Lecture # 1 (IC 152)

Review

* **Vector space**

$(V(\mathbb{F}), +, \cdot)$, V = set of vectors, $\mathbb{F}$ – field of scalars

operations $+, \cdot$ namely

vector addition and scalar multiplication satisfying

1) $\alpha, \beta \in V$, $\alpha + \beta \in V$, $\alpha + \beta = \beta + \alpha$ commutativity

2) $\alpha, \beta, \gamma \in V$, $(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$ associativity

3) $\exists!$ vector '0' (zero) in V such that

$$\alpha + 0 = 0 + \alpha = \alpha.$$

4) For every $\alpha \in V$, $\exists$ '$-\alpha$' s.t.

$$\alpha + (-\alpha) = 0 \text{ (zero vector)}$$

5) $c \in \mathbb{F}$ & $\alpha \in V$, $c \cdot \alpha \in V$ $\forall\, c \in \mathbb{F}$ & $\alpha \in V$

(closedness under scalar multiplication)

$1 \cdot \alpha = \alpha$ $\forall\, \alpha \in V$ (1 is the multiplicative identity in $\mathbb{F}$)

... scalars $c_1, c_2 \in \mathbb{F}$, $\alpha \in V$

6) For any scalars $c_1, c_2 \in \mathbb{F}$,

scalar multiplication

$$(c_1 c_2)\alpha = c_1(c_2\alpha)$$

field operation

scalar multiplication

7) $c(\alpha+\beta) = c\alpha + c\beta$, $\forall c \in \mathbb{F}$, $\alpha, \beta \in V$

8) For any c_1 $c_2 \in \mathbb{F}$, $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha$.

<u>Definition</u>: A subset $B \subset V$, V vector space over $\mathbb{F}$ is called basis of V if

i) B is linearly independent set

ii) B spans V.

> Dimension of a vector space # elements in basis

Linear Transformation

Let $T: V \longrightarrow W$ be a map, V, W are vector spaces over the same field $\mathbb{F}$. Then T is called linear transformation if

for any $\alpha, \beta \in V$

$$T(\alpha+\beta) = T\alpha + T\beta$$

& $c \in \mathbb{F}$, $\quad T(\underset{V}{c\alpha}) = c\, T\alpha$

Let $\dim V = n$,

Then, Nullity of $T = \dim(\text{Null space of } T)$

$$= \dim\{\alpha \in V : T\alpha = 0\} \subseteq V$$

Rank of $T = \dim(\text{range space of } T) = \dim\{T\alpha : \alpha \in V\} \subseteq W$

$$\Rightarrow \text{Rank}(T) + \text{Nullity}(T) = \dim V$$

* <u>Linear Operator</u> if $V = W$.

We know that if $\dim V = n$, then $T : V \to V$ has a matrix representation $(n \times n)$ relative to an ordered basis of V.

<u>Definition</u>: Let V be a vector space over a field F and $T: V \to V$ be a linear operator on V. Then ... lue / characteristic

$T: V \longrightarrow V$ … eigenvalue / characteristic
a scalar $c \in \mathbb{F}$ is called eigenvalue / characteristic root of T if $\exists$ a nonzero vector $\alpha \in V$ such that
$T(\alpha) = c\alpha$.
The corresponding vector α is called eigenvector associated with eigenvalue c.

Guess the eigen values of $T \equiv 0$ & $T = Id$ linear operators. For example, take $T: \mathbb{R}^2 \to \mathbb{R}^2$.
Let $c \in \mathbb{F}$ is an eigenvalue of $T \equiv 0$
$\exists$ a nonzero $(x, y) \in \mathbb{R}^2$ satisfying

$$(0,0) = 0(x,y) = T(x,y) = c(x,y)$$

$$\Rightarrow \quad cx = 0, \; cy = 0 \Rightarrow c = 0$$ if either of x & y is nonzero

Similarly $(x, y) = Id(x, y) = c(x, y)$

$$\Rightarrow x = cx, \quad y = cy$$

$$\Rightarrow c = 1$$

Note that if α is an eigenvector corresponding … linear transformation T

to an eigenvalue c of ...

Then any nonzero scalar multiple of α is an eigenvector of T corresponding to the eigenvalue c.