Lecture #8 (IC152)

Recall: (minimal polynomial) $T: V \to V$, $\dim V < \infty$, F

- monic least degree annihilating polynomial of T
$$\downarrow$$
$$p(T) = 0,$$

Notation: minimal polynomial as $m(x) \in P(\mathbb{F})$

Theorem:- Let $T: V \longrightarrow V$, $\dim V = n$, linear operator on a vector space V. Then, the characteristic polynomial & minimal polynomial of T have the same roots except multiplicities.

Proof: Assume that $m(x)$ is a minimal polynomial.
Let $p(x)$ be the characteristic polynomial of T and 'c' is an eigenvalue of T, $p(c) = 0$
if $m(c) = 0 \Rightarrow c$ is a root of minimal polynomial.

If f is a polynomial

As $T\alpha = c\alpha$ for some $\alpha \neq 0$
(As c is an eigen value of T)

If f is a poly... & $T\alpha = c\alpha$ $\Rightarrow f(T)\alpha = f(c)\alpha$

$$\Rightarrow m(T)\alpha = m(c)\alpha$$

$$\Rightarrow 0 = m(c)\alpha$$

As $\alpha \neq 0 \Rightarrow m(c) = 0$

Conversely assume that 'c' is a root of $m(x)$
then $p(c) = 0$?

Since 'c' is a root of m,

$$m(x) = (x-c)\,q(x)$$

with degree of $q <$ deg m

Is $q(T) = 0$? No as it will contradict that $m(x)$ is minimal.

Thus $q(T) \neq 0 \Rightarrow \exists\ \beta \in V$ s.t $q(T)\beta \neq 0$

Let $\underline{\alpha} := q(T)\beta$, $\alpha \neq 0$.

$$m(T)\beta = 0$$

$$(T-cI)q(T)\beta = 0$$

$$\Rightarrow (T-cI)\alpha = 0 \quad \text{for } \alpha \neq 0$$

$$\Rightarrow T\alpha = c\alpha \quad \text{for } \alpha \neq 0$$

$\Rightarrow$ c is an eigenvalue with α an eigenvector.

$\Rightarrow$ $p(c) = 0$.

Let T be diagonalizable linear operator on a finite dimensional vector space.

$\boxed{\text{Let } c_1, c_2, \dots, c_k \text{ be the distinct}}$ eigenvalues of T, then

Claim: $m(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$

$T: V \to V$

$\ast$ $m(T) = 0$

$T\alpha_i = ?$

If $m(T)\alpha = 0 \ \forall \alpha \in \mathcal{B}$, basis of V

$\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

Then $m(T) = 0$ $\qquad T\alpha = T(\sum c_i \alpha_i) = \sum c_i T\alpha_i$

$$\Rightarrow \quad m(T)\alpha = (T - c_1 I)(T - c_2 I)\cdots(T - c_k I)\alpha$$

$$= 0 \quad \text{(As } \mathcal{B} \text{ consists of eigenvectors of } T\text{)}$$

$\alpha \to c_i$

$(T - c_i I)\alpha = 0$

$$\Rightarrow \quad m(T) = 0$$

* degree of $m(x)$ is least by previous theorem.

What about converse ?

Converse is also true.

Theorem :- If p be any annihilating polynomial of T ($T : V \to V$, $\dim V < \infty$,) then minimal polynomial $m(x)$ divides p.

Proof. To show: $p(x) = q(x)\, m(x)$!! $q \in P(\mathbb{F})$.

As $\quad p(x) = q(x)\, m(x) + r(x)$, with

$\deg r(x) < \deg m(x)$ or $r(x) = 0$!!

(By polynomial division).

Thus, $p(T) = q(T)\,m(T) + r(T)$

$0 = 0 + r(T) \Rightarrow r(T) = 0$

$\Rightarrow$ $r(x)$ is annihilating polynomial of T of degree less than degree of m. which is a contradiction unless $r(x) = 0$

$\Rightarrow$ $p(x) = q(x)\,m(x)$

or $m(x)$ divides $p(x)$.

<u>Theorem</u>: (Cayley-Hamilton)

Let T be a linear operator on a finite dimensional vector space V. Then characteristic polynomial of T annihilates T, i.e. if $p(x)$ is characteristic polynomial, then $p(T) = 0$.

Or

minimal polynomial of T divides it's characteristic polynomial.

Example:- $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$ ✓ (Diagonalizable)

Char. poly. $(x-3)^2(x-5)$

what will be it's minimal polynomial

$m(x) = (x-3)(x-5)$?

* $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (Not diagonalizable)

Char. polynomial $(x-3)^2(x-4)$

minimal polynomial: ✓ $(x-3)(x-4)$ ✓?

or ✓ $(x-3)^2(x-4)$. ✓

(Cayley-Hamilton)

$(x-3)^2(x-4)(x-5)^2$

$(x-3)(x-4)(x-5)$

$(x-3)^2(x-4)(x-5)$

* $(x-3)(x-4)(x-5)^2$

$A^2 = 3A$ ✓

$A^2 - 3A = 0$

$* \ (x-3)(x-4)(x-5)$ ✓

$p(x) = (x-3x)$ is ... of degree '2'

$= x(x-3)$

$A \neq 3I$

A

As $m(x) / p(x)$,

$m(x) = x$, or $m(x) = x-3$ or $\underline{x(x-3)}$

$m(A) = 0 \Rightarrow A = 0$

if yes

$m(x) = x$,

$A = 3I$ ✓

if yes

$m(x) = x-3$

if not

$$\underline{\underline{m(x) = x(x-3)}}$$