Homework 2: Tree-based Models

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Problem 1 for X1

$$P(Y = 1 | X_1 = 0) = 0$$

$$P(Y = 2 | X_1 = 0) = \frac{1}{2}$$

$$P(Y = 3 | X_1 = 0) = \frac{1}{2}$$

$$H(Y | X_1 = 0) = -[P(Y = 1 | X_1 = 0) \log_2 P(Y = 1 | X_1 = 0) + P(Y = 2 | X_1 = 0) \log_2 P(Y = 2 | X_1 = 0) + P($$

$$= -\left[0 \cdot \log_2(0) + \frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{2} \log_2 \left(\frac{1}{2} \right) \right]$$

$$= 1$$

$$P(Y = 1 | X_1 = 1) = \frac{1}{2}$$

$$P(Y = 2 | X_1 = 1) = \frac{1}{4}$$

$$P(Y = 3 | X_1 = 1) = \frac{1}{4}$$

$$H(Y | X_1 = 1) = -[P(Y = 1 | X_1 = 1) \log_2 P(Y = 1 | X_1 = 1) + P(Y = 2 | X_1 = 1) \log_2 P(Y = 2 | X_1 = 1) + P($$

$$= -\left[\frac{1}{2} \log_2 \left(\frac{1}{2} \right) + \frac{1}{4} \log_2 \left(\frac{1}{4} \right) + \frac{1}{4} \log_2 \left(\frac{1}{4} \right) \right]$$

$$= 1.5$$

$$P(X_1 = 1) = \frac{4}{6}$$

$$P(X_1 = 0) = \frac{2}{6}$$

$$H(Y | X_1) = P(X_1 = 1) H(Y | X_1 = 1) + P(X_1 = 0) H(Y | X_1 = 0)$$

$$= \frac{4}{6} \cdot 1.5 + \frac{2}{6} \cdot 1$$

$$= 1 + 0.3333$$

$$= 1.3333$$

ANSWER FOR X1: $H(Y | X_1) = 1.3333$

Problem 1 for X2

$$P(Y = 1 | X_2 = 0) = 0$$

$$P(Y = 2 | X_2 = 0) = \frac{1}{3}$$

$$P(Y = 3 | X_2 = 0) = \frac{2}{3}$$

$$H(Y | X_2 = 0) = -[P(Y = 1 | X_2 = 0) \log_2 P(Y = 1 | X_2 = 0) + P(Y = 2 | X_2 = 0) \log_2 P(Y = 2 | X_2 = 0) + P($$

$$= -\left[0 \cdot \log_2(0) + \frac{1}{3} \log_2 \left(\frac{1}{3} \right) + \frac{2}{3} \log_2 \left(\frac{2}{3} \right) \right]$$

$$= 0.9183$$

$$P(Y = 1 | X_2 = 1) = \frac{2}{3}$$

$$P(Y = 2 | X_2 = 1) = \frac{1}{3}$$

$$P(Y = 3 | X_2 = 1) = 0$$

$$H(Y | X_2 = 1) = -[P(Y = 1 | X_2 = 1) \log_2 P(Y = 1 | X_2 = 1) + P(Y = 2 | X_2 = 1) \log_2 P(Y = 2 | X_2 = 1) + P($$

$$= -\left[\frac{2}{3} \log_2 \left(\frac{2}{3} \right) + \frac{1}{3} \log_2 \left(\frac{1}{3} \right) + 0 \cdot \log_2(0) \right]$$

$$= 0.9183$$

$$P(X_2 = 1) = \frac{1}{2}$$

$$P(X_2 = 0) = \frac{1}{2}$$

$$H(Y | X_2) = P(X_2 = 1) H(Y | X_2 = 1) + P(X_2 = 0) H(Y | X_2 = 0)$$

$$= \frac{1}{2} \cdot 0.9183 + \frac{1}{2} \cdot 0.9183$$

$$= 0.9183$$

ANSWER FOR X2: $H(Y | X_2) = 0.9183$

Problem 2 infogain

$$P(Y_1) = \frac{2}{6} = \frac{1}{3}$$
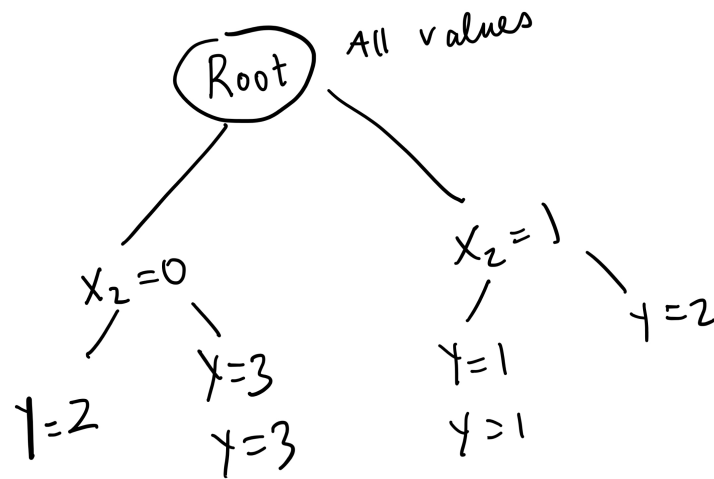$$P(Y_2) = \frac{2}{6} = \frac{1}{3}$$
$$P(Y_3) = \frac{2}{6} = \frac{1}{3}$$
$$H(Y) = -\sum_{i=1}^{3} P(Y_i) \log_2 P(Y_i)$$
$$= -3 \left[\frac{1}{3} \log_2 \left(\frac{1}{3} \right) \right]$$
$$= 1.585$$

Therefore, $H(Y) = 1.585$ now use it for IG.

$$IG(Y, X_1) = H(Y) - H(Y|X_1)$$
$$= 1.585 - 1.333$$
$$= 0.252$$
$$IG(Y, X_2) = H(Y) - H(Y|X_2)$$
$$= 1.585 - 0.918$$
$$= 0.667$$

Problem 3 splits and tree

From problem 2, since $IG(Y, X_2)$ is higher than X1, we use $IG(Y, X_2)$.
Here is the decision tree using the split

All values

Root

$X_2 = 0$

$X_2 = 1$

$Y = 2$

$X = 3$

$Y = 3$

$Y = 1$

$Y = 1$

$Y = 2$

Problem 4 classification for test example

$X_1 = 0$ and $X_2 = 1$ We follow the tree.
since $X_2 = 1$, go right from root
Then majority vote. Since more $Y = 1$,
test example is $Y = 1$