MA202: Calculus II

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Lecture Notes

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Module 3

Lecture 6

Review of Riemann Integration

Recall the following concepts of Riemann integration of a bounded function $f : [a, b] \to \mathbb{R}$

1. Partition of an interval $[a, b]$,
2. The quantities $L(P, f)$ and $U(P, f)$ (upper and lower Riemann sum)
3. Refinement of a partition,
4. How the integration is defined as a limit of sum (Geometrically),
5. Monotone, continuous functions are Riemann integrable etc.

Double Integral

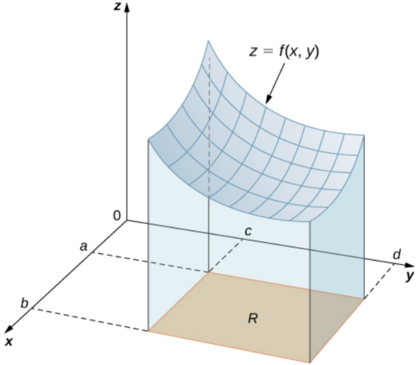
1. Recall the concepts of Riemann integration of single variable functions $y = f(x)$ over $[a, b]$ which represents the area under the curve $y = f(x)$ bounded by $x = a$ and $x = b$.
2. In a similar way the double integration of a function $z = f(x, y)$ represents the volume under the surface $z = f(x, y)$

Double Integral on a Rectangle:

- First we will see the double integration of a bounded function over a rectangle in $\mathbb{R}^2$ (Rectangle is the direct generalisation of closed interval in $\mathbb{R}$.)
- Let us consider the rectangle $R = [a, b] \times [c, d]$ in $\mathbb{R}^2$ where $a < b$ and $c < d$. Also let $f : R \to \mathbb{R}$ be a bounded function.

Double Integral on a Rectangle

- The graph of f represents a surface above the xy-plane with equation $z = f(x, y)$ where z is the height of the surface at the point (x, y).



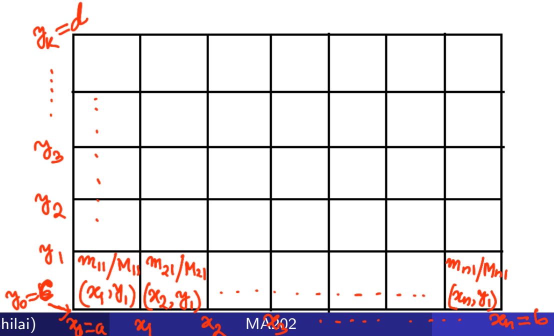
Double Integral on a Rectangle

- We divide R into some small non-overlapping rectangles.
- Let $n, \ k \in \mathbb{N}$ and consider the partition P of R as

$$P = \{(x_i, y_j) : i = 0, 1, \cdots, n, \ \& \ j = 0, 1, \cdots, k\}$$

where $a = x_0 < x_1 < \cdots < x_n = b$, and $c = y_0 < y_1 < \cdots < y_k = d$.

- R is divided into nk number of non-overlapping sub-rectangles $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ where $i = 1, \cdots, n$ and $j = 1, \cdots, k$.



Double Integral on a Rectangle

- Define for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$

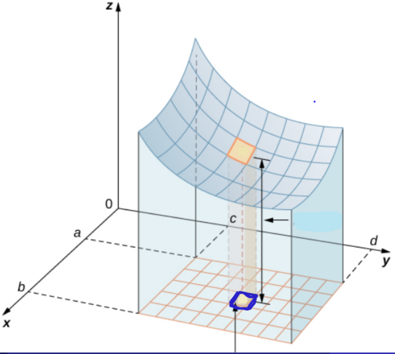
$$
\begin{aligned}
m_{ij}(f) &= \inf\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\} \\
M_{ij}(f) &= \sup\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\} \\
m(f) &= \inf\{f(x,y) : (x,y) \in R\} \\
M(f) &= \sup\{f(x,y) : (x,y) \in R\}
\end{aligned}
$$

- Define respectively the lower double sum and upper double sum of f with respect to P by

$$
\begin{aligned}
L(P,f) &= \sum_{i=1}^{n}\sum_{j=1}^{k} m_{ij}(f)(x_i - x_{i-1})(y_j - y_{j-1}), \\
U(P,f) &= \sum_{i=1}^{n}\sum_{j=1}^{k} M_{ij}(f)(x_i - x_{i-1})(y_j - y_{j-1}),
\end{aligned}
$$

- If we consider $A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$ be the area of the small sub-rectangle formed by $[x_i, x_{i-1}] \times [y_j - y_{j-1}]$ then the quantities $L(P,f)$ and $U(P,f)$ can also be written as

$$L(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{k} m_{ij}(f)A_{ij}, \quad U(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{k} \underline{M_{ij}(f)}A_{ij}.$$



Double Integral on a Rectangle

- Since $m(f) \leq m_{ij}(f) \leq M_{ij}(f) \leq M(f)$ for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$, the following relation can be derived easily

$$m(f)(b-a)(d-c) \leq L(P, f) \leq U(P, f) \leq M(f)(b-a)(d-c).$$

$$\sum_i \sum_j m(f)(x_i - x_{i-1})(y_j - y_{j-1}) \leq L(P, f) \leq U(P, f) \leq$$

$$\sum_i \sum_j M(f)(x_i - x_{i-1})(y_j - y_{j-1})$$

Double Integral on a Rectangle

- Note that if we refine the partition P of the rectangle $[a, b] \times [c, d]$, that means if we increase the number of sub-rectangles in P then the new partition P^* is called the refinement of P and in this case

$$L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f).$$

- This shows that the quantities $L(P, f)$ and $U(P, f)$ are bounded when the partition varies.
- Define $L(f) = \sup_{P} L(P, f)$, $U(f) = \inf_{P} U(P, f)$.
- It is easy to check that $L(f) \leq U(f)$.

Double integrable function on rectangle

A bounded function $f : R \to \mathbb{R}$ is said to be double integrable on R if $L(f) = U(f)$.

In this case the double integral of f on the rectangle R is the value $L(f) = U(f)$ and it is written as

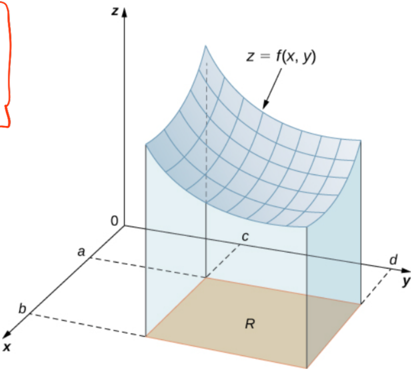
$$\iint_R f(x, y) d(x, y), \text{ or } \iint_R f.$$

Double Integral on a Rectangle

Geometrical Interpretation: Let $f : R \to \mathbb{R}$ be integrable on R and non-negative. The double integral of f on R gives the volume of the solid formed under the surface $z = f(x, y)$ and above the rectangle R. In other words it gives the volume of the following set

$$S = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R, \ \& \ 0 \leq z \leq f(x, y)\}.$$

$$\boxed{\text{Volume}(S) = \iint_R f}$$



1. If $f(x,y) = 1$ for all $(x,y) \in R$ then $m_{ij} = M_{ij} = 1$ for all i and j and also $L(P,f) = (b-a)(d-c) = U(P,f)$ for all partition P. Hence $\sup_p L(P,f) = (b-a)(d-c) = \inf_P U(P,f)$ and consequently

$$\iint_R f(x,y)d(x,y) = (b-a)(d-c).$$

2. Let $f : R \to \mathbb{R}$ is defined as $f(x,y) = 1$ when both x and y are rational and $f(x,y) = 0$ for any other values of x, and y in R. Clearly f is bounded and $m_{ij} = 0,\ M_{ij} = 1$ for all i,j. Therefore $L(P,f) = 0$ and $U(P,f) = (b-a)(d-c)$ (check!). Hence

$$L(f) = \sup_P L(P,f) = 0 \neq U(f) = \inf_P U(P,f) = (b-a)(d-c).$$

Hence f is not double-integrable on R.

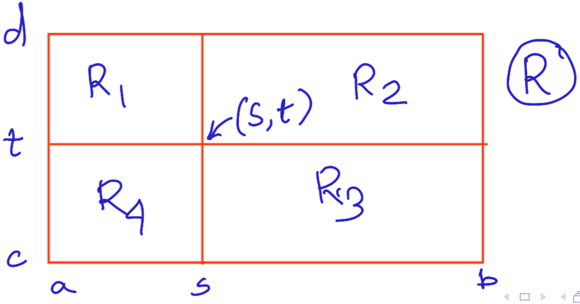
The function is known as bivariate Dirichlet function.

Double Integral on a Rectangle : Few results

Theorem (Riemann condition)

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b] \times [c, d]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

The proof is similar as single variable case.



Double Integral on a Rectangle : Few results

(Domain Additivity)

Let $R := [a, b]\times[c, d]$, and let $f : R \to \mathbb{R}$ be a bounded function. Let $s \in (a, b)$, $t \in (c, d)$. Then f is integrable on R if and only if f is integrable on the four subrectangles $[a, s]\times[c, t]$, $[a, s]\times[t, d]$, $[s, b]\times[c, t]$ and $[s, b]\times[t, d]$. In this case, the integral of f on R is the sum of the integrals of f on the four subrectangles.

Some Conventions: The following conventions are made on the double integral of a function f on the rectangle R,

1. If $a = b$ or $c = d$, then $\iint_R f = 0$,
2. $\iint_{[b,a]\times[c,d]} f = -\iint_{[a,b]\times[c,d]} f$, and $\iint_{[a,b]\times[d,c]} f = -\iint_{[a,b]\times[c,d]} f$,
3. $\iint_{[b,a]\times[d,c]} f = \iint_{[a,b]\times[c,d]} f$,

Let $R := [a, b] \times [c, d]$. If $f, g : R \to \mathbb{R}$ are integrable, then

(i) $f + g$ is integrable, and $\iint_R (f + g) = \iint_R f + \iint_R g$,

(ii) αf is integrable, and $\iint_R \alpha f = \alpha \iint_R f$ for all $\alpha \in \mathbb{R}$,

(iii) $f \cdot g$ is integrable,

(iv) If there is $\delta > 0$ such that $|f(x, y)| \geq \delta$ for all $(x, y) \in R$ (so that $1/f$ is bounded), then $1/f$ is integrable,

(v) If $f \leq g$, then $\iint_R f \leq \iint_R g$,

(vi) $|f|$ is integrable, and $|\iint_R f| \leq \iint_R |f|$.

Double Integral on a Rectangle : Few results

Let $f : \mathbf{R} \to \mathbb{R}$. Suppose that for each fixed $x \in [a, b]$

$$\phi(x) := \int_c^d f(x, y)dy$$

exists. If ϕ is Riemann integrable on $[a, b]$ then

$$\int_a^b \phi(x)dx = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

is called an iterated integral of f over $\mathbf{R}$.

Similarly $\int_c^d \left(\int_a^b f(x, y)dx \right) dy$, when exists, is another iterated integral of f over $\mathbf{R}$.

1. The important question is that how to evaluate the double integral on R?
2. Our main approach is to repeatedly apply Riemann integration when one variable kept fixed, i,e., in terms of iterated integral.
3. In this regard we have the following theorem.

Double Integral on a Rectangle : Evaluation

Theorem (Fubini Theorem on a Rectangle)

Let $R := [a,b]\times[c,d]$, and let $f : R \to \mathbb{R}$ be integrable.
Let I denote the double integral of f on R.

(i) If for each fixed $x \in [a,b]$, the Riemann integral $\int_c^d f(x,y)dy$ exists, then the **iterated integral** $\int_a^b \left(\int_c^d f(x,y)dy\right)dx$ exists and is equal to I.

(ii) If for each fixed $y \in [c,d]$, the Riemann integral $\int_a^b f(x,y)dx$ exists, then the **iterated integral** $\int_c^d \left(\int_a^b f(x,y)dx\right)dy$ exists and is equal to I.

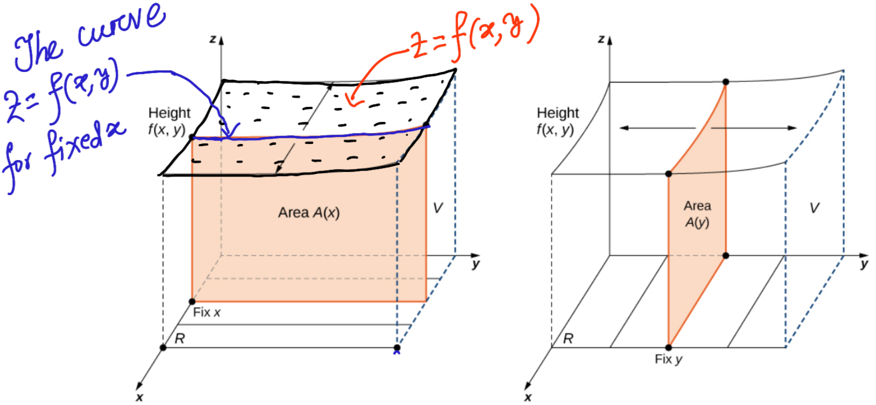
(iii) If the hypotheses in both (i) and (ii) above hold, and in particular, if f is continuous on R, then

$$\int_a^b \left(\int_c^d f(x,y)dy\right)dx = I = \int_c^d \left(\int_a^b f(x,y)dx\right)dy.$$

Geometrical Interpretation of Fubini Theorem:

- If f be a non-negative function on $R = [a, b] \times [c, d]$ then the volume D under the surface $z = f(x, y)$ and above the rectangle R (in other words the double integral $\iint_R f$) can be obtained by finding the area of the cross section of D perpendicular to x-axis (by keeping x variable fixed) or by finding the area of the cross-section of D perpendicular to y-axis (by keeping y variable fixed) and then sum up all such possible cross-sections,
- That means by calculating the iterated integrals as follows:
- For $x \in [a, b]$, we find the area $A(x) = \int_c^d f(x, y)dy$ of the cross-section of D perpendicular to x-axis or,
- For $y \in [c, d]$, we find the area $A(y) = \int_a^b f(x, y)dx$ of the cross-section of D perpendicular to y-axis.
- Then by Fubini Theorem

$$Vol(D) = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$


The curve
z= f(x,y)
for fixed x
z=f(x,y)
z
Height
f(x, y)
Area A(x)
V
y
Fix x
R
x
z
Height
f(x, y)
Area
A(y)
V
y
R
Fix y
x

Double Integral on a Rectangle : Evaluation

- Then by Fubini Theorem

$$Vol(D) = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

also

$$Vol(D) = \int_c^d A(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Important

To apply Fubini's Theorem, f needs to be integrable on R.

(i) Let $R := [0,1]\times[0,1]$, and $f(x,y) := (x+y)^2$, $(x,y)\in R$. Then f is continuous on R. The double integral of f on R is

$$\begin{aligned}\int_0^1\left(\int_0^1 (x+y)^2 dx\right) dy &= \frac{1}{3}\int_0^1 (x+y)^3\Big|_0^1 dy \\ &= \frac{1}{3}\int_0^1 \left((1+y)^3 - y^3\right) dy = \frac{7}{6}.\end{aligned}$$

$R = [0,1]\times[0,1]$ and $f: R \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ 2y & \text{when } x \text{ is irrational.} \end{cases}$$

f is double integrable or not. Calculate the iterated integral

Double Integral on a Rectangle : Examples

(ii) Let $R := [0,1]\times[0,1]$, $f(0,0) := 0$, and for $(x,y) \neq (0,0)$, let $f(x,y) := xy(x^2-y^2)/(x^2+y^2)^3$. For $x \in [0,1]$, $x \neq 0$,

$$A(x) := \int_0^1 f(x,y)dy = \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}dy = \frac{x}{2(1+x^2)^2}.$$

(Substitute $x^2+y^2 = u$.) Also, $A(0) = \int_0^1 0dy = 0$. Hence

$$\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = \int_0^1 A(x)dx = \int_0^1 \frac{x}{2(1+x^2)^2}dx = \frac{1}{8}.$$

By interchanging x and y, $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = -\frac{1}{8}$.

Thus the two iterated integrals exist, but they are not equal.

Note that since $f(1/n, 1/2n) = 24n^2/125$ for all $n \in \mathbb{N}$, the function f is not bounded on R, and so it is not integrable on R. Thus Fubini's theorem is not applicable.