MA202: Calculus II

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Lecture Notes

Department of Mathematics

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Module 2
Lecture 3

Absolute Extremum

- An absolute maximum point is a point where the function obtains its greatest possible value.
- Similarly, an absolute minimum point is a point where the function obtains its least possible value.

Theorem

If D be a closed and bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is continuous, then f is bounded function and attains its maximum and minimum, i.e., there exist $(x_1, y_1), (x_2, y_2) \in D$ such that

1. $f(x_1, y_1) = \min_{(x,y)\in D} f(x, y)$,
2. $f(x_2, y_2) = \max_{(x,y)\in D} f(x, y)$,

Maxima and Minima

With the help of the above result and from the necessary condition (first derivative condition) we immediately obtain the following result

Theorem

Let D be a non-empty, closed and bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be continuous. The absolute minimum and the absolute maximum of f is attained either at a critical point of f or a boundary point of D.

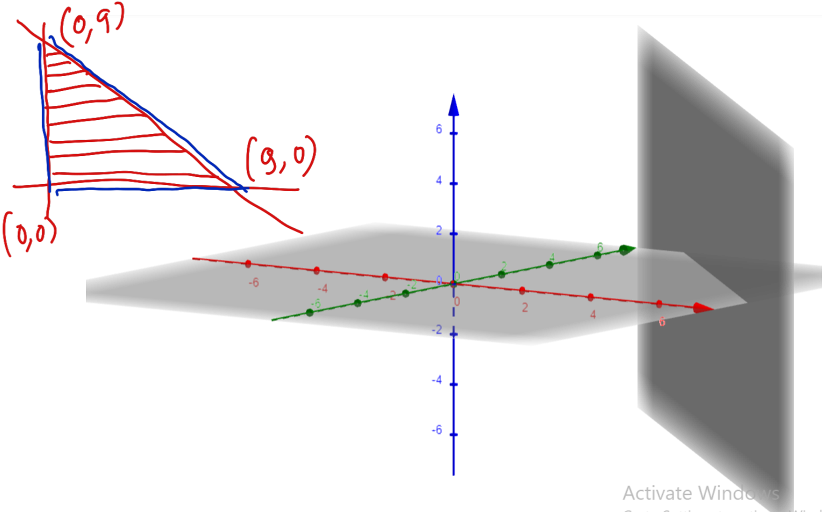
Proof: Let at the point $(x_1, y_1) \in D$ f attains local minimum. If $(x_1, y_1) \in \partial D$, then we are done. If $(x_1, y_1) \notin \partial D$ then (x_1, y_1) is an interior point of D, and f has a local minimum at (x_1, y_1). If $\bigtriangledown f(x_1, y_1)$ does not exist, then (x_1, y_1) is a critical point of f. If $\bigtriangledown f(x_1, y_1)$ exists, then necessarily $\bigtriangledown f(x_1, y_1) = (0, 0)$, and so (x_1, y_1) is a critical point of f. So local minimum attained either at boundary or at critical points. Similar argument follows for the local maximum.

Procedure to find the absolute extrema

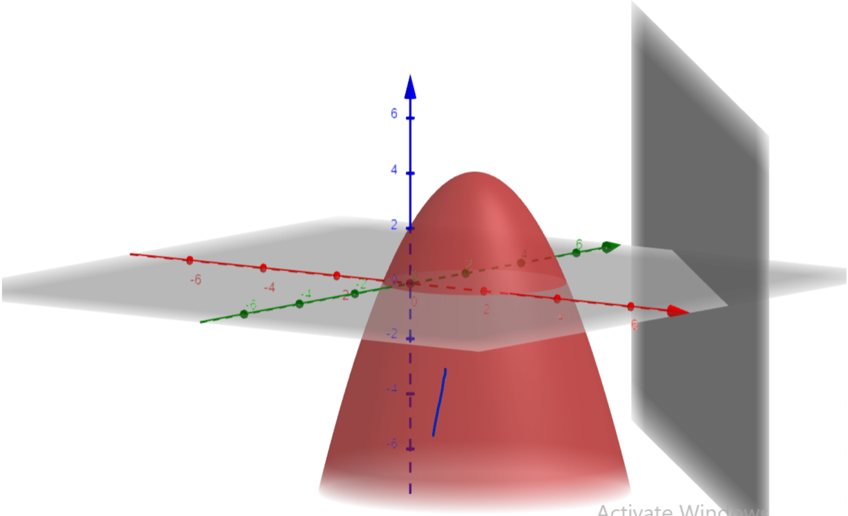
Let D be a nonempty closed and bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function

- Find the boundary of D and determine the absolute extrema of f on the boundary. (This is equivalent to one variable problem)
- Determine the critical points of f in D.
- Compare the values of f at the critical points and at the extreme values of f on the boundary
- The largest among these is the absolute maximum and smallest among these is the absolute minimum.

Let $D = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0,\ x + y \leq 9\}$ and $f : D \to \mathbb{R}$ is defined as
$$f(x, y) = 2 + 2x + 2y - x^2 - y^2.$$



Absolute Extrema : Example



Absolute Extrema : Example

- Clearly f is differentiable for all $(x, y) \in D$. Then $f_x(x, y) = 0 = f_y(x, y)$ gives only one critical point which is $(1, 1)$ and $f(1, 1) = 4$.
- The boundary of D, ∂D consists of three one dimension components which are (i) $x = 0, y \in [0, 9]$; (ii) $y = 0, x \in [0, 9]$; (iii) $x + y = 9, x, y \geq 0$. We have to check on each component about the extreme values of f
    - Component 1: On the line $x = 0$ we have $f(0, y) = g_1(y) = 2 + 2y - y^2$ for $y \in [0, 9]$. Then $g_1'(y) = 0$ gives $y = 1$. Hence $f(0, 1) = 3$. Also $f(0, 0) = 2$ and $f(0, 9) = -61$.
    - Component 2: On the line $y = 0$ we have $f(x, 0) = g_2(x) = 2 + 2x - x^2$ for $x \in [0, 9]$. Then $g_2'(x) = 0$ gives $x = 1$. Hence $f(1, 0) = 3$. Also $f(0, 0) = 2$ and $f(9, 0) = -61$.
    - Component 3: On the line $x + y = 9,\ x, y \geq 0$ we have $f(x, 9 - x) = g_3(x) = -61 + 18x - 2x^2$ for $x \in [0, 9]$. Then $g_3'(x) = 0$ gives $x = 9/2$ and consequently $y = 9 - x = 9/2$. Hence $f(9/2, 9/2) = -\frac{41}{2}$. Also $f(0, 9) = -61$ and $f(9, 0) = -61$.

Now we write all the values obtained above in the following table

$$\begin{pmatrix} (x,y): & (1,1) & (0,1) & (1,0) & (9,0) & (0,9) & (0,0) & (9/2,9/2) \\ f(x,y): & 4 & 3 & 3 & -61 & -61 & 2 & -\frac{41}{2} \end{pmatrix}.$$

Hence the absolute maximum of f on D is 4 and attained at $(1,1)$. The absolute minimum of f on D is -61 and it attained at $(9,0)$ and $(0,9)$.

Constrained Extrema

- Let $D \subset \mathbb{R}^2$ and $f, g : D \to \mathbb{R}$. We are now interested in the problem of finding the maximum and minimum of f on D subject to the constraint $g = 0$.
- $g(x, y) = 0$ describes a curve in $\mathbb{R}^2$ plane implicitly.
- One procedure to solve such problem is to solve the equation $g(x, y) = 0$ in terms of y (or x). Then the whole problem reduces to one variable problem.
- There is also another way to handle such problems.

max/min $f(x,y)$
subject to $g(x,y)=0$

Constrained Extrema

Let (x_0, y_0) be an interior point of D such that $g(x_0, y_0) = 0$. Suppose f has a local extremum at (x_0, y_0) subject to the constraint $g(x, y) = 0$. We would like to show that the gradient vectors $(\nabla f)(x_0, y_0)$ and $(\nabla g)(x_0, y_0)$ are parallel.

Let us see a step by step prove of the above statement

Assume that we are able to solve the equation $g(x, y) = 0$ for y in terms of x near x_0, that is, there a function η defined near x_0 such that $g(x, \eta(x)) = 0$ and $\eta(x_0) = y_0$. If $\eta'(x_0)$ exists, then $g_x(x_0, y_0) + g_y(x_0, y_0)\eta'(x_0) = 0$ by the chain rule (ii).

Consider the function $\phi(x) := f(x, \eta(x))$ for x near x_0. Now ϕ has a local extremum at x_0, and so $\phi'(x_0) = 0$, that is, $f_x(x_0, y_0) + f_y(x_0, y_0)\,\eta'(x_0) = 0$ again by the chain rule (ii).

Constrained Extrema

It follows that $f_x(x_0, y_0)g_y(x_0, y_0) = f_y(x_0, y_0)g_x(x_0, y_0)$, that is, the gradient vectors $(\nabla f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$ and $(\nabla g)(x_0, y_0) = (g_x(x_0, y_0), g_y(x_0, y_0))$ are parallel.

In fact, if $g_y(x_0, y_0) \neq 0$, then $(\nabla f)(x_0, y_0) = \lambda_0(\nabla g)(x_0, y_0)$, where $\lambda_0 := f_y(x_0, y_0)/g_y(x_0, y_0)$.

Similarly, if $g(x, y) = 0$ can be solved for x in terms of y near y_0 and if $g_x(x_0, y_0) \neq 0$, then $(\nabla f)(x_0, y_0) = \lambda_0(\nabla g)(x_0, y_0)$, where $\lambda_0 := f_x(x_0, y_0)/g_x(x_0, y_0)$.

$$\boxed{g_y(x_0, y_0) \neq 0, \quad g_x(x_0, y_0) \neq 0}$$

Lagrange Multiplier Theorem:

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f, g : D \to \mathbb{R}$ have continuous partial derivatives in a neighbourhood of (x_0, y_0). Let $C := \{(x, y) \in D : g(x, y) = 0\}$. Suppose (i) $g(x_0, y_0) = 0$, (ii) $(\nabla g)(x_0, y_0) \neq (0, 0)$, and (iii) the function f, when restricted to C, has a local extremum at (x_0, y_0). Then there is $\lambda_0 \in \mathbb{R}$ such that

$$(\nabla f)(x_0, y_0) = \lambda_0 (\nabla g)(x_0, y_0).$$

The real number λ_0 is called a **Lagrange multiplier**.

Constrained Extrema

Procedure to solve problems using Lagrange Multiplier Method:

Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$ is closed and bounded. The problem is to find the absolute extremum of f subject to $g(x, y) = 0$.

- Introduce a new variable λ in the problem which is the Lagrange multiplier
- Find the solutions of the equations $\boxed{\nabla f(x, y) = \lambda g(x, y)}$ and $\boxed{g(x, y) = 0}$ simultaneously where $\boxed{\nabla g(x, y) \neq (0, 0)}$. $g_x \neq 0, g_y \neq 0$
- compare the values of f at these solutions to conclude the absolute maximum or minimum of f.

Note

The points where $\nabla g(x, y) = (0, 0)$ have to be considered separately.

Constrained Extrema

Functions of three variables

Lagrange Multiplier Method can be generalized to the functions of three variables also. In this case we have to find the simultaneous solutions of $\bigtriangledown f(x,y,z) = \lambda g(x,y,z)$ and $g(x,y,z) = 0$ where $\bigtriangledown g(x,y,z) \neq (0,0)$.

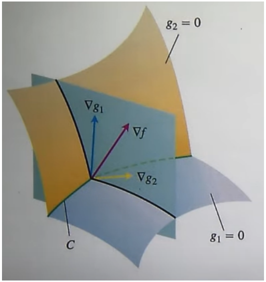
More than two constraints

The Lagrange Multiplier Method can be extended when two or more constraints are involved namely $g = 0$ and $h = 0$. In this case, we compare the values of f at the simultaneous solutions of the following equations

$$\bigtriangledown f = \lambda \bigtriangledown g + \mu \bigtriangledown h$$

and $g = 0 = h$ with $\bigtriangledown g \neq 0$, $\bigtriangledown h \neq 0$ and $\bigtriangledown g$ and $\bigtriangledown h$ are not parallel.

Constrained Extrema



- ∇g_1 is perpendicular to the surface $g_1 = 0$ and ∇g_2 is perpendicular to the surface $g_2 = 0$.
- The curve C is the intersection of the surfaces $g_1 = 0$ and $g_2 = 0$. Therefore both ∇g_1 and ∇g_2 are perpendicular to the curve C.
- Since ∇g_1 and ∇g_2 are not parallel so they are linearly independent and span the a plane. Each vector in the plane is of the form $\lambda \nabla g_1 + \mu \nabla g_2$ and it is perpendicular to C.

Constrained Extrema

Suppose f has a global extremum on the set $\{(x, y) \in D : g(x, y) = 0\}$. (For example, when this set is nonempty, closed and bounded, and f is continuous on it). Then it is also a local extremum of f, and so it is attained either at a simultaneous solution (x_0, y_0) of the above two equations where $\nabla g(x_0, y_0) \neq (0, 0)$, or at a point (x_1, y_1) where $g(x_1, y_1) = 0$ and $(\nabla g)(x_1, y_1) = (0, 0)$.

Procedure to solve problems using Lagrange Multiplier Method for more constraints:

Let f is defined in a closed bounded region of $\mathbb{R}^2$ and the problem is to find the extreme values of the function f subject to the constraints $g = 0$ and $h = 0$.

- Introduce a new variables λ and μ in the problem which is the Lagrange multiplier
- Find the solutions of the equations $\bigtriangledown f(x, y) = \lambda \bigtriangledown g(x, y) + \mu \bigtriangledown h(x, y)$ and $g(x, y) = 0$, $h(x, y) = 0$ simultaneously where $\bigtriangledown g(x, y) \neq (0, 0)$, $\bigtriangledown g(x, y) \neq (0, 0)$ also $\bigtriangledown g(x, y)$ is not parallel to $\bigtriangledown h(x, y)$.
- compare the values of f at these solutions to conclude the absolute maximum or minimum of f.

Note

The points where $\bigtriangledown g(x, y) = (0, 0)$ or $\bigtriangledown h(x, y) = (0, 0)$ or $\bigtriangledown g(x, y)$ and $\bigtriangledown h(x, y)$ are parallel have to be considered separately.

$f(x, y) := x\,y$, subject to $x^2 + y^2 - 1 = 0$.

Let $g(x, y) := x^2 + y^2 - 1$ for $(x, y) \in \mathbb{R}^2$. Note that the set $\{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$, that is, the unit circle, is nonempty, closed and bounded, and f is continuous on it.

Now $(\nabla f)(x, y) = \lambda\,(\nabla g)(x, y)$ and $g(x, y) = 0$ means

$$y = 2\lambda x, \quad x = 2\lambda y, \quad \text{and} \quad x^2 + y^2 - 1 = 0.$$

Then $y\,x = 4\lambda^2 xy$, that is, $4\lambda^2 = 1$, since $x\,y \neq 0$. Thus $\lambda = \pm 1/2$, and the simultaneous solutions of the above equations are given by $(x, y) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Also, $(\nabla g)(x, y) \neq (0, 0)$ whenever $g(x, y) = 0$.

Thus the hypotheses of the Lagrange Multiplier Theorem are satisfied. Hence the maximum of f on the unit circle is $f\,(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/2$, while the minimum of f on the unit circle is $f\,(1/\sqrt{2}, -1/\sqrt{2}) = f\,(-1/\sqrt{2}, 1/\sqrt{2}) = -1/2$.

Constrained Extrema

Let us find the extremum values of the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.

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End of Module - 2