Lecture # 11 (IC152)

Definition (Inner product on vector space over a field $\mathbb{F}$ of $\mathbb{R}$ or $\mathbb{C}$)

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$$

$$(\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle \in \mathbb{F}$$

Properties, $\forall\, \alpha, \beta, \gamma \in V$, $c \in \mathbb{F}$.

(Positivity) i) $\langle \alpha, \alpha \rangle \geq 0 \;\forall\, \alpha \in V$ & $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$

(linearity) ii) $\langle \underline{\alpha + \beta}, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

iii) $\langle c\alpha, \beta \rangle = c \langle \alpha, \beta \rangle$

(conjugate symmetry) iv) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$

Ex. $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R} / \mathbb{C}$

$x = (x_1, x_2, \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$

$\sum^{n} x_i \bar{y}_i$ defines an inner product on the $\langle x, y \rangle = \sum_{j=1} x_j \bar{y_j}$ ✓ ,

$\mathbb{F}^n$: (Standard inner product)

$\langle x, y \rangle_T = \langle Tx, Ty \rangle$

Ex $\quad M_{m\times n}(\mathbb{F})$, $\quad A, B \in M_{m\times n}(\mathbb{F})$

$$\langle A, B \rangle = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}\, \overline{b_{jk}}$$ ✓ ✓

Ex: $V = C([0,1]; \mathbb{F})$ — V. space of continuous functions on $[0,1]$

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\,dx$$ is an inner product

$f, g \in V$

$(f+g)(x) = f(x) + g(x)$

Verify: 1) $\langle f, f \rangle = \int_0^1 f(x)\overline{f}(x)\,dx = \int_0^1 |f(x)|^2 dx \geq 0$

but $\langle f, f \rangle = 0 \Rightarrow \int_0^1 |f(x)|^2 dx = 0 \Rightarrow f \equiv 0$

2) $\langle f+g, h \rangle$

$= \int_0^1 (f+g)(x)\,\overline{h}(x)\,dx$

$\because \int_0^1 g(x)\,dx = 0$

* $g(x) \geq 0$
* g is contin

$$= \int_0^1 (f(x)+g(x))\overline{h(x)}$$

$$= \int_0^1 f(x)\overline{h(x)} + \int_0^1 g(x)\overline{h(x)}\,dx$$

$$= \langle f, h\rangle + \langle g, h\rangle$$

$\Rightarrow g = 0$

If not, $x_0 \in [0,1]$

$g(x_0) \neq 0$

By continuity

$[x_0-\delta, x_0+\delta]$

s.t. $g(x) > 0$ in $[x_0-\delta, x_0+\delta]$

$$\int_0^1 g(x)dx \geqslant \int_{x_0-\delta}^{x_0+\delta} g(x) > 0$$

contradiction

3) $$\overline{\langle f, g\rangle} = \overline{\int f(x)\overline{g(x)}\,dx}$$

$$= \int \overline{f(x)\overline{g(x)}}\,dx$$

$$= \int \overline{f}(x)\, g(x)\,dx$$

$$= \int g(x)\overline{f}(x)\,dx$$

$$= \langle g, f\rangle$$

4) $$\langle cf, g\rangle = \int (cf)(x)\overline{g(x)}\,dx$$

V.

$(cf)(x) = cf(x)$

$$= \int cf(x)\overline{g(x)}\,dx$$

$$= c\int f(x)\overline{g(x)}\,dx$$

$$= c\langle f, g\rangle$$

Ex: V. space of polynomials over a field $F = \mathbb{R}$ or $\mathbb{C}$

Let $f, g \in P(F)$ $\quad \langle f, g\rangle = \int_0^1 f(x)\,\overline{g(x)}\,dx$ ✓

Definition ("length")

It is defined as a function on V → by

"norm" $\leftarrow \|\cdot\| : V \longrightarrow \mathbb{R}^+$ as

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha\rangle}$$

$V = \mathbb{R}^2$,

$$\|(2,3)\| = \langle (2,3),(2,3)\rangle^{1/2}$$

$$= (2\cdot 2 + 3\cdot 3)^{1/2}$$

$$= \sqrt{13}$$

Definition (Inner product space)

A vector space equipped with an inner product is known as inner product space (ips).

$(V, \langle \cdot \rangle)$

<u>Example</u>: $V = \mathbb{R}^n$, $\langle x, y \rangle = \sum_{j=1}^{n} x_j \bar{y}_j$ ✓

$(V, \langle \, , \, \rangle)$ is an ips.

Let us see $\langle \alpha, \beta \rangle = \mathrm{Re}\langle \alpha, \beta \rangle + i\,\mathrm{Im}\langle \alpha, \beta \rangle$ ✓

$$\langle \alpha, \beta \rangle = \mathrm{Re}\langle \alpha, \beta \rangle + i\,\mathrm{Re}(-i\langle \alpha, \beta \rangle)$$

$$\langle \alpha, \beta \rangle = \mathrm{Re}\langle \alpha, \beta \rangle + i\,\mathrm{Re}\langle \alpha, i\beta \rangle$$ ✓ ✓

$z = x + iy$

$y = \mathrm{Im}(z)$

$-iz = -ix + y$

$\mathrm{Re}(-iz) = +y$

$\Rightarrow \mathrm{Im}(z) = \mathrm{Re}(-iz)$

Observe that

$$\|\alpha + \beta\|^2 = \langle \alpha + \beta, \alpha + \beta \rangle$$ ✓

$$= \langle \alpha, \alpha + \beta \rangle + \langle \beta, \alpha + \beta \rangle$$

$$\overset{?}{=} \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle$$

$$= \|\alpha\|^2 + \langle \alpha, \beta \rangle + \overline{\langle \alpha, \beta \rangle} + \|\beta\|^2$$

$\langle \alpha, \alpha + \beta \rangle$

$= \overline{\langle \alpha + \beta, \alpha \rangle}$

$= \overline{\langle \alpha, \alpha \rangle} + \overline{\langle \beta, \alpha \rangle}$

$= \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle$

$$\|\alpha + \beta\|^2 = \|\alpha\|^2 + 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \|\beta\|^2$$

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 - 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \|\beta\|^2$$ ✓

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$= 2\,\mathrm{Re}(z)$

If $\mathbb{F} = \mathbb{R}$ then

$$4\,\mathrm{Re}\langle\alpha,\beta\rangle = 4\langle\alpha,\beta\rangle = \|\alpha+\beta\|^2 - \|\alpha-\beta\|^2$$

$$\boxed{\langle\alpha,\beta\rangle = \frac{1}{4}\|\alpha+\beta\|^2 - \frac{1}{4}\|\alpha-\beta\|^2}$$

Polarization identity

If $\mathbb{F} = \mathbb{C}$

$$\langle\alpha,\beta\rangle = \mathrm{Re}\langle\alpha,\beta\rangle + i\,\mathrm{Re}\langle\alpha, i\beta\rangle$$

$$4\,\mathrm{Re}\langle\alpha, i\beta\rangle = \|\alpha+i\beta\|^2 - \|\alpha-i\beta\|^2$$

$$\boxed{\langle\alpha,\beta\rangle = \frac{1}{4}\|\alpha+\beta\|^2 - \frac{1}{4}\|\alpha-\beta\|^2 + \frac{i}{4}\|\alpha+i\beta\|^2 - \frac{i}{4}\|\alpha-i\beta\|^2}$$

Properties of "norm" or "length" :

V – inner product space $(V, \langle\cdot,\cdot\rangle)$

i) $\boxed{\|c\alpha\| = |c|\,\|\alpha\|}$

2) $\boxed{\|\alpha\| > 0}$ if $\alpha \neq 0$ $\quad \|\alpha\| = 0$ if $\alpha = 0$

$\mathbb{C} \ni z = x + iy$

$|z|^2 = z\bar{z}$

3) $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$ (Cauchy-Schwartz inequality)

4) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ (triangle inequality)

$= (x+iy)x$
$(x-iy)$

$|z|^2 = x^2 + y^2$

$|z| = \sqrt{x^2+y^2}$

$\mathbb{R} \ni z$, $|z| = z$ if $z \geq$
$= -z$ if $z <$

Proof: 1) $\|c\alpha\|^2 = \langle c\alpha, c\alpha \rangle$

$= c \langle \alpha, c\alpha \rangle = c\bar{c} \langle \alpha, \alpha \rangle$

$= |c|^2 \|\alpha\|^2$

$\|c\alpha\| = |c| \, \|\alpha\|$

2) $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$

> 0 if $\alpha \neq 0$

$\Rightarrow \|\alpha\| > 0$ if $\alpha \neq 0$ and $\|\alpha\| = 0$ if $\alpha = 0$

Let us assume 3) is correct !!

4) $\|\alpha + \beta\|^2 = \langle \alpha + \beta, \alpha + \beta \rangle$

$= \|\alpha\|^2 + 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \|\beta\|^2$

$\leq \|\alpha\|^2 + 2\,|\langle \alpha, \beta \rangle| + \|\beta\|^2$

$\leq \|\alpha\|^2 + 2\|\alpha\| \, \|\beta\| + \|\beta\|^2$

$(\|\alpha\| + \|\beta\|)^2$

$\left.\begin{array}{l} \mathrm{Re}(z) \leq |z| \\ \mathrm{im}(z) \leq |z| \end{array}\right\}$

$$= (\|\alpha\| + \|\beta\|)$$

$$\Rightarrow \|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

$\langle \alpha, 0 \rangle = 0 \;\forall\, \alpha \in V$ – ips. ? ✓

$\left\{ \begin{array}{l} \langle \alpha, \beta \rangle = 0 \;\forall\, \beta \text{ in } V \\ \quad \Rightarrow \alpha = 0 \end{array} \right.$

As 3) holds good for $\alpha = 0$, we can start with $\alpha \neq 0$

$$\boxed{\gamma = \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2} \alpha}$$

$$0 \leq \langle \gamma, \gamma \rangle = \left\langle \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha \,,\, \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha \right\rangle$$

$$= \left\langle \gamma \,,\, \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha \right\rangle$$

$$= \langle \gamma, \beta \rangle - \overline{\frac{\langle \beta, \alpha \rangle}{\|\alpha\|}} \langle \gamma, \alpha \rangle \checkmark$$

$$\langle \gamma, \alpha \rangle = \langle \beta, \alpha \rangle - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2} \ldots$$

$0\alpha = 0$?

$\langle \alpha, 0 \rangle$

$= 0$ by polarization ide

$\langle \alpha, 0 + 0 \rangle$

$= \langle \alpha, 0 \rangle + \langle \alpha, 0 \rangle$

$\langle \alpha, 0 \rangle = 2 \langle \alpha, 0 \rangle$

$\Rightarrow \langle \alpha, 0 \rangle = 0$

$$\left\{ \left\langle \beta - \frac{\langle \beta,\alpha\rangle}{\|\alpha\|^2}\alpha , \alpha \right\rangle = \langle \beta,\alpha\rangle - \frac{\langle \beta,\alpha\rangle}{\cancel{\|\alpha\|^2}} \cancel{\langle \alpha,\alpha\rangle} = 0 \right\}$$

$$\Rightarrow 0 \le \langle \gamma,\gamma\rangle = \langle \gamma,\beta\rangle = \left\langle \beta - \frac{\langle \beta,\alpha\rangle}{\|\alpha\|^2}\alpha, \beta \right\rangle$$

$$= \langle \beta,\beta\rangle - \frac{\langle \beta,\alpha\rangle}{\|\alpha\|^2}\langle \alpha,\beta\rangle$$

$$= \|\beta\|^2 - \frac{\overline{\langle \alpha,\beta\rangle}}{\|\alpha\|^2}\langle \alpha,\beta\rangle$$

$$0 \le \langle \gamma,\gamma\rangle = \|\beta\|^2 - \frac{|\langle \alpha,\beta\rangle|^2}{\|\alpha\|^2}$$

$$\Rightarrow \|\beta\|^2 - \frac{|\langle \alpha,\beta\rangle|^2}{\|\alpha\|^2} \ge 0$$

$$\Rightarrow |\langle \alpha,\beta\rangle|^2 \le \|\alpha\|^2\|\beta\|^2$$

$$\Rightarrow |\langle \alpha,\beta\rangle| \le \|\alpha\|\,\|\beta\|$$