Homework 1

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Problem 1. *Let A, B and C be subsets of a universal set U. Prove that*

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C).$$

Proof. To show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$, we must show that:

(i) $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$

(ii) $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$

We will begin with (i). Assume that $(A \cup B) \setminus C$ is not empty in which case (i) is vacuously true. Let $x \in (A \cup B) \setminus C$. From the complement, we have that $x \in A \cup B$ and $x \notin C$. Therefore, $x \in A$ or $x \in B$. We will consider two cases.

1. Assume that $x \in A$. Note that from before we have that $x \notin C$. Because $x \in A$ and $x \notin C$, we have that $x \in A \setminus C$.

2. Assume that $x \in B$. Note that from before we have that $x \notin C$. Because $x \in B$ and $x \notin C$, we have that $x \in B \setminus C$.

Therefore, we have that $x \in A \setminus C$ or $x \in B \setminus C$. Thus, $x \in (A \setminus C) \cup (B \setminus C)$. It now follows that $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

We will now handle (ii). Assume that $(A \setminus C) \cup (B \setminus C)$ is not empty in which case (ii) is vacuously true. Let $x \in (A \setminus C) \cup (B \setminus C)$. From the union, we have that $x \in A \setminus C$ or $x \in B \setminus C$. We will consider two cases.

1. Assume that $x \in A \setminus C$. From the complement, we have that $x \in A$ and $x \notin C$.

2. Assume that $x \in B \setminus C$. From the complement, we have that $x \in B$ and $x \notin C$.

First, note that from the two cases $x \in A$ or $x \in B$. Therefore, $x \in A \cup B$. Second, note that in both cases $x \notin C$. Because $x \in A \cup B$ and $x \notin C$, it follows that $x \in (A \cup B) \setminus C$. Thus, $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$. Because, we have now proven both (i) and (ii), it follows that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$. ☐

$U \setminus (A \setminus B) = (U \setminus A) \cup B$.

Problem 2. *Let A and B be subsets of a universal set U. Prove*

$$U \setminus (A \setminus B) = (U \setminus A) \cup B.$$

Proof. To show that $U \setminus (A \setminus B) = (U \setminus A) \cup B$, we must show that:

(i) $U \setminus (A \setminus B) \subseteq (U \setminus A) \cup B$.

(ii) $(U \setminus A) \cup B \subseteq U \setminus (A \setminus B)$.

We will begin with (i). Assume that $U \setminus (A \setminus B)$ is not empty which case (i) is vacuously true. Let $x \in U \setminus (A \setminus B)$. From the complement, we have that $x \in U$ and $x \notin A \setminus B$. From the complement again, we have that it is not true that $x \in A$ and $x \notin B$. By DeMorgan's Law, we now have that $x \notin A$ or $x \in B$. We will consider two cases.

1. Assume that $x \notin A$. We already know that $x \in U$. From the definition of set complement, we have that $x \in U \setminus A$.

2. In this case, we assume that $x \in B$ and that will suffice.

From the two cases, we see that $x \in U \setminus A$ or $x \in B$. Therefore, $x \in (U \setminus A) \cup B$. Therefore, it now follows that $U \setminus (A \setminus B) \subseteq (U \setminus A) \cup B$.

We will now handle (ii). Assume that $(U \setminus A) \cup B$ is not empty in which case (ii) is vacuously true. Let $x \in (U \setminus A) \cup B$. From the union, we have that $x \in U \setminus A$ or $x \in B$. We will consider two cases.

1. Assume that $x \in U \setminus A$. From the complement, we have that $x \in U$ and $x \notin A$. Consider the set $A \setminus B$. For any element $y \in A \setminus B$, we have that $y \in A$ and $y \notin B$. Therefore, because $x \notin A$, it follows that $x \notin A \setminus B$.

2. Assume that $x \in B$. Consider the set $A \setminus B$. For any element $y \in A \setminus B$, we have that $y \in A$ and $y \notin B$. Therefore, because $x \in B$, it follows that $x \notin A \setminus B$. Because $B \subset U$, also follows that $x \in U$.

Note that in both cases, we have $x \in U$ and $x \notin A \setminus B$. Thus, $x \in U \setminus (A \setminus B)$. Thus, $(U \setminus A) \cup B \subseteq U \setminus (A \setminus B)$. Because, we have now proven both (i) and (ii), $U \setminus (A \setminus B) = (U \setminus A) \cup B$. □