Lecture #6 (IC 152)

Theorem: Let $T: V \to V$, $\dim V < \infty$ be a linearly operator & $c_1, c_2, \ldots c_k$ be the distinct eigenvalues. Let $E_1, E_2, \ldots E_k$ be the ~~eigen~~space corresponding to eigenvalues $c_1, c_2, \ldots c_k$. Then T is diagonalizable iff

$$\checkmark \dim V = \dim E_1 + \dim E_2 + \ldots \dim E_k$$

Recall

Eigenvectors corresponding to distinct eigen values are linearly independent.

$T: V \to V$, $\dim V = n$

$c_1, c_2, c_3 \ldots c_k$ are distinct eigenvalues

$\dim E_{c_1} = 2 = \langle \alpha, \beta \rangle$

$\{ \alpha, \beta, \underbrace{\alpha_3, \alpha_4 \ldots \alpha_k}_{k+1 \text{ vectors !!}} \}_{v}$

Are they linearly independent?

Proof :- If T is diagonalizable

$$\Rightarrow f(t) = (x-c_1)^{d_1}(x-c_2)^{d_2} \cdots (x-c_k)^{d_k}$$

$$\Rightarrow \dim E_i = d_i$$

As characteristic polynomial is of degree $= \dim V$

$$d_1 + d_2 + \ldots d_k = \dim V .$$

$$\sum_{i=1}^{k} \dim E_i = \dim V.$$

Conversely,

Let $\dim V = \sum_{i=1}^{k} \dim E_i$

Let us think of $E_1 + E_2 + \cdots E_k =: E \subseteq V$

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_k$ are bases of $E_1, E_2 \ldots E_k$ respectively.

Now E is spanned by $\{\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}, \ldots \underline{\mathcal{B}_k}\} = \mathcal{B}$

dim V

If $\mathcal{B}$ is linearly independent then $\mathcal{B}$ forms a basis for E. But by assumption $\underline{d_1 + d_2 + \cdots d_k} = \dim V$ implies $\mathcal{B}$ forms a basis for V.

So only thing to prove is $\mathcal{B}$ is linearly independent set.

$\mathcal{B}_1 = \{\alpha_1, \alpha_2 \ldots \alpha_{d_1}\}$

$\mathcal{B}_2 = \{\beta_1, \beta_2, \ldots \beta_{d_2}\}$

$$\vdots$$

$$\mathcal{B}_k = \{\gamma_1, \gamma_2, \dots \gamma_{d_k}\}$$

A linear combination is

$$\boxed{a_1\alpha_1 + a_2\alpha_2 + \dots a_{d_1}\alpha_{d_1}} + \boxed{b_1\beta_1 + b_2\beta_2 + \dots b_{d_2}\beta_{d_2}} + \dots$$
$$\boxed{l_1\gamma_1 + l_2\gamma_2 + \dots l_{d_k}\gamma_{d_k}} = 0$$

To show $a_i = b_i = \dots = l_i = 0 \quad \forall i$

$$w_1 + w_2 + \dots w_k = 0 \checkmark \quad \text{where } w_i \in E_i \ \forall i = 1, 2, \dots k$$

If we could show that $w_i = 0 \ \forall i$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots a_{d_1}\alpha_{d_1} = 0 \Rightarrow a_1 = a_2 = \dots = a_{d_1} = 0$$

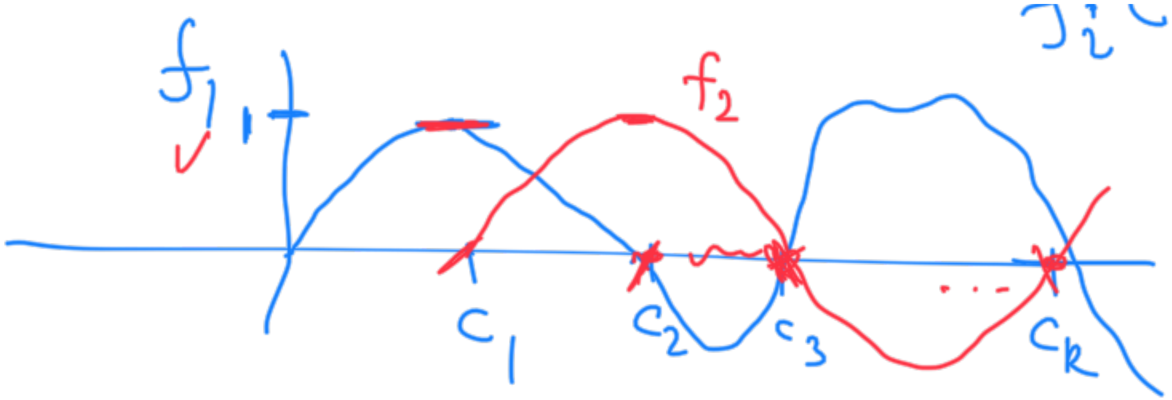
as $\{\alpha_1, \alpha_2 \dots \alpha_{d_1}\}$ is linearly indep set.

✓ "If f is a polynomial and $T\alpha = c\alpha$, then $f(T)\alpha = f(c)\alpha$"

Let us define $f_1, f_2, \dots f_k$ polynomials satisfying

$$f_j(c_i) = \{1 \text{ if } i = j\} \quad \forall i = 1, 2, \dots k$$

$f_i(c_j) = \begin{cases} \\ 0 & i \neq j \end{cases}$ $\quad j = 1, 2, \dots k$



$$\left\{ \begin{array}{l} g_1(x) = (x - c_2)(x - c_3) \cdots (x - c_k) \\ g_1(c_i) = 0 \quad \forall\ i = 2, \dots k \\ f_1(x) = \dfrac{1}{\boxed{g_1(c_1)}}\, g_1(x) \end{array} \right.$$

$g_1(c_1) \neq 0$

Now

$$0 = f_i(T)\, 0$$

$$= f_i(T)(w_1 + w_2 + \cdots w_k)$$

$$= \sum_{j=1}^{k} f_i(T)\, w_j = \sum_{j=1}^{k} f_i(c_j)\, w_j$$

$$= 0 + 0 \cdot + \underset{i^{th}}{1 w_i} + 0 + \cdots 0$$

$$\boxed{0 = w_i} \quad \forall\ i = 1, 2, \dots k$$

This proves that $\mathcal{B}$ forms a basis for V (consisting of eigenvectors of T). Hence T is diagonalizable.

Application of diagonalizability.

Solutions of system of differential equations

$$\left.\begin{aligned} x'(r) &= x + y \\ y'(r) &= 4x + y \end{aligned}\right\}$$

$x = x(r)$, $y = y(r)$, $r \in \mathbb{R}$.

✓ Construct $X(r) = \begin{pmatrix} x(r) \\ y(r) \end{pmatrix}$, $X'(r) = \begin{pmatrix} x'(r) \\ y'(r) \end{pmatrix}$

The above system becomes !!

$$\begin{pmatrix} x'(r) \\ y'(r) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x(r) \\ y(r) \end{pmatrix}$$

$$X' = AX, \text{ where}$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

... diagonalizable, then

Now if A is diagonalizable, then

$A = QDQ^{-1}$ for some invertible matrix Q.

$$\Rightarrow \quad X' = QDQ^{-1}X$$

$$\Rightarrow \quad Q^{-1}X' = DQ^{-1}X$$

$$(Q^{-1}X)' = D\,\underline{\underline{Q^{-1}X}}$$

Denote $Q^{-1}X = Y$ ←

$$\Rightarrow \quad Y' = DY, \checkmark$$

Here D is a diagonal matrix,

the system $Y' = DY$ is de-coupled and hence each equation of the system $Y' = DY$ can be solved independently.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \quad y_1' = d_1 y_1 \leftarrow$$

$$y_2' = d_2 y_2 \leftarrow$$

which leads to

$X = QY$, a solution to given system.

Characteristic polynomial for A

$$f(x) = x^2 - 2x - 3$$

Eigenvalues

$$f(x) = 0 \Rightarrow (x+1)(x-3) = 0$$

$$x = 3, -1$$

As eigenvalues are distinct, A is diagonalizable

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

We need to find E_3 & E_1 to construct Q

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x - y = 0 \Rightarrow y = 2x$$
$$-4x + 2y = 0$$

$$\Rightarrow \quad E_3 = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$$

Similarly $E_{-1} = \left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$

$$\Rightarrow \quad Q = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

Now $\quad Y' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} Y \Rightarrow \begin{array}{l} y_1' = 3y_1 \Rightarrow y_1 = c_1 e^{3r} \\ y_2' = -y_2 \Rightarrow y_2 = c_2 e^{-r} \end{array}$

$$Y = \begin{pmatrix} c_1 e^{3r} \\ c_2 e^{-r} \end{pmatrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3r} \\ c_2 e^{-r} \end{pmatrix}$$

$$x(r) = c_1 e^{3r} + c_2 e^{-r}$$

$$y(x) = 2c_1 e^{\,\cdots} - 2c_2 e \qquad \}$$

<u>Example</u>: (Fibonacci's Rabbits)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

x - pairs of juvenile
y - pairs of adult ✓ rabbits.

Let initially, $x=1$, $y=0$ ✓

$$(x, y) \xrightarrow[\text{year}]{\text{After 1}} (y, x+y)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow$$ After 1 year, 0 pairs of juvenile & 1 pair of adult rabbit.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

After 2 years, 1 pair of juvenile & 1 pair of adult rabits

Thus the population after n-years {starting $\underline{I} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$} will be

$$A^n \underline{I}, \text{ where}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Question: How can we compute A^n with least cost?

We know that if A is diagonalizable then $A = QDQ^{-1}$ for some diagonal matrix D & invertible matrix Q

$$\Rightarrow A^n = Q D^n Q^{-1} \checkmark \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$D^n = \begin{pmatrix} d_1^n & 0 \\ 0 & d_2^n \end{pmatrix}$$

Eigenvalues of A, $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

Eigenspaces of T, let v_1, v_2

$$A^n = \begin{pmatrix} v_1 & v_2 \end{pmatrix}_{2\times 2} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}$$

A^2, A^3, A^4 . . . . .

adult rabbits

Observe that the pairs of rabits form a Fibonacci's series.

$$\overset{+}{\overline{1, 2}}$$

$0, 1, 1, 2, 3 \ldots \ldots \ldots$

$$\underset{+}{\underline{2, 3}}$$