

Signale und Systeme Boxen

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1 Motivation, Wiederholung und Überblick

a

1

$$\begin{aligned}\underline{U}_1 &= U_1 \angle \varphi_1 = \frac{30}{\sqrt{2}} \angle \frac{\pi}{3} \\ \underline{Z}_R &= R = 1000 \\ \underline{Z}_C &= \frac{1}{j\omega C} = \frac{1}{sC} = \frac{1000}{s}\end{aligned}$$

2

$$H(s) := \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_C}{\underline{Z}_R + \underline{Z}_C} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC} = \frac{1}{1 + s}$$

3

$$\begin{aligned}|H(j\omega)| &= \left| \frac{1}{1 + j\omega RC} \right| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + \omega^2}} \\ \angle H(j\omega) &= \angle \frac{1}{1 + j\omega RC} = \angle \frac{1 - j\omega RC}{1 + (\omega RC)^2} = \arctan \left(\frac{\frac{-\omega RC}{1 + (\omega RC)^2}}{\frac{1}{1 + (\omega RC)^2}} \right) \\ &= \arctan(\omega RC) = -\arctan(\omega) \\ \underline{U}_2 &= H(s) \cdot \underline{U}_1 = |H(j\omega)| \cdot |\underline{U}_1| \angle \varphi_1 \angle H(j\omega) \\ &= \frac{1}{\sqrt{1 + \omega^2}} \cdot |\underline{U}_1| \angle \left(\frac{\pi}{3} - \arctan(\omega) \right)\end{aligned}$$

4

$$\begin{aligned}
 a) \quad \underline{U}_2 &= \frac{1}{\sqrt{1 + (2\pi \cdot 0,5)^2}} \cdot 21,2 \angle \frac{\pi}{3} - \arctan(2\pi \cdot 0,5) \approx 6,43 \angle -0,21 \\
 &\Rightarrow u_2(t) = 6,43 \cdot \sqrt{2} \cdot \sin(2\pi \cdot 0,5 \cdot t - 0,21) \approx 9,09 \text{ V} \cdot \sin(\pi t - 0,21) \\
 b) \quad \underline{U}_2 &\approx 0,67 \angle -0,49 \Rightarrow u_2(t) \approx 0,95 \text{ V} \cdot \sin(10\pi t - 0,49) \\
 c) \quad \underline{U}_2 &\approx 0,0067 \angle -0,523 \Rightarrow u_2(t) \approx 9,55 \text{ mV} \cdot \sin(100\pi t - 0,523)
 \end{aligned}$$

5

$$\begin{aligned}
 \text{Für } x(t) &= 30 \text{ V} \sin(\pi t + \pi/3) \quad \text{ist} \quad \mathcal{H}\{x(t)\} = 9,09 \text{ V} \sin(\pi t - 0,21) \\
 \text{Für } y(t) &= 30 \text{ V} \sin(10\pi t + \pi/3) \quad \text{ist} \quad \mathcal{H}\{y(t)\} = 0,95 \text{ V} \sin(10\pi t - 0,49)
 \end{aligned}$$

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$$\begin{aligned}
 u_1(t) &= 15 \text{ V} \sin(\pi t + \pi/3) + 60 \text{ V} \sin(10\pi t + \pi/3) = 0,5x(t) + 2y(t) \\
 \text{und damit } a &= 0,5, b = 2 \text{ und} \\
 u_2(t) &:= \mathcal{H}\{u_1(t)\} = \mathcal{H}\{0,5x(t) + 2y(t)\} \stackrel{??}{=}
 \end{aligned}$$

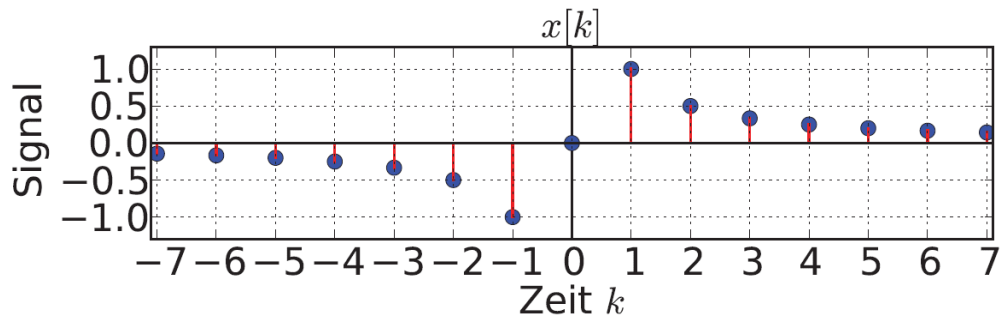
7

$$\begin{aligned}
 \mathcal{H}\{ax(t) + by(t) + cz(t)\} &= \mathcal{H}\{ax(t) + 1 \cdot (by(t) + cz(t))\} \\
 &\stackrel{??}{=} a\mathcal{H}\{x(t)\} + 1 \cdot \mathcal{H}\{by(t) + cz(t)\} \stackrel{??}{=} a\mathcal{H}\{x(t)\} + b\mathcal{H}\{y(t)\} + c\mathcal{H}\{z(t)\}
 \end{aligned}$$

8

$$x[-\infty], \dots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \dots, x[\infty]$$

9



10

- $x[-k]$ die Spiegelung von $x[k]$ an der Signalpegel-Achse
- $x[k + k_0]$ die Verschiebung von $x[k]$ um k_0 nach links
- $x[k - k_0]$ die Verschiebung von $x[k]$ um k_0 nach rechts

2 Diskrete Signale

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$$\begin{aligned} (b) \quad x[-k] &= \begin{cases} -\frac{1}{k}, k/ne0 \\ 0, k = 0 \end{cases} \\ (c) \quad x[k + k_0] &= x[k + 3] = \begin{cases} \frac{1}{k+3}, k/ne-3 \\ 0, k = -3 \end{cases} \\ (d) \quad x[k - k_0] &= x[k - 3] = \begin{cases} \frac{1}{k-3}, k/ne3 \\ 0, k = 3 \end{cases} \end{aligned}$$

12

$$\begin{aligned} x[k_0 - k] &= x[-(k - k_0)] \\ &= x[(-k) + k_0] \end{aligned}$$

13

$$\text{mit } x[k_0 - k] = x[3 - k] = \begin{cases} \frac{1}{3-k}, k \neq 3 \\ 0, k = 3 \end{cases}$$

14

- $x[k]$ heißt gerades Signal, falls $x[k] = x[-k] \forall k \in \mathbb{Z}$ gilt.
- $x[k]$ heißt ungerades Signal, falls $x[k] = -x[-k] \forall k \in \mathbb{Z}$ gilt.

15

$$x[-k] = \begin{cases} \frac{1}{-k}, k \neq 0 \\ 0, k = 0 \end{cases} = \begin{cases} -\frac{1}{k}, k \neq 0 \\ 0, k = 0 \end{cases} = -x[k]$$

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$$y[-k] = \begin{cases} \frac{1}{(-k)^2}, k \neq 0 \\ 0, k = 0 \end{cases} = \begin{cases} -\frac{1}{k^2}, k \neq 0 \\ 0, k = 0 \end{cases} = y[k]$$

17

- $x[k]$ heißt kausales Signal, falls gilt: $x[k] = 0 \forall k < 0$
- $x[k]$ heißt nicht-kausales Signal, falls gilt $\exists k < 0 : x[k] \neq 0$
- $x[k]$ heißt anti-kausales Signal, falls $x[-k-1]$ kausal ist, d.h. falls gilt: $x[k] = 0 \forall k \leq 0$

18

- $x[k]$ ist nicht-kausal
- $u[k]$ ist kausal
- $v[k]$ ist anti-kausal

19

$$\delta[k] := \begin{cases} 1, k = 0 \\ 0, k \neq 0 \end{cases}$$

20

$$\epsilon[k] := \begin{cases} 1, k \geq 0 \\ 0, k < 0 \end{cases}$$

21

$$\delta[k - k_0] = \begin{cases} 1, k = k_0 \\ 0, k \neq k_0 \end{cases}$$

bzw.

$$\delta[k + k_0] = \begin{cases} 1, k \neq -k_0 \\ 0, k = -k_0 \end{cases}$$

22

$$x[k] \cdot \delta[k - i] = \begin{cases} x[i], k = i \\ 0, k \neq i \end{cases}$$

$$= x[i] \cdot \delta[k - i] \quad (2.1)$$

Siebeigenschaft

23

$$x[k] = \sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k - i] \quad \text{für alle } k \in \mathbb{Z}$$

24

$$x[k] = \sum_{i=-K}^K x[i] \cdot \delta[k - i]$$

25

$$u[k] = \delta[k + 2] + \delta[k + 1] + \delta[k] + \delta[k - 1]$$

$$v[k] = 2 \cdot \delta[k + 3] + \delta[k + 1] - \delta[k - 1] - 2 \cdot \delta[k - 3]$$

26

$$\text{sgn}[k] := \epsilon[k] - \epsilon[-k] = \begin{cases} 1, k > 0 \\ 0, k = 0 \\ -1, k < 0 \end{cases}$$

27

$$\text{III}[k] := \epsilon[k] + \epsilon[-k - 1] = 1 \text{ für alle } k \in \mathbb{Z}$$

28

$$\text{rect}_{k_1, k_2}[k] := \epsilon[k - k_1] - \epsilon[k - k_2 - 1] = \begin{cases} 1, k_1 \leq k \leq k_2 \end{cases}$$

29

$$x[k] = q^k \cdot \epsilon[k]$$

30

$$x[k] : 0, \dots, 0, x[0] = 1, x[1] = -0.7, x[2] = 0.49, x[3] = 0.343, \dots$$

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$$x[k] : 0, \dots, 0, x[0] = 1, x[1] = -0.8, x[2] = 0.64, x[3] = -0.512, \dots$$

32

$$\begin{aligned} x[k] + y[k] &: x[-\infty] + y[-\infty], \dots, x[0] + y[0], x[1] + y[1], \dots, x[\infty] + y[\infty] \\ x[k] \cdot y[k] &: x[-\infty] \cdot y[-\infty], \dots, x[0] \cdot y[0], x[1] \cdot y[1], \dots, x[\infty] \cdot y[\infty] \\ c \cdot x[k] &: c \cdot x[-\infty], \dots, c \cdot x[0], c \cdot x[1], \dots, c \cdot x[\infty] \end{aligned}$$

33

$$S_{k_1, k_2} := \{\vec{x} \in S | x[k] = 0 \forall k < k_1 \text{ oder } k > k_2\}$$

34

$$\begin{aligned}\vec{x} &= (0 \quad 3 \quad 2 \quad 5 \quad 0 \quad 0) \\ \vec{y} &= (0 \quad 0 \quad 2 \quad -3 \quad 0 \quad 2) \\ \vec{x} + \vec{y} &= (0 \quad 3 \quad 4 \quad 2 \quad 0 \quad 2) \\ \vec{x} - \vec{y} &= (0 \quad 3 \quad 0 \quad 8 \quad 0 \quad -2) \\ \vec{x} \cdot \vec{y} &= (0 \quad 0 \quad 4 \quad -15 \quad 0 \quad 0) \\ c + \vec{x} &= (0 \quad 15 \quad 10 \quad 25 \quad 0 \quad 0)\end{aligned}$$

35

$$(x * y)[k] := \sum_{i=-\infty}^{\infty} x[i] \cdot y[k-i]$$

36

$$\begin{array}{ccc} i=0 & i=0 & i=0 \\ \downarrow & \downarrow & \downarrow \\ x[i] = (3 \quad 2 \quad 1), & y[i] = (1 \quad -1 \quad 2) \text{ bzw. } & z[0-i] = (2 \quad -1 \quad 1) \end{array}$$

37

	$x[i] =$	$3 \quad 2 \quad 1$	$\sum x[i]y[k-i] =$	$(x * y)[k]$
$k=0$	$y[k-i] =$	$2 \quad -1 \quad 1$	$3 \cdot 1$	$= 3$
$k=1$		$2 \quad -1 \quad 1$	$3 \cdot (-1) + 2 \cdot 1$	$= -1$
$k=2$		$2 \quad -1 \quad 1$	$3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1$	$= 5$
$k=3$		$2 \quad -1 \quad 1$	$2 \cdot 2 + 1 \cdot (-1)$	$= 3$
$k=4$		$2 \quad -1 \quad 1$	$1 \cdot 2$	$= 2$

38

$$x[k] * y[k] = 3\delta[k] - \delta[k-1] + 5\delta[k-2] + 3\delta[k-3] + 2\delta[k-4]$$

39

$$\begin{aligned}
 & i = -43 \\
 & \downarrow \\
 & x[i] = (-1 \quad 3 \quad -2) \text{ und} \\
 & i = 19 \\
 & \downarrow \\
 & y[i] = (1 \quad -2 \quad 4 \quad -1) \text{ bzw. } y[-i] = (-1 \quad 4 \quad -2 \quad 1)
 \end{aligned}$$

40

$i = -43$	k	$x[i]$			-1	3	-2					$(x * y)[k]$
	-24	$y[k - i] =$	-1	4	-2	1						-1
	-23			-1	4	-2	1					$2 + 3 = 5$
	-22				-1	4	-2	1				$-4 - 6 - 2 = -12$
	-21					-1	4	-2	1			$1 + 12 + 4 = 17$
	-20						-1	4	-2	1		$-3 - 8 = -11$
	-19							-1	4	-2	1	2

41

$$\begin{aligned}
 (x * y)[k] &= -\delta[k + 24] + 5\delta[k + 23] - 12\delta[k + 22] + 17\delta[k + 21] \\
 &\quad - 11\delta[k + 20] + 2\delta[k + 19]
 \end{aligned}$$

42

$$x[k] * y[k] \in \mathcal{S}_{a+c, b+d} \quad \text{und hat Lange} \quad n + m - 1.$$

43

- I) Kommutativität: $x * y = y * x$
 II) Assoziativität: $w * (x * y) = (w * x) * y$ und $c \cdot (x * y) = (c \cdot) * y$
 III) Distributivität: $w * (x + y) = w * x + w * y$
 IV) Neutrales Element: $x * \delta = x$
 V) Verschiebung: $x[k] * \delta[k_0 - k] = x[k - k_0]$
 VI) Zeitinvarianz: $x[k] * y[k - k_0] = (x[k] * y[k])[k - k_0]$
 VII) Linearität: $(c \cdot x + d \cdot y) * w = c \cdot (x * w) + d \cdot (y * w)$

44

$$p(z) := a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n$$

45

$$x[k] = a_0 \delta[k] + a_1 \delta[k - 1] + a_2 \delta[k - 2] + \dots + a_n \delta[k - n]$$

46

$$p(z) \cdot q(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_{2n} z^{2n} \quad \text{Mit Koeffizienten } c_k = (c * y)[k]$$

47

$$p(z) = 3 + 2z + z^2 \text{ und } q(z) = 1 - z + 2z^2$$

48

$$\begin{aligned} p(z) \cdot q(z) &= (3 + 2z + z^2) \cdot (2z^2 - z + 1) \\ &= 3 \cdot 1 + z(3 \cdot (-1) + 2 \cdot 1) + z^2(3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1) \\ &\quad + z^3(2 \cdot 2 + 1 \cdot (-1)) + z^4(1 \cdot 2) \\ &= 3 - z + 5z^2 + 3z^3 + 2z^4 \end{aligned}$$

49

$$E_x := \sum_{i=-\infty}^{\infty} |x[i]|^2$$

50

$$P_x := \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{i=-K}^K |x[i]|^2$$

51

$$\langle x[k], y[k] \rangle_E := \sum_{k=-\infty}^{\infty} x^*[k] \cdot y[k]$$

52

$$\langle x[k], y[k] \rangle_P := \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{k=-K}^K x^*[k] \cdot y[k]$$

53

$$\begin{aligned} ||x[k]||_E &:= \sqrt{\langle x[k], x[k] \rangle_E} = \sqrt{E_x} \text{ bzw.} \\ ||x[k]||_P &:= \sqrt{\langle x[k], x[k] \rangle_P} = \sqrt{P_x} \end{aligned}$$

54

$$\cos \Phi = \frac{\langle x[k], y[k] \rangle}{||x[k]|| \cdot ||y[k]||}$$

55

$$\varphi_{xy}[\kappa] := \langle x[k], y[k + \kappa] \rangle$$

56

$$\varphi_{xx}[\kappa] := \langle x[k], x[k + \kappa] \rangle$$

57

$$\varphi_{xy}^E[\kappa] = x^*[-\kappa] * y[\kappa] \text{ bzw. } \varphi_{xy}^P[\kappa] = \lim_{K \rightarrow \infty} \frac{1}{2K+1} x_K^*[-\kappa] * y_K[\kappa]$$

3 Diskrete Systeme

Inhalt...

58

$$y[k] = \mathcal{H}\{x[k]\}$$

59

entwickelt sich nun das Guthaben des Sparbuchs wie folgt:
 zu Beginn: $y[0] = x_0$
 nach 1 Jahr: $y[1] = x_0 + p \cdot x_0 = (1 + p) \cdot x_0$ nach 2 Jahren: $y[2] = (1 + p)x_0 + p \cdot (1 + p) \cdot x_0 = (1 + p) \cdot (1 + p) \cdot x_0 = (1 + p)^2 \cdot x_0$
 nach 3 Jahren: $y[3] = \dots = (1 + p)^3 \cdot x_0$
 nach i Jahren: $y[i] = (1 + p)^i \cdot x_0$
 D.h. das Ausgangssignal ist die kausale Exponentialfolge $y[k] = x_0 \cdot (1 + p)^k \cdot \epsilon[k]$

60

$$y[k + 1] = y[k] \cdot (1 + p) + x[k + 1] \quad (3.1)$$

Das heißt $y[k + 1]$ ergibt sich aus dem verzinnten Guthaben $y[k]$ des vorigen Jahres und zusätzlich den neuen Einzahlungen $x[k + 1]$.

61

$$\mathcal{H}\{c \cdot x_1[k] + d \cdot x_2[k]\} = c \cdot \mathcal{H}\{x_1[k]\} + d \cdot \mathcal{H}\{x_2[k]\}$$

62

$$y[0] = x[0] = c \cdot x_1[0] + d \cdot x_2[0]$$

63

$$\begin{aligned} y[k+1] &\stackrel{(3.1)}{=} y[k] \cdot (1+p) + x[k+1] \\ &\stackrel{(I.V.)}{=} (cy_1[k] + d \cdot y_2[k]) \cdot (1+p) + c \cdot x_1[k+1] + d \cdot x_2[k+1] \\ &= c \cdot (y_1[k] \cdot (1+p) + x_1[k+1]) + d \cdot (y_2[k] \cdot (1+p) + x_2[k+1]) \\ &\stackrel{(3.1)}{=} c \cdot y[k+1] + d \cdot y_2[k+1] \end{aligned}$$

64

$$\mathcal{H}\{x[k - k_0]\} = y[k - k_0]$$

65

$$\begin{aligned} z[k_0] &= x[k_0 - k_0] = x[k_0] = y[0] = y[k_0 - k_0] \\ \text{und } z[k] &= 0 = y[k - k_0] \text{ f\"ur } k < k_0 \end{aligned}$$

66

$$\begin{aligned} z[k+1] &\stackrel{(3.1)}{=} z[k] \cdot (1+p) + x[k+1 - k_0] \\ &\stackrel{(I.V.)}{=} y[k - k_0] \cdot (1+p) + x[k - k_0 + 1] \\ &\stackrel{(3.1)}{=} y[k - k_0 + 1] \end{aligned}$$

67

... wenn der Ausgabewert $y[k_0]$ zur Zeit k_0 nur von fr\"uheren Eingabewerten $x[k], k \leq k_0$ abh\"angig ist.

68

$$|x[k]| < C \forall k \Rightarrow |y[k]| < D \forall k$$

69

$$y[k] = x_0 \cdot (1 + p)^k \cdot \epsilon[k] \rightarrow \infty \text{ für } k \rightarrow \infty$$

70

..., wenn der Ausgang $y[k]$ zur Zeit k nur vom Eingang $x[k]$ zur Zeit k abhängt.

71

..., falls $y[k]$ nur von $x[\kappa]$ für $|\kappa - k| \leq L$ abhängt.

72

$$h[k] := \mathcal{H}\{\delta[k]\}$$

73

$$\begin{aligned} y[k] = \mathcal{H}\{x[k]\} &\stackrel{(2.6)}{=} \mathcal{H}\left\{\sum_{i=-\infty}^{\infty} x[i] \cdot \delta[k-i]\right\} \\ &= \sum_{i=-\infty}^{\infty} x[i] \mathcal{H}\{\delta[k-i]\} \\ &= \sum_{i=-\infty}^{\infty} x[i] \cdot h[k-i] \\ &= x[k] * h[k] \end{aligned}$$

74

$$y[k] = x[k] * h[k] \text{ für alle } x[k] \in \mathcal{S}$$

75

$$h[k] := \mathcal{H}\{\delta[k]\} = (1 + p)^k \epsilon[k]$$

76

$$\begin{aligned}
y[k] &= h[k] * x[k] = \sum_{i=-\infty}^{\infty} (1+p)^i \epsilon[i] \cdot x[k-i] \\
&= \sum_{i=0}^{\infty} (1+p)^i \cdot x[k-i]
\end{aligned}$$

77

$$y[k] = \sum_{i=0}^k (1+p)^i \cdot x[k-i]$$

78

$$\sum_{i=-\infty}^{\infty} |h[i]| < \infty$$

79

$$\begin{aligned}
|y[k]| &= |h[k] * x[k]| = \left| \sum_{i=-\infty}^{\infty} h[i] x[k-i] \right| \stackrel{DUG}{\leq} \sum_{i=-\infty}^{\infty} |h[i] \cdot x[k-i]| \\
&= \sum_{i=-\infty}^{\infty} |h[i]| \cdot |x[k-i]| < M \sum_{i=-\infty}^{\infty} |h[i]| < M \cdot C < \infty
\end{aligned}$$

80

$$x[k] := \text{sgn}(h[-k]) = \begin{cases} 1, & h[-k] > 0 \\ 0, & h[-k] = 0 \\ -1, & h[-k] < 0 \end{cases}$$

81

$$x[k] \cdot h[-k] = \text{sgn}(h[-k]) \cdot h[-k] = |h[-k]| \geq 0$$

82

$$|x[0]| = |(x \cdot h)[0]| = \left| \sum_{i=-\infty}^{\infty} x[i] \cdot h[-i] \right| = \sum_{i=-\infty}^{\infty} |h[-i]| = \sum_{i=-\infty}^{\infty} |h[i]| = \infty$$

83

$$\sum_{i=-\infty}^{\infty} |h[i]| = \sum_{i=-\infty}^{\infty} (1+p)^i \cdot \epsilon[i] = \sum_{i=0}^{\infty} (1+p)^i$$

84

$$|1+p| < 1 \quad \text{bzw. äquivalent für} \quad -2 < p < 0$$

85

$$h[k] = 0, \quad \forall k < 0$$

86

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^{\infty} h[i] \cdot x[k-i]$$

87

Lösung: Folgendes Blockschaltbild realisiert die Rekursion $y[k+1] = y[k] \cdot (1+p) + x[k+1]$ von (3.1) auf Seite 40: $y[k] = y[k-1] \cdot (1+p) + x[k]$
Hier tikz einfügen!

88

$$y[k] = h[k] * x[k] = \sum_{i=-\infty}^{\infty} h[i] \cdot x[k-i] = \sum_{i=0}^n \beta^i \cdot x[k-i]$$

89

$$y[k] = \frac{1}{\alpha_0} \cdot \left(\sum_{i=0}^N \beta_i \cdot x[k-i] - \sum_{i=0}^N \alpha_i \cdot y[k-i] \right)$$

90

$$\begin{aligned} \vec{v}[k+1] &= f v(\vec{v}[k], \vec{x}[k]) \\ \vec{y}[k] &= f y(\vec{v}[k], \vec{x}[k]) \end{aligned}$$

91

$$\begin{aligned} \text{I) } \vec{y}[k_0] &= f x(\vec{v}[k_0], \vec{x}[k_0]) & \text{II) } \vec{v}[k_0 + 1] &= f v(\vec{v}[k_0], \vec{x}[k_0]) \\ \text{III) } \vec{y}[k_0 + 1] &= f y(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1]) & \text{IV) } \vec{v}[k_0 + 2] &= f v(\vec{v}[k_0 + 1], \vec{x}[k_0 + 1]) \\ \text{V) } \vec{y}[k_0 + 2] &= \dots \end{aligned}$$

92

$$\vec{v}[k] : (x[k-1] \quad x[k-2] \quad x[k-3] \quad \dots \quad x[k-L])$$

93

$$\begin{aligned} \vec{v}[k+1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot v[k] + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot x[k] \\ y[k] &= (a_1 \quad a_2 \quad a_3 \quad a_4) \cdot \vec{v}[k] + (a_0) \cdot x[k] \end{aligned}$$

94

$$X(z) := \mathcal{Z}\{x[k]\} := \sum x[k] \cdot z^{-k}$$

95

$$a := \limsup_{h \rightarrow \infty} \sqrt[h]{|x[k]|}$$

$$b := \frac{1}{\limsup_{h \rightarrow \infty} \sqrt[h]{|x[-k]|}}$$

96

$$X^+(z) := \mathcal{Z}\{x[k]\} := \sum_{h=0}^{\infty} x[k] \cdot z^{-k}$$

97

$$a) \quad \mathcal{Z}\{\delta[k]\} = \sum_{k=-\infty}^{\infty} \delta[k] \cdot z^{-k} = z^{-0} = 1 \text{ für } z \in \mathbb{C}$$

$$b) \quad \mathcal{Z}\{\delta[k-i]\} = \sum_{k=-\infty}^{\infty} \delta[k-i] \cdot z^{-k} = z^{-i} \text{ für } 0 < |z| < \infty$$

$$c) \quad \mathcal{Z}\{\epsilon[k]\} = \sum_{k=-\infty}^{\infty} \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{k}\right)^k$$

geom. Reihe $\frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$ für $\left|\frac{1}{z}\right| < 1$ bzw. $|z| > 1$

$$d) \quad \mathcal{Z}\{a^k \cdot \epsilon[k]\} = \sum_{k=-\infty}^{\infty} a^k \cdot \epsilon[k] \cdot z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{z}{z-a} \text{ für } \left|\frac{a}{z}\right| < 1 \Leftrightarrow |z| > |a|$$

$$e) \quad \mathcal{Z}\{a^k \cdot \epsilon[-k-1]\} = \sum_{k=-\infty}^{\infty} -a^k \cdot \epsilon[-k-1] \cdot z^{-k} = - \sum_{k=-\infty}^{-1} a^k \cdot z^{-k}$$

$$= - \sum_{k=1}^{\infty} a^{-k} \cdot z^k = - \sum_{k=1}^{\infty} \left(\frac{a}{z}\right)^k = -\frac{z}{a} \cdot \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^k = \frac{z}{a} \cdot \frac{1}{1 - \frac{z}{a}}$$

$$= \frac{z}{z-a} \text{ für } \left|\frac{z}{a}\right| < 1 \Leftrightarrow |z| < |a|$$

98

$$\begin{aligned}
\mathcal{Z}\{\alpha x[k] + \beta y[k]\} &\stackrel{Def.}{=} \sum_{k=-\infty}^{\infty} (\alpha x[k] + \beta y[k]) \cdot z^{-k} \\
&= \alpha \left(\sum_{k=-\infty}^{\infty} x[k] z^{-k} \right) + \beta \left(\sum_{k=-\infty}^{\infty} y[k] z^{-k} \right) \\
&= \alpha X(z) + \beta Y(z)
\end{aligned}$$

99

$$\begin{aligned}
\mathcal{Z}\{x[k + k_0]\} &\stackrel{Def.}{=} \sum_{k=-\infty}^{\infty} x[k + k_0] z^{-k} \stackrel{k'=k+k_0}{(k=k'-k_0)} \sum_{k'=-\infty}^{\infty} \underbrace{x[k'] z^{-k'+k_0}}_{=z^{-k'} \cdot z^{k_0}} \\
&= z^{k_0} \cdot \underbrace{\sum_{k'=-\infty}^{\infty} x[k'] z^{-k'}}_{=X(z)} = z^{k_0} \cdot X(z)
\end{aligned}$$

100

$$\mathcal{Z}\{a^k \cdot x[k]\} \stackrel{Def.}{=} \sum_{k=-\infty}^{\infty} \underbrace{a^k}_{=(\frac{1}{\alpha})^{-k}} \cdot x[k] \cdot z^{-k} = \sum_{k=-\infty}^{\infty} x[k] \cdot \left(\frac{z}{\alpha}\right)^{-k} = X\left(\frac{z}{\alpha}\right)$$

101

$$\mathcal{Z}\{x[-k]\} = \sum_{k=-\infty}^{\infty} x[-k] z^{-k} \stackrel{k'=-k}{(k=-k')} \sum_{k'=-\infty}^{\infty} x[k'] z^{k'} = \sum_{k'=-\infty}^{\infty} x[k'] \cdot \left(\frac{1}{z}\right)^{-k'} = X\left(\frac{1}{z}\right)$$

102

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

103

$$\lim_{k \rightarrow \infty} x[k] = \lim_{z \rightarrow 1} (z - 1) \cdot X(z)$$

104

$$\mathcal{Z}\{\alpha^{k-1} \cdot \epsilon[k-1]\} = z^{-1} \cdot \mathcal{Z}\{\alpha^k \cdot \epsilon[k]\} = z^{-1} \cdot \frac{z}{z - \alpha} = \frac{1}{z - \alpha}$$

105

$$\mathcal{Z}\{A \cdot \alpha^{k-k_0} \cdot \epsilon[k-k_0]\} = A \cdot z^{-k_0} \cdot \mathcal{Z}\{\alpha^k \epsilon[k]\} = A \cdot z^{-k_0} \cdot \frac{z}{z - \alpha} = \frac{A \cdot z^{-(k_0-1)}}{z - \alpha}$$

106

$$\begin{aligned} \mathcal{Z}\{k \cdot \alpha^k \cdot \epsilon[k]\} &= -z \cdot \frac{d}{dz} \mathcal{Z}\{\alpha^k \epsilon[k]\} = -z \cdot \left(\frac{z}{z - \alpha} \right)' \\ &= -z \cdot \left(\frac{1 \cdot (z - \alpha) - z \cdot 1}{(z - \alpha)^2} \right) = \frac{\alpha \cdot z}{(z - \alpha)^2} \end{aligned}$$

107

$$Y(z) = H(z) \cdot X(z)$$

108

$$H(z) = \frac{Y(z)}{X(z)}$$

109

$$\begin{aligned} h[k] &= (1 + p)^k \cdot \epsilon[k] \\ H(z) &= \frac{z}{z - (1 + p)} \end{aligned}$$

110

$$y[k+1] = y[k] \cdot (1+p) + x[k+1]$$



$$Y(z) = Y(z) \cdot (1+p) + zX(z) \Leftrightarrow Y(z)(z - (1+p)) = zX(z)$$

111

$$H(z) := \frac{Y(z)}{X(z)} = \frac{z}{z - (1+p)}$$

112

$$x[k] = x_0 \cdot \epsilon[k]$$



$$X(z) = x_0 \cdot \frac{z}{z-1}$$

113

$$Y(z) = H(z) \cdot X(z) = \frac{z}{z - (a+p)} \cdot x_0 \cdot \frac{z}{z-q} = x_0 \cdot z^2 \cdot \frac{1}{(z - (a+p)) \cdot (z-1)}$$

114

$$\frac{1}{(z - (a+p)) \cdot (z-1)} = \frac{p^{-1}}{z - (a+p)} - \frac{p^{-1}}{z-1}$$


115

$$\frac{p^{-1}}{z - (a+p)} \bullet \text{---} \circ p^{-1}(a+p)^{K-1} \epsilon[K-1] \text{ und}$$

$$\frac{p^{-1}}{z-1} \bullet \text{---} \circ p^{-1} 1^{K-1} \epsilon[K-1] = p^{-1} \epsilon[K-1]$$

117

$$\sum_{i=0}^n \alpha_i y[k-i] = \sum_{i=N-M}^N \beta_i x[k-i] \quad (a_0 \neq 0, \beta_{N-M} \neq 0)$$



$$\sum_{i=0}^N \alpha_i \cdot z^{-i} \cdot Y(z) = \sum_{i=N-M}^N \beta_i \cdot z^{-i} \cdot X(z)$$

$$\Leftrightarrow Y(z) \cdot \sum_{i=0}^N \alpha_i \cdot z^{-i} = X(z) \cdot \sum_{i=N-M}^N \beta_i \cdot z^{-i}$$

118


$$\begin{aligned} H(z) &:= \frac{Y(z)}{X(z)} = \frac{\sum_{i=N-M}^N \beta_i \cdot z^{-i}}{\sum_{i=0}^N \alpha_i \cdot z^{-i}} \\ &= \frac{\sum_{i=N-M}^N \beta_i \cdot z^{N-i}}{\sum_{i=0}^N \alpha_i \cdot z^{N-i}} \\ &= \frac{\beta_{N-M} \cdot z^M + \beta_{N-M+1} \cdot z^{M-1} + \dots + \beta_N}{\alpha_0 \cdot z^N + \alpha_1 \cdot z^{N-1} + \dots + \alpha_N} \end{aligned}$$

119

$$y(k) = \frac{1}{\alpha_0} \left(\sum_{i=N-M}^N \beta_i x[k-i] - \sum_{i=1}^N \alpha_i y[k-i] \right)$$

120

$$h[k] = 3 \cdot \left(\frac{1}{5}\right)^k \cdot \epsilon[k] + 2 \cdot \left(\frac{1}{2}\right)^k \cdot \epsilon[k]$$



$$\begin{aligned} H(z) &= 3 \cdot \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}} = \frac{15z}{5z-1} + \frac{4z}{2z-1} \\ &= \frac{15z(2z-1) + 4z(5z-1)}{(5z-1)(2z-1)} = \frac{50z^2 - 19z}{10z^2 - 7z + 1} \end{aligned}$$

121

$$\alpha_0 = a_2 = 10, \alpha_1 = -7, \alpha_1 = 1 \text{ und } \beta_0 = 50, \beta_1 = -19, \beta_2 = 0$$

122

$$10y[k] - 7y[k-1] + 1y[k-2] = 50x[k] - 19x[k-1] \text{ bzw. äquivalent}$$

$$y[k] = \frac{1}{10} \cdot (50x[k] - 19x[k-1] + 7y[k-1] - y[k-2])$$

123

$$h[k] = 3 \cdot 5^{-(k+2)} \epsilon[k+2] + 2 \cdot 2^{-k} \epsilon[k]$$

$$\begin{aligned} & \circ \\ & | \\ & \bullet \\ H(z) &= 3 \cdot z^2 \frac{z}{z - \frac{1}{5}} + 2 \cdot \frac{z}{z - \frac{1}{2}} \\ &= \frac{15z^3}{5z-1} + \frac{4z}{2z-1} = \frac{30z^4 - 15z^3 + 20z^2 - 4z}{10z^2 - 7z + 1} \\ &(\Rightarrow M=4, N=2) \end{aligned}$$

124

$$\alpha_0 = a_{2-0} = 10, \alpha_1 = -7, \alpha_2 = 1 \text{ und} \\ \beta_{-2} = b_4 = 30, \beta_{-1} = 15, \beta_0 = 20, \beta_1 = -4$$

125

$$10y[k] - 7y[k-1] + y[k-2] = 30x[k+2] - 15[k+1] + 20x[k] - 4x[k-1]$$

$$y[k] = \frac{1}{10} (30x[k+2] - 15[k+1] + 20x[k] - 4x[k-1] + 7y[k-1] - y[k-2])$$

126

$$y[k-2] = \frac{1}{10} (30x[k+2] - 15[k-1] + 20x[k-2] - 4x[k-3] + 7y[k-3] - y[k-4])$$

127

$$x[k] = \mathcal{Z}^{-1} \{X(z)\} = \mathcal{Z}^1 \left\{ \sum_{k=-\infty}^{\infty} x[k]z^{-k} \right\}$$

128

$$Y(z) = 1 \cdot z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{4}z^{-4} + \dots + \frac{(-1)^{k+1}}{k}z^{-k} + \dots$$

129

$$y[k] = \begin{cases} 0 & , k \leq 0 \\ (-1)^{k+1} \cdot \frac{1}{k} & , k > 0 \end{cases}$$

130

$$Y(z) \stackrel{!}{=} \frac{A_1}{z - \lambda_1} + \frac{A_2}{z - \lambda_2} + \dots + \frac{A_n}{z - \lambda_n}$$

131

$$Y(z) \stackrel{!}{=} \frac{A_1(\lambda_2) \dots (z - \lambda_N) + \dots + A_N(z - \lambda_1) \dots (z - \lambda_{N-1})}{HN}$$

132

$$\begin{aligned} Y(z) &= \frac{1}{(z - (1 + P)) \cdot (z - 1)} = \frac{A(z - 1) + B(z - (1 + P))}{HN} \\ &= \frac{(A + B) \cdot z - A - B(a + P)}{HN} \end{aligned}$$

133

$$\begin{array}{lll} -A & -(a+p)B = 1 & (1) \\ A & +B = 0 & (2) \end{array}$$

134

$$(-1 - p + 1) \cdot B = 1 \text{ bzw. } B = -\frac{1}{p}$$

135

$$A = -B = \frac{1}{p}$$

136

$$Y(z) = \frac{p^{-1}}{z - (a + p)} - \frac{p^{-1}}{z - 1}$$

$$\begin{array}{c} \bullet \\ | \\ \circ \end{array}$$

$$y[k] = p^{-1} \cdot ((1 + p)^{k-1} - 1) \cdot \epsilon[k - 1]$$

137

$$\begin{aligned} Y(z) &= \frac{A}{z - 5} + \frac{B}{z - 3} + \frac{C}{(z - 3)^2} = \frac{A(z - 3)^2 + B(z - 5)(z - 3)C(z - 5)}{HN} \\ &= \frac{A(z^2 - 6z + 9) + B(z^2 - 8z + 15) + C(z - 5)}{HN} \\ &= \frac{z^2(A + B) + z(-6A - 8B + C) + 9A + 15B - 5C}{HN} \end{aligned}$$

138

$A +$	B	$= 2(1)$
$- 6A$	$- 8B + 1C$	$= -9(2)$
$9A$	$+ 15B - 5C$	$= 3(3)$
$5 \cdot (2) + (3) : - 21A$	$- 25B$	$= -42(4)$
$21 \cdot (1 + 4) :$	$- 4B$	$= 0(5)$

139

$$Y(z) = \frac{2}{z-5} + \frac{3}{(z-3)^2}$$



$$y[k] = 2 \cdot 5^{k-1} \epsilon[k-1] + (k-1)3^{k-1} \epsilon[k-1]$$

140

$$Y(z) = s(z) + \frac{r(z)}{q(z)}$$

141

$$s(z) = s_0 + s_1 z + \dots + s_{M-N} Z^{M-N}$$



$$s[k] = s_0 \delta[k] + s_1 \delta[k+1] + \dots + s_{M-N} \delta[k+M-N]$$

142

$$Y(z) = 3z^2 - 2z + 1 \frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$

143

$$Y(z) = 3z^2 - 2z + 1 \frac{2z^2 - 9z + 3}{(z-5)(z-3)^2}$$



$$y[k] = 3\delta[k+2] - 2\delta[k+1] + \delta[k] + 2 \cdot 5^{k-1} \epsilon[k-1] + (k-1)3^{k-1} \epsilon[k-1]$$

144

$$Y(z) = \sum_{i=0}^{M-N} s_i z^i + \sum_{i=1}^Q \sum_{v=1}^{n_i} \frac{A_{i,v}}{(z - \lambda_i)^v}$$



$$y[k] = \sum_{i=0}^{M-N} s_i \delta[k+i] + \sum_{i=1}^Q \sum_{v=1}^{n_i} A_{i,v} \cdot \binom{k-1}{v-1} \lambda_i^{k-v} \epsilon[k-1]$$

147

$$A = \frac{1}{0!} \left(\frac{d}{dz} \right)^0 [(z-1) \cdot \frac{z}{z-1} (z+1)]|_{z=1} = \frac{z}{z+1}|_{z=1} = \frac{1}{2}$$

$$B = \frac{1}{0!} \left(\frac{d}{dz} \right)^0 [(z+1) \cdot \frac{z}{z-1} (z+1)]|_{z=-1} = \frac{z}{z-1}|_{z=-1} = \frac{1}{2}$$

4 Kontinuierliche Signale

Inhalt...

171

$$\int_{-\infty}^t \epsilon(\tau) d\tau = \begin{cases} 0 & , t \leq 0 \\ \int_0^1 1 d\tau = [\tau]_0^t = t & , t > 0 \end{cases} = t \cdot \epsilon(t) =: \text{ramp}(t)$$

172

$$\int_{-\infty}^t \delta(\tau) d\tau = \epsilon(t)$$

173

$$\delta(t) := \frac{d}{dt} \epsilon(t) = \lim_{T \rightarrow 0} \frac{1}{T} \text{rect}\left(\frac{t}{T}\right) = \lim_{T \rightarrow 0} \frac{1}{T} \text{tri}\left(\frac{t}{T}\right) = \lim_{T \rightarrow 0} \frac{2}{T} \text{si}\left(s\pi \frac{t}{T}\right)$$

174

- I) $x(\tau) \cdot \delta(\tau - t) = x(t) \cdot \delta(\tau - t)$
 II) $\int_{-\infty}^{\infty} \delta(\tau - t) d\tau = 1$
 III) $x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$ für alle $t \in \mathbb{R}$

175

$$\int_{-\infty}^{\infty} \delta(\tau - t) d\tau = \int_{-\infty}^{\infty} \delta(\lambda) d\lambda = \epsilon(\infty) = 1$$

176

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(\tau)\delta(\tau-t)d\tau \stackrel{(I)}{=} \int_{-\infty}^{\infty} x(t)\delta(\tau-t)d\tau \\ &= x(t) \int_{-\infty}^{\infty} \delta(\tau-t)d\tau \stackrel{(II)}{=} x(t) \cdot 1 = x(t)\end{aligned}$$

177

$$(x * y)(t) := \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

a)

$$\begin{aligned}\epsilon(t) * \epsilon(t) &= \int_{-\infty}^{\infty} \epsilon(\tau)\epsilon(t-\tau)d\tau = \int_0^{\infty} \epsilon(t-\tau)d\tau = \int_0^t 1 \cdot d\tau \stackrel{\lambda:=t-\tau}{=} \\ &\quad - \int_t^{-\infty} \epsilon(\lambda)d\lambda = \int_{-\infty}^0 \epsilon(\lambda)d\lambda \stackrel{(4.2)}{=} t\epsilon(t) = \text{ramp}(t)\end{aligned}$$

b)

178

$$\begin{aligned}\text{rect}(t) * \epsilon(t) &= \int_{-\infty}^{\infty} \text{rect}(\tau)\epsilon(t-\tau)d\tau \\ &= \int_{-0,5}^{0,5} \epsilon(t-\tau)d\tau \stackrel{\lambda:=t-\tau}{=} \int_{t-(-0,5)}^{t-0,5} \epsilon(\lambda)d\lambda = - \int_{t-0,5}^{t+0,5} \epsilon(\lambda)d\lambda \\ &\stackrel{(4.2)}{=} [\lambda\epsilon(\lambda)]_{t-0,5}^{t+0,5} = \text{ramp}(t+0,5) - \text{ramp}(t-0,5) \\ &= \begin{cases} 0 & , t \leq -0,5 \\ t+0,5 & , -0,5 \leq t \leq 0,5 \\ 1 & , t \geq 0,5 \end{cases} =: \text{sramp}\end{aligned}$$

c)

179

$$\begin{aligned}\text{rect}(t) * \epsilon(t) &= (\epsilon(t+0,5) - \epsilon(t-0,5)) * \epsilon(t) \\ &\stackrel{(III)}{=} \epsilon(t+0,5) * \epsilon(t) - \epsilon(t-0,5) * \epsilon(t) \\ &\stackrel{(VI.a)}{=} \text{ramp}(t+0,5) - \text{ramp}(t-0,5) = \text{sramp}(t)\end{aligned}$$

180

$$\cos \Phi = \frac{\langle x(t), y(t) \rangle}{\|x(t)\| \cdot \|y(t)\|}$$

181

$$E_{\delta(t-\tau)} = \|\delta(t-\tau)\|^2 = \int_{-\infty}^{\infty} \delta^2(t-\tau) dt = \lim_{T \rightarrow \infty} \int_{\tau-\frac{T}{2}}^{\tau+\frac{T}{2}} \frac{a}{T^2} dt = \lim_{T \rightarrow \infty} \frac{a}{T} = \infty$$

$$\cos \Phi = \frac{\langle \delta(t-\tau_1), \delta(t-\tau_2) \rangle}{\|\delta(t-\tau_1)\| \cdot \|\delta(t-\tau_2)\|} = \frac{\int_{-\infty}^{\infty} \delta(t-\tau_1) \cdot \delta(t-\tau_2) dt}{\infty \cdot \infty} = \frac{\int_{-\infty}^{\infty} 0 dt}{\infty} = 0$$

182

$$\begin{aligned} \|\text{rect}(t-\tau)\|^2 &= \int_{-\infty}^{\infty} \text{rect}^2(t-\tau) dt \stackrel{\lambda=t-\tau}{=} \int_{-\infty}^{\infty} \text{rect}(\lambda) d\lambda \stackrel{(Def.rect)}{=} 1 \\ \cos \Phi &= \frac{\langle \text{rect}(t-\tau_1), \text{rect}(t-\tau_2) \rangle}{\|\text{rect}(t-\tau_1)\| \cdot \|(t-\tau_2)\|} \\ &= \frac{\int_{-\infty}^{\infty} \text{rect}(\lambda) \cdot \text{rect}(\lambda - ((\tau_2 - \tau_1))) d\lambda}{1 \cdot 1} \\ &\stackrel{(symmrect)}{=} \int_{-\infty}^{\infty} \text{rect}(\lambda) \text{rect}((\tau_2 - \tau_1) - \lambda) d\lambda \\ &\stackrel{(Def.Faltung)}{=} (\text{rect} * \text{rect})(\tau_2 - \tau_1) \\ &\stackrel{S91, Bsp.d}{=} \text{tri}(\tau_2 - \tau_1) \\ \implies \Phi &= \arccos \text{tri}(\tau_2 - \tau_1) \end{aligned}$$

183

$$\begin{aligned} \langle \sin(n\omega_0 t), \sin(m\omega_0 t) \rangle_T &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2j} (e^{jn\omega_0 t} - e^{-jn\omega_0 t}) \cdot \frac{1}{2j} (e^{jm\omega_0 t} - e^{-jm\omega_0 t}) dt \\ &= -\frac{1}{4} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(n+m)\omega_0 t} - e^{j(m-n)\omega_0 t} - e^{j(n-m)\omega_0 t} + e^{-j(n+m)\omega_0 t} dt \\ &\stackrel{(4.12)}{=} -\frac{T}{4} (\delta[n+m] - \delta[m-n] - \delta[n-m] - \delta[-(n+m)]) \\ &= \frac{T}{4} \cdot (2\delta[n-m] - 2\delta[n+m]) = \frac{T}{2} \delta[n-m] = \begin{cases} 0 & , n \neq m \\ \frac{T}{2} & , n = m \neq 0 \end{cases} \end{aligned}$$