# Wave - Particle Duality | Quantum Mechanics

From Einstein's Special Theory of Relativity, we get $\boxed{\mathcal{E} = mc^2}$.

$\mathcal{E} \to$ Total Energy, $m \to$ mass, $c \to$ Speed of light (universal constant)

Further $\boxed{\mathcal{E} = \mathcal{E}_k + m_0 c^2}$, where

$\mathcal{E}_k \to$ Kinetic energy, $m_0 \to$ rest mass.

Relation between $m$ and $m_0$ $\Rightarrow$ $\boxed{m = \dfrac{m_0}{\sqrt{1-(v/c)^2}}}$

$\boxed{m_0 c^2 \to \text{Rest energy}} \to \Rightarrow \boxed{\mathcal{E} = mc^2 = \dfrac{m_0 c^2}{\sqrt{1-(v/c)^2}}}$

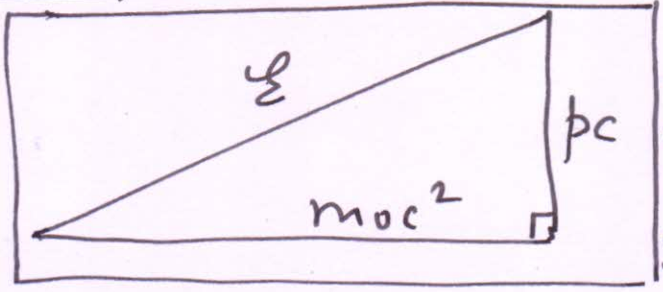
Writing $\boxed{\gamma = \dfrac{1}{\sqrt{1-(v/c)^2}}}$, we write

$\boxed{\mathcal{E} = \gamma m_0 c^2} \to$ Total energy of a particle.

Momentum $\to \boxed{p = mv} \Rightarrow \boxed{p = \gamma m_0 v}$.

The Energy-Momentum Equation:

$$\boxed{\mathcal{E}^2 = (pc)^2 + (m_0 c^2)^2}$$




A Pythagorean Relationship

P.T.O.

For a "massless" particle $\boxed{m_0 = 0}$.

$\Rightarrow \boxed{\mathcal{E}^2 = (pc)^2} \Rightarrow \boxed{\mathcal{E} = pc} \Rightarrow \boxed{\frac{pc}{\mathcal{E}} = 1}$

Now $\boxed{p = mv}$ and $\boxed{\mathcal{E} = mc^2}$

$\therefore \frac{pc}{\mathcal{E}} = \frac{mvc}{mc^2} = 1 \Rightarrow \frac{v}{c} = 1 \Rightarrow \boxed{v = c}$

$\Rightarrow$ Massless particles travel at the speed of light. Since $m_0$ is the rest mass, with $\boxed{m_0 = 0}$, particles can never rest. They travel at the speed of light. Photons (particles of light) have such a property.

Now, for a photon, from Planck's Quantum Theory, $\boxed{\mathcal{E} = h\nu = \frac{hc}{\lambda}}$

Since $\boxed{p = \frac{\mathcal{E}}{c}}$ when $\boxed{m_0 = 0}$, we get $p = \frac{h\nu}{c} = \frac{h\cancel{c}}{\cancel{c}\lambda} \Rightarrow \boxed{p = \frac{h}{\lambda}}$.

$\therefore$ Photon energy $\boxed{\mathcal{E} = h\nu}$, Photon momentum $\boxed{p = h/\lambda}$.

(P.T.O.)

Of the two equations, $\boxed{\mathcal{E} = h\nu}$, $\boxed{p = \frac{h}{\lambda}}$ the left-hand side has $\mathcal{E}$ (energy) and $p$ (momentum), which are particle attributes, while the right-hand side has $\nu$ (frequency) and $\lambda$ (wavelength), which are wave attributes. The bridge between the two is $h$ (Planck's constant). For light $\boxed{\lambda\nu = c}$. With $c$ being a universal constant, knowing $\lambda$ gives $\nu$ and vice-versa. Thereafter both $\mathcal{E}$ and $p$ can be known.

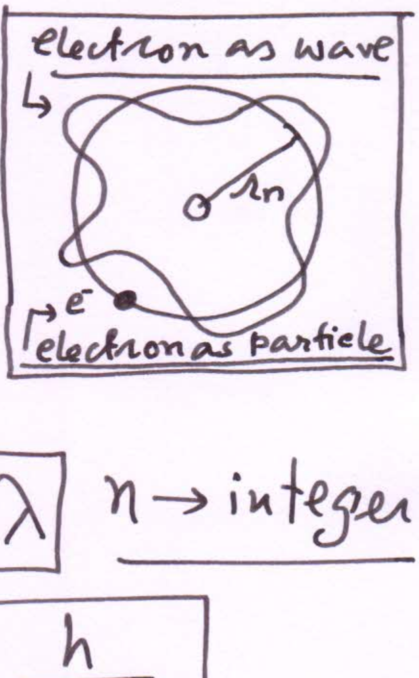
Hence, light, which was seen to be an electromagnetic wave (Maxwell) is now to be seen also as a particle without any rest (Einstein, Planck).

∴ If light waves can have particle-like properties, then material particles (with non-zero rest mass, can have wave-like properties — Wave-Particle Duality (De Broglie)

(P.T.O.)

If the particle has a velocity $v$, then $\boxed{\lambda\nu = v}$. Unlike for light, $v$ is not a universal constant. Hence, just knowing $\lambda$ will not give $\nu$, and vice-versa. At least two quantities out of $\lambda$, $\nu$ and $v$ have to be prescribed, to know $E$ and $p$.

## The Electron as a Wave




The electron is a standing wave around the circumference of a Bohr orbit.

$$\boxed{\text{Circumference} = 2\pi r_n = n\lambda} \quad n \to \text{integer}$$

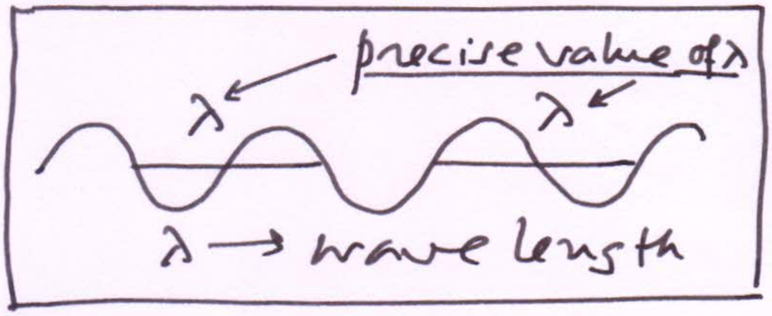
De Broglie: $\boxed{\lambda = \frac{h}{p}} \Rightarrow \boxed{\lambda = \frac{h}{mv}}$

$$\therefore \boxed{2\pi r_n = \frac{nh}{mv}} \Rightarrow \boxed{mvr_n = n\frac{h}{2\pi}}$$

$\Rightarrow \boxed{mvr_n = n\hbar}$ $(\hbar = h/2\pi) \Rightarrow$ Angular momentum $\boxed{mvr_n}$ is an integral multiple of $\hbar$. Bohr's second postulate.

# Wave Formations

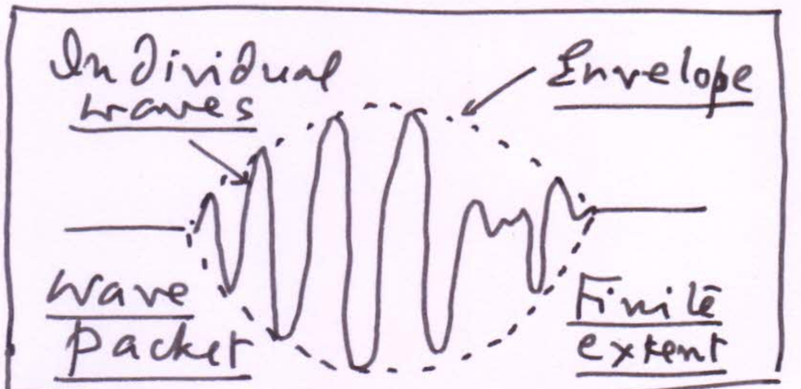
1/. **Wave Train:**




The wavelength is precisely known. Hence, $\boxed{p = h/\lambda}$ ∴ Momentum is also precisely known.

However, the wave is spread over all space up to infinity (fully imprecise)

2/. **Wave Group:**

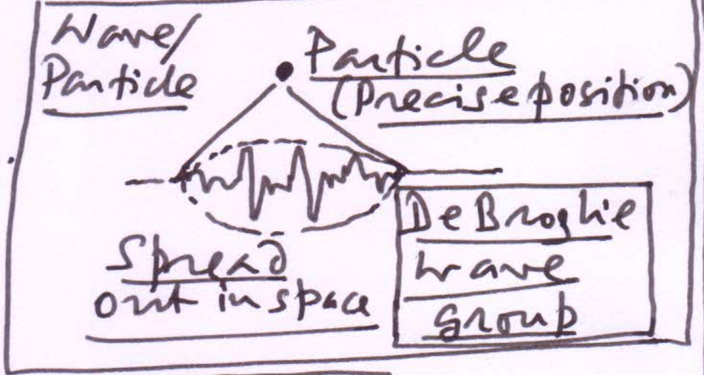



If the wave is gathered into a finite region of space, then it becomes localized into a wave group (wave packet).

The individual waves do not have the same wavelength. Hence there is imprecision in λ and also p. The localization of the wave group, however, makes the wave spread somewhat more precisely known. (no longer infinite).

# The Heisenberg Uncertainty Principle

1/. A point-like particle has a precise location.




2/. By the De Broglie hypothesis, the particle can be a wave, with $\boxed{\lambda = h/p}$.

3/. As a wave group, the position of the particle is imprecisely known, because the wave is now spread out in space.

The wave number $\boxed{k = 2\pi/\lambda}$.

Now, $\boxed{\Delta x \, \Delta k_x \geq 1/2}$ The right hand side has the equality sign for a Gaussian wave group only.

$\Delta x \rightarrow$ Uncertainty of the position of the particle in the $x$-direction.

$\Delta k_x \rightarrow$ Uncertainty in the wave number in the $x$-direction.

Now $\boxed{p_x = \frac{h}{\lambda_x}}$ $\longrightarrow$ Momentum along the $x$ direction.

$\therefore$ $\boxed{p_x = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda_x} = \hbar k_x}$ Momentum in terms of the wave number.

(P.T.O.)

Uncertainty in $p_x$ is $\boxed{\Delta p_x = \hbar \, \Delta k_x}$

$\therefore \boxed{(\hbar \, \Delta k_x)\Delta x \geq \hbar/2} \Rightarrow \boxed{\Delta p_x \, \Delta x \geq \hbar/2}$

Position-momentum uncertainty in the $x$-coordinate. $\boxed{x, p_x}$ are conjugate variables.

Likewise, the position-momentum uncertainty in the $y$ and $z$-coordinates are $\boxed{\Delta p_y \, \Delta y \geq \hbar/2}$ and $\boxed{\Delta p_z \, \Delta z \geq \hbar/2}$.

$\boxed{y, p_y}$ and $\boxed{z, p_z}$ are also conjugate variables.

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Similarly, the angular frequency $\boxed{\omega = 2\pi\nu}$ and $\boxed{\Delta\omega \, \Delta t \geq 1/2}$ → The equality sign on the right hand side applies only to a Gaussian wave group

$\Delta t \rightarrow$ Uncertainty in time.

$\Delta\omega \rightarrow$ Uncertainty in the angular frequency.

Now, $\boxed{E = h\nu = \frac{h}{2\pi} \cdot 2\pi\nu = \hbar\omega}$. Uncertainty in $E$ is $\boxed{\Delta E = \hbar \, \Delta\omega}$ $\therefore \boxed{(\hbar \, \Delta\omega)\Delta t \geq \hbar/2}$

$\Rightarrow \boxed{\Delta E \, \Delta t \geq \hbar/2}$ → Energy-time uncertainty.

$\boxed{E, t}$ are conjugate variables. The uncertainty principle is due to the wave nature of matter.