IC153: Calculus 1
(Final Lecture)

by

Anurag Singh

IIT Bhilai

January 05, 2022

Recap of the previous lecture

- Riemann criterion for integrability
- Uniform continuity
- continuous on $[a, b] \implies$ uniform continuous
- Continuous on $[a, b] \implies$ integrable on $[a, b]$
- Monotone on $[a, b] \implies$ integrable on $[a, b]$

Some more properties of integral

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ *be integrable functions and* $c \in \mathbb{R}$. *Then*

1. $f + g$ *is integrable and* $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
2. $f \cdot g$ *is integrable.*
3. cf *is integrable and* $\int_a^b (cf)(x)dx = c\int_a^b f(x)dx$.
4. *If* $f(x) \leq g(x)$ *for all* $x \in [a, b]$ *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
5. $|f|$ *is integrable and* $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$.
6. *If* $|f(x)| \leq M$ *for all* $x \in [a, b]$ *then* $\left| \int_a^b f(x)dx \right| \leq M(b - a)$.

First Fundamental Theorem of Calculus

Theorem

Let f be integrable on $[a, b]$. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)dt$. Then

1. *F is continuous on $[a, b]$.*
2. *If f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof: let $M = \sup\{ |f(x)| : x \in [a,b]\}$

and let $a \leq x < y \leq b$.

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right|$$

$$|F(y) - F(x)| \leq \int_x^y |f(t)|\, dt$$

$$|F(y) - F(x)| \leq M\, |y - x| \quad \text{— (1)}$$

(1) is true $\forall\ a \leq x < y \leq b$.

Claim: (1) $\implies$ F is unif. continuous.

Given $\epsilon > 0$, we have to find $\delta = \epsilon/M$

s.t. $|x-y| < \delta = \frac{\epsilon}{M} \implies |F(x) - F(y)| < M \cdot \frac{\epsilon}{M}$

$= \epsilon$

(ii). $$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right| = \left|\frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt - f(x_0)\right|$$

$$= \left|\frac{1}{(x-x_0)}\int_{x_0}^{x} f(t)\,dt - \frac{f(x_0)}{x-x_0}\int_{x_0}^{x} 1\,dt\right|$$

$$= \left|\frac{1}{x-x_0}\int_{x_0}^{x} \underbrace{\big(f(t)-f(x_0)\big)}_{<\,\epsilon}\,dt\right|$$

$$< \left|\frac{1}{x-x_0}\cdot\epsilon\int_{x_0}^{x} 1\,dt\right| = \epsilon$$

$\Rightarrow$ F is differentiable at x_0 & $F'(x_0) = f(x_0)$

Note: Suppose f is continuous on $[a, b]$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$, that is,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ for all } x \in [a, b].$$

Antiderivative

Definition

Let $a < b$ and $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. We say that f has an antiderivative/primitive on $[a, b]$ if there exists a differentiable function $F : [a, b] \longrightarrow \mathbb{R}$ such that $F' = f$.

Proposition

If $f : [a, b] \longrightarrow \mathbb{R}$ has an antiderivative F, then F is unique up to addition by a constant.

Proof: If $F' = f = G'$, then $(F - G)' = 0$ on the interval $[a, b]$, and hence $F - G$ is a constant function (by the MVT).

Observation: *(From first FTC) If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then f has an antiderivative given by $F(x) := \int_a^x f(t)dt$ for $x \in [a, b]$.*

Second Fundamental Theorem of Calculus

Theorem

Let f be integrable on $[a,b]$. If there is a differentiable function F on $[a,b]$ such that $F' = f$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$

By MVT, there is $c_i \in (x_{i-1}, x_i)$ s.t.

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i)$$

Clearly;

$$f(c_i) \geq m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

$$f(c_i) \leq M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

$$m_i \leq f(c_i) \leq M_i$$

$$\sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(P, f) \leq \underbrace{\sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) \right)} \leq U(P, f)$$

$$= F(x_n) - F(x_0)$$

$$= F(b) - F(a)$$

$$\Rightarrow \underline{\int_a^b} f\,dx \leq F(b) - F(a) \leq \overline{\int_a^b} f\,dx$$

Since f is integrable $\Rightarrow$

$$\int_a^b f\,dx = F(b) - F(a)$$

f'

Note: If f is differentiable and is integrable on $[a, b]$, then

$$\int_a^b \left(\frac{d}{dx} f(x) \right) dx = f(b) - f(a).$$

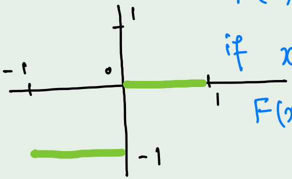
Thus the operations of differentiation and integration are kind of 'inverse' to each other.

Theorem

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function and $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$. Moreover $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Example

1. Let $f(x) := [x]$ for $x \in [-1,1]$. Define $F(x) := \int_{-1}^{x} [t]dt$ for $x \in [-1,1]$.
   Check that $F(x) = -1 - x$ if $x \in [-1, 0)$, and $F(x) = -1$ if $x \in [0,1]$.



$$F(x) = \int_{-1}^{x} [t]\, dt$$

if $x \in [-1, 0)$

$$F(x) = \int_{-1}^{x} (-1)\, dt = [-x]_{-1}^{x} = (-x) - (-(-1)) = -x - 1$$

$x \in [0,1]$

$$F(x) = \int_{-1}^{0} -1\, dt + \int_{0}^{x} 0\, dt = -1 + [c]_0^x = -1 + c - c = -1$$

Integration by parts

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function such that f' is integrable. Suppose $g : [a, b] \longrightarrow \mathbb{R}$ is integrable and has an antiderivative G on $[a, b]$. Then

$$\begin{aligned}\int_a^b f(x)g(x)dx &= f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx \\ &= f(x)G(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)G(x)dx.\end{aligned}$$

Proof: Define $H(x) = f(x) \cdot G(x)$

$$\begin{aligned}H'(x) &= f(x)\, G'(x) + f'(x)\, G(x) \\ &= f(x)\, g(x) + f'(x)\, G(x)\end{aligned}$$

$\Rightarrow$ H' is integrable on $[a,b]$. Hence by second FTC

$$\int_a^b H'(x)\,dx = H(b) - H(a) = f(b)G(b) - f(a)\,G(a)$$

$$\int_a^b \left(f(x)\, g(x) + f'(x)\, G(x) \right) dx = f(x)\, G(x) \Big|_{x=a}^{x=b}$$

$$\int_a^b f(x)\, g(x)\, dx = f(x)\, G(x) \Big|_{x=a}^{x=b} - \int_a^b f'(x)\, G(x)\, dx$$

Questions?

All the best!