Homework 2

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Problem 1. *Let J be a non-empty index set and let $\{A_j : j \in J\}$ be an indexed family of sets and suppose B is a set. Prove that*

$$B \setminus \Big[\bigcap_{j \in J} A_j\Big] = \bigcup_{j \in J} (B \setminus A_j)$$

Proof. In order to prove that $B \setminus [\cap_{j \in J} A_j] = \cup_{j \in J} (B \setminus A_j)$, we must show that:

(i) $B \setminus \left[\bigcap_{j \in J} A_j\right] \subseteq \bigcup_{j \in J} (B \setminus A_j)$

(ii) $\bigcup_{j \in J} (B \setminus A_j) \subseteq B \setminus \left[\bigcap_{j \in J} A_j\right]$

We will begin with (i). Assume that $B \setminus [\cap_{j \in J} A_j]$ is nonempty because otherwise (i) is vacuously true. Therefore, let $x \in B \setminus [\cap_{j \in J} A_j]$. From the complement, we have that $x \in B$ and $x \notin [\cap_{j \in J} A_j]$. Thus, it is not true that for all $j \in J$, we have $x \in A_j$. Consequently, there must exist a $j \in J$ such that $x \notin A_j$. Because $x \in B$ and $x \notin A_j$, we have that $x \in B \setminus A_j$. In other words, there exists a $j \in J$ such that $x \in B \setminus A_j$. Therefore, it follows that $x \in \cup_{j \in J} (B \setminus A_j)$. It now follows that $B \setminus [\cap_{j \in J} A_j] \subseteq \cup_{j \in J} (B \setminus A_j)$.

We will now handle (ii). Assume that $\cup_{j \in J} (B \setminus A_j)$ is nonempty because otherwise (ii) is vacuously true. Therefore let $x \in \cup_{j \in J} (B \setminus A_j)$. Therefore, there exists a $j \in J$ such that $x \in B$ and $x \notin A_j$. Thus, $x \in B$. Moreover, it is not true that for all $j \in J$, we have that $x \in A_j$. Therefore, $x \notin \cap_{j \in J} A_j$. From the definition of set complement, we now have that $x \in B \setminus [\cap_{j \in J} A_j]$. It now follows that $\cup_{j \in J} (B \setminus A_j) \subseteq B \setminus [\cap_{j \in J} A_j]$. Because we have proven (i) and (ii), it follows that $B \setminus [\cap_{j \in J} A_j] = \cup_{j \in J} (B \setminus A_j)$.

□

Problem 2. *Prove or give a counterexample:*

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Proof. In order to prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, we must show that:

(i) $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

(ii) $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

We will begin with (i). Assume that $A \times (B \cap C)$ is nonempty because otherwise (i) is vacuously true. Therefore, let the ordered pair $(x, y) \in A \times (B \cap C)$. From the definition of Cartesian product, we have that $x \in A$ and $y \in B \cap C$. From the definition of intersection, we know that $y \in B$ and $y \in C$. Because $x \in A$ and $y \in B$, it follows that $(x, y) \in A \times B$. Similarly, because $x \in A$ and $y \in C$, we have that $(x, y) \in A \times C$. As both $x \in A \times B$ and $x \in A \times C$, combining both, we have $x \in (A \times B) \cap (A \times C)$. Therefore, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

We will now handle (ii). Assume that $(A \times B) \cap (A \times C)$ is nonempty because otherwise (ii) is vacuously true. Therefore, let the ordered pair $(x, y) \in (A \times B) \cap (A \times C)$. From the intersection, we have that $(x, y) \in A \times B$ and $(x, y) \in A \times C$. From the former, we get that $x \in A$ and $y \in B$. From the latter, we get that $x \in A$ and $y \in C$. Because $y \in B$ and $y \in C$, we know that $y \in B \cap C$. As $x \in A$ and $y \in B \cap C$, it follows that $x \in A \times (B \cap C)$. Thus, we have that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Because we have proven (i) and (ii), it now follows that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$