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Stat 104: Quantitative Methods

Class 9: Independence and 2x2 Tables

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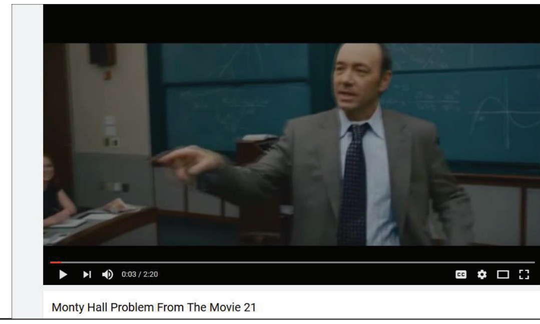
Famous Problem



https://www.youtube.com/watch?v=Zr_xWfThjJ0

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Talking Elmo Example

We're selling Talking Elmo Dolls, and we want to know how effective our ad is. Of the people who saw our ad, what percent bought the product? Of those who didn't see the ad?

		Y		
		Saw Ad	**Didn't See Ad**	
X	**Bought**	0.2	0.1	0.3
	Didn't Buy	0.3	0.4	0.7
		0.5	0.5	1

Is this a legal probability table (why) ?

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Marginal probabilities are also called **unconditional probabilities**.

From the table, what is the probability that someone bought an Elmo doll ? 0.3

The question of interest to Fisher-Price : *is advertising effective* ?

That is, what if we first discover that someone saw an ad for Talking Elmo. What is the chance now that the person bought a doll ?

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We are looking for

P(Bought Elmo|saw ad)

which is read as the probability the consumer buys an Elmo *given* the consumer saw an ad for Elmo.

Ideally we want this probability to be above 30%, indicating that the advertising is effective.

This type of probability is called a **conditional probability** and is always denoted with the vertical bar "|".

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Definition of conditional probability

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Out of all the times B happens, how often does A also happen ?

P(Bought elmo|saw ad) = 0.2/0.5 = 0.4

P(Bought elmo|didn't see ad) = 0.1/0.5 = 0.2

These results indicate that advertising is not working working

Example

- Given you like Sprite, what is the probability that you're a male ?

	Sierra Mist	Sprite
Male	0.4	0.15
Female	0.22	0.23

P(male|sprite) = 0.15/.38=0.39

P(male) = 0.55

Conditioning changed the probability

Independence

One of the objectives of calculating conditional probability is to determine whether two events are related.

In particular, we would like to know if two events are *independent*, that is, if the probability of one event is *not affected* by the occurrence of the other event.

Two events A and B are said to be ***independent*** if

P(A|B) = P(A) or P(B|A) = P(B)

If A and B are not independent, they are called **dependent.**

Note P(A|B)=P(A) is the same as P(A and B) = P(A)P(B)

Does the Joint Factor

- To see if events A and B are independent we check if P(A|B)=P(A)
- This is the same as checking if
  - P(A and B)=P(A)P(B).
- Technically we say the joint factors into the product of the marginals. Phew.

Conditional Probability

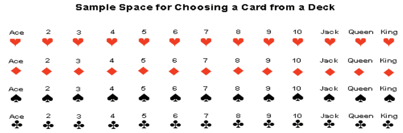
Dependence and independence

- If Pr(A|D)>Pr(A) then A and D are positively related.
- If Pr(A|D)<Pr(A) then A and D are negatively related.
- If Pr(A|D)=Pr(A) then A and D are independent.

Example

- What is the probability you will draw a king, given you have drawn a red card?

Sample Space for Choosing a Card from a Deck



P(King)=4/52=1/13

P(king|red)=P(king and red)/P(red) =(2/52)/(26/52)=1/13

Knowing you drew a red card gives you no new info.

Recall Talking Elmo Example

Since

P(bought | Seen ad) = 40% and P(bought) = 30% we can see that having seen the ad *effects* the probability of buying a Talking Elmo. They are dependent

In fact we can conclude that seeing the ad is effective in increasing sales and would say that the probability of buying a Talking Elmo *depends* on whether or not the customer saw an ad.

Look at Sprite data differently

- Now consider the following organization of our soft drink data

	Sierra Mist	Sprite	
Under 20	0.31	0.19	0.5
Over 20	0.31	0.19	0.5
	0.62	0.38	

Is there a relationship between drink preference and age ?

What about drink preference and gender (previous table)

P(sprite|under 20) = 0.19/.50=~~0.30~~ 0.38
P(sprite) = 0.38
Conclusion?

Activity and Grade Level

- Based on this table, are activity and grade level dependent or independent?

Grade	Snowboarding	Skiing	Ice Skating	
6th	0.12	0.08	0.08	**0.28**
7th	0.15	0.10	0.13	**0.39**
8th	0.11	0.14	0.09	**0.33**
	0.39	**0.31**	**0.30**	

For tables like this, we need to **check independence for each outcome**.

Even if one combination does not suffice -> Dependent

More on Independence

- Independence is a very strong assumption, so <u>be careful and verify</u> it is true before you assume it holds for two different business processes.
- What is the problem in assuming independence ?
- <u>Probabilities usually don't multiply!</u>

Multiplication rule, part 1

- If A and B are independent, then
  P(A |B) = P(A).
- But P(A|B) = P(A and B)/P(B) so this means that **<u>IF A and B are independent</u>,**
  P(A and B) = P(A)P(B).
- That is, you just multiply the marginal probabilities together.

Multiplication Rule for Independent Events

If *E* and *F* are independent events, then

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

This can be generalized to more than two independent events.

More Practice: A few ideas at once

During the 1979 baseball season, Pete Rose of the Cincinnati Reds set a National League record by hitting safely in 44 consecutive games. Assume that Rose is a .300 hitter and that he comes to bat 4 times each game.

If each at-bat is assumed to be an independent event, what probability might reasonably be associated with a hitting streak of that length ?

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Let A_i denote the event "Rose hits safely in the ith game".

Then P(Rose hits safely in 44 consecutive games)=

P(A_1 and A_2 and .... and A_{44})= P(A_1)P(A_2) .... P(A_{44}) (why??)
We now need to find P(A_1). It will be easier to think of the complement of A_1.

$$\begin{aligned} P(A_1) &= 1 - P(\text{Rose does not hit safely in game 1}) \\ &= 1 - P(\text{Rose makes four outs}) \\ &= 1 - (.7)^4 \quad \text{why??} \\ &= 0.76 \end{aligned}$$

So, $P(\text{Rose hits safely in 44 consecutive games}) =$

$(0.76)(0.76)\cdots(0.76) = (0.76)^{44} = 0.0000057$ Yowsa!!

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Warning

- P(A and B) is called a joint probability
- In general, joint probabilities are often difficult to find, so assuming independence can be very helpful, because then P(A and B) = P(A)P(B).
- However, as the next example shows, bad things can happen when one blindly assumes independence.

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Actual court case

People v. Collins, 68 Cal.2d 319 (1968)

- On June 18, 1964, Juanita Brooks was attacked in an alley near her home in LA and her purse stolen
- A witness reported that a woman running from the scene was blond, had a pony tail, dressed in dark clothes and fled from the scene in a yellow car driven by a black man with a beard and mustache.
- Police arrested a couple, Janet and Mark Collins, which fit the description

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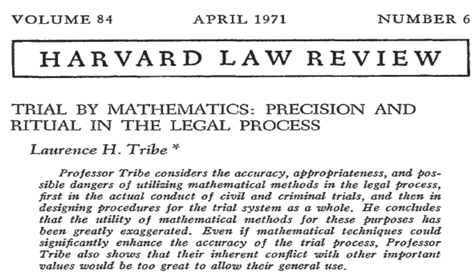
Fantastic Read (on web site)



VOLUME 84 APRIL 1971 NUMBER 6

HARVARD LAW REVIEW

TRIAL BY MATHEMATICS: PRECISION AND RITUAL IN THE LEGAL PROCESS

Laurence H. Tribe *

Professor Tribe considers the accuracy, appropriateness, and possible dangers of utilizing mathematical methods in the legal process, first in the actual conduct of civil and criminal trials, and then in designing procedures for the trial system as a whole. He concludes that the utility of mathematical methods for these purposes has been greatly exaggerated. Even if mathematical techniques could significantly enhance the accuracy of the trial process, Professor Tribe also shows that their inherent conflict with other important values would be too great to allow their general use.

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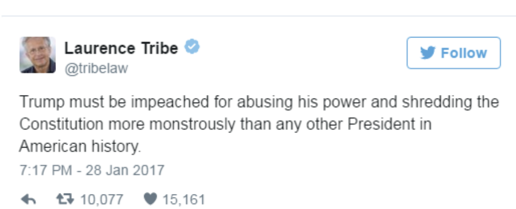
Follow him on twitter



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People vs. Collins

- During 7-day trial, prosecution ran into difficulties
  - Juanita Brooks could not identify either defendant
  - Witness admitted at a preliminary hearing that he was uncertain of his identification of Mr. Collins in a police lineup
- Math instructor at a state college took the stand as an expert witness
- Was asked, "What is the chance that Mr. and Mrs. Collins are innocent given that they match the descriptions of the perpetrators on all six characteristics?"
- **The expert witness testified that the probability of a combination of characteristics, or their *joint probability,* is given by the product of their individual probabilities**

People vs. Collins

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- The prosecution provided the following probabilities
- Prosecution multiplied these probabilities to claim that the probability that a randomly selected couple would have all these characteristics was 1 in 12 million
- Prosecutor concluded that the chance that the defendants were innocent was only 1 in 12 million
- The jury convicted the Collinses of second-degree robbery.

Evidence	Probability
Girl with blonde hair	1/3
Girl with ponytail	1/10
Partly yellow car	1/10
Man with mustache	1/4
Black man with beard	1/10
Interracial couple in car	1/1,000

People vs. Collins

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- Defense appealed and the California Supreme Court reversed the conviction on four grounds:
  - The probabilities lacked "evidentiary foundation." They were merely estimates. subjective probab.
  - **Multiplying the six probabilities assumes independence, for which there is no proof**
  - The prosecutor assumed that the six characteristics were certain

The multiplication rule, part 2

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The multiplication rule is used to calculate the joint probability of two events. It is based on the formula for conditional probability defined earlier:

$$\boxed{P(A \mid B) = \frac{P(A\ and\ B)}{P(B)}}$$

- If we multiply both sides of the equation by P(B) we have: P(A and B) = P(A | B)P(B).
- Likewise, P(A and B) = P(B | A)P(A)

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General Multiplication Rule

The probability that two events *A* and *B* both occur is

$$P(\text{A and B}) = P(A) \cdot P(B|A)$$

Example: Pair of Kings

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- Select two cards from the standard deck of 52 cards without replacement. Find the probability of selecting two kings.

The Secrets of the 2x2 Table

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- Many probability problems can be written in the form of a 2x2 table-once you understand the ins and outs of working with the 2x2, most problems are easy to solve.

	B	$\bar{B}$
A	$P(A \text{ and } B)$ $= P(B)P(A \mid B)$	$P(A \text{ and } \bar{B})$ $= P(\bar{B})P(A \mid \bar{B})$
$\bar{A}$	$P(\bar{A} \text{ and } B)$ $= P(\bar{A})P(B \mid \bar{A})$	$P(\bar{A} \text{ and } \bar{B})$ $= P(\bar{A})P(\bar{B} \mid \bar{A})$

$$P(A) = P(A \text{ and } B) + P(A \text{ and } \bar{B})$$
$$= P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})$$

Called the rule of total probability

Example:

Suppose an applicant for a job has been invited for an interview.

The chance that

•He is nervous is $P(N) = 0.7$

•The interview is successful if he is nervous $P(S \mid N) = 0.2$

•The interview is successful if he is not nervous $P(S \mid \bar{N}) = 0.9$

What is the probability the interview is successful ? P(S)

Solution

- Always Draw a Table!

	N	N'	
S	0.2*0.7=0.14	0.9*0.3=0.27	0.41
S'	0.56	0.03	0.59
	0.7	0.3	

P(S) = 0.41

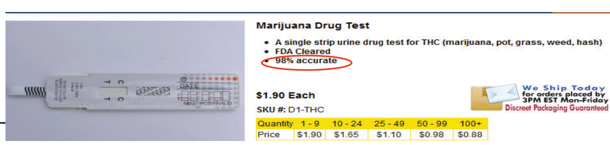
Know what you are conditioning on

- One important point about conditional probability is that (usually) P(A|B) **does not equal** P(B|A).
- Some books make a big deal out of this but its not that fancy a concept.
- The usual example concerns drug testing; it's a good example to study since it focuses us on why its important to understand what one is conditioning on.

Urine or You're Out

- Suppose the Incredibly Rapid Rooter Plumbing Company wants to drug test their employees.
- They purchase a test with the following characteristic :

P(test + | actual drug user) = 98%.



What do we want to condition on?

- You always condition on *what you know*, so for an employer's legal department, they need to know
  P(actual drug user | test +).
- That is, what is the chance the test will be correct ?
- Just because P(A|B) is near 100% doesn't imply P(B|A) has to be near 100%.
- That is, P(test+|Druggie)=.98 doesn't mean P(Druggie|test+) = 0.98

Important point

Some More Assumptions

- Suppose also that P(test - | non drug user) = 90%
- Suppose also that Rapid Rooter also has good reason to suspect that 10% of their employees are drug users. [P(drug user) = 10%]
- Using the multiplication rule we can create the usual 2x2 table.

P(test+ I drug user) = 0.98

Solution

- The 2x2 table

	Test +	Test -	
Drug User	0.98*0.1=0.098	0.002	0.1
Not Drug User	0.09	0.9*0.9=0.81	0.9
	0.188	0.812	

Calculate

- From the table, calculate P(drug user | test +)

0.098/0.188 = 0.52

Not a good test. If you test +ve, it's 50:50 that you're a drug user. Not helpful.

- What is the interpretation of this probability ?
- Will the lawyers be happy ?

Let's Make a Deal (Monty Hall)

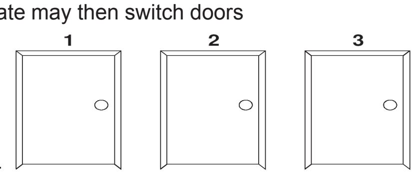
The Rules:

•Three doors – one prize, two blanks

•Candidate selects one door

•Showmaster reveals one losing door

•Candidate may then switch doors



Should you switch ?

Events of interest:

•A – choose winning door at the beginning

•W – win the prize

Strategy : Switch doors (S)

Know : $P_S(W \mid A) = 0 \quad P_S(A) = 1/3$

$P_S(W \mid \bar{A}) = 1 \quad P_S(\bar{A}) = 2/3$

Find $P_S(W)$

- Always Draw a Table!

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Events of interest:

- •A – choose winning door at the beginning
- •W – win the prize

$\boxed{\text{Strategy : Do not switch doors (N)}}$

Know : $P_N(W \mid A) = 1 \quad P_N(A) = \frac{1}{3}$

$P_N(W \mid \bar{A}) = 0 \quad P_N(\bar{A}) = \frac{2}{3}$

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Find $P_N(W)$

- Always Draw a Table!

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So the best strategy is to (choose one)

- Always Switch
- Never Switch

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According to the *New York Times,* this problem and [Marilyn vos Savant's] solution were 'debated in the halls of the Central Intelligence Agency and the barracks of fighter pilots in the Persian Gulf' and 'analyzed by mathematicians at the Massachusetts Institute of Technology and computer programmers at Los Alamos National Laboratory in New Mexico.'"

Most people (including some mathematicians) think it doesn't matter

- "You blew it, and you blew it big! I'll explain: After the host reveals a goat, you now have one-in-two chance of being correct. Whether you change your answer or not, the odds are the same...Shame!"

Scott Smith, Ph.D., University of Florida
(*Parade*, 12/2/90)

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Actually, many of Dr. Smith's professional colleagues are sympathetic. **Persi Diaconis**, a former professional magician who was a Harvard University professor specializing in probability and statistics, said there was no disgrace in getting this one wrong.

"I can't remember what my first reaction to it was," he said, "because I've known about it for so many years. I'm one of the many people who have written papers about it. But I do know that my first reaction has been wrong time after time on similar problems. ***Our brains are just not wired to do probability problems very well, so I'm not surprised there were mistakes."***

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Summary of Probability Rules

For events A and B,

Rule 1: $P(A^c) = 1 - P(A)$ or $P(A) + P(A^c) = 1$

Rule 2: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

If A and B are *mutually exclusive,*

$$P(A \text{ or } B) = P(A) + P(B)$$

Rule 3: $P(A \text{ and } B) = P(A)P(B \mid A) = P(B)P(A \mid B)$

If A and B are *independent,*

$$P(A \text{ and } B) = P(A)P(B)$$

Rule 4:

$$P(A \mid \mathbf{B}) = \frac{P(A \text{ and } B)}{P(\mathbf{B})} \quad \text{or} \quad P(B \mid \mathbf{A}) = \frac{P(A \text{ and } B)}{P(\mathbf{A})}$$

Things you should know

- ❑What is probability
- ❑Random experiment, events
- ❑Joint Probability, Marginal Probability
- ❑Addition rule
- ❑Conditional Probability, Independence
- ❑Multiplication Rules
- ❑Independence
- ❑Creating 2x2 tables