{0,5} \epsilon(t-\tau)d\tau \stackrel{\lambda:=t-\tau}{=} -\int_{t-(-0,5)}^{t-0,5} \epsilon(\lambda)d\lambda = -\int_{t-0,5}^{t+0,5} \epsilon(\lambda)d\lambda$$

$$\stackrel{(4.2)}{=} [\lambda\epsilon(\lambda)]_{t-0,5}^{t+0,5} = \mathrm{ramp}(t+0,5) - \mathrm{ramp}(t-0,5)$$

$$= \begin{cases} 0 & ,t \le -0,5 \\ t+0,5 & ,-0,5 \le t \le 0,5 \\ 1 & ,t \ge 0,5 \end{cases} =: \mathrm{sramp}$$

$$\boxed{179}$$

c)

$$\mathrm{rect}(t)*\epsilon(t) = (\epsilon(t+0,5) - \epsilon(t-0,5))*\epsilon(t)$$

$$\stackrel{(III)}{=} \epsilon(t+0,5)*\epsilon(t) - \epsilon(t-0,5)*\epsilon(t)$$

$$\stackrel{(VI.a)}{=} \mathrm{ramp}(t+0,5) - \mathrm{ramp}(t-0,5) = sramp(t)$$