Problem 1: Correlation between Z and Z^2

Question Let Z be a random variable following the standard normal distribution,

$$Z \sim \mathcal{N}(0, 1).$$

We want to compute the correlation between Z and Z^2, and determine whether they are independent.

Solution

1. Definitions. Recall that the correlation between two random variables X and Y is given by

$$\operatorname{Corr}(X, Y) \;=\; \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}.$$

Hence we need

$$\operatorname{Cov}(Z, Z^2) \;=\; \mathbb{E}[Z \cdot Z^2] \;-\; \mathbb{E}[Z]\,\mathbb{E}[Z^2].$$

Because $Z^2 \cdot Z = Z^3$, this is

$$\operatorname{Cov}(Z, Z^2) \;=\; \mathbb{E}[Z^3] \;-\; \mathbb{E}[Z]\,\mathbb{E}[Z^2].$$

2. Moments of a standard normal. For $Z \sim \mathcal{N}(0, 1)$ we know:

$$\mathbb{E}[Z] \;=\; 0, \qquad \mathbb{E}[Z^2] \;=\; 1, \qquad \mathbb{E}[Z^3] \;=\; 0.$$

Plugging these into our covariance expression yields

$$\operatorname{Cov}(Z, Z^2) \;=\; 0 - \; 0 \times 1 \;=\; 0.$$

Therefore,

$$\operatorname{Corr}(Z, Z^2) \;=\; \frac{0}{\sqrt{\operatorname{Var}(Z)\operatorname{Var}(Z^2)}} \;=\; 0.$$

Since $\operatorname{Var}(Z) = 1$, it only remains to note that $\operatorname{Var}(Z^2)$ is finite and nonzero, so the correlation is indeed 0.

3. Independence Even though Z and Z^2 are uncorrelated (their correlation is 0), they are *not* independent. Intuitively, knowing Z^2 gives you information about the magnitude of Z (it tells you $|Z|$), which influences probabilities about Z being large or small, thus ruling out independence.

Final Answer.

$$\operatorname{Corr}(Z, Z^2) \;=\; 0, \qquad \text{but } Z \text{ and } Z^2 \text{ are not independent.}$$