Toy model to solve the inverse problem of EEG source analysis using regularization

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November 29, 2022

Abstract

Solving the inverse problem in EEG to identify the source of a signal requires a number of assumptions to constrain the solution space. This report provides a very simple modelling approaches to generate source activation signals of each source points inside the brain. We again employ a toy model of how the signals from the sources in the brain is captured as EEG signals at the electrodes at the scalp of the head a.k.a head model. Then we try to calculate the backwards the solution of the inverse problem of finding the source activations responsible for the EEG signals using the method of regularization. The method of regularization for undetermined set of equations of our model is equivalent to and is reformulated in terms of simple matrix algebra.

1 Introduction

In the wide range of STEM fields, modeling refers to given a set of parameters, we predict the results and so this is referred to as forward modelling. But in the case of inverse problem, we start from the results or the measured data and then try to infer the parameters that *caused* those measurements. In contrast to the forward problem, the inverse problem does not have unique solution, more specifically there can be multiple set of parameters that might explain the same measurement data.[8]

In the field of neuroscience, the EEG inverse problem is to find the *source localization* which is the cause and is responsible for measured EEG data. It means to identify where in the brain a particular type of activity originates based on surface EEG recording. The application of this is for patients suffering from brain disorders like epilepsy, we would like to know the brain regions/site of seizure initiation or to understand the origin of various stimulus responses.[9]

In this report, we present initially the forward model to generate EEG signals. We consider the brain as a sphere which is a source space and embedded inside it are the source points from which the activation signals arise. We then construct an electrode helmet encapsulating the source space with the electrodes evenly placed on it. The source activations are modelled in terms of Vector Auto regression model of p = 2 order(VAR-2)[2][4] and assume that each source points depends on its activations of the past and are independent of other source points. With this gather the EEG signals.

To solve the Inverse problem to identify the source of an EEG signal we need three things:[10][6]

1. Source model - The source model tells us the 3D positions i.e. xyz coordinates of the source points, which is also known as source space. It is assumed that EEG signals are generated by these sources.
2. Head model - It is used to describe how the signals from sources flow to the outer surface/scalp of the head where the electrodes are placed.
3. EEG data - It is obtained from the forward model.

In our project, the source model and Head model are common to both the forward model and inverse model and is explained in detail in the methods section. The only assumption made in this project for the Head model is that the signal obtained from the electrodes depends only on the distance of these electrodes from the source points. This assumption doesn't take into consideration that the source points being as dipoles or the conductivity inside the head as seen in vast literature[7][6]. In this aspect, our model is a crude model which describes the flow of signals through the source space from the source points to the scalp of the head.

2 Methods

2.1 Brain-Electrode Spherical toy model(Head model)

A simplistic way to model the brain is in the form of sphere. The electrode helmet that is placed on the head is modelled as a hemisphere with the electrodes evenly placed on the surface using the *sunflower spiral algorithm*[1] which is generalised to 3D. An image of hemispherical helmet of radius = 1 with 32 evenly placed electrodes(marked in red and annotated from $E0$ to $E31$) is shown in the Fig. 1a as an example.

The brain is modelled as a sphere and is considered as *source space* as it embeds within it the *source points* responsible for the EEG signals. The placement of these source points inside the sphere is as follows:

1. Consider a cube whose side length is equal to the diameter of the sphere and which is also encapsulating the sphere.
2. By distinctly considering the points evenly spaced along the x, y, z directions, the cube is remodelled as a cubic lattice which can also be visualized as voxels grouped together to form a cube.
3. The center of each voxel is then considered as source point and out of all the source points only those which lie inside the sphere are considered as the correct source points of the brain.

The Fig. 1b is shown as an example of brain considered as a sphere of radius = 1. It is voxellated into 56 cubes with their voxel centers which is being considered as source points are evenly placed inside the sphere.

For the case of simplicity, as shown in the Fig. 1c we have considered an electrode helmet of radius = 1 whose center is at $(0, 0, 0)$ with just 5 electrodes $E_i, i \in \{0, 1, 2, 3, 4\}$ evenly placed on it. The spherical brain is of radius 0.5 and is z-axis shifted with respect to electrode-helmet coordinates by 0.45, thus the center of the spherical brain is $(0, 0, 0.45)$. An imaginary cube encapsulating the spherical brain is voxellated(4 lattice points per each side), and the number of centers of these voxels lying inside the sphere are 12, which becomes the source points.

2.2 Source model and EEG data

The EEG is modelled in terms of simplified linear model[2][4] by approximating it as a linear mixture of cortical source activations as given by the Eq. 1

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) \tag{1}$$

where:

1. $\mathbf{y}$(t) is vector of EEG signals at time instant t observed at the electrodes, whose dimension is m where m refers to number of electrodes.
2. $\mathbf{A}$ is the mixing matrix of dimension (m, n) which transforms every source activation $s(t) of\, dim(n, 1)$ into observable EEG signal. Here n refers to number of source points.
3. $\mathbf{s}$(t) is the source activation of n source points at time t. $dim(\mathbf{s}(t)) = n$.

The entries of the matrix $\mathbf{A}$ is modelled as inverse square of the euclidean distance between electrodes and the source points. It is given by:

$$a_{ij} = 1/d_{ij}^2, \tag{2}$$

where,

1. d_{ij} is the euclidean distance between electrode i and source point j
2. $i \in \{0, 1, 2..., m-1\}$ and $j \in \{0, 1, ...n-1\}$

The matrix A is then scaled to values between 0 and 1.

The source activations $\mathbf{s}(t)$ are modeled as linear *Vector Auto-Regressive VAR-p Processes*[2][4] with p = 2 being order of dependence on its past.

$$\mathbf{s}(t=0) = \mathbf{c}_0 * sin(2\pi \mathbf{f} t)|_{t=0} + \mathbf{c}_1 * sin(2\pi \mathbf{f} t)|_{t=1} + \mathbb{N}(0, 0.1) \tag{3}$$

$$\mathbf{s}(t=1) = \mathbf{c}_0 * sin(2\pi \mathbf{f} t)|_{t=1} + \mathbf{c}_1 * \mathbf{s}(0) + \mathbb{N}(0, 0.1) \tag{4}$$

$$\mathbf{s}(t) = \mathbf{c}_0 * \mathbf{s}(t-1) + \mathbf{c}_1 * \mathbf{s}(t-2) + \mathbb{N}(0, 0.1), \quad \forall \quad t > 2 \tag{5}$$

where we have:

1. the coefficients terms: $\{C_0, C_1\} \sim \mathbb{U}(0, 0.6)$, and $\{C_0, C_1\} \in \mathbb{R}$. $dim(C_0) = dim(C_1) = n$. With the assumption that all the entries of C_0 are less than or equal to elements of C_1. These coefficients encapsulate that dependence of signal activation $s_i(t)$ becomes lesser and lesser consecutively on its past values.
2. the noise term is given by Gaussian with its zero mean and low variance.
3. f is vector of frequencies of $dim(n)$ in the range of $[10, 50]$ since the source frequency is usually assumed to lie within

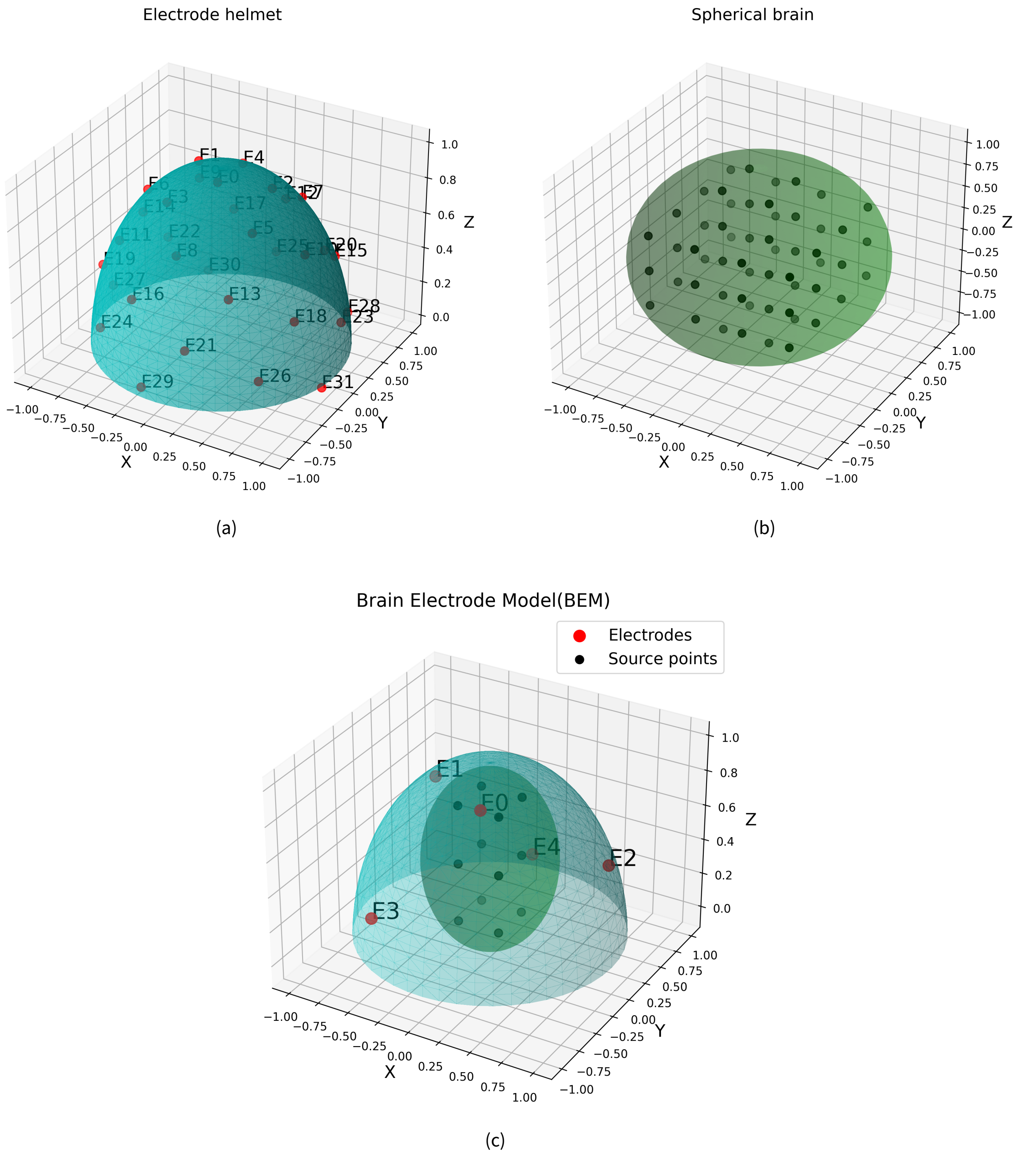

Figure 1. (a) Electrode helmet modelled as an hemisphere of radius = 1. Here in the example figure, the 32 electrodes are placed evenly on the surface using sunflower spiral algorithm generalized to 3D. **(b)** The brain is modelled as a sphere. This sphere which embeds the source points is called a source space. The source points are actually the centers of the voxels where this sphere is voxellated(divided into equal voxels) The side length of the voxel or square lattice cube is given by (2*r/(nlattice - 1)) with nlattice=6 are the square lattice encapsulating the sphere. **(c)** A toy Brain Electrode Model with just 5 electrodes(marked in red) and a brain as a sphere smaller in size relative to the electrode helmet. The source space has 12 source points(marked in black) evenly distributed inside it. The position of brain is adjusted along z axis in relation to the electrode helmet's coordinates.

this range.

The initial states of the source activations is assumed and are modelled in terms of sinusoidal waves whose frequencies lie within range of [10, 50] as seen in Eqns. 3. Note that in the above equations, time is in units and later the step size was chosen to be equal to $\Delta t = 1/f_s$ where $f_s \approx 512Hz$ = sampling frequency of the electrodes for the EEG signals.

2.3 Solving the inverse problem

Given the equation concerning our problem[6]:

$$\mathbf{AX} = \mathbf{b} \tag{6}$$

where:

1. $\mathbf{A}$ is the physical model describing the relation between the electrodes and the source points as defined in the Eq. 2
2. $\mathbf{b}$ is the preprocessed EEG signals.
3. $\mathbf{X}$ of $dim = n$ is the vector representing the source localization that we are supposed to find and results in the EEG signals $\mathbf{b}$.

It has the following properties:

1. It is Undetermined set of linear equations i.e. $m < n$, there are more variables than equations.
2. $\mathbf{X}$ is underspecified, i.e. there are many choices of X leading to the same b.
3. Given the matrix $rank(\mathbf{A}) = m$ i.e. full rank, for each $b \in \mathbb{R}^m$, we have set of solutions of the form $\{X|AX = b\} = \{X_p + X_n\}|X_n \in \mathbb{N}(A)$.
4. $dim(\mathbb{N}(A)) = n - m$

X_p is any particular solution such that $AX_p = b$. X_n characterizes available choices in solution. The process to do is we need to search for solutions of X so as to minimize X_n. It is a minimization problem given by:

$$min||X||_2^2 \tag{7}$$

subjected to the constraint given by Eq. 6. This optimization problem in literature is reformulated using Lagrange multipliers using matrix right inverse of $\mathbf{A}$ i.e. $\mathbf{A}^T(\mathbf{AA}^T)^{-1}$. The solution is given by:

$$\mathbf{X} = \mathbf{A}^T(\mathbf{AA}^T)^{-1}\mathbf{b} \tag{8}$$

It is shown and well established in the literature[3] that the Eq. 7 and Eq. 8 are equivalent.

3 Results

3.1 Simulated EEG data from VAR-2 model

The EEG data was simulated by using source data obtained from VAR-2 model given by Eq. 3 and using transformation matrix A, which transforms the source activations to EEG raw signals given by Eq. 1. The raw EEG data observed from the 5 electrodes $\{E0, E1, E2, E3, E4\}$ is shown in the Fig. 2.

3.2 Preprocessing of raw EEG data

Band pass filtering: The relevant range of frequencies so as to remove the noise associated with the EEG raw data signals is between 1Hz to 50 Hz approximately. A band pass filter with low cut = 1Hz and high cut = 50 Hz along with frequency of sampling of the filter equal to one fourth of sampling frequency of the EEG electrodes i.e. $f_s = 512Hz/4$ was applied to raw EEG data seen in Fig. 2. The built-in functions of *scipy.signal* module of python like *butter_bandpass_filter, lfilter* were used to achieve it.

Re-referencing the signals: Re referencing of the signals is done using average referencing. It is done by computing the average of the signal at all EEG electrodes and subtracting it from the EEG signal at every electrode for every time point. After filtering the eeg signals using the band pass filtering as explained previously, the same filtered signals were re-referenced using the average reference.

The resulting clean EEG signals is shown in the Fig. 3

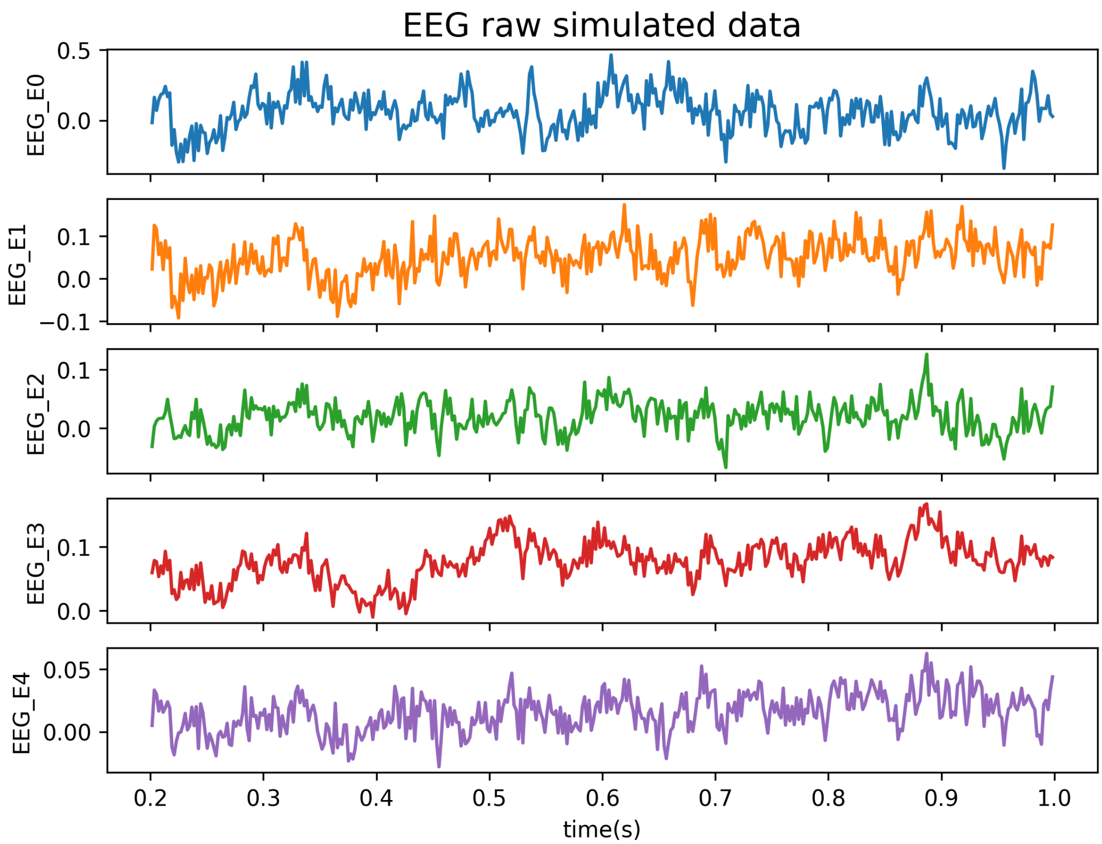

Figure 2. Raw EEG simulated data.

3.3 Solution of Inverse problem

Using Eq. 8, with the pre-processed EEG data as **b** and **A** which is already found from Eq. 2, we can compute **X** which are one among the many values of source localization points. The matrix **A** was found to have full rank and thus using the method of regularization and the solution to the inverse problem stated in 8 is valid for our case.

When we consider the pearson correlation of the source activations time series from Eq. 3, we observe that source points are truly independent of each other in accordance to the constructed model which is shown in Fig. 4a. However, when the source time series values were obtained through the solution of inverse problem of regularization, it was observed that there exists some correlations among the source points as seen in Fig 4b which is contrary to the Fig. 4a. The correlation coefficient between these two pearson correlation matrices i.e. Fig. 4b and Fig. 4a is 0.4507.

4 Conclusions

We here summarize the main conclusions of this report. A simple model of brain as single sphere was considered with the electrode helmet encapsulating the brain/source space. The brain electrode model was crudely considered with the signals obtained at the electrodes being proportional to inverse squared euclidean distance from the source points. A simple VAR-2 model was considered by making assumption that the activations of a given source $s_i(t)$ depends only on its own past i.e. $s_i(t-1)$ and $s_i(t-2)$ not on other sources, thus neglecting the Granger causality. The pearson correlation of the source activation time series from VAR-2 has 1 along the diagonal and ~ 0 elsewhere and is in accordance with the model constructed for source activations. However, pearson correlation of the source localizations found as solution to the inverse problem do not exactly match with the one found earlier since we also observe positive and negative correlations among different sources. This is explained from the fact that the sources closer to each other in terms of euclidean distance, have higher correlations than otherwise.

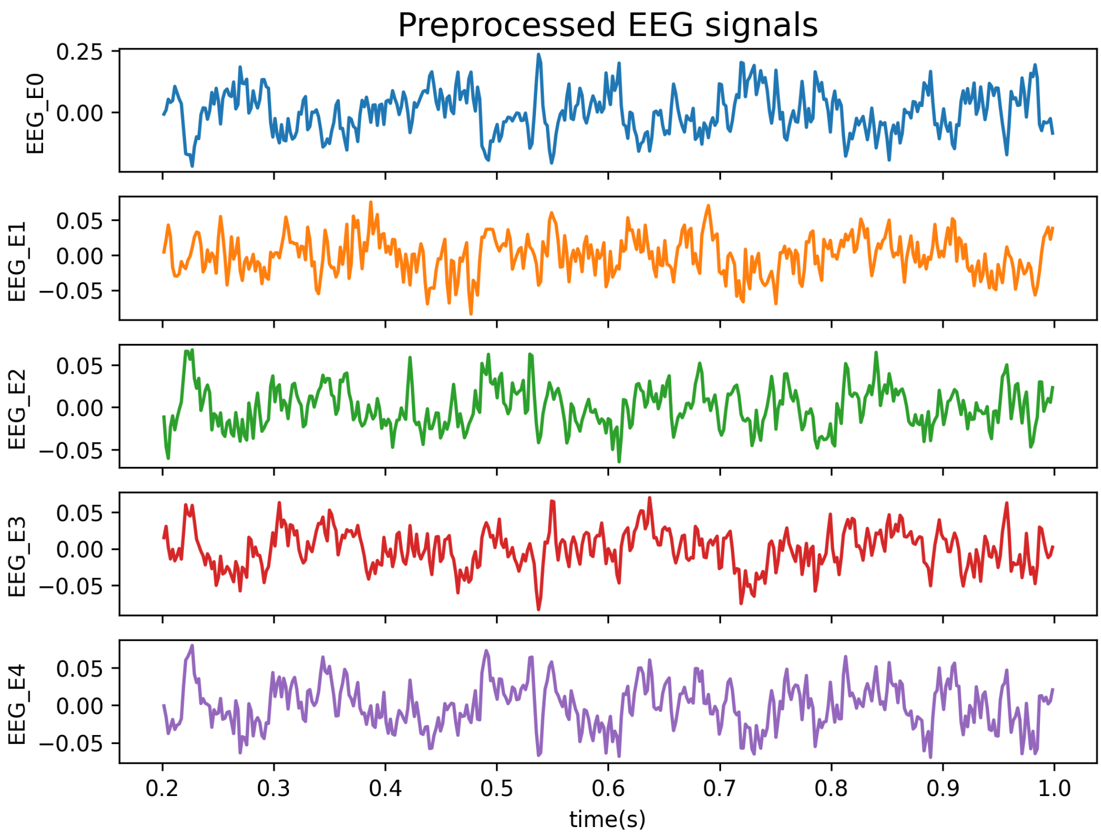

Figure 3. Band pass filtered with low-cut = 1Hz, high-cut = 50Hz, fs=512Hz/4 where 512Hz is optimal sampling frequency of the Electrodes EEG. EEG signals were re-referenced with respect to average EEG signals considering all the electrodes.

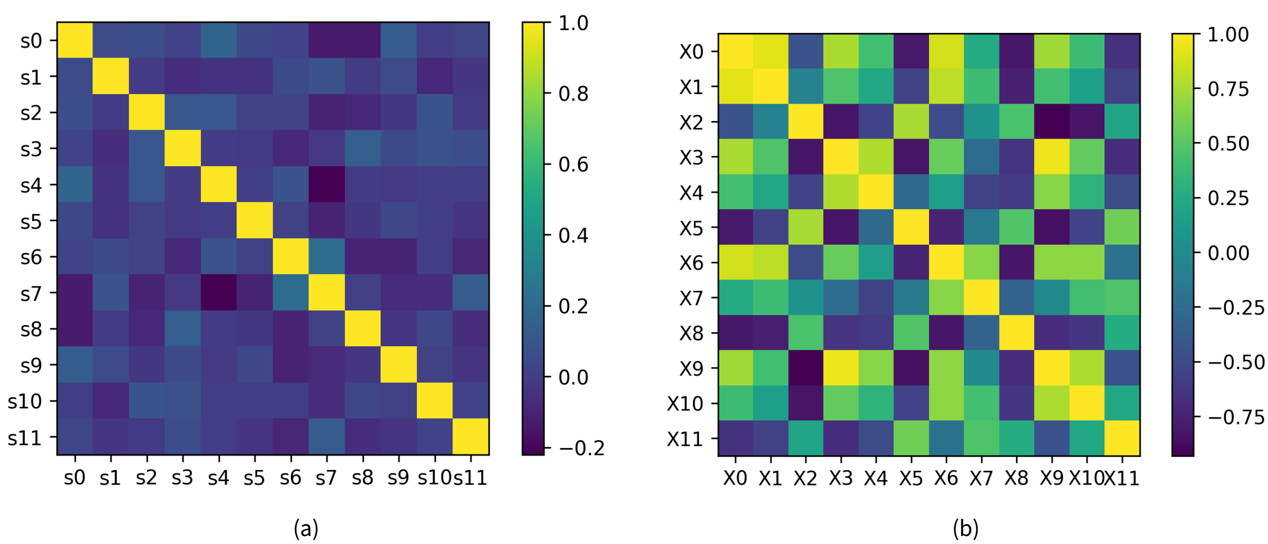

Figure 4. (a) Pearson product-moment correlation of source activations from VAR-2 model found from set of Eqns. 3. **(b)** Pearson product-moment correlation of source localizations found as solution to the Inverse problem from Eq. 8

5 Future perspectives:

- A proper forward modelling approach as described by et al.[7] to generate EEG signals can be further researched to improve upon the model. In the toy model presented in this report, we never explored on the possibility of considering the source points as dipoles with a potential being measured at the electrodes.

- The model suggested in this report is simply a 2 layered toy model with a layer encompassing the source points and a layer of scalp with electrodes on it. However, *Vatta et al.*[11] have proposed a *Realistic and Spherical Head Modeling for EEG Forward Problem Solution* which could taken as inspiration for Head model.
- Our model for the forward modelling consists of using just the very simple VAR-2 model where we have assumed that the source activations of the source points depends only on its own past but not on each other. This is quite a strong assumption because we are considering that the source points are independent of each other, which is not true as there are regions of brain which influence each other and are synchronous [5]. As mentioned in the paper by *Chostekova et al.*[4], a Multivariate Granger causality Auto regressive model can be employed instead of VAR-2 model for generating source activations.

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