Numerical Methods of Differential Equations.

1. Calculating First Derivatives.

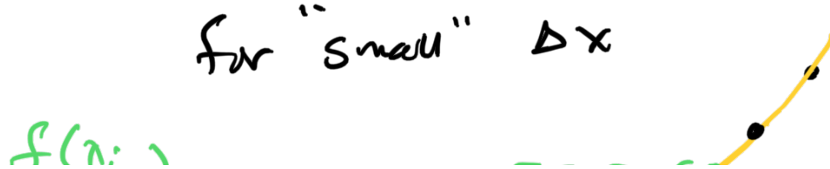
$$f'(x) \equiv \lim_{\Delta h \to 0} \frac{f(x+\Delta h) - f(x)}{\Delta h}$$

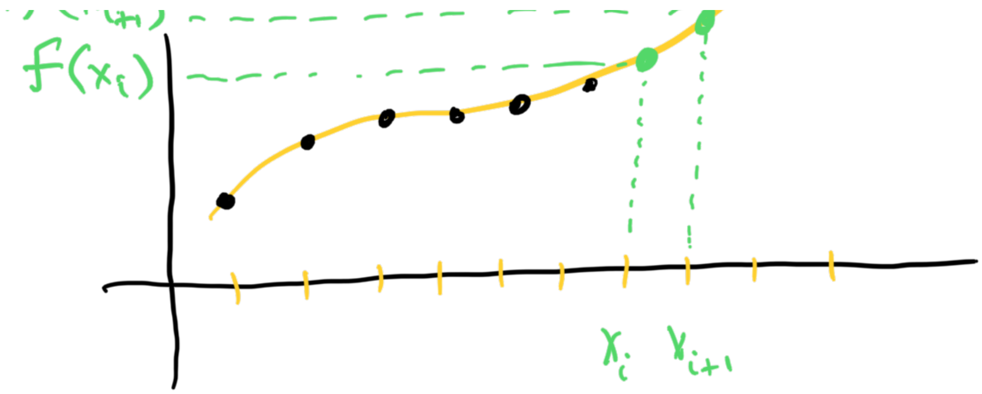
(i.e. the fundametal theorem of calculus)

(i) Forward Difference Method

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for "small" Δx





$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

Notation:

$f(x_i) \to f_i$

$f(x_{i+1}) \to f_{i+1}$

$f'(x_i) \to f'_i$

$$\boxed{f'_i = \frac{f_{i+1} - f_i}{\Delta x}}$$

Note:

... discrete array

If the x-axis is a
which runs from index ϕ to index N, then f' will only be defined from index ϕ to index $N-1$! (Since we need f_{i+1} to calculate it)

(ii) Backward Difference.

$$f_i' = \frac{f_i - f_{i-1}}{\Delta x}$$

→ Here, f' will only be defined for index 1 to index I

Example 1.

A traveling wave (i.e. pulse) in one dimension.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

→ u is the wavefunction

→ x is position

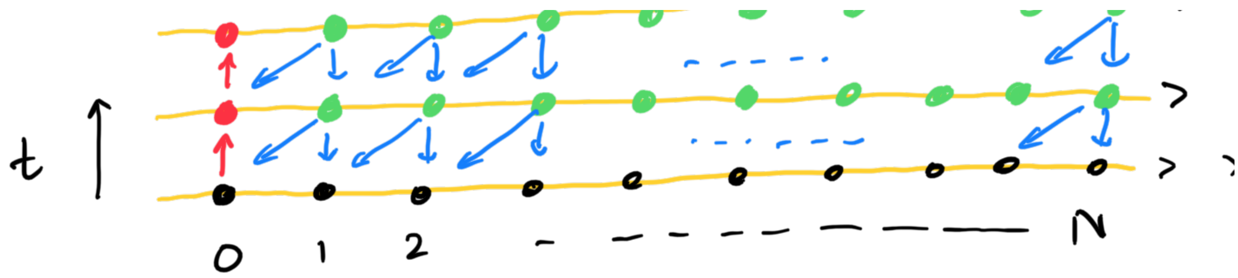
→ t is time

Method: Use the <u>forward</u> difference for time derivative, and the <u>backward</u> difference for the spatial derivative.

$$\frac{u_i^{n+1} - u_i^{n}}{\Delta t} + c\left(\frac{u_i^{n} - u_{i-1}^{n}}{\Delta x}\right) = ($$

$$\downarrow$$

$$\boxed{u_i^{n+1} = u_i^{n} - \frac{c\,\Delta t}{\Delta x}\left(u_i^{n} - u_{i-}\right.}$$



(i.e. f_0 never changes)

$\boxed{\text{Important}}$ → need

$$\boxed{\frac{c\,\Delta t}{\Delta x} \le 1}$$

$\therefore$ there is a trade-off ...
cannot make Δt and Δx arbitrarily small.

Let $x = [0, 20\,m]$

$c = 25\ m/s$

for a pulse traveling to the right, it should take $t = \frac{20\,m}{25\,m/s} \sim 0.8\,s$

to traverse the distance.

Let $t = [0, 0.625\,s]$

Divide the x-axis into 400 slices. $\therefore i = 0, \ldots, 400$

$N_x = 401$

Let $\frac{c\,\Delta t}{\Delta x} = 1$ $\quad \Delta t = \frac{\Delta x}{c}$

$$\Delta t = \frac{20\,m/400}{25\,m/s}$$

$$\boxed{\Delta t = 0.002\,s}$$

Call this $\boxed{\Delta t_{max}}$

→ we must choose a value of Δt

that is smaller than this.

→ let's choose $\Delta t = 0.00025\,s$

then $N_t = \frac{0.625\,s}{0.00025\,s} + 1 = \underline{\underline{2500}}$ $+1 = 2501$

So, we will discretize in the following way.

$x = [0, 20\,m]$ $N_x = 401$

$t = [0, 0.625\,s]$ $N_t = 2500$

$c = 25\,m/s$

with $\boxed{\frac{c\,\Delta t}{\Delta x} = 0.125}$

We see that is some situations, this

[illegible] work!

does not work.

Let's try $\frac{c\,\Delta t}{\Delta t} = 1$

let $t = [0, 0.620\,s]$

$\frac{c\,\Delta t}{\Delta x} < 1$ $\Delta t = \frac{\Delta x}{c} = \frac{20m/400}{25m/s}$

$= 0.002s$

$N_t = \frac{0.620\,s}{0.002\,s} + 1 = 310+1$

$= \boxed{311}$

Now, it all works perfectly, and we simulate the traveling wave well.

Lessons Learned:

→ the translation from continuous to discrete functions, necessary for computational calculus, is

filled with pits of vipers and
dark monsters and nastiness!

What weapons can we manufacture to help us fight the monsters?

The reason that in the method above that we unlocked the monster is rooted in how we calculated the derivatives!

Math to the rescue!

Taylor series expansion:

$$f(x) = f(x_0) + f'(x)(x-x_0) + \frac{f''(x)(x-x_0)^2}{2} + \frac{f'''(x)(x-x_0)^3}{6} + \dots$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2$$

$$+ \frac{f'''(x)}{6} h^3 + \cdots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2} h^2$$

$$- \frac{f'''(x)}{6} h^3 + \cdots$$

[Before : $f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$

or $f'(x) = \frac{f(x) - f(x+h)}{h} + O(h)$

i.e. ignore terms of $O(h^2)$ (in f(x

and higher]

Now :

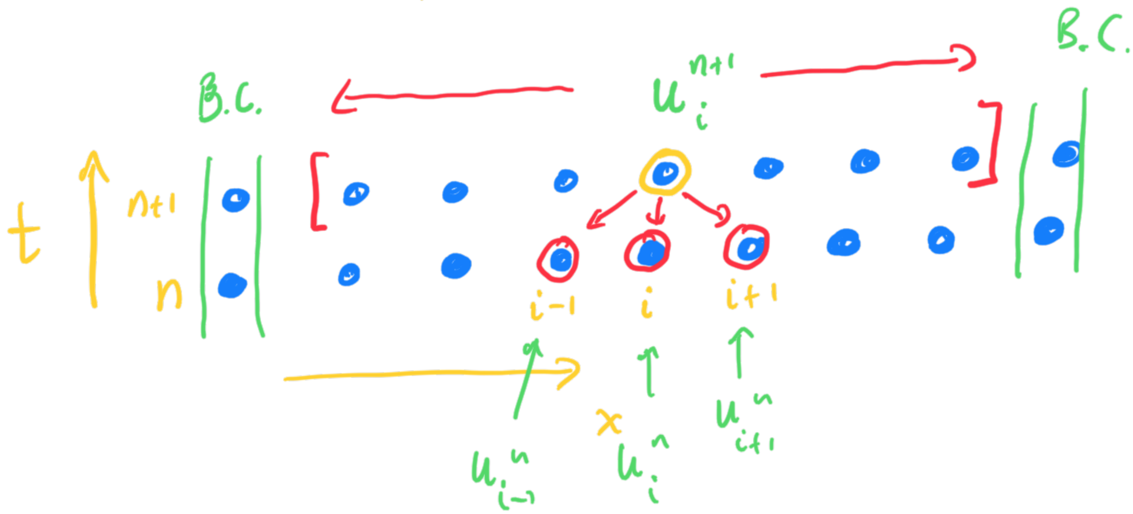
$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

write $\frac{\partial u}{\partial x} = \frac{u^n_{i+1} - u^n_{i-1}}{2\Delta x}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

$$\boxed{u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)}$$

Centered Difference Solution to the Advection Equation.



Lax Method :

→ Replace u_i^n with average of u_{i+1}^n and u_{i-1}^n

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)}$$

Lax – Wendroff Method :

– Add a second order term:

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \underbrace{\frac{c^2\Delta t^2}{2\Delta x^2}\left(u_{i+1}^n + u_{i-1}^n - 2u_i^n\right)}_{\text{related to the 2nd Derivative.}}$$

$$f(x) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2}\Delta x^2$$

$$f^+ = f + f'\Delta x + \frac{f''}{2}\Delta x^2$$

$$f^- = f - f'\Delta x + \frac{f''}{2}\Delta x^2$$

$$f^+ + f^- - 2f = f + f'\Delta x + \frac{f''}{2}\Delta x^2$$

$$+ f - f'\Delta x + \frac{f''}{2}\Delta x^2$$

$$- 2f$$

$$= f''\Delta x^2$$

$$\therefore \boxed{f'' = \frac{f^+ + f^- - 2f}{\Delta x^2}}$$