§ 1.5. RELATION

In order to express a relation from the set A to the set B, we always need a **statement** which connects the elements of A with the elements of B.

For example. Suppose $A = \{1, 3, 5, 9\}$, $B = \{0, 2, 4, 8\}$. Now suppose a relation from the set A to the set B is expressed by the statement **'is less than'**.

Now taking the first and second co-ordinates of the elements of the sets A and B respectively, the ordered pairs satisfying the statement **'is less than'** are as follows :

$$(1, 2), (1, 4), (1, 8), (3, 4), (3, 8), (5, 8).$$

The set R of these ordered pairs given by

$$R = \{(1, 2), (1, 4), (1, 8), (3, 4), (3, 8), (5, 8)\}$$

expresses a relation from the set A to the set B.

Clearly the set R is a **subset** of $A \times B$.

Definition. Let A and B be two non-empty sets. A **relation** from A to B is a subset of $A \times B$ and is denoted by R. Thus, R is a relation from A to $B \Rightarrow R \subseteq A \times B$.
Symbolically, we write

$$R = \{(x, y) : x \in A, y \in B \text{ and } xRy\}$$

xRy indicates that x is R related to y. **[Ravishankar 1992S]**

Example 1. Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{0, 2, 4, 8\}$ be two given sets. Now suppose a relation from the set A to B is expressed by the statement 'is less than'.

Let us take the first and second co-ordinates of the elements of the sets A and B respectively. Then the ordered pairs satisfying the statement 'is less than' are as follows :

$$(1, 2), (1, 4), (1, 8), (3, 4), (3, 8), (5, 8), (7, 8).$$

Then the set R of these ordered pairs given by

$$R = \{(1, 2), (1, 4), (1, 8), (3, 4), (3, 8), (5, 8), (7, 8)\}$$

expresses a relation from the set A to the set B.

Clearly $R \subseteq A \times B$.

It may be observe that the relation of set theoretic language is quite analogous to the relation of the language of conversation. For example, if we consider the phrase "is father of" of the general language, A is a well-defined set of persons and R is a subset of $A \times A$ defined as follows :

$$R = \{(x, y) : x, y \in A \text{ and } x \text{ is father of } y\}$$

then we infer that in every ordered pair of the set R, first co-ordinate is the father of second co-ordinate; *i.e.*, first co-ordinate is R related to second co-ordinate, symbolically, xRy. Therefore, R is a relation from A to A itself. Clearly $R \subseteq A \times A$. In such case we also say that R is a relation in A. Thus R is called relation of the set theoretic language.

Notes : (1) If the set A has m elements and the set B has n elements, then $A \times B$ will have $m \cdot n$ elements. Therefore, power set of $A \times B$ will have 2^{mn} elements. Since every subset of $A \times B$ may be a relation from A to B, and hence there may be 2^{mn} different relation from A to B.

(2) Since null set ϕ is a subset of every set, if follows that $\phi \subset A \times B$. Hence ϕ is also a relation from A to B which is called **Null relation** or **Void relation** from A to B.

(3) Since every set is subset of itself and so $A \times B \leq A \times B$. Hence $A \times B$ is also a relation which is called **Universal relation** from A to B.

(4) If $B = A$, then ϕ and $A \times A$ are called **null relation** and **universal relation**, respectively, in A.

(5) If x is R related to y, then we write x before R and y after R, *i.e.*, xRy.

(6) If $R \subseteq A \times B$ and $a \in A$, $b \in B$ are such that $(a, b) \notin R$, then we say that a is not R related to b. Symbolically, we write $a \not R\, b$.

◈ § 1.6. DOMAIN AND RANGE OF A RELATION

Let $R = \{(x, y) : x \in A, y \in B \text{ and } x\ R\ y\}$ be a relation from A to B. Then the set of first co-ordinates of every element of R is called **domain of R** and the set of second co-ordinates of its every element is called **range of R**. Symbolically,

$$d(R) = \text{domain of } R = \{x : x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$$

and $$r(R) = \text{range of } R = \{y : y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}$$

◈ § 1.7. INVERSE RELATION

Definition. If $R \subseteq A \times B$ is a relation from the set A to the set B, then the inverse of R, denoted by R^{-1}, is a relation from B to A defined by

$$R^{-1} = \{(y, x) : (x, y) \in R, x \in A, y \in B\}.$$

Thus to find R^{-1}, we write in reverse order all ordered pair belonging to R.

$\therefore$ Range of R^{-1} = Domain of R

and Domain of R^{-1} = Range of R.

Example. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$ and let $R = \{(1, a), (2, a), (2, b), (3, a), (3, b)\}$ be a relation from A to B, then $R^{-1} = \{(a, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$ will be a relation from B to A. Here,

$$d(R) = \{1, 2, 3\}, \qquad r(R) = \{a, b\},$$
$$d(R^{-1}) = \{a, b\} \text{ and } r(R^{-1}) = \{1, 2, 3\}$$

Hence it is clear that

$$d(R) = r(R^{-1}) = \{1, 2, 3\} \text{ and } d(R^{-1}) = r(R) = \{a, b\}.$$

◈ § 1.8. COMPOSITE RELATION

Definition. Let A, B, C be three non-empty sets and R be a relation from A to B and S be a relation from B to C; *i.e.*, $R \subseteq A \times B$, $S \subseteq B \times C$. Then the **composite relation** of the two relations R and S is a relation from A to C, denoted by $S \text{ o } R$ and is defined as follows :

$$S \text{ o } R = \{(x, z) : \exists \text{ an element } y \in B \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

where $x \in A, z \in C$.

Hence $(x, y) \in R, (y, z) \in S \Rightarrow (x, z) \in S \text{ o } R$.

Consequently, by definition of inverse of a relation, we have $(z, x) \in (S \text{ o } R)^{-1}$.

Theorem 1. *If R^{-1} and S^{-1} are inverse of the relations R and S respectively, then*

$$(S \text{ o } R)^{-1} = R^{-1} \text{ o } S^{-1}.$$ **[Bilaspur 2000; Sarguja 2010, 2011]**

Proof. Let A, B, C be three non-empty sets. R and S be relations from A to B and from B to C, respectively, then by definition of relations and composite of relations, we have $R \subseteq A \times B$, $S \subseteq B \times C$ and $S \text{ o } R \subseteq A \times C$. It, then, follows from the definition of inverse relation,

$$(S \text{ o } R)^{-1} \subseteq C \times A.$$

Let $(z, x) \in (S \text{ o } R)^{-1}$ be arbitrary. Then

$$\begin{aligned}(z, x) \in (S \text{ o } R)^{-1} &\Rightarrow (x, z) \in S \text{ o } R \\ &\Rightarrow \exists\ y \in B : (x, y) \in R, (y, z) \in S \\ &\Rightarrow (y, x) \in R^{-1}, (z, y) \in S^{-1} \\ &\Rightarrow (z, y) \in S^{-1}, (y, x) \in R^{-1} \\ &\Rightarrow (z, x) \in R^{-1} \text{ o } S^{-1}.\end{aligned}$$

Thus $(z, x) \in (S \text{ o } R)^{-1} \Rightarrow (z, x) \in R^{-1} \text{ o } S^{-1} \ \forall (z, x) \in (S \text{ o } R)^{-1}$.

Hence $$(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}. \quad ...(1)$$

Similarly, we can prove that

$$R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}. \quad ...(2)$$

Hence, by (1) and (2), we have

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

◈ § 1.9. BINARY RELATION

Definition. Let A be a non-empty set and R be a relation in A; *i.e.*, $R \subseteq A \times A$. Then R is called a **binary relation on A.**

◈ § 1.10. IDENTITY RELATION IN A SET

Let A be a non-empty set and I_A be a relation in A defined by $I_A = \{(x, y) : x, y \in A \text{ and } x = y\}$, then I_A is called an identity relation in A. More explicitly, an identity relation in a set of all ordered pairs (x, y) of $A \times A$ for which $x = y$.

Example. If $A = \{1, 2, 3, 4\}$ be a set than the set $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is an identity relation in A.

Note. It is clear from the definition of the identity relation I_A in A that

$$d(I_A) = r(I_A) = A.$$

◈ § 1.11. DIFFERENT TYPES OF BINARY RELATIONS

1. Reflexive Relation :

Let A be a non-empty set and R be a binary relation in A *i.e.*, $R \subseteq A \times A$, then the relation R is called **reflexive relations** if $x\,R\,x$; *i.e.*, every element of A is R related to itself.

Symbolically, $\quad \forall\ x \in A, (x, x) \in R.$

Example 1. If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$ then R is a reflexive relation in A since every element of A; *i.e.*, 1, 2, 3 all are R related to themselves; *i.e.*,

$$1\,R\,1 \Rightarrow (1, 1) \in R$$
$$2\,R\,2 \Rightarrow (2, 2) \in R$$
and
$$3\,R\,3 \Rightarrow (3, 3) \in R.$$

Hence R is a reflexive relation in A.

Example 2. If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3), (2, 2)\}$, then R is not reflexive because $3 \in A$ but 3 is not R related to 3; *i.e.*, $(3, 3) \notin R$.

Example 3. Let S be a set and let $P(S)$ be the power set of S; *i.e.*, $P(S) = \{A : A \subseteq S\}$. Then the relation R defined as follows :

$$(A, B) \in R \quad \text{if} \quad A \leq B$$

is reflexive, since $\quad A \subseteq A \ \forall\ A \in P(S).$

2. Symmetric Relation :

If R is a relation in the set A, then R is called symmetric relation if a is R-related to b then b is also R-related to a

i.e., $(a, b) \in R \Rightarrow (b, a) \in R$

i.e., $a \; R \; b \Rightarrow b \; R \; a$ where $a, b \in A$.

Note. Clearly a relation R will be symmetric if $R = R^{-1}$.

Example 1. *If* $A = \{2, 4, 5, 6\}$ *and*

(*a*) $R_1 = \{(2, 4), (4, 2), (4, 5), (5, 4), (6, 6)\}$,

(*b*) $R_2 = \{(2, 4), (2, 6), (6, 2), (5, 4), (4, 5)\}$.

Solution. (a) Relation R_1 is symmetric since

$$(2, 4) \in R_1 \Rightarrow (4, 2) \in R_1;$$
$$(4, 5) \in R_1 \Rightarrow (5, 4) \in R_1;$$
$$(6, 6) \in R_1 \Rightarrow (6, 6) \in R_1$$

i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ is true.

(b) R_2 is not symmetric since

$$(2, 4) \in R_2 \Rightarrow (4, 2) \notin R_2.$$

3. Anti-symmetric Relation :

If R is a relation in the set A, then B is called anti-symmetric if

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b$$

i.e., $a \; R \; b$ and $b \; R \; a \Rightarrow a = b$

where $a, b \in A$.

Example 1. In the set of natural numbers, the relation 'a divides b' is anti-symmetric, since 'a divides b' and 'b divides a' is possible only when $a = b$, *i.e.*, if the given relation be denoted R then

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b.$$

Example 2. In the set of integers, the relation '$\leq$' is anti-symmetric, since

$$a \leq b \text{ and } b \leq a \Rightarrow a = b.$$

Example 3. In the set of sets the relation '$\subseteq$' is anti-symmetric since.

Let A and B are any two sets, then

$$A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B.$$

4. Transitive Relation :

If R is a relation in the set A, then R is called transitive relation if a is R-related to b and b is R-related to c, then a is also R-related to c

i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

i.e., $a \; R \; c$ and $b \; R \; c \Rightarrow a \; R \; c$ where $a, b, c \in A$.

Example. If $A = \{1, 3, 5\}$ and $R = \{(1, 3), (1, 5), (3, 5)\}$, then R is transitive since $(1, 3) \in R$ and $(3, 5) \in R \Rightarrow (1, 5) \in R$.

Remarks : (i) Every identity relation is reflexive but its converse is not true.

(ii) If in a set A identity relation is I_A and reflexive relation is R, then $I_A \subset R$.

UNIT-I

CHAPTER 2

PARTIALLY ORDERED SETS

§ 2.1. PARTIAL ORDER RELATION

Definition. A relation R on a set A is said to be **partial order relation** if it is

(PO_1). **Reflexive.** For each $a \in A$, $a\, R\, a$.

(PO_2). **Anti-symmetric.** If $a, b \in A$, then

$$a\, R\, b \text{ and } b\, R\, a \Rightarrow a = b.$$

(PO_3). **Transitive.** If $a, b, c \in A$, then

$$a\, R\, b \text{ and } b\, R\, c \Rightarrow a\, R\, c.$$

Definition. The set A together with the partial order R is called partially order set (poset). **[Ravishankar 2008]**

Problem. Define and given example of a partially order set.

Illustration. The relation '$\leq$' in the usual sense "less than or equal to" is a partial order relation on the set of real numbers **R**.

Example 1. *The set of integers Z with usual ordering '$\leq$' read as "Less than or equal to" is a poset.*

Solution. The set of integers Z is poset if it satisfying the following properties :

(a) Reflexive : Since $a \leq a$ for every integer a.

$\therefore$ $\leq$ is reflexive.

(b) Antisymmetric : If $a \leq b$ and $b \leq a$ where $a, b \in Z$ then $a = b$.
Hence $\leq$ is antisymmetric.

(c) Transitive : If $a \leq b$ and $b \leq c$ where $a, b, c \in Z$ then $a \leq c$

$\Rightarrow$ $\leq$ is a transitive relation.

Hence $(Z, \leq)$ is a poset.

Example 2. *The set S be any collection of sets. The relation $\subseteq$ read as "is a subset of" is a partial ordering of S.*

Solution. The set S is poset if it satisfying the following properties :

(a) Reflexive : Since $A \subset A$ for any subset A of S.

$\therefore$ $\subseteq$ is reflexive.

(b) Anti-symmetric : If $A \subset B$ and $B \subseteq A$, $\forall A, B \in S$ then $A = B$.

$\therefore \subseteq$ is anti-symmetric.

(c) Transitive : If $A \subseteq B$ and $B \subseteq A$ for any sets $A, B, C \in S$, then $A \subseteq C$.

$\therefore \subseteq$ is transitive.

Hence $(Z, \leq)$ is a poset.

Example 3. *Consider the integer Z^+ of positive integers. We say "a divides b" written as $a \mid b$ iff there is an integer c such that $ac = b$.*

Solution. The set Z^+ of positive integer is posets if

(a) Reflexive : Since $a \mid a$ for every positive integer $a \in Z^+$. $\therefore$ '$\mid$' is reflexive.

(b) Anti-symmetric : If $a \mid b$ and $b \mid a$ $\forall a, b \in Z^+$ then $ac = b$ and $bd = a$, for some $b, d \in Z^+$

$\Rightarrow \quad bdc = b \Rightarrow b(dc) = b \Rightarrow dc = 1 \quad [\because b \neq a]$

$\Rightarrow \quad d = 1$ and $c = 1 \quad \therefore a = b.$

Hence divisibility relation is anti-symmetric.

(c) Transitive : If $a \mid b$ and $b \mid c$ $\forall a, b, c \in Z^+$

$\Rightarrow \quad ax = b, by = c$ for some $x, y \in Z^+$

$\Rightarrow \quad c = by = a(xy)$

$\Rightarrow \quad a(xy) = c$, for $xy \in Z^+$

$\Rightarrow \quad a \mid c. \quad \therefore$ "$\mid$" is transitive.

Hence $(Z^+, \mid)$ is a poset.

Example 4. *The relation of divisibility is not a partial order relation on the set Z of integer.*

Solution. The relation is not anti-symmetric. Since

$$2 \mid -2 \text{ and } -2 \mid 2 \text{ but } 2 \neq -2$$

Example 5. *Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be the relation in A which is defined by a divides b then R is partial order in A.*

Example 6. *Prove that "being a subset of" is a partial order relation on the power set of non-empty set A.*

Solution. Let $P(A) = 2^A = X$ be the power set of A, *i.e.*, X is the set of all subsets of A. For any U, V, W in X, set $U \leq V \Leftrightarrow U \subseteq V$.

(a) Reflexive : Since $U \subseteq U, U \leq U$.

(b) Anti-symmetric : Suppose $U \leq V$ and $V \leq U$ then

$$U \subseteq V \text{ and } V \subseteq U \Rightarrow U = V.$$

(c) Transitive : Suppose $U \leq V$ and $V \leq W$, then $U \subseteq V$ and $V \subseteq W \Rightarrow U \subseteq W$.

Hence $\quad U \leq W$.

Thus $(x, \leq)$ is partial order.

Example 7. *If R is partially ordered relation on a set X and $A \leq X$, show that $R \cap (A \times A)$ is a partial ordering relation on A.*

Solution. Denote $R \cap (A \times A)$ by R'.

(a) Reflexive : Let $x \in A$, then $(x, x) \in A \times A$.

Since R is reflexive, $(x, x) \in R \Rightarrow xRx$.

Therefore $\quad (x, x) \in R \cap (A \times A) = R'$.

(b) Anti-symmetric : Suppose $(x, y) \in R'$ and $(y, x) \in R'$
$\Rightarrow$ $(x, y) \in R \cap (A \times A)$ and $(y, x) \in R \cap (A \times A)$.
Since R is anti-symmetric, $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$.

(c) Transitive : Suppose $(x, y) \in R' = R \cap (A \times A)$ and
$(y, z) \in R' = R \cap (A \times A) \Rightarrow (x, y), (y, z) \in R$ and $(A \times A)$
R is transitive, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$. Since $R \subset A \times A$.
$\therefore$ $(x, z) \in A \times A$ and hence $(x, z) \in R \cap (A \times A) = R'$.
Thus $R' = R \cap (A \times A)$ is partial ordering relation on A.

2.1.1. Definition (Comparable). Two elements a and b in a poset $(S, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$. Thus, a and b are called in comparable if neither $a \leq b$ nor $b \leq a$.

Illustration 1. In the poset $(Z^+, |)$, the integer 3 and 9 are comparable. Since $3 | 9$. But the integers 5 and 7 are in comparable because neither $5 | 7$ nor $7 | 5$.

Illustration 2. The set $(Z^+, |)$ is not linearly ordered. Since 4 and 7 are incomparable as neither $4 | 7$ nor $7 | 4$. But $A = \{2, 6, 12, 36\}$ is linearly ordered subset of Z^+. Since $2 | 6$, $6 | 12$ and $12 | 36$.

2.1.2. Definition (Lexicographic Ordering). The lexicographic ordering $\leq$ on $S_1 = A_1 \times S_2 = A_2$ is defined by indicating that one pair is less than a second pair if the first entry of first pair is less than (in $S_1 = A_1$) the first entry of the second pair or if the first entries are equal but the second entry of this pair is less than (in $S_2 = A_2$) the second entry of the second pair *i.e.*, $(a_1, a_2) \leq (b_1, b_2)$ either if $a_1 \leq b_1$ or if $a_1 = b_1$ and $a_2 \leq b_2$ for posets $(S_1 = A_1 \leq_1)$ and $(S_2 = A_2 \leq_2)$.

Illustration 3. $(3, 5) < (4, 8)$ is lexicographic ordering constructed from the usual relation $\leq$ on z.

Illustration 4. Let **C** be the set of all complex numbers. If $z = x + iy$, $w = u + iv \in \mathbf{C}$, then the relation Υ defined on **C** by

$$z \Upsilon w \text{ iff } x \leq u \text{ and } y \leq v$$

where $\leq$ has usual meaning for the real numbers, is a partial order relation on **C**.

§ 2.2. TOTAL ORDERING RELATION OR LINEARLY ORDERED

Definition. A relation R on a set A is said to be **total ordering relation** if the relation R is reflexive, anti-symmetric, transitive and satisfies the following relation.

(PO_4): **Law of Dichotomy.** For each $a, b \in A$, either $a \leq b$ or $b \leq a$; *i.e.*, any two elements of A are comparable (or related).

Illustration 1. The usual ordering '$\leq$' in the set of all real numbers **R**; *i.e.*, for $x, y \in \mathbf{R}$

$$x \leq y \text{ iff } x \text{ is less than or equal to } y$$

is a total ordering relation on **R**.

Illustration 2. The binary relation '$|$' on the set $A = \{5, 5^2, 5^3, 5^4, \ldots\}$ which is defined by

$$x | y \text{ iff } x \text{ divides } y \text{ or } x \, D \, y \text{ iff } x \text{ divides } y.$$

is a total ordering relation on A.

Problem. Define a totally ordered set. Give an example of a P.O.S. which is not totally ordered.

◈ § 2.3. PARTIALLY ORDERED SET

Definition. A set A with a partial ordering relation $\leq$ on A is called a **partially ordered set** (or in brief **poset**) and symbolically, it is denoted by $(A, \leq)$.

Note. We generally denote a partial order by the symbol $\leq$ in place of R. We read $\leq$ as less than or equal to. The symbol $\leq$ does not necessary mean usual "Less than or equal to" as read for real numbers. If $\leq$ is partially order on A. Then the order pair $(A, \leq)$ is called a partially ordered set or simply **poset**.

Illustration 1. The usual ordering relation '$\leq$' (less than or equal to) on the set of real numbers **R** is a partial order relation. Thus, $(\mathbf{R}, \leq)$ is a partially ordered set (or a poset).

Illustration 2. The binary relation '$|$' on the set of all positive integers $\mathbf{I}^+$ defined by

$$x \mid y \text{ iff } x \text{ divides } y$$

is a partial order relation. Thus, $(\mathbf{I}^+, |)$ is a partially ordered set (or a poset).

Note that if a binary relation R is defined on a set A, we can have a simple graphical representation. We represent the elements in A by points and use arrows to represent the ordered pairs in R. For example, the binary relation on the set $\{a, b, c, d\}$ in Fig. 2.1 (a) is graphically represented in Fig. 2.1 (b).

	a	b	c	d
a	√	√	√	√
b			√	
c				√
d			√	

Fig. 2.1. (a)

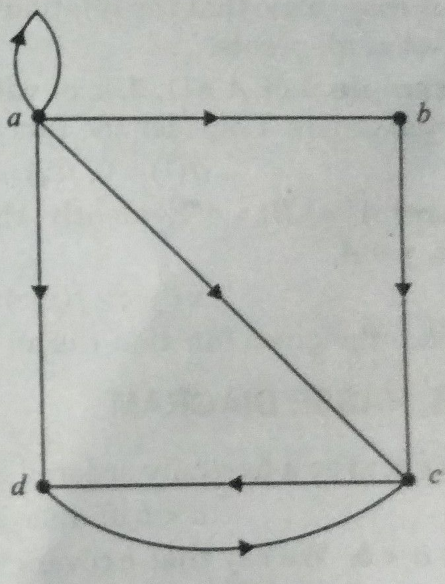


Fig. 2.1. (b)

When a binary relation is a partial ordering relation, the graphical representation can be simplified. Since the relation is reflexive, we can omit arrows from points back to themselves. Since the relation is transitive, we can omit arrows between points that are connected by sequence of arrows. For example, such a simplified representation for the partial ordering relation in Fig. 2.2 (a) is shown in Fig. 2.2 (b). In case, when the graphical representation is so oriented that all arrow heads point in one direction (upward, downward, left to right or right to left), we can omit the arrow heads as in Fig. 2.2 (c).

UNIT-I

CHAPTER 3

LATTICES

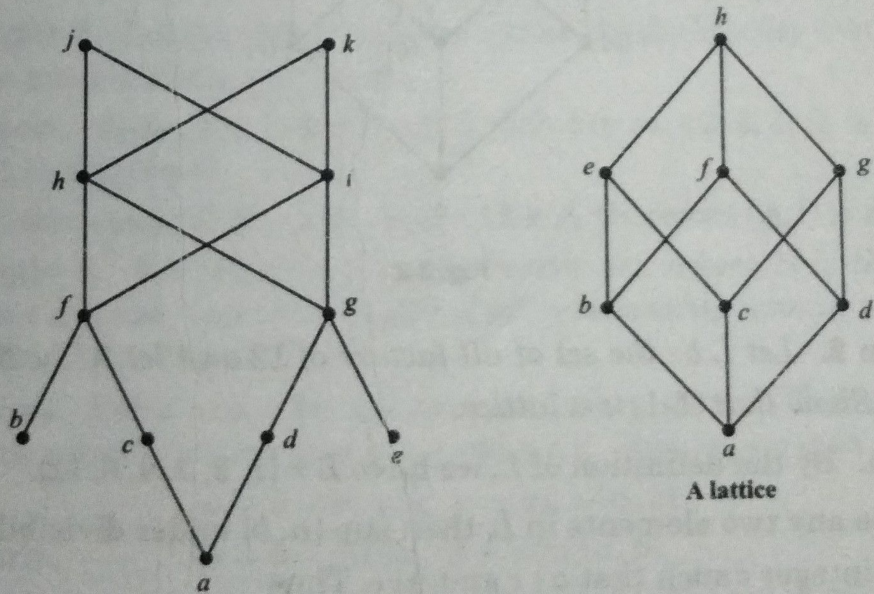
◈ § 3.1. LATTICE

Definition. A partially ordered set $(L, \leq)$ is said to be a **lattice** if every two elements in the set L has a unique least upper bound (sup) and a unique greatest lower bound (inf). In other words, poset $(L, \leq)$ is a lattice if for every $a, b \in L$, sup $\{a, b\}$ and inf $\{a, b\}$ exist in L.

We denote sup $\{a, b\}$ by $a \vee b$ and call it the **join** of a and b and inf $\{a, b\}$ by $a \wedge b$ and call it the **meet** of a and b. Other notations like $a \cup b$ and $a \cap b$ or $a + b$ and $a \,.\, b$ are also used for sup $\{a, b\}$ and inf $\{a, b\}$ respectively.

Examples on Lattice :

Example 1. The partially ordered set in Fig. 3.1 (a) is not a lattice, whereas the one in Fig. 3.1 (b) is because in Fig. 3.1 (a) elements a, b, c, d, e, f and g are all lower bounds of h and i, while f and g are greatest lower bounds of h and i; *i.e.*, the pair of elements h and i do not have a unique greatest lower bound.



Not a lattice
(a)

A lattice
(b)

Fig. 3.1

Without loss of generality, we may assume that $a \leq b$. Then

$$a \vee b = b \text{ and } a \wedge b = a.$$

Thus both $a \vee b$ and $a \wedge b$ exist in L. Hence every chain is a lattice.

◈ § 3.2. ALGEBRAIC SYSTEM DEFINED BY A LATTICE

We know that a lattice is a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound. For example Fig. 3.4 below shows a lattice. There is a natural way to define an algebraic system with two operations corresponding to a given lattice.

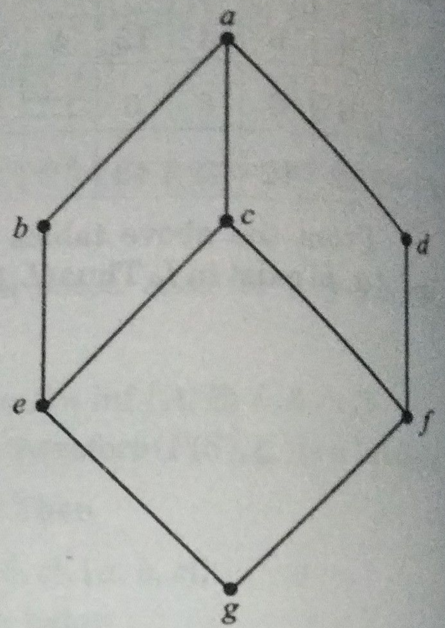


Fig. 3.4

Definition. Let $(L, \leq)$ be a lattice. Let $\vee$ and $\wedge$ be two binary operations on L such that for $a, b \in L$,

$$a \vee b = \text{least upper bound of } a \text{ and } b$$

and

$$a \wedge b = \text{greatest lower bound of } a \text{ and } b.$$

Then the algebraic system $(L, \vee, \wedge)$ is called the **algebraic system** defined by the lattice $(L, \leq)$.

For example, the algebraic system defined by lattice in Fig. 19 is shown in Fig. 3.5 below :

$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	d	e	c	e
f	a	a	c	d	e	f	f
g	a	b	c	d	e	f	g

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	c	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

Fig. 3.5

Again, by Theorem 1 above, we have

$$a \wedge c \le a \text{ and } a \wedge c \le c. \qquad \text{...(9)}$$

Now, $\quad a \wedge c \le a$ and $a \le b \Rightarrow a \wedge c \le b.$

Also, $\quad a \wedge c \le c$ and $c \le d \Rightarrow a \wedge c \le d.$

Thus, $a \wedge c$ is a lower bound of b and d.

Since $b \wedge d$ is the greatest lower bound of b and d, we have

$$a \wedge c \le b \wedge d.$$

§ 3.3. PRINCIPLE OF DUALITY

Let $(L, \le)$ be a partially ordered set. Let $\le_R$ be a binary relation on L such that for a and b in L,

"$a \le_R b$ if and only if $b \le a$".

We can easily show that $(L, \le_R)$ is also a partially ordered set. Furthermore, if $(L, \le)$ is a lattice then $(L, \le_R)$ is also a lattice. It follows that the lattices $(L, \le)$ and $(L, \le_R)$ are closely related and so are the algebraic systems defined by them. We note that the join operation of the algebraic system defined by $(L, \le)$ is the meet operation of the algebraic system defined by $(L, \le_R)$, and the meet operation of the algebraic system defined by $(L, \le)$ is the join operation of the algebraic system defined by $(L, \le_R)$. Furthermore, it is clear that

"$a \le_R b$ if and only if $a \ge b$".

Principle of Duality for Lattices

Statement. Let $(L, \le)$ be a lattices. For any given valid statement concerning the general properties of lattices we can obtain another valid statement by replacing the relation $\le$ with $\ge$, the term join operation with the term meet operation, and the term meet operation with the term join operation. This is known as the *principle of duality for lattices*.

Dual Statement

Definition. Let $(L, \le)$ be a lattice and let it defines the algebraic system $(L, \vee, \wedge)$. Then the dual of any statement in the algebraic system $(L, \vee, \wedge)$ is defined to be the statement that is obtained by interchanging $\vee$ and $\wedge$. For example, the dual of the statement

$$a \wedge (b \vee a) = a \vee a \text{ is } a \vee (b \wedge a) = a \wedge a.$$

§ 3.4. DUAL LATTICE

Definition. Let $(L, \le)$ be a poset. For any $a, b \in L$, the converse of the relation $\le$, denoted by $\ge$, is defined as

$$a \ge b \Leftrightarrow b \le a.$$

Then $(L, \ge)$ is also a poset called **dual poset** of $(L, \le)$. It may be observe that if $(L, \le)$ is a lattice then $(L, \ge)$ is also a lattice. For if $a, b \in L$ then least upper bound of a and b in $(L, \le)$ exist. Further, the greatest lower bound of a and b in $(L, \ge)$ is equal to the least upper bound of a and b in $(L, \le)$. Hence the greatest lower bound of a and b exists in $(L, \ge)$. Similarly the greatest lower bound of a and b in $(L, \le)$ is equal to the least upper bound of a and b in $(L, \ge)$.

§ 3.5. SOME PROPERTIES OF LATTICES

Theorem 1. (Principle of Duality). *The dual of any theorem in a lattice is also a theorem.*

Proof. The proof of this theorem follows from the fact that the dual theorem can be proved by using the dual of each step of the proof of original theorem.

In this section, we shall discuss some of the basic properties possessed by algebraic systems defined by lattices. Let $(L, \vee, \wedge)$ be an algebraic system defined by a lattice $(L, \leq)$.

Theorem 2. (Commutative Law). *Both the join and meet operations are commutative.*

Or

Let $(L, \vee, \wedge)$ be an algebraic system defined by a lattice $(L, \leq)$. Then, for a, b in L,

(i) $a \vee b = b \vee a$ *and* (ii) $a \wedge b = b \wedge a$.

Proof. (i) From the definition of the least upper bound of two elements in the lattice $(L, \leq)$, we have

$$\begin{aligned} a \vee b &= \text{lub } \{a, b\} \\ &= \text{lub } \{b, a\} \\ &= b \vee a. \end{aligned}$$

(ii) Next, from the definition of the greatest lower bound of two elements in the lattice $(L, \leq)$, we have

$$\begin{aligned} a \wedge b &= \text{glb } \{a, b\} \\ &= \text{glb } \{b, a\} \\ &= b \wedge a. \end{aligned}$$

Theorem 3. (Associative Law). *Both the join and meet operations are associative.*

Or

Let $(L, \vee, \wedge)$ be an algebraic system defined by a lattice $(L, \leq)$. Then, for a, b, c in L,

(i) $a \vee (b \vee c) = (a \vee b) \vee c$ *and* (ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Or

State and prove associative law for a lattice.

Proof. (i) Let $a \vee (b \vee c) = g$ and $(a \vee b) \vee c = h$.

Now, $g = a \vee (b \vee c) \Rightarrow g = \text{lub } \{a, b \vee c\}$

$\Rightarrow g$ is an upper of a and $b \vee c$

$\Rightarrow a \leq g$ and $b \vee c \leq g$.

Furthermore,

$b \vee c \leq g \Rightarrow \text{lub } \{b, c\} \leq g$

$\Rightarrow g$ is an upper bound of b and c

$\Rightarrow b \leq g$ and $c \leq g$.

Now, $a \leq g$ and $b \leq g \Rightarrow g$ is an upper bound of a and b. ...(1)

Since the join of a and b is the least upper bound of a and b, it follows from (1) that

$$a \vee b \leq g.$$

Again, $a \vee b \leq g$ and $c \leq g \Rightarrow g$ is an upper bound of $a \vee b$ and c

$$\Rightarrow (a \vee b) \vee c \leq g$$

$$\Rightarrow h \leq g. \quad ...(2)$$

In a similar manner we can show that

$$g \leq h. \quad ...(3)$$

Now, by the antisymmetry property of a partial ordering relation, we have

$$h \leq g \text{ and } g \leq h \Rightarrow g = h.$$

Therefore, we conclude that

$$a \vee (b \vee c) = (a \vee b) \vee c.$$

(ii) According to the principle of duality, the meet operation $\wedge$ is also associative; *i.e.*,

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

Theorem 4. (Idempotent Law). *Let* $(L, \leq)$ *be a lattice. For every* a *in* L,

(*i*) $a \vee a = a$ *and* (*ii*) $a \wedge a = a$.

Proof. (i) For any a, b in L, we have

$$a \leq a \vee b. \quad ...(1)$$

Taking $b = a$ in (1), we obtain

$$a \leq a \vee a. \quad ...(2)$$

Since the partial order relation $\leq$ is reflexive, we have

$a \leq a$

$\Rightarrow$ a is an upper bound of a and a

$\Rightarrow$ lub $\{a, a\} \leq a$

$\Rightarrow$ $a \vee a \leq a$. ...(3)

Now, by antisymmetry property of partial order relation $\leq$, we have

$$a \leq a \vee a, \; a \vee a \leq a \Rightarrow a \vee a = a.$$

Therefore, for every a in L, we have

$$a \vee a = a. \quad ...(4)$$

(ii) According to the principle of duality, we also have

$$a \wedge a = a. \quad ...(5)$$

The results in Theorem 3 are known as the *idempotent property* of the join and meet operations.

Theorem 5. (Absorption Law). *Let* $(L, \leq)$ *be a lattice. For any* a *and* b *in* L,

(*i*) $a \vee (a \wedge b) = a$ *and* (*ii*) $a \wedge (a \vee b) = a$.

Proof. (i) Since $a \vee (a \wedge b)$ is the join of a and $a \wedge b$, we have

$a \vee (a \wedge b) = \text{lub } \{a, a \wedge b\}$

$\Rightarrow$ $a \vee (a \wedge b)$ is an upper bound of a and $a \wedge b$ ...(1)

$\Rightarrow$ $a \leq a \vee (a \wedge b)$.

Since the partial ordering relation $\leq$ is reflexive, we have

$$a \leq a. \quad \text{...(2)}$$

We have

$$a \wedge b \leq a. \quad \text{...(3)}$$

Since

$$a \leq a \text{ and } a \wedge b \leq a$$

We have

$$a \vee (a \wedge b) \leq a \vee a$$

$$\Rightarrow \quad a \vee (a \wedge b) \leq a \qquad [\because a \vee a = a] \quad \text{...(4)}$$

Combining (1) and (4), and noting the fact that partial ordering relation $\leq$ is antisymmetric, we have

$$a \vee (a \wedge b) = a. \quad \text{...(5)}$$

(ii) According to the principle of duality, it follows from (5) that

$$a \wedge (a \vee b) = a. \quad \text{...(6)}$$

The results in Theorem 4 are known as the *absorption property* of the join and meet operations.

Theorem 6. *Let $(L, \leq)$ be a lattice and let $\wedge$ and $\vee$ denote the operations of meet and join in L. Then for any $a, b \in L$*

(*i*) $a \leq b \Leftrightarrow a \wedge b = a$ (*ii*) $a \leq b \Leftrightarrow a \vee b = b$.

Proof. (i) Suppose $a \wedge b = a$. Since $a \wedge b = \inf\{a, b\}$ it follows that $a \wedge b \leq b$. Now

$$a \wedge b \leq b \Rightarrow a \leq b \qquad [\because a \wedge b = a]$$

Conversely, suppose that $a \leq b$. Since the relation $\leq$ is reflexive, we have

$$a \leq a.$$

Now, $a \leq b$ and $a \leq a \Rightarrow a$ is a lower bound of $\{a, b\}$

$$\Rightarrow a \leq \inf\{a, b\} = a \wedge b.$$

Since $a \wedge b$ is infimum of $\{a, b\}$, we have

$$a \wedge b \leq a.$$

Thus, by anti-symmetry of the relation $\leq$, we have

$$a \leq a \wedge b \text{ and } a \wedge b = a \Rightarrow a \wedge b = a.$$

Hence we conclude that

$$a \leq b \Leftrightarrow a \wedge b = a.$$

(ii) Suppose that $a \vee b = b$. Since $a \vee b = \sup\{a, b\}$, we have

$$a \leq a \vee b.$$

Now,

$$a \leq a \vee b \text{ and } a \vee b = b \Rightarrow a \leq b.$$

Conversely, suppose that $a \leq b$. By the reflexivity of the relation $\leq$, we must have $b \leq b$.

Now, $a \leq b, b \leq b \Rightarrow b$ is an upper bound of $\{a, b\}$

$$\Rightarrow \sup\{a, b\} \leq b$$

$$\Rightarrow a \vee b \leq b.$$

But from the definition of $a \vee b = \sup\{a, b\}$ we have $b \leq a \vee b$. By the anti-symmetry of the relation $\leq$, we have

$$a \vee b \leq b \text{ and } b \leq a \vee b \Rightarrow a \vee b = b.$$

Corollary. *Let $(L, \leq)$ be a lattice and $a, b \in L$. Then*

$$a \wedge b = a \Leftrightarrow a \vee b = b.$$

Proof. From Theorem 6 above, we have

$$a \wedge b = a \Leftrightarrow a \leq b$$
$$\Leftrightarrow a \vee b = b.$$

Therefore, $a \wedge b = a$ if and only if $a \vee b = b$.

Theorem 7. *Let $(L, \leq)$ be a lattice and $a, b, c \in L$. Then the following implications hold :*

(*i*) $a \leq b$ and $a \leq c \Rightarrow a \leq b \vee c$

(*ii*) $a \leq b$ and $a \leq c \Rightarrow a \leq b \wedge c$.

Proof. (i) Suppose $a \leq b$ and $a \leq c$. From the definition of join operation in lattice $(L, \leq)$, we have

$$b \vee c = \sup\{b, c\}$$

$\Rightarrow$ $b \vee c$ is an upper bound of $\{b, c\}$

$\Rightarrow$ $b \leq b \vee c$.

Now, by transitivity of the relation $\leq$, we have

$$a \leq b \text{ and } b \leq b \vee c \Rightarrow a \leq b \vee c.$$

(ii) Suppose $a \leq b$ and $a \leq c$. Then

$a \leq b$ and $a \leq c \Rightarrow a$ is a lower bound of $\{b, c\}$

$\Rightarrow a \leq \text{glb}\{b, c\}$

$\Rightarrow a \leq b \wedge c$ $\qquad [\because \text{glb}\{b, c\} = b \wedge c]$

Corollary. *Let $(L, \leq)$ be a lattice and $(L, \geq)$ be its dual. Then for $a, b, c \in L$,*

(*i*) $a \geq b$ *and* $a \geq c \Rightarrow a \geq b \wedge c$

(*ii*) $a \geq b$ *and* $a \geq c \Rightarrow a \geq b \vee c$.

Proof. Applying principle of duality on Theorem 7, we get the results.

Theorem 8. *Let $(L, \leq)$ be a lattice and $a, b, c, d \in L$. Then the following implications hold :*

(*i*) $a \leq b$ *and* $c \leq d \Rightarrow a \vee c \leq b \vee d$

(*ii*) $a \leq b$ *and* $c \leq d \Rightarrow a \wedge c \leq b \wedge d$.

Proof. (i) Suppose that $a \leq b$ and $c \leq d$. By the definition of join operation □ in lattice $(L, \leq)$, we have

$$b \leq b \vee d \text{ and } d \leq b \vee d.$$

Now, by transitivity of the relation $\leq$, we have

$$a \leq b \text{ and } b \leq b \vee d \Rightarrow a \leq b \vee d.$$

Similarly, $\qquad c \leq d$ and $d \leq b \vee d \Rightarrow c \leq b \vee d$.

Now,

$a \leq b \vee d$ and $c \leq b \vee d \Rightarrow b \vee d$ is an upper bound of $\{a, c\}$

$\Rightarrow \text{lub}\{a, c\} \leq b \vee d$

$\Rightarrow a \vee c \leq b \vee d$ $\qquad [\because a \vee c = \text{lub}\{a, c\}]$

(ii) Suppose that $a \leq b$ and $c \leq d$. By the definition of meet operation $\wedge$ in lattice $(L, \leq)$, we have

$$a \wedge c \leq a \text{ and } a \wedge c \leq c.$$

By transitivity of the relation $\leq$, we have

$$a \wedge c \leq a \text{ and } a \leq b \Rightarrow a \wedge c \leq b.$$

Similarly,

$$a \wedge c \leq c \text{ and } \text{ and } c \leq d \Rightarrow a \wedge c \leq d.$$

Now,

$$a \wedge c \leq b \text{ and } a \wedge c \leq d \Rightarrow a \wedge c \text{ is a lower bound of } \{b, d\}$$
$$\Rightarrow a \wedge c \leq \text{glb}\{b, d\}$$
$$\Rightarrow a \wedge c \leq b \wedge d \qquad [\because\ b \wedge d = \text{glb}\{b, d\}]$$

Corollary. *Let $(L, \leq)$ be a lattice. Then for any $a, b, c \in L$, the following implications hold :*

(i) $a \leq b \Rightarrow a \vee c \leq b \vee c$ (ii) $a \leq b \Rightarrow a \wedge c \leq b \wedge c$.

Proof. The proofs of (i) and (ii) follows by taking $d = c$ in Theorem 8 (i) and 5 (ii) above.

Theorem 9. *Let $(L, \leq)$ be a lattice. Then for any $a, b, c \in L$, the following inequalities hold :*

(i) $a \wedge (b \wedge c) \geq (a \wedge b) \vee (a \wedge c)$ (ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

These inequalities are known as distributive inequalities. They are also called **semi-distributive laws.**

Proof. (i) By the definition of meet operation $\wedge$ in lattice $(L, \leq)$, we have

$$a \wedge b = \inf\{a, b\} \leq a$$

and

$$a \wedge b = \inf\{a, b\} \leq b.$$

Further, by the definition of join operation $\vee$ is lattice $(L, \leq)$, we have

$$b \leq \sup\{b, c\} = b \vee c$$

and

$$c \leq \sup\{b, c\} = b \vee c.$$

Also, by transitivity of the relation $\leq$, we have

$$a \wedge b \leq b \text{ and } b \leq b \vee c \Rightarrow a \wedge b \leq b \vee c.$$

Now,

$$a \wedge b \leq a \text{ and } a \wedge b \leq b \vee c \Rightarrow a \wedge b \text{ is a lower bound of } \{a, b \vee c\}$$
$$\Rightarrow a \wedge b \leq \text{glb}\{a, b \vee c\}$$
$$\Rightarrow a \wedge b \leq a \wedge (b \vee c) \quad [\because\ a \wedge (b \vee c) = \text{glb}\{a, b \vee c\}] \qquad ...(1)$$

Again, $a \wedge c \leq a$ and $a \wedge c \leq c \leq b \vee c \Rightarrow a \wedge c \leq a \wedge (b \vee c)$. ...(2)

From (1) and (2), we have

$$a \wedge (b \vee c) \text{ is an upper bound of } \{a \wedge b, a \wedge c\}$$
$$\Rightarrow \text{lub}\{a \wedge b, a \wedge c\} \leq a \wedge (b \vee c)$$
$$\Rightarrow (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c). \qquad ...(3)$$

(ii) Apply the principle of duality, we immediately obtain

$$(a \vee b) \wedge (a \vee c) \geq a \vee (b \wedge c)$$

i.e.,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \qquad ...(4)$$

by interchanging $\vee$ and $\wedge$ and replacing $\leq$ by $\geq$ in (3).

Theorem 10. *Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$, the following holds :*

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

This is known as the **modular inequality.**

Proof. We know that for any $a, b, c \in L$,

$$a \le c \Leftrightarrow a \vee c = c \qquad ...(1)$$

and
$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c). \qquad ...(2)$$

From (1) and (2), we have

$$a \le c \Leftrightarrow a \vee (b \wedge c) \le (a \vee b) \wedge c \qquad [\because a \vee c = c]$$

ILLUSTRATIVE EXAMPLES

Example 1. *Let* N *denote the set natural numbers. For any* $a, b \in$ N, *show that*

$$max\,\{a, min\,\{a, b\}\} = a$$

and
$$min\,\{a, max\,\{a, b\}\} = a.$$

Solution. We know that (N, $\le$) is a lattice where N is the set of natural numbers and $\le$ is the "less than or equal to" relation on N. In this lattice, the operations join $\vee$ and meet $\wedge$ are given by

$$a \vee b = \max\,\{a, b\}$$

and
$$a \wedge b = \min\,\{a, b\}.$$

Now applying absorption properties

$$a \wedge (a \vee b) = a \qquad ...(1)$$

and
$$a \vee (a \wedge b) = a \qquad ...(2)$$

In this lattice, we get

$$\begin{aligned} \min\,\{a, \max\,\{a, b\}\} &= \min\,\{a, a \vee b\} \\ &= a \wedge (a \vee b) \\ &= a \qquad [\text{by (1)}] \end{aligned}$$

and
$$\begin{aligned} \max\,\{a, \min\,\{a, b\}\} &= \max\,\{a, a \wedge b\} \\ &= a \vee (a \wedge b) \\ &= a \qquad [\text{by (2)}] \end{aligned}$$

for any positive integers a and b.

Example 2. *For any positive integers a and b, show that*

$$lcm\,\{a, gcd\,(a, b)) = a$$

and
$$gcd\,(a, lcm\,(a, b)) = a.$$

Solution. We know that (N, $\le$) is a lattice where N is the set of natural numbers and the relation $\le$ is defined as

$$a \le b \text{ if and only if } a \mid b.$$

In this lattice, the operations join $\vee$ and meet $\wedge$ are given by

$$a \vee b = \text{lcm}\,(a, b)$$

and
$$a \wedge b = \text{gcd}\,(a, b).$$

Now applying absorption laws

$$a \wedge (a \vee b) = a \qquad ...(1)$$

and
$$a \vee (a \wedge b) = a \qquad ...(2)$$