Kepler's Laws of Planetary Motion

1/. Planets move in elliptical orbits, which have the Sun at one focus.

2/. The radial vector from the Sun to a planet, sweeps out equal areas in equal times.

3/. The square of the period of revolution of a planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Consequence of Kepler's Laws:

The three laws lead to the law of gravitation between the Sun and the planets, discovered by Isaac Newton.

Newton's Law of Universal Gravitation

$$\boxed{\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}}$$

=> Any two objects of mass m_1 and m_2, separated by a distance r, are attracted towards each other by a force, $\vec{F} = F(r)\hat{r}$.

Kepler's Laws and Newton's Dynamics

1. Planetary Orbits from Gravity and vice-versa

For a particle moving on a plane,

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}.$$

If the motion is under a central force, $\boxed{\vec{F} = F(r)\hat{r}}$, we write $\boxed{m\vec{a} = F(r)\hat{r}}$.

Hence, $\boxed{m(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = F(r)\hat{r}}$

Balancing components on both sides we write $\boxed{m(\ddot{r} - r\dot{\theta}^2) = F(r)}$

and $\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$. Consider the second equation above first. We get

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}(\underline{2r\dot{r}\dot{\theta} + r^2\ddot{\theta}}) = 0$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0} \Rightarrow \boxed{r^2\dot{\theta} = h \text{ (constant)}}$$

$\therefore \boxed{\dot{\theta} = h/r^2}$ since $\boxed{u = \frac{1}{r}}$, $\Rightarrow \boxed{\dot{\theta} = hu^2}$.

(P.T.O.)

Now we write $\boxed{\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}}$

Also, $\boxed{r = u^{-1}} \Rightarrow \boxed{dr = -1\,u^{-2}\,du}$.

$\therefore \boxed{\dot{r} = -\frac{1}{\cancel{u^2}}\frac{du}{d\theta}\cdot h\cancel{u^2} = -h\frac{du}{d\theta}}$ (Since $\dot{\theta} = hu^2$)

$$\Rightarrow \ddot{r} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\cdot hu^2$$

$\Rightarrow \boxed{\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}}$. Using these results in the $\hat{r}$ component, $\boxed{m(\ddot{r} - r\dot{\theta}^2) = F(r)}$,

We get, $m\left(-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4\right) = F(1/u)$

$$\Rightarrow -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = F(1/u)$$

$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{-F(1/u)}{mh^2u^2}}$ Transform of the radial equation.

As per Kepler's First Law, planets move in elliptical orbits. The orbit equation is $\boxed{r = \frac{l}{1 + \epsilon\cos\theta} = \frac{1}{u}}$

If $0 < \epsilon < 1$, then ellipse.

(P. T. O.)

$$\therefore \boxed{u = \frac{1+\epsilon\cos\theta}{l}} \Rightarrow \boxed{\frac{du}{d\theta} = -\frac{\epsilon}{l}\sin\theta}$$

and $\boxed{\dfrac{d^2u}{d\theta^2} = -\dfrac{\epsilon}{l}\cos\theta = \dfrac{1}{l} - u}$.

Hence, $\dfrac{d^2u}{d\theta^2} + u = \dfrac{1}{l} - \cancel{u} + \cancel{u} = \dfrac{-F(1/u)}{mh^2u^2}$

$\therefore \boxed{F(1/u) = -\dfrac{mh^2}{l}u^2}$ Since, $\boxed{u = 1/r}$

$\boxed{F(1/u) = F(r) = -\dfrac{mh^2}{l}\cdot\dfrac{1}{r^2}}$. m, h and l are constants

We write $\boxed{K = \dfrac{mh^2}{l}}$ $\therefore$ $\boxed{F(r) = -\dfrac{K}{r^2}}$

$F(r)$ is an inverse-square law force!
If a planetary orbit is a conic section, specifically an ellipse (Kepler), then the force driving the planet around the Sun, is an inverse-square law force (Newton).

Newton's gravity follows ~~fo~~ from Kepler's First Law.

(P.T.O.)

Orbits of Planets from Newton's Gravity

If $\boxed{F(r) = -\frac{k}{r^2}}$ → Newton's inverse square law force.

We write $\boxed{F(1/u) = -ku^2}$. Hence

$$\frac{d^2u}{d\theta^2} + u = \frac{-F(1/u)}{mh^2u^2} = \frac{-(-k\cancel{u^2})}{mh^2\cancel{u^2}}$$

$\therefore \boxed{\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}}$ Define $\boxed{\xi = u - \frac{k}{mh^2}}$.

$\therefore \boxed{\frac{d\xi}{d\theta} = \frac{du}{d\theta}}$ and $\boxed{\frac{d^2\xi}{d\theta^2} = \frac{d^2u}{d\theta^2}}$

Hence we transform ~~write~~ $\boxed{\frac{d^2u}{d\theta^2} + u - \frac{k}{mh^2} = 0}$

as $\boxed{\frac{d^2\xi}{d\theta^2} + \xi = 0}$ → The simple harmonic oscillator equation.

The solution of the simple harmonic oscillator equation is $\boxed{\xi = C\cos(\theta - \theta_0)}$ in which C and θ_0 are constants of the integration. We can verify the solution. (P.T.O.)

Check: $\boxed{\xi_s = C\cos(\theta-\theta_0)}$. Hence,

$$\boxed{\frac{d\xi_s}{d\theta} = -C\sin(\theta-\theta_0)} \ \& \ \boxed{\frac{d^2\xi_s}{d\theta^2} = -C\cos(\theta-\theta_0)}$$

$$\therefore \frac{d^2\xi_s}{d\theta^2} = -\xi_s \Rightarrow \boxed{\frac{d^2\xi_s}{d\theta^2} + \xi_s = 0}$$ $\therefore$ The solution is valid

Hence, $\boxed{\xi_s = u - \frac{K}{mh^2} = C\cos(\theta-\theta_0)}$.

$$\Rightarrow u = \frac{K}{mh^2} + C\cos(\theta-\theta_0) = \frac{1}{r}$$

$$\Rightarrow r = \frac{1}{(K/mh^2) + C\cos(\theta-\theta_0)}$$

$$\Rightarrow \boxed{r = \frac{(mh^2/K)}{1 + (Cmh^2/K)\cos(\theta-\theta_0)}}$$

Polar equation of a Conic Section

Compare with the polar equation of a conic section, $\boxed{r = \frac{l}{1+\epsilon\cos\theta}}$.

We can choose an axis where $\boxed{\theta_0 = 0}$.

$\therefore \boxed{l = \frac{mh^2}{K}}$ and $\boxed{\epsilon = \frac{Cmh^2}{K}}$ $\therefore$ Driven by an inverse-square law force, a planet moves along a conic section (ellipse).

$U \to$ Potential Energy

The Energy Conservation Equation

Energy Conservation: $\boxed{\frac{1}{2}mv^2 + U(r) = \mathcal{E}}$

Now $\boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2}$ and $\boxed{\dot{r} = -h\frac{du}{d\theta}}$

$$\Rightarrow \boxed{\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \mathcal{E}} \quad \text{and } \boxed{\dot{\theta} = hu^2}$$

$\boxed{u = 1/r}$

$$\frac{1}{2}m\left[h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}h^2u^4\right] + U(r) = \mathcal{E} \quad \boxed{r = 1/u}$$

$$\Rightarrow \frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U(1/u) = \mathcal{E}$$

$$\Rightarrow \boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\left[\mathcal{E} - U\left(\frac{1}{u}\right)\right]}{mh^2}}$$

(In u and θ Coordinates)

Also $\boxed{\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{h}{r^2}}$

$$\Rightarrow \boxed{\dot{r}^2 = \left(\frac{dr}{d\theta}\right)^2\frac{h^2}{r^4}} \quad \text{and} \quad \boxed{\dot{\theta}^2 = h^2/r^4}$$

$$\Rightarrow \frac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2\frac{h^2}{r^4} + r^2\frac{h^2}{r^4}\right] + U(r) = \mathcal{E}$$

$$\Rightarrow \boxed{\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] + U(r) = \mathcal{E}}$$

(In r and θ Coordinates)

Characterising the Orbit by ϵ

For a central inverse-square law force, $\boxed{F(r) = -k/r}$ or $\boxed{F(1/u) = -ku}$,

the orbit is a conic section,

$$\boxed{u = \frac{k}{mh^2} + C\cos(\theta - \theta_0)}$$ C, θ_0 are unknown constants.

With a suitable choice of an axis, we can set $\boxed{\theta_0 = 0}$. Hence, we have

$$\boxed{u = \frac{k}{mh^2} + C\cos\theta} \Rightarrow \boxed{\frac{du}{d\theta} = -C\sin\theta}$$

Consider $\boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\left[\mathcal{E} - U(1/u)\right]}{mh^2}}$

The left hand side (L.H.S.) is

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = C^2\sin^2\theta + \left(\frac{k}{mh^2} + C\cos\theta\right)^2$$

$$= C^2\sin^2\theta + \frac{k^2}{m^2h^4} + C^2\cos^2\theta + \frac{2Ck\cos\theta}{mh^2}$$

$$= C^2 + \frac{k^2}{m^2h^4} + \frac{2Ck\cos\theta}{mh^2} \quad \left(\text{since } \sin^2\theta + \cos^2\theta = 1\right)$$

(P.T.O.)

The <u>right hand side</u> (<u>R.H.S.</u>) is

$$\frac{2\left[\mathcal{E} - U(1/u)\right]}{mh^2} = \frac{2\mathcal{E}}{mh^2} - 2\frac{U(1/u)}{mh^2}$$

POTENTIAL ENERGY

Now $\boxed{\vec{F} = -\vec{\nabla} U}$ $\therefore$ $\boxed{F(r)\hat{r} = -\frac{dU}{dr}\hat{r}}$

Since, $\boxed{F(r) = \frac{-k}{r^2}}$ $\Rightarrow$ $\boxed{-\frac{dU}{dr} = -\frac{k}{r^2}}$

$\Rightarrow$ $\boxed{U = \int k r^{-2}\,dr = \frac{-k}{r}}$ $\Rightarrow$ $\boxed{U = -ku}$

<u>Potential</u>

$$\therefore \frac{2\mathcal{E}}{mh^2} - \frac{2U}{mh^2} = \frac{2\mathcal{E}}{mh^2} + \frac{2ku}{mh^2}$$

$$= \frac{2\mathcal{E}}{mh^2} + \frac{2k}{mh^2}\left(\frac{k}{mh^2} + C\cos\theta\right)$$

$$= \frac{2\mathcal{E}}{mh^2} + \frac{2k^2}{m^2h^4} + \frac{2kC\cos\theta}{mh^2}$$

Balancing <u>L.H.S.</u> and <u>R.H.S.</u>

$$C^2 + \frac{k^2}{m^2h^4} + \cancel{\frac{2Ck\cos\theta}{mh^2}} = \frac{2\mathcal{E}}{mh^2} + \frac{2k^2}{m^2h^4} + \cancel{\frac{2kC\cos\theta}{mh^2}}$$

$$\Rightarrow \boxed{C^2 = \frac{2\mathcal{E}}{mh^2} + \frac{k^2}{m^2h^4}}$$

(<u>P.T.O.</u>)

The eccentricity, $\boxed{\epsilon = \dfrac{Cmh^2}{k}}$

$$\therefore \frac{C^2 m^2 h^4}{k^2} = 1 + \frac{2\mathcal{E}}{mk^2} \cdot \frac{m^2 h^4}{k^2}$$

$$\Rightarrow \boxed{\frac{C^2 m^2 h^4}{k^2} = 1 + \frac{2\mathcal{E} m h^2}{k^2} = \epsilon^2}$$

$$\Rightarrow \boxed{\epsilon = \sqrt{1 + \frac{2\mathcal{E} m h^2}{k^2}}}$$

$\mathcal{E} \to$ Total energy
$h \to$ Angular momentum
$m \to$ mass of planet

Now $\boxed{F(r) = \dfrac{-k}{r^2} = -\dfrac{GMm}{r^2}}$ $\Rightarrow \boxed{k = GMm}$

$M \to$ Mass of Sun

$$\therefore \frac{2\mathcal{E} m h^2}{k^2} = \frac{2\mathcal{E} m h^2}{G^2 M^2 m^2} = \frac{2\mathcal{E} h^2}{G^2 M^2 m}$$ $\boxed{M, m, h > 0}$

$$\Rightarrow \boxed{\epsilon = \sqrt{1 + \frac{2\mathcal{E} h^2}{G^2 M^2 m}}}$$

i.) $\boxed{\text{If } \mathcal{E} > 0, \text{ then } \epsilon > 1}$ $\Rightarrow$ Hyperbolic orbit.

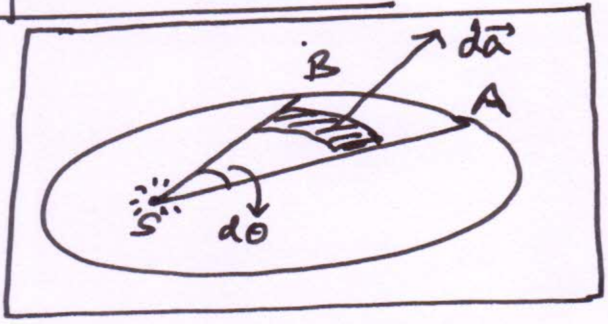
ii.) $\boxed{\text{If } \mathcal{E} = 0, \text{ then } \epsilon = 1}$ $\Rightarrow$ Parabolic Orbit

iii.) $\boxed{\text{If } -\dfrac{G^2 M^2 m}{2h^2} < \mathcal{E} < 0, \text{ then } 0 < \epsilon < 1.}$

$\Rightarrow$ Elliptical Orbit (as planets have).

2). Areal Velocity of Planets

In (r,θ) coordinates



$$\boxed{d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}}$$

↳ line element

∴ area element, $\boxed{d\vec{a} = r\,dr\,d\theta\,(\hat{r}\times\hat{\theta})}$

In the figure the shaded area element is $\boxed{r'\,dr'\,d\theta}$ (magnitude) in which r' is a dummy variable. The full area of the sector $\overline{ABS}$ is to be formed by integrating r' from $\boxed{r'=0}$ (at S, the position of the Sun, to $\boxed{r'=r}$ (at A or B, the position of the planet on its elliptical orbit, with S at one focus.

$$\therefore \boxed{d\vec{A} = \left(\int_0^r r'\,dr'\right) d\theta\,(\hat{r}\times\hat{\theta})}.$$

$\vec{A}$ is the area vector traced out by the orbital radius ($\overline{SA}$ or $\overline{SB}$), (P.T.O.)

as the planet moves from A to B.

$$\therefore d\vec{A} = \left(\frac{r'^2}{2} \Big|_0^r \right) d\theta \, \hat{z} \quad \left(\text{Since } \hat{r} \times \hat{\theta} = \hat{z} \right)$$

$$\Rightarrow \boxed{d\vec{A} = \frac{r^2}{2} d\theta \, \hat{z}}$$

. If this area is swept out in a time dt, then the areal velocity is $\boxed{\frac{d\vec{A}}{dt} = \frac{r^2}{2} \frac{d\theta}{dt} \hat{z}}$.

Now, $\boxed{r^2 \frac{d\theta}{dt} = r^2 \dot{\theta} = h}$ (a constant)

$$\Rightarrow \boxed{\frac{d\vec{A}}{dt} = \dot{\vec{A}} = \frac{h}{2} \hat{z}} \Rightarrow \text{Areal velocity is a constant vector.}$$

($\hat{z}$ points perpendicular to the orbital plane)

1/. Hence, equal areas are swept out by the orbital radius, in equal time intervals (Kepler's second law)

2/. Closer to the Sun, a planet moves faster along the orbit, than when the planet is farther away from the Sun.

3/. Orbital Time Period

Area of an ellipse is $\boxed{\pi ab}$,

$a \rightarrow$ semi-major axis, $b \rightarrow$ semi-minor axis.

Areal velocity $\boxed{\dot{A} = h/2}$ (magnitude).

Time taken to sweep out the full orbital area, $\boxed{T = \dfrac{\pi ab}{h/2}} \rightarrow$ Orbital Time Period.

Now, $\boxed{b = a\sqrt{1-\epsilon^2}} \rightarrow$ (From the theory of the ellipse)

$\Rightarrow \boxed{T = \dfrac{2\pi a^2\sqrt{1-\epsilon^2}}{h}}$. Further, the semi-latus rectum,

$\boxed{l = a(1-\epsilon^2)}$ and $\boxed{l = \dfrac{mh^2}{K}}$.

$$\Rightarrow \sqrt{1-\epsilon^2} = \sqrt{\frac{mh^2}{K}} \Rightarrow T = \frac{2\pi a^2}{h}\sqrt{\frac{mh^2}{Ka}}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{m}{K}a^3}} \Rightarrow \boxed{T^2 = 4\pi^2\frac{m}{K}a^3}$$

Since, $\boxed{K = GMm} \Rightarrow \boxed{T^2 = \dfrac{4\pi^2}{GM}a^3}$, which proves Kepler's Third Law $\longrightarrow \boxed{T^2 \propto a^3}$