Signale und Systeme Boxen

Florian Lubitz & Steffen Hecht

21. Januar 2019

1 Motivation, Wiederholung und Überblick

a

$\boxed{1}$

$$\underline{U}_1 = U_1 \angle \varphi_1 = \frac{30}{\sqrt{2}} \angle \frac{\pi}{3}$$
$$\underline{Z}_R = R = 1000$$
$$\underline{Z}_C = \frac{1}{j\omega C} = \frac{1}{sC} = \frac{1000}{s}$$

$\boxed{2}$

$$H(s) := \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_C}{\underline{Z}_R + \underline{Z}_C} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{1+s}$$

$\boxed{3}$

$$|H(j\omega)| = \left|\frac{1}{1 + j\omega RC}\right| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} = \frac{1}{\sqrt{1+\omega^2}}$$
$$\sphericalangle H(j\omega) = \sphericalangle \frac{1}{1 + j\omega RC} = \sphericalangle \frac{1 - j\omega RC}{1 + (\omega RC)^2} = \arctan\left(\frac{\frac{-\omega RC}{1+(\omega RC)^2}}{\frac{1}{1+(\omega RC)^2}}\right)$$
$$= \arctan(\omega RC) = -\arctan(\omega)$$
$$\underline{U}_2 = H(s) \cdot \underline{U}_1 = |H(j\omega)| \cdot |\underline{U}_1| \angle \varphi_1 \sphericalangle H(j\omega)$$
$$= \frac{1}{\sqrt{1+\omega^2}} \cdot |\underline{U}_1| \angle (\frac{\pi}{3} - \arctan(\omega))$$

$\boxed{4}$

$$a)\quad \underline{U}_2 = \frac{1}{\sqrt{1+(2\pi\cdot 0,5)^2}} \cdot 21.2\angle\frac{\pi}{3} - \arctan(2\pi\cdot 0,5) \approx 6,43\angle -0,21$$

$$\Rightarrow u_2(t) = 6,43\cdot\sqrt{2}\cdot\sin(2\pi\cdot 0,5\cdot t - 0,21) \approx 9{,}09\,\mathrm{V}\cdot\sin(\pi t - 0,21)$$

$$b)\quad \underline{U}_2 \approx 0,67\angle -0,49 \Rightarrow u_2(t) \approx 0{,}95\,\mathrm{V}\cdot\sin(10\pi t - 0,49)$$

$$c)\quad \underline{U}_2 \approx 0,0067\angle -0,523 \Rightarrow u_2(t) \approx 9{,}55\,\mathrm{mV}\cdot\sin(100\pi t - 0,523)$$

$\boxed{5}$

Für $x(t) = 30\,\mathrm{V}\sin(\pi t + \pi/3)$ ist $\mathcal{H}\{x(t)\} = 9{,}09\,\mathrm{V}\sin(\pi t - 0,21)$

Für $y(t) = 30\,\mathrm{V}\sin(10\pi t + \pi/3)$ ist $\mathcal{H}\{y(t)\} = 0{,}95\,\mathrm{V}\sin(10\pi t - 0,49)$

$\boxed{6}$

$$u_1(t) = 15\,\mathrm{V}\sin(\pi t + \pi/3) + 60\,\mathrm{V}\sin(10\pi t + \pi/3) = 0,5x(t) + 2y(t)$$

und damit $a = 0,5, b = 2$ und

$$u_2(t) := \mathcal{H}\{u_1(t)\} = \mathcal{H}\{0,5x(t) + 2y(t)\} \overset{??}{=}$$

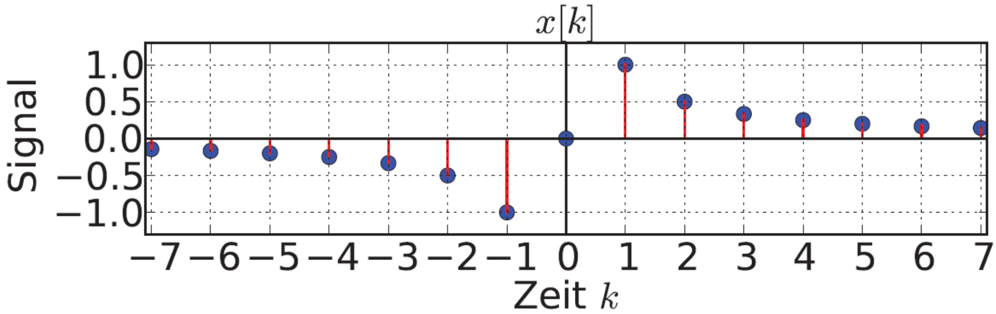
$\boxed{7}$

$$\mathcal{H}\{ax(t) + by(t) + cz(t)\} = \mathcal{H}\{ax(t) + 1\cdot(by(t) + cz(t)\}$$
$$\overset{(??)}{=} a\mathcal{H}\{x(t)\} + 1\cdot\mathcal{H}\{by(t) + cz(t)\} \overset{(??)}{=} a\mathcal{H}\{x(t)\} + b\mathcal{H}\{y(t)\} + c\mathcal{H}\{z(t)\}$$

$\boxed{8}$

$$x[-\infty], \ldots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \ldots, x[\infty]$$

$\boxed{9}$



$\boxed{10}$

- $x[-k]$ die Spiegelung von $x[k]$ an der Signalpegel-Achse
- $x[k+k_0]$ die Verschiebung von $x[k]$ um k_0 nach links
- $x[k-k_0]$ die Verschiebung von $x[k]$ um k_0 nach rechts

2 Diskrete Signale

11

$$(b) \quad x[-k] = \begin{cases} -\frac{1}{k}, & k \neq 0 \\ 0, & k = 0 \end{cases}$$

$$(c) \quad x[k+k_0] = x[k+3] = \begin{cases} \frac{1}{k+3}, & k \neq -3 \\ 0, & k = -3 \end{cases}$$

$$(d) \quad x[k-k_0] = x[k-3] = \begin{cases} \frac{1}{k-3}, & k \neq 3 \\ 0, & k = 3 \end{cases}$$

12

$$\begin{aligned} x[k_0 - k] &= x[-(k-k_0)] \\ &= x[(-k)+k_0] \end{aligned}$$

13

$$\text{mit} \quad x[k_0 - k] = x[3-k] = \begin{cases} \frac{1}{3-k}, & k \neq 3 \\ 0, & k = 3 \end{cases}$$

14

- $x[k]$ heißt <u>gerades Signal</u>, falls $x[k] = x[-k] \; \forall k \in \mathbb{Z}$ gilt.
- $x[k]$ heißt <u>ungerades Signal</u>, falls $x[k] = -x[-k] \; \forall k \in \mathbb{Z}$ gilt.

15

$$x[-k] = \begin{cases} \frac{1}{-k}, & k \neq 0 \\ 0, & k = 0 \end{cases} = \begin{cases} -\frac{1}{k}, & k \neq 0 \\ 0, & k = 0 \end{cases} = -x[k]$$

16

$$y[-k] = \begin{cases} \frac{1}{(-k)^2}, & k \neq 0 \\ 0, & k = 0 \end{cases} = \begin{cases} -\frac{1}{k^2}, & k \neq 0 \\ 0, & k = 0 \end{cases} = y[k]$$

17

- $x[k]$ heißt kausales Signal, falls gilt: $x[k] = 0 \ \forall k < 0$
- $x[k]$ heißt nicht-kausales Signal, falls gilt $\exists k < 0 : x[k] \neq 0$
- $x[k]$ heißt anti-kausales Signal, falls $x[-k-1]$ kausal ist, d.h. falls gilt: $x[k] = 0 \ \forall k \leqslant 0$

18

- $x[k]$ ist nicht-kausal
- $u[k]$ ist kausal
- $v[k]$ ist anti-kausal

19

$$\delta[k] := \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

20

$$\epsilon[k] := \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

21

$$\delta[k-k_0] = \begin{cases} 1, & k = k_0 \\ 0, & k \neq k_0 \end{cases}$$

bzw.

$$\delta[k+k_0] = \begin{cases} 1, & k \neq -k_0 \\ 0, & k = -k_0 \end{cases}$$

22

$$x[k] \cdot \delta[k-i] = \begin{cases} x[i], & k = i \\ 0, & k \neq i \end{cases}$$
$$= x[i] \cdot \delta[k-i] \qquad (2.1)$$

Siebeigenschaft

23

$$x[k] = \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k-i] \quad \text{für alle} \quad k \in \mathbb{Z}$$

24

$$x[k] = \sum_{i=-K}^{K} x[i] \cdot \delta[k-i]$$

25

$$u[k] = \delta[k+2] + \delta[k+1] + \delta[k] + \delta[k-1]$$
$$v[k] = 2 \cdot \delta[k+3] + \delta[k+1] - \delta[k-1] - 2 \cdot \delta[k-3]$$

26

$$sgn[k] := \epsilon[k] - \epsilon[-k] = \begin{cases} 1, & k > 0 \\ 0, & k = 0 \\ -1, & k < 0 \end{cases}$$

27

$$Ш[k] := \epsilon[k] + \epsilon[-k-1] = 1 \text{ für alle } k \in \mathbb{Z}$$

28

$$rect_{k_1,k_2}[k] := \epsilon[k - k1] - \epsilon[k - k_2 - 1] = \begin{cases} 1, & k_1 \leqslant k \leqslant k_2 \end{cases}$$

29

$$x[k] = q^k \cdot \epsilon[k]$$

30

$$x[k] : 0, ..., 0, x[0] = 1, x[1] = -0.7, x[2] = 0.49, x[3] = 0.343, ...$$

31

$$x[k] : 0, ..., 0, x[0] = 1, x[1] = -0.8, x[2] = 0.64, x[3] = -0.512, ...$$

32

$$\begin{aligned} x[k] + y[k] &: x[-\infty] + y[-\infty]..., x[0] + y[0], x[1] + y[1], ..., x[\infty] + y[\infty] \\ x[k] \cdot y[k] &: x[-\infty] \cdot y[-\infty]..., x[0] \cdot y[0], x[1] \cdot y[1], ..., x[\infty] \cdot y[\infty] \\ c \cdot x[k] &: c \cdot x[-\infty]..., c \cdot x[0], c \cdot x[1], ..., c \cdot x[\infty] \end{aligned}$$

33

$$S_{k_1,k_2} := \{\overrightarrow{x} \in S | x[k] = 0 \ \forall k < k_1 \text{ oder } k > k_2\}$$

34

$$\begin{aligned}
\overrightarrow{x} &= (0 \quad 3 \quad 2 \quad 5 \quad 0 \quad 0\ ) \\
\overrightarrow{y} &= (0 \quad 0 \quad 2 \quad -3 \quad 0 \quad 2\ ) \\
\overrightarrow{x} + \overrightarrow{y} &= (0 \quad 3 \quad 4 \quad 2 \quad 0 \quad 2\ ) \\
\overrightarrow{x} - \overrightarrow{y} &= (0 \quad 3 \quad 0 \quad 8 \quad 0 \quad -2) \\
\overrightarrow{x} \cdot \overrightarrow{y} &= (0 \quad 0 \quad 4 \quad -15 \quad 0 \quad 0\ ) \\
c + \overrightarrow{x} &= (0 \quad 15 \quad 10 \quad 25 \quad 0 \quad 0\ )
\end{aligned}$$

35

$$(x * y)[k] := \sum_{i=-\infty}^{\infty} x[i] \cdot y[k-i]$$

36

$$\underset{\substack{\downarrow \\ i=0}}{} x[i] = (\overset{\substack{i=0\\ \downarrow}}{3} \quad 2 \quad 1), \quad y[i] = (\overset{\substack{i=0\\ \downarrow}}{1} \quad -1 \quad 2) \text{ bzw. } \quad z[0-i] = (2 \quad -1 \quad \overset{\substack{i=0\\ \downarrow}}{1})$$

37

	$x[i] =$			3	2	1			$\sum x[i]y[k-i] =$	$(x*y)[k]$
$k=0$	$y[k-i] =$	2	-1	1					$3 \cdot 1$	$= 3$
$k=1$			2	-1	1				$3 \cdot (-1) + 2 \cdot 1$	$= -1$
$k=2$				2	-1	1			$3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1$	$= 5$
$k=3$					2	-1	1		$2 \cdot 2 + 1 \cdot (-1)$	$= 3$
$k=4$						2	-1	1	$1 \cdot 2$	$= 2$

38

$$x[k] * y[k] = 3\delta[k] - \delta[k-1] + 5\delta[k-2] + 3\delta[k-3] + 2\delta[k-4]$$

39

$$
\begin{array}{l}
\qquad\quad i=-43 \\
\qquad\quad\ \downarrow \\
x[i] = (-1 \quad 3 \quad -2) \text{ und} \\
\qquad\quad i = 19 \\
\qquad\quad \downarrow \\
y[i] = (1 \quad -2 \quad 4 \quad -1) \text{ bzw. } y[-i] = (-1 \quad 4 \quad -2 \quad 1)
\end{array}
$$

40

$i = -43$

k	$x[i]$				-1	3	-2				$(x*y)[k]$
-24	$y[k-i] =$	-1	4	-2	1						-1
-23			-1	4	-2	1					$2+3=5$
-22				-1	4	-2	1				$-4-6-2=-12$
-21					-1	4	-2	1			$1+12+4=17$
-20						-1	4	-2	1		$-3-8=-11$
-19							-1	4	-2	1	2

41

$$
\begin{aligned}
(x*y)[k] = &-\delta[k+24] + 5\delta[k+23] - 12\delta[k+22] + 17\delta[k+21] \\
&- 11\delta[k+20] + 2\delta[k+19]
\end{aligned}
$$

42

$$x[k] * y[k] \in \mathcal{S}_{a+c,b+d} \quad \text{und hat Länge} \quad n+m-1.$$

43

I) Kommutativität: $x * y = y * x$

II) Assoziativität: $w * (x * y) = (w * x) * y$ und $c \cdot (x * y) = (c\cdot) * y$

III) Distributivität: $w * (x + y) = w * x + w * y$

IV) Neutrales Element: $x * \delta = x$

V) Verschiebung: $x[k] * \delta[k_0 - k] = x[k - k_0]$

VI) Zeitinvarianz: $x[k] * y[k - k_0] = (x[k] * y[k])[k - k_0]$

VII) Linearität: $(c \cdot x + d \cdot y) * w = c \cdot (x * w) + d \cdot (y * w)$

44

$$p(z) := a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_n z^n$$

45

$$x[k] = a_0 \delta[k] + a_1 \delta[k-1] + a_2 \delta[k-2] + ... + a_n \delta[k-n]$$

46

$$p(z) \cdot q(z) = c_0 + c1_z + c_2 z^2 + ... + c_{2n} z^{2n} \quad \text{Mit Koeffizenten } c_k = (c * y)[k]$$

47

$$p(z) = 3 + 2z + z^2 \text{ und } q(z) = 1 - z + 2z^2$$

48

$$\begin{aligned} p(z) \cdot q(z) &= (3 + 2z + z^2) \cdot (2z^2 - z + 1) \\ &= 3 \cdot 1 + z(3 \cdot (-1) + 2 \cdot 1) + z^2(3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1) \\ &\quad + z^3(2 \cdot 2 + 1 \cdot (-1)) + z^4(1 \cdot 2) \\ &= 3 - z + 5z^2 + 3z^3 + 2z^4 \end{aligned}$$

$\boxed{49}$

$$E_x := \sum_{i=-\infty}^{\infty} |x[i]|^2$$

$\boxed{50}$

$$P_x := \lim_{K\to\infty} \frac{1}{2K+1} \sum_{i=-K}^{K} |x[i]|^2$$

$\boxed{51}$

$$\langle x[k], y[k] \rangle_E := \sum_{k=-\infty}^{\infty} x^*[k] \cdot y[k]$$

$\boxed{52}$

$$\langle x[k], y[k] \rangle_P := \lim_{K\to\infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x^*[k] \cdot y[k]$$

$\boxed{53}$

$$||x[k]||_E := \sqrt{\langle x[k], x[k] \rangle_E} = \sqrt{E_x} \text{ bzw.}$$
$$||x[k]||_P := \sqrt{\langle x[k], x[k] \rangle_P} = \sqrt{P_x}$$

$\boxed{54}$

$$\cos \Phi = \frac{\langle x[k], y[k] \rangle}{||x[k]|| \cdot ||y[k]||}$$

$\boxed{55}$

$$\varphi_{xy}[\kappa] := \langle x[k], y[k+\kappa] \rangle$$

$\boxed{56}$

$$\varphi_{xx}[\kappa] := \langle x[k], x[k+\kappa] \rangle$$

$\boxed{57}$

$$\varphi_{xy}^{E}[\kappa] = x^*[-\kappa] * y[\kappa] \text{ bzw. } \varphi_{xy}^{P}[\kappa] = \lim_{K\to\infty} \frac{1}{2K+1} x_K^*[-\kappa] * y_K[\kappa]$$

3 Diskrete Systeme

Inhalt...

58

$$y[k] = \mathcal{H}\{x[k]\}$$

59

$$x[k] = x_0 \cdot \delta[k] = \begin{cases} x_0, k = 0 \\ 0, k \neq 0 \end{cases}$$

entwickelt sich nun das Guthaben des Sparbuchs wie folgt:
zu Beginn: $y[0] = x_0$
nach 1 Jahr: $y[1] = x_0 + p \cdot x_0 = (1 + p) \cdot x_0$ nach 2 Jahren: $y[2] = (1 + p)x_0 + p \cdot (1 + p) \cdot x_0 = (1 + p) \cdot (1 + p) \cdot x_0 = (1 + p)^2 \cdot x_0$
nach 3 Jahren: $y[3] = ... = (1 + p)^3 \cdot x_0$
nach i Jahren: $y[i] = (1 + p)^i \cdot x_0$
D.h. das Ausgangssignal ist die kausale Exponentialfolge $y[k] = x_0 \cdot (1 + p)^k \cdot \epsilon[k]$

60

$$y[k + 1] = y[k] \cdot (1 + p) + x[k + 1] \tag{3.1}$$

Das heißt $y[k+1]$ ergibt sich aus dem verzinsten Guthaben $y[k]$ des vorigen Jahres und zusätzlich den neuen Einzahlungen $x[k + 1]$.

61

$$\mathcal{H}\{c \cdot x_1[k] + d \cdot x_2[k]\} = c \cdot \mathcal{H}\{x_1[k]\} + d \cdot \mathcal{H}\{x_2[k]\}$$

62

$$y[0] = x[0] = c \cdot x_1[0] + d \cdot x_2[0]$$

63

$$\begin{aligned} y[k+1] &\overset{(3.1)}{=} y[k] \cdot (1+p) + x[k+1] \\ &\overset{(I.V)}{=} (cy_1[k] + d \cdot y_2[k]) \cdot (1+p) + c \cdot x_1[k+1] + d \cdot x_2[k+1] \\ &= c \cdot (y_1[k] \cdot (1+p) + x_1[k+1]) + d \cdot (y_2[k] \cdot (1+p) + x_2[k+1]) \\ &\overset{(3.1)}{=} c \cdot y[k+1] + d \cdot y_2[k+1] \end{aligned}$$

64

$$\mathcal{H}\{x[k-k_0]\} = y[k-k_0]$$

65

$$\begin{aligned} z[k_0] = x[k_0 - k_0] = x[k_0] = y[0] = y[k_0 - k_0] \\ \text{und } z[k] = 0 = y[k-k_0] \text{ für } k < k_0 \end{aligned}$$

66

$$\begin{aligned} z[k+1] &\overset{(3.1)}{=} z[k] \cdot (1+p) + x[k+1-k_0] \\ &\overset{(I.V.)}{=} y[k-k_0] \cdot (1+p) + x[k-k_0+1] \\ &\overset{(3.1)}{=} y[k-k_0+1] \end{aligned}$$

67 Ein System $\mathcal{H}$ heißt <u>kausal</u>, wenn der Ausgabewert $y[k_0]$ zur Zeit k_0 nur von früheren Eingabewerten $x[k], k \leq k_0$ abhängig ist.

68

$$|x[k]| < C \forall k \Rightarrow |y[k]| < D \forall k$$

69

$$y[k] = x_0 \cdot (1+p)^k \cdot \epsilon[k] \to \infty \text{ für } k \to \infty$$

70 Ein System heißt gedächtnislos, wenn der Ausgang $y[k]$ zur Zeit k nur vom Eingang $x[k]$ zur Zeit k abhängt.

71 Dagegen hat ein System ein Gedächtnis der Länge L, falls $y[k]$ nur von $x[\kappa]$ für $|\kappa - k| \leq L$ abhängt.

72

$$h[k] := \mathcal{H}\{\delta[k]\}$$

73

$$\begin{aligned} y[k] = \mathcal{H}\{x[k]\} &\overset{(2.6)}{=} \mathcal{H}\left\{\sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k-i]\right\} \\ &= \sum_{i=-\infty}^{\infty} x[i]\mathcal{H}\{\delta[k-i]\} \\ &= \sum_{i=-\infty}^{\infty} x[i] \cdot h[k-i] \\ &= x[k] * h[k] \end{aligned}$$

74

$$y[k] = x[k] * h[k] \text{ für alle } x[k] \in \mathcal{S}$$

75

$$h[k] := \mathcal{H}\{\delta[k]\} = (1+p)^k \epsilon[k]$$

$\boxed{76}$

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} (1+p)^i \epsilon[i] \cdot x[k-i] = \sum_{i=0}^{\infty} (1+p)^i \cdot x[k-i]$$

$\boxed{77}$

$$y[k] = \sum_{i=0}^{k} (1+p)^i \cdot x[k-i]$$

$\boxed{78}$

$$\sum_{y=-\infty}^{\infty} |h[i]| < \infty$$

$\boxed{79}$

$$|y[k]| = |h[k] * x[k]| = |\sum_{i=-\infty}^{\infty} h[k]x[k-i]| \overset{DUG}{\leq} \sum_{i=-\infty}^{\infty} |h[i] \cdot x[k-i]|$$
$$= \sum_{i=-\infty}^{\infty} |h[i]| \cdot |x[k-i]| < M \sum_{i=-\infty}^{\infty} |h[i]| < M \cdot C < \infty$$

$\boxed{80}$

$$x[k] := sgn(h[-k]) = \begin{cases} 1, & h[-k] > 0 \\ 0, & h[-k] = 0 \\ -1, & h[-k] < 0 \end{cases}$$

$\boxed{81}$

$$x[k] \cdot h[-k] = sgn(h[-k]) \cdot h[-k] = |h[-k]| \geq 0$$

82

$$|x[0]| = |(x \cdot h)[0]| = \left| \sum_{i=-\infty}^{\infty} x[i] \cdot h[-i] \right| = \sum_{i=-\infty}^{\infty} |h[-i]| = \sum_{i=-\infty}^{\infty} |h[i]| = \infty$$

83

$$\sum_{i=-\infty}^{\infty} |h[i]| = \sum_{i=-\infty}^{\infty} (1+p)^i \cdot \epsilon[i] = \sum_{i=0}^{\infty} (1+p)^i$$

84

$$|1+p| < 1 \quad \text{bzw. äquivalent für} \quad -2 < p < 0$$

85

$$h[k] = 0, \quad \forall k < 0$$

86

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^{\infty} h[i] \cdot x[k-i]$$

87

Lösung: Folgendes Blockschaltbild realisiert die Rekursion $y[k+1] = y[k] \cdot (1+p) + x[k+1]$ von (3.1) auf Seite 40: $y[k] = y[k-1] \cdot (1+p) + x[k]$
Hier tikz einfügen!

88

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^{n} \beta i \cdot x[k-i]$$

89

$$y[k] = \frac{1}{\alpha_0} \cdot \left(\sum_{i=0}^{N} \beta i \cdot x[k-i] - \sum_{i=0}^{N} \alpha i \cdot y[k-i] \right)$$

90

$$\begin{aligned} \vec{v}[k+1] &= fv(\vec{v}[k], \vec{x}[k]) \\ \vec{y}[k] &= fy(\vec{v}[k], \vec{x}[k]) \end{aligned}$$

91

I) $\vec{y}[k_0] = fx(\vec{v}[k_0], \vec{x}[k_0]$ II) $\vec{v}[k_0 + 1] = fv(\vec{v}[k_0], \vec{x}[k_0])$ III) $\vec{y}[k_0 + 1] = fy(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1])$ IV) $\vec{v}[k_0 + 2] = fv(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1])$ V) $\vec{y}[k_0 + 2] = \ldots$

92

$$\vec{v}[k] : \begin{pmatrix} x[k-1] & x[k-2] & x[k-3] & \ldots & x[k-L] \end{pmatrix}$$

93

$$\vec{v}[k+1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot v[k] + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot x[k]$$

$$y[k] = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \cdot \vec{v}[k] + (a_0) \cdot x[k]$$

94

$$X(z) := \mathcal{Z}\{x[k]\} := \sum x[k] \cdot z^{-k}$$

95

$$a := \limsup_{h\to\infty} \sqrt[k]{|x[k]|}$$
$$b := \frac{1}{\limsup\limits_{h\to\infty} \sqrt[k]{|x[-k]|}}$$

96

$$X^{+}(z) := \mathcal{Z}\{x[k]\} := \sum_{h=0}^{\infty} x[k] \cdot z^{-k}$$

97

$$
\begin{aligned}
a)\quad & \mathcal{Z}\{\delta[k]\} && = \sum_{k=-\infty}^{\infty} \delta[k] \cdot z^{-k} = z^{-0} = 1 \text{ für } z \in \mathbb{C} \\
b)\quad & \mathcal{Z}\{\delta[k-i]\} && = \sum_{k=-\infty}^{\infty} \delta[k-i] \cdot z^{-k} = z^{-i} \text{ für } 0 < |z| < \infty \\
c)\quad & \mathcal{Z}\{\epsilon[k]\} && = \sum_{k=-\infty}^{\infty} \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right)^{k} \\
& && \overset{geom.}{\underset{Reihe}{=}} \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1} \text{ für } \left|\frac{1}{z}\right| < 1 \text{ bzw. } |z| > 1 \\
d)\quad & \mathcal{Z}\{a^k \cdot \epsilon[k]\} && = \sum_{k=-\infty}^{\infty} a^k \cdot \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^{k} = \frac{1}{1-\frac{a}{z}} \\
& && = \frac{z}{z-a} \text{ für } \left|\frac{a}{z}\right| < 1 \Leftrightarrow |z| > |a| \\
e)\quad & \mathcal{Z}\{a^k \cdot \epsilon[-k-1]\} && = \sum_{k=-\infty}^{\infty} -a^k \cdot \epsilon[-k-1] \cdot z^{-k} = -\sum_{k=-\infty}^{-1} a^k \cdot z^{-k} \\
& && = -\sum_{k=1}^{\infty} a^{-k} \cdot z^{k} = -\sum_{k=1}^{\infty} \left(\frac{a}{z}\right)^{k} = -\frac{z}{a} \cdot \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{k} = \frac{z}{a} \cdot \frac{1}{1-\frac{z}{a}} \\
& && = \frac{z}{z-a} \text{ für } \left|\frac{z}{a}\right| < 1 \Leftrightarrow |z| < |a|
\end{aligned}
$$

98

$$
\begin{aligned}
\mathcal{Z}\{\alpha x[k] + \beta y[k]\} &\overset{Def.}{=} \sum_{k=-\infty}^{\infty} (\alpha x[k] + \beta y[k]) \cdot z^{-k} \\
&= \alpha \left(\sum_{k=-\infty}^{\infty} x[k] z^{-k} \right) + \beta \left(\sum_{k=-\infty}^{\infty} y[k] z^{-k} \right) \\
&= \alpha X(z) + \beta Y(z)
\end{aligned}
$$

99

$$
\mathcal{Z}\{x[k+k_0]\} \overset{Def.}{=} \sum_{k=-\infty}^{\infty} x[k+k_0] z^{-k} \underset{(k=k'-k_0)}{\overset{k'=k+k_0}{=}} \sum_{k'=-\infty}^{\infty} \underbrace{x[k'] z^{-k'+k_0}}_{=z^{-k'} \cdot z^{k_0}}
$$

$$
= z^{k_0} \cdot \underbrace{\sum_{k'=-\infty}^{\infty} x[k'] z^{-k'}}_{=X(z)} = z^{k_0} \cdot X(z)
$$

100

$$
\mathcal{Z}\{a^k \cdot x[k]\} \overset{Def.}{=} \sum_{k=-\infty}^{\infty} \underbrace{a^k}_{=(\frac{1}{\alpha})^{-k}} \cdot x[k] \cdot z^{-k} = \sum_{k=-\infty}^{\infty} x[k] \cdot \left(\frac{z}{\alpha}\right)^{-k} = X\left(\frac{z}{\alpha}\right)
$$

101

$$
\mathcal{Z}\{x[-k]\} = \sum_{k=-\infty}^{\infty} x[-k] z^{-k} \underset{(k=-k')}{\overset{k'=-k}{=}} \sum_{k'=-\infty}^{\infty} x[k'] z^{k'} = \sum_{k'=-\infty}^{\infty} x[k'] \cdot \left(\frac{1}{z}\right)^{-k'} = X\left(\frac{1}{z}\right)
$$

102

$$
x[0] = \lim_{z \to \infty} X(z)
$$

103

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)\cdot X(z)$$

104

$$\mathcal{Z}\{\alpha^{k-1}\cdot\epsilon[k-1]\} = z^{-1}\cdot\mathcal{Z}\{\alpha^k\cdot\epsilon[k]\} = z^{-1}\cdot\frac{z}{z-\alpha} = \frac{1}{z-\alpha}$$

105

$$\mathcal{Z}\{A\cdot\alpha^{k-k_0}\cdot\epsilon[k-k_0]\} = A\cdot z^{-k}\cdot\mathcal{Z}\{\alpha^k\epsilon[k]\} = A\cdot z^{-k_0}\cdot\frac{z}{z-\alpha} = \frac{A\cdot z^{-(k_0-1)}}{z-\alpha}$$

106

$$\mathcal{Z}\{k\cdot\alpha^k\cdot\epsilon[k]\} = -z\cdot\frac{d}{dz}\mathcal{Z}\{\alpha^k\epsilon[k]\} = -z\cdot\left(\frac{z}{z-\alpha}\right)' = -z\cdot\left(\frac{1\cdot(z-\alpha)-z\cdot 1}{(z-\alpha)^2}\right) = \frac{\alpha\cdot z}{(z-\alpha)^2}$$

107

$$Y(z) = H(z)\cdot X(z)$$

108

$$H(z) = \frac{Y(z)}{X(z)}$$

109

$$h[k] = (1+p)^k\cdot\epsilon[k]$$
$$H(z) = \frac{z}{z-(1+p)}$$

110

$$\begin{array}{l} y[k+1] = y[k]\cdot(1+p) + x[k+1] \\ \quad\circ \\ \quad| \\ \quad\bullet \\ Y(z) = Y(z)\cdot(1+p) + zX(z) \Leftrightarrow Y(z)(z-(1+p)) = zX(z) \end{array}$$

111

$$H(z) := \frac{Y(z)}{X(z)} = \frac{z}{z-(1+p)}$$

112

$$\begin{array}{c} x[k] = x_0\cdot\epsilon[k] \\ \circ \\ | \\ \bullet \\ X(z) = x_0\cdot\dfrac{z}{z-1} \end{array}$$

113

$$Y(z) = H(z)\cdot X(z) = \frac{z}{z-(a+p)}\cdot x_0\cdot\frac{z}{z-q} = x_0\cdot z^2\cdot\frac{1}{(z-(a+p))\cdot(z-1)}$$

114

$$\frac{1}{(z-(a+p))\cdot(z-1)} = \frac{p^{-1}}{z-(a+p)} - \frac{p^{-1}}{z-1}$$

115

$$\frac{p^{-1}}{z-(a+p)} \bullet\!\!-\!\!\!-\!\!\circ\; p^{-1}(a+p)^{K-1}\epsilon[K-1] \text{ und } \frac{p^{-1}}{z-1} \bullet\!\!-\!\!\!-\!\!\circ\; p^{-1}1^{K-1}\epsilon[K-1] = p^{-1}\epsilon[K-1]$$

116

$$Y(z) = x_0 \cdot z^2 \cdot \left(\frac{p^{-1}}{z - (1+p)} - \frac{p^{-1}}{z-1} \right)$$

$$\bullet\!\!-\!\!\circ$$

$$\begin{aligned} y[k] &= x_0 \cdot (p^{-1}(1+p)^{k+1} \cdot \epsilon[k+1]) \cdot p^{-1} - \epsilon[k+1] \\ &= \frac{x_0}{p} \cdot ((1+p)^{k+1} - 1) \cdot \epsilon[k+1] \end{aligned}$$

117

$$\sum_{i=0}^{n} \alpha_i y[k-i] = \sum_{i=N-M}^{N} \beta_i x[k-i] \qquad (a_0 \neq 0,\ \beta_{N-M} \neq 0)$$

$$\circ\!\!-\!\!\bullet$$

$$\begin{aligned} \sum_{i=0}^{N} \alpha_i \cdot z^{-i} \cdot Y(z) &= \sum_{i=N-M}^{N} \beta_i \cdot z^{-i} \cdot X(z) \\ \Leftrightarrow Y(z) \cdot \sum_{i=0}^{N} \alpha_i \cdot z^{-i} &= X(z) \cdot \sum_{i=N-M}^{N} \beta_i \cdot z^{-i} \end{aligned}$$

118

$$\begin{aligned} H(z) := \frac{Y(z)}{X(z)} &= \frac{\sum_{i=N-M}^{N} \beta_i \cdot z^{-i}}{\sum_{i=0}^{N} \alpha_i \cdot z^{-i}} \\ &= \frac{\sum_{i=N-M}^{N} \beta_i \cdot z^{N-i}}{\sum_{i=0}^{N} \alpha_i \cdot z^{N-i}} \\ &= \frac{\beta_{N-M} \cdot z^M + \beta_{N-M+1} \cdot z^{M-1} + ... + \beta_N}{\alpha_0 \cdot z^N + \alpha_1 \cdot z^{N-1} + ... + \alpha_N} \end{aligned}$$

119

$$y(k) = \frac{1}{\alpha_0} \left(\sum_{i=N-M}^{N} \beta_i x[k-i] - \sum_{i=1}^{N} \alpha_i y[k-i] \right)$$

120

$$h[k] = 3 \cdot (\frac{1}{5})^k \cdot \epsilon[k] + 2 \cdot (\frac{1}{2})^k \cdot \epsilon[k]$$

$$\circ\!\!-\!\!\bullet$$

$$H(z) = 3 \cdot \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}} = \frac{15z}{5z-1} + \frac{4z}{2z-1}$$
$$= \frac{15z(2z-1) + 4z(5z-1)}{(5z-1)(2z-1)} = \frac{50z^2 - 19z}{10z^2 - 7z + 1}$$

121

$\alpha_0 = a_2 = 10, \alpha_1 = -7, \alpha_1 = 1$ und $\beta_0 = 50, \beta_1 = -19, \beta_2 = 0$

122

$10y[k] - 7y[k-1] + 1y[k-2] = 50x[k] - 19x[k-1]$ bzw. äquivalent

$$y[k] = \frac{1}{10} \cdot (50x[k] - 19x[k-1] + 7y[k-1] - y[k-2])$$

123

$$h[k] = 3 \cdot 5^{-(k+2)}\epsilon[k+2] + 2 \cdot 2^{-k}\epsilon[k]$$

$$\circ\!\!-\!\!\bullet$$

$$H(z) = 3 \cdot z^2 \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}}$$
$$= \frac{15z^3}{5z-1} + \frac{4z}{2z-1} = \frac{30z^4 - 15z^3 + 20z^2 - 4z}{10z^2 - 7z + 1}$$

$(\Rightarrow M = 4, N = 2)$

124

$\alpha_0 = a_{2-0} = 10, \alpha_1 = -7, \alpha_2 = 1$ und

$\beta_{-2} = b_4 = 30, \beta_{-1} = 15, \beta_0 = 20, \beta_1 = -4$

125
$$10y[k]-7y[k-1]+y[k-2]=30x[k+2]-15[k+1]+20x[k]-4x[k-1]$$
$$y[k]=\frac{1}{10}\left(30x[k+2]-15[k+1]+20x[k]-4x[k-1]+7y[k-1]-y[k-2]\right)$$

126
$$y[k-2]=\frac{1}{10}\left(30x[k+2]-15[k-1]+20x[k-2]-4x[k-3]+7y[k-3]-y[k-4]\right)$$

127
$$x[k]=\mathcal{Z}^{-1}\left\{X(z)\right\}=\mathcal{Z}^{1}\left\{\sum_{k=-\infty}^{\infty}x[k]z^{-k}\right\}$$

128
$$Y(z)=1\cdot z^{-1}-\frac{1}{2}z^{-2}+\frac{1}{3}z^{-3}-\frac{1}{4}z^{-4}+...+\frac{(-1)^{k+1}}{k}z^{-k}+...$$

129
$$y[k]=\begin{cases}0 & ,k\leq 0\\ (-1)^{k+1}\cdot\frac{1}{k} & ,k>0\end{cases}$$

130
$$Y(z)\overset{!}{=}\frac{A_1}{z-\lambda_1}+\frac{A_2}{z-\lambda_2}+...+\frac{A_n}{z-\lambda_n}$$

131
$$Y(z)\overset{!}{=}\frac{A_1(\lambda_2)...(z-\lambda_N)+...+A_N(z-\lambda_1)...(z-\lambda_{N-1})}{HN}$$

132

$$
\begin{aligned}
Y(z) = \frac{1}{(z-(1+P))\cdot(z-1)} &= \frac{A(z-1)+B(z-(1+P))}{HN} \\
&= \frac{(A+B)\cdot z - A - B(a+P)}{HN}
\end{aligned}
$$

133

$$
\begin{aligned}
-A-(a+p)B &= 1 \quad (1) \\
A+B &= 0 \quad (2)
\end{aligned}
$$

134

$$(-1-p+1)\cdot B = 1 \text{ bzw. } B = -\frac{1}{p}$$

135

$$A = -B = \frac{1}{p}$$

136

$$Y(z) = \frac{p^{-1}}{z-(a+p)} - \frac{p^{-1}}{z-1}$$

$$\bullet\!-\!\!\!-\!\circ$$

$$y[k] = p^{-1}\cdot((1+p)^{k-1}-1)\cdot\epsilon[k-1]$$

137

$$
\begin{aligned}
Y(z) = \frac{A}{z-5} + \frac{B}{z-3} + \frac{C}{(z-3)^2} &= \frac{A(z-3)^2+B(z-5)(z-3)C(z-5)}{HN} \\
&= \frac{A(z^2-6z+9)+B(z^2-8z+15)+C(z-5)}{HN} \\
&= \frac{z^2(A+B)+z(-6A-8B+C)+9A+15B-5C}{HN}
\end{aligned}
$$

138

$$
\begin{array}{rrcrcrcrl}
 & A & + & B & & & = & 2 & (1) \\
 & -6A & - & 8B & + & 1C & = & -9 & (2) \\
 & 9A & + & 15B & - & 5C & = & 3 & (3) \\
5 \cdot (2) + (3): & -21A & - & 25B & & & = & -42 & (4) \\
21 \cdot (1) + (4): & & & -4B & & & = & 0 & (5)
\end{array}
$$

139

$$Y(z) = \frac{2}{z-5} + \frac{3}{(z-3)^2}$$

$$\bullet\!\!-\!\!\!-\!\!\circ$$

$$y[k] = 2 \cdot 5^{k-1}\epsilon[k-1] + (k-1)3^{k-1}\epsilon[k-1]$$

140

$$Y(z) = s(z) + \frac{r(z)}{q(z)}$$

141

$$s(z) = s_0 + s_1 z + ... + s_{M-N}Z^{M-N}$$

$$\bullet\!\!-\!\!\!-\!\!\circ$$

$$s[k] = s_0\delta[k] + s_1\delta[k+1] + ... + s_{M-N}\delta[k+M-N]$$

142

$$Y(z) = 3z^2 - 2z + 1\frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$

143

$$Y(z) = 3z^2 - 2z + 1\frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$

$$\bullet\!-\!\circ$$

$$y[k] = 3\delta[k+2] - 2\delta[k+1] + \delta[k] + 2 \cdot 5^{k-1}\epsilon[k-1] + (k-1)3^{k-1}\epsilon[k-1]$$

144

$$Y(z) = \sum_{i=0}^{M-N} s_i z^i + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} \frac{A_{i,v}}{(z-\lambda_i)^v}$$

$$\bullet\!-\!\circ$$

$$y[k] = \sum_{i=0}^{M-N} s_i \delta[k+i] + \sum_{i=1}^{Q} \sum_{v=1}^{n_i} A_{i,v} \cdot \binom{k-1}{v-1} \lambda_i^{k-v} \epsilon[k-1]$$

145

$$Y(z) = \frac{z}{(z+1)(z-1)} = \frac{A}{(z-1)} + \frac{B}{(z+1)}$$

146

$$A = \frac{1}{0!}\left(\frac{d}{dz}\right)^0 [(z-1) \cdot \frac{z}{z-1}(z+1)]|_{z=1} = \frac{z}{z+1}|_{z=1} = \frac{1}{2}$$

$$B = \frac{1}{0!}\left(\frac{d}{dz}\right)^0 [(z+1) \cdot \frac{z}{z-1}(z+1)]|_{z=-1} = \frac{z}{z-1}|_{z=-1} = \frac{1}{2}$$

147

$$\begin{aligned}\frac{A}{z \cdot \alpha} + \frac{B}{z \cdot \beta} &= \frac{\frac{a\cdot\alpha+b}{\alpha-\beta}(z-\beta) + \frac{a\cdot\beta+b}{\beta-\alpha}(z-\alpha)}{(z-a)(z-\beta)} \\ &= \frac{z \cdot (a \cdot \alpha + b) - (a \cdot \beta + b)) - \beta(a \cdot \alpha + b) + \alpha(a \cdot \beta + b)}{(\alpha-\beta)(z-\alpha)(z-\beta)} \\ &= \frac{z \cdot a \cdot (\alpha - \beta) + b \cdot (\alpha - \beta)}{(\alpha-\beta)(z-\alpha)(z-\beta)} = \frac{a \cdot z + b}{(z-\alpha)(z-\beta)}\end{aligned}$$

148

$$A = \frac{a \cdot \alpha + b}{\alpha - \beta} = \frac{1 \cdot 1 + 0}{1 - (-1)} = \frac{1}{2} B \qquad = \frac{a \cdot \beta + b}{\alpha - \beta} = \frac{1 \cdot (-1) + 0}{-1 - 1)} = \frac{1}{2}$$

149

$$Y(z) = \frac{A}{z - \alpha} + \frac{B}{z - \beta} = \frac{0.5}{z - 1} + \frac{0.5}{z + 1}$$

150

$$|\lambda_i| < 1 \quad \forall i = 1, \ldots, N$$

151

$$y[k] = y[k-1] + y[k-2] + x[k]$$

$$\circ\!\!-\!\!\!-\!\!\bullet$$

$$Y(z) = z^{-1} Y(z) + z^{-2} Y(z) + X(z) \Leftrightarrow Y(z) \cdot (1 - z^{-1} - z^{-2}) = X(z)$$

152

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - z^{-1} - z^{-2})} = \frac{z^2}{z^2 - z - 1}$$

153

$$\lambda_{1/2} = \frac{-1 \pm \sqrt{1^2 + 4}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

154

$$Y(z) = H(z) \cdot X(z) = \frac{z^2}{z^2 - z - 1} \cdot z^{-1} = \frac{z}{z^2 - z - 1} = \frac{z}{(z - \lambda_1)(z - \lambda_2)}$$

155

$$Y(z) = \frac{1z + 0}{(z-\lambda_1)(z-\lambda_2)} = \frac{A}{z-\lambda_1} + \frac{B}{z-\lambda_2}$$

156

$$\begin{aligned} A &= \frac{a \cdot \alpha + b}{\alpha - \beta} = \frac{\lambda_1}{\lambda_1 - \lambda_2} = \frac{1+\sqrt{5}}{2\sqrt{5}} \\ B &= \frac{a \cdot \beta + b}{\alpha - \beta} = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{1-\sqrt{5}}{-2\sqrt{5}} \end{aligned}$$

157

$$Y(z) = \frac{A}{z-\lambda_1} + \frac{B}{z-\lambda_2}$$

$$\bullet\!-\!\!\circ$$

$$\begin{aligned} y[k] &= A \cdot \lambda_1^{k-1} \epsilon[k-1] + B \cdot \lambda_2^{k-1} \epsilon[k-1] \\ &= \frac{1+\sqrt{5}}{2\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \epsilon[k-1] + \frac{1-\sqrt{5}}{-2\sqrt{5}} \\ &\quad - \frac{1-\sqrt{5}}{-2\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \epsilon[k-1] \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} \right) \epsilon[k-1] \end{aligned}$$

158

$$\begin{aligned} &U(z) = H_1(z) \cdot X(z) \text{ und} \\ &Y(z) = H_2(z) \cdot U(z) = H_2(z) \cdot H_1(z) \cdot X(z) \end{aligned}$$

$\boxed{159}$

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \cdot H_2(z)$$

$$\bullet\!\!-\!\!\circ$$

$$h[k] = h_1[k] * h_2[k]$$

$\boxed{160}$

$$\begin{aligned} V(z) &= H_1(z) \cdot X(z) \text{ und} \\ W(z) &= H_2(z) \cdot X(z) \text{ und also } Y(z) = V(z) + W(z) = H_1(z) \cdot X(z) + H_2(z) \cdot X(z) \\ &= (H_1(z) + H_2(z)) \cdot X(z) \end{aligned}$$

$\boxed{161}$

$$H(z) := \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$

$$\bullet\!\!-\!\!\circ$$

$$h[k] = h_1[k] + h_2[k]$$

$\boxed{162}$

$$\begin{aligned} V(z) &= X(z) - W(z) \text{ und } W(z) = H_2(z) \cdot Y(z) \text{ und} \\ Y(z) &= V(z) \cdot H_1(z) = H_1(z) \cdot (X(z) \cdot H_2(z) \cdot Y(z)) \\ &= H_1(z) \cdot X(z) - H_1(z) \cdot H_2(z) \cdot Y(z) \\ &\Leftrightarrow Y(z) \cdot (1 + H_1(z) + H_2(z)) = H_1(z) \cdot X(z) \end{aligned}$$

$\boxed{163}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z) + H_2(z)}$$

164

$$H(z) = H_1(z) \cdot H_2(z) = \frac{(z+3)(z-1)}{(z^2-1)(z+3)(z-1)} = \frac{1}{(z-1)^2} = \frac{1}{z^2-2z-1}$$

165

$$y[k] - 2y[k-1] + y[k-2] = x[k-2]$$

166

$$y[k] = 2y[k] - y[k-2] + x[k-2]$$

167

$$\begin{aligned} H(z) = H_1(z) + H_2(z) &= \frac{z+3}{z^2-1} + \frac{z+1}{(z+3)(z-1)} = \frac{(z+3)^2+(z+1)^2}{(z^2-1)(z+3)} \\ &= \frac{2z^2+8z+10}{z^3+3z^2-z-3} \text{ (Systemfunktion)} \\ y[k] + 3y[k-1] - y[k-2] - 3y[k-3] &= 2x[k-1] + 8x[k-2] + 10x[k-3] \text{ (DLG)} \\ y[k] &= 2x[k-1] + 8x[k-2] + 10x[k-3] - 3y[k-1] + y[k-2] + 3y[k-3] \text{ (Alg.)} \end{aligned}$$

168

$$\begin{aligned} H(z) &= \frac{H_1(z)}{1+H_1(z)+H_2(z)} = \frac{\frac{z+3}{z^2-1}}{1+\frac{(z+3)(z+1)}{(z^2-1)(z+3)(z-1)}} = \frac{\frac{z+3}{z^2-1}}{\frac{(z-1)^2+1}{(z-1)^2}} = \frac{\frac{z+3}{z+1}}{\frac{z^2-2z+2}{z-1}} \\ &= \frac{(z+3)(z-1)}{(z+1)(z^2-2z+2)} = \frac{z^2+2z-3}{z^3-z^2+2} \text{ (Systemfunktion)} \\ y[k] - y[k-1] + 2y[k-3] &= x[k-1] + 2x[k-2] + 3x[k-3] \text{ (DLG)} \\ y[k] &= x[k-1] + 2x[k-2] - 3x[k-3] + y[k-1] - 2y[k-3] \text{ (Alg.)} \end{aligned}$$

169

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i \cdot \Delta t_n)$$

170

$$\Delta t_n = \frac{b-a}{n}$$

4 Kontinuierliche Signale

Inhalt...

171

$$\int_{-\infty}^{t} \epsilon(\tau)d\tau = \begin{cases} 0 & ,t \leq 0 \\ \int_0^1 1d\tau = [\tau]_0^t = t & ,t > 0 \end{cases} \quad = t \cdot \epsilon(t) =: ramp(t)$$

172

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \epsilon(t)$$

173

$$\delta(t) := \frac{d}{dt}\epsilon(t) = \lim_{T\to 0} \frac{1}{T}\mathrm{rect}(\frac{t}{T}) \qquad = \lim_{T\to 0} \frac{1}{T}\mathrm{tri}(\frac{t}{T}) = \lim_{T\to 0} \frac{2}{T}\mathrm{si}(s\pi\frac{t}{T})$$

174

I) $x(\tau) \cdot \delta(\tau - t) = x(t) \cdot \delta(\tau - t)$

II) $\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = 1$

III) $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ für alle $t \in \mathbb{R}$

175

$$\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = \int_{-\infty}^{\infty} \delta(\lambda)d\lambda = \epsilon(\infty) = 1$$

176

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(\tau-t)d\tau \overset{(I)}{=} \int_{-\infty}^{\infty} x(t)\delta(\tau-t)d\tau$$
$$= x(t)\int_{-\infty}^{\infty} \delta(\tau-t)d\tau \overset{(II)}{=} x(t)\cdot 1 = x(t)$$

177

$$(x*y)(t) := \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

178

a) $$\epsilon(t)*\epsilon(t) = \int_{-\infty}^{\infty} \epsilon(\tau)\epsilon(t-\tau)d\tau = \int_{0}^{\infty} \epsilon(t-\tau)d\tau = \int_{0}^{t} 1\cdot d\tau \overset{\lambda:=t-\tau}{=}$$
$$-\int_{t}^{-\infty} \epsilon(\lambda)d\lambda = \int_{-\infty}^{0} \epsilon(\lambda)d\lambda \overset{(42)}{=} t\epsilon(t) = \mathrm{ramp}(t)$$

b) $$\mathrm{rect}(t)*\epsilon(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\epsilon(t-\tau)d\tau$$
$$= \int_{-0,5}^{0,5} \epsilon(t-\tau)d\tau \overset{\lambda:=t-\tau}{=} -\int_{t-(-0,5)}^{t-0,5} \epsilon(\lambda)d\lambda = -\int_{t-0,5)}^{t+0,5} \epsilon(\lambda)d\lambda$$
$$\overset{(4.2)}{=} [\lambda\epsilon(\lambda)]_{t-0,5}^{t+0,5} = \mathrm{ramp}(t+0,5) - \mathrm{ramp}(t-0,5)$$
$$= \begin{cases} 0 & ,t \le -0,5 \\ t+0,5 & ,-0,5 \le t \le 0,5 \\ 1 & ,t \ge 0,5 \end{cases} =: \mathrm{sramp}$$

179

c)

$$\mathrm{rect}(t)*\epsilon(t) = (\epsilon(t+0,5) - \epsilon(t-0,5))*\epsilon(t)$$
$$\overset{(III)}{=} \epsilon(t+0,5)*\epsilon(t) - \epsilon(t-0,5)*\epsilon(t)$$
$$\overset{(VI.a)}{=} \mathrm{ramp}(t+0,5) - \mathrm{ramp}(t-0,5) = sramp(t)$$

180

$$\cos\Phi = \frac{\langle x(t), y(t)\rangle}{\|x(t)\| \cdot \|y(t)\|}$$

181

$$E_{\delta(t-\tau)} = \|\delta(t-\tau)\|^2 = \int_{-\infty}^{\infty} \delta^2(t-\tau)dt = \lim_{T\to\infty} \int_{\tau-\frac{T}{2}}^{\tau+\frac{T}{2}} \frac{a}{T^2} dt = \lim_{T\to\infty} \frac{a}{T} = \infty$$

$$\cos\Phi = \frac{\langle \delta(t-\tau_1), \delta(t-\tau_2)\rangle}{\|\delta(t-\tau_1)\| \cdot \|\delta(t-\tau_2)\|} = \frac{\int_{-\infty}^{\infty} \delta(t-\tau_1)\cdot\delta(t-\tau_2)dt}{\infty\cdot\infty} = \frac{\int_{-\infty}^{\infty} 0dt}{\infty} = 0$$

182

$$\|\mathrm{rect}(t-\tau)\|^2 = \int_{-\infty}^{\infty} \mathrm{rect}^2(t-\tau)dt \overset{\lambda=t-\tau}{=} \int_{-\infty}^{\infty} \mathrm{rect}(\lambda)d\lambda \overset{(Def.rect)}{=} 1$$

$$\begin{aligned}
\cos\Phi &= \frac{\langle \mathrm{rect}(t-\tau_1), \mathrm{rect}(t-\tau_2)\rangle}{\|\mathrm{rect}(t-\tau_1)\| \cdot \|\mathrm{rect}(t-\tau_2)\|} \\
&= \frac{\int_{-\infty}^{\infty} \mathrm{rect}(\lambda)\cdot\mathrm{rect}(\lambda-((\tau_2-\tau_1)))d\lambda}{1\cdot 1} \\
&\overset{(symmrect)}{=} int_{-\infty}^{\infty}\mathrm{rect}(\lambda)\mathrm{rect}((\tau_2-\tau_1)-\lambda)d\lambda \\
&\overset{(Def.Faltung)}{=} (\mathrm{rect} * \mathrm{rect})(\tau_2-\tau_1) \\
&\overset{S91,Bsp.d}{=} \mathrm{tri}(\tau_2-\tau_1)
\end{aligned}$$

$$\Longrightarrow \Phi = \arccos\mathrm{tri}(\tau_2-\tau_1)$$

183

$$\langle \sin(n\omega_0 t), \sin(m\omega_0 t)\rangle_T = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \cdot \frac{1}{2j}(e^{jm\omega_0 t} - e^{-jm\omega_0 t})dt$$

$$= -\frac{1}{4}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(n+m)\omega_0 t} - e^{j(m-n)\omega_0 t} - e^{j(n.m)\omega_0 t} + e^{-j(n+m)\omega_0 t}dt$$

$$\overset{(4.12)}{=} -\frac{T}{4}(\delta[n+m] - \delta[m-n] - \delta[n-m] - \delta[-(n+m)])$$

$$= \frac{T}{4}\cdot(2\delta[n-m] - 2\delta[n+m]) = \frac{T}{2}\delta[n-m] = \begin{cases} 0 & ,n \neq m \\ \frac{T}{2} & ,n = m \neq 0 \end{cases}$$

184

$$\vec{k} = B^{-1} \cdot \vec{v}$$

185

$$\vec{b_1} \cdot \vec{b_2} = 2 - 4 + 2 = 0; \quad \vec{b_1} \cdot \vec{b_3} = 2 + 2 - 4 = 0; \quad \vec{b_2} \cdot \vec{b_2} = 4 - 2 - 2 = 0;$$

mit gleichen quadrierten (eukl.) Längen

$$\|b_1\|^2 = 1 + 4 + 4 = 9; \quad \|b_2\|^2 = 4 + 4 + 1 = 9; \quad \|b_3\|^2 = 4 + 1 + 4 = 9$$

186

$$B^{-1} = \overline{\begin{pmatrix} \frac{\vec{b_1}}{\|\vec{b_1}\|^2} & \frac{\vec{b_2}}{\|\vec{b_2}\|^2} & \frac{\vec{b_3}}{\|\vec{b_3}\|^2} \end{pmatrix}}^T = \frac{1}{9} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

187

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = B^{-1} \cdot \vec{v} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \vec{v}$$

188

$$\vec{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \vec{v}$$

189

$$B = \begin{pmatrix} \frac{1}{2}(t) & \cos(\omega_0 t) & \sin(\omega_0 t) & \cos(2\omega_0 t) & \sin(2\omega_0 t) & \cdot \end{pmatrix}$$

$\boxed{190}$

$$B^{-1} = \begin{pmatrix} \frac{1}{T/4} \cdot \frac{1}{2}(^T t) \\ \frac{1}{T/2} \cdot \cos(\omega_0 t)^T \\ \frac{1}{T/2} \cdot \sin(\omega_0 t)^T \\ \frac{1}{T/2} \cdot \cos(2\omega_0 t)^T \\ \frac{1}{T/2} \cdot \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix} = \frac{2}{T} \begin{pmatrix} 1(t)^T \\ \cos(\omega_0 t)^T \\ \sin(\omega_0 t)^T \\ \cos(2\omega_0 t)^T \\ \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix}$$

$\boxed{191}$

$$\begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \end{pmatrix} = B^{-1} \cdot \big(x(t)\big) = \frac{2}{T} \begin{pmatrix} 1(t)^T \\ \cos(\omega_0 t)^T \\ \sin(\omega_0 t)^T \\ \cos(2\omega_0 t)^T \\ \sin(2\omega_0 t)^T \\ \vdots \end{pmatrix} \cdot \big(x(t)\big) = \frac{2}{T} \begin{pmatrix} \langle 1(t)^T, x(t) \rangle \\ \langle \cos(\omega_0 t), x(t) \rangle \\ \langle \sin(\omega_0 t), x(t) \rangle \\ \langle \cos(2\omega_0 t), x(t) \rangle \\ \langle \sin(2\omega_0 t), x(t) \rangle \\ \vdots \end{pmatrix}$$

$\boxed{192}$

$$\begin{aligned} a_0 &= \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt \\ a_1 &= \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \cos(\omega_0 t)dt & b_1 &= \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \sin(\omega_0 t)dt \\ a_2 &= \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \cos(2\omega_0 t)dt & b_2 &= \frac{2}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cdot \sin(2\omega_0 t)dt \\ &\vdots & &\vdots \end{aligned}$$

193

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(n\omega_0 t)dt \\
&= \frac{2}{T}\int_{-\frac{T_i}{2}}^{\frac{T_i}{2}} \hat{x}\cos(n\omega_0 t)dt = \frac{2\hat{x}}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{-\frac{T_i}{2}}^{\frac{T_i}{2}} \\
&= \frac{2\hat{x}}{T}\left(\sin(n\omega_0\frac{T_i}{2}) - \sin(-n\omega_0\frac{T_i}{2})\right) \\
&= \frac{4\hat{x}}{T\cdot\frac{2}{T_i}}\left(\frac{\sin(n\omega_0\frac{T_i}{2})}{n\omega_0\frac{T_i}{2}}\right) = 2\hat{x}\frac{T_i}{T}\mathrm{si}\left(n\frac{2}{T}\cdot\frac{T_i}{2}\right) \\
&= 2\hat{x}\frac{T_i}{T}\mathrm{si}\left(n\cdot\pi\cdot\frac{T_i}{T}\right)
\end{aligned}
$$

194

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)dt \\
&= \frac{2}{T}\int_{-\frac{T_i}{2}}^{\frac{T_i}{2}} \hat{x}\sin(n\omega_0 t)dt = \frac{2\hat{x}}{T}\left[\frac{-\cos(n\omega_0 t)}{n\omega_0}\right]_{-\frac{T_i}{2}}^{\frac{T_i}{2}} \\
&= \frac{2\hat{x}}{Tn\omega_0}\left(-\cos(n\omega_0\frac{T_i}{2}) + \cos(-n\omega_0\frac{T_i}{2})\right) = 0
\end{aligned}
$$

195

$$
a_0 = 2\hat{x}\frac{T_i}{T}, \quad 2\hat{x}\frac{T_i}{T}(n\cdot\pi\cdot\frac{T_i}{T}), \quad b_n = 0
$$

196

$$
\begin{aligned}
(X_F(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(n\omega_0 t) \\
&= \hat{x}\frac{T_i}{T} + \sum_{n=1}^{\infty} 2\hat{x}\frac{T_i}{T}\mathrm{si}(n\pi\frac{T_i}{T}) \cdot \cos(n\omega_0 t) \\
&= 2\hat{x}\frac{T_i}{T}(\frac{1}{2} + \mathrm{si}(\pi\frac{T_i}{T}) \cdot \cos(\omega_0 t) + \mathrm{si}(2\pi\frac{T_i}{T}) \cdot \cos(2\omega_0 t) \\
&\quad + \mathrm{si}(3\pi\frac{T_i}{T}) \cdot \cos(3\omega_0 t) + \dots)
\end{aligned}
$$

197

$$
\begin{aligned}
x_F(t) \approx \frac{0.5}{2} &+ 0.45\cos(\omega_0 t) + 0.15\cos(3\omega_0 t) \\
&+ 0 \cdot \cos(4\omega_0 t) - 0.09\cos(5\omega_0 t)
\end{aligned}
$$

198

$$
\begin{aligned}
r_n \cos(\underbrace{n\omega_0 t}_{\alpha} - \underbrace{\varphi_n}_{\beta}) &= \underbrace{r_n \cdot \cos(\varphi_n)}_{:=a_n} \cdot \cos(n\omega_0 t) + \underbrace{r_n \cdot \sin(\varphi_n)}_{:=b_n} \cdot \sin(n\omega_0 t) \\
&= a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)
\end{aligned}
$$

199

$$
r_n = \sqrt{a_n^2 + b_n^2} \text{ und } \varphi_n = sgn(b_n) \cdot \arccos\frac{a_n}{r_n}
$$

$$
(\text{bzw. } \varphi_n = \arctan\left(\frac{b_n}{a_n}\right) + \pi \cdot (1 - \epsilon(a_n) \cdot sgn(b_n)))
$$

200

$$
\begin{aligned}
r_0 &= \frac{a_0}{2} = \hat{x}\frac{T_i}{T} \\
r_n &= \sqrt{a_n^2 + b_n^2} = |a_n| = 2\hat{x}\frac{T_i}{T} \cdot \left| si\left(n\pi\frac{T_i}{T}\right)\right|
\end{aligned}
$$

201

$$X(f) = \sum_{n=0}^{\infty} r_n \cdot \delta(f - nf_0)$$
$$= \hat{x}\frac{T_i}{T} \cdot \delta(f) + \sum_{n=1}^{\infty} 2\hat{x}\frac{T_i}{T} \cdot \left| si\left(n\pi\frac{T_i}{T}\right)\right| \cdot \delta(f - nf_0)$$

202

$$X_H(f) = 2\hat{x}\frac{T_i}{T} \cdot \left| si\left(n\pi\frac{T_i}{T}\right)\right| = 2\hat{x}\frac{T_i}{T} \cdot \left| si\left(\pi\frac{T_i}{T} \cdot \frac{f}{f_0}\right)\right|$$

203

$$f \in \{4f_0, 8f_0, 12f_0, \dots\}$$

204

$$c_0 := \frac{a_0}{2},\ c_n := \frac{1}{2}(a_n - jb_n),\ c_{-n} := \frac{1}{2}(a_n + jb_n) = c_n^*$$

205

$$a_0 = 2c_0,\ a_n = c_n + c_{-n} = 2Re(c_n),\ b_n = j(c_n - c_{-n}) = -2Im(c_n)$$

206

$$c_n := \frac{1}{2}(a_n - jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)dt - j\frac{1}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\underbrace{(\cos(n\omega_0 t) + j\sin(-n\omega_0 t))}_{e^{-jn\omega_0 t}}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-jn\omega_0 t}dt$$
$$c_{-n} = c_n^* = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{jn\omega_0 t}dt$$

207

$$
\begin{aligned}
c_0 &= \frac{a_0}{2} = \hat{x}\frac{T_i}{T} \\
c_n &= \frac{1}{2}(a_n - jb_n) = \frac{a_n}{2} = \hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(n\pi\frac{T_i}{T}\right) \\
c_{-n} &= (c_n)^* = c_n
\end{aligned}
$$

208

$$c_k = \hat{x}\frac{T_i}{T} \cdot \mathrm{si}\left(k\pi\frac{T_i}{T}\right)$$

209

$$X_F(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega_0 t} \text{ für } c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt$$

210

$$
\begin{aligned}
X_F(t) &= \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} x(t')e^{-j\omega_k t'}dt' \cdot e^{j\omega_k t} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} x(t')e^{-j\omega_k t'}dt'\right] \cdot e^{j\omega_k t} \cdot \Delta\omega
\end{aligned}
$$

211

$$X_F(t) \overset{(T\to\infty)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t'}\right]}_{=:X(\omega)} e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

212

$$x(t) = \delta(t) \circ\!\!-\!\!\bullet\ X(f) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j2\pi f t}dt = e^0 = 1$$

$\boxed{213}$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = \int_{-\infty}^{\infty} e^{j2\pi f t} df$$

$\boxed{214}$

$$\begin{aligned} x(t) \circ\!\!-\!\!\bullet\ X(\omega) &= \int_{-\infty}^{\infty} \hat{x}\cdot \mathrm{rect}(\frac{t}{T_i}) e^{-j\omega t} dt = \hat{x}\int_{-T_i/2}^{T_i/2} e^{-j\omega t} dt = \hat{x}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_i/2}^{T_i/2} \\ &= -\frac{\hat{x}}{j\omega}\left(e^{-j\omega\frac{T_i}{2}} - e^{j\omega\frac{T_i}{2}}\right) = -\frac{\hat{x}}{j\omega}\cdot 2j\cdot Im(e^{-j\omega\frac{T_i}{2}}) \\ &= -\frac{2\hat{x}}{\omega}\sin(-\omega\frac{T_i}{2}) = \hat{x}T_i\cdot \mathrm{si}(\omega\frac{T_i}{2}) \overset{(\omega=2\pi f)}{=} \hat{x}\cdot T_i\cdot \mathrm{si}(\pi f T_i) \end{aligned}$$

$\boxed{215}$

$$X(f) = \delta(f-f_0) \bullet\!\!-\!\!\circ\ x(t) = \int_{-\infty}^{\infty} \delta(f-f_0)\cdot e^{j2\pi f t} df = e^{j2\pi f_0 t}$$

$\boxed{216}$

$$c_1 x_1(t) + c_2 x_2(t) \circ\!\!-\!\!\bullet\ c_1 X_1(\omega) + c_2 X_2(\omega)$$

$\boxed{217}$

$$\mathrm{rect}(\frac{t}{2t}) \circ\!\!-\!\!\bullet\ 2T\cdot \mathrm{si}(T\omega) \text{ und } \mathrm{rect}(\frac{t}{4t}) \circ\!\!-\!\!\bullet\ 4T\cdot \mathrm{si}(2T\omega)$$

$\boxed{218}$

$$\begin{aligned} x(t) &= 2\mathrm{rect}(\frac{t}{2t}) + 0.5\mathrm{rect}(\frac{t}{4t}) \\ &\circ\!\!-\!\!\bullet \\ X(\omega) &= 2\cdot 2T\cdot \mathrm{si}(T\omega) + 0.54T\cdot \mathrm{si}(2T\omega) \\ &= 4T\mathrm{si}(T\omega) + 2T\mathrm{si}(2T\omega) = 4T\mathrm{si}(\pi 2Tf) + 2T\mathrm{si}(\pi 4Tf) \end{aligned}$$

$\boxed{219}$

$$\cos(2\pi f_0 t) = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t} \bullet\!\!-\!\!\circ\ \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$
$$\sin(2\pi f_0 t) = \frac{1}{2j}e^{j2\pi f_0 t} + \frac{1}{2j}e^{-j2\pi f_0 t} \bullet\!\!-\!\!\circ\ \frac{1}{2j}\delta(f - f_0) + \frac{1}{2j}\delta(f + f_0)$$
$$= \frac{j}{2}\delta(f + f_0) - \frac{j}{2}\delta(f - f_0)$$

$\boxed{220}$

$$x(t - t_0) \circ\!\!-\!\!\bullet\ e^{-j\omega t_0}X(\omega)$$

$\boxed{221}$

$$x(t) = \mathrm{rect}(\frac{t - t_0}{T}) \circ\!\!-\!\!\bullet\ X(\omega) = e^{-j\omega_0 t} \cdot T \cdot \mathrm{si}(\pi f T)$$

$\boxed{222}$

$$x(at) \circ\!\!-\!\!\bullet\ \frac{1}{|a|}X(\frac{\omega}{a})$$

insbesondere also $x(-t) \circ\!\!-\!\!\bullet\ X(-\omega)$

$\boxed{223}$

$$x_2(t) = \mathrm{rect}(0,5t) \circ\!\!-\!\!\bullet\ X_2(f) := 2\mathrm{si}(\pi\frac{f}{0,5}) \text{ und}$$
$$x_3(t) = rect(2t) \circ\!\!-\!\!\bullet\ X_3(f) := \frac{1}{2}\mathrm{si}(\pi\frac{f}{2})$$

$\boxed{224}$

$$\delta(at) \circ\!\!-\!\!\bullet\ \frac{1}{|a|} \bullet\!\!-\!\!\circ\ \frac{1}{|a|}\delta(t), \text{ d.h. } \delta(at) = \frac{1}{|a|}\delta(t)$$

225 I) Falls $x(t) \circ\!\!-\!\!\bullet\ X_\omega(\omega)$ gilt, dann gilt auch $X_\omega(t) \circ\!\!-\!\!\bullet\ 2\pi x(-\omega)$

II) Falls $x(t) \circ\!\!-\!\!\bullet\ X_f(f)$ gilt, dann gilt auch $X_f(t) \circ\!\!-\!\!\bullet\ x(-f)$

226

$$1 \circ\!\!-\!\!\bullet\ 2\pi\delta(-\omega) = 2\pi\delta(-\omega) \text{ bzw. } 1 \circ\!\!-\!\!\bullet\ \delta(-f) = \delta(f)$$

227

$$\begin{aligned} X(t) = T'\mathrm{si}(\pi T' t) &\circ\!\!-\!\!\bullet\ \mathrm{rect}(\frac{-f}{T'}) = rect(\frac{f}{T'}) \\ \overset{(T'=\frac{1}{T})}{\Longleftrightarrow} \frac{1}{T}\mathrm{si}(\pi\frac{t}{T}) &\circ\!\!-\!\!\bullet\ \mathrm{rect}(T\cdot f) \\ \overset{(lin.)}{\Longleftrightarrow} \mathrm{si}(\pi\frac{t}{T}) &\circ\!\!-\!\!\bullet\ T\mathrm{rect}(Tf) \end{aligned} \tag{4.1}$$

228

$$\frac{d}{dt}x(t) \circ\!\!-\!\!\bullet\ j\omega X(\omega)$$

229

$$\begin{array}{c} x''(t) + 3x'(t) + x(t) = \mathrm{rect}(t) \\ \circ \\ | \\ \bullet \\ (j\omega)^2 X(\omega) + 3j\omega X(\omega) + X(\omega) = \mathrm{si}(\frac{\omega}{2}) \end{array}$$

230

$$X(\omega) = \frac{\mathrm{si}(\frac{\omega}{2})}{(j\omega)^2 + 3j\omega + 1}$$

231

$$\mathrm{sgn}'(t) := \epsilon'(t) - (\epsilon(-t))' = \delta(t) - \delta(-t)\cdot(-1) = 2\delta(t)$$

$$\begin{array}{c}\circ\\ |\\ \bullet\end{array}$$

$$j\omega Y(\omega) = 2\cdot 1 \Longleftrightarrow Y(\omega) = \frac{2}{j\omega} \overset{(\omega=2\pi f)}{=} \frac{1}{j\pi f}$$

232

$$\epsilon(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

$$\begin{array}{c}\circ\\ |\\ \bullet\end{array}$$

$$X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \overset{(5.29)}{=} \pi\delta(\omega) + \frac{1}{j\omega}$$

233

$$\begin{aligned} e^{at}\cdot\epsilon(t) \circ\!\!-\!\!\!-\!\!\bullet\ & \int_{-\infty}^{\infty} e^{at}\epsilon(t)\cdot e^{-j\omega t}dt = \int_{0}^{\infty} e^{(a-j\omega)t}dt \\ &= \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_0^{\infty} \overset{(a<0)}{=} 0 - \frac{1}{a-j\omega} \\ &= \frac{1}{j\omega - a} = \frac{1}{j2\pi f - a} \end{aligned}$$

234

I) Faltungstheorem: $x(t) * y(t) \circ\!\!-\!\!\!-\!\!\bullet\ X(\omega)\cdot(\omega)$

II) Multiplikationstheorem: $x(t)\cdot y(t) \circ\!\!-\!\!\!-\!\!\bullet\ \frac{1}{2\pi}X(\omega) * Y(\omega)$ (bzw. $\circ\!\!-\!\!\!-\!\!\bullet\ X_f(f) * Y_f(f)$)

235

$$Y(f) = X(f)\cdot \mathrm{rect}(\frac{f}{2f_g}) = \begin{cases} X(f) & , |f| < f_g \\ 0 & , |f| > f_g \end{cases}$$

236

$$Y(f) = X(f) \cdot \mathrm{rect}(\frac{f}{2f_g})$$

$$\bullet\!\!-\!\!\circ$$

$$y(t) = x(t) * 2f_g \cdot (s\pi f_g t)$$

237

$$\int_{-\infty}^{t} x(\tau)d\tau \circ\!\!-\!\!\bullet \frac{1}{j\omega}X(\omega) + \pi \cdot X(0) \cdot \delta(\omega)$$
$$= \frac{1}{j2\pi f}X_f(f) + \frac{1}{2}X_f(0) \cdot \delta(f)$$

238

$$e^{j\omega t} \cdot x(t) \circ\!\!-\!\!\bullet X(\omega - \omega_0) \quad (= X_f(f - f_0))$$

239

$$X(j\omega) = \int_{-\infty}^{\infty} e^{5t}\epsilon(t) \cdot e^{-j\omega t}dt = \int_{0}^{\infty} e^{(5-j\omega)\cdot t}dt = \left[\frac{e^{(5-j\omega)\cdot t}}{5 - j\omega}\right]_0^{\infty} \to \infty$$

240

$$x(t) \cdot e^{-\sigma t} = e^{5t} \cdot \epsilon(t) \cdot e^{-6t} = e^{-t} \cdot \epsilon(t)$$

241

$$\mathcal{L}\left\{x(t)\right\} := \mathcal{F}\left\{x(t) \cdot \epsilon^{-\sigma t}\right\}$$

$\boxed{242}$

$$\mathcal{L}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t} \cdot e^{j\omega t} dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t} dt$$
$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt =: X_{\mathcal{L}}(s)$$

$\boxed{243}$

$$x(t)e^{-dt} = \mathcal{F}^{-1}\left\{X_{\mathcal{L}}(s)\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{\mathcal{L}}(\sigma + j\omega) \cdot e^{j\omega t} d\omega$$

$\boxed{244}$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_{\mathcal{L}}(\sigma + j\omega) \cdot e^{(\sigma + j\omega) \cdot t} d\omega = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X_{\mathcal{L}}(s) \cdot e^{st} ds$$

$\boxed{245}$

$$\delta(t) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt = e^{-s \cdot 0} = 1 \text{ für } s \in \mathbb{C}$$

$\boxed{246}$

$$e^{at}\epsilon(t) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} e^{at}\epsilon(t)e^{-st} dt = \int_{0}^{\infty} e^{(a-s)t} dt$$
$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\infty} \overset{\mathrm{Re}(a-s)<0}{=} e^{-\infty} - \frac{e^0}{a-s} = \frac{1}{s-a} \quad \text{für} \quad \mathrm{Re}(s) > \mathrm{Re}(a)$$

247

$$x(t) = \cos(\omega_0 t) \cdot \epsilon(t) = \frac{1}{2} e^{j\omega_0 t}\epsilon(t) + \frac{1}{2} e^{-j\omega_0 t}\epsilon(t)$$

$$\circ\!\!-\!\!\bullet$$

$$\begin{aligned} X(s) &= \frac{1}{2}\frac{1}{s - j\omega_0} + \frac{1}{2}\frac{1}{s + j\omega_0} = \frac{1}{2}\frac{s - j\omega_0 + s + j\omega_0}{(s + j\omega_0)(s - j\omega_0)} \\ &= \frac{s}{s^2 + \omega_0^2} \qquad \text{für} \quad \mathrm{Re}(s) > 0 \end{aligned}$$

248

$$u(t) \quad = R \cdot i(t) \quad \text{bzw} \quad u(t) \quad = L \cdot \frac{d}{dt} i(t) \quad \text{bzw} \quad i(t) \quad = C \cdot \frac{d}{dt} u(t)$$

$$\circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet$$

$$U(s) \quad = R \cdot I(s) U(s) \quad = L \cdot sI(s) I(s) \quad = C \cdot s \cdot U(s)$$

249

$$H_R(s) := \frac{U(s)}{I(s)} = R \quad \text{bzw} \quad H_L(s) := \frac{U(S)}{I(S)} = sL \quad \text{bzw} \quad H_C(s) := \frac{U(s)}{I(s)} = \frac{1}{sC}$$

250

$$\text{MR} : u_1(t) + u_2(t) + ... + u_n(t) = 0$$

$$\circ\!\!-\!\!\bullet$$

$$U_1(t) + U_2(t) + ... + U_n(t) = 0$$

$$\text{KR} : i_1(t) + i_2(t) + ... + i_n(t) = 0$$

$$\circ\!\!-\!\!\bullet$$

$$I_1(t) + I_2(t) + ... + I_n(t) = 0$$

251

a) $u_1(t) = 10\epsilon(t) \circ\!\!-\!\!\!-\!\!\bullet U_1(s) = \frac{10}{s}$
b) $u_1(t) = 3\delta(t) \circ\!\!-\!\!\!-\!\!\bullet U_1(s) = 3$
c) $u_1(t) = 2e^{4t} \cdot \epsilon(t) \circ\!\!-\!\!\!-\!\!\bullet U_1(s) = \frac{2}{s-4}$
$H_R(s) = 2000 = R, H_C(s) = \frac{1}{sC} = \frac{1000}{s}$

252

$$H(s) := \frac{U_2(s)}{U_1(s)} = \frac{H_R(s)}{H_R(s) + H_C(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{RCs}{RCs + 1} = \frac{2s}{2s + 1}$$

253

a) $U_2(s) = H(s) \cdot U_1(s) = \frac{2s}{2s+1} \cdot \frac{10}{s} = \frac{20}{2s+1}$
b) $U_2(s) = H(s) \cdot U_1(s) = \frac{2s}{2s+1} \cdot 3 = \frac{6s}{2s+1}$
c) $U_2(s) = H(s) \cdot U_1(s) = \frac{2s}{2s+1} \cdot \frac{2}{s-4} = \frac{4-s}{(2s+1)(s-4)}$

254

a) $U_2(s) = \frac{20}{2s+1} = 10\frac{1}{s+0{,}5} \bullet\!\!-\!\!\!-\!\!\circ\ u_2(t) = 10e^{-0{,}5t}\epsilon(t)$
b) $U_2(s) = \frac{6s}{2s+1} = 3s\frac{1}{s+0{,}5} \bullet\!\!-\!\!\!-\!\!\circ\ 3 \cdot \frac{d}{dt}\left[e^{-0{,}5t}\epsilon(t)\right]$
$\Longrightarrow\ u_2(t) = 3 \cdot (e^{-0{,}5t} \cdot -0{,}5 \cdot \epsilon(t) + e^{-0{,}5t} \cdot \delta(t)) = 3\delta(t) - 1{,}5e^{-0{,}5t}\epsilon(t)$ c)
$U_2(s) \bullet\!\!-\!\!\!-\!\!\circ\ u_2(t) =$ ... s.h. Übungen mit PBZ

5 Analoge Signale

Inhalt...

255

$$\mathcal{H}\left\{c \cdot x_1(t) + d \cdot x_2(t)\right\} = c \cdot \mathcal{H}\left\{x_1(t)\right\} + d \cdot \mathcal{H}\left\{x_2(t)\right\}$$

256

$$y(t - t_0) = \mathcal{H}\left\{x(t - t_0)\right\}$$

257

$$y(t) = x(t) * h(t) \quad \text{für alle } x(t) \in \mathcal{V}$$

258

$$Y(s) = X(s) \cdot H(s) \quad \text{für alle } x(t) \in \mathcal{V}$$

259

$$H(s) = \frac{Y(s)}{X(s)}$$

260

$$y(t) = x(t) * h(t) \quad \text{für alle } x(t) \in \mathcal{V}$$

$$Y(s) = X(s) \cdot H(s) \quad \text{für alle } X(s) \circ\!\!-\!\!\bullet\; x(t) \in \mathcal{V}$$

261

$$\begin{aligned} y(t) = \mathcal{H}\{x(t)\} &= \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)d\tau\right\} && \text{(wegen Satz 4.4.III)} \\ &= \int_{-\infty}^{\infty} x(\tau)\cdot H\{\delta(t-\tau)d\tau\} && \text{(Linarität)} \\ &= \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)d\tau \\ &= x(t) * h(t) && \text{(Def. Faltung, s.h. Seite 89)} \end{aligned}$$

262

$$H(s) = \frac{RCs}{RCs+1} = s \cdot \frac{1}{s+\frac{1}{RC}}$$

$$\begin{aligned} h(t) &= \frac{d}{dt}\left[e^{-\frac{t}{RC}}\cdot\epsilon(t)\right] = e^{-\frac{t}{RC}}\cdot -\frac{1}{RC}\cdot\epsilon(t) + e^{-\frac{t}{RC}}\cdot\delta(t) \\ &= \delta(t) - \frac{1}{RC}\cdot e^{-\frac{t}{RC}}\epsilon(t) = \delta(t) - \frac{1}{2}e^{-\frac{t}{2}}\epsilon(t) \end{aligned}$$

263

$$\begin{aligned} y(t) = h(t) * x(t) &= (\delta(t) - \frac{1}{RC}\cdot e^{-\frac{t}{RC}}\epsilon(t)) * x(t) \\ &= x(t) - -\frac{1}{RC}\cdot\int_{-\infty}^{\infty} e^{-\frac{\tau}{RC}}\epsilon(\tau)\cdot x(t-\tau)d\tau \\ &= x(t) - -\frac{1}{RC}\cdot\int_{0}^{\infty} e^{-\frac{\tau}{RC}}\epsilon(\tau)\cdot x(t-\tau)d\tau \\ &= x(t) - -\frac{1}{2}\cdot\int_{0}^{\infty} e^{-\frac{\tau}{2}}\epsilon(\tau)\cdot x(t-\tau)d\tau \end{aligned}$$

264

$$\begin{aligned}
y(t) &= \epsilon(t) - \frac{1}{RC}\int_0^{\infty} e^{-\frac{\tau}{RC}} \cdot \epsilon(t-\tau)d\tau \\
&= \epsilon(t) - \frac{1}{RC}\int_0^{\max(t,0)} e^{-\frac{\tau}{RC}} d\tau \\
&= \epsilon(t) - \frac{1}{RC}\left[\frac{e^{-\frac{\tau}{RC}}}{-\frac{1}{RC}}\right]^{\max(t,0)} = \epsilon(t) + \left(e^{-\frac{t}{RC}} - 1\right)\cdot\epsilon(t) = e^{-\frac{t}{RC}}\epsilon(t) = e^{-\frac{t}{2}}\epsilon(t)
\end{aligned}$$

265

$$i_R(t) = \frac{1}{R}\cdot u_R(t) \quad \text{btw.} \quad i_C(t) = C \cdot \frac{d}{dt}u_C(t)$$

266

$$u_1(t) = u_R(t) + u_C(t)$$

267

$$x(t) = RC\frac{d}{dt}y(t) + y(t)$$

268

$$x(t) = RC\frac{d}{dt}y(t) + y(t)$$

$$\circ\!\!-\!\!\!\bullet$$

$$X(s) = RC\cdot s\cdot Y(s) + Y(s)$$

269

$$\begin{aligned}
&X(s) = RC\cdot s\cdot Y(s) + Y(s) \Leftrightarrow X(s) = Y(s)\cdot(RCs+1) \\
&\implies H(s) = \frac{Y(s)}{X(s)} = \frac{1}{RCs+1}
\end{aligned}$$

270

$$Y(s) = H(s) \cdot X(s) = \frac{1}{RCs+1} \cdot \frac{1}{s} = \frac{1}{(RCs+1)s} = \frac{1}{RC} \cdot \frac{1}{(s+\frac{1}{RC}) \cdot s}$$

271

$$Y(s) = \frac{1}{RC} \cdot (\frac{A}{s-\alpha} + \frac{B}{s-\beta}) = \frac{1}{RC} \cdot (\frac{-RC}{s-\frac{1}{RC}} + \frac{RC}{s}) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}$$

272

$$Y(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}$$

$$\bullet\!\!-\!\!\!-\!\!\circ$$

$$y(t) = \epsilon(t) - e^{-\frac{t}{RC}} \cdot \epsilon(t) = (a - e^{-\frac{t}{RC}}) \cdot \epsilon(t)$$

274

$$\sum_{i=0}^{N} a_i y^{(i)}(t) = \sum_{i=0}^{M} b_i x^{(i)}(t)$$

$$\circ\!\!-\!\!\!-\!\!\bullet$$

$$\sum_{i=0}^{N} a_i s^i Y(s) = \sum_{i=0}^{M} b_i s^i X(s) \Leftrightarrow Y(s) \cdot \sum_{i=0}^{N} a_i s^i = X(s) \cdot \sum_{i=0}^{M} b_i s^i$$

275

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{M} b_i s^i}{\sum_{i=0}^{N} a_i s^i} = \frac{b_0 + b_1 s + b_M s^M}{a_0 + a_1 s + ... + a_N s^N}$$

276
$$y(t) = \frac{d}{dt}x(t)$$
$$\circ\!\!-\!\!\bullet$$
$$Y(s) = s \cdot X(s) \implies D(s) := \frac{Y(s)}{X(s)} = s$$

277
$$y(t) = \frac{b_0 x^{(0)}(t) + b_1 x^{(1)}(t) + ... + b_N x^{(N)}(t) - a_0 x^{(0)}(t) - a_1 x^{(1)}(t) - ... - a_N x^{(N)}(t)}{a_0}$$

278
$$y^{(n)} = \frac{d}{dt}y^{(n-1)} \quad \text{bzw} \quad x^{(n)} = \frac{d}{dt}x^{(n-1)}$$

279
$$\int^{(0)} y(t) := y(t) \text{ und } \int^{(n)} y(t) = \int_{-\infty}^{t} \int^{(n-1)} y(\tau) d\tau$$

280
$$\int^{(n)} y(m)t = \int^{(n-m)} y(t)$$

281
$$\begin{aligned} a_0 \int^{(N)} y(t) + a_1 \int^{(N-1)} y(t) + ... + a_N \int^{(0)} y(t) \\ = b_0 \int^{(N)} x(t) + b_1 \int^{(N-1)} x(t) + ... + b_N \int^{(0)} x(t) \end{aligned}$$

282

$$H_{RC}(s) = \frac{1}{RCs+1}$$

283

$$b_0 = 1, a_0 = 1, a_1 = RC$$

284

Zeichung

285

$$H(s) := \frac{Y(s)}{X(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{1 + RCs + LCs^2}$$

286

$$b_0 = 1, a_0 = 1, a_1 = RC, a_2 = LC$$

287

Zeichnung

290

a) $H(s) = \frac{1}{s+3} \cdot (s+2) \cdot \frac{1}{s} \cdot \frac{1}{s-1} \cdot \frac{1}{s+1} \cdot (s-5) = \frac{(s+2)\cdot(s-5)}{s(s-1)(s+1)(s+3)}$ b)$H(s) = \frac{1}{s+3} \cdot \left(s + 2 + \frac{1}{s} \cdot \frac{s+1}{s+5}\right) = \frac{(s+2)(s+5)\cdot s+s+1}{(s+5)(s+3)\cdot s}$

291

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

292

$$H(s) = \frac{1}{s-\lambda} \circ\!\!-\!\!\bullet\; e^{\lambda t}\epsilon(t) =: h(t)$$

293

$$h(t) = e^{(\sigma+j\omega)t}\epsilon(t) = e^{\sigma t} \cdot e^{j\omega t} \cdot \epsilon(t)$$

294

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left|e^{\sigma t}\right| \cdot \left|e^{j\omega t}\right| \cdot |\epsilon(t)| = \int_{0}^{\infty} e^{\sigma t} = \left[\frac{e^{\sigma t}}{\sigma}\right]_0^{\infty}$$
$$= 0 - \frac{e^0}{\sigma} = -\frac{1}{\sigma} = \frac{1}{|\sigma|} < \infty$$

295

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{\sigma t} dt = \infty$$

296

$$H(s) = \frac{1}{(s-\lambda)^n} \circ\!\!-\!\!\bullet\; \frac{t^{n-1}}{(n-1)!} e^{\lambda t}\epsilon(t) =: h(t)$$

297

$$h(t) = \frac{t^{n-1}}{(n-1)!} e^{j\omega t}\epsilon(t)$$

298

$$H(s) = \frac{b_M}{a_M} \cdot \frac{(s-\kappa_1)(s-\kappa_2)(s-\kappa_3)...(s-\kappa_M)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)...(s-\lambda_M)} = \sum_{i=1}^{Q}\sum_{v=1}^{n_i} \frac{A_{i,v}}{(s-\lambda_i)^v}$$

$$\circ\!\!-\!\!\bullet$$

$$h(t) = \sum_{i=1}^{Q}\sum_{v=1}^{n_i} A_{i,v} \cdot \frac{t^{v-1}}{(v-1)!} e^{\lambda_i t}\epsilon(t)$$

299

$Re\lambda_i < 0$ für alle $i = 1, 2, ..., N$

300

$Re\lambda_i \geq 0$ für ein $i = \{1, 2, ..., N\}$

301

a) $H_1(s) = \dfrac{3s}{s^2-9} = \dfrac{3s}{(s+3)(s-3)}$ mit Nullstellen: $\kappa_1 = 0$

und Polstellen: $\lambda_1 = 3, \lambda_2 = -3 \Rightarrow$ Instabil wegen $Re(\lambda_1) = 3 > 0$

b) Aus MNF folgt $\lambda_1 1, 2 = \dfrac{-5 \pm 1}{2} \Rightarrow \lambda_1 = -3, \lambda_2 = -2$

Damit $H_2(s) = \dfrac{2(s-3)}{(s+3)(s+2)}$ NS: $\kappa_1 = 3 \Rightarrow$ Stabil, denn $Re(\lambda_i) < 0$ für $i = 1, 2$

c) $H_3(s) = 2(s-4)$ NS: $\kappa_1 = 4$, PS: $\lambda_1 = 2j, \lambda_2 = 2j, \lambda_3 = 2$

$\Rightarrow$ Grenzstabil, da $Re(\lambda_i) \leq 0$ aber $Re(\lambda_1) = Re(\lambda_2) = 0$ und keine doppelte PS

302

Zeichnung

303

$$a_i > 0 \text{ für alle } i = 1, 2, 3, ..., N$$

304

$$a_i = \sum_{k_1,...,k_{N-i}} \alpha_{k_1} \cdot \alpha_{k_2} \cdot ... \cdot \alpha_{k_{N-i}}$$

305

Für das Nennerpolynom $a(s) = 4s^3 + 3s^2 - 2s + 1$ ist das Sytem instabil,
da der Koeffizient $a_1 = -2 < 0$
Für den Nenner$a(s) = 4s^3 + 3s^2 + 2s + 1$ könnte das System stabil sein.
Es sind aber noch weitere Tests notwendig.

306

307

$$H_1 = |a_2| = a_2$$
$$H_2 = \left| \begin{pmatrix} a_2 & a_0 \\ a_3 & a_1 \end{pmatrix} \right| = a_2 a_1 - a_0 a_3$$
$$H_3 = |H| = a_2 a_1 a_0 - a_0^2 a_3$$

308

a) $H(s) = \frac{7s^2-2s+3}{2s^3+5s^2+4s+1}$
Lösung: Für das Nennerpolynom gilt $N = 3, a_3 = 2, a_2 = 5, a_1 = 4, a_0 = 1$.
Nach Satz ?? testen wir:

$$a_i > 0 \forall i = 0,1,2,3$$
$$a_1 a_2 - a_0 a_3 = 4 \cdot 5 - 1 \cdot 2 = 18 > 0$$

also ist das System stabil.
b) $H(s) = \frac{3s-2}{s^5+3s^4+2s^3+3s^2+5s+7}$
Lösung: Für das Nennerpolynom gilt $N = 5, a_5 = 1, a_4 = 3, a_3 = 2, a_2 = 3, a_1 = 5, a_0 = 7$.
Wir testen:

$$\begin{aligned}
&a_i > 0 \forall i = 0,1,2,3,4,5\\
&a_3 a_4 - a_2 a_5 = 2 \cdot 3 - 3 \cdot 1 = 3 > 0\\
&(a_1 a_2 - a_0 a_3)(a_3 a_4 - a_2 a_5) - (a_1 a_4 - a_0 a_5)^2 = (5 \cdot 3 - 7 \cdot 2) \cdot 3 - (5 \cdot 3 - 7 \cdot 1)^2\\
&\quad = 3 - 64 = -61 < 0
\end{aligned}$$

also ist das System instabil.

309

$$y(t) = H(\alpha) \cdot e^{\alpha t}$$

310

$$y(t) = H(\alpha) \cdot e^{\alpha t}$$

311

$$\begin{aligned}
y(t) = h(t) * x(t) = h(t) * e^{\alpha t} = \int_{-\infty}^{\infty} h(\tau) e^{\alpha(t-\tau)} d\tau = e^{\alpha t} \int_{-\infty}^{\infty} h(\tau) e^{-\alpha\tau} d\tau\\
= e^{\alpha t} \int_{-\infty}^{\infty} h(t') e^{-st'} dt' |_{s=\alpha}^{-\infty} = e^{\alpha t} \cdot H(\alpha)
\end{aligned}$$

$\boxed{312}$

$$\mathcal{H}\{X_{\mathbb{F}}(f)e^{j2\pi ft}\} = H_{\mathcal{F}}(f) \cdot X_{\mathcal{F}}(f)e^{j2\pi ft}$$

$\boxed{313}$

$$y(t) = \mathcal{H}\{\int_{-\infty}^{\infty} X_{\mathcal{F}}(f)e^{j2\pi ft}df\} = \int_{-\infty}^{\infty} \mathcal{H}\{X_{\mathcal{F}}(f)e^{j2\pi ft}df\} = \int_{-\infty}^{\infty} H_{\mathcal{F}}(f) \cdot X_{\mathcal{F}}e^{j2\pi ft}df$$

$\boxed{314}$

$$\begin{aligned} X_{\mathcal{F}}(f) \cdot e^{j2\pi ft} &= |X_{\mathcal{F}}| \cdot e^{j\sphericalangle X_f(f)} \cdot e^{j2\pi ft} = |X_{\mathcal{F}}| \cdot e^{j(2\pi ft + \sphericalangle X_f(f))} \\ H_{\mathcal{F}}(f) &= \left| H_{\mathcal{F}}(f) \cdot e^{j\sphericalangle H_{\mathcal{F}}(f)} \right| \\ Y_{\mathcal{F}}(f) \cdot e^{j2\pi ft} &= H_{\mathcal{F}}(f) \cdot X_{\mathcal{F}}(f) \cdot e^{j2\pi ft} = |H_{\mathcal{F}}(f)| \cdot |X_{\mathcal{F}}(f)| \cdot e^{j(2\pi ft + \sphericalangle X_{\mathcal{F}}(f) + \sphericalangle H_{\mathcal{F}}(f))} \end{aligned}$$

$\boxed{315}$

$$H_{\mathcal{F}}(f) = H(s)|_{s=j2\pi f}$$

$\boxed{316}$

$$|H_{\mathcal{F}}(f)| = \sqrt{(ReH_{\mathcal{F}}(f))^2 + (ImH_{\mathcal{F}}(f))^2}$$

$\boxed{317}$

$$\phi(f) := \sphericalangle H_{\mathcal{F}}(f) = \widetilde{\operatorname{sgn}}(ImH_{\mathcal{F}}(f)) \cdot \arccos \frac{ReH_{\mathcal{F}}(f)}{|H_{\mathcal{F}}(f)|}$$

$\boxed{318}$

$$H(f) = H(s)|_{s=j2\pi f} = \frac{K_p}{1 + j2\pi Tf} = \frac{K_p}{1 + j\frac{f}{f_g}} \text{ mit } f_g := \frac{1}{2\pi T}$$

319

$$|H(f)| = \left| \frac{K_p}{1 + j\frac{f}{f_g}} \right| = \frac{|K_p|}{\sqrt{1 + (\frac{f}{f_g})^2}}$$

320

$$H(f) = \frac{K_p \cdot (1 - j\frac{f}{f_g})}{(1 + j\frac{f}{f_g}) \cdot (1 - j\frac{f}{f_g})} = \frac{K_p}{1 + (\frac{f}{f_g})^2} \cdot \left(1 - j\frac{f}{f_g}\right)$$

321

$$\phi(f) = \sphericalangle H(f) = \arctan(\frac{ImH(f)}{ReH(f)}) = \arctan\frac{-\frac{f}{f_g}}{1} = -\arctan(\frac{f}{f_g})$$

322

f	0	f_g	∞
$\lvert H(f)\rvert$	K_p	$\frac{K_p}{\sqrt{2}}$	0
$\sphericalangle H(f)$	0	-$\frac{\pi}{4}$	$-\frac{\pi}{2}$

323

$$x(t) = A \cdot \cos(2\pi f_0 t) \quad \left(= \frac{A}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\right)$$

324

$$\begin{aligned} y(t) &= |H(f_0)| \cdot A \cdot \cos(2\pi f_0 t + \varphi(f_0)) \\ &= |H(f_0)| \cdot A \cdot \cos(2\pi f_0 (t + \frac{\varphi(f_0)}{2\pi f_0})) = |H(f_0)| \cdot x(t - t_{ph}) \end{aligned}$$

325

$$t_{ph}(f) := -\frac{\varphi(f)}{2\pi f}$$

326

$$t_g(f) := -\frac{1}{2\pi f}\frac{d}{df}\varphi(f) = -\frac{1}{2\pi}\varphi'(f)$$

327

$$t_{ph}(f) := -\frac{\varphi(f)}{2\pi f} = \frac{\arctan(\frac{f}{f_g})}{2\pi f}$$

328

$$t_g(f) := -\frac{1}{2\pi f}\frac{d}{df}(-\arctan(2\pi T f)) = \frac{T}{1+(2\pi f T)^2} = \frac{1}{2\pi f_g}\cdot\frac{1}{1+(\frac{f}{f_g})^2}$$

329

f	0	f_g	∞
$t_{ph}(f)$	$\frac{1}{2\pi f_g}$	$\frac{1}{8 f_g}$	0
$t_g(f)$	$\frac{1}{2\pi f_g}$	$\frac{1}{4\pi f_g}$	0

330

$$a(f) := 20\log|H(f)| = 10\log|H(f)|^2 \quad \text{in dB}$$

331

$$a(f) = 20\log|H(f)| = 10\log|H(f)|^2 = 10\log\frac{|Y(f)|^2}{|X(f)|^2}$$

332

$$a(f) = 20\log\left|\frac{(s+2)(s+3)}{(s+7)(s+5)}\right|_{s=j2\pi f}$$
$$= 20\log|s+2| + 20\log|s+3| - 20\log|s+7| - 20\log|s+5|\,|_{s=j2\pi f}$$

333

$$H_{TPf_g}(f) = \mathrm{rect}\left(\frac{f}{2f_g}\right)$$

334

$$\begin{aligned} H_{HPf_g}(f) &= 1 - H_{TPf_g}(f), \\ H_{BPf_uf_o}(f) &= H_{TPf_o}(f) - H_{TPf_u}(f), \\ H_{BSf_uf_o}(f) &= 1 - H_{BPf_uf_o}(f) \end{aligned}$$

335

$$x[k] = x(t)|_{t=kT}, k \in \mathbb{Z}$$

336

$$f_a = \frac{1}{T}$$

337

$$Ш_T(t) := \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

338

$$x_a(t) := x(t) \cdot Ш_T(t)$$

339

$$x_a(t) = x(t)\cdot Ш_T(t) = \sum_{k=-\infty}^{\infty} x(kT)\cdot\delta(t-kT) = \sum_{k=-\infty}^{\infty} x[k]\cdot\delta(t-kT)$$

$$\circ\!\!-\!\!\bullet\ \mathcal{F}$$

$$X_a(f) = X(f) * \frac{1}{T}Ш_{1/T}(f) = X(f) * \left(\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T})\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f)*\delta(f-\frac{k}{T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-\frac{k}{T})$$

340

$$x_a(t) := x(t)\cdot Ш_T(t) \circ\!\!-\!\!\bullet X_a(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(f-\frac{k}{T})$$

341

$$H_{IP}(f) := T\cdot \mathrm{rect}(\frac{f}{2f_g^{IP}}) = T\cdot\mathrm{rect}(Tf) \bullet\!\!-\!\!\circ\ si(\pi\frac{t}{T}) =: h_{IP}(t)$$

342

$$X(f) = X_a(f)\cdot H_{IP}(f)$$

$$\bullet\!\!-\!\!\circ$$

$$x(t) = x_a(t) * h_{IP}(t) = \left[\sum_{k=-\infty}^{\infty} x[k]\cdot\delta(t-kT)\right] * si(\pi\frac{t}{T}) = \sum_{k=-\infty}^{\infty} x[k]\cdot si(\pi\frac{t-kT}{T})$$

343

$$x(kT) = x[k] = x(kT) \quad \forall k\in\mathbb{Z}$$

$\boxed{344}$

$$X(f) = 0 \text{ für alle } f > f_g$$

$\boxed{345}$

$$f_a > 2f_g$$

$\boxed{346}$

$$x[k] = \frac{1}{\Delta T}\int_{kT-\frac{\Delta T}{2}}^{kT+\frac{\Delta T}{2}} x(t)dt = x(t) * \frac{1}{\Delta T}\mathrm{rect}(\frac{t}{\Delta T})|_{t=kT}$$

$\boxed{347}$

$$x_a(t) = \left[x(t) * \frac{1}{\Delta T}\mathrm{rect}(\frac{t}{\Delta T})\right] \cdot Ш_T(t)$$

$$X_a(f) = [X(f) \cdot si(\pi \Delta T f)] * \frac{1}{T}Ш_{\frac{1}{T}}(f)$$

$\boxed{348}$

$$\frac{1}{\Delta T} \overset{!}{\gg} \frac{1}{T} = f_a \text{ bzw. } \Delta T \overset{!}{\ll} T$$

$\boxed{349}$

$$x_{tr}(t) = x_a(t) * \frac{1}{T}\mathrm{rect}(\frac{t}{T}) - \frac{1}{2})$$

$$X_{tr}(f) = X_a(f) \cdot si(\pi T f) \cdot e^{-j\pi T f}$$

350

$$\frac{1}{T} > \frac{1}{2T} \overset{!}{\gg} f_g$$

351 $f_g \approx 20\,\text{kHz}$ (menschlicher Hörbereich) mit Abtastfrequenz $f_a = 44{,}1\,\text{kHz} \rightarrow$ Übergangsbereich von $f_g \approx 20\,\text{kHz}$ bis $f_a - f_g = 24{,}1\,\text{kHz}$ d.h. relative Breite von $\frac{24{,}1\,\text{kHz}}{20\,\text{kHz}} \approx 120\%$

352

$$X_a(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f - \frac{n}{T})$$

353

$$\begin{aligned}
X_a(f) &\overset{(\#)}{=} \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]\delta(t-kT)e^{-j2\pi ft}dt \\
&= \sum_{k=-\infty}^{\infty} x[k]\int_{-\infty}^{\infty}\delta(t-kT)e^{-j2\pi ft}dt \overset{(\#\#)}{=} \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fTk} \\
&= \sum_{k=-\infty}^{\infty} x[k]z^{-k}|_{z=e^{j2\pi ft}} \text{ (Z-Trafo von } x[k] \text{ an der Stelle } z = e^{j2\pi ft}) \\
&= X_z(z)|_{z=e^{j2\pi ft}} =: X_{\mathcal{F}_z}(f)
\end{aligned}$$

354

$$\Omega := 2\pi\frac{f}{f_a} = 2\pi Tf$$

355

$$X_{\mathcal{F}_z}(\Omega) := \mathcal{F}_z\{x[k]\} := \sum_{k'=-\infty}^{\infty} x[k]e^{-j\Omega k} = X_z(z)|_{z=e^{j\Omega}}$$

356

$$\Omega := 2\pi T f \text{ bzw. } \frac{d\Omega}{df} = 2\pi T$$

357

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{\mathcal{F}_z}(\Omega)e^{j\Omega k}d\Omega &\overset{(Def7.6)}{=} \sum_{l=-\infty}^{\infty}\frac{x[l]}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(k-l)}d\Omega \\ &\overset{(n=k-l)}{=} \sum_{l=-\infty}^{\infty} x[l]\delta[k-l] = x[k]\end{aligned}$$

358

$$x(t) = \cos(w\pi f_0 t) \circ\!\!-\!\!\!-\!\!\bullet X_{\mathcal{F}}(f) = \frac{1}{2}[\delta(f+f_0) + \delta(f-f_0)]$$

359

$$X_{\mathcal{F}_z}(f) = \frac{1}{2T}[\delta(f+f_0) + \delta(f-f_0)], \qquad -\frac{1}{2T} < f < \frac{1}{2T}$$

360

$$\begin{aligned}\delta(f \pm f_0) &= \delta\left(\frac{\Omega \pm \Omega_0}{2\pi T}\right) = 2\pi T \cdot \delta(\Omega \pm \Omega_0) \\ X_{\mathcal{F}_z}(\Omega) &= \frac{1}{2T}[2\pi T \cdot \delta(\Omega + \Omega_0) + 2\pi T \cdot \delta(\Omega - \Omega_0)] \\ &= \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)], \qquad -\pi < \Omega < \pi\end{aligned}$$

361

$$\begin{gathered} y[k] = h[k] * x[k] \\ \Big\updownarrow \mathcal{F}_z \\ Y_{\{_z}(\Omega) = \mathcal{H}_{\mathcal{F}_z}(\Omega) \cdot X_{\mathcal{F}_z}(\Omega) \text{ bzw.} \\ Y_{\{_z}(f) = \mathcal{H}_{\mathcal{F}_z}(f) \cdot X_{\mathcal{F}_z}(f) \end{gathered}$$

362

$$y[k] = h[k] * x[k] \overset{z}{\circ\!\!-\!\!\!-\!\!\bullet} Y_Z(z) = H_z(z) \cdot X_z(z)$$

363

$$Y_z(e^{j\Omega}) = H_z(e^{j\Omega}) \cdot X_z(e^{j\Omega}) \text{ bzw.}$$
$$\text{äquivalent } Y_{\{_z}(\Omega) = \mathcal{H}_{\mathcal{F}_z}(\Omega) \cdot X_{\mathcal{F}_z}(\Omega)$$

364

$$\begin{aligned} h[k] &= \frac{1}{3}\left(\delta[k+1] + \delta[k] + \delta[k-1]\right) \\ &\;\Big\downarrow \mathcal{F}_z \\ H(\Omega) &= H_z(z)|_{z=e^{j\Omega}} \\ &= \frac{1}{3}(z + 1 + z^{-1})|_{z=e^{j\Omega}} \\ &= \frac{1}{3}(e^{j\Omega} + 1 + e^{-j\Omega}) = \frac{1}{3} \cdot (1 + 2Re(e^{j\Omega})) \\ &= \frac{1}{3}(1 + 2\cos(\Omega)) \end{aligned}$$

365

$$x[k] := \cos(\Omega_0 k) \circ\!\!-\!\!\!-\!\!\bullet X(\Omega) := \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)], \qquad -\pi < \Omega < \pi$$

366

$$\begin{aligned} Y(\Omega) &= H(\Omega) \cdot X(\Omega) = H(\Omega) \cdot \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)] \\ &= \pi H(\Omega) \cdot \delta(\Omega + \Omega_0) + \pi \cdot H(\Omega) \cdot \delta(\Omega - \Omega_0) \\ &= \pi H(-\Omega_0) \cdot \delta(\Omega + \Omega_0) + H(\Omega_0) \cdot \delta(\Omega - \Omega_0) \\ &= \pi H(\Omega_0)[\cdot\delta(\Omega + \Omega_0) + \cdot\delta(\Omega - \Omega_0)] \end{aligned}$$

367

$$Y(\Omega) = \pi H(\Omega_0)[\cdot\delta(\Omega+\Omega_0) + \cdot\delta(\Omega-\Omega_0)]$$

$$\circ\!\!-\!\!\bullet \; \mathcal{F}_z$$

$$y[k] = H(\Omega_0)\cdot x[k] = \frac{1}{3}[1+2\cos(\Omega_0)]\cdot\cos(\Omega_0 k)$$

368

$$|H(\Omega)| = |\frac{1}{3}[1+2\cos(\Omega_0)]| = \frac{1}{3}|1+2\cos(\Omega_0)|$$

$$\varphi(\Omega) = \sphericalangle H(\Omega) = 0 \qquad (\text{da } H(\Omega) \text{ reell})$$

$$t_G(\Omega) = -\frac{d}{d\Omega}\varphi(\Omega) = 0$$

369

$$y(t) = h(t) * x(t)$$

$$\circ\!\!-\!\!\bullet \; \mathcal{F}$$

$$Y_{\mathcal{F}}(f) = H_{\mathcal{F}}(f)\cdot X_{\mathcal{F}}(f)$$

370

$$y^D[k] = h^D[k] * x[k]$$

$$\circ\!\!-\!\!\bullet \; \mathcal{F}_z$$

$$Y_a^D(f) = H_{\mathcal{F}_z}^D(f)\cdot X_{\mathcal{F}_z}(f)$$

Todo list