IC200 : Introduction to Astronomy & Astrophysics **Instructor : Dr. Mahavir Sharma**
Tutorial-4

1. Calculate the upper limit of the distance from the centre of a black hole from which even the light can not escape. What is its value for a solar mass black hole. What is its values for the mass of the earth (imagine hypothetically if the entire mass of earth is confined to a point).

2. Calculate and plot the mass distribution corresponding to the density profile for the dark matter of a galaxy given by

$$\rho = \frac{\rho_0}{x(1+x^2)}; \qquad x = \frac{r}{r_s}$$

   where ρ_0 and r_s are constants. How would you derive the potential corresponding to this?

3. Show that if the Universe is composed only of matter then the expansion factor $a(t) \propto t^{2/3}$, where t is the time since the beginning of the Universe. If it is composed of only photons then show that $a(t) \propto \sqrt{t}$.

4. The light we receive from a far-away galaxy is redshifted such that redshift, $z = 6$. Think and then try to derive/explain how far the galaxy is from us? How fast the galaxy is moving away from us?

5. Think and describe on the basis of what you learned in the lecture whether the Hubble constant H changes with time?

6. The light we receive from Andromeda is redshifted or blueshifted. How much do you think is the shift? How would you calculate?

① Without using GTR, we have an approximate relation for the escape velocity, given as:

$$\boxed{\frac{1}{2}v^2 = \frac{GM}{r}}$$, M is mass of object from which the matter escapes.

$$\Rightarrow \quad r = \frac{2GM}{v^2}$$

For light, $v = c$ & hence

$$\boxed{r = \frac{2GM}{c^2}}$$

So, i). For a solar mass BH,

$$M = M_\odot = 2 \times 10^{30}\,\text{Kg}$$

$$\text{then } r = \frac{2 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{9 \times 10^{16}}$$

$$= \frac{4 \times 6.67 \times 10^{3}}{9} \simeq 3 \times 10^{3}\,\text{m} \cong 3\,\text{Km}.$$

ii). For Earth with $M = 6 \times 10^{24}$ Kg

$$r = \frac{2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{9 \times 10^{16}} = 9 \times 10^{-3}\,\text{m}.$$

$$\cong 9\,\text{mm}.$$

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$$\rho = \frac{\rho_0}{x(1+x^2)} \quad ; \quad x = \frac{r}{r_s}$$

So, $$M(r) = \int_0^r 4\pi r^2 \rho \, dr$$

$$= \int 4\pi \left(\frac{r}{r_s}\right)^2 \cdot r_s^2 \cdot \frac{\rho_0}{x(1+x^2)} \cdot r_s \, d\left(\frac{r}{r_s}\right)$$

$$M\left(\frac{r}{r_s}\right) = 4\pi r_s^3 \rho_0 \int \frac{1 \cdot x^2}{x(1+x^2)} \, dx$$

$$= \left[\frac{4\pi \rho_0 r_s^3}{2} \cdot \ln(1+x^2) \right]_0^{r/r_s}$$

So, $$M = \frac{4\pi \rho_0 r_s^3}{2} \cdot \ln\left[1 + \left(\frac{r}{r_s}\right)^2\right]$$

Now, potential $$\phi = -\int \frac{GM}{r^2} \, dr$$

$$= \frac{4\pi \rho_0 r_s^3 \cdot G}{2} \int \frac{\ln\left[1 + \left(\frac{r}{r_s}\right)^2\right]}{\left(\frac{r}{r_s}\right)^2 r_s^2} \cdot r_s \cdot d\left(\frac{r}{r_s}\right)$$

$$= \frac{4\pi G \rho_0 r_s^2}{2} \cdot \int \frac{\ln(1+x^2)}{x^2} \cdot dx$$ (integration by parts)

$\ln(1+x^2) \to$ II

$x^2 \to$ I

So,
$$\boxed{\phi = 2\pi G \rho_0 r_s^2 \left[2\tan^{-1}x - \frac{\ln(1+x^2)}{x} \right] + C}$$

at $x \to \infty$, $\phi \to 0$ so

$$0 = 2\pi G \rho_0 r_s^2 \left[2 \cdot \frac{\pi}{2} - 0 \right] + C$$

$$\Rightarrow \quad \boxed{C = -2\pi^2 G \rho_0 r_s^2}.$$

(3) We know,

$$r_{physical} = a\, r_{comoving}$$

$$\Rightarrow \quad v = \frac{dr_{ph}}{dt} = \frac{d(a r_{co})}{dt} = \dot{a} r_{co} + \underbrace{a \dot{r}_{co}}_{\to 0}$$

$$\dot{r}_{co} = 0$$

$$\Rightarrow \quad v = \frac{\dot{a}\, r_{ph}}{a} = \frac{\dot{a}}{a} \cdot r_{ph}$$

$$\Rightarrow \quad \boxed{v = \frac{\dot{a}}{a} \cdot r_{ph}} \quad \text{or} \quad \boxed{v = H r} \longrightarrow (1)$$

(Hubble's law)

Now, I^{st} law of thermodynamics $\rightarrow$

[matter-energy density $\Rightarrow E = \rho c^2$]

$$\frac{d \rho c^2}{dt} + \vec{\nabla} \cdot \left[(\rho c^2 + P) \vec{v} \right] = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + \vec{\nabla} \cdot \left[\left(\rho + \frac{P}{c^2} \right) \vec{v} \right] = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + (\vec{\nabla} \cdot \vec{v}) \left(\rho + \frac{P}{c^2} \right) = 0$$

$$\Rightarrow \quad \frac{d\rho}{dt} + \left(\vec{\nabla} \cdot \frac{\dot{a}}{a} \vec{r} \right) \left(\rho + \frac{P}{c^2} \right) = 0.$$

$$\Rightarrow \frac{d\rho}{dt} + \frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)(\nabla \cdot \cancel{\vec{r}})^{3} = 0$$

Now, from Kinetic theory of Gases

$$P = \frac{1}{3}\rho v^2$$

So, $$\boxed{\dot{\rho} = -\frac{3\dot{a}}{a}\rho\left[1 + \frac{v^2}{3c^2}\right]} \quad \text{— (2)}$$

Now, for matter dominated $\rightarrow$ $v << c$

& for radiation dominated $\rightarrow$ $v \approx c$

Also, we know from FRW Metric,

$$\boxed{\dot{a} \propto a\sqrt{\rho}} \quad \text{— (3)}$$

So, for matter dominated in eq^n (2),

$$\boxed{\dot{\rho} = -\frac{3\dot{a}}{a}\rho} \quad \text{— (4)}$$

$\rightarrow$ Integrating (4), we have $\rho \propto \frac{1}{a^3}$, then by eq^n (3)

$\dot{a} \propto \frac{a}{a^{3/2}} \Rightarrow \dot{a} \propto \frac{1}{\sqrt{a}}$. Also $\frac{\dot{a}}{a} = \frac{1}{t}$ (from eq^n (1))

So, $a^{3/2} \propto t$ or $\boxed{a \propto t^{2/3}}$.

Also, for radiation dominated in eqn (2),

$$\dot{\rho} = -\frac{3\dot{a}}{a}\rho\left[\frac{4}{3}\right] = -\frac{4\dot{a}}{a}\rho \quad \text{--- (5)}$$

Integrating (5), we get $\rho \propto \frac{1}{a^4}$

$\Rightarrow$ From (3), we have

$$\dot{a} \propto a\sqrt{\rho}$$

$\Rightarrow$ So, $\dot{a} \propto \frac{a}{a^2} \Rightarrow a^2 \propto t$

or $\boxed{a \propto t^{1/2}}$.

(4) Redshift $\rightarrow$

$$1 + Z = \frac{\lambda_{observed}}{\lambda_{emitted}}.$$

From special theory of relativity, the Doppler shift

$$1 + Z = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

$$\Rightarrow (1 + Z)^2 = \frac{1 + v/c}{1 - v/c}$$

or $$v/c = \frac{(1 + Z)^2 - 1}{(1 + Z)^2 + 1}$$

$$\Rightarrow v = \frac{(Z+1)^2 - 1}{(Z+1)^2 + 1} \cdot c$$

Given $Z = 6$,

$$v = \frac{7^2 - 1}{7^2 + 1} \cdot c = \frac{48}{50} \cdot c = \frac{96}{100} \cdot c$$

So, $\boxed{v = 0.96\, c}$

*For finding out distance:

$$a = \frac{1}{1+Z}$$ So, $$a = \frac{1}{7}$$ (for our case)

Note: (Use the 2^{nd} Friedman's eq^n)

Equations

There are two independent Friedmann equations for modelling a homogeneous, isotropic universe. The first is:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3}$$

which is derived from the 00 component of Einstein's field equations. The second is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \quad \text{— (1)}$$

which is derived from the first together with the trace of Einstein's field equations (the dimension of the two equations is time^{-2}).

So, from here, we will have $a(t)$ (by solving the differential eq^n (1))

Now, $$r_{co} = \int_{a=1/7}^{a=1} dr = \int_{a=1/7}^{a=1} \frac{c\,dt}{a(t)} \quad \text{— (2)}$$

Also, for now $r_{co} = r_{phy}$, Since $a = 1$

So, solve (2) to get r_{phy} using integration from (1)

(which will get us: $a(t)$)

(5) Hubble constant, $H = \frac{\dot{a}}{a}$

For matter dominated, $a \propto t^{2/3}$

$\dot{a} \propto t^{-1/3}$

or $\frac{\dot{a}}{a} \propto \frac{1}{t}$

For radiation dominated, $a \propto t^{1/2}$

$\dot{a} \propto t^{-1/2}$

or $\frac{\dot{a}}{a} \propto \frac{1}{t}$

So, $H = \frac{\dot{a}}{a}$, not a constant with time but constant everywhere in space for specific time.

Also, since $H \propto \frac{1}{t}$, so if $t \rightarrow$ very large then change in H is very low.

(Using $\Delta H \propto -t^{-2} \cdot \Delta t$)

(6) Andromeda has a blue shift with respect to our Milky Way. It can be calculated by knowing the speed with which Andromeda is approaching us, or by studying the radiation spectra.

(from internet)

& then use, $1+Z = \sqrt{\dfrac{1+v/c}{1-v/c}}$.