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Answer 1

a) Base Case: For $n = 1$, we have $2^{3(1)} - 3^1 = 8 - 3 = 5$, which is divisible by 5.

Inductive Step: Assume that $2^{3k} - 3^k$ is divisible by 5 for some arbitrary positive integer k. Now, we want to show that $2^{3(k+1)} - 3^{k+1}$ is also divisible by 5.

$$\begin{aligned} 2^{3(k+1)} - 3^{k+1} &= 2^{3k+3} - 3 \cdot 3^k \\ &= 2^3 \cdot 2^{3k} - 3 \cdot 3^k \\ &= 8 \cdot (2^{3k} - 3^k) - 3 \cdot 3^k \end{aligned}$$

By our inductive assumption, $2^{3k} - 3^k$ is divisible by 5. Let $2^{3k} - 3^k = 5m$ for some integer m.

$$\begin{aligned} &= 8 \cdot (5m) - 3 \cdot 3^k \\ &= 40m - 3 \cdot 3^k \end{aligned}$$

Now, notice that $40m$ is clearly divisible by 5, and $3 \cdot 3^k$ is also divisible by 5. Therefore, the entire expression is divisible by 5.

Thus, by mathematical induction, $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$.

b) Base Case: For $n = 2$, we have $4^2 - 7(2) - 1 = 16 - 14 - 1 = 1$, which is greater than 0.

Inductive Step: Assume that $4^k - 7k - 1 > 0$ for some arbitrary positive integer $k \geq 2$. Now, we want to show that $4^{k+1} - 7(k+1) - 1 > 0$.

$$\begin{aligned} 4^{k+1} - 7(k+1) - 1 &= 4 \cdot 4^k - 7k - 7 - 1 \\ &= (4^k - 7k - 1) - 7 + 3 \cdot 4^k \end{aligned}$$

By our inductive assumption, $4^k - 7k - 1 > 0$, so let $4^k - 7k - 1 = m$ for some positive integer m.

$$= m + 3 \cdot 4^k - 7$$

$$m \geq 1$$

since our base value is 1.

Therefore, $m + 3 \cdot 4^k - 7$ is a positive integer (because $m \geq 1$ and also $3 \cdot 4^k - 7 \ > 0$ for every $k > \log_4(\frac{7}{3})$ as our $k > 2$), the entire expression is positive.

Thus, by mathematical induction, $4^n - 7n - 1 > 0$ for all integers $n \geq 2$.

Answer 2

a) How many bit strings of length 10 have at least seven 1s in them?

Let's consider the different cases:

1. **Exactly 7 ones:** There are $\binom{10}{7}$ ways to choose the positions of the seven 1s, and the remaining three positions must be 0s.

2. **Exactly 8 ones:** Similarly, there are $\binom{10}{8}$ ways to choose the positions of the eight 1s, and the remaining two positions must be 0s.

3. **Exactly 9 ones:** There are $\binom{10}{9}$ ways to choose the positions of the nine 1s, and the remaining position must be 0.

4. **Exactly 10 ones:** There is only one way to have all positions filled with 1s.

Now, add up the results from each case:

$$\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + 1 = 176$$

b) We have 4 identical Discrete Mathematics textbooks and 5 identical Statistical Methods textbooks. In how many ways can you make a collection of 4 books from these 9 textbooks with the condition that at least one Discrete Mathematics textbook and at least one Statistical Methods textbook must be in the collection?

Using combinations with identical objects, the total number of ways to choose 4 books from 9, with the condition that at least one Discrete Mathematics textbook and at least one Statistical Methods textbook, which means that we need to find how many ways there are to select 2 book from 3 identical Discrete Mathematics textbooks and 4 identical Statistical Methods textbooks is given by:

$$\binom{n+r-1}{r}$$

, where r = n = 2.It equals to

$$\binom{3}{2} = 3$$

So, there are 3 ways to choose 4 books from the given textbooks, ensuring at least one from each category.

c)

The total number of functions from a set with 5 elements to a set with 3 elements is 3^5 because each of the 5 elements in the domain can be mapped to any of the 3 elements in the codomain.

Now, let's use the Inclusion-Exclusion Principle:

$$\text{Surjective functions} = \text{Total functions} - \binom{3}{1} \times \text{Functions missing 1 element} - \binom{3}{2} \times \text{Functions missing 2 ele}$$

Where we must also consider the Inclusion-Exclusion Principle on "Functions missing 1 element" $= \text{Total function} - \binom{2}{1} \times \text{Functions missing 1 elements}$

$$= 2^5 - \binom{2}{1} \times \text{Functions missing 1 elements}$$

Therefore our first equality becomes :

$$= 3^5 - \binom{3}{1} \times 2^5 + 3 \times 1^5$$

$$= 243 - 3 \times 32 + 3 \times 1$$

$$= 243 - 96 + 3 = 150$$

Therefore, there are 150 surjective (onto) functions from a set with 5 elements to a set with 3 elements.

Answer 3

Solution:
Consider an equilateral triangle with side length 500 meters. We want to place the five kids within this triangle. Divide the triangle into four smaller congruent equilateral triangles by drawing lines from each vertex to the midpoint of the opposite side. Now, each of these four smaller triangles has a side length of 250 meters.

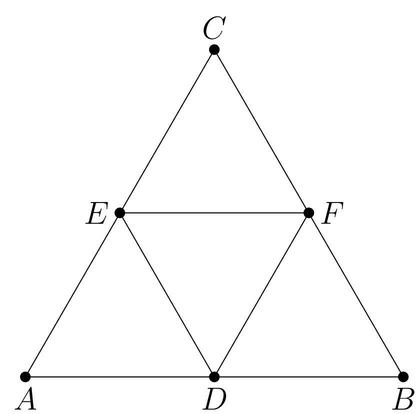

Figure 1: Equilateral triangle divided into four smaller triangles.

Now, consider the five kids as "pigeons" and the four smaller triangles as "pigeonholes." According to the pigeonhole principle, if you distribute five pigeons into four pigeonholes, at least one pigeonhole must contain more than one pigeon.

In the context of this problem, placing five kids within the equilateral triangle is analogous to placing pigeons into pigeonholes. Each of the four smaller triangles serves as a pigeonhole. Since there are five kids, at least two of them must be in the same smaller triangle.

Now, each smaller triangle has a side length of 250 meters. So, no matter where the two kids are placed within that smaller triangle, the maximum distance between them is 250 meters. This proves that there will always be two kids within 250 meters of each other, regardless of how the five kids are distributed within the equilateral triangle-shaped circus.

Answer 4

a. Homogeneous Solution:

The homogeneous part of the solution corresponds to the recurrence relation without the non-homogeneous term 5^{n-1}. The homogeneous part satisfies $a_n^{(h)} = 3a_{n-1}^{(h)}$. The characteristic equation for the homogeneous part is obtained by assuming $a_n^{(h)} = r^n$:

$$r^n = 3r^{n-1}$$

Divide both sides by r^{n-1}:

$$r = 3$$

So, the characteristic root is $r = 3$, and the homogeneous solution is $a_n^{(h)} = C \cdot 3^n$, where C is a constant.

b. Particular Solution:

For the particular solution, we'll guess that $a_n^{(p)}$ has the form $A \cdot 5^{n-1}$. Substituting this into the original recurrence relation, we get:

$$A \cdot 5^{n-1} = 3 \cdot A \cdot 5^{n-2} + 5^{n-1}$$

Solving for A, we find $A = \frac{5}{2}$. Therefore, the particular solution is $a_n^{(p)} = \frac{5}{2} \cdot 5^{n-1}$. Moreover we can use initial condition to find the C constant

$$a_1 = C \cdot 3^1 + \frac{5}{2} \cdot 5^{1-1}$$
$$4 = C \cdot 3 + \frac{5}{2}$$
$$C = \frac{1}{2}$$

The homogeneous solution is given by $a_n^{(h)} = \frac{7}{6} \cdot 3^n$, and the particular solution is $a_n^{(p)} = \frac{1}{2} \cdot 5^{n-1}$. So, the complete solution is:

$$a_n = a_n^{(h)} + a_n^{(p)} = \frac{1}{2} \cdot 3^n + \frac{5}{2} \cdot 5^{n-1}$$

Mathematical Induction:

Base Case: For $n = 1$,

$$a_1 = \frac{1}{2} \cdot 3^1 + \frac{5}{2} \cdot 5^{1-1} = \frac{1}{2} \cdot 3 + \frac{5}{2} = 4.$$

Inductive Step: Assume that

$$a_k = \frac{1}{2} \cdot 3^k + \frac{5}{2} \cdot 5^{k-1}$$

for some k. We want to show that this implies

$$a_{k+1} = \frac{1}{2} \cdot 3^{k+1} + \frac{5}{2} \cdot 5^k$$

is also true.

Starting with the recurrence relation:

$$\begin{aligned}
a_{k+1} &= 3a_k + 5^k \\
&= 3\left(\frac{1}{2} \cdot 3^k + \frac{5}{2} \cdot 5^{k-1}\right) + 5^k \\
&= \frac{3}{2} \cdot 3^k + \frac{3}{2} \cdot 5^k + 5^k. \\
&= \frac{1}{2} \cdot 3^{k+1} + \frac{5}{2} \cdot 5^k.
\end{aligned}$$

This matches the assumed form, and thus, by mathematical induction, the solution is valid for all $n \geq 1$.