Lecture #9 (IC152)

Recall :-

* Annihilating polynomial of $T \in L(V,V)$ (dim $V < \infty$)

  $p(T) = 0$ (zero linear operator on V)

* minimal polynomial of T
  i) monic
  ii) annihilating
  iii) least degree

* Char poly & minimal polynomial have same roots except multipl…
* minimal polynomial divides any annihilating polynomial of T
* char. poly. is annihilating polynomial of T. (Cayley-Hamilton Theorem)
* T is diagonalizable $\Longleftrightarrow$ minimal polynomial is product of distinct linear factors.

Proof (Cayley-Hamilton Theorem)

Recall $B \,\mathrm{adj}(B) = |B| I$ for any matrix (square) B

Take $B = (xI - A)$, (A is given matrix / linear ...)

Let us try to compute $\operatorname{adj}(B) = \operatorname{adj}(xI - A)$ ✓

(Observe that entries of B, being a polynomial are not of degree more than $n-1$, if $A \in M_{n\times n}(\mathbb{F})$)

Therefore

$$\operatorname{adj}(B) = C_0 + C_1 x + C_2 x^2 + \cdots C_{n-1} x^{n-1} \checkmark$$

for $C_0, C_1, \ldots C_{n-1} \in M_{n\times n}(\mathbb{F})$ ✓

(Note that $\operatorname{adj}(B) \in M_{n\times n}(P_{n-1}(\mathbb{F}))$)

(Char Poly) ✓ Let us write $\underline{\det(xI - A) = a_0 + a_1 x + \cdots x^n}$ $(\because a_n = 1)$

Let us substitute all these expressions in

$$(xI - A)\operatorname{adj}(xI - A) = |xI - A|\, I$$

$$(xI - A)\left(C_0 + xC_1 + x^2 C_2 + \cdots \underline{x^{n-1} C_{n-1}}\right)$$

$$= (a_0 + a_1 x + \cdots + x^n)\, I$$

Comparing powers of 'x' to get

Ix ✓ $-AC_0 = a_0 I$?

$I \times \quad -AC_0 = a_0 I$

$A \times \quad C_0 - AC_1 = a_1 I$

$A^2 \times \quad C_1 - AC_2 = a_2 I$

$\vdots \qquad \vdots$

$A^{n-1} \times \quad C_{n-2} - AC_{n-1} = a_{n-1} I$

$A^n \times \quad C_{n-1} = I$

Now after multiply suitable powers of A in above system

$$-\cancel{AC_0} = a_0 I$$
$$\cancel{AC_0} - \cancel{A^2C_1} = a_1 A$$
$$\cancel{A^2C_1} - A^3C_2 = a_2 A^2$$
$$\vdots$$
$$\cancel{A_{n-1}C_{n-2}} - \cancel{A^n C_{n-1}} = a_{n-1} A^{n-1}$$
$$\cancel{A^n C_{n-1}} = A^n$$

zero matrix $\leftarrow 0 = a_0 I + a_1 A + \cdots A^n$

of order $n \times n$

$\Rightarrow$ A satisfies (annihilated by) $a_0 + a_1 x + a_2 x^2 + \cdots \; x^n$ which is characteristic polynomial for A.

Remark: $p(x) :=$ char poly of A

$\Rightarrow \; p(x) = |xI - A|$

Put $x = A$ to get $p(A) = |AI - A| = |A - A| = |0| \neq 0$ $\in F$

Is it correct ??

Not correct

Example:–

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Char. poly $:= (x-2)^3(x-5)$

minimal polynomial

$(x-2)(x-5)$ $\quad (x-2)^2(x-5)$ $\quad (x-2)^3(x-5)$

$(x-2)^4$

$(x-2)^3(x-5)(x-3)$

$(A-2I)^3(A-5I)(A-3I) = ($

$(x-2)(x-5)$ ✓ $\quad (x-2)(x-5)$ ✓ $\quad \rightarrow (x-2)\ (x-2)^2$

$A = M_{3\times 3}(F)$

$A^2 = A$

$\Rightarrow x^2 - x$ is annihilating poly

$m(x) = \cancel{x},\ \cancel{(x-1)},\ x(x-1)$ ✓

$A \neq 0$ then $m(x) \neq x$

$A \neq I$ then $m(x) \neq x-1$

i) $(A-2I)(A-5I) = 0$ (zero matrix)

if yes $\Longrightarrow$ min poly $(x-2)(x-5)$ & A is diagonalizable

ii) If Not

$(A-2I)^2(A-5I) \overset{\text{if}}{=} 0$

then $m(x) = (x-2)^2(x-5)$

Not diagonalizable.

iii) If ii) does not hold

then $m(x) = (x-2)^3(x-5)$ (Cayley-Hamilton theorem)

& A is not diagonalizable.

Example: Let $p(x) = (x-3)(x-4)$ be such that $p(A) = 0$, then find out possibilities of A

$p(x)$ is annihilating polynomial $\Rightarrow m(x) = x-3$ or $m(x) = x-4$ or $m(x) = (x-3)(x-4)$

i) if $m(x) = x-3 \Rightarrow A = 3I$ ✓

ii) if $m(x) = x-4 \Rightarrow A = 4I$

iii) if $m(x) = (x-3)(x-4) \Rightarrow A$ = similar to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ ✓

Example: Let $A^2 = I$, then what will be minimal polynomial.

minimal polynomial can be $(x-1), (x+1), (x^2-1)$ ✓

Example: Similar matrices have same minimal polynomial. what about converse?

As $A = Q^{-1}BQ \Rightarrow f(A) = Q^{-1} f(B) Q$

Find out counter example !!

Example :- Let $A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$ $a, b, c \in \mathbb{F}$

Char poly = min poly

Char poly $= \begin{vmatrix} x & 0 & -c \\ -1 & x & -b \\ 0 & -1 & x-a \end{vmatrix}$

$$= x(x^2 - ax) - bx - c$$

$$= x^3 - ax^2 - bx - c \checkmark$$

Choices of minimal polynomial

i) char poly (obvious)

ii) One degree poly (monic) of the type $x - d$

but $A - dI \neq 0$ for any d.

iii) Two degree monic polynomial of the type.

$$x^2 + dx + e \quad \checkmark \quad \checkmark$$

Observation $B = A^2 + dA + eI$

then observe some specific entry in B which can not be zero.

$$\Rightarrow \quad B \neq 0$$

Check $\left[(B)_{31} = 1 \neq 0 \Rightarrow B \neq 0 \right]$!!

$\Rightarrow$ Only choice is char polynomial.