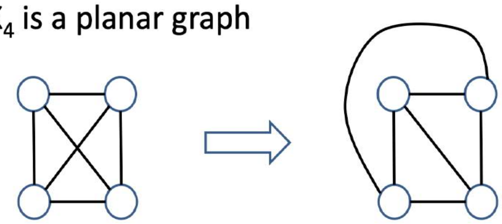
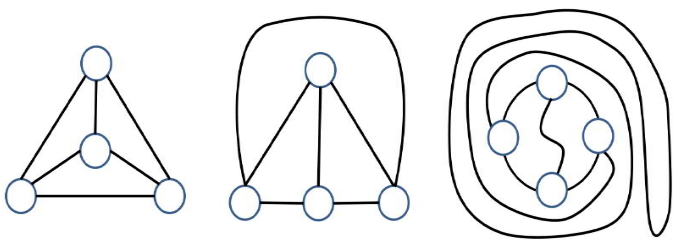
Definition : A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

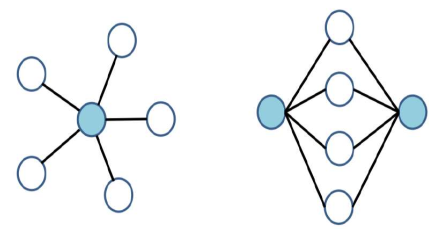
- Ex : K_4 is a planar graph



Ex : Other planar representations of K_4

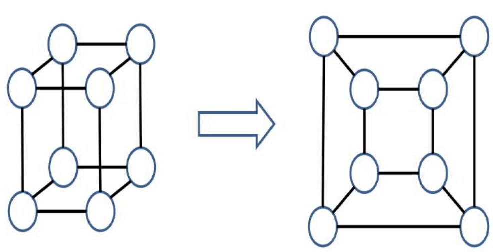


Ex : $K_{1,n}$ and $K_{2,n}$ are planar graphs for all n



$K_{1,5}$ $K_{2,4}$

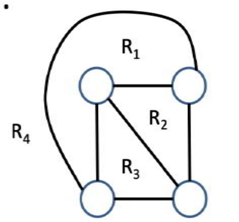
Ex : Q_3 is a planar graph



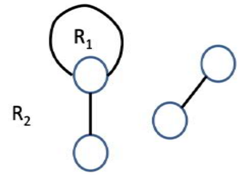
2 Euler's formula

A planar graph with cycles divides the plane into a set of regions, also called *faces*. Each region is bounded by a simple cycle of the graph: the path bounding each region starts and ends at the same vertex and uses each edge only once. Notice that, by convention, we also count the unbounded area outside the whole graph as one region.

- **Ex :**



4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

Euler's Planar Formula

Suppose that G is a connected simple planar graph, with v vertices, e edges, and f faces. Then Euler's formula states that:

$$v - e + f = 2$$

We can now prove Euler's formula ($v - e + f = 2$) works in general, for any connected simple planar graph.

Proof: by induction on the number of edges in the graph.

Base: If $e = 0$, the graph consists of a single vertex with a single region surrounding it. So we have $1 - 0 + 1 = 2$ which is clearly right.

Induction: Suppose the formula works for all graphs with no more than n edges. Let G be a graph with $n + 1$ edges.

Case 1: G doesn't contain a cycle. So G is a tree and we already know the formula works for trees. **(WHY THIS STATEMENT IS TRUE??)**

Case 2: G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G'.

Since the cycle separates the plane into two regions, the regions to either side of p must be distinct. When we remove the edge p, we merge these two regions. So G' has one fewer regions than G.

Since G' has n edges, the formula works for G' by the induction hypothesis. That is $v' - e' + f' = 2$. But $v' = v$, $e' = e - 1$, and $f' = f - 1$. Substituting, we find that

$$v - (e - 1) + (f - 1) = 2$$

So

$$v - e + f = 2$$

- **Theorem:** For a simple connected planar graph with $v \geq 3$ vertices and e edges, $e \leq 3v - 6$.

- Proof: Let r be the number of regions in a planar representation of the graph, and for a region R, let $\deg(R)$ be the number of edges that are adjacent to the region, so each edge is adjacent to two regions.
  We know that $\deg(R) \geq 3$ for each region since the graph doesn't have multiple edges (for interior regions) and has at least three vertices (for the exterior region). Since each edge is adjacent to two regions,
  $2e = \sum \deg(R) \geq 3r$.
  We can use this with Euler's formula ($r = e - v + 2$) to get
  $3r3(e-v+2)e \leq 2e \leq 2e \leq 3v-6$. ■

- This theorem isn't if-and-only-if, so be careful.
  - It does show that K_5 is non-planar: $v=5$, $e=10$.
  - But for $K_{3,3}$, we have $v=6$ and $e=9$. It satisfies the inequality, but is non-planar.

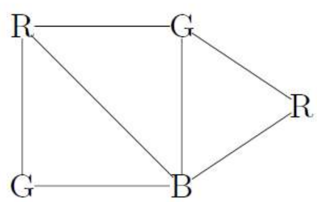
- **Corallary:** A simple connected planar graph with $v \geq 3$ has a vertex of degree five or less.
  Proof: Suppose every vertex has degree 6 or more. Then the total number of edges is $2e \geq 6v$.
  But, because the graph is planar,
  $\sum \deg(v) = 2e \leq 6v - 12$.
  We have a contradiction. ■

Graph coloring

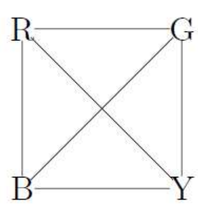
Remember that two vertices are *adjacent* if they are directly connected by an edge.

A *coloring* of a graph G assigns a color to each vertex of G, with the restriction that two adjacent vertices never have the same color. The *chromatic number* of G, written $\chi(G)$, is the smallest number of colors needed to color G.

For example, only three colors are required for this graph:



But K_4 requires 4:



We can also prove a useful general fact about colorability:

Claim 1 *If all vertices in a graph G have degree $\leq D$, then G can be colored with $D+1$ colors.*

Proof: by induction on the number of vertices in G.

Base: The graph with just one vertex has maximum degree 0 and can be colored with one color.

Induction: Suppose that any graph with $\leq k$ vertices and maximum vertex degree $\leq D$ can be colored with $D+1$ colors.

Let G be a graph with $k+1$ vertices and maximum vertex degree D. Remove some vertex v (and its edges) from G to create a smaller graph G'.

The maximum vertex degree of G' is no larger than D, because removing a vertex can't increase the degree. So, by the inductive hypothesis, G' can be colored with $D+1$ colors.

Because v has at most D neighbors, its neighbors are only using D of the available colors, leaving a spare color that we can assign to v. The coloring of G' can be extended to a coloring of G with $D+1$ colors.

Coloribility of Planar Graphs

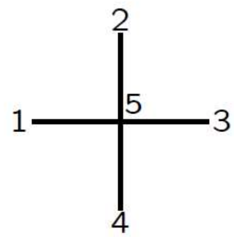
Theorem 2. *Every planar graph can be 5-colored.*

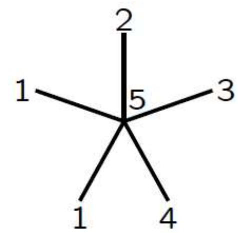
Proof: As you might expect, we will again do this by induction on the number of vertices.

Base case: The simplest connected planar graph consists of a single vertex. Pick a color for that vertex. we are done.

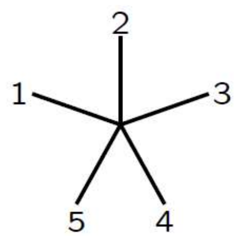
Induction step: Assume $k \geq 1$, and assume that every planar graph with k or fewer vertices can be 5-colored. Now consider a planar graph with $k+1$ vertices. From above, we know that the graph has a vertex of degree 5 or fewer. Remove that vertex (and all edges connected to it). By the induction hypothesis, we can 5-color the remaining graph. Put the vertex (and edges) back in. We have a graph with every vertex colored (without conflicts) except for the one.

If the vertex has degree less than 5, or if it has degree 5 and only 4 or fewer colors are used for vertices connected to it, we can pick an available color for it, and we are done (numbers represent colors).

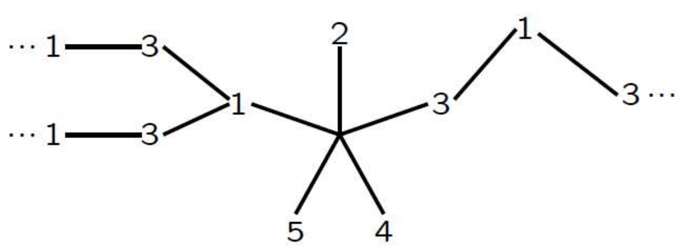




If the vertex has degree 5, and all 5 colors are connected to it, we have a little more work to do. In this case, using numbers 1 through 5 to represent colors, we label the vertices adjacent to the "special" (degree 5) vertex 1 through 5 (in order).

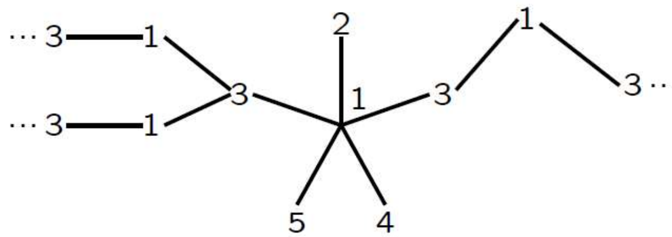


Now make a subgraph out of all the vertices colored 1 or 3 which are connected to the 1 and 3 colored vertices adjacent to the "special" vertex.



If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in this subgraph, simply exchange the colors 1 and 3 throughout the subgraph connected to the vertex colored 1.

This will leave color 1 available to color the "special" vertex, and we are done.



On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the same "subgraph" process with vertices colored 2 and 4 adjacent to the "special" vertex. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3. Now we can exchange the colors 2 and 4 in the subgraph connected to the adjacent vertex labeled 2. This will leave color 2 for the "special" vertex.