An Analytical Solution of the Inverted Pendulum*

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A rigorous solution of the inverted pendulum requires several mathematical techniques which are not in the repetoire of the typical student of classical mechanics at the senior level. However, the problem is tractable by Lagrange's equations and the physics of the situation is sufficiently simple so that it is an excellent example to use to extend the student's analytical abilities. Two unexpected dividends are derived from this problem: (1) the vertically driven case is an example of a time-dependent potential energy function, a rare situation in mechanics; (2) it is a vivid example of a classical system undergoing strong focusing and therefore can be used to introduce a discussion of the strong-focusing synchrotron.

INTRODUCTION AND DEFINITIONS

A PHYSICAL pendulum in the shape of a long thin rod is supported at its base. The supporting pivot is driven in a sinusoidal fashion. Under certain conditions this results in a pendulum-like motion about the vertical, unstable-equilibrium position. The problem is set up for four cases: (1) The vertically driven case; (2) The horizontally driven case; (3) The general case in two dimensions; and (4) The case in which the supporting pivot is near the rim of a fly wheel. For cases (1) and (2) complete solutions are derived.

We consider Fig. 1 and define the following quantities:

m = mass of the pendulum,

I_0 = moment of inertia relative to the center of mass,

(X_0, Y_0) = coordinates of the point of support,

(X, Y) = coordinates of the center of mass,

l = distance from the support to the center of mass,

T = kinetic energy,

V = potential energy,

q = generalized coordinate,

θ = the angle through which the pendulum oscillates.

We take the zero of the potential energy to correspond with the x axis.

The problem would be "fast unmöglich" by Newton's equations but yields readily to Lagrange's equations of motion. Defining T and V

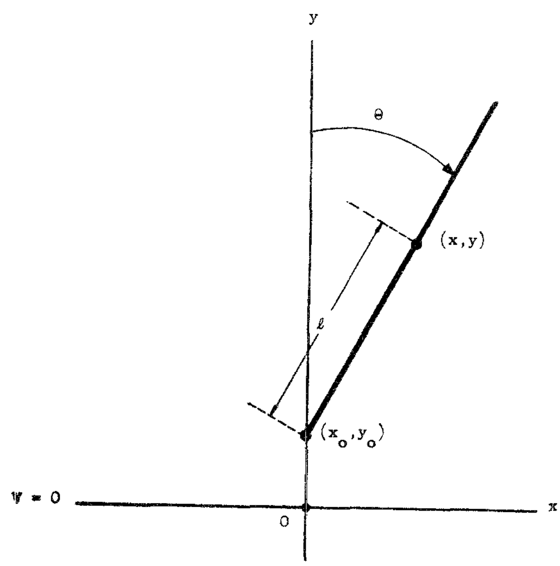

FIG. 1. The coordinates of the point of support (x_0,y_0) and of the center of mass (x,y) are separated by the distance l. The bob makes an angle θ with respect to the vertical. The x axis is taken as the zero of potential.

* The solution of the problem of the inverted pendulum is not original with the authors but was worked out independently by one of them (J.H.H.) as a "Special Study Project" during his senior year (1959–1960) at Kalamazoo College. One of the authors (F.M.P.) has made an extensive literature survey to locate references to this problem. The only one known is in Corben and Stehle *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1960), 2nd ed. There seems to be no mention of the problem in *Physics Abstracts* going back to 1898 Volume 1. However, Professor D. M. Dennison of the University of Michigan has used this problem for some years in his course in Classical Mechanics at the graduate level.

relative to a fixed coordinate system, Lagrange's equation in generalized coordinates is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)-\frac{\partial T}{\partial q}+\frac{\partial V}{\partial q}=0. \tag{1}$$

This problem has only one degree of freedom, namely θ.

CASE 1

The Vertically-Driven Pendulum

For the moment we assume that the point of support is oscillated harmonically only in the vertical direction. Then

$$X_0=0, \tag{2}$$

$$Y_0=A\cos\omega t, \tag{3}$$

where ω is the angular-driving frequency, t is the time and A is the amplitude of the oscillation.

The kinetic and potential energies for this case are

$$T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+\tfrac{1}{2}I_0\dot{\theta}^2, \tag{4}$$

$$V=mgy. \tag{5}$$

Note that it is necessary to employ I_0 in the expression for T because the rotational kinetic energy relative to the center of mass must be reckoned in order that Lagrange's equation hold.

From Fig. 1 we see that

$$x=l\sin\theta, \tag{6}$$

$$y=A\cos\omega t+l\cos\theta. \tag{7}$$

On differentiating Eqs. (6) and (7), squaring and substituting into Eq. (4) we have

$$T=\tfrac{1}{2}m[l^2\dot{\theta}^2+2Al\omega\sin\omega t\sin\theta\dot{\theta}+A^2\omega^2\sin^2\omega t]+\tfrac{1}{2}I_0\dot{\theta}^2, \tag{8}$$

$$V=mg(A\cos\omega t+l\cos\theta). \tag{9}$$

To simplify we may use the parallel axis theorem to set

$$I=I_0+ml^2, \tag{10}$$

where I is the moment of inertia relative to (x_0,y_0). Upon substitution into Eq. (8), the kinetic energy becomes

$$T=\tfrac{1}{2}I\dot{\theta}^2+Alm\omega\sin\omega t\sin\theta\dot{\theta}+\tfrac{1}{2}mA^2\omega^2\sin^2\omega t, \tag{11}$$

from which we easily derive

$$(\partial T/\partial\dot{\theta})=I\dot{\theta}+Alm\omega\sin\omega t\sin\theta, \tag{12}$$

$$(\partial T/\partial\theta)=Alm\omega\sin\omega t\cos\theta\dot{\theta}, \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)=I\ddot{\theta}+Alm\omega^2\cos\omega t\sin\theta+Alm\omega\sin\omega t\cos\theta\dot{\theta}, \tag{14}$$

$$\partial V/\partial\theta=-mgl\sin\theta. \tag{15}$$

Substituting Eqs. (13), (14), and (15) into (1) and rearranging gives, for the equation of motion,

$$\ddot{\theta}+(I^{-1}Alm\omega^2\cos\omega t-I^{-1}mgl)\sin\theta=0. \tag{16}$$

Just as an intuitive check on the validity of Eq. (16) we transform to the case of the simple pendulum. The angle θ becomes $\theta-\pi$ and we assume it is small with respect to the stable equilibrium position. Then since

$$\sin(\theta-\pi)=-\sin\theta\approx-\theta, \tag{17}$$

and also neglecting the forcing term, since it is zero for a simple pendulum, Eq. (16) becomes

$$\ddot{\theta}+I^{-1}mgl\theta=0, \tag{18}$$

the equation of motion of a simple pendulum, as expected. Eq. (18) does not prove that Eq. (16) is correct but it implies strongly that it is probably correct.

We set

$$\omega_0^2=I^{-1}mgl, \tag{19}$$

where ω_0 is the natural frequency of the pendulum. On assuming that θ is small, Eq. (16) becomes

$$\ddot{\theta}+(-\omega_0^2+I^{-1}Alm\omega^2\cos\omega t)\theta=0. \tag{20}$$

This is a Mathieu equation of the general form

$$(d^2u/dz^2)+(\theta_0+2\theta_1\cos 2z)u=0. \tag{21}$$

We return to Eq. (20) and solve it completely after setting up the remaining cases.

CASE 2

The Horizontally-Driven Pendulum

We assume the point of support is oscillated harmonically and find

$$x=A\cos\omega t+l\sin\theta, \tag{22}$$

$$y=l\cos\theta. \tag{23}$$

Taking derivatives of Eqs. (22) and (23) with respect to time, squaring, substituting into Eq. (4) and using Eq. (10) gives

$$T=\tfrac{1}{2}A^2m\omega^2\sin^2\omega t-Alm\omega\sin\omega t\cos\theta\dot\theta+\tfrac{1}{2}I\dot\theta^2, \quad (24)$$

$$V=mgl\cos\theta. \quad (25)$$

On taking the appropriate derivatives of Eqs. (24) and (25) and substituting into Eq. (1) we find for the equation of motion

$$\ddot\theta-I^{-1}Alm\omega^2\cos\omega t\cos\theta-I^{-1}mgl\sin\theta=0. \quad (26)$$

As it stands, Eq. (26) cannot be solved in closed form. However, letting θ be small yields the important simplification $\sin\theta\approx\theta$ and $\cos\theta\approx 1$ so that Eq. (26) becomes

$$\ddot\theta-\omega_0^2\theta=C\cos\omega t \quad (27)$$

and

$$C=I^{-1}Alm\omega^2. \quad (28)$$

We return to Eq. (27) later and solve it completely after setting up cases 3 and 4.

CASE 3

The Pendulum Driven in Two Dimensions

We assume the pendulum is driven vertically with amplitude A and frequency ω and horizontally with amplitude B and frequency ω'. Then

$$x=B\cos(\omega' t+\Phi)+l\sin\theta, \quad (29)$$

$$y=A\cos\omega t+l\cos\theta. \quad (30)$$

Upon performing the usual manipulations we find

$$T=Alm\omega\sin\omega t\sin\theta\dot\theta-Blm\omega'\sin(\omega' t+\Phi)\cos\theta\dot\theta$$
$$+\tfrac{1}{2}A^2m\omega^2\sin^2\omega t+\tfrac{1}{2}B^2m\omega'^2\sin^2(\omega' t+\Phi)+\tfrac{1}{2}I\dot\theta^2, \quad (31)$$

$$V=mg(A\cos\omega t+l\cos\theta). \quad (32)$$

On deriving the Legrange equation of motion from Eqs. (31) and (32) and making the usual approximation that θ is small, we find

$$\ddot\theta+(-\omega_0^2+I^{-1}Alm\omega^2\cos\omega t)\theta$$
$$=I^{-1}Blm\omega'^2\cos(\omega' t+\Phi). \quad (33)$$

On comparison of Eq. (33) with Eq. (20) we see that Eq. (33) is a forced Mathieu equation. Comments on possible solutions are presented later.

CASE 4

The Pendulum Support is Attached to the Rim of a Flywheel of Radius A

The previous, general, two-dimensional situation reduces to

$$x=A\cos\omega t+l\sin\theta, \quad (34)$$

$$y=A\sin\omega t+l\cos\theta. \quad (35)$$

The long derivation of Lagrange's equation may be avoided if we note the following:

$$\sin\omega t=\cos(\omega t-\tfrac{1}{2}\pi), \quad (36)$$

which, upon substitution into Eq. (35), gives essentially Eq. (30).

Letting

$$A\rightarrow B, \quad (37)$$

$$\omega\rightarrow\omega' \quad (38)$$

in Eq. (34), we then have, essentially, Eq. (29). Thus, using Eq. (33) as a guide we may write down Lagrange's equation for this case with great ease. The result is

$$\ddot\theta+(-\omega_0^2+I^{-1}Alm\omega^2\sin\omega t)\theta$$
$$=I^{-1}Alm\omega^2\cos\omega t. \quad (39)$$

Eq. (39) is seen to be a forced Mathieu equation slightly more amenable to solution than Eq. (33).

SOLUTIONS TO THE PROBLEMS

Case 4

The homogeneous equation obtained from Eq. (39) by equating the left side to zero can be solved in terms of known functions since it is simply Mathieu's equation. If a trial substitution can be found which will give a particular integral for Eq. (39) then the general solution will be in hand. The complete solution is not discussed further here.

Case 3

If we set $B=A$, $\omega'=\omega$, $\Phi=0$ and let $\theta \rightarrow \theta-\pi$, then Eq. (33) collapses,

$$\ddot{\theta}+I^{-1}mgl\theta=0, \tag{40}$$

and we have the equation for the simple pendulum. The oscillations take place along the 45° line and the point of support is taken above the center of mass.

Case 2

We wish to solve Eq. (27). Let us assume a solution of the form

$$\theta=\alpha\cos\omega t+\beta\sin\omega t. \tag{41}$$

Differentiating, substituting into Eq. (27) and collecting the coefficients of the sin and cos terms gives

$$(\alpha\omega^2+\alpha\omega_0^2+C)\cos\omega t+(\beta\omega^2+\beta\omega_0^2)\sin\omega t=0. \tag{42}$$

This equation must hold for all values of t; thus the coefficients of the $\sin\omega t$ and $\cos\omega t$ terms must separately vanish. This requires

$$\alpha\beta\omega^2+\alpha\beta\omega_0^2+\beta C=0, \tag{43}$$

$$-\alpha\beta\omega^2-\alpha\beta\omega_0^2=0. \tag{44}$$

Adding gives

$$\beta C=0, \tag{45}$$

from which

$$\beta=0 \quad \text{since} \quad C\neq 0. \tag{46}$$

Thus from Eq. (42) we have

$$\alpha=-C/(\omega^2+\omega_0^2) \tag{47}$$

and

$$\theta=-Alm\omega^2\cos\omega t/(I\omega^2-mgl), \tag{48}$$

where use has been made of Eqs. (19) and (28).

Case 1

The problem to be solved, Eq. (20), is the Mathieu equation. Suitable transformations put it in the standard form of Eq. (21). This is a useful stratagem for then the solution may be used in other problems. Clearly we require $2Z=\omega t$, from which

$$\dot{Z}=\omega/2. \tag{49}$$

Using the chain rule

$$\dot{\theta}=(d\theta/dZ)\dot{Z}, \tag{50}$$

$$\ddot{\theta}=(d^2\theta/dZ^2)(\dot{Z})^2+(d\theta/dZ)\ddot{Z}. \tag{51}$$

On letting $\theta\rightarrow u$ the substitution of Eqs. (49), (50), and (51) into (20) gives

$$\frac{\omega^2}{2^2}\frac{d^2u}{dZ^2}+\left[\frac{Alm\omega^2}{I}\cos 2Z-\omega_0^2\right]u=0. \tag{52}$$

Or defining

$$\theta_0=-4\omega_0^2/\omega^2 \tag{53}$$

and

$$\theta_1=2Alm/I, \tag{54}$$

Eq. (52) becomes

$$\frac{d^2u}{dZ^2}+[\theta_0+2\theta_1\cos 2Z]u=0. \tag{55}$$

This is the standard form of the Mathieu equation. It appears frequently in physical problems involving elliptic boundary conditions and was solved originally by Mathieu in 1868 while he was studying the modes of vibration of elliptic drum heads. Mathieu found that the most effective power series for solving the Mathieu equation is the doubly periodic series

$$u=e^{i\lambda Z}\sum_{n=-\infty}^{n=+\infty}b_n e^{2inZ}, \tag{56}$$

where λ is a constant. The term $e^{i\lambda Z}$ describes the periodic oscillation of the inverted pendulum as a whole and the summation represents the driving frequency and all of its harmonics. However, we know from past experience that the effect of the overtones dies off very rapidly and therefore the summation must be a rapidly convergent series, i.e., for n large, $b_{n+1}\ll b_n$. We also note that for the summation to be periodic, n must be an integer.

On differentiating Eq. (56) and substituting the results into Eq. (55) and using the fact that

$$\cos 2Z = \tfrac{1}{2}(e^{2iZ}+e^{-2iZ}), \qquad (57)$$

Eq. (55) becomes

$$\sum_{n=-\infty}^{n=+\infty} [b_n(\theta_0-[2n+\lambda]^2)e^{i(2n+\lambda)Z}+\theta_1 b_n e^{i(2n+\lambda+2)Z} + \theta_1 b_n e^{i(2n+\lambda-2)Z}]=0. \qquad (58)$$

In the second term we let $l=n+1$; in the third term we let $m=n-1$; and then, since l and m are dummy symbols, we replace each of them by n. The result is

$$\sum_{n=-\infty}^{n=+\infty} [b_n[\theta_0-(2n+\lambda)^2] + \theta_1 b_{n-1}+\theta_1 b_{n+1}]e^{i(2n+\lambda)}=0. \qquad (59)$$

This equation will be zero only if the coefficients of all the exponentials vanish. This means that we have an infinity of three term equations which must all go to zero simultaneously.

We note that as $n\to\infty$ the coefficient of $b_n\to\infty$ also. This is, however, easily taken care of as it can be considered analogous to a pole of second order. Dividing each term by θ_0-4n^2 will keep all coefficients of the b's finite as $n\to\pm\infty$. The coefficients of the $e^{i(2n+\lambda)}$ become

$$\frac{\theta_1 b_{n-1}}{\theta_0-4n^2}+\frac{[\theta_0-(\lambda+2n)^2]b_n}{\theta_0-4n^2}+\frac{\theta_1 b_{n+1}}{\theta_0-4n^2}=0, \qquad (60)$$

$$n=-\infty\cdots-2,\ -1,\ 0,\ 1,\ 2,\ \cdots+\infty.$$

We may solve this infinite set by solving the infinite determinant $\Delta_{(\lambda)}=0$. The determinant $\Delta_{(\lambda)}=0$ is called Hill's determinant after Hill who first used it in order to give an approximate analytical solution to the problem of Lunar perturbations. A few rows of $\Delta_{(\lambda)}$ "around the center" are

$$\begin{array}{c|ccccccc|}
 & n=-3 & n=-2 & n=-1 & n=0 & n=1 & n=2 & n=3 \\
\hline
n=-2 & \cdots\frac{\theta_1}{\theta_0-16} & \frac{\theta_0-(\lambda-4)^2}{\theta_0-16} & \frac{\theta_1}{\theta_0-16} & 0 & 0 & 0 & 0\cdots \\
n=-1 & \cdots\ 0 & \frac{\theta_1}{\theta_0-4} & \frac{\theta_0-(\lambda-2)^2}{\theta_0-4} & \frac{\theta_1}{\theta_0-4} & 0 & 0 & 0\ \cdots \\
n=0 & \cdots\ 0 & 0 & \frac{\theta_1}{\theta_0} & \frac{\theta_0-\lambda^2}{\theta_0} & \frac{\theta_1}{\theta_0} & 0 & 0\ \cdots \\
n=1 & \cdots\ 0 & 0 & 0 & \frac{\theta_1}{\theta_0-4} & \frac{\theta_0-(\lambda+2)^2}{\theta_0-4} & \frac{\theta_1}{\theta_0-4} & 0\ \cdots \\
n=2 & \cdots\ 0 & 0 & 0 & 0 & \frac{\theta_1}{\theta_0-16} & \frac{\theta_0-(\lambda+4)^2}{\theta_0-16} & \frac{\theta_1}{\theta_0-16}\cdots
\end{array} = 0. \qquad (61)$$

Equating this determinant to zero generates eigenvalues (allowed values) of λ. From the Mathieu equation it may be seen that for λ real the solution is stable and for λ imaginary the solution is unstable.

In Appendix I it is shown that the roots of Hill's determinant are roots of the equation

$$\Delta_{(0)}\sin^2\tfrac{1}{2}\pi(\theta_0)^{\frac{1}{2}}=\sin^2\tfrac{1}{2}\pi\lambda, \qquad (62)$$

where $\Delta_{(0)}$ is Hill's determinant with $\lambda=0$.

For $\theta_0\ll 1$ we may neglect θ_0 in all terms of this determinant except in the row $n=0$. The determinant then becomes

2nd order $\Delta_{(0)}{}^2$

1st order $\Delta_{(0)}{}^1$

0th order $\Delta_{(0)}{}^0$

$$\left|\begin{array}{ccccccccc} \cdots & \dfrac{\theta_1}{-16} & 1 & \dfrac{\theta_1}{-16} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \dfrac{\theta_1}{-4} & 1 & \dfrac{\theta_1}{-4} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \dfrac{\theta_1}{\theta_0} & \boxed{1} & \dfrac{\theta_1}{\theta_0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \dfrac{\theta_1}{-4} & 1 & \dfrac{\theta_1}{-4} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \dfrac{\theta_1}{-16} & 1 & \dfrac{\theta_1}{-16} & \cdots \end{array}\right| = 0. \quad (63)$$

Clearly it is impossible to expand this determinant when it is infinite. However, since the series from which it is obtained is rapidly convergent, it should be possible to take successive approximations to the infinite determinant and obtain as exact a solution as we require. Consider, therefore, the 0th-, 1st- and 2nd-order determinants as indicated in Eq. (63). The required successive approximations are

$$\Delta_{(0)}{}^0 = 1, \quad (64)$$

$$\Delta_{(0)}{}^1 \begin{vmatrix} 1 & -\dfrac{\theta_1}{4} & 0 \\ \dfrac{\theta_1}{\theta_0} & 1 & \dfrac{\theta_1}{\theta_0} \\ 0 & -\dfrac{\theta_1}{4} & 1 \end{vmatrix} = 1 + \frac{\theta_1{}^2}{2\theta_0}, \quad (65)$$

$$\Delta_{(0)}{}^2 = 1 + \frac{\theta_1{}^2}{2\theta_0} - \frac{\theta_1{}^2}{32} - \frac{\theta_1{}^4}{128\theta_0} - \frac{\theta_1{}^4}{4096}. \quad (66)$$

For a typical inverted pendulum $\theta_1 = 2Alm/I \approx \frac{1}{50}$ so that the term $\theta_1{}^2/32$ and all higher terms in Eq. (66) are negligible. Since θ_0 is small the term $\theta_1{}^2/2\theta_0$ must be retained. Hence, in the work to follow, the first-order solution $\Delta_{(0)}{}^1$ of Hill's determinant is used. Since $\theta_0 \ll 1$,

$$\sin^2 \tfrac{1}{2}\pi(\theta_0)^{\frac{1}{2}} = \tfrac{1}{4}\pi^2\theta_0, \quad (67)$$

so that upon substitution of Eq. (65) into Eq. (62) we have

$$\left\{1 + \frac{\theta_1{}^2}{2\theta_0}\right\} \tfrac{1}{4}\pi^2\theta_0 = \sin^2 \tfrac{1}{2}\pi\lambda \quad (68)$$

or

$$\sin^2 \tfrac{1}{2}\pi\lambda = \tfrac{1}{4}\pi^2 \left[\frac{2A^2l^2m^2}{I^2} - \frac{4\omega_0{}^2}{\omega^2}\right] = Q. \quad (69)$$

We know that λ is real. Therefore $\sin \frac{1}{2}\pi\lambda$ must vary from 0 to $+1$. The stability condition must be such that

$$0 < Q < 1. \quad (70)$$

Thus for a given inverted pendulum subject to a given forcing frequency ω, we can analytically determine the limits on the amplitude A of the oscillations.

The lower limit of stability occurs for $Q=0$. Hence from Eq. (69) we have

$$A=\sqrt{2}I\omega_0/lm\omega. \tag{71}$$

The upper limit of stability occurs for $Q=1$. Hence from Eq. (69) we have

$$A=\sqrt{2}I(\pi^2\omega_0^2+\omega^2)^{\frac{1}{2}}/(\pi lm\omega). \tag{72}$$

To obtain a pendulum which is stable in the inverted position we select some convenient value of the amplitude between the two limits and use it to compute λ. Then

$$\lambda=\frac{2}{\pi}\operatorname{Arcsin}\left\{\frac{\pi}{2}\left(\frac{2A^2l^2m^2}{I^2}-\frac{4\omega_0^2}{\omega^2}\right)^{\frac{1}{2}}\right\}. \tag{73}$$

THE COEFFICIENTS b_n

We now turn our attention to the problem of obtaining expressions for the coefficients b_n of the power series in Eq. (56). Since the b's represent the amplitudes of the fundamental frequency and all of the overtones we can safely say that for large values of positive or negative n;

$$b_n\approx 0. \tag{74}$$

The recursion relation for the b_n is Eq. (60). Letting

$$C_n=\theta_0-(\lambda+2n)^2, \tag{75}$$

we have for $n\geqslant 0$

$$\begin{array}{llll}
n=0, & \theta_1b_1+\theta_1b_{-1}=-C_0b_0, & (76.1) & \\
n=1, & \theta_1b_2+\theta_1b_0=-C_1b_1, & (76.2) & \\
n=2, & \theta_1b_3+\theta_1b_1=-C_2b_2, & (76.3) & (76)\\
n=3, & \theta_1b_4+\theta_1b_2=-C_3b_3, & (76.4) & \\
n=4, & \theta_1b_5+\theta_1b_3=-C_4b_4, & (76.5) & \\
\vdots & \vdots \quad \vdots & \vdots &
\end{array}$$

We assume that $b_5\ll b_4$ and that $b_5\approx 0$. Then Eq. (76.5) becomes on solving for b_4

$$b_4=-\theta_1b_3/C_4, \tag{77}$$

which may be substituted into Eq. (76.4) to give

$$b_3=-\theta_1b_2\bigg/\left(C_3-\frac{\theta_1^2}{C_4}\right), \tag{78}$$

which may be substituted into Eq. (76.3) to give

$$b_2=-\cfrac{\theta_1b_1}{C_2-\cfrac{\theta_1^2}{C_3-\cfrac{\theta_1^2}{C_4}}}, \tag{79}$$

which may be substituted into Eq. (76.2) to give;

$$b_1=-\cfrac{\theta_1b_0}{C_1-\cfrac{\theta_1^2}{C_2-\cfrac{\theta_1^2}{C_3-\cfrac{\theta_1^2}{C_4}}}}. \tag{80}$$

$$\vdots$$

Now that we can see how the process goes we can generalize and obtain the expression for b_n,

$$b_n=\cfrac{-\theta_1b_{n-1}}{C_n-\cfrac{\theta_1^2}{C_{n+1}-\cfrac{\theta_1^2}{C_{n+2}-\cfrac{\theta_1^2}{C_{n+3}-\cfrac{\theta_1^2}{C_{n+4}-\ddots}}}}} \tag{81}$$

We may obtain any value of b_n as accurately as we please by taking enough terms in the continued fraction. For negative n an identical expansion is obtained except that C_1 is replaced by C_{-1}, C_2, by C_{-2}, and C_n by C_{-n}.

THE EXACT EQUATION OF MOTION

The solution to the Mathieu equation is the power series given in Eq. (56). We note that

$$e^{i\lambda Z}=\cos\lambda Z+i\sin\lambda Z, \tag{82}$$

$$e^{2inZ}=\cos 2nZ+i\sin 2nZ. \tag{83}$$

Since the imaginary part lacks any physical significance we drop them and the solution becomes

$$u=\cos\lambda Z\sum_{n=-\infty}^{n=+\infty}b_n\cos 2nZ. \tag{84}$$

Remembering that $b_0=A$ and $Z=\omega t/2$, the solution for u can be written out term by term. The terms for, $n=0 \pm 1, \pm 2$, yield

$$u=\cos\frac{\lambda\omega t}{2}\left\{\begin{aligned}&\left[A-\left(\cfrac{\theta_1 A}{C_1-\cfrac{\theta_1^2}{C_2-\cfrac{\theta_1^2}{C_3-\cfrac{\theta_1^2}{\vdots}}}}+\cfrac{\theta_1 A}{C_{-1}-\cfrac{\theta_1^2}{C_{-2}-\cfrac{\theta_1^2}{C_{-3}-\cfrac{\theta_1^2}{\vdots}}}}\right)\cos\omega t\right.\\ &+\left(\left(\cfrac{-\theta_1}{C_2-\cfrac{\theta_1^2}{C_3-\cfrac{\theta_1^2}{C_4-\cfrac{\theta_1^2}{\vdots}}}}\right)\left(\cfrac{-\theta_1 A}{C_1-\cfrac{\theta_1^2}{C_2-\cfrac{\theta_1^2}{C_3-\cfrac{\theta_1^2}{\vdots}}}}\right)\right.\\ &\left.\left.+\left(\cfrac{-\theta_1}{C_{-2}-\cfrac{\theta_1^2}{C_{-3}-\cfrac{\theta_1^2}{C_{-4}-\cfrac{\theta_1^2}{\vdots}}}}\right)\left(\cfrac{-\theta_1 A}{C_{-1}-\cfrac{\theta_1^2}{C_{-2}-\cfrac{\theta_1^2}{C_{-3}-\cfrac{\theta_1^2}{\vdots}}}}\right)\right)\cos 2\omega t\right.\\ &+[\text{worse and worse terms of higher order}]\end{aligned}\right\}. \quad (85)$$

Since $\theta_1 \ll 1$ and unless λ is such that C_1 or C_{-1} is very small, we can neglect all terms in the general solution with coefficients b_2 or b_{-2} or with greater $|n|$. Therefore, for an amplitude near the lower limit of stability we have for u, the actual motion of the pendulum

$$u=A\cos\frac{\lambda\omega t}{2}\left\{1-\left(\frac{\theta_1}{C_1}+\frac{\theta_1}{C_{-1}}\right)\cos\omega t+\cdots\right\}. \quad (86)$$

A check on Eq. (86) may be made as follows: From Eq. (75) we may write

$$\left.\begin{aligned}C_0&=\theta_0-\lambda^2, &&(87.1)\\ C_1&=\theta_0-4\lambda-\lambda^2-4, &&(87.2)\\ C_{-1}&=\theta_0+4\lambda-\lambda^2-4. &&(87.3)\end{aligned}\right\} \quad (87)$$

From Eq. 76.1 we see that the equation for $n=0$ can be written

$$\theta_1(b_1+b_{-1})=-C_0b_0=-C_0A. \quad (88)$$

But from Eq. 85, to a first approximation, we find

$$b_1+b_{-1}=-\frac{\theta_1 A}{C_1}-\frac{\theta_1 A}{C_{-1}}. \quad (89)$$

Thus

$$-\theta_1^2 A\left[\frac{1}{C_1}+\frac{1}{C_{-1}}\right]=-C_0A, \quad (90)$$

or

$$\theta_1^2(C_{-1}+C_1)=C_0C_1C_{-1}, \quad (91)$$

$$\theta_1^2(2\theta_0-2\lambda^2-8)=(\theta_0-\lambda^2)(\theta_0-4\lambda-\lambda^2-4)\times(\theta_0+4\lambda-\lambda^2-4). \quad (92)$$

This is a 6th-degree equation in λ and in principle could be solved as accurately as desired by Horner's or Newton's method. Obviously such a problem is worse than solving the Mathieu equation in the first place. Eq. (92), however, may be used as a check on the previous work. We know that θ_0 is a constant but θ_1 is a rather strong function of λ. Therefore, by arbitrarily assigning λ any value between 0 and 1 and by assuming that this is a valid equation for the generation of λ, we should be able to generate the same values for u by using Eq. (92) and by using Hill's method, Eq. (62).

We let $\lambda=1$ and $\theta_0=(1/1000)$.

Hill's method gives

$$\sin\tfrac{1}{2}\pi\lambda=1=\tfrac{1}{4}\pi^2\theta_0\left[1+\frac{\theta_1{}^2}{2\theta_0}\right], \tag{93}$$

from which we find

$$\theta_1\approx 0.906. \tag{94}$$

Hunter's method gives

$$\theta_1{}^2(2\theta_0-10)=(\theta_0-1)^2(\theta_0-9), \tag{95}$$

$$\theta_1\approx 0.947, \tag{96}$$

good agreement considering the approximations used. For $\lambda=0.77$ the agreement is exact to three decimals

$$\theta_1=0.871, \tag{97}$$

and for $\lambda=0$ the agreement is exact

$$\theta_1=[(5)^{\frac{1}{2}}/50]=0.0447. \tag{98}$$

APPENDIX I

Proof that the roots of Hill's determinant are roots of the equation:

$$\Delta_{(0)}\sin^2\tfrac{1}{2}\pi(\theta_0)^{\frac{1}{2}}=\sin^2\tfrac{1}{2}\pi\lambda. \tag{99}$$

It is desirable to make the elements of the main diagonal of Hill's determinant equal to unity. We can accomplish this by multiplying the nth row of Eq. (61) by

$$(\theta_0-4n^2)/[\theta_0-(\lambda+2n)^2] \quad \text{for all } n.$$

Then the left side of Eq. (61) becomes

$$\begin{vmatrix}
\cdots 0 & \frac{\theta_1}{\theta_0-(\lambda-4)^2} & 1 & \frac{\theta_1}{\theta_0-(\lambda-4)^2} & 0 & 0 & 0 & 0 & 0\cdots \\
\cdots 0 & 0 & \frac{\theta_1}{\theta_0-(\lambda-2)^2} & 1 & \frac{\theta_1}{\theta_0-(\lambda-2)^2} & 0 & 0 & 0 & 0\cdots \\
\cdots 0 & 0 & 0 & \frac{\theta_1}{\theta_0-\lambda^2} & 1 & \frac{\theta_1}{\theta_0-\lambda^2} & 0 & 0 & 0\cdots \\
\cdots 0 & 0 & 0 & 0 & \frac{\theta_1}{\theta_0-(\lambda+2)^2} & 1 & \frac{\theta_1}{\theta_0-(\lambda+2)^2} & 0 & 0\cdots \\
\cdots 0 & 0 & 0 & 0 & 0 & \frac{\theta_1}{\theta_0-(\lambda+4)^2} & 1 & \frac{\theta_1}{\theta_0-(\lambda+4)^2} & 0\cdots
\end{vmatrix}. \tag{100}$$

In effect, Hill's determinant, which is an infinite matrix, has been multiplied by a second infinite matrix which has nonzero elements only along the main diagonal. According to matrix theory

$$[A_{ij}]_n=C[B_{ij}]_n \tag{101}$$

if

$$C[A_b]^n{}_{b=1}=[B_b]^n{}_{b=1}. \tag{102}$$

That is,

$$Ca_{ij}=b_{ij} \tag{103}$$

or

$$Ca_{12}=b_{12} \tag{104}$$

$$\vdots$$

etc.

Thus $\Delta_{(\lambda)}$ has become a new determinant $\Delta_{i(\lambda)}$ defined by the equation

$$\prod_{n=-\infty}^{n=+\infty} \frac{\theta_0-4n^2}{\theta_0-(\lambda+2n)^2}[\Delta_{(\lambda)}]=\Delta_{i(\lambda)}, \tag{105}$$

from which

$$\Delta_{(\lambda)}=\Delta_{i(\lambda)}\prod_{n=-\infty}^{n=+\infty}\frac{\theta_0-(\lambda+2n)^2}{\theta_0-4n^2}; \tag{106}$$

now

$$\frac{\theta_0-(\lambda+2n)^2}{\theta_0-4n^2}=\frac{[(\theta_0)^{\frac{1}{2}}-(\lambda+2n)][(\theta_0)^{\frac{1}{2}}+(\lambda+2n)]}{\theta_0-(2n)^2}, \tag{107}$$

and shifting the brackets in Eq. (107) and dividing by $(2n)^2$ we have for the product in Eq. (106)

$$\prod_{n=-\infty}^{n=+\infty}\left(1+\frac{\lambda+\theta_0^{\frac{1}{2}}}{2n}\right)\left(1+\frac{\lambda-\theta_0^{\frac{1}{2}}}{2n}\right)\left[1-\left(\frac{\theta_0^{\frac{1}{2}}}{2n}\right)^2\right]^{-1}. \tag{108}$$

From the Theory of Complex Variables we may derive the equation

$$\frac{\sin Z}{Z}=\prod_{\substack{n=-\infty\\ n\neq 0}}^{n=+\infty}\left(1-\frac{Z}{n\pi}\right)e^{Z/n\pi}. \tag{109}$$

Eq. (109) may be expanded easily as follows:

$$\frac{\sin Z}{Z}=\prod_{n=-1}^{n=-\infty}\left(1-\frac{Z}{n\pi}\right)\times\prod_{n=-1}^{n=-\infty}e^{Z/n\pi}\prod_{n=1}^{n=\infty}\left(1-\frac{Z}{n\pi}\right)\prod_{n=1}^{n=\infty}e^{Z/n\pi}, \tag{110}$$

but

$$\prod_{n=-1}^{n=-\infty}e^{Z/n\pi}=e^{-Z/\pi}e^{-Z/2\pi}e^{-Z/3\pi}\cdots, \tag{111}$$

and

$$\prod_{n=1}^{n=+\infty}e^{Z/n\pi}=e^{Z/\pi}e^{Z/2\pi}e^{Z/3\pi}\cdots, \tag{112}$$

so that the second and fourth products in Eq. (110) when taken together give just 1. Finally,

$$\prod_{n=-1}^{n=-\infty}\left(1-\frac{Z}{n\pi}\right)=\left(1+\frac{Z}{\pi}\right)\left(1+\frac{Z}{2\pi}\right)\left(1+\frac{Z}{3\pi}\right)\cdots \tag{113}$$

$$=\prod_{n=1}^{n=\infty}\left(1+\frac{Z}{n\pi}\right), \tag{114}$$

so that on substituting Eqs. (111), (112), and (114) into Eq. (109) we have, finally,

$$\frac{\sin Z}{Z}=\prod_{n=1}^{n=\infty}\left[1-\frac{Z}{n\pi}\right]\left[1+\frac{Z}{n\pi}\right]=\prod_{n=1}^{n=\infty}\left[1-\left(\frac{Z}{n\pi}\right)^2\right]. \tag{115}$$

Since n is squared in this result we could integrate from $n=-1$ to $n=-\infty$ and obtain exactly the same result. Hence

$$\frac{\sin Z}{Z}=\prod_{n=-1}^{n=-\infty}\left[1-\left(\frac{Z}{n\pi}\right)^2\right] \tag{116}$$

and using Eqs. (115) and (116) we have

$$\frac{\sin^2 Z}{Z^2}=\prod_{\substack{n=-\infty\\ n\neq 0}}^{n=+\infty}\left[1-\left(\frac{Z}{n\pi}\right)^2\right]. \tag{117}$$

With these expressions for $\sin Z/Z$ and $\sin^2 Z/Z^2$, we should be able to express the modified Hill's determinant $\Delta_{i(\lambda)}$ in terms of them. Therefore we return to Eq. (108), and after an expansion similar to that employed in Eq. (109), we find

$$\prod_{n=-\infty}^{n=+\infty}\left(1+\frac{\lambda-\theta_0^{\frac{1}{2}}}{2n}\right)\left(1+\frac{\lambda+\theta_0^{\frac{1}{2}}}{2n}\right)\left[1-\left(\frac{\theta_0^{\frac{1}{2}}}{2n}\right)^2\right]^{-1}=\frac{\prod_{n=1}^{\infty}\left[1-\left(\frac{\lambda+\theta_0^{\frac{1}{2}}}{2n}\right)^2\right]\prod_{n=1}^{\infty}\left[1-\left(\frac{\lambda-\theta_0^{\frac{1}{2}}}{2n}\right)^2\right]}{\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left[1-\left(\frac{\theta_0^{\frac{1}{2}}}{2n}\right)^2\right]}. \tag{118}$$

Letting

$$Z_1/n\pi=[\lambda+\theta_0^{\frac{1}{2}}]/2n, \tag{119.1}$$

$$Z_2/n\pi=[\lambda-\theta_0^{\frac{1}{2}}]/2n, \tag{119.2}$$

$$Z_3/n\pi=\theta_0^{\frac{1}{2}}/2n, \tag{119.3}$$

and using Eqs. (116) and (117) we may write Eq. (118) as

$$\prod_{\substack{n=-\infty\\n\neq 0}}^{n=+\infty}\frac{\left(1+\dfrac{\lambda+\theta_0^{\frac{1}{2}}}{2n}\right)\left(1+\dfrac{\lambda-\theta_0^{\frac{1}{2}}}{2n}\right)}{1-\left(\dfrac{\theta_0}{2n}\right)^2}=\frac{\left(\dfrac{\sin Z_1}{Z_1}\right)\left(\dfrac{\sin Z_2}{Z_2}\right)}{\dfrac{\sin^2 Z_3}{Z_3}}=\frac{\dfrac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})}{\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})}\cdot\dfrac{\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}}{\dfrac{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}{(\frac{1}{2}\pi)^2\theta_0}}. \tag{120}$$

Multiplying Eq. (120) by $(\theta_0-\lambda^2)/\theta_0$ we obtain

$$\prod_{\substack{n=-\infty\\n\neq 0}}^{n=+\infty}\frac{\left(1+\dfrac{\lambda+\theta_0^{\frac{1}{2}}}{2n}\right)\left(1+\dfrac{\lambda-\theta_0^{\frac{1}{2}}}{2n}\right)}{1-\left(\dfrac{\theta_0^{\frac{1}{2}}}{2n}\right)^2}\left(\frac{\theta_0-\lambda^2}{\theta_0}\right)=-\frac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}. \tag{121}$$

This is the product portion of the expression for $\Delta_{(\lambda)}$, i.e.

$$\Delta_{(\lambda)}=-\Delta_{i(\lambda)}\frac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}. \tag{122}$$

Louville's theorem states that a function which is analytic for all finite values of the independent variable and is everywhere bounded is a constant. We therefore define a new function D which satisfies Louville's theorem by means of the equation

$$D=\Delta_{1(\lambda)}-K[\cot\tfrac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})-\cot\tfrac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})]; \tag{123}$$

D is a constant. It is also a completely general function. Therefore λ may assume any value in D which we please. If $\lambda=\infty$ then $D=1$ because all off-diagonal terms in Eq. (100) vanish and the product of the main diagonal is 1. Consequently Eq. (123) becomes

$$\Delta_{1(\lambda)}=1+K[\cot\tfrac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})-\cot\tfrac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})], \tag{124}$$

and setting $i=1$ and substituting Eq. (124) into Eq. (122) we find

$$\Delta_{(\lambda)}=-\frac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}\times\{1+K[\cot\tfrac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})-\cot\tfrac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})]\}. \tag{125}$$

The cotangents in Eq. (125) may be expanded in terms of cosines and sines. After a great deal of simplifying we finally arrive at the form

$$\Delta_{(\lambda)}=-\frac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}+2K\cot\tfrac{1}{2}\pi(\theta_0^{\frac{1}{2}}). \tag{126}$$

If we put $\lambda=0$ we can determine K. We have

$$\Delta_{(0)}=1+2K\cot\tfrac{1}{2}\pi(\theta_0^{\frac{1}{2}}). \tag{127}$$

Thus

$$2K=[\Delta_{(0)}-1]/\cot\tfrac{1}{2}\pi(\theta_0^{\frac{1}{2}}), \tag{128}$$

and $\Delta_{(\lambda)}$ becomes

$$\Delta_{(\lambda)}=\Delta_0-1-\frac{\sin\frac{1}{2}\pi(\lambda+\theta_0^{\frac{1}{2}})\sin\frac{1}{2}\pi(\lambda-\theta_0^{\frac{1}{2}})}{\sin^2\frac{1}{2}\pi(\theta_0^{\frac{1}{2}})}. \tag{129}$$

The sines in Eq. (129) may be expanded and yield after some simplifying

$$\Delta_{(\lambda)}=\Delta_{(0)}-[\sin^2\tfrac{1}{2}\pi\lambda/\sin^2\tfrac{1}{2}\pi(\theta_0)^{\frac{1}{2}}]. \tag{130}$$

Since Hill's determinant $\Delta_{(\lambda)}=0$, the roots of Eq. (130) are given by

$$\Delta_{(0)}\sin^2\tfrac{1}{2}\pi(\theta_0^{\frac{1}{2}})=\sin^2\tfrac{1}{2}\pi\lambda. \tag{131}$$

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