Conditional Probability Example Problems

[NOTE-all these examples are done via tree diagrams-I prefer 2x2 tables.]

A particular test correctly identifies those with a certain serious disease 94% of the time and correctly diagnoses those without the disease 98% of the time. A friend has just informed you that he has received a positive result and asks for your advice about how to interpret these probabilities. He knows nothing about probability, but he feels that because the test is quite accurate, the probability that he does have the disease is quite high, likely in the 95% range. You want to use your knowledge of probability to address your friend's concerns. What is the probability your friend actually has the disease? We'll tackle this problem a little later using Bayes' Theorem. Right now, let's focus our attention on ideas that lead us to Bayes' Theorem. Specifically, we'll look at conditional probability and the multiplication rule for two dependent events.

The **conditional probability** of an event *B* in relationship to an event *A* is the probability that event *B* occurs after event *A* has already occurred.

We denote "probability of event *B* given event *A* has occurred" by: $P\left(B \mid A\right)$

Multiplication Rule (two dependent events):

$$P\left(A \text{ and } B\right) = P\left(A\right)\cdot P\left(B \mid A\right) = P\left(B \text{ and } A\right) = P\left(B\right)\cdot P\left(A \mid B\right)$$

The multiplication rule gives us a method for finding the probability that events *A* and *B* both occur, as illustrated by the next two examples.

Example 1:
In a class with 3/5 women and 2/5 men, 25% of the women are business majors. Find the probability that a student chosen from the class at random is a female business major.

Define the relevant events: W = the student is a woman
B = the student is a business major

Express the given information and question in probability notation:

"class with 3/5 women" $\Rightarrow P\left(W\right) = 3/5 = 0.60$

"25% of the women are business majors" is the same as saying "the probability a student is a business major, given the student is a woman is 0.25" $\Rightarrow P\left(B \mid W\right) = 0.25$

"probability that a student chosen from the class at random is a female business major" is the same as saying "probability student is a woman and a business major" $\Rightarrow P\left(W \text{ and } B\right)$

Use the multiplication rule to answer the question:

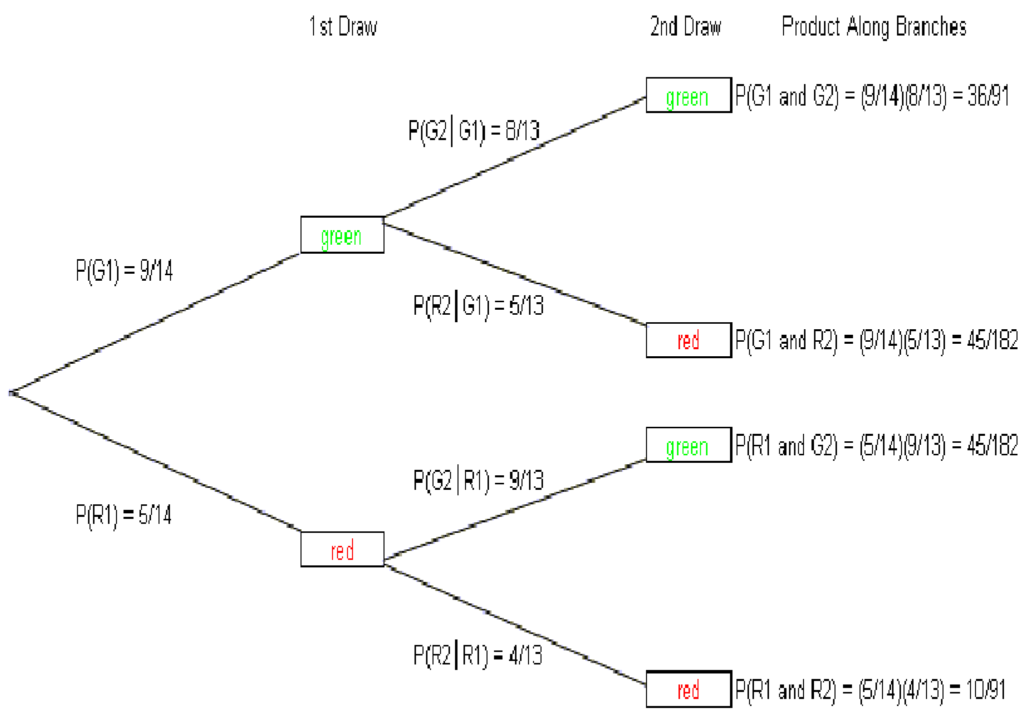
$$P\left(W \text{ and } B\right) = P\left(W\right)\cdot P\left(B \mid W\right) = \left(0.60\right)\left(0.25\right) = 0.15$$

Example 2:
A box contains 5 red balls and 9 green balls. Two balls are drawn in succession **without** replacement. That is, the first ball is selected and its color is noted but it is not replaced, then a second ball is selected. What is the probability that:

a. the first ball is green and the second ball is green?
b. the first ball is green and the second ball is red?
c. the first ball is red and the second ball is green?
d. the first ball is red and the second ball is red?

Solutions:

We will construct a tree diagram to help us answer these questions.



Using the tree diagram, we see that:

a. the probability the first ball is green and the second ball is green = $P(G1 \text{ and } G2) = \dfrac{36}{91}$

b. the probability the first ball is green and the second ball is red = $P(G1 \text{ and } R2) = \dfrac{45}{182}$

c. the probability the first ball is red and the second ball is green = $P(R1 \text{ and } G2) = \dfrac{45}{182}$

d. the probability the first ball is red and the second ball is red = $P(R1 \text{ and } R2) = \dfrac{10}{91}$

Formula for Conditional Probability:
The probability that the second event B occurs given that the first event A has occurred can be found by:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}, \quad \text{where } P(A) \neq 0$$

Note: This formula is obtained from the Multiplication Rule for two dependent events.
(Using algebra, we solve for $P(B|A)$ by dividing both sides of the equation by $P(A)$)

<u>The key to solving conditional probability problems is to</u>:

1. Define the events.
2. Express the given information and question in probability notation.
3. Apply the formula.

<u>Example 3</u>:
The probability that Sam parks in a no-parking zone and gets a parking ticket is 0.06. The probability that Sam has to park in a no-parking zone (he cannot find a legal parking space) is 0.20. Today, Sam arrives at school and has to park in a no-parking zone. What is the probability that he will get a parking ticket?

<u>Solution</u>:

<u>Define the events</u>: N = Sam parks in a no-parking zone, T = Sam gets a parking ticket

<u>Express the given information and question in probability notation</u>:
"probability that Sam parks in a no-parking zone and gets a parking ticket is 0.06"
tells us that $P(N \text{ and } T) = 0.06$.

"probability Sam has to park in the no-parking zone is 0.20" tells us that $P(N) = 0.20$

"Today, Sam arrives at school and has to park in a no-parking zone. What is the probability that he will get a parking ticket?" is the same as "What is the probability he will get a parking ticket, <u>given</u> that he has to park in a no-parking zone" That is, we want to find $P(T|N)$.

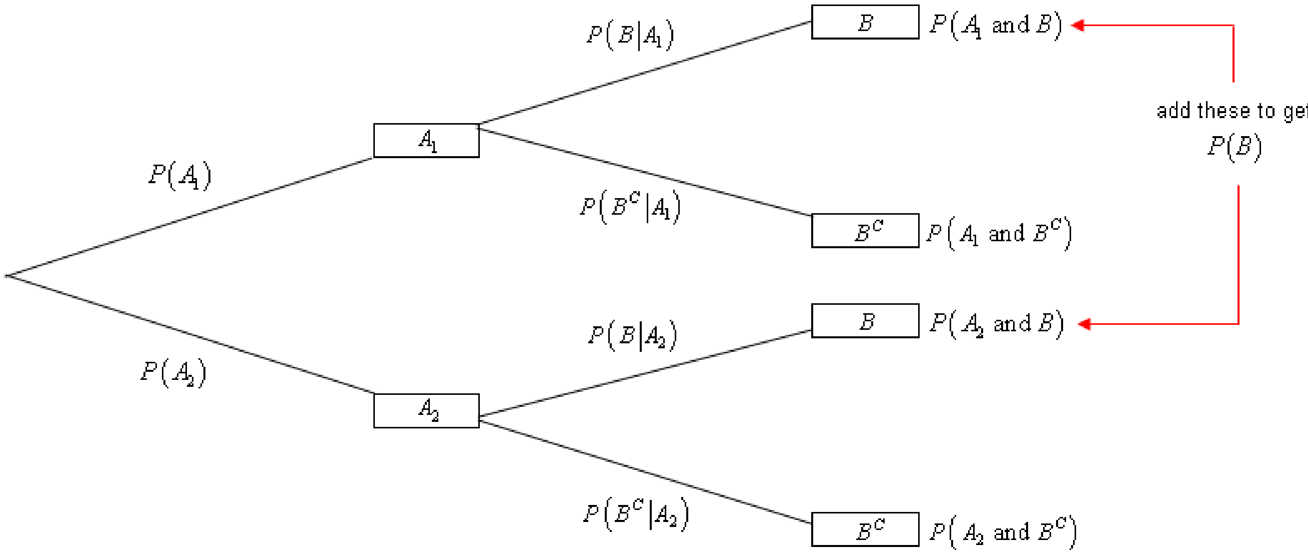
<u>Apply the formula</u>: $P(T|N) = \dfrac{P(N \text{ and } T)}{P(N)} = \dfrac{0.06}{0.20} = 0.30$

<u>Note</u>: Students seem to have difficulties understanding that the question asks us to find $P(T|N)$, not $P(N \text{ and } T)$. They think the answer is 0.06. They fail to consider that Sam could park in a no-parking zone but not receive a ticket. It might be useful to construct a Venn diagram.

<u>The Law of Total Probability</u>:

If A_1 and A_2 are mutually exclusive events with $P(A_1) + P(A_2) = 1$, then for any event B,

$$\begin{aligned} P(B) &= P(A_1 \text{ and } B) + P(A_2 \text{ and } B) \\ &= P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) \end{aligned}$$



Notes: B^C represents the complement of event B

Also, $P(A_1)+P(A_2)=1$, $P(B|A_1)+P(B^C|A_1)=1$, $P(B|A_2)+P(B^C|A_2)=1$,

and $P(A_1 \text{ and } B)+P(A_1 \text{ and } B^C)+P(A_2 \text{ and } B)+P(A_2 \text{ and } B^C)=1$

More generally, if $A_1, A_2, \ldots, A_k$ are mutually exclusive events with $P(A_1)+P(A_2)+\cdots+P(A_k)=1$, then for any event B,

$$P(B) = P(A_1 \text{ and } B)+P(A_2 \text{ and } B)+\cdots+P(A_k \text{ and } B)$$
$$= P(A_1)\cdot P(B|A_1)+P(A_2)\cdot P(B|A_2)+\cdots+P(A_k)\cdot P(B|A_k)$$

Example 4: An automobile dealer has kept records on the customers who visited his showroom. Forty percent of the people who visited his dealership were women. Furthermore, his records show that 37% of the women who visited his dealership purchased an automobile, while 21% of the men who visited his dealership purchased an automobile.

a. What is the probability that a customer entering the showroom will buy an automobile?

b. Suppose a customer visited the showroom and purchased a car. What is the probability that the customer was a woman?

c. Suppose a customer visited the showroom but did not purchase a car. What is the probability that the customer was a man?

Define the events: $A_1 =$ customer is a woman

$A_2 =$ customer is a man

$B =$ customer purchases an automobile

$B^C =$ customer does not purchase an automobile

Express the given information and question in probability notation:

"Forty percent of the people who visited his dealership were women" $\Rightarrow P(A_1)=0.40$

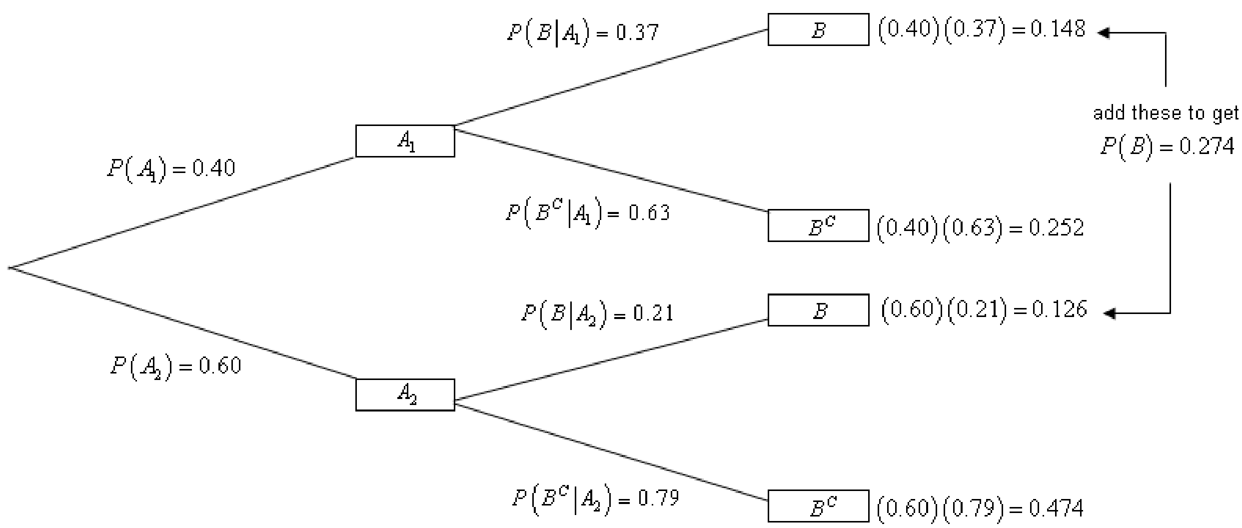
this statement also tells us that 60% of the customers must be men $\Rightarrow P(A_2)=0.60$

"37% of the women who visited his dealership purchased an automobile" $\Rightarrow P(B|A_1) = 0.37$

"21% of the men who visited his dealership purchased an automobile" $\Rightarrow P(B|A_2) = 0.21$

"What is the probability that a customer entering the showroom will buy an automobile?" $\Rightarrow P(B) = ?$

Create a tree diagram:



Use your tree diagram and the Law of Total Probability to answer the question: $P(B) = 0.274$

Solution to part b:

"Suppose a customer visited the showroom and purchased a car. What is the probability that the customer was a woman?"

Express the question in probability notation:

We can rewrite the question as, "What is the probability that the customer was a woman, *given* that the customer purchased an automobile." That is, we want to find $P(A_1|B)$

We can use Bayes' Theorem to help us compute this conditional probability.

Bayes' Theorem (Two-Event Case):

$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2)} = \frac{P(A_1 \text{ and } B)}{P(A_1 \text{ and } B) + P(A_2 \text{ and } B)} = \frac{P(A_1 \text{ and } B)}{P(B)}$$

where A_1 and A_2 are mutually exclusive events with $P(A_1) + P(A_2) = 1$
and B is any event with $P(B) \neq 0$

Note: the denominator is determined by the Law of Total Probability

Solution to part b (continued):

Use Bayes' Theorem and your tree diagram to answer the question:

$$P(A_1|B)=\frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1)+P(A_2)\cdot P(B|A_2)}=\frac{P(A_1\text{ and }B)}{P(A_1\text{ and }B)+P(A_2\text{ and }B)}=\frac{0.148}{0.148+0.126}=\frac{0.148}{0.274}\approx 0.54$$

The probabilities needed for the computation are easily obtained from our tree diagram.
We already found $P(A_1\text{ and }B)+P(A_2\text{ and }B)$, which is $P(B)$, for part a.) of this example and $P(A_1\text{ and }B)$ is obtained by following the tree diagram path $\rightarrow A_1 \rightarrow B$, the product of the corresponding probabilities is 0.148.

Solution to part c:

"Suppose a customer visited the showroom but did not purchase a car. What is the probability that the customer was a man?

Express the question in probability notation:

We can rewrite the question as, "What is the probability that the customer was a man, *given* that the customer did not purchase an automobile." That is, we want to find $P(A_2|B^C)$

Use Bayes' Theorem and your tree diagram to answer the question:

$$P(A_2|B^C)=\frac{P(A_2)\cdot P(B^C|A_2)}{P(A_2)\cdot P(B^C|A_2)+P(A_1)\cdot P(B^C|A_1)}=\frac{0.474}{0.474+0.252}=\frac{0.474}{0.726}\approx 0.653$$

Again, the probabilities needed for the computation are easily obtained from our tree diagram.

Additional Notes:

The probabilities $P(A_1)$ and $P(A_2)$ are called prior probabilities because they are initial or *prior* probability estimates for specific events of interest. When we obtain new information about the events we can update the prior probability values by calculating revised probabilities, referred to as posterior probabilities. The conditional probabilities $P(A_1|B)$, $P(A_2|B)$, $P(A_1|B^C)$, and $P(A_2|B^C)$ are posterior probabilities. Bayes' Theorem enables us to compute these posterior probabilities.

Example 5:

Let's return to the scenario that began our discussion: A particular test correctly identifies those with a certain serious disease 94% of the time and correctly diagnoses those without the disease 98% of the time. A friend has just informed you that he has received a positive result and asks for your advice about how to interpret these probabilities. He knows nothing about probability, but he feels that because the test is quite accurate, the probability that he does have the disease is quite high, likely in the 95% range. Before attempting to address your friend's concern, you research the illness and discover that 4% of men have this disease. What is the probability your friend actually has the disease?

Define the events: $A_1 =$ a man has this disease

$A_2 =$ a man does not have this disease

$B =$ positive test result

$B^C =$ negative test result

Express the given information and question in probability notation:

"test correctly identifies those with a certain serious disease 94% of the time" $\Rightarrow P(B|A_1) = 0.94$

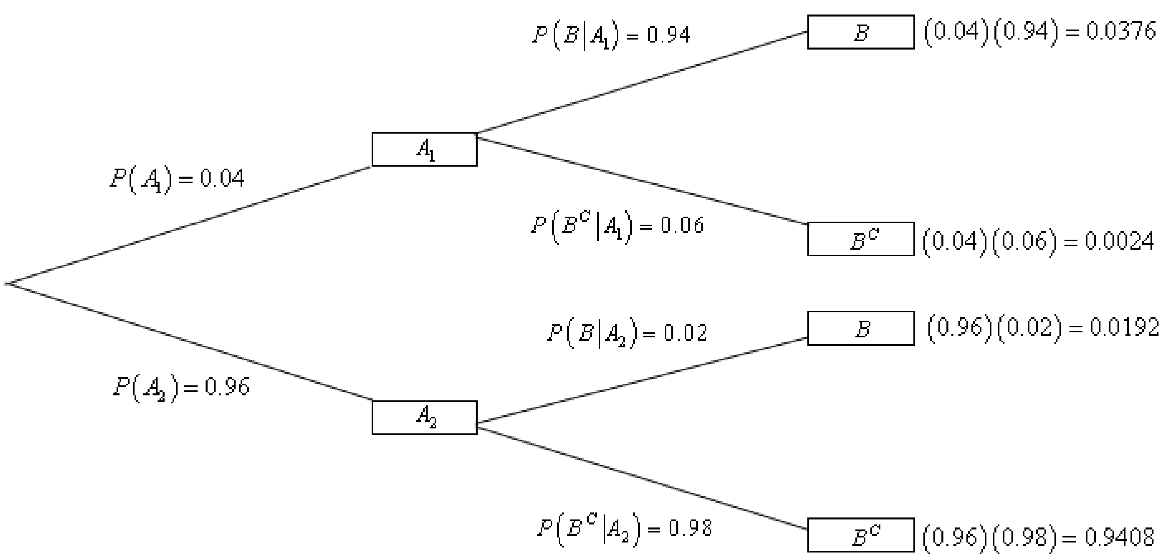
"test correctly diagnoses those without the disease 98% of the time" $\Rightarrow P(B^C|A_2) = 0.98$

"you discover that 4% of men have this disease" $\Rightarrow P(A_1) = 0.04$

this statement also tells us that 96% of men do not have the disease $\Rightarrow P(A_2) = 0.96$

"What is the probability your friend actually has the disease (*given* a positive result)?" $\Rightarrow P(A_1|B) = ?$

Construct a tree diagram:



Use Bayes' Theorem and your tree diagram to answer the question:

$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_2)}{P(A_1)\cdot P(B|A_2)+P(A_2)\cdot P(B|A_2)} = \frac{0.0376}{0.0376+0.0192} \approx 0.662$$

There is a 66.2% probability that he actually has the disease. The probability is high, but considerably lower than your friend feared.