IC105: Probability and Statistics **2021-22-M**

Lecture 9: Moment Generating Function

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Theorem 9.1. *Let X be a continuous (discrete) r.v. Then*

$$E(X) = \int_0^\infty P(X > y)\mathrm{d}y - \int_{-\infty}^0 P(X < y)\mathrm{d}y,$$

provided $E(X)$ is finite.

Proof. We will provide the proof for the case when X is a continuous r.v. with p.d.f., say f. We have

$$\begin{aligned}
E(X) &= \int_{-\infty}^\infty x f(x)\mathrm{d}x \\
&= \int_{-\infty}^0 x f(x)\mathrm{d}x + \int_0^\infty x f(x)\mathrm{d}x \\
&= -\int_{-\infty}^0 \int_x^0 f(x)\mathrm{d}y\mathrm{d}x + \int_0^\infty \int_0^x f(x)\mathrm{d}y\mathrm{d}x \\
&= -\int_{-\infty}^0 \int_{-\infty}^y f(x)\mathrm{d}x\mathrm{d}y + \int_0^\infty \int_y^\infty f(x)\mathrm{d}x\mathrm{d}y = -\int_{-\infty}^0 P(X < y)\mathrm{d}y + \int_0^\infty P(X > y)\mathrm{d}y.
\end{aligned}$$

This completes the proof. □

Corollary 9.2. *(a) Suppose that X is a continuous (discrete) r.v. with $P(X \geq 0) = 1$. Then*

$$E(X) = \int_0^\infty P(X > y)\mathrm{d}y.$$

(b) Suppose that $P(X \in \{0, \pm 1, \pm 2, \dots\}) = 1$. Then

$$E(X) = \sum_{n=1}^\infty P(X \geq n) - \sum_{n=1}^\infty P(X \leq -n).$$

(c) Suppose that $P(X \in \{0, 1, 2, \dots\}) = 1$. Then $E(X) = \sum_{n=1}^\infty P(X \geq n)$.

Proof. Exercise. □

The following theorem suggests that for any r.v. X and any function $h : \mathbb{R} \to \mathbb{R}$, $E(h(X))$ can be directly found using p.m.f. / p.d.f. of X.

Theorem 9.3. *(a) Let X be a discrete r.v. with p.m.f $f(\cdot)$ and support S. Let $h : \mathbb{R} \to \mathbb{R}$ be a given function and let $Z = h(X)$. Then*

$$E(Z) = \sum_{x \in S} h(x) f(x) \text{ provided } \sum_{x \in S} |h(x)| f(x) < \infty.$$

(b) Let X be a continuous r.v. with p.d.f $f(\cdot)$ and let $h : \mathbb{R} \to \mathbb{R}$ be a given function. If $Z = h(X)$, then

$$E(Z) = \int_{-\infty}^{\infty} h(x) f(x) \mathrm{d}x, \text{ provided } \int_{-\infty}^{\infty} |h(x)| f(x) \mathrm{d}x < \infty.$$

Proof. We will provide the proof of (a) only. The proof of (b) follows on similar lines. The support of $Z = h(X)$ is $T = h(S)$. We have

$$\begin{aligned}
E(Z) = \sum_{t \in T} tP(Z = t) &= \sum_{t \in T} tP(h(X) = t) \\
&= \sum_{t \in T} t \sum_{\{x \in S : h(x) = t\}} P(X = x) \\
&= \sum_{\{x \in S : h(x) = t\}} \sum_{t \in T} tP(X = x) \\
&= \sum_{\{x \in S : h(x) = t\}} \sum_{t \in T} h(x) P(X = x) \\
&= \sum_{t \in T} \sum_{\{x \in S : h(x) = t\}} h(x) P(X = x) \\
&= \sum_{\bigcup_{t \in T} \{x \in S : h(x) = t\}} h(x) P(X = x) = \sum_{x \in S} h(x) P(X = x).
\end{aligned}$$

This completes the proof. □

Example 9.4. *(a) Let the r.v. X have the p.m.f.*

$$f(x) = \begin{cases} 1/6, & \text{if } x = -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise}. \end{cases}$$

Find $E(X^2)$.

(b) Let the r.v. X have the p.d.f.

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Find $E(X^3)$.

Solution: (a) $E(X^2) = \sum_{x \in S} x^2 f(x) = 4 \times \frac{1}{6} + 1 \times \frac{1}{6} + 0 \times \frac{1}{6} + 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} = \frac{19}{6}$.

(b) $E(X^3) = \int_{-\infty}^{\infty} x^3 f(x) \mathrm{d}x = 2 \int_0^1 x^4 \mathrm{d}x = \frac{2}{5}$.

Theorem 9.5. *Let X be a discrete or continuous r.v. with p.m.f./ p.d.f. f and support S. Let $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$ be given functions.*

(a) Then, for real constants $c_1, c_2, \ldots, c_m$

$$E\left(\sum_{i=1}^{m} c_i h_i(X)\right) = \sum_{i=1}^{m} c_i E(h_i(X)),$$

provided involved expectations are finite.

(b) Let $h_1(x) \leq h_2(x),\ \forall\, x \in S$. Then,

$$E(h_1(X)) \leq E(h_2(X)),\ \textit{ provided involved expectations are finite}.$$

In particular, if $E(X)$ is finite and $P(a \leq X \leq b) = 1$, for some real constants a and b ($a < b$) then $a \leq E(X) \leq b$.

(c) If $P(X \geq 0) = 1$ and $E(X) = 0$, then $P(X = 0) = 1$.

(d) If $E(X)$ is finite then $|E(X)| \leq E(|X|)$.

(e) Let a and b be two real constants. Then,

$$E(aX + b) = aE(X) + b,\ \textit{ provided involved expectations are finite}.$$

Proof. The proofs of (a), (b) and (e) follows from the definition of expectation of a r.v.

(c) We will provide the proof for the case when X is a continuous r.v. Then

$$\begin{aligned}
P(X > 0) &= P\left(\bigcup_{n=1}^{\infty} \left\{X \geq \frac{1}{n}\right\}\right) \\
&= \lim_{n\to\infty} P\left(X \geq \frac{1}{n}\right), \quad \left(\left\{X \geq \frac{1}{n}\right\} \uparrow\right) \\
&= \lim_{n\to\infty} \int_{1/n}^{\infty} f(x)\mathrm{d}x \\
&\leq \lim_{n\to\infty} \int_{1/n}^{\infty} nxf(x)\mathrm{d}x, \quad (x \in [1/n, \infty) \implies nx \geq 1) \\
&\leq \lim_{n\to\infty} \left[n \int_{0}^{\infty} xf(x)\mathrm{d}x\right] \\
&= \lim_{n\to\infty} [nE(X)] = 0 \implies P(X = 0) = 1.
\end{aligned}$$

(d) We have

$$-|X| \leq X \leq |X| \implies E(-|X|) \leq E(X) \leq E(|X|) \implies |E(X)| \leq E(|X|).$$

This completes the proof. □

Some Special Expectations:

(i) $h(x) = x,\ E(X) = \mu_1'$= mean of X.

(ii) $h(x) = x^r,\ r = \{1, 2, \ldots\},\ E(X^r) = \mu_r'$= rth moment of X about origin.

(iii) $h(x) = |x|^r,\ r = \{1, 2, \ldots\},\ E(|X|^r) = r$th absolute moment of X about origin.

(iv) $h(x) = (x - \mu_1')^r,\ r = \{1, 2, \ldots\}$, $E(X - \mu_1')^r = \mu_r$= rth moment of X about its mean or rth central moment.

(v) $\mu_2 = E(X - \mu_1')^2 = \sigma^2$= variance of X. We also denote it by $\mathrm{Var}(X)$. And, $\sqrt{\mu_2} = \sqrt{E(X - \mu_1')^2} = \sigma$ is called the standard deviation of X (positive square root of the variance of r.v. X).

Remark 9.6. *(i)* $\mathrm{Var}(X) = \sigma^2 = E(X - \mu_1')^2 = E(X^2 - 2\mu_1' X + (\mu_1')^2) = E(X^2) - 2(\mu_1')^2 + (\mu_1')^2 = E(X^2) - (E(X))^2$.

(ii) Since $(X - \mu_1')^2 \geq 0$, *we have*

$$\mathrm{Var}(X) = E(X - \mu_1')^2 \geq 0 \implies E(X^2) \geq (E(X))^2.$$

(iii) $\mathrm{Var}(X) = 0 \iff E(X - \mu_1')^2 = 0 \iff P(X = E(X)) = 1$.

Theorem 9.7. *Let X be a r.v. such that $E(|X|^s) < \infty$, for some $s > 0$. Then, $E(|X|^r) < \infty$, $\forall\, 0 < r < s$.*

Proof. Note that $|X|^r \leq \max\{|X|^s, 1\} \leq |X|^s + 1$. This implies that $E(|X|^r) \leq E(|X|^s + 1) = E(|X|^s) + 1 < \infty$. Thus, the result follows. □

9.1. Moment Generating Function

Let X be a r.v with d.f. F and p.d.f. / p.m.f. $f(\cdot)$.

Definition 9.8. *We say that the moment generating function (m.g.f.) of X (denoted by $M_X(\cdot)$) exists and equals*

$$M_X(t) = E(e^{tX}), \text{ provided } E(e^{tX}) \text{ is finite in } (-h, h) \text{ for some } h > 0.$$

Remark 9.9. *(i)* $M_X(0) = 1$, *thus* $A = \{t \in \mathbb{R} : E(e^{tX}) \text{ is finite}\} \neq \phi$.

(ii) $M_X(t) > 0$, $\forall\, t \in A = \{s \in \mathbb{R} : E(e^{sX}) \text{ is finite}\}$.

(iii) Suppose that $M_X(t)$ exists and is finite on $(-h, h)$ for some $h > 0$. For real constants c and d, let $Y = cX + d$. Then, the m.g.f. of Y also exists and is finite on $\left(-\frac{h}{|c|}, \frac{h}{|c|}\right)$ *(with the convention that $\pm\frac{a}{0} = \pm\infty$, if $a > 0$). Moreover,*

$$M_Y(t) = E(e^{t(cX+d)}) = e^{td} M_X(ct), \quad t \in \left(-\frac{h}{|c|}, \frac{h}{|c|}\right).$$

(iv) The name m.g.f. to the transform M_X is motivated by the fact that M_X can be used to generate moments of any r.v., as illustrated in the following theorem.

Theorem 9.10. *Let X be a r.v. with m.g.f. M_X that is finite on $(-h, h)$, $h > 0$. Then,*

(a) For each $r \in \{1, 2, \dots\}$, $\mu_r' = E(X^r)$ is finite;

(b) For each $r \in \{1, 2, \dots\}$, $\mu_r' = E(X^r) = M_X^{(r)}(0)$, where

$$M_X^{(r)}(0) = \left[\frac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(t)\right]_{t=0}, \text{ the } r\text{th derivative of } M_X \text{ at the point } 0;$$

(c) $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$, $t \in (-h, h)$, *so that μ_r' is equal to coefficient of $\frac{t^r}{r!}$ $(r = 1, 2, \dots)$ in the Maclaurin's series expansion of $M_X(t)$ around $t = 0$.*

Proof. (a) We have

$$
\begin{aligned}
& E(e^{tX}) < \infty,\ \ \forall\, t \in (-h,h) \\
\Longrightarrow\ & \int_{-\infty}^{0} e^{tx} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \ \text{ and } \int_{0}^{\infty} e^{tx} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \\
\Longrightarrow\ & \int_{-\infty}^{0} e^{-t|x|} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \ \text{ and } \int_{0}^{\infty} e^{t|x|} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \\
\Longrightarrow\ & \int_{-\infty}^{0} e^{|t||x|} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \ \text{ and } \int_{0}^{\infty} e^{|t||x|} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h) \\
\Longrightarrow\ & \int_{-\infty}^{\infty} e^{|tx|} f(x)\mathrm{d}x < \infty,\ \ \forall\, t \in (-h,h);
\end{aligned}
$$

here $f(\cdot)$ denotes the p.d.f. of r.v. X.

Fix $r \in \{1,2,\dots\}$ and $t \in (-h,h) - \{0\}$. Then, $\lim_{|x|\to\infty} \frac{|x|^r}{e^{|tx|}} = 0$ and therefore $\exists$ a positive real number $A_{r,t}$ such that $|x|^r < e^{|tx|}$, $\forall\, |x| > A_{r,t}$. Therefore

$$
\begin{aligned}
E(|X|^r) &= \int_{-\infty}^{\infty} |x|^r f(x)\mathrm{d}x \\
&= \int_{|x|\le A_{r,t}} |x|^r f(x)\mathrm{d}x + \int_{|x|>A_{r,t}} |x|^r f(x)\mathrm{d}x \\
&\le A_{r,t}^r \int_{|x|\le A_{r,t}} f(x)\mathrm{d}x + \int_{|x|>A_{r,t}} e^{|tx|} f(x)\mathrm{d}x \\
&\le A_{r,t}^r + \int_{-\infty}^{\infty} e^{|tx|} f(x)\mathrm{d}x < \infty, \quad r = 1,2,\dots.
\end{aligned}
$$

(b) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\mathrm{d}x$, $M_X^{(r)}(t) = \frac{\mathrm{d}^r}{\mathrm{d}t^r} \int_{-\infty}^{\infty} e^{tx} f(x)\mathrm{d}x$, $r = 1,2,\dots$.

Using the arguments of advanced calculus it can be shown that of $M_X(t) = E(e^{tX}) < \infty$, $\forall\, t \in (-h,h)$, then the derivative can be passed through the integral sign. Therefore,

$$
M_X^{(r)}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^r}{\mathrm{d}t^r} \left(e^{tx} f(x)\right) \mathrm{d}x = \int_{-\infty}^{\infty} x^r e^{tx} f(x)\mathrm{d}x, \ \ r = 1,2,\dots
$$

and

$$
M_X^{(r)}(0) = \int_{-\infty}^{\infty} x^r f(x)\mathrm{d}x = E(X^r).
$$

(c) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \left(\sum_{r=0}^{\infty} \frac{t^r x^r}{r!} \right) f(x)\mathrm{d}x.$

Under the assumption that $M_X(t) = E(e^{tX}) < \infty$, $\forall\, t \in (-h,h)$, using arguments of advanced calculus, it can be shown that the summation sign can be passed through the integral sign. Thus,

$$
M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x)\mathrm{d}x = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r), \ \ r = 1,2,\dots.
$$

This completes the proof. □

Corollary 9.11. *Under the notation and assumption of the above theorem define* $\psi_X(t) = \ln(M_X(t)),\ \ t \in (-h, h)$. *Then,*

$$\mu_1' = \mu = E(X) = \psi_X^{(1)}(0) \text{ and } \mu_2 = \sigma^2 = \mathrm{Var}(X) = \psi_X^{(2)}(0).$$

Proof. For $t \in (-h, h)$

$$\psi_X^{(1)}(t) = \frac{M_X^{(1)}(t)}{M_X(t)} \implies \psi_X^{(1)}(0) = M_X^{(1)}(0) = E(X) \quad (\text{since } \ M_X(0) = 1).$$

Also,

$$\psi_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - (M_X^{(1)}(t))^2}{(M_X(t))^2} \implies \psi_X^{(2)}(0) = M_X^{(2)}(0) - (M_X^{(1)}(0))^2 = E(X^2) - (E(X))^2 = \mathrm{Var}(X).$$

This completes the proof. □

Example 9.12. *(a) Let X be a discrete r.v. with p.m.f.*

$$f_X(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\lambda > 0$. Show that the m.g.f. of X exists and is finite on whole $\mathbb{R}$. Find $M_X(t)$, mean, variance of X and $E(X^3)$.

(b) Let X be a continuous r.v. with p.d.f.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\lambda > 0$. Find m.g.f., mean, variance of X and $E(X^r)$, $r = 1, 2, \ldots$ (provided they exist).

(c) Let X be a continuous r.v. having the p.d.f. $f(x) = \dfrac{1}{\pi(1 + x^2)}$, $-\infty < x < \infty$ (called Cauchy p.d.f. and corresponding probability distribution is called Cauchy distribution). Show that the m.g.f. of X does not exist.

Solution: (a) We have

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t - 1)},\ \forall\, t \in \mathbb{R}.$$

Thus, m.g.f. of X exists and finite on whole of $\mathbb{R}$ and $M_X(t) = e^{\lambda(e^t-1)},\ t\ \in \mathbb{R}$.

Now $\psi_X(t) = \ln(M_X(t)) = \lambda(e^t - 1) \implies \psi_X^{(1)}(t) = \lambda e^t = \psi_X^{(2)}(t),\ \forall\, t \in \mathbb{R}$.

Thus, $E(X) = \psi_X^{(1)}(0) = \lambda$ and $\mathrm{Var}(X) = \psi_X^{(2)}(0) = \lambda$. Again,

$$\begin{aligned} &M_X^{(1)}(t) = \lambda e^t e^{\lambda(e^t-1)} = \lambda e^t M_X(t) \implies M_X^{(1)}(0) = E(X) = \lambda, \\ &M_X^{(2)}(t) = \lambda e^t M_X^{(1)}(t) + \lambda e^t M_X(t) \implies M_X^{(2)}(0) = E(X^2) = \lambda^2 + \lambda, \\ &M_X^{(3)}(t) = \lambda e^t M_X^{(2)}(t) + 2\lambda e^t M_X^{(1)}(t) + \lambda e^t M_X(t) \implies M_X^{(3)}(0) = E(X^3) = \lambda^3 + 3\lambda^2 + \lambda. \end{aligned}$$

Alternatively, for $t \in \mathbb{R}$,

$$\begin{aligned} M_X(t) &= e^{\lambda(e^t-1)} \\ &= 1 + \lambda(e^t - 1) + \frac{\lambda^2(e^t-1)^2}{2!} + \frac{\lambda^3(e^t-1)^3}{3!} + \cdots \\ &= 1 + \lambda\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\right) + \frac{\lambda^2}{2!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\right)^2 + \frac{\lambda^3}{3!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}\right)^3 + \cdots \\ &= 1 + \lambda t + t^2\left(\frac{\lambda}{2!} + \frac{\lambda^2}{2!}\right) + t^3\left(\frac{\lambda}{3!} + \frac{2\lambda^2}{(2!)^2} + \frac{\lambda^3}{3!}\right) + \cdots \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= \text{coefficient of } t \text{ in the expansion of } M_X(t) = \lambda, \\ E(X^2) &= \text{coefficient of } \frac{t^2}{2!} \text{ in the expansion of } M_X(t) = \lambda^2 + \lambda, \\ E(X^3) &= \text{coefficient of } \frac{t^3}{3!} \text{ in the expansion of } M_X(t) = \lambda^3 + 3\lambda^2 + \lambda. \end{aligned}$$

(b) $\int_{-\infty}^{\infty} e^{tx} f_X(x)\mathrm{d}x = \lambda \int_{-\infty}^{\infty} e^{-\lambda(1-t/\lambda)x}\mathrm{d}x < \infty$, if $t < \lambda$. Thus the m.g.f. of X exists and, for $t < \lambda$,

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots + \frac{t^r}{\lambda^r} + \cdots.$$

For $r = 1, 2, \ldots$

$$\mu_r' = E(X^r) = \text{coefficient of } \frac{t^r}{r!} \text{ in the expansion of } M_X(t) = \frac{r!}{\lambda^r}, \ \ r \in \{1, 2, \ldots\}.$$

Alternatively,

$$M_X^{(1)}(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2}, \ M_X^{(2)}(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3} \ \text{ and } \ M_X^{(r)}(t) = \frac{r!}{\lambda^r}\left(1 - \frac{t}{\lambda}\right)^{-(r+1)}, \ \ t < \lambda.$$

This implies

$$E(X^r) = M_X^{(r)}(0) = \frac{r!}{\lambda^r}, \ \ r = 1, 2, \ldots \text{and } \ \mathrm{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

(c) Since $E(X)$ is not finite, the m.g.f. of X does not exist.

Definition 9.13 (Equality in Distribution)**.** *Let X and Y be two r.v.'s with d.f.'s F_X and F_Y, respectively. We say that X and Y have the same distribution (written as $X \stackrel{d}{=} Y$) if $F_X(x) = F_Y(x), \ \forall\, x \in \mathbb{R}$.*

Remark 9.14. *(i) Let X and Y be two discrete r.v.'s with p.m.f.'s f_X and f_Y, respectively. Then,*

$$X \stackrel{d}{=} Y \iff f_X(x) = f_Y(x), \ \forall\, x \in \mathbb{R}.$$

(ii) Let X and Y be two continuous r.v.'s. Then, $X \stackrel{d}{=} Y$ iff there exist versions of p.d.f.'s f_X and f_Y of X and Y, respectively, such that $f_X(x) = f_Y(x), \ \forall\, x \in \mathbb{R}$.

(iii) Suppose $X \stackrel{d}{=} Y$, then for any Borel measurable function $h : \mathbb{R} \to \mathbb{R}, \ h(X) \stackrel{d}{=} h(Y)$ and hence $E(h(X)) = E(h(Y))$.

Theorem 9.15. *Let X and Y be r.v.'s such that for some $c > 0$, $M_X(t) = M_Y(t)$, $\forall\, t \in (-c, c)$. Then, $X \stackrel{d}{=} Y$.*

Proof. Special Case: Suppose that X and Y are discrete r.v.'s with support $S_X = S_Y = \{1, 2, \dots\}$, $p_k = P(X = k)$ and $q_k = P(Y = k)$, $k = 1, 2, \dots$. Then

$$\begin{aligned} &M_X(t) = M_Y(t),\ \forall\, t \in (-c, c), \text{for some } c > 0 \\ \implies &\sum_{k=1}^{\infty} e^{kt} p_k = \sum_{k=1}^{\infty} e^{kt} q_k,\ \ \forall\, t \in (-c, c) \\ \implies &\sum_{k=1}^{\infty} \Lambda^k p_k = \sum_{k=1}^{\infty} \Lambda^k q_k,\ \ \forall\, \Lambda \in (e^{-c}, e^{c}) \\ \implies &p_k = q_k,\ \ \forall\, k = 1, 2, \dots, \end{aligned}$$

since if two power series are equal over an interval then their coefficients are the same. Thus, $X \stackrel{d}{=} Y$. □

Example 9.16. *For any $p \in (0, 1)$ and positive integer n, let $X_{p,n}$ be a discrete r.v. with p.m.f.*

$$f_{p,n}(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{if } x = \{0, 1, \dots, n\}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, $p \in (0, 1)$ and $n \in \mathbb{N}$. (Such a r.v. or probability distribution is called binomial r.v. or distribution with n trails and probability of success p). Define $Y_{p,n} = n - X_{p,n}$. Using the m.g.f. of $X_{p,n}$, show that $Y_{p,n} \stackrel{d}{=} X_{1-p,n}$. Find $E(X_{1/2,n})$.

Solution: We have

$$M_{X_{p,n}}(t) = E\left(e^{tX_{p,n}}\right) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} = (1 - p + pe^t)^n,\ \ t \in \mathbb{R}.$$

Now

$$\begin{aligned} M_{Y_{p,n}}(t) = E\left(e^{tY_{p,n}}\right) &= E\left(e^{t(n - X_{p,n})}\right) \\ &= e^{nt} M_{X_{p,n}}(-t) = e^{nt}(1 - p + pe^{-t})^n \\ &= (p + (1-p)e^t)^n = (1 - (1-p) + (1-p)e^t)^n = M_{X_{1-p,n}}(t),\ \ \forall\, t \in \mathbb{R}. \end{aligned}$$

Thus, $Y_{p,n} \stackrel{d}{=} X_{1-p,n}$.

Alternatively,

$$\begin{aligned} f_{Y_{p,n}}(y) &= P(Y_{p,n} = y) \\ &= P(X_{p,n} = n - y) \\ &= \begin{cases} \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}, & \text{if } n - y = \{0, 1, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \binom{n}{y} (1-p)^y (1 - (1-p))^{n-y}, & \text{if } y = \{0, 1, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases} \\ &= f_{X_{1-p,n}}(y),\ \ \forall\, y \in \mathbb{R}. \end{aligned}$$

Thus, $Y_{p,n} \stackrel{d}{=} X_{1-p,n}$.

Now for $p = 1/2$, $X_{1/2,n} \stackrel{d}{=} n - X_{1/2,n}$. Thus, $E(X_{1/2,n}) = E(n - X_{1/2,n}) \implies E(X_{1/2,n}) = n/2$.

Example 9.17. *Let X be a r.v. with p.d.f. $f_X(x) = \dfrac{e^{-|x|}}{2}$, $-\infty < x < \infty$ and let $Y = -X$. Show that $Y \stackrel{d}{=} X$ and hence show that $E(X) = 0$.*

Solution: We have

$$M_Y(t) = E(e^{tY}) = E(e^{-tX}) = \int_{-\infty}^{\infty} e^{-tx} \frac{e^{-|x|}}{2} \mathrm{d}x = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2} \mathrm{d}x = M_X(t), \;\; \forall\, t \in (-1, 1).$$

$$\Bigg[M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2} \mathrm{d}x = \int_{-\infty}^{0} e^{tx} \frac{e^{x}}{2} \mathrm{d}x + \int_{0}^{\infty} e^{tx} \frac{e^{-x}}{2} \mathrm{d}x$$
$$= \frac{1}{2} \left(\int_{0}^{\infty} e^{-(1+t)x} \mathrm{d}x + \int_{0}^{\infty} e^{-(1-t)x} \mathrm{d}x \right)$$
$$= \frac{1}{2} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) = \frac{1}{1-t^2}, \;\; \forall\, t \in (-1, 1) \implies X \stackrel{d}{=} Y. \Bigg]$$

Alternatively, the p.d.f. of Y is

$$f_Y(y) = \frac{e^{-|y|/2}}{2} = f_X(y), \;\; \forall - \infty < y < \infty \implies X \stackrel{d}{=} Y.$$

Thus, $E(Y) = E(X) \implies E(-X) = E(X) \implies E(X) = 0$ (since $\displaystyle\int_{-\infty}^{\infty} |x| f_X(x) \mathrm{d}x < \infty$).