Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Tutorial Sheet 5

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1. Find the matrix of the following inner products relative to given ordered basis $\mathcal{B}$

   (i) $V = P_2(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $\mathcal{B} = \{1, x, x^2\}$ Let M the matrix of inner product, then $M_{ij} =< \alpha_j, \alpha_i >$, where $\{\alpha_1, \alpha_2, \alpha_3\}$ is the given ordered basis of $P_2(\mathbb{R})$. This results into the following matrix

$$M = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/4 \\ 1/2 & 1/4 & 1/5 \end{bmatrix}$$

   (ii) $V = \mathbb{C}^3(\mathbb{C})$ with $< \alpha, \beta >= \sum_{j=1}^{3} \alpha_j \bar{\beta}_j$, with any ordered basis $\mathcal{B}$ of V. (You can choose any ordered basis of your choice)

2. Apply Gram-Schmidt process to the following subsets S of inner product space V to get an orthonormal basis for span of S.

   (i) $V = \mathbb{R}^3$ with standard inner product, $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$The Gram-Schmidt formula helps to construct, from any given linearly independent set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, an orthogonal set $\{\beta_1 = \alpha_1, \beta_2, \cdots, \beta_n\}$ (spanning the same set as spanned by $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$) in the following way

$$\beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{< \alpha_k, \beta_j >}{\|\beta_j\|^2} \beta_j, \quad k = 2, 3, \cdots n$$

   Applying the above formula, we get a required basis for span of S as $\{(1, 0, 1), \frac{1}{2}(-1, 2, 1), \frac{1}{3}(1, 1, -1)\}$.

   (ii) $V = P_2(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $S = \{1, x, x^2\}$.

   (iii) $V = \mathbb{C}^4(\mathbb{C})$, with standard inner product, $S = \{(1, i, 2 - i, -1), (2 + 3i, 3i, 1 - i, 2i), (-1 + 7i, 6 + 10i, 11 - 4i, 3 + 4i)\}$.

   (iv) $V = M_{2\times 2}(\mathbb{R})$ with inner product defined as $< A, B >= \sum_{i,j=1}^{2} A_{ij} B_{ij}$,

$$S = \left\{ \begin{bmatrix} 3 & 5 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 9 \\ 5 & -1 \end{bmatrix}, \begin{bmatrix} 7 & -17 \\ 2 & -6 \end{bmatrix} \right\}$$

3. Prove the following

(i) Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be an orthonormal basis for an inner product space V. Prove that for any $\alpha, \beta \in V$

$$< \alpha, \beta >= \sum_{i=1}^{n} < \alpha, \alpha_i > \overline{< \beta, \alpha_i >}$$

Observe that any vector α can be expressed uniquely as $\alpha = \sum_{i=1}^{n} < \alpha, \alpha_i > \alpha_i$ and similarly $\beta = \sum_{i=1}^{n} < \beta, \alpha_i > \alpha_i$ relative to given orthonormal basis. Now

$$< \alpha, \beta >=< \sum_{i=1}^{n} < \alpha, \alpha_i > \alpha_i, \sum_{j=1}^{n} < \beta, \alpha_j > \alpha_j >= \sum_{i=1}^{n} < \alpha, \alpha_i > \overline{< \beta, \alpha_i >},$$

using orthonormality of the given ordered basis.

(ii) Let $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be an orthonormal subset of an inner product space V. Prove that for any $\alpha \in V$

$$\|\alpha\|^2 \geq \sum_{i=1}^{n} | < \alpha, \alpha_i > |^2$$

Done in the class

4. Compute $S^{\perp}$ for $S = \{(1, 0, i), (1, 2, 1)\}$ in $\mathbb{C}^3$.
Any vector $c = (c_1, c_2, c_3) \in S^{\perp}$ if c solves the following system of equations

$$c_1 - ic_3 = 0$$
$$c_1 + 2c_2 + c_3 = 0.$$

The solution set $(S^{\perp})$ is the span of $(i, -\frac{1+i}{2}, 1)$.

5. Find the orthogonal projections of the following vectors on the given subspace of the specified inner product space

(i) $V = \mathbb{R}^3$ with standard inner product, $\alpha = (2, 1, 3)$, and $W = \{(x, y, z) : x + 3y - 2z = 0\}$. If $\mathcal{B} = \{v_1, v_2, \cdots, v_k\}$ be an orthonormal basis for W then orthogonal projection of α on W is given by $u = \sum_{i=1}^{k} < \alpha, v_i > v_i$. To find an orthonormal basis, we first need to find a basis for W, which is easy to see as $\{(2, 0, 1), (-3, 1, 0)\}$ spans W and is linearly independent. Now we use Gram-Schmidt process to find an orthogonal basis, which is $\{(2, 0, 1), (-3/5, 1, 6/5)\}$. An orthonormal basis for W is $\mathcal{B} = \{\frac{1}{\sqrt{5}}(2, 0, 1), \frac{1}{\sqrt{70}}(-3, 1, 0)\}$. Therefore $u = \frac{1}{14}(29, 17, 40)$.

(ii) $V = P(\mathbb{R})$ with the inner product $< f, g >= \int_0^1 f(x)g(x)dx$, $h(x) = 4 + 3x - 2x^2$, $W = P_1(\mathbb{R})$.

6. Let V be an inner product space, S and S_0 are the subsets of V and W is a finite dumensional subspace of V. Prove the following results

(i) $S_0 \subseteq S$ implies that $S^{\perp} \subseteq S_0^{\perp}$. For any $x \in S^{\perp}$, $< x, y >= 0$ for all $y \in S$. As $S_0 \subseteq S$, $< x, y >= 0$ for all $y \in S_0$. Therefore $x \in S_0^{\perp}$.

(ii) $S \subseteq (S^{\perp})^{\perp}$ implies span $(S) \subseteq (S^{\perp})^{\perp}$ Take $x \in S$ and $y \in S^{\perp}$, then $< x, y >= 0$ for all $y \in S^{\perp}$ which implies $x \in (S^{\perp})^{\perp}$ or $S \subseteq (S^{\perp})^{\perp}$. As $(S^{\perp})^{\perp}$ is a subspace and hence span$(S) \subseteq (S^{\perp})^{\perp}$

(iii) $W = (W^{\perp})^{\perp}$ From part (ii), $W \subseteq (W^{\perp})^{\perp}$. For the converse, take $x \in (W^{\perp})^{\perp}$ such that $x \notin W$, then from Problem 7 below, we get $y \in W^{\perp}$ such that $< x, y > \neq 0$ but this is a contradiction (as $x \in (W^{\perp})^{\perp}$ and $< x, y >= 0$ for all $y \in W^{\perp}$). Thus $(W^{\perp})^{\perp} \subseteq W$ which implies $W = (W^{\perp})^{\perp}$.

(iv) $V = W \oplus W^{\perp}$ We know that any $y \in V$ can be expressed uniquely as $y = u + v$, where $u \in W$ and $v \in W^{\perp}$. Moreover $W \cap W^{\perp} = \{0\}$ (if $x \in W \cap W^{\perp}$ implies $< x, x >= 0$ or $x = 0$). Therefore $V = W \oplus W^{\perp}$.

7. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^{\perp}$, but $< x, y > \neq 0$.
As $V = W \oplus W^{\perp}$, there exists unique vectors u and v such that $x = u + v$. Observe that $v \neq 0$, otherwise $x \in W$. Now choose $y = v$, to get $< x, y >=< u, v > + < v, v >= 0 + \|v\|^2 \neq 0$.

8. Let $V = C([-1, 1]; \mathbb{R})$ be an inner product space with inner product defined as

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx.$$

Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Then prove that $W_e^{\perp} = W_0$.
As product of even and odd function is odd function, we have for $g \in W_0$ and for every $f \in W_e$, $< f, g >= 0$. Thus $W_0 \subset W_e^{\perp}$. Now assume, if possible, some $h \in W_e^{\perp}$ but $h \notin W_0$. As every h can be written as sum of even and odd functions, say $h = \psi + \phi$, where ψ is even and ϕ is odd, $< h, \psi >=< \psi, \psi > + < \phi, \psi > \implies \psi = 0$. Hence $h \in W_0$. Thus $W_e^{\perp} = W_0$