IC153: Calculus 1
(Lecture 13)

by

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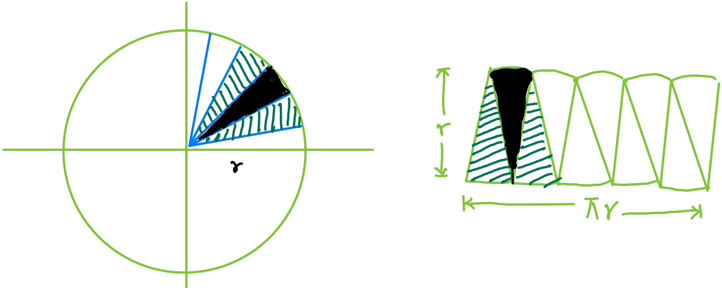
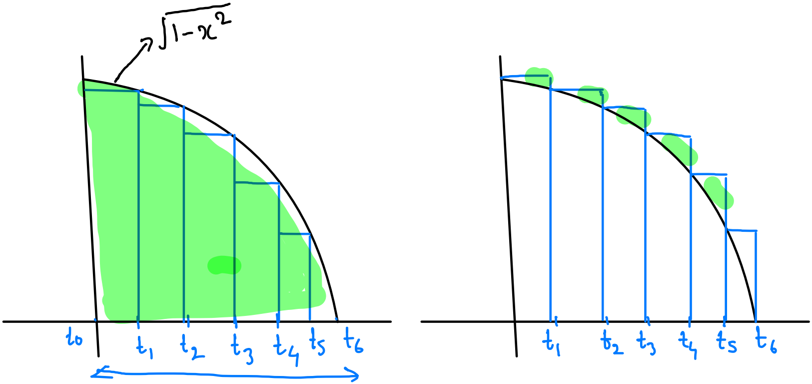
Riemann Integration: Motivation



Area of circle $\longrightarrow$ area of parallelogram

as number of sectors increase



A: area of the region R bounded by $x=0$, $y=0$ & $y=\sqrt{1-x^2}$

s: sum of areas of rectangles inside R.

S: sum of areas of rectangles covering region R.

$$\Rightarrow \qquad s \leq A \leq S$$

Lower sum and upper sum

Partition: Let $a < b$. A partition of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a and one of which is b.

written as $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

\# Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function and $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$.

let
$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \}$$
$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i \} .$$

The $\boxed{\text{Lower sum}}$ of f for P, denoted $L(P,f)$, is defined as

$$L(P,f) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$$

The $\boxed{\text{upper sum}}$ of f for P, denoted $U(P,f)$, is defined as

$$U(P,f) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$$

Observations: ① For any partition P of $[a,b]$,

$$L(P,f) \leq U(P,f)$$

②. If P_1 & P_2 are any two partitions of $[a,b]$, then

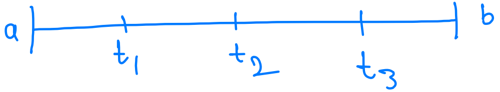
$$L(P_1,f) \leq U(P_2,f)$$

Refinement of partition

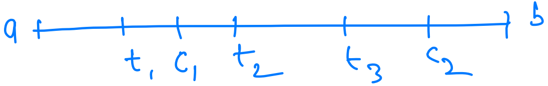
Definition: A partition P_2 of $[a,b]$ is said to be finer than a partition P_1 if $P_1 \subset P_2$.

If P_2 is finer than P_1 then we say P_2 is a refinement of P_1.

$P_1 =$



$P_2 =$



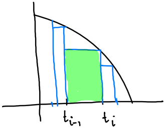
Let $a < c < d$ then $\inf\{f(x) : x \in [a,d]\} \le \inf\{f(x) : x \in [a,c]\}$

Theorem

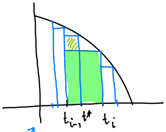
Let P_2 be a refinement of P_1 then $L(P_1, f) \le L(P_2, f)$ and $U(P_2, f) \le U(P_1, f)$.

Proof: let $P_1 : \quad a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$

$P_2 : \quad a = t_0 < t_1 < t_2 < \dots < t_{i-1} < t^* < t_i < \dots < t_n = b$



P_1



P_2

$$L(P_1, f) = \sum_{k=1}^{n} m_i (t_k - t_{k-1})$$

$$L(P_2, f) = \sum_{k=1}^{i-1} m_i (t_k - t_{k-1}) +$$

$\inf\{f(x) : x \in [t_{i-1}, t^*]\}$
$\quad ''$

$$m_1^* (t^* - t_{i-1}) + m_2^* (t_i - t^*) + \sum_{k=i}^{n} m_i (t_k - t_{k-1})$$

$$L(P_2, f) - L(P_1, f) = m_1^* (t^* - t_{i-1}) + m_2^* (t_i - t^*) - m_i (t_i - t_{i-1})$$

$$\ge m_i (t^* - t_{i-1}) + m_i (t_i - t^*) - m_i (t_i - t_{i-1}) \ge 0$$

Theorem

$\inf_P U(P, f) \geq \sup_P L(P, f)$.

Notation: $\sup_P L(P,f) = \underline{\int_a^b} f\,dx$

$\inf_P U(P,f) = \overline{\int_a^b} f\,dx$

Proof: Let P_1 & P_2 partitions.

take $P = P_1 \cup P_2$. Clearly $P_1 \cup P_2$ is finer than P_1 & P_2

$$L(P_1, f) \leq L(P, f) \leq U(P,f) \leq U(P_2, f)$$

$$\Rightarrow \quad L(P_1, f) \leq U(P_2, f)$$

$$\Rightarrow \quad \sup_{P_1} L(P_1, f) \leq U(P_2, f) \Rightarrow \underline{\int_a^b} f\,dx \leq U(P_2, f)$$

$$\Rightarrow \quad \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$$

Riemann Integral

Definition: A bounded $f : [a,b] \longrightarrow \mathbb{R}$ is called integrable if

$$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$$

Example: (1). $f : [a,b] \longrightarrow \mathbb{R}$

$f(x) = c$ constant function.

let $P : a < t_1 < \ldots < t_{n-1} < t_n = b$ be a partition

$$m_i = \inf_{x \in [t_{i-1}, t_i]} f(x) = c$$

$$M_i = \sup_{x \in [t_{i-1}, t_i]} f(x) = c$$

$$L(P, f) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) = \sum_{i=1}^{n} c\,(t_i - t_{i-1})$$

$$= c\,(b-a)$$

$$U(P, f) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}) = c\,(b-a)$$

$$\Rightarrow \quad \underline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = c\,(b-a)$$

$\Rightarrow$ f is integrable &

$$\int_a^b f\,dx = c\,(b-a)$$

Example ②: $f(x) = x$.

$P: a = t_0 < t_1 < \cdots < t_n = b$

$$m_i = \inf_{x \in [t_{i-1}, t_i]} f(x) = t_{i-1}$$

$$M_i = \sup_{x \in [t_{i-1}, t_i]} f(x) = t_i$$

$$L(P, f) = \sum_{i=1}^{n} t_{i-1} (t_i - t_{i-1})$$

$$U(P, f) = \sum_{i=1}^{n} t_i (t_i - t_{i-1})$$

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (t_i - t_{i-1})^2$$

$$U(P, f) + L(P, f) = \sum_{i=1}^{n} (t_i^2 - t_{i-1}^2) = t_n^2 - t_0^2 = b^2 - a^2$$

$$U(P,f) = \frac{b^2-a^2}{2} + \sum_{i=1}^{n} (t_i - t_{i-1})^2/2$$

$$L(P,f) = \frac{b^2-a^2}{2} - \sum_{i=1}^{n} (t_i - t_{i-1})^2/2$$

$$\Rightarrow \quad \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \frac{b^2-a^2}{2}$$

Is every bounded function integrable?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

$L(P, f) = 0$ for any partition $\Rightarrow \underline{\int_a^b} f\,dx = 0$

$U(P, f) = b-a$ for any partition $\Rightarrow \overline{\int_a^b} f\,dx = b-a$

$\Rightarrow$ f is not integrable.

Note: There are non continuous integrable functions.

Questions?