Department of Mathematics
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IC152: Linear Algebra-II
Tutorial Sheet 1

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1. Find the eigenvalues of zero and identity linear operators on a $n-$dimensional vector space by exhibiting the characteristic polynomials.
   The matrices of zero linear operator and identity linear operator are zero and identity matrices respectively and therefore characteristic polynomials are $x^n = 0$ and $(x-1)^n = 0$ respectively. Thus 0 and 1 are the only eigenvalues for zero and identity linear operators respectively.

2. Suppose $T : V \to V$ be a linear operator on a vector space over a field $\mathbb{F}$ such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.
   As every vector is an eigenvector, we have $T\alpha = f(\alpha)\alpha$, where $f : V \to \mathbb{F}$. It will be sufficient to prove the claim that f is constant. In fact, when α and β are linearly dependent, i.e., for some $c \in \mathbb{F}$, $\alpha = c\beta$, then $f(\alpha)\alpha = T\alpha = T(c\beta) = cT\beta = cf(\beta)\beta = f(\beta)c\beta = f(\beta)\alpha$ which implies $f(\alpha) = f(\beta)$ as $\alpha \neq 0$ being an eigenvector. Next if α and β are linearly independent then $T(\alpha+\beta) = f(\alpha+\beta)(\alpha+\beta) = f(\alpha+\beta)\alpha + f(\alpha+\beta)\beta$. On the other hand $T(\alpha+\beta) = T(\alpha) + T(\beta) = f(\alpha)\alpha + f(\beta)\beta$ which implies $f(\alpha+\beta)\alpha + f(\alpha+\beta)\beta = f(\alpha)\alpha + f(\beta)\beta$ or $(f(\alpha+\beta) - f(\alpha))\alpha + (f(\alpha+\beta) - f(\beta))\beta = 0$ which implies $f(\alpha+\beta) = f(\alpha) = f(\beta)$ as α and β are linearly independent. This proves the claim.

3. Let U and T are linear operators on a finite dimensional vector space V over a field $\mathbb{F}$. Prove that UT and TU have the same eigenvalues. What if V is not of finite dimension?
   Suppose that c is an eigenvalue of UT then there exists some nonzero $\alpha \in V$ such that $UT\alpha = c\alpha$. Note that if $c \neq 0$, then $T\alpha \neq 0$ otherwise $U(T\alpha) = 0$ while $c\alpha \neq 0$, a contradiction. Thus $TU(T\alpha) = T(UT\alpha) = T(c\alpha) = cT\alpha$ implies c is an eigenvalue for TU with eigenvector $T\alpha$. Thus any nonzero eigenvalue of UT is an eigenvalue of TU and vice-versa.
   Next we suppose that $c = 0$. In this case if $T\alpha \neq 0$, we will get 0 as an eigenvalue of TU with eigenvector $T\alpha$ as above. In case when $T\alpha = 0$, we have following two cases.
   - If $\mathrm{Null}(U) = \{0\}$, then U is invertible (as V is finite dimensional) and therefore $U^{-1}\alpha \neq 0$ turns out to be the eigenvector for eigenvalue 0 as $TU(U^{-1}\alpha) = T\alpha = 0\alpha = 0 = 0U^{-1}\alpha$.
   - If $\mathrm{Null}(U) \neq \{0\}$,, then for any $0 \neq \beta \in \mathrm{Null}(U)$, we have $TU\beta = T0 = 0 = 0\beta$ asserts that 0 is an eigenvalue for TU.

This result is not true if $V$is not assumed to be finite dimensional. Consider the following counter example. Take $T, S : \mathbb{F}^\infty \to \mathbb{F}^\infty$ defined as $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$ and $S(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots)$ then ST is the identity and hence has only eigenvalue 1, while TS has 0 as an eigenvalue with eigenvector $(1, 0, 0, \cdot\cdot)$.

4. Find the eigenvalues and eigenvectors of the following operators

   (i) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined as $T(x, y) = (x + y, x)$.

   (ii) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, defined as $T(f(x)) = f(x) + (x + 1)f'(x)$.

   (iii) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, defined as $T(A) = A^t$.

   (iv) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$, defined as $T(x, y) = (y, -x)$.

5. Does $T : \mathcal{C}(\mathbb{R}; \mathbb{R}) \to \mathcal{C}(\mathbb{R}; \mathbb{R})$, where $\mathcal{C}(\mathbb{R}; \mathbb{R})$ denotes the space of continuous real valued functions defined on $\mathbb{R}$, defined as

$$(Tf)(x) = \int_0^x f(t)dt$$

has an eigenvector?

Answer is NO. If possible, assume $f \neq 0$ is an eigenvector of T corresponding to some eigenvalue c then $(Tf)(x) = (cf)(x)$, i.e.,

$$\int_0^x f(t)dt = cf(x).$$

Note that from calculus, the function $F(x) := \int_0^x f(t)dt$ is differentiable and $F'(x) = f(x)$ with $F(0) = 0$. Thus we have

$$cF'(x) = F(x),\ F(0) = 0.$$

If $c = 0$, we get $F(x) = 0$ and $f(x) = F'(x) = 0$, a contradiction. If $c \neq 0$, 0n solving the IVP, we get $F(x) = ae^{x/c}$ for some constant a to be evaluated using initial condition $F(0) = 0$ which implies $a = 0$ and thus $F(x) = 0$ leading to $f(x) = F'(x) = 0$, a contradiction.

6. Let T be a linear operator on a vector space over a filed $\mathbb{F}$ with $T\alpha = c\alpha$ for some $c \in \mathbb{F}$, then for any polynomial f over the field F, $f(T)\alpha = f(c)\alpha$.

   Proof follows by observing the following. If T has an eigenvector α corresponding to an eigenvalue λ, then for any positive integer k and $c \in \mathbb{F}$, α is an eigenvector for T^k and (cT) as well corresponding to the eigenvalue λ^k and $c\lambda$ respectively.

7. Define determinant and trace of a linear operator on a finite dimensional vector space. Justify that your definitions are well defined.

   Let $\dim(V) = n$ and $T : V \to V$ be a linear operator. Then relative to an ordered basis

$\mathcal{B}$ of V, T can be expressed by a matrix of the size $n \times n$, denoted as $[T]_{\mathcal{B}}$. Thus we can define $\det(T) := \det[T]_{\mathcal{B}}$ and $tr(T) := tr[T]_{\mathcal{B}}$. We claim that these two definitions are independent of choices of underlying ordered bases and hence well defined. The claim follows as for any two ordered bases $\mathcal{B}$ and $\mathcal{B}'$ of V, $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ are similar matrices and similar matrices have the equal determinant and trace.

8. Let $A \in M_{n\times n}(\mathbb{R})$ with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots \cdot a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Is it possible for an invertible matrix to have any of its eigen value equal to 0?

Note that $f(t) = \det(A - tI)$. Thus $a_0 = f(0) = \det(A)$. No, as roots (eigenvalues) of f(t), say λ_i satisfy

$$\Pi_{i=1}^n \lambda_i = \frac{a_0}{(-1)^n}$$

which implies $\det A = (-1)^n \Pi_{i=1}^n \lambda_i$ will be zero if any of the λ_i is zero giving a contradiction for an invertible matrix.