Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Tutorial Sheet 5

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1. Find the matrix of the following inner products relative to given ordered basis $\mathcal{B}$

   (i) $V = P_2(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $\mathcal{B} = \{1, x, x^2\}$

   (ii) $V = \mathbb{C}^3$ with $< \alpha, \beta >= \sum_{j=1}^{3} \alpha_j \bar{\beta}_j$, with any ordered basis $\mathcal{B}$ of $\mathbb{C}^3$. (You can choose any ordered basis of your choice)

2. Apply Gram-Schmidt process to the following subsets S of inner product space V to get an orthonormal basis for span of S.

   (i) $V = \mathbb{R}^3$ with standard inner product, $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$

   (ii) $V = P_2(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $S = \{1, x, x^2\}$.

   (iii) $V = \mathbb{C}^4$, with standard inner product, $S = \{(1, i, 2 - i, -1), (2 + 3i, 3i, 1 - i, 2i), (-1 + 7i, 6 + 10i, 11 - 4i, 3 + 4i)\}$.

   (iv) $V = M_{2\times 2}(\mathbb{R})$ with standard inner product

$$S = \left\{ \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix}, \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \right\}$$

3. Prove the following

   (i) Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be an orthonormal basis for an inner product space V. Prove that for any $\alpha, \beta \in V$

$$< \alpha, \beta >= \sum_{i=1}^{n} < \alpha, \alpha_i > \overline{< \beta, \alpha_i >}$$

   (ii) Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be an orthonormal subset of an inner product space V. Prove that for any $\alpha \in V$

$$\|\alpha\|^2 \geq \sum_{i=1}^{n} | < \alpha, \alpha_i > |^2$$

4. Compute $S^{\perp}$ for $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$.

5. Find the orthogonal projections of the following vectors on the given subspace of the specified inner product space

   (i) $V = \mathbb{R}^3$ with standard inner product, $\alpha = (2, 1, 3)$, and $W = \{(x, y, z) : x + 3y - 2z = 0\}$.

(ii) $V = P(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $h(x) = 4 + 3x - 2x^2$, $W = P_1(\mathbb{R})$.

6. Let V be an inner product space, S and S_0 are the subsets of V and W is a finite dumensional subspace of V. Prove the following results

   (i) $S_0 \subseteq S$ implies that $S^{\perp} \subseteq S_0^{\perp}$.

   (ii) $S \subseteq (S^{\perp})^{\perp}$ implies span $(S) \subseteq (S^{\perp})^{\perp}$

   (iii) $W = (W^{\perp})^{\perp}$

   (iv) $V = W \oplus W^{\perp}$

7. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^{\perp}$, but $< x, y > \neq 0$.

8. Let $V = C([-1, 1]; \mathbb{R})$ be an inner product space with inner product defined as

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx.$$

   Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Then prove that $W_e^{\perp} = W_0$.