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Answer 1

Firstly we can divide Q_{n+1} into 2 pieces of Q_n and additional edges that connect corresponding vertices of 2 Q_n. We can also conclude this from 3D cube. Moreover, it consist of two square which one of them is above the other and 4 edges that connects eight corresponding vertices of squares. Therefore, we can conclude the recurrence relation as follows:

$$a_n = 2 \cdot a_{n-1} + 2^{n-1}$$

To make it more clear, $2 \cdot a_{n-1}$ stands for 2 Q_{n-1} and we have additional 2^{n-1} because 2^{n-1} vertices of Q_{n-1} will be connected with 2^{n-1} vertices of the other Q_{n-1} and this produces 2^{n-1} new edges. So the recurrence relation is : $a_n = 2 \cdot a_{n-1} + 2^{n-1}$.

Answer 2

To find generatic function of $< 1, 4, 7, 10, 13, ... >$ in closed form we can try to manipulate our known generatic function $< 1, 1, 1, 1, ... > \leftrightarrow \frac{1}{1-x}$. Now let's solve this step by step.

1-) $< 1, 1, 1, 1, ... > \leftrightarrow \frac{1}{1-x}$ (Known Generitic Function)

2-) $< 1, 2, 3, 4, 5, ... > \leftrightarrow \frac{1}{(1-x)^2}$ (Take Derivative): every exponent $(x)^n$ comes down and becomes $n(x)^{n-1}$

3-) $< 0, 1, 2, 3, 4, ... > \leftrightarrow \frac{x}{(1-x)^2}$ (Shift Sequence One Times To Right): when we shift k times to right we multiply function by $(x)^k$

4-) $< 0, 3, 6, 9, 12, ... > \leftrightarrow \frac{3x}{(1-x)^2}$ (Multiply Sequence with Three): we multiply function with k when we multiply every element with k

Now we observe that if we sum up generitic functions that we obtained at step 1 and 4 we can get $< 1, 4, 7, 10, 13, ... >$ in closed form funciton. To see, $< 1, 1, 1, 1, 1... > + < 0, 3, 6, 9, 12, ... > = < 1, 4, 7, 10, 13, ... >$. From the theorem of summing we know that closed form of two summed sequence equal to summation of their closed form so our answer is:

$$\frac{1}{1-x} + \frac{3x}{(1-x)^2} = \frac{1+2x}{(1-x)^2}$$

Answer 3

To solve the given recurrence relation $a_n = a_{n-1} + 2^n$, $n \geq 1$ with the initial condition $a_0 = 1$ using generating functions, we'll define the generating function $G(x)$ for the sequence $\{a_n\}$.

The generating function is defined as:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Now, we'll express the given recurrence relation in terms of the generating function. Multiply both sides by x^n and sum over all $n \geq 1$:

$$G(x) - a_0 = \sum_{n=1}^{\infty} (a_{n-1} + 2^n) x^n$$

Use the fact that $a_n = a_{n-1} + 2^n$:

$$G(x) - 1 = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} 2^n x^n$$

Now, manipulate the series to make them match the form of the generating functions:

$$G(x) - 1 = x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} 2^n x^n$$

Recognize that $\sum_{n=0}^{\infty} a_n x^n$ is just $G(x)$ and $\sum_{n=1}^{\infty} 2^n x^n$ is $\frac{1}{1-2x} - 1$:

$$G(x) = xG(x) + 1 + \frac{1}{1-2x} - 1$$

Now, solve for $G(x)$:

$$G(x) = xG(x) + \frac{1}{(1-2x)}$$

Now, solve for $G(x)$:

$$G(x)(1-x) = \frac{1}{(1-2x)}$$

$$G(x) = \frac{1}{(1-2x)(1-x)}$$

Now, decompose the fraction into partial fractions:

$$G(x) = \frac{2}{(1-2x)} - \frac{1}{(1-x)}$$

Now, recall that the generating function for $< 1, 1, 1, 1, ... >$ is $\frac{1}{1-x}$, and the generating function for $< 2, 2 \cdot 2^1, 2 \cdot 2^2, 2 \cdot 2^3, ..., 2 \cdot 2^n, ... > = \frac{2}{1-2x}$. Now by summing this to sequence's n^{th} position we can find a_n and first sequence's n^{th} position is -1 and the n^{th}position is $2 \cdot 2^n$ in the second sequence.

So, the solution is:

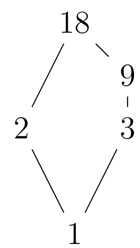
$$a_n = 2^{n+1} - 1$$

Answer 4

a) **Draw the Hasse diagram of R:**

The Hasse diagram is a graphical representation of a partially ordered set, where elements are represented as points, and the ordering relation is represented by line segments between the points.

For the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on $A = \{1, 2, 3, 9, 18\}$, the Hasse diagram is as follows:



b) **Give the matrix representation for R:**

The matrix representation for a relation R on a set A is an $|A| \times |A|$ matrix where the entry in row i and column j is 1 if $(a_i, a_j) \in R$ and 0 otherwise.

For $R = \{(a, b) \mid a \text{ divides } b\}$ on $A = \{1, 2, 3, 9, 18\}$, the matrix representation is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) **Is (A, R) a lattice? Explain your answer:**

A lattice is a partially ordered set in which every pair of elements has both a greatest lower bound (meet) and a least upper bound (join). In this case, (A, R) is a lattice because for any two elements in A, there exists a greatest lower bound and a least upper bound. We can show it through a table

Pairs	GLB	LUB
(1,2)	1	2
(1,3)	1	3
(1,9)	1	9
(1,18)	1	18
(2,3)	1	18
(2,9)	1	18
(2,18)	2	18
(3,9)	3	9
(3,18)	3	18
(9,18)	9	18

d) **Give the matrix representation for R_s, where R_s is the symmetric closure of R. Explain your answer:**

The symmetric closure R_s of a relation R includes both (a, b) and (b, a) whenever $(a, b) \in R$. Therefore, the matrix representation for R_s is obtained by taking the union of the original matrix with its revers ;formula of symmetric closure $R_s = R \vee R^{-1}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

e) **In (A, R), are the integers 2 and 9 comparable? Are 3 and 18 comparable? Explain your answer:**

In (A, R), two elements a and b are comparable if either a divides b or b divides a.

-For 2 and 9: 2 does not divide 9, so they are incomparable.
-For 3 and 18: 3 divides 18, so they are comparable.

The integers 2 and 9 are incomparable because in the partially ordered set, pair (2,9) does not exists .However 3 and 18 are comparable and 18 has higher priority because pair (3,18) is in partially ordered set. In general, two elements in (A, R) are comparable if one divides the other, and from the Hasse diagram, you can see the direction of divisibility relationships.

Answer 5

Reflexive and Symmetric Relations:

For a relation to be reflexive, it must contain the pair (a, a) for every a in A. For a relation to be symmetric, if it contains (a, b), it must also contain (b, a). Now, let's analyze how many choices we have for constructing such a relation using a matrix. A relation to be reflexive ,in its matrix representation it's diagonal entries must be 1 so that it contains the pair (a, a) for every a in A.

$$\begin{bmatrix} 1 & x_{12} & x_{13} & \dots \\ x_{21} & 1 & x_{23} & \dots \\ \vdots & \vdots & \ddots & \dots \\ x_{n1} & x_{n2} & x_{n3} & 1 \end{bmatrix}$$

In addition, if a relation is symmetric then x_{ij} must be equal to x_{ji} .

a) **Reflexivity and Symmetric:** For each element a in A, we have the choice to include (a, a) in the relation in order to make it reflexive. If (a, b) is in the relation, we must include (b, a) to make it symmetric. However, if (a, b) is not in the relation, we can either include or not include (b, a) without affecting symmetry.Since we have two choices (include or not include) for every entrie except diagonal entries since it is reflexive. So we have $(n^2 - n)/2$ entry, since crosswise entries need to be same we can just check half of this entries and for each entry we have 2 case 1 or 0. This means we have $2^{n(n-1)/2}$ choices for symmetric pairs. .

Therefore, the total number of different binary relations on A that are both reflexive and symmetric is $2^{n^2/2-n/2}$.

b) **Reflexive and Antisymmetric Relations:**

For a relation to be reflexive, it must contain the pair (a, a) for every a in A. An antisymmetric relation is one in which if (a, b) and (b, a) are both in the relation, then a must be equal to b. In other words, if there is any pair (a, b) where $a \neq b$, it must not contain (b, a). Now, let's analyze the choices:

This case is similar to the case we have in part (a) in a way that we have same options for diagonals but this time we have 3 options instead of 2 since we can choose crosswise entries to be (1,0), (0,1) or (0,0) .

Therefore, the total number of different binary relations on A that are both reflexive and antisymmetric is $3^{n(n-1)/2}$.

Answer 6

The transitive closure of an antisymmetric relation is **NOT** always antisymmetric. Let's consider a counterexample to illustrate this. Consider a relation R with the pairs $\{(a, b), (b, c), (c, a)\}$. This relation is antisymmetric because it doesn't contain the pair (a, b) along with its reverse (b, a). Now, let's find the transitive closure of R. The transitive closure includes all pairs (x, y) such that there exists a sequence $(x, a_1), (a_1, a_2), \dots, (a_n, y)$ where each pair in the sequence is in R. When we examine the transitive closure, denoted as R^+, we encounter an interesting scenario.

In R^+, we have (b, c) and (c, a), which logically implies (b, a) should be present in the transitive closure due to the transitive property. However, this clashes with the original relation's property that aRb and bRa together imply $a = b$. In the case of R^+, (a, b) and (b, a) coexist, breaking the antisymmetry.

This counter-example highlights that while the original relation R might be antisymmetric, its transitive closure R^+ can deviate from this property, illustrating the complexity of combining antisymmetry and transitivity in certain cases.