STA511 Homework #3

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1. $f(x)$ was given as the density of interest and $c_\theta g_\theta(x)$ was given as a possible candidate for dominating curve, where c_θ depends only on θ and $\theta > 0$ is a design parameter.

   (a) $c_\theta g_\theta(x)$ is a piecewise function, where $f(x) \le c_\theta g_\theta(x)$ is true for both of the sub-functions, independent from θ. This can be visualized by plotting both parts of the $c_\theta g_\theta(x)$ with $f(x)$ function.

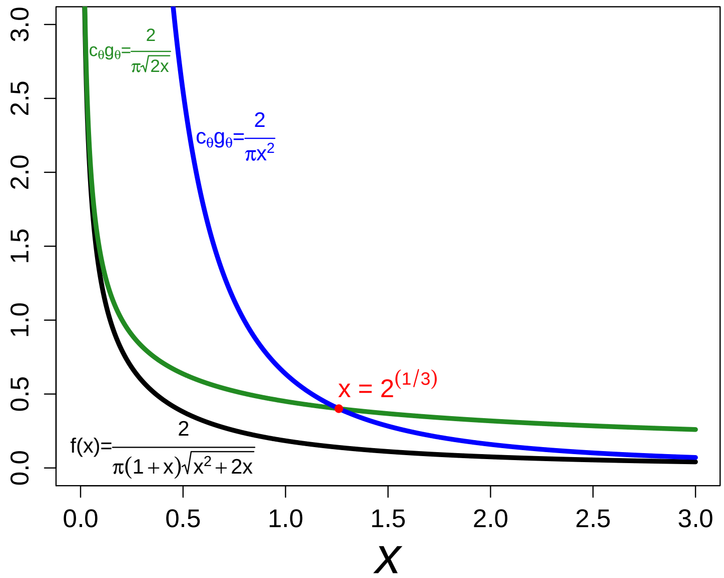

Figure 1: Density function of $f(x)$ and both sub-functions of $c_\theta g_\theta(x)$. It can be seen that for both of the sub-functions, $f(x)$ remains less or equal. Equations that are associated with the respective curve are plotted with same colors. Additionally, the $c_\theta g_\theta(x)$ functions intercept at $x = 2^{1/3}$.

   (b) As shown in Figure 1, distance of sub-function $c_\theta g_\theta(x) = \frac{2}{\pi\sqrt{2x}}$ is much smaller to $f(x)$ than $c_\theta g_\theta(x) = \frac{2}{\pi x^2}$ for $0 < x \le 2^{1/3}$, where for the interval $2^{1/3} \le x < \infty$ $c_\theta g_\theta(x) = \frac{2}{\pi x^2}$ is closer to $f(x)$. Therefore, it can already be seen that the intercept $x = 2^{1/3}$ would be the optimal θ, as it allows the minimum distance from $f(x)$ for $c_\theta g_\theta(x)$. Hence, c_θ would be minimal (or optimal) for $\theta = 2^{1/3}$. Optimal value of c_θ can be found by obtaining the c_θ function through integration of the $c_\theta g_\theta(x)$ function, where $g_\theta(x)$ is defined as a PDF and therefore integration of g_θ would be equal to 1. $c_\theta g_\theta(x)$ integration can be computed as follows:

$$\int_0^\theta \frac{2}{\pi\sqrt{2x}}dx + \int_\theta^\infty \frac{2}{\pi x^2}dx$$

   This integration would give c_θ as a function of θ:

$$c_\theta = \frac{2}{\pi}\left(\sqrt{2\theta} + \frac{1}{\theta}\right)$$

c_θ function can be plotted against possible values of θ and it can be seen that the minimal point of the c_θ function is achieved for $\theta = 2^{1/3}$.

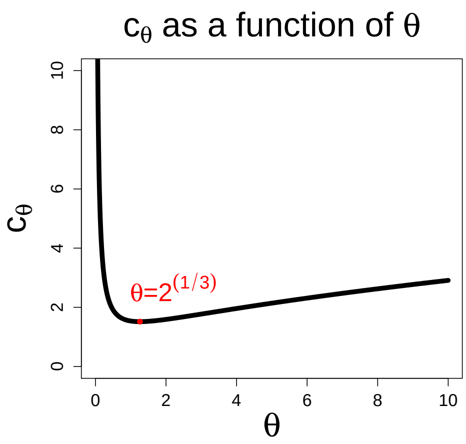

Figure 2: c_θ was plotted as a function of θ. It can be seen that c_θ is minimum for $\theta = 2^{1/3}$

c_θ function is undifferentiable; however, the minimal point can be verified by the `min()` function in `R`.

```
c_theta <- function(theta){(2/pi)*(sqrt(2*theta)+(1/theta))}
tseq <- seq(0, 10, length=10000)[-1]

min(c_theta(tseq))
> 1.515856

a=2^(1/3)
c_theta(a)
> 1.515856
```

(c) Generalized rejection to obtain 500 observations from $f(x)$ was performed. For the results a histogram of the accepted observations with the PDF $f(x)$ superimposed was generated.

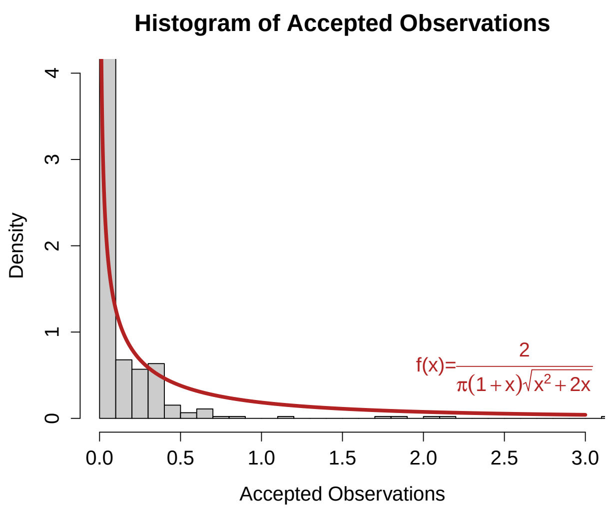

Figure 3: Histogram of the accepted observations are presented. Red curve on the histogram represents the PDF $f(x)$.

2. Laplace random variables from a given PDF $f(x)$ was generated using the given Cauchy distribution $g(x)$ as the dominating density.

   (a) The optimal μ cannot be obtained from $g(x)$ mathematically, because the roots of $\frac{d}{dx}g(x)$ consist of imaginary numbers, which cannot be computed by `R`. However, for $\mu = 3$, $c = sup\frac{f(x)}{g(x)}$ was computed as `4.712`. This can also be verified by plotting $f(x)$, $g(x)$ and $ag(x)$, where $a = 4.712$ as shown in Figure 4.

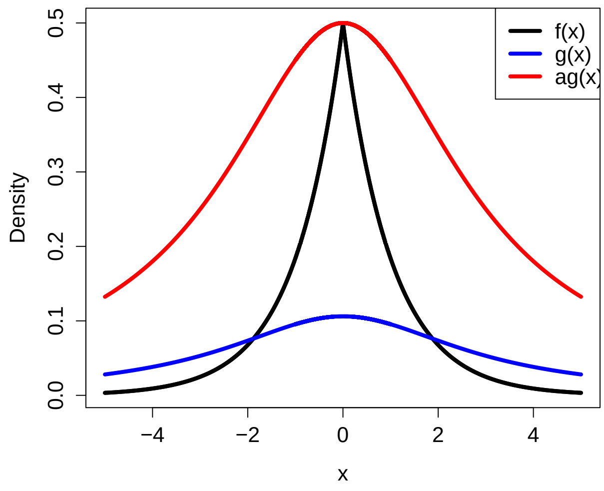

Figure 4: Densities of $f(x)$, $g(x)$ and $ag(x)$, where $a = sup\frac{f(x)}{g(x)}$

   (b) Random variables from the given Cauchy distribution, $g(x)$ can be generated via inversion method, where CDF of $g(x)$ was generated by integrating $f(x)$ and inverse of CDF function was taken and random variables from $U(0,1)$ was used for random variable generation. CDF of $g(x)$ is obtained by integrating $g(x)$:

$$\int_{-\infty}^{x} \frac{3}{\pi(9+x^2)} dx$$

CDF was determined as:

$$G(x) = \frac{\arctan\frac{x}{3}}{\pi}$$

Inverse CDF was determined as:

$$G(x)^{-1} = 3\tan\pi x$$

R code is given as:

```
inv_g_cdf <- function(x){3*tan(pi*x)}
set.seed(84)
uni <- runif(1000, 0,1)
xc <- inv_g_cdf(uni)
```

In order to validate the method, a histogram of generated Cauchy random variables was produced and PDF Cauchy, $g(x)$ was superimposed (Figure 5).

   (c) 1000 observations were generated from the Laplace distribution (θ=1) using a generalized rejection algorithm. A histogram of the accepted observations was constructed with the superimposed *pdf* of the Laplace distribution (Figure 6).

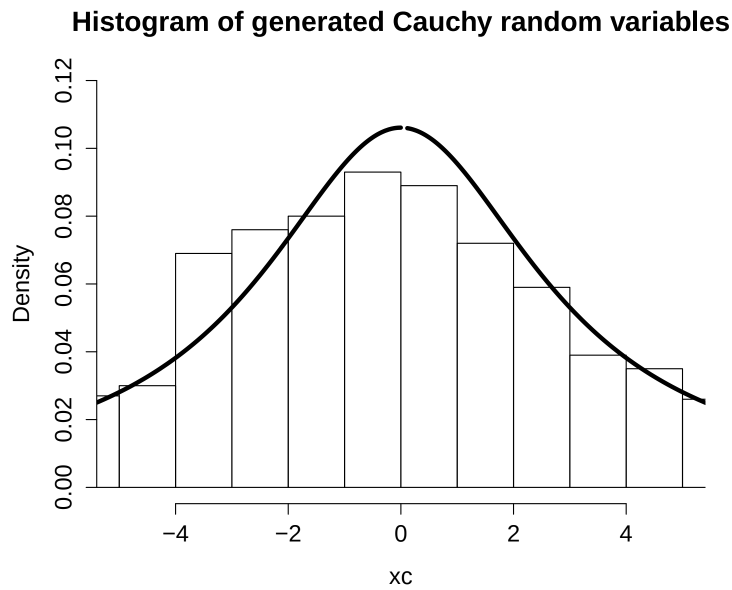

Figure 5: Histogram of generated Cauchy random variables with PDF of Cauchy, $g(x)$.

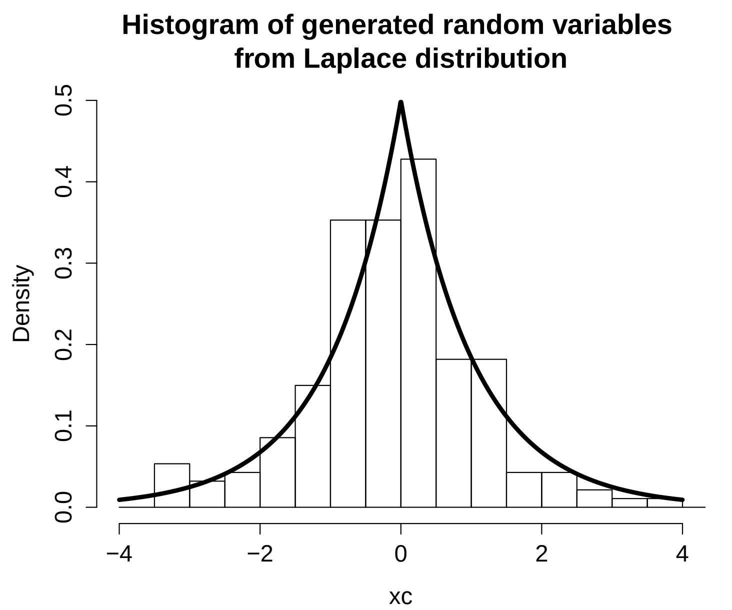

Figure 6: Histogram of generated Laplace random variables using generalized rejection algorithm with *pdf* of Cauchy, $g(x)$ is assumed for the dominating density.

3. The beta distribution with parameters $\alpha > 0, \beta > 0$ has a continuous density for $0 < x < 1$.

(a) Plots of the target density when $(\alpha, \beta) = (0.5, 0.5), (1, 0.5), (0.5, 1), (1, 1), (2, 2), (5, 10)$ were generated in R.

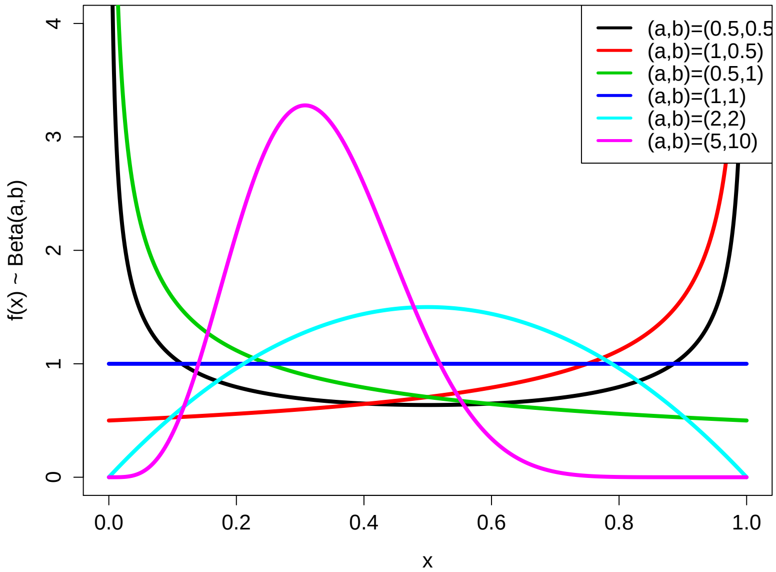

Figure 7: Beta functions with different α and β parameters.

(b) In this and the following questions, α and β parameters were accepted as $(\alpha > 1, \beta < 1)$. To show that $f(x)$ is increasing for $x \le (\alpha - 1)/(\alpha + \beta - 2)$ and decreasing for $x \ge (\alpha - 1)/(\alpha + \beta - 2)$ $Beta(\alpha, \beta)$ distribution with different parameters were plotted with $f((\alpha - 1)/(\alpha + \beta - 2))$ points (Figure 8). It can be seen that the $f((\alpha - 1)/(\alpha + \beta - 2))$ is always giving the maximum density of the associated Beta function, meaning that the slope at $f((\alpha - 1)/(\alpha + \beta - 2))$ is equal to 0, and the part of the function that is before this point increases and the other part of the function decreases upon passing this point.

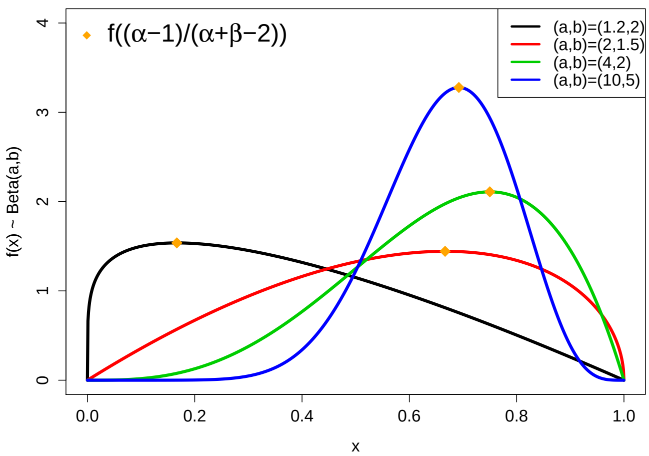

Figure 8: Beta functions with different α and β parameters, where the $f((\alpha - 1)/(\alpha + \beta - 2))$ points are plotted on the graph.

(c) Depending on the parameter setting (i.e. whether α and β are larger than 0 or α is an integer) different rejection algorithm approaches can be used. For a Beta density where x is defined for a certain interval (i.e. $0 \leq x \geq 1$) and where both α and β parameters are larger than 1, Beta random variables can be generated by using Uniform distribution, where $\alpha + \beta - 1$ number random variables are generated from U(0,1) and the α th order statistic is returned as a Beta random variable. For a Beta distribution where α and β parameters are non-integers, gamma distribution can be used to generate random variables from the Beta distribution using the rejection method. Additionally for Beta distributions where $\alpha = \beta$, a normal distribution can be used as a dominating density, where $c_{optimal}$ can be simply computed by

$$c_{optimal} = \frac{f_{\alpha\beta}((\alpha-1)/(\alpha+\beta-2))}{g_{\alpha\beta}((\alpha-1)/(\alpha+\beta-2)}$$

μ of the normal distribution can be estimated by:

$$\mu = (\alpha_{f(x)} - 1)/(\alpha_{f(x)} + \beta_{f(x)} - 2)$$

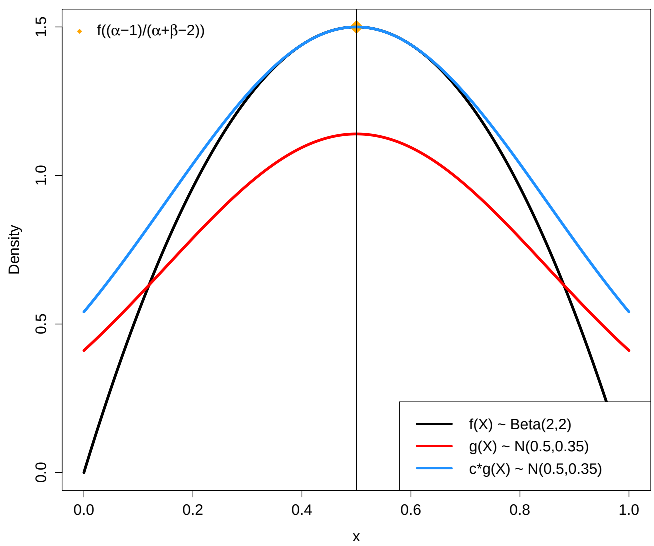

Figure 9: Beta function with $\alpha = \beta$ parameters and $g(x)$ Normal where $\mu = 0.5$, $\sigma = 0.35$, and $cg(x)$ is determined as dominating curve.

(d) Accept/rejection algorithm in R for the parameter (2,2) was implemented for different parameter setting specific rejection schemes and a generalized rejection where N(0.5, 0.35) was selected as a dominating curve. Histograms were generated for generated variables and $f(x)$ was superimposed (Figure 10). It can be seen that rejection with Uniform and Gamma distribution schemes provide a much better sampling than other methods.

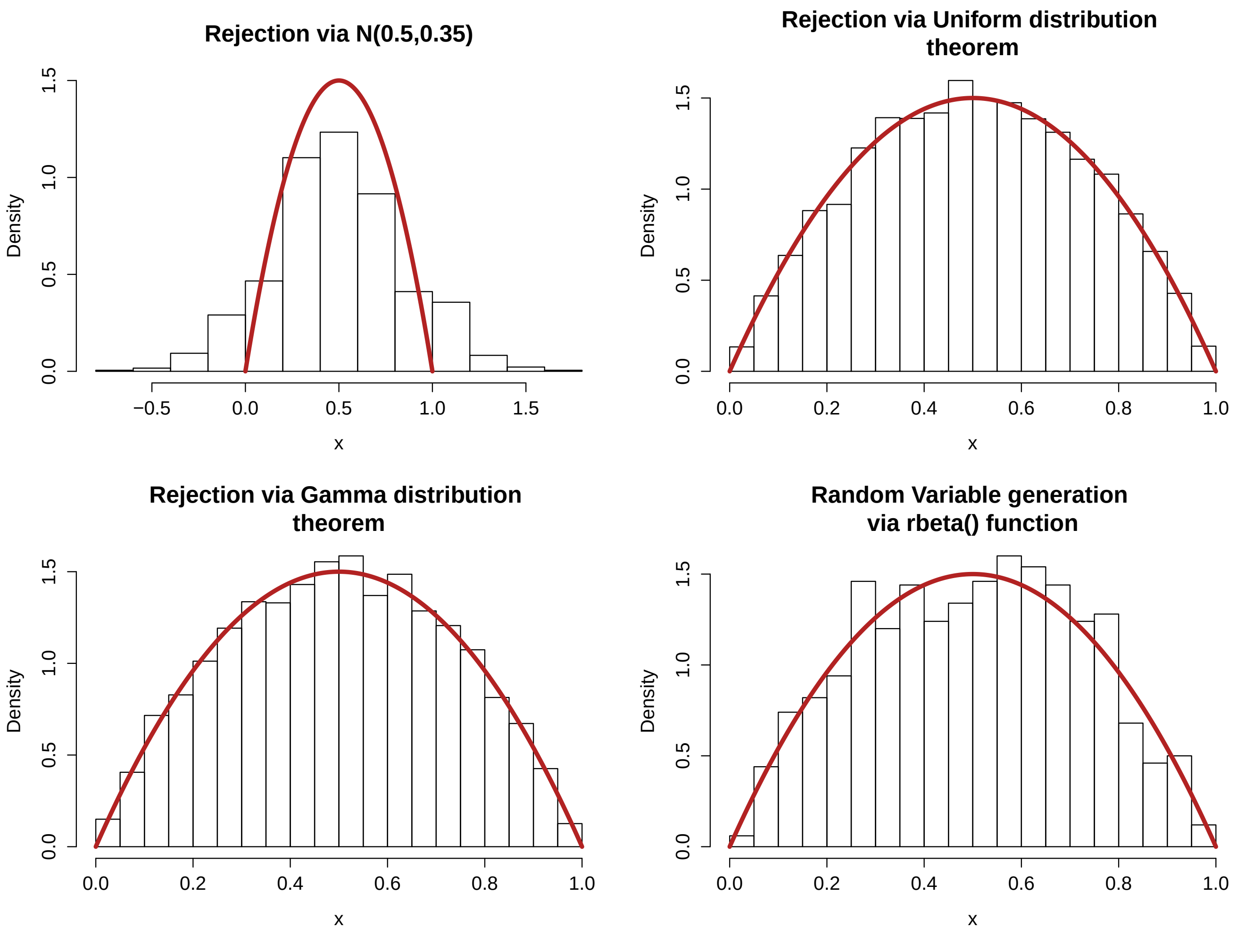

Figure 10: Histograms of generated variables with different rejection schemes, where $f(x)$ was superimposed.