Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Quiz-II

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1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ be a real matrix. Then which of the following is NOT correct

   (a) Eigenvalues of A are 2, 2

   (b) A is diagonalizable

   (c) minimal polynomial and characteristic polynomial of A are same.

   (d) minimal polynomial of A is of degree 1

   Option (c) is the answer as minimal polynomial is $x - 2$ which is one degree. The options (a) and (b) are abviously not the answers as 2, 2 are eigenvalues of A and A is diagonal matrix hence diagonalizable.

2. Let $f(T) = 0$ for any polynomial $f \in P(\mathbb{F})$ and $T \in L(V, V)$, $\dim V(\mathbb{F}) < \infty$. Assume $p(x)$ and $m(x)$ $(\neq f(x))$ be the characteristic and minimal polynomial for T respectively. Then which of the following statements are correct

   (a) roots of $f(x)$ and $m(x)$ are same except multiplicity

   (b) roots of $m(x)$ and $p(x)$ are same except multiplicity

   (c) degree of $m(x) \leq$ degree of $f(x)$

   (d) degree of $m(x) \leq$ degree of $p(x)$.

   Options $(b), (c)$ and (d) are correct. Counter example for (a) is: I, (identity operator) satisfies $f(x) = x^2 - x$ which has roots 0 and 1 while the minimal polynomial for I is $m(x) = x - 1$ which does not have 0 as a root.

3. Answer the following.

   (a) Find the minimal polynomial for the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

   $$T(x, y, z) = (2x + y, 2y, 2z)$$

   (b) Is T diagonalizable? Justify your answer.

(a) The matrix of T relative to the standard ordered basis of $\mathbb{R}^3$ namely $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The characteristic polynomial for T is $(x-2)^3$. Thus minimal polynomial has three choices, $x-2$, $(x-2)^2$ and $(x-2)^3$. As $[T]_{\mathcal{B}} \neq 2I$ and $([T]_{\mathcal{B}} - 2I)^2 = 0$, we have minimal polynomial as $(x-2)^2$.

(b) T is not diagonalizable as minimal polynomial is not the product of distinct linear factors.

4. Let T be a linear operator on a finite dimensional vector space satisfying $T^3 = T$. Show that T is diagonalizable.

Observe that $f(x) := x^3 - x = x(x-1)(x+1)$ is an annihilating polynomial for T. As minimal polynomial divides any annihilating polynomial, the choices for minimal polynomial are $x, x-1, x+1, x(x-1), x(x+1), (x-1)(x+1), x(x-1)(x+1)$. It is obvious to see that minimal polynomial is a product of distinct linear factors for all the above choices and hence T must be diagonailzable.

5. Find out Hermitian matrices B and C such that the following matrix A can be written as $A = B + iC$

$$A = \begin{bmatrix} 1+i & 2 \\ 2+i & 1-i \end{bmatrix}.$$

We know (From Tutorial 3, Problem 4) that choices are $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$. Thus $B = \begin{bmatrix} 1 & 2-i/2 \\ 2+i/2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1/2 \\ 1/2 & -1 \end{bmatrix}$.