MA202: Calculus II

Instructor: Dr. Arnab Patra

Lecture Notes

Department of Mathematics
Indian Institute of Technology Bhilai

Module 2

Lecture 3

Important Things

Let $D \subseteq \mathbb{R}^2$.

1. Closed Set: D is a closed set if D contains all its limit points.
2. Boundary point: A point $(x_0, y_0) \in \mathbb{R}^2$ is said to be a boundary point of D if every open disk centred at the point contains some points of D ans also contains some points outside of D. The set of all boundary points is called boundary of D and denoted by ∂D.
3. A closed set contains all of its boundary points. (why?)

Examples: Let $D_1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $D_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then D_1 is closed, but D_2 is not. In fact, $\partial D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \partial D_2$.

Maxima and Minima

1. Recall the single variable case: To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line.
2. For this we are interested in the points where the derivative vanishes. This makes the tangent horizontal (why?)
3. At such points, we then look for local maxima, local minima etc.
4. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent plane. (Think: Which conditions will make the tangent plane horizontal, in the form $z = c$)
5. At such points, we then look for local maxima, local minima, and saddle points.

Maxima and Minima

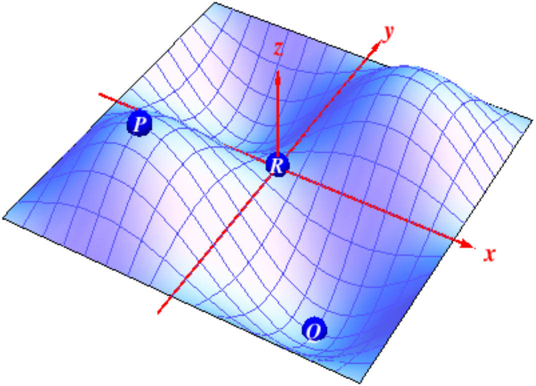
Local maximum and minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk cenetred at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk cenetred at (a, b).

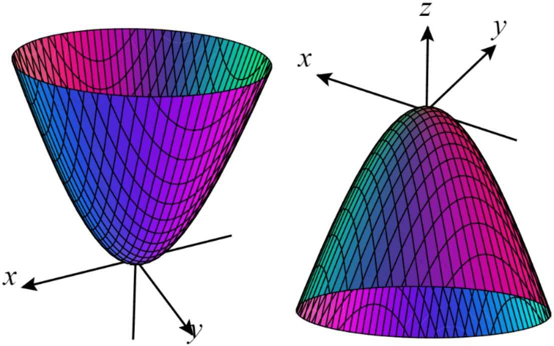
A local extremum of a function is the point at which it has either local maximum or minimum.

Maxima and Minima



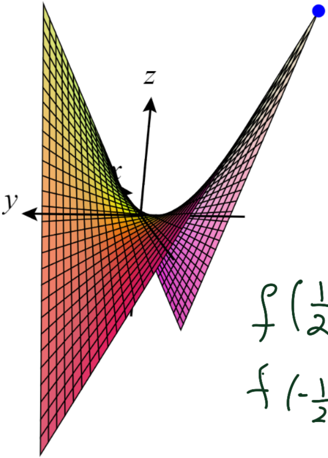
Maxima and Minima

1. $f(x, y) = (x^2 + y^2)$, $(x, y) \in \mathbb{R}^2$. The function f has a local minimum at $(0, 0)$.
2. $f(x, y) = -(x^2 + y^2)$, $(x, y) \in \mathbb{R}^2$. The function f has a local maximum at $(0, 0)$.



Maxima and Minima

1. $f(x, y) = xy$, $(x, y) \in \mathbb{R}^2$. The function f has neither a local minimum nor a local maximum at $(0, 0)$.



$(x, y) \in N$

$\boxed{f(0,0) \geq f(x,y)}$

$(0,0)$ N

$\boxed{(x_1, y_1) \in N}$

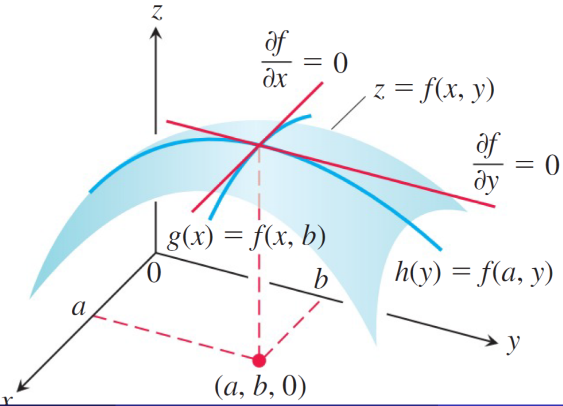
$\boxed{f(0,0) \leq f(x_1,y_1)}$

$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} > f(0,0)$

$f\left(-\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{4} < f(0,0)$

Necessary condition (first derivative test):

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if both the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.



Proof (Sketch): If $f(x, y)$ has a local extremum at (a, b), then the function $g(x) = f(x, b)$ has a local extremum at $x = a$. Therefore by the result for single variable functions $g'(a) = 0$. Now $g'(a) = f_x(a, b)$, so $f_x(a, b) = 0$. A similar argument with the function $h(y) = f(a, y)$ shows that $f_y(a, b) = 0$.

Note

Equation of the tangent plane of $z = f(x, y)$ at (a, b, c) is given by

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We are interested in a horizontal tangent plane at (a, b, c). In other words the equation of the tangent plane, if exists at (a, b, c) is of the form $z = c$. Think under which condition this will happen!

Critical Point

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f.

- If the function is differentiable at a critical point then the total derivative vanishes at that point. Similar as single variable case.
- Now consider the function $f(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$.
- $f_x(x,y) = y$ and $f_y(x,y) = x$ for all $(x,y) \in \mathbb{R}^2$. Therefore $f_x = 0 = f_y$ only at the point $(0,0)$.

$(0,0)$ $\nabla f = (0,0)$. $\boxed{(0,0)}$ $\left(\frac{1}{n}, \frac{1}{n}\right)$

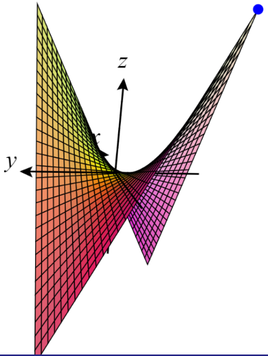
$\left(-\frac{1}{n}, \frac{1}{n}\right)$

$f\left(\frac{1}{n}, \frac{1}{n}\right) > f(0,0)$

$f\left(-\frac{1}{n}, \frac{1}{n}\right) < f(0,0)$

Maxima and Minima

- But see that $(0,0)$ neither a local maximum nor a local minimum. Any neighbourhood of $(0,0)$ contains points (x,y) where both the x and y component are positive, so $f(x,y) > f(0,0)$ this shows that $(0,0)$ is not a local maximum and also the neighbourhood contains points (x,y) such that any of the one component is negative. In this case $f(x,y) < f(0,0)$. This shows that $(0,0)$ is not a local minimum.



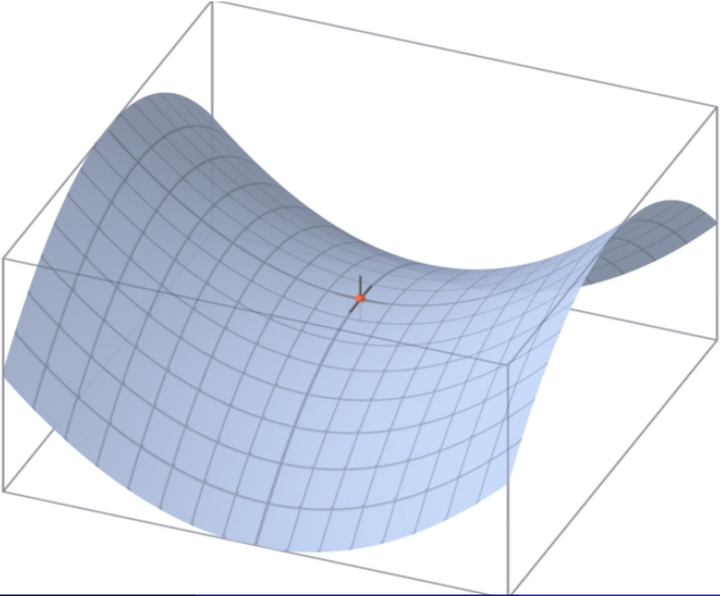
Saddle point

An interior point (a, b) of the domain D of a function $f : D \to \mathbb{R}$ is called a saddle point of f if

1. both $f_x(a, b)$ and $f_y(a, b)$ exist and equal to zero,
2. f does not have a local extremum at (a, b).

1. A saddle point is a critical point
2. If f has a saddle point at (a, b) then ANY neighbourhood of (a, b) (open disk centred at (a, b)) contains points (x, y) such that $f(x, y) > f(a, b)$ and also contains points (x, y) such that $f(x, y) < f(a, b)$. In other words f does not have any local maximum or local minimum.

Maxima and Minima



The main question is after obtaining all the critical points, how to check for local maximum or local minimum or saddle point.

1. Remember the second derivative test for single variable case.
2. What is the term for functions of several variables which is analogues to the second derivative of a function of single variable? H
3. Refer to Taylor's Theorem for functions of two variables

Hessian Matrix

$\boxed{H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}}$. This is defined for a function having continuous first and second order partial derivatives.

The following quantity plays a crucial role in this sequel

$$\det H = f_{xx} f_{yy} - \boxed{f_{xy}^2}.$$

Sufficient Condn

Second Derivative Test

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) (neighbourhood of (a, b)) and that $f_x(a, b) = f_y(a, b) = 0.$ Then

1. f has a local maximum at (a, b) if $f_{xx}(a, b) < 0$ and $\det H > 0$ at (a, b)
2. f has a local minimum at (a, b) if $f_{xx}(a, b) > 0$ and $\det H > 0$ at (a, b)
3. f has a saddle point at (a, b) if $\det H < 0$ at (a, b)
4. test is inconclusive if $\det H = 0$. In this case we have to find some other way to determine the behaviour of f at (a, b).

Remark: In the last case of the above result, all the three possibilities can happen provided $f_x(a,b) = 0 = f_y(a,b)$. Consider the following examples (0, 0)

1. Let $f(x,y) = -(x^4 + y^4)$ for all $(x,y) \in \mathbb{R}^2$. Then $f(x,y) = -(x^4 + y^4) < 0 = f(0,0)$ for all (x,y) in any neighbourhood of $(0,0)$. Hence f has local maximum at $(0,0)$. Check that $\det H = 0$ at $(0,0)$.
2. Let $f(x,y) = (x^4 + y^4)$. Check as above that f has a local minimum at $(0,0)$ but $\det H = 0$ at $(0,0)$.
3. Let $f(x,y) = x^3y^3$. Then f has a saddle point at $(0,0)$ but $\det H = 0$ at $(0,0)$.

Example

1. Let $f(x, y) = 4xy - x^4 - y^4$ for $(x, y) \in \mathbb{R}^2$.
2. Then we have form the necessary conditions

$$\triangledown f(x, y) = 0 \Rightarrow y - x^3 = 0 = x - y^3.$$

The solutions are $(0, 0), (1, 1), (-1, -1)$. These are the critical points. Now we apply sufficient conditions.

$$f_{xx} = -12x^2, \; f_{yy} = -12y^2, f_{xy} = f_{yx} = 4.$$

3. At $(1, 1)$: $f_{xx} = -12 < 0$ and the Hessian Matrix at $(1, 1)$ is given by

$$H = \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}, \quad \det H = 144 - 16 > 0$$

Hence f has a local maximum at $(1, 1)$.

Example

1. At $(-1,-1)$: $f_{xx} = -12 < 0$ and the Hessian Matrix at $(-1,-1)$ is given by

$$H = \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}, \quad \det H = 144 - 16 > 0$$

Hence f has a local minimum at $(-1,-1)$.

2. At $(0,0)$: The Hessian Matrix at $(0,0)$ is given by

$$H = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \quad \det H = -16 < 0$$

Hence f has a saddle point at $(0,0)$.

Example