# Energy Conservation under Central Forces

Central Force: $\boxed{\vec{F}(\vec{r}) = F(r)\,\hat{r}}$.

By Newton's second law: $\boxed{\vec{F}(\vec{r}) = m\dfrac{d\vec{v}}{dt}}$

Now $\vec{F}\cdot\vec{v} = m\dfrac{d\vec{v}}{dt}\cdot\vec{v}$. Also $\boxed{\vec{v} = \dfrac{d\vec{l}}{dt}}$.

Hence, $\boxed{\vec{F}\cdot\dfrac{d\vec{l}}{dt} = m\,\vec{v}\cdot\dfrac{d\vec{v}}{dt} = \dfrac{m}{2}\dfrac{d(v^2)}{dt}}$.

$\therefore$ $\boxed{d\left(\frac{m}{2}v^2\right) - \vec{F}\cdot d\vec{l} = 0}$. Since we know that $\vec{F}(\vec{r})$ is an irrotational vector, we can write $\boxed{\vec{F}(\vec{r}) = -\vec{\nabla}U}$, in which $U = U(\vec{r})$ is a scalar potential function.

Hence by integration we can get,

$$\boxed{\frac{mv^2}{2} + \int_{\vec{a}}^{\vec{r}} \vec{\nabla}U\cdot d\vec{l} = \text{constant}}$$

By the theorem of gradients we can write

$$\boxed{\int_{\vec{a}}^{\vec{r}} \vec{\nabla}U\cdot d\vec{l} = U(\vec{r}) - U(\vec{a})}$$

in which $\vec{a}$ is a reference position. (P.T.O.)

$\boxed{\mathcal{U}(\vec{r}) - \mathcal{U}(\vec{a})}$ is the potential difference at $\vec{r}$, with respect to a reference value $\mathcal{U}(\vec{a})$ at position $\vec{a}$.

$$\therefore \quad \boxed{\frac{m}{2}v^2 + \mathcal{U}(\vec{r}) = \text{Constant} + \mathcal{U}(\vec{a})}$$

The terms on the right hand side add up to a total constant $\mathcal{E}$.

Hence, $\boxed{\frac{m}{2}v^2 + \mathcal{U}(\vec{r}) = \mathcal{E}}$. $\mathcal{E}$ is the total energy.

This gives Conservation of energy.

## Work Done under a Central Force

By Stokes's Curl Theorem we write,

$$\boxed{\oint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \oint_P \vec{F} \cdot d\vec{l}}$$

S (surface integral) P (line integral)

$d\vec{a} \rightarrow$ area element.
$d\vec{l} \rightarrow$ line element.

But $\vec{F}$ (the central force field) is an irrotational Vector. $\therefore \boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$.

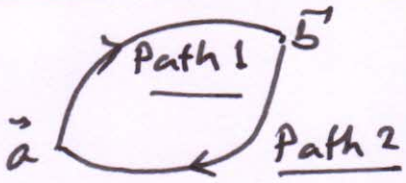
$\therefore \boxed{\oint \vec{F} \cdot d\vec{l} = 0}$, $\boxed{\vec{F} \cdot d\vec{l} \rightarrow \text{Work done}}$. In a closed path it is zero.

## Work is Path-Independent

For a central force, $\boxed{\oint \vec{F}\cdot d\vec{l} = 0}$.

$$\therefore \boxed{\oint \vec{F}\cdot d\vec{l} = \underset{(\text{Path } 1)}{\int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l}_{(1)}} + \underset{(\text{Path } 2)}{\int_{\vec{b}}^{\vec{a}} \vec{F}\cdot d\vec{l}_{(2)}} = 0}$$

$$\Rightarrow \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l}_{(1)} - \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l}_{(2)} = 0$$




$$\Rightarrow \boxed{\underset{(\text{Path } 1)}{\int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l}_{(1)}} = \underset{(\text{Path } 2)}{\int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l}_{(2)}}}$$

Work done is the same along either path

$\therefore$ In a central force field, the work done is independent of the path between two points.

## Angular Momentum Conservation under a Central Force

Central Force: $\boxed{\vec{F} = f(r)\,\hat{r}} \rightarrow \boxed{F(r) = f(r)}$

$$\therefore \boxed{\vec{r}\times\vec{F} = r\,\hat{r}\times f(r)\,\hat{r} = r\,f(r)(\hat{r}\times\hat{r}) = \vec{0}}$$

(Since $\hat{r}\times\hat{r} = \vec{0}$ by the rules of cross product)

Now we write, $\boxed{\vec{F} = m\dfrac{d\vec{v}}{dt}}$

$$\Rightarrow \vec{r}\times\vec{F} = m\left(\vec{r}\times\frac{d\vec{v}}{dt}\right) = \vec{0} \quad \left(\text{since } \vec{r}\times\vec{F} = 0\right)$$

(P.T.O.)

Writing $\boxed{\frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{v}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d\vec{v}}{dt}}$

we see that $\boxed{\vec{v}\times\frac{d\vec{r}}{dt} = \vec{v}\times\vec{v} = \vec{0}}$ $\left(\underline{\text{Since } \frac{d\vec{r}}{dt} = \vec{v}}\right)$

$\therefore \boxed{\frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{r}\times\frac{d\vec{v}}{dt} = \vec{0}}$ $\left(\underline{\text{Since } \vec{r}\times\vec{F} = \vec{0}}\right)$.

$\Rightarrow \boxed{\vec{r}\times\vec{v} = \vec{h}}$, $\underline{\vec{h} \text{ is a constant vector.}}$

$\therefore \boxed{m(\vec{r}\times\vec{v}) = m\vec{h}}$ $\Rightarrow m\vec{h}$, the angular momentum is conserved.

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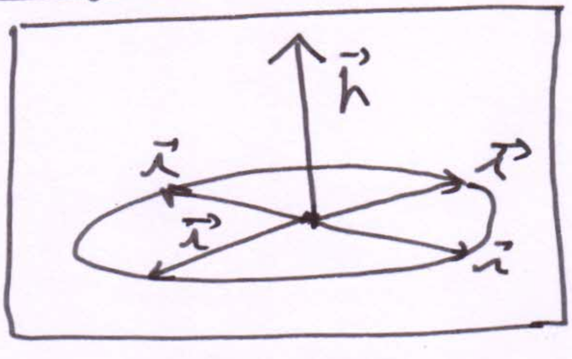
## Motion on a Plane in a Central Force

Use the vector identity of triple product

$\boxed{\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}}$ $\vec{a},\vec{b},\vec{c}$ are general vectors.

$\therefore \boxed{\vec{r}\cdot(\vec{r}\times\vec{v}) = \vec{r}\cdot\vec{h} = \underbrace{(\vec{r}\times\vec{r})}_{=\vec{0}}\cdot\vec{v} = 0}$ $\rightarrow$ Scalar

$\Rightarrow \boxed{\vec{r}\cdot\vec{h} = 0}$ Hence, $\vec{r}$ is perpendicular to $\vec{h}$ always. Since $\vec{h}$ is a constant vector, $\vec{r}$ must lie on a plane perpendicular to $\vec{h}$.




(P. T. O.)

Hence, the radial position of a particle moving under a central force is always on a plane. Motion under a central force takes place on a plane.

For a particle moving on the $x$-$y$ plane, the velocity, $\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$ in polar coordinates, $(r, \theta)$. Hence, we get

1/. $\boxed{\vec{r}\times\vec{v} = \vec{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}$ $(\vec{r} = r\hat{r})$

$\Rightarrow \vec{r}\times\vec{v} = r\dot{r}(\hat{r}\times\hat{r}) + r^2\dot{\theta}(\hat{r}\times\hat{\theta})$

But $\boxed{\hat{r}\times\hat{r} = \vec{0}}$ $\therefore$ $\boxed{\vec{r}\times\vec{v} = r^2\dot{\theta}(\hat{r}\times\hat{\theta})}$

But $\boxed{\hat{r}\times\hat{\theta} = \hat{z}}$ $\Rightarrow$ $\boxed{\vec{r}\times\vec{v} = \vec{h} = r^2\dot{\theta}\hat{z}}$, i.e. directed perpendicular to the $x$-$y$ plane.

2/. Total energy: $\boxed{\frac{1}{2}mv^2 + U(\vec{r}) = \mathcal{E}}$

$\therefore$ $\boxed{v^2 = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})\cdot(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}$ [since $v^2 = \vec{v}\cdot\vec{v}$]

$\Rightarrow$ $\boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2}$ $\boxed{\text{Since } \hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1 \text{ and } \hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{r} = 0}$.

Hence in polar coordinates, energy conservation is $\boxed{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(\vec{r}) = \mathcal{E}}$

Since, $\boxed{|\vec{h}| = h = r^2\dot{\theta} = \text{Constant}}$, we write $\boxed{\dot{\theta} = h/r^2}$. Using this in the energy conservation condition, we get, $\boxed{\frac{1}{2} m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + \mathcal{U}(\vec{r}) = \mathcal{E}}$

Also, $\boxed{\text{work done} = \int \vec{F}(\vec{r}) \cdot d\vec{l} = \int F(\vec{r})\,\hat{r} \cdot d\vec{l}}$

Now $\boxed{d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}}$ $\Rightarrow$ $\boxed{\hat{r} \cdot d\vec{l} = dr}$
(in polar coordinates) $(\because \hat{r} \cdot \hat{\theta} = 0)$

Hence, $\boxed{\text{work done} = \int F(\vec{r})\,dr}$ $\rightarrow$ Only the radial variation.

## Physical Implications of Central Forces

1/. Acts along the radial vector ($\hat{r}$), and its magnitude depends only on $r$.

2/. Total energy is conserved.

3/. Work done is path-independent between two points. It is zero in a closed cycle.

4/. Angular momentum is conserved.

5/. Particle motion occurs on a plane.