Quiz.

Solutions.

① Let $A_1, A_2, \ldots, A_n$ be a partition of the sample space Ω...

Here,
$$P[E\cap F] = \sum_{i=1}^{n} P[E\cap F | A_i] P[A_i] \quad \text{(by total probability thm).}$$
$$= \sum_{i=1}^{n} P[E|A_i] P[F|A_i] P[A_i] \quad \text{(as E and F are conditionally independent)}$$
$$= \sum_{i=1}^{n} P[E|A_i] P[F] P[A_i] \quad (\text{as } F \perp\!\!\!\perp A_i \ \forall\, i=1,2,\ldots,n)$$
$$= P[F] \sum_{i=1}^{n} P[E|A_i] P[A_i]$$
$$= P[F] P[E] \quad \text{(by total probability thm)}$$

$\Rightarrow E \perp\!\!\!\perp F \Rightarrow E^c \perp\!\!\!\perp F \Rightarrow$ E and F cannot be disjoint.

② Suppose 25% of the population are smokers.

S : Smoker
L : Develop lung cancer.

$P(S) = \frac{1}{4}$, $P[L|S] = 27 P[L|S^c]$

$$P[S|L] = \frac{P[L|S]P[S]}{P[L|S]P[S] + P[L|S^c]P[S^c]} \quad \text{(by Baye's thm)}$$
$$= \frac{27 P[L|S^c]\cdot 1/4}{27 P[L|S^c]\cdot 1/4 + P[L|S^c]\cdot 3/4} = \frac{27}{30} = \frac{9}{10}.$$

③ Bob plays a game.....

$$P(\text{Bob wins}) = P(\{HH\}\cup\{HTHH\}\cup\{HTHTHH\}\cup\ldots) + P(\{THH\}\cup\{THTHH\}\cup\{THTHTHH\}\cup\ldots).$$
$$= P(HH) + P(HTHH) + P(HTHTHH) + \ldots$$
$$+ P(THH) + P(THTHH) + P(THTHTHH) + \ldots$$
$$= p^2 + p^3(1-p) + p^4(1-p)^2 + \ldots$$
$$+ p^2(1-p) + p^3(1-p)^2 + p^4(1-p)^3 + \ldots$$
$$= p^2(1 + p(1-p) + p^2(1-p)^2 + \ldots) + p^2(1-p)(1 + p(1-p) + p^2(1-p)^2 + \ldots)$$
$$= (p^2 + p^2(1-p))(1 + p(1-p) + p^2(1-p)^2 + \ldots)$$
$$= p^2(2-p)\big/ 1-p+p^2.$$

④ In a quiz competition . . . .

$P(\text{Person quit at even times}) = P(\{RW\} \cup \{RRRW\} \cup \{RRRRRW\} \cup \ldots\})$

$= P(RW) + P(RRRW) + P(RRRRRW) + \cdots$

Here, $R \to$ Right answer.
$W \to$ Wrong answer.

$= p(1-p) + p^3(1-p) + p^5(1-p) + \cdots$

$= p(1-p) \cdot \frac{1}{1-p^2} = \frac{p}{1+p}$.

Given, $\alpha = \frac{p}{1+p} \Rightarrow \alpha + \alpha p = p \Rightarrow p = \frac{\alpha}{1-\alpha} \leq 1$

$\Rightarrow \alpha \leq 1 - \alpha$

$\Rightarrow \boxed{\alpha \leq 0.5} \leq 0.7$.

⑤ In Raipur, . . .

Given, $P(R) = \frac{1}{4}$, $P(T|R) = \frac{1}{3}$, $P(T^c|R) = \frac{2}{3}$, $P(R^c) = \frac{3}{4}$

$P(T|R^c) = \frac{1}{5}$, $P(T^c|R^c) = \frac{4}{5}$,

$P(L|R \cap T) = \frac{3}{4}$, $P(L|R^c \cap T^c) = \frac{1}{8}$,

$P(L|R \cap T^c) = \frac{1}{4}$, $P(L|R^c \cap T) = \frac{1}{4}$.

$P(L) = P(L|R\cap T)P(R\cap T) + P(L|R^c\cap T^c)P(R^c \cap T^c) + P(L|R\cap T^c)P(R \cap T^c)$
$+ P(L|R^c \cap T)P(R^c \cap T)$

$= \frac{3}{4} \cdot P(T|R)P(R) + \frac{1}{8} \cdot P(T^c|R^c)P(R^c) + \frac{1}{4}P(T^c|R)P(R)$
$+ \frac{1}{4} \cdot P(T|R^c)P(R^c)$.

$= \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{4}{5} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{3}{4}$.

$= \frac{5}{48} + \frac{18}{160} = \frac{5}{48} + \frac{9}{80} = \frac{400 + 432}{48 \times 80}$

$= \frac{\cancel{832}\ \cancel{104}\ 13}{\cancel{48}_{6} \times \cancel{80}_{10}} = \frac{13}{60}$.

⑥ Alia's Question. G_A : Family has a girl named Alia.

$$P(GG \mid G_A) = \frac{P(G_A \mid GG)\,P(GG)}{P(G_A \mid GG)\,P(GG) + P(G_A \mid BG)\,P(BG) + P(G_A \mid GB)\,P(GB) + P(G_A \mid BB)\,P(BB)}$$

$$= \frac{\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\right)\cdot 1/4}{\left(\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\right)\cdot 1/4 + \frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4} + 0\cdot\frac{1}{4}}$$

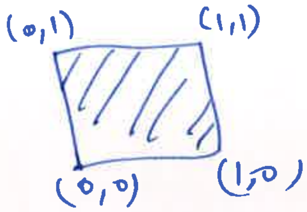
$$= \frac{\frac{3}{4}\cdot\frac{1}{4}}{\frac{3}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}} = \frac{3}{7}.$$

* $P(G_A \mid GG)$ = P(given that family has two girls ~~what the~~ at least one girl has name Alia).

⑦

$P(A) = 0 \Rightarrow A = \emptyset$ is false.

$P(A) = 1 \Rightarrow A = \Omega$ is false.



$= \Omega$.

Let $A = \Omega \setminus \{(0,0)\}$

$P(A) = 1$ but $A \neq \Omega$.

$P(\{(0,0)\}) = 1$ but $\{(0,0)\} \neq \emptyset$.

⑧ G : selecting a girl child at random.

$$P(GG \mid G) = \frac{P(G \mid GG)\,P(GG)}{P(G \mid GG)\,P(GG) + P(G \mid BG)\,P(BG) + P(G \mid GB)\,P(GB) + P(G \mid BB)\,P(BB)}$$

$$= \frac{1\cdot 1/4}{1\cdot 1/4 + \frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}+0\cdot\frac{1}{4}} = \frac{2}{4} = \frac{1}{2}.$$

(9) Given $P(A) = P(B)$

$$p_1(1-p_2) + p_2(1-p_1) + \cancel{p_1 p_2} = \cancel{p_1 p_2}$$

$$p_1 - p_1 p_2 + p_2 - p_1 p_2 = 0$$

$$\Rightarrow \quad p_1 + p_2 = 2 p_1 p_2 \quad \Rightarrow \quad \frac{1}{p_1} + \frac{1}{p_2} = 2$$

(10)

$$\Omega_1 = \{H, TH, TTH, TTTH, \ldots\}$$

$$\Omega_2 = \{HHTHHTHHTTTT\ldots, \; THTHTHTT\ldots, \; HTTTHT\ldots, \}$$

$$= \{H, T\} \times \{H, T\} \times \{HT\} \times \{HT\} \times \ldots$$

Ω_1 is a countably infinite set.

Ω_2 is a uncountable set.