

Q.1. Find the matrix of the following inner products relative to given ordered basis  $B$  -

i)  $V = P_2(\mathbb{R})$  with the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx, \quad B = \{1, x, x^2\}.$$

Sol  $\rightarrow$  Let  $M$  be the matrix of inner product then we know that  $M_{ij} = \langle \alpha_j, \alpha_i \rangle$  where  $\alpha_1 = 1, \alpha_2 = x, \alpha_3 = x^2$ . Thus,

$$M = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/5 \end{pmatrix}.$$

ii)  $V = \mathbb{C}^3(\mathbb{C})$  with  $\langle \alpha, \beta \rangle = \sum_{j=1}^3 \alpha_j \bar{\beta}_j$  with any ordered basis  $B$ .

Sol  $\rightarrow$  Suppose  $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  then,

$$\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1).$$

$$M = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_3, \alpha_1 \rangle \\ \langle \alpha_1, \alpha_2 \rangle & \langle \alpha_2, \alpha_2 \rangle & \langle \alpha_3, \alpha_2 \rangle \\ \langle \alpha_1, \alpha_3 \rangle & \langle \alpha_2, \alpha_3 \rangle & \langle \alpha_3, \alpha_3 \rangle \end{pmatrix}$$

where  $\alpha_1 = 1$ ,  $\alpha_2 = x$ ,  $\alpha_3 = x^2$ . Thus,

$$M = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/5 \end{pmatrix}$$

ii)  $V = \mathbb{C}^3(\mathbb{C})$  with  $\langle \alpha, \beta \rangle = \sum_{j=1}^3 \alpha_j \bar{\beta}_j$  with any ordered basis  $B$ .

Sol  $\rightarrow$  Suppose  $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  then,

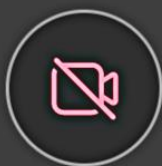
$$\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1).$$

$$M = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_3, \alpha_1 \rangle \\ \langle \alpha_1, \alpha_2 \rangle & \langle \alpha_2, \alpha_2 \rangle & \langle \alpha_3, \alpha_2 \rangle \\ \langle \alpha_1, \alpha_3 \rangle & \langle \alpha_2, \alpha_3 \rangle & \langle \alpha_3, \alpha_3 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Apply Gram Schmidt process to the following subset  $S$  of inner product

Speaking: Tutorial 3 Classroom



space  $V$  to get an orthonormal basis for span of  $S$ .

i)  $V = \mathbb{R}^3$  with standard inner product.  
 $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$ . ✓

Sol → We know that the Gram-Schmidt formula is,

$$\beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{\langle \alpha_k, \beta_j \rangle}{\|\beta_j\|^2} \beta_j, \quad k=2, 3, \dots, n.$$

$$\therefore \beta_1' = \alpha_1 = (1, 0, 1), \quad \therefore \beta_1 = \frac{\beta_1'}{\|\beta_1'\|}$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1' \rangle}{\|\beta_1'\|^2} \beta_1', \quad \therefore \beta_2 = \frac{\beta_2'}{\|\beta_2'\|}$$

$$= (0, 1, 1) - \frac{\langle (0, 1, 1), (1, 0, 1) \rangle}{2} (1, 0, 1)$$

$$= \frac{1}{2} (-1, 2, 1)$$



Speaking: Shreya Nukala, Tutorial 3 Clas...

ii)  $V = \mathcal{P}_2(\mathbb{R})$  with inner product  $\langle f, g \rangle$   
 $= \int_0^1 f(x) g(x) dx, S = \{1, x, x^2\}.$

Sol  $\rightarrow$  Here,  $\alpha_1 = 1, \alpha_2 = x$  &  $\alpha_3 = x^2.$

Thus,

$$\beta_1' = \alpha_1 = 1.$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1.$$

$$= x - \frac{\langle x, 1 \rangle}{\|1\|^2} \cdot 1.$$

$$= x - 1 \quad \beta_1'$$

$$\beta_3' = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2.$$

$$= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} - \frac{\langle x^2, x-1 \rangle}{\|x-1\|^2} (x-1)$$

$$= x^2 - \frac{1}{2} + \frac{1}{4} (x-1)$$

$$= x^2 + \frac{1}{4}x - \frac{3}{4}$$



Q4.  $S^\perp$  for  $S = \{(1,0,i), (1,2,1)\}$

$A = (a,b,c) \in S^\perp$  & if  $A$  solves the following system of eq,

$$\langle A, (1,0,i) \rangle = 0$$

$$\Rightarrow \underline{a - ic = 0}$$

$$\langle A, (1,2,1) \rangle = 0$$

$$\underline{a + 2b + c = 0}$$

$$\underline{S^\perp = \left\{ \left( i, \frac{i-1}{2}, 1 \right) \right\}}$$

Since,

$$\dim V = \dim S + \dim S^\perp$$

$$\therefore \dim S^\perp = 1$$

&  $A$  will span  $S^\perp$ .





$$\therefore \dim S^\perp = 1$$

& A will span  $S^\perp$ .

Q.5.  $\alpha = (2, 1, 3)$

$W = \{(x, y, z) : x + 3y - 2z = 0\}$

Projection of  $\alpha$  over  $W = ?$

Sol  $\rightarrow$  If  $B = \{v_1, v_2, v_3\}$  is orthonormal basis for  $W$  then orthogonal projection of  $\alpha$

on  $W$  is,

$$\underline{u} = \sum_{i=1}^3 \langle \alpha, v_i \rangle v_i$$

So we have to find  $B = ?$

$$\dim W = 2$$

Because,

$$\underline{x + 3y = 2z}$$

$$B = \{(2, 0, 1), (-3, 1, 0)\}$$

on  $W$  is,

$$\underline{u} = \sum_{i=1}^2 \langle \underline{u}, \underline{v}_i \rangle \underline{v}_i$$

So we have to find  $B = ?$

$$\dim W = 2$$

Because,

$$\underline{2+3y} = \underline{x_2}$$

$$\underline{B} = \{ (2, 0, 1), (-3, 1, 0) \}$$

$$B = \left\{ \frac{1}{\sqrt{5}}(2, 0, 1), \frac{1}{\sqrt{70}}(-3, 1, 0) \right\}$$

$$u = \frac{1}{\sqrt{5}} \langle (2, 1, 3), \frac{1}{\sqrt{5}}(2, 0, 1) \rangle (2, 0, 1)$$

$$= \frac{1}{\sqrt{70}} \langle (2, 1, 3), \frac{1}{\sqrt{70}}(-3, 1, 0) \rangle (-3, 1, 0)$$

$$= \frac{1}{14} (29, 17, 40)$$



Sol → Here,  $\alpha_1 = 1, \alpha_2 = x$  &  $\alpha_3 = x^2$ .

Thus,

$$\beta_1' = \alpha_1 = 1.$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1.$$

$$= x - \frac{\langle x, 1 \rangle}{\|1\|^2} \cdot 1.$$

$$= x - 1. \quad \beta_2' = \frac{\beta_2}{\|\beta_2\|}.$$

$$\beta_3' = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2.$$

$$= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} - \frac{\langle x^2, x-1 \rangle}{\|x-1\|^2} (x-1)$$

$$= x^2 - \frac{1}{3} + \frac{1}{4} (x-1)$$

$$= x^2 + \frac{1}{4}x - \frac{3}{4}$$

$\mathbb{C}^4(\mathbb{C})$  with standard inner product,  
Speaking: Shreya Nukala, Tutorial 3 Clas...

$$v = \frac{1}{2} (1-i, 2-i, -1) \quad (2+2i, 2, 1-i, 2i)$$





$$= \frac{1}{14} (29, 17, 40)$$

$$ii) V = \mathcal{P}(\mathbb{R}) \quad \langle f, g \rangle = \int_0^1 f(x) g(x) dx$$

$$h(x) = 4 + 3x - 2x^2, \quad \omega = \underline{\mathcal{P}_2(\mathbb{R})}$$

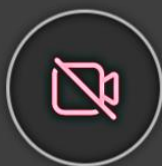
$$\overline{B} = \{1, x\}$$

$$B = 1, 3 \left(2 - \frac{1}{2}\right)$$

$$\left\langle 2 - \frac{1}{2}, 2 - \frac{1}{2} \right\rangle$$

$$x^2 - x + \frac{1}{4}$$

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$



$$g = \left( 1, \sqrt{3} \left( x - \frac{1}{2} \right) \right)$$

$$u = \langle 4 + 3x - 2x^2, 1 \rangle \perp$$

$$+ \langle 4 + 3x - 2x^2, \sqrt{3} \left( x - \frac{1}{2} \right) \rangle$$

$$u = ?$$

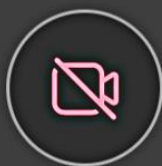
Q.6  $S_0 \subseteq S$  implies  $S^\perp \subseteq S_0^\perp$ .

For any  $x \in S^\perp$ ,  $\langle x, y \rangle = 0 \quad \forall y \in S$ .

Now  $S_0 \subseteq S$ , that implies  $\langle x, y \rangle = 0 \quad \forall y \in S_0$ .

$$\Rightarrow x \in S_0^\perp$$

ii)  $\underline{S} \subseteq \underline{(S^\perp)^\perp}$  implies  $\underline{\text{span}(S)} \subseteq \underline{(S^\perp)^\perp}$



Now  $S_0 \subset S$ , then implies  $\underline{\text{span}}(S) \subset (S^\perp)^\perp$

$y \in S_0$ .

$$\Rightarrow x \in S_0^\perp$$

$$\text{ii) } \underline{S \subset (S^\perp)^\perp} \text{ implies } \underline{\text{span}(S) \subset (S^\perp)^\perp}$$

Take  $\underline{x \in S}$  then for any  $\underline{y \in S^\perp}$

$$\langle x, y \rangle = 0 \quad \forall y \in S^\perp$$

$$\underline{x \in (S^\perp)^\perp}$$

$$\underline{S \subset (S^\perp)^\perp}$$

Since  $\underline{(S^\perp)^\perp}$  is a subspace &

\* if  $A \subset W$  where  $W$  is a subspace  
then  $\underline{\text{span } A} \subset W$ .

$$\underline{\text{span } S} \subset (S^\perp)^\perp$$

Since  $(S^\perp)^\perp$  is a subspace &

\* if  $A \subset W$  where  $W$  is a subspace  
then  $\text{span } A \subset W$ .

$$\Rightarrow \boxed{\text{span } S \subset (S^\perp)^\perp} \leftarrow 0$$

$$\text{If } x, y \in (S^\perp)^\perp$$

$$\text{then } x+y \in (S^\perp)^\perp.$$

$$\text{then } \langle x, z \rangle = 0 \text{ for any } z \in S^\perp$$

$$\& \langle y, z \rangle = 0 \text{ for any } z \in S^\perp$$

$$\langle x+y, z \rangle = 0 \text{ for any } z \in S^\perp.$$

$$\underline{x+y \in (S^\perp)^\perp.}$$

$$\text{If } A \subset W$$

Speaking: Tutorial 3 Classroom



$$\underline{x+y \in (S^\perp)^\perp}$$

$$\begin{array}{c} \text{If } A \subset W \\ \downarrow \\ \text{subspace} \end{array}$$

$$\underline{\text{Claim} - \text{span } A \subset W}$$

because let  $a, b \in A \xrightarrow{C^W} \text{subspace}$  then

$$\underline{\alpha a + \beta b \in \text{span } A} \text{ for any } \alpha, \beta \in \mathbb{F}$$

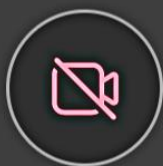
$$\text{span } A \subset W$$

$$c) W = (W^\perp)^\perp$$

$$\text{From ii), } W \subset (W^\perp)^\perp$$

$$\text{For proving } \underline{(W^\perp)^\perp \subset W}$$

$$\text{Let } x \in (W^\perp)^\perp \text{ such that } x \in$$





$$\text{If } A \subset W$$

↓  
subspace

Claim —  $\text{span } A \subset W$ .

because let  $a, b \in A \xrightarrow{\subset W \rightarrow \text{subspace}}$  then

$$\underline{\alpha a + \beta b \in \text{span } A} \text{ for any } \alpha, \beta \in \mathbb{F}.$$

$$\text{span } A \subset W.$$

$$c) W = (W^\perp)^\perp$$

$$\text{From ii), } W \subset (W^\perp)^\perp$$

For proving  $\underline{(W^\perp)^\perp \subset W}$ .

Let  $x \in (W^\perp)^\perp$  such that  $x \notin W$ .  
then there will be some  $y \in W^\perp$  s.t.

$\langle x, y \rangle \neq 0$  but this is contradiction

as  $x \in (W^\perp)^\perp$  then  $\langle x, y \rangle = 0 \forall y \in W^\perp$ .

Let  $x \in (W^\perp)^\perp$  such that  $x \notin W$ .  
 then there will be some  $y \in W^\perp$  s.t.  
 $\langle x, y \rangle \neq 0$ . but this is contradiction  
 as  $x \in (W^\perp)^\perp$  then  $\langle x, y \rangle = 0 \forall y \in W^\perp$ .

d)  $V = W \oplus W^\perp$ .

For any  $y \in V$  we have  $y = u + v$   
 where  $u \in W$  &  $v \in W^\perp$  & also  
 $W \cap W^\perp = \{0\}$  because,

if  $x \in W$  &  $x \in W^\perp$  s.t.  $x \neq 0$  then

$$\underline{\langle x, x \rangle = 0}$$

$$\underline{\text{but } x \neq 0.}$$

This contradicts def<sup>n</sup> of inner  
 product. Hence  $\{0\} = W \cap W^\perp$ .



where  $\underline{u} \in W$  &  $\underline{v} \in W^\perp$  & also  
 $W \cap W^\perp = \{0\}$  because,

if  $x \in W$  &  $x \in W^\perp$  s.t.  $x \neq 0$  then

$$\underline{\langle x, x \rangle = 0}$$

$$\underline{\text{but } x \neq 0.}$$

This ~~is~~ contradicts def<sup>n</sup> of inner product. Hence  $\{0\} = W \cap W^\perp$ .

$$\therefore V = W \oplus W^\perp.$$

Q. 7.  $V \rightarrow$  I.P.S.

$W$  is subspace of  $V$ .

If  $\underline{x} \notin W$ ,  $\exists$  T.  $y \in V$  s.t.  $\underline{y} \in W^\perp$  but

$$\langle x, y \rangle \neq 0.$$




  
 $y \perp w$ .  $V = W \oplus W^\perp$ .

$W$  is subspace of  $V$ .

If  $\underline{x} \notin W$ ,  $\exists$   $y \in V$  s.t.  $\underline{y} \in W^\perp$  but

$$\langle x, y \rangle \neq 0.$$

Sol  $\rightarrow$  As  $V = W \oplus W^\perp$ , there are

unique vectors  $u$  &  $v$  s.t.;

$$x = u + v$$

Now we know that  $x \notin W$

$$\therefore v \neq 0.$$

Because if  $v = 0$  then  $x = u \in W$ .

Now suppose  $y = v$  then,

$$\begin{aligned}
 \langle x, y \rangle &= \langle u + v, y \rangle = \langle u + v, v \rangle \\
 &= \langle u, v \rangle + \langle v, v \rangle \\
 &= 0 + \langle v, v \rangle \neq 0.
 \end{aligned}$$

Because  $v \neq 0$ .



$$= \langle u, v \rangle + \langle v, u \rangle$$

$$= 0 + \langle u, u \rangle \neq 0$$

Because  $u \neq 0$ .

Q.8  $V = C([-1, 1], \mathbb{R})$

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx$$

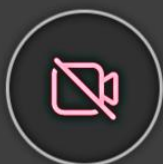
Suppose  $W_e$  &  $W_o$  denotes the subspaces  
of  $V$   $\downarrow$  even  $\downarrow$  odd  
funct. funct.

P.T.  $W_e^\perp = W_o$ .

$$\langle f, g \rangle = \int_{-1}^1 \underbrace{f(x)}_{\text{even}} \underbrace{g(x)}_{\text{odd}} dx.$$

$$\int_{-a}^a f(x) dx = 0$$

if  $f$  is odd.





$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx$$

Suppose  $W_e$  &  $W_o$  denotes the subspaces  
of  $V$   $\downarrow$  even  $\downarrow$  odd  
funct.  $\downarrow$  funct.

P.T.  $W_e^\perp = W_o$ .

$$\begin{aligned} \langle f, g \rangle &= \int_{-1}^1 \underbrace{f(n)}_{\substack{\downarrow \\ \text{even}}} \underbrace{g(n)}_{\substack{\downarrow \\ \text{odd}}} dx. \\ &= \int_{-1}^1 \underbrace{h(n)}_{\substack{\downarrow \\ \text{odd}}} dx = 0. \end{aligned}$$

$$\star \int_{-a}^a f(n) dx = 0 \quad \text{odd}$$

if  $f$  is odd.

$$\int_{-a}^a f(n) = \int_{-a}^0 f(n) + \int_0^a f(n)$$

$$= \int_0^a f(-n) + \int_0^a f(n)$$

$$= - \int_0^a f(n) + \int_0^a f(n) = 0$$

$$= 0$$

As we know that product of odd function & even function is odd therefore if  $f \in W_e$ ,  $g \in W_o$  then  $\langle \underline{f}, \underline{g} \rangle = 0$ . Thus  $W_o \subset W_e^\perp$ .  
 $W_e^\perp \subset W_o$

\* Now if possible let us take some  $\underline{h} \in W_e^\perp$  but  $\underline{h} \notin W_o$ . Then we know that  $\underline{h} = \underline{\psi} + \underline{\phi}$  where  $\phi$  is

is odd &  $\psi$  is even.

Now,

$$\begin{aligned} \langle \underline{h}, \underline{\psi} \rangle &= \langle \underline{\psi}, \underline{\psi} \rangle + \langle \underline{\phi}, \underline{\psi} \rangle \\ \underline{h} &\in W_e^\perp \text{ even} \quad \underline{\psi} \in W_e \text{ even} \\ &= \langle \underline{\psi}, \underline{\psi} \rangle + 0 \neq 0. \end{aligned}$$

This is contradiction.

$$\therefore \underline{h} \notin W_o$$