Assignment-2

Vector Space and Inner Product space

(1) Show directly that the set V of real valued continuous functions on the interval $[0,1]$ is a vector space under the pointwise operations we discussed in class.

(2) Decide if the following two subsets of V are subspaces. If a subspace, prove it. If not a subspace, say why explicitly.

- W^{even} is the set of real polynomials of even degree.
- $W^{\text{even power}}$ is the set of real polynomials in which every term has even degree.

(3)

Find the span of the subset $S = \{w_1, w_2, w_3, w_4\}$ in $\mathbb{R}^3$, where

$$w_1 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad w_3 = \begin{pmatrix} 10 \\ 10 \\ 25 \end{pmatrix}, \qquad w_4 = \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix}.$$

(4)

Show that the set $S = \{v_1, v_2, v_3\}$ is a linearly independent subset of $\mathbb{R}^4$, where

$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ 10 \\ 3 \\ 25 \end{pmatrix}.$$

(5)

Suppose $V_1, \ldots, V_m$ are inner product spaces. Show that the equation

$$\langle (u_1, \ldots, u_m), (v_1, \ldots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.

[*In the expression above on the right, $\langle u_1, v_1 \rangle$ denotes the inner product on V_1, …, $\langle u_m, v_m \rangle$ denotes the inner product on V_m. Each of the spaces $V_1, \ldots, V_m$ may have a different inner product, even though the same notation is used here.*]

(6)

Show that the square of an average is less than or equal to the average of the squares. More precisely, show that if $a_1, \dots, a_n \in \mathbf{R}$, then the square of the average of $a_1, \dots, a_n$ is less than or equal to the average of ${a_1}^2, \dots, {a_n}^2$.

(7)

Suppose V is a real inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

(8)

Suppose $u, v \in V$. Prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbf{R}$ if and only if $\|u\| = \|v\|$.

(9)

Suppose $e_1, \dots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \dots, e_m)$.

(10)

Suppose x, y, z are linearly independent. Is it true that x+y, y+z, z+x are also linearly independent?