Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Tutorial Sheet 3

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1. Show that the following matrix A is Hermitian

$$A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}.$$

Prove that there exists a unitary matrix U such that A can be written as $A = UDU^{-1}$ for a diagonal matrix D.

Observe that $A^* = A$ and hence A is Hermitian. The chaacteristic polynomial for A is $(x+1)(x+2)(x-6)$. As eigenvalues are distinct, A is diagonalizable. The eigenspaces corresponding to distinct eigenvalues are $E_{-1} =< (-1, 1+2i, 1)^t >, E_{-2} =< (1+3i, -2-i, 5)^t >$ and $E_6 =< (1-22i, 6-9i, 13)^t >$. It can be checked that for unitary matrix $U = \begin{bmatrix} -1 & 1+3i & 1-22i \\ 1+2i & -2-i & 6-9i \\ 1 & 5 & 13 \end{bmatrix}$ and a diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, A can be written as $A = UDU^-1$.

2. Let A be an $n \times n$ complex matrix. Prove that A is Hermitian if and only if X^*AX is real for all vectors X in $\mathbb{C}^n$.

Assume A is Hermitian, then $(X^*AX)^* = X^*A^*X = X^*AX$ and hence X^*AX is real. Conversely, if X^*AX is real then, $(X^*AX)^* = X^*AX$ which implies, $X^*(A^* - A)X = 0$ which implies $A^* = A$.

3. Find out a real symmetric matrix B and a real skew-symmetric matrix C such that the following matrix A can be written as $A = B + iC$

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

Can every Hermitian matrix A can be written in a similar fashion?

Every Hermitian matrix A can be written as $A = \frac{A+\bar{A}}{2} + i\frac{A-\bar{A}}{2i}$, where $B = \frac{A+\bar{A}}{2}$ and $C = \frac{A-\bar{A}}{2i}$ are real (as $\bar{B} = B$) symmetric ($B^t = B$) and real ($\bar{C} = C$) skew-symmetric ($C^t = -C$) matrices. For given Hermitian matrix A, we can construct in a similar way.

4. Find out Hermitian matrices B and C such that the following matrix A can be written as $A = B + iC$

$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}.$$

Generalize it for any complex $n \times n$ matrix.
Every complex matrix A can be written as $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$, where $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$ are Hermitian matrices. For given complex matrix A, we can construct in a similar way.

5. Find the minimal polynomial for the following linear operators

   (i) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined as $Tf = f'$.

   (ii) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, defined as $T(A) = A^t$

6. Let V be an n-dimensional vector space and let T be a linear operator on V . Suppose that there exists some positive integer k so that $T^k = 0$. Prove that $T^n = 0$.
   As $T^k = 0$ for some integer k, it means x^k is an annihilating polynomial for T. As minimal polynomial divides any annihilating polynomial, the minimal polynomial of T will be x^m, $1 \leq m \leq k$, i.e. $T^m = 0$. Moreover, by Cayley-Hamilton theorem, $1 \leq m \leq n$. Thus $T^n = T^{n-m}T^m = 0$.

7. Find a minimal polynomial of the following matrix without finding characteristic polynomial
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
   Clearly $A^2 = 0$ but $A \neq 0$. Therefore the minimal polynomial dividing annihilating polynomial x^2 is either x or x^2. But x can not be the minimal polynomial as $T \neq 0$. Hence minimal polynomial is x^2.

8. Let $a, b, c \in \mathbb{R}$, then show that for the following matrix characteristic and minimal polynomials are same
$$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$$
   The characteristic polynomial for the given matrix is $x^3 - ax^2 - bx - c$. The choices for minimal poynomial are

   (i) A two degree monic polynomial of the type $x^2 + px + q$

   (ii) One degree monic polynomial of the type $x + p$

   (iii) Characteristic polynomial

   Note that the choice (ii) is not possible as A is not a scalar multiple of identity for any choices of $a, b, c \in \mathbb{R}$. Moreover, upon computation, the matrix $B = A^2 + pA + qI \neq O$ for any choice of $p, q \in \mathbb{R}$ as $B_{31} = 1 \neq 0$

9. Prove that if $T \in L(V, V)$ is annihilated by a polynomial over $\mathbb{C}$ having distinct roots, then T is diagonalizable. As a direct application of this result, show the following

   (a) Let T be a linear operator on a complex vector space such that $T^k = I$ for some positive integer k. Then T is diagonalizable.

   (b) Prove that every matrix A satisfying $A^2 = A$ is diagonalizable.

   Let $p(x) = (x - c_1)(x - c_2) \cdots (x - c_k)$, where $c_1, c_2, \cdots c_k$ are distinct, be an annihilating polynomial for T. As minimal polynomial divides any annihilating polynomial, the minimal polynomial must be the product of distinct linear factors and therefore T must be diagonalizable.

   (a) As $x^k - 1$ annihilates T and has distinct roots in $\mathbb{C}$, T must be diagonalizable

   (b) As $x^2 - x = x(x-1)$ is an annihilating polynomial for T with the distinct roots $0, 1$ and hence T must be diagonalizable.

10. Compute the minimal polynomial for the following matrices

$$(i)\ \ A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad (ii)\ \ B = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

   (i) Characteristic polynomial for the matrix A is $(x-3)(x-2)^2$. Thus we have two choices for the minimal polynomial: $(x-3)(x-2)$ or $(x-3)(x-2)^2$. Upon computation we see that $(A - 3I)(A - 2I) = O$ hence $(x-3)(x-2)$ is the minimal polynomial.

   (ii) Characteristic polynomial for the matrix B is $(x-2)^3$. We have three choice for the minimal polynomial, namely, $(x-2), (x-2)^2$ and $(x-2)^3$. Observe that the matrix $B \neq 2I$. Thus $x - 2$ can not be the minimal polynomial. Upon computation, we find that $B^2 \neq 0$, hence $(x-2)^3$ is the minimal polynomial.

11. Verify Cayley-Hamilton theorem for the following

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (2x + 5y, 6x + y)$

   (b) $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

12. Let characteristic polynomial of a matrix A be $x^2 - x + 1$. Compute A^3 and A^5.
   As A satisfies it's characteristic polynomial (by Cayley-Hamilton theorem), we get $A^2 - A + I = 0$ which implies $A^2 = A - I$. Now $A^3 = AA^2 = A(A - I) = A^2 - A = A - I - A = -I$. Similarly, $A^5 = A^3A^2 = -I(A - I) = I - A$