Tutorial 3 (Solutions).

1. (a) Given $X \geq 0$. To show: $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$.

Note that $P(X \geq k) = \sum_{i=k}^{\infty} f_X(i)$.

Thus,

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} \sum_{i\geq k}^{\infty} f_X(i)$$

$1 \leq k < \infty$, $k \leq i < \infty$ | $1 \leq i < \infty$, $1 \leq k \leq i$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} f_X(i)$$

$$= \sum_{i=1}^{\infty} i f_X(i)$$

$$= E(X).$$

(b) Given

$$f_Y(y) = \begin{cases} \frac{1}{b-a+1}, & y = a, a+1, \dots, b, \\ 0, & \text{otherwise}. \end{cases}$$

, and a, b are such that $b > a \geq 0$

$$P(Y \geq k) = \begin{cases} 1, & k \leq a, \\ \frac{b-k+1}{b-a+1}, & a+1 \leq k \leq b, \\ 0, & k \geq b+1. \end{cases}$$

So,

$$\sum_{k=1}^{\infty} P(Y \geq k) = \sum_{k=1}^{a} 1 + \sum_{k=a+1}^{b} \frac{b-k+1}{b-a+1}$$

$a+1 \leq k \leq b$

$-a-1 \geq -k \geq -b$

$b-a \geq \underbrace{b-k+1}_{l} \geq 1$

$$= a + \frac{1}{b-a+1} \sum_{l=1}^{b-a} l$$

$$= a + \frac{1}{(b-a+1)} \frac{(b-a)(b-a+1)}{2} = \frac{b+a}{2}.$$

Therefore, $E[Y] = \frac{b+a}{2}$.

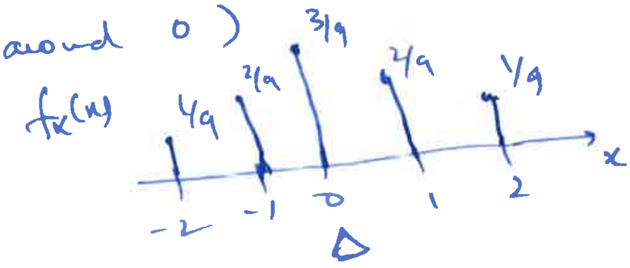
(2)

(a) Note that $S_X = \{ -2, -1, 0, 1, 2 \}$.

$$f_X(x) = \begin{cases} \frac{1}{9} & , x = -2, 2, \\ \frac{2}{9} & , x = -1, 1, \\ \frac{3}{9} & , x = 0, \\ 0 & , \text{otherwise}. \end{cases}$$

$$E(X) = \sum_{x=-2}^{2} x f_X(x) = -2 \times \frac{1}{9} + 2 \times \frac{1}{9} - \frac{2}{9} + \frac{2}{9} + 0 \times \frac{3}{9} + 0 = 0.$$

(Note that pmf f_X is symmetric around 0)

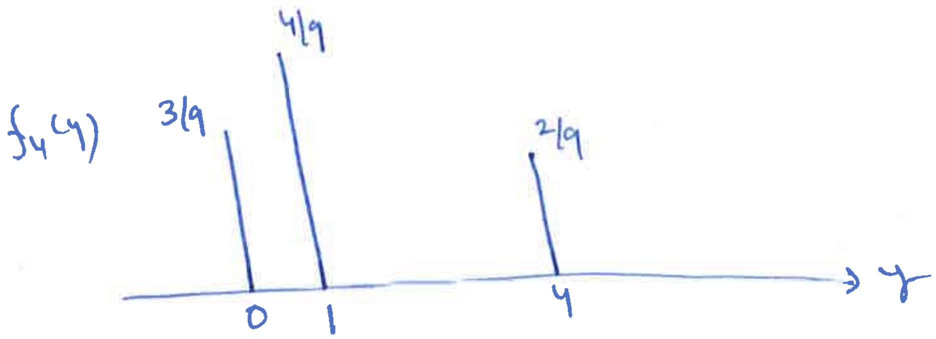


Also, $$E(X^2) = \sum_{x=-2}^{2} x^2 f_X(x)$$
$$= 4 \times \frac{1}{9} + 4 \times \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + 0 \times \frac{3}{9} + 0 = \frac{12}{9} = \frac{4}{3}.$$

and $$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{4}{3} - 0^2 = \frac{4}{3}.$$

(b) Let $Y = X^2$. So, $S_Y = \{0, 1, 4\}$

$$f_Y(y) = \begin{cases} 2/9 & , y = 4, \\ 4/9 & , y = 1, \\ 3/9 & , y = 0, \\ 0 & , \text{otherwise}. \end{cases}$$



(3) Let I_k be the reward paid at time k. We have

$$E(I_k) = P(I_k = 1) = P(T \text{ at time } k \text{ and } H \text{ at time } k-1)$$
$$= p(1-p).$$

$$R = \sum_{k=1}^{n} I_k \quad \Rightarrow \quad E(R) = E\left(\sum_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} E(I_k) = np(1-p).$$

* I_k's are not all independent (Why?).

$$E(I_k^2) = p(1-p).$$

$$E(I_k I_{k+1}) = 0 \quad \text{(Why?)}$$

$$E(I_k I_{k+l}) = E(I_k)E(I_{k+l}) = p^2(1-p)^2 \quad \text{for } l \geq 2.$$

used — Recall: (If X and Y are independent then $E(XY) = E(X)E(Y)$)

$$\therefore \; E(R^2) = E\left(\left(\sum_{k=1}^{n} I_k\right)^2\right)$$
$$= E\left[\left(I_1 + I_2 + \dots + I_n\right)^2\right]$$
$$= E\left[\sum_{k=1}^{n} I_k^2 + 2\sum_{k=1}^{n-1} I_k I_{k+1} + 2\sum_{l-k \geq 2} I_k I_l\right]$$
$$= \sum_{k=1}^{n} E(I_k^2) + 2\sum_{k=1}^{n-1} E(I_k I_{k+1}) + 2\sum_{l-k \geq 2} E(I_k I_l).$$
$$= np(1-p) + 2 \times \left[0 + \left(\frac{n(n-1)}{2} - (n-1)\right) p^2(1-p)^2\right]$$
$$= np(1-p) + (n-1)(n-2)\, p^2(1-p)^2.$$

$$\therefore \; Var(R) = E(R^2) - (E(R))^2$$
$$= np(1-p) + (n^2 - 3n + 2)\, p^2(1-p)^2 - n^2 p^2(1-p)^2$$
$$= np(1-p) - (3n-2)\, p^2(1-p)^2$$

D.