IC105: Probability and Statistics
Tutorial 2 (Solutions)

① (a) It is given that F is the distribution function of rv X. So, F is right continuous, hence $F(20) = F(20+)$.

$$\Rightarrow 16k^2 - 16k + 3 = 0 \quad \Rightarrow k = \frac{1}{4} \text{ or } k = \frac{3}{4} \longrightarrow ①$$

Also, F is non-decreasing. So,

$$F(5-) \leq F(5) \Rightarrow \frac{2}{3} \leq \frac{7-6k}{6}$$

$$\Rightarrow 6k \leq 3$$

$$\Rightarrow k \leq \frac{1}{2} \longrightarrow ②$$

(b) From ① and ② we get $k = \frac{1}{4}$.

$$\therefore F(x) = \begin{cases} 0, & x < 2, \\ \frac{2}{3}, & 2 \leq x < 5, \\ \frac{11}{12}, & 5 \leq x < 9, \\ \frac{91}{96}, & 9 \leq x < 14, \\ 1, & x \geq 14. \end{cases}$$

The set of discontinuity points of F is $D = \{2, 5, 9, 14\}$.

Also, $P(X=2) = F(2) - F(2-) = 2/3$, $P(X=5) = F(5) - F(5-) = 1/4$,

$P(X=9) = F(9) - F(9-) = \frac{1}{32}$, $P(X=14) = F(14) - F(14-) = 5/96$.

$$\therefore P(X \in D) = P(X=2) + P(X=5) + P(X=9) + P(X=14) = 1.$$

Therefore, X is a discrete r.v. with support $D = \{2, 5, 9, 14\}$.

(c) The p.m.f of X is given by

$$f_X(x) = \begin{cases} P(X=x), & x \in D \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 2/3, & x = 2, \\ 1/4, & x = 5, \\ 1/32, & x = 9, \\ 5/96, & x = 14, \\ 0, & \text{otherwise}. \end{cases}$$

② (a) As f_X is given to be p.m.f., we have

$$\sum_{x \in \{-3,-2,\ldots,3\} = S} f_X(x) = 1 \Rightarrow \sum_{x=-3}^{3} f_X(x) = 1 \Rightarrow \boxed{a = 28}.$$

(b) The p.m.f of $Z = X^2$ is ~~given by~~ obtained as follows:

x	-3	-2	-1	1	2	3
$f_X(x)$	$9/28$	$1/7$	$1/28$	$1/28$	$1/7$	$9/28$
f_Z						

z	1	4	9
$f_Z(z)$	$1/14$	$2/7$	$9/14$

Note that $h(S) = \{1, 4, 9\}$ where $h: \mathbb{R} \to \mathbb{R}$ is $h(x) = x^2$.

$h^{-1}(\{1\}) = \{-1, 1\}$, $h^{-1}(\{4\}) = \{-2, 2\}$, $h^{-1}(\{9\}) = \{-3, 3\}$.

$$\therefore f_Z(z) = \begin{cases} \displaystyle\sum_{x \in h^{-1}(\{z\})} f_X(x), & z \in h(S), \\ 0, & \text{otherwise}. \end{cases}$$

$$= \begin{cases} 1/14, & z = 1, \\ 2/7, & z = 4, \\ 9/14, & z = 9, \\ 0, & \text{otherwise}. \end{cases}$$

(3) As $Y = |X|$ its p.d.f can be obtained as

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Also, note that $Y \geq 0$.

(a) It is given that $f_X(x) = \begin{cases} \frac{1}{3}, & -2 < x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$

$$f_Y(y) = \begin{cases} 1/3 + 1/3 = 2/3, & 0 \leq y \leq 1, \\ 1/3, & 1 < y \leq 2 \\ 0, & \text{otherwise}. \end{cases}$$

(b) Here, we have $f_X(x) = \begin{cases} 2e^{-2x}, & x > 0, \\ 0, & \text{otherwise}. \end{cases}$

As, $X > 0$ there are no negative values of X that need to be considered. Thus,

$$f_Y(y) = f_X(y) = \begin{cases} 2e^{-2y}, & y > 0, \\ 0, & \text{otherwise}. \end{cases}$$

(c) For general $f_X(x)$, we have.

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y); & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

(4) Bernoulli trial: A random experiment with exactly two ~~trials~~ possible outcomes "success" and "failure".

$P(\text{success}) = p$, $P(\text{failure}) = 1-p$. $(0 < p \leq 1)$.

Let X be r.v. that denotes the number of Bernoulli trials (independents). needed to get one success.

$\therefore \quad S_X = \{1, 2, 3, \ldots\}$.

So,

$$f_X(x) = \begin{cases} P(X=x) & x \in S_X, \\ 0 & \text{, otherwise} \end{cases} = \begin{cases} (1-p)^{x-1} p & , x \in S_X, \\ 0 & , \text{otherwise}. \end{cases}$$

Expected no. of trials $= E(X) = \sum_{x=1}^{\infty} x f_X(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$

$= p + 2(1-p)p + 3(1-p)^2 p + \cdots$

$= \frac{1}{p}$.

Note that the probability of getting a new grade when you have already gotten i grades is $= \left\{ \frac{8-i}{8}, \quad i \in \{0, 1, 2, \ldots, 7\} \right.$

$= p_i$.

Expected no. of test $= \frac{1}{p_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_7}$.

$= 1 + \frac{8}{7} + \frac{8}{6} + \frac{8}{5} + \frac{8}{4} + \frac{8}{3} + \frac{8}{2} + 8$

$= 1 + 8 \times \frac{363}{140} = \frac{761}{35} \approx 22. \quad (21.7428)$

(5) (a) For each round, the probability that both Alice and Bob have a loss is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

Let X represent the total number of rounds played until the first time where they both have a loss.

As done ~~Proba~~ in the previous question, i.e., Q. 4 we get

$$f_X(x) = \begin{cases} P(X = x) & , x \in \{1, 2, \dots\} = S_X, \\ 0 & , \text{otherwise}. \end{cases}$$

$$= \begin{cases} (1-p)^{x-1} p & , x = 1, 2, \dots, \\ 0 & , \text{otherwise} \end{cases}$$

$$= \begin{cases} \left(\frac{8}{9}\right)^{x-1} \frac{1}{9} & , x = 1, 2, \dots, \\ 0 & , \text{otherwise}. \end{cases}$$

(b) Given Z is defined as the random variable that denotes the time at which Bob has his third loss. (

Let Y be the number of games played by Bob until his 3^{rd} loss.

$\therefore \; S_Y = \{3, 4, 5, \dots\}$. (support of S_Y).

$$f_Y(k) \overset{= P(Y=k)}{=} \begin{cases} \binom{k-1}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{k-1-2} \cdot \frac{1}{3} = \binom{k-1}{2} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{k-3} & , k = 3, 4, 5, \dots \\ 0 & , \text{otherwise}. \end{cases}$$

Note that ~~for~~ $Z \geq 24$ $\therefore$ $S_Z = \{6, 8, 10, \cdots\}$.

$$\therefore f_Z(z) = \begin{cases} \binom{\frac{z}{2}-1}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{\frac{z}{2}-3}, & z = 6, 8, 10, \cdots, \\ 0, & \text{otherwise} \end{cases}.$$

(c) Let A : event that Alice wins, B : event that Bob wins.

$A \cup B$: event that either Alice wins or Bob wins or both Alice and Bob win.

$A \cap B$: event that both Alice and Bob win.

U : be a random variable indicating the number of rounds we pee until at least one of them wins.

$$p = P(A \cup B) = 1 - P(A^c \cap B^c) = 1 - \frac{1}{3}\cdot\frac{1}{3} = \frac{8}{9}.$$

$$f_U(u) = \begin{cases} \left(\frac{8}{9}\right)\left(\frac{1}{9}\right)^{u-1}, & u = 1, 2, \cdots, \\ 0, & \text{o/w}. \end{cases} \quad \text{and } E(U) = \frac{9}{8}.$$

Let V : be a random variable representing the number of additional rounds we have to observe until the other wins.

(Note: 1. If both Alice and Bob win at the Uth round then $V = 0$).

and $P(A \cap B \mid A \cup B) = \dfrac{\frac{2}{3}\cdot\frac{2}{3}}{\frac{8}{9}} = \frac{1}{2}$.

2. If Alice wins the Uth round, then the time V_1 until Bob wins has the pmf:

$$f_{V_1}(v) = \begin{cases} \left(\frac{1}{3}\right)^{v-1}\left(\frac{2}{3}\right), & v = 1, 2, \cdots, \\ 0, & \text{o/w} \end{cases}. \quad E(V_1) = \frac{3}{2}$$

and $P(A \cap B^c \mid A \cup B) = \dfrac{\frac{1}{3}\cdot\frac{2}{3}}{\frac{8}{9}} = \frac{1}{4}$.

3. If Bob wins the Uth round, then the time V_2 until Alice wins has the pmf:

$$f_{W_2}(w) = \begin{cases} \left(\frac{1}{3}\right)^{w-1}\left(\frac{2}{3}\right) & , w = 1, 2, \ldots, \\ 0 & , \text{o/w}. \end{cases} \qquad E(W_2) = \frac{3}{2}$$

and $P(A^c \cap B \mid A \cup B) = \dfrac{\frac{1}{3}\cdot\frac{2}{3}}{\frac{8}{9}} = \frac{1}{4}$.

Let N be the number of rounds until Alice and Bob has won at least once is

$$N = U + V$$

So, $E(N) = E(U) + E(V)$

$$= \cancel{\frac{1}{\frac{8}{9}} + 0 \cdot P(A \cap B \mid A \cup B) + \frac{1}{2}}$$

$$= \frac{9}{8} + 0 \cdot P(A \cap B \mid A \cup B) + \frac{3}{2} P(A \cap B^c \mid A \cup B)$$

$$+ \frac{3}{2} P(A^c \cap B \mid A \cup B)$$

$$= \frac{9}{8} + \frac{3}{2}\cdot\frac{1}{4} + \frac{3}{2}\cdot\frac{1}{4} = \frac{15}{8}.$$

$\square$