Department of Mathematics
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IC152: Linear Algebra-II
Tutorial Sheet 4

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1. Let V be a vector space over $\mathbb{F}$ and $< \cdot, \cdot >$ be an inner product on V then show that

   (i) $< 0, \alpha >= 0$ for all $\alpha \in V$.

   (ii) if $< \alpha, \beta >= 0$ for all $\beta \in V$ then $\alpha = 0$.

   (ii) if $< \alpha, \beta >=< \gamma, \beta >$ for all $\beta \in V$ then $\alpha = \gamma$.

   (iv) $< \alpha, \beta >= 0$ if and only if $\|\alpha\| \leq \|\alpha + c\beta\|$ for all $c \in \mathbb{F}$.

   (i) Observe, by linearity, $< 0, \alpha >=< 0 + 0, \alpha >=< 0, \alpha > + < 0, \alpha >$ which implies $< 0, \alpha >= 0$

   (ii) As $< \alpha, \beta >= 0$ all $\beta \in V$ then it is true for $\beta = \alpha$, which implies $< \alpha, \alpha >= 0$ or $\alpha = 0$.

   (iii) Take $\delta = \alpha - \gamma$, we have $< \delta, \beta >= 0$ all $\beta \in V$ which implies from part (ii), $\delta = 0$, equivalently $\alpha = \gamma$.

   (iv) First let $< \alpha, \beta >= 0$, then following the definition of inner product, we get $\|\alpha + c\beta\|^2 = \|\alpha\|^2 + 2\text{Re}(c < \beta, \alpha >) + \|c\beta\|^2 = \|\alpha\|^2 + \|c\beta\|^2 \geq \|\alpha\|^2$. As length is always positive, we get $\|\alpha\| \leq \|\alpha + c\beta\|$ for arbitrary choice of $c \in \mathbb{F}$. Conversely, assume $\|\alpha\| \leq \|\alpha + c\beta\|$ for all $c \in \mathbb{F}$. If $\beta = 0$ then $< \alpha, \beta >= 0$. Assume $\beta \neq 0$. Using the above inequality, we have $2\text{Re}(c < \beta, \alpha >) + \|c\beta\|^2 \geq 0$. Now if we choose $c = -\frac{<\alpha,\beta>}{\|\beta\|^2}$ we get $-2\frac{|<\alpha,\beta>|^2}{\|\beta\|^2} + \frac{|<\alpha,\beta>|^2}{\|\beta\|^2} \geq 0$ which holds true only if $< \alpha, \beta >= 0$ as $\beta \neq 0$.

2. Let $< \cdot, \cdot >$ be the standard inner product on $\mathbb{R}^2$. Find $\alpha \in \mathbb{R}^2$ if $< (1, 2), \alpha >= -1$ and $< (-1, 1), \alpha >= 3$.
   Let $\alpha = (\alpha_1, \alpha_2)$. Using the standard inner product definition we get a system of equations as $\alpha_1 + 2\alpha_2 = -1, \quad -\alpha_2 + \alpha_2 = 3$ which on solving gives $\alpha_1 = -\frac{7}{3}$ and $\alpha_2 = \frac{2}{3}$

3. Which of the following are inner product?

   (i) For any $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, define $< \alpha, \beta >= \alpha_2(\alpha_1 + 2\beta_1) + \beta_2(2\alpha_1 + 5\beta_2)$.

   (ii) For any $A, B \in M_{n\times n}(\mathbb{C})$ define $< A, B >= trace(A\bar{B})$.

   (iii) For any $A, B \in M_{n\times n}(\mathbb{R})$ define $< A, B >= trace(A + B)$.

(iv) For any $f, g \in \mathrm{P}(\mathbb{R})$ define $< f, g >= \int_0^1 f'(x)g(x)dx$.

(i) It is not as for $\alpha = \beta = (1, 0)$, we have $< \alpha, \alpha >= 0$ but $\alpha \neq 0$.

(ii) It is not as for $n = 2$, choose $A = B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ then $A\bar{A} = A^2 = 0$ and hence $< A, A >= trace(A\bar{A}) = trace(0) = 0$ but $A \neq 0$.

(iii) It is not. For example if we choose $A = B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $< A, A >= trace(A + A) = 0$ but $A \neq 0$.

(iv) It is not as for $f = g = 1$, we get $< f, f >= 0$ but $f \neq 0$.

4. Compute $< \alpha, \beta >, \|\alpha\|, \|\beta\|, \|\alpha + \beta\|$ for the following vectors in the specified inner product spaces and verify the triangle and Cauchy Schwartz inequlaity.

(i) $V = \mathbb{C}^3$ with standard inner product and $\alpha = (2, 1 + i, i), \beta = (2 - i, 2, 1 + 2i)$

(ii) $V = C([0, 1]; \mathbb{R})$ with $< f, g >= \int_0^1 f(x)g(x)dx$ and $\alpha = x, \beta = e^x$

(iii) $V = M_{2\times2}(\mathbb{C})$ with standard inner product and $\alpha = \begin{bmatrix} 1 & 2 + i \\ 3 & i \end{bmatrix}, \beta = \begin{bmatrix} 1 + i & 0 \\ i & -i \end{bmatrix}$

5. Suppose $u, v \in V$ are such that $\|u\| = 3, \|u + v\| = 4, \|u - v\| = 6$. Then what will be $\|v\|$?

Using parallelogram law, we get $2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2 - 2\|u\|^2 = 34$ which implies $\|v\| = \sqrt{17}$.

6. Find the matrix of standard inner product $< \cdot, \cdot >$ on $\mathbb{R}^3$ relative to an ordered basis $\mathcal{B} = \{\alpha_1 = (1, -1, 2), \alpha_2 = (2, 0, 1), \alpha_3 = (-1, 2, -1)\}$.

The matrix of inner product relative to $\mathcal{B}$ (say M) is given by

$$M = \begin{bmatrix} < \alpha_1, \alpha_1 > & < \alpha_2, \alpha_1 > & < \alpha_3, \alpha_1 > \\ < \alpha_1, \alpha_2 > & < \alpha_2, \alpha_2 > & < \alpha_3, \alpha_2 > \\ < \alpha_1, \alpha_3 > & < \alpha_2, \alpha_2 > & < \alpha_3, \alpha_3 > \end{bmatrix} = \begin{bmatrix} 6 & 4 & -5 \\ 4 & 5 & -3 \\ -5 & -3 & 6 \end{bmatrix}$$

7. Let $(V, < \cdot, \cdot >)$ be an inner product space. Then prove that, for any orthogonal vectors $\alpha, \beta \in V$

$$\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2.$$

Using the definition of inner product, we get $\|\alpha + \beta\|^2 = \|\alpha\|^2 + 2\mathrm{Re} < \alpha, \beta > +\|\beta\|^2$. As α, β are orthogonal, we have $< \alpha, \beta >= 0$, which gives $\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$

8. Use standard inner product on $\mathbb{R}^2$ over $\mathbb{R}$ to prove the following statement: "A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other."
   Let adjacent sides of the parallelogram be represented by vectors α and β, then the diagonals will be $\alpha + \beta$ and $\alpha - \beta$. Now if diagonals are perpendicular, we have $0 =< \alpha+\beta, \alpha-\beta >= \|\alpha\|^2 - \|\beta\|^2$ which implies $\|\alpha\| = \|\beta\|$ and hence parallegram is a rhombus. Conversely, if parallegram is a rhombus, then $< \alpha+\beta, \alpha-\beta >= \|\alpha\|^2 - \|\beta\|^2 = 0$ and hence diagonals are perpendicular.