Definition (Field). Let $\mathbb{F}$ be a nonempty set and let "+" and "$\cdot$" (called addition and multiplication, respectively) be two binary operations. The set $\mathbb{F}$ is called a field if it satisfies the following axioms:

(A1) Addition (+) is associative:

$$(a+b)+c = a+(b+c) \quad \forall\ a, b, c \in \mathbb{F}.$$

(A2) Existence of additive identity: There exists an identity element with respect to addition, denoted by 0 such that

$$a+0 = 0+a = a \quad \forall\ a \in \mathbb{F}.$$

(A3) Existence of additive inverse: For each element $a \in \mathbb{F}$, there exists an element $b \in \mathbb{F}$ such that

$$a+b = 0 = b+a.$$

This element "b" is denoted by "$-a$".

(A4) Addition is commutative:

$$a+b = b+a \quad \forall\ a, b \in \mathbb{F}.$$

(M1) Multiplication is associative:

$$(a\cdot b)\cdot c = a\cdot(b\cdot c) \quad \forall\ a, b, c \in \mathbb{F}.$$

(M2) Existence of multiplicative identity: There exist a multiplicative identity $e \in \mathbb{F}$ such that

$$a\cdot e = a = e\cdot a \quad \forall\ a \in \mathbb{F}.$$

(M3) Existence of multiplicative inverse: For every element $a \in \mathbb{F} \setminus \{0\}$, there exists an element $b \in \mathbb{F} \setminus \{0\}$ such that

$$a \cdot b = e = b \cdot a.$$

This "b" is denoted by "a^{-1}".

(M4) Multiplication is commutative:

$$a \cdot b = b \cdot a \quad \forall\, a, b \in \mathbb{F}.$$

(D) Distributive property:

$$(a+b)\cdot c = a\cdot c + b\cdot c \quad \forall\, a, b, c \in \mathbb{F}$$

and $$a\cdot(b+c) = a\cdot b + a\cdot c \quad \forall\, a, b, c \in \mathbb{F}.$$

<u>Examples</u> (a) The set $\mathbb{Q}$ of rational numbers, the set $\mathbb{R}$ of real numbers, the set $\mathbb{C}$ of complex numbers with respect to usual addition and multiplication are fields.

(b) The set $\mathbb{Z}$ of integers is not a field (why?).

(c) The set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field, where the binary operations "+" and "$\cdot$" is defined as follows:

$$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}.$$

$$(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2}.$$

Remark: Most of the times, we simply write ab in place of $a \cdot b$.

Theorem. If $\mathbb{F}$ is a field, then $a \cdot 0 = 0 \ \forall\, a \in \mathbb{F}$.

Proof. Let $a \cdot 0 = b$. Then

$$b = a \cdot 0 = a(0+0) \qquad [\text{since } 0+0=0]$$

$$= a \cdot 0 + a \cdot 0 \qquad [\text{Distributive property}]$$

$$= b + b.$$

Thus

$$b = b + b,$$

and so

$$0 = b + (-b) = (b+b) + (-b)$$

$$= b + [b + (-b)] \qquad [\text{associative property}]$$

$$= b + 0$$

$$= b.$$

Thus $b = 0$, that is

$$a \cdot 0 = 0.$$

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