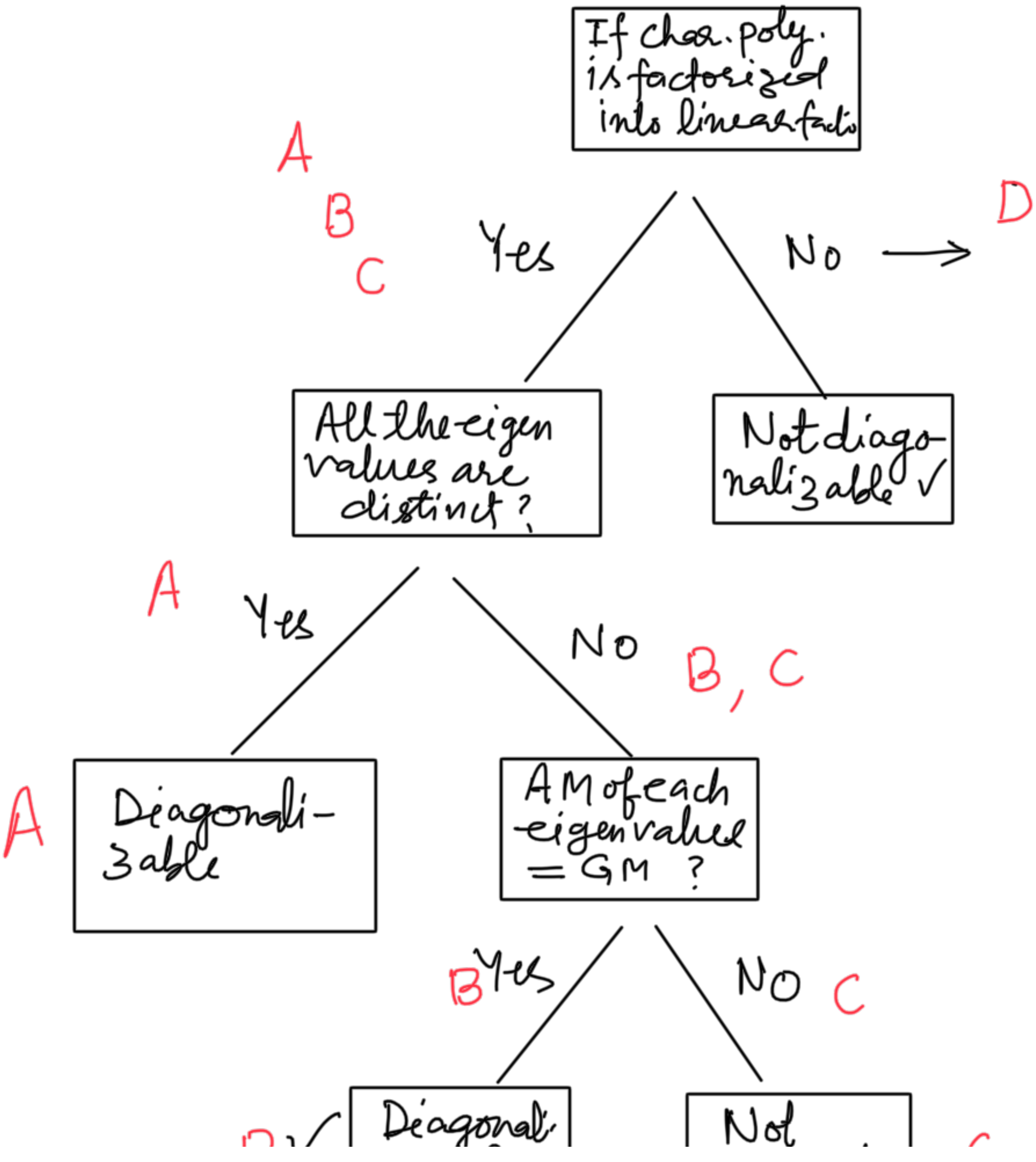
Lecture #4 (IC152)



$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (x+y, y+z, z+x)$

$\mathcal{B} = \{e_1, e_2, e_3\}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [T]_{\mathcal{B}}$$

Char polynomial

$f(x) = (x-2)(x^2 - x + 1)$

$x = 2$ ✓, complex roots

Thus $x = 2$ is the only eigenvalue.

Algebraic multiplicty of eigenvalue $2 = 1$

$\therefore$ Geometric multiplicity $= 1$

Algebraic Multiplicty (AM)

Geometric multiplicity (GM)

B ✓ Diagonalizable — Diagonalizable — C

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Characteristic polynomial $= f(x) = \det(xI - A)$

$$f_A(x) = (x-3)(x+5)(x-6)$$

Eigenvalues are $3, -5, 6$

All of them are distinct and hence A or T_A is diagonalizable.

B, $(x-3)^2(x-5)$ ✓

C, $(x-3)^2(x-4)$

A, $(x-3)(x+5)(x-6)$

Now,

$$f_B = (x-3)^2(x-5)$$

Eigenvalues are $3, 3, 5$

Let us find out eigenspace corresponding to eigenvalue 3, E_3.

Consider

$$(3I - B) = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix}$$

E_3 = Null space of $(3I - B)$

= solution space of $\begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} -x - z &= 0 \\ -2x - 2z &= 0 \\ -x - z &= 0 \end{aligned}, \quad x = -z \checkmark, \quad y \text{ is free}$$

Thus solution space is spanned by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\dim E_3 = 2 =$ A.M. of 3

Hence B or T_B is diagonalizable.

Characteristic polynomial for C

$$f_c(x) = (x-3)^2(x-4)$$

Eigenvalues are $3, 3, 4$

Check if E_3 is 2-dimensional or not?

$E_3 =$ solution space of

$$(3I - C)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0$, $z = 0$, x free

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\dim E_3 = 1$

G.M. of $3 <$ A.M. of 3

Hence not diagonalizable.

Eigenvalues & properties of some special matrices

Let $A \in M_{n\times n}(\mathbb{C})$, then A is called Hermitian if

$$A = A^* := \bar{A}^t = \text{transpose of complex conjugate of } A$$

Example: $A = \begin{bmatrix} 1 & \cancel{3}3-i \\ 3+i & 2 \end{bmatrix}$

$z \in \mathbb{C}$

$z = a+ib$

$\bar{z} = a-ib$

$A = (a_{ij}),\ a_{ij} \in \mathbb{C}$

$\bar{A} = (\bar{a}_{ij})$

$z = z^*$ ✓

$a+ib = a-ib$

$\Rightarrow b = 0$

$z \in \mathbb{R}$.

Property: Every eigenvalue of Hermitian matrix is real.

Proof:- Observation: u^*Au is real number if A is Hermitian ($u \in \mathbb{C}^n$, $A \in M_{n\times n}(\mathbb{C})$)

$$(u^*Au)^* = u^*A^*u = u^*Au$$

$$\Rightarrow u^*Au \in \mathbb{R}$$

$A \in M_{n\times n}(\mathbb{C})$

$u \in \mathbb{C}^n$

$u^*Au \in \mathbb{C}$

$1\times n,\ n\times n,\ n\times 1$

Observation

Let λ be an eigenvalue of A then

Let λ ...

$\exists\ v \neq 0$ s.t.

$$Av = \lambda v$$

$$(Av)^* = (\lambda v)^*$$

$$v^* A^* = v^* \lambda^* \checkmark$$

$$v^* \underline{\underline{A v}} = v^* \lambda^* v \checkmark \leftarrow$$

$$v^* \lambda v = \lambda^* v^* v$$

$$\underline{(\lambda - \lambda^*)}\,\underline{\underline{v^* v}} = 0 \implies \lambda = \lambda^* \text{ as } v \neq 0$$

$$\implies \lambda \in \mathbb{R}$$

$\lambda \in \mathbb{C}$

$\checkmark\ \bar{z} z = |z|^2 \in \mathbb{R}$

$z^* z = \|z\|^2$

$\because \|z\| =$ length of vector

$z = a + ib$

$|z|^2 = a^2 + b^2$

<u>Corollary</u> :- Characteristic polynomial of a Hermitian matrix splits into linear factors on $\mathbb{R}$.

<u>Property</u> :- Hermitian matrices are diagonalizable.

<u>Definition</u> :- $A \in M_{n \times n}(\mathbb{C})$, then A is skew-Hermitian if $A^* = -A$.

$[\ 0 \quad i\ ]$

Example: $\begin{bmatrix} -i & 0 \\ & \end{bmatrix}$

Observe that diagonal entries of skew-Hermitian matrices are either zero or purely imaginary.

Property:- Eigenvalues of skew-Hermitian matrix are either zero or purely imaginary.

Proof. Let λ be a eigenvalue of A (skew Hermi

then $A v = \lambda v$ for some $v \neq 0$

$$(Av)^* = (\lambda v)^*$$

$$v^* A^* = v^* \lambda^*$$

$$-v^* A = v^* \lambda^*$$

$$-v^* \underline{\underline{A v}} = v^* \lambda^* v$$

$$-v^* \lambda v = v^* \lambda^* v$$

$$\Rightarrow (\lambda - \lambda^*) v^* v = 0$$

$$\Rightarrow \lambda + \lambda^* = 0 \quad \text{as } v \neq 0$$

imaginary or

$\Rightarrow \lambda$ is purely imaginary

$\lambda = 0$

Property: Skew-Hermitian matrices are diagonalizable.

Unitary Matrix

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called unitary if $A^*A = AA^* = I$.

Property: If λ is an eigenvalue of a unitary matrix A then $|\lambda| = 1$.

Proof: Let λ be an eigenvalue & v be the eigenvector for λ,

$$Av = \lambda v$$

$$v^*A^* = v^*\lambda^*$$

$$v^*A^*Av = v^*\lambda^*Av$$

$$v^*v = v^*\lambda^*\lambda v$$

$$v^* v = \lambda^* \lambda \, v^* v$$

$$(1 - |\lambda|) v^* v = 0$$

$$|\lambda| = 1 \quad \text{as } v \neq 0.$$

Property. Unitary matrices are diagonalizable.

Remark:

- Every real symmetric matrix is Hermitian.
- Every real skew-symmetric matrix is skew-Hermitian.
- Every real orthogonal matrix is unitary.