IC105: Probability and Statistics **2021-22-M**

Lecture 14: Normal (Gaussian) Distribution

Instructor: Dr. Kuldeep Kumar Kataria *Scribe:*

14.0.1. Gamma and Related Distributions

Gamma Function: $\Gamma : (0,\infty) \to (0,\infty)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t, \quad \alpha > 0.$$

It converges for any $\alpha > 0$. Integration by parts yields $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\alpha > 0$ and $\Gamma(1) = 1$. For any $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)!, \quad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} \mathrm{d}t = 2\int_0^\infty e^{-x^2} \mathrm{d}x$$

This implies

$$\begin{aligned}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 &= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\mathrm{d}y \\ &= 4\int_0^\infty \int_0^{\pi/2} re^{-r^2} \mathrm{d}\theta\mathrm{d}r, \quad (x = r\cos\theta,\ y = r\sin\theta) \\ &= \pi \implies \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\end{aligned}$$

Also,

$$\begin{aligned}&\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},\ \ \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1\cdot 3\sqrt{\pi}}{2^2}, \\ &\Gamma\left(\frac{2n+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n} = \frac{(2n)!}{n!4^n}\sqrt{\pi},\ \ n \in \mathbb{N}.\end{aligned}$$

Clearly,

$$\int_0^\infty e^{-x/\theta} x^{\alpha-1} \mathrm{d}x = \theta^\alpha \Gamma(\alpha),\ \ \alpha > 0,\ \theta > 0.$$

Definition 14.1. *A r.v. X is said to have a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$ (written as $X \sim GAM(\alpha,\theta)$ if its p.d.f. is given by*

$$f(x|\alpha,\theta) = \begin{cases} \dfrac{e^{-x/\theta}x^{\alpha-1}}{\theta^\alpha\Gamma(\alpha)}, & x > 0, \\ 0, & \textit{otherwise}. \end{cases} \to \textit{family of distributions } \{GAM(\alpha,\theta), \alpha > 0, \theta > 0\}.$$

Let $X \sim GAM(\alpha, \theta) \implies \dfrac{X}{\theta} \sim GAM(\alpha, 1)$ (θ is called scale parameter since the distribution of $\dfrac{X}{\theta}$ does not depend on θ). The p.d.f. of $Z \sim GAM(\alpha, 1)$ is $f(z) = \begin{cases} \dfrac{e^{-z}z^{\alpha-1}}{\Gamma(\alpha)}, & z > 0, \\ 0, & \text{otherwise.} \end{cases}$

Also,

$$\begin{aligned} E(Z^r) = \frac{1}{\Gamma(\alpha)} \int_0^\infty z^{\alpha+r-1}e^{-z}\mathrm{d}z &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}, \ \ r > -\alpha, \ \alpha > 0, \\ &= \alpha(\alpha+1)\cdots(\alpha+r-1), \ \text{ if } \ r \in \mathbb{N}. \end{aligned}$$

$$\begin{aligned} &\text{Mean} = \mu_1' = E(X) = \alpha\theta, \ \ \mu_2' = E(X^2) = \alpha(\alpha+1)\theta^2, \ \ \mu_2 = \sigma^2 = \mathrm{Var}(X) = \alpha\theta^2, \\ &\mu_3 = E((X-\mu_1')^3) = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = 2\alpha\theta^3, \\ &\mu_4 = E((X-\mu_1')^4) = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4 = 3\alpha(\alpha+2)\theta^4, \\ &\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \frac{2}{\sqrt{\alpha}}, \ \ \text{Kurtosis} = \nu_1 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{6}{\alpha}. \end{aligned}$$

For $0 < \alpha \le 1$, $f(x|\alpha,\theta) \downarrow$ and for $\alpha > 1$, $f(x|\alpha,\theta) \uparrow$ in $(0, (\alpha-1)\theta)$ and $\downarrow$ in $((\alpha-1)\theta, \infty)$.

$$\begin{aligned} \text{m.g.f. } M_X(t) &= E(e^{tX}) = E(e^{t\theta Z}), \quad (Z = X/\theta) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{t\theta z}e^{-z}z^{\alpha-1}\mathrm{d}z = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1-t\theta)z}z^{\alpha-1}\mathrm{d}z = (1-t\theta)^{-\alpha}, \ \ t < \frac{1}{\theta}. \end{aligned}$$

Theorem 14.2. *Let $X_1, X_2, \ldots, X_k$ be independent r.v.'s such that $X_i \sim GAM(\alpha_i, \theta)$, for some $\alpha_i > 0$, $\theta > 0$, $i = 1, 2, \ldots, k$. Then $Y = \sum_{i=1}^k X_i \sim GAM(\sum_{i=1}^k \alpha_i, \theta)$.*

Proof. Note that

$$M_Y(t) = \prod_{i=1}^k M_{X_i}(t) = \prod_{i=1}^k (1-t\theta)^{-\alpha_i} = (1-t\theta)^{-\sum_{i=1}^k \alpha_i}, \ \ t < \frac{1}{\theta} = \text{m.g.f. of } GAM(\sum_{i=1}^k \alpha_i, \theta).$$

This completes the proof. ☐

Theorem 14.3 (Relationship beween Gamma and Poisson distribution)**.** *For $n \in \mathbb{N}$, $\theta > 0$ and $t > 0$, let $X \sim GAM(n, \theta)$ and $Y \sim Po(t/\theta)$. Then $P(X > t) = P(Y \le n-1)$, i.e.*

$$\frac{1}{(n-1)!\theta^n} \int_t^\infty e^{-x/\theta}x^{n-1}\mathrm{d}x = \sum_{j=0}^{n-1} \frac{e^{-t/\theta}(t/\theta)^j}{j!}.$$

Proof. Use integration by parts. ☐

Remark 14.4. *For $n \in \mathbb{N}$ and $\theta > 0$, let $X \sim GAM(n, \theta)$. Then*

$$\sum_{j=n}^\infty \frac{e^{-X/\theta}(X/\theta)^j}{j!} \sim U(0,1) \ \ \textit{and} \ \ \sum_{j=0}^{n-1} \frac{e^{-X/\theta}(X/\theta)^j}{j!} \sim U(0,1) \ \ \ (U \sim U(0,1) \implies 1 - U \sim U(0,1)).$$

Definition 14.5. *For a $\theta > 0$, a $GAM(1, \theta)$ distribution is called exponential distribution with scale parameter θ (denoted by $Exp(\theta)$).*

The p.d.f. of $T \sim Exp(\theta)$ is given by

$$f_T(t) = \begin{cases} \frac{1}{\theta} e^{-t/\theta}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

and its d.f. is given by

$$F_T(t) = P(T \leq t) = 1 - P(T > t) = \begin{cases} 0, & t \leq 0, \\ 1 - e^{-t/\theta}, & t > 0. \end{cases}$$

Mean= $E(T) = \theta$, variance= θ^2, $\mu_r' = E(T^r) = r!\theta^r$, $r \in \mathbb{N}$, coefficient of skewness=$\beta_1 = 2$, Kurtosis=$\nu_1 = 9$. M.g.f.=$M_T(t) = (1 - t\theta)^{-1}$, $t < 1/\theta$ and

$$P(T > t) = \begin{cases} 1, & t \leq 0, \\ e^{-t/\theta}, & t > 0. \end{cases}$$

For $s > 0, t > 0$

$$P(T > s + t | T > s) = \frac{P(T > s + t)}{P(T > s)} = e^{-t/\theta} = P(T > t)$$
$$\implies P(T > s + t) = P(T > s)P(T > t), \ \forall\, s, t > 0 \to \text{Lack of Memory Property.}$$

Let T denote the lifetime of a system. Given that the system has survived $s(> 0)$ units of time the probability that it will survive t additional units of time is the same as the probability that a fresh system (of age 0) will survive t units of time. In other words, the system has no memory of its current age or it is not ageing with time.

Theorem 14.6. *Let Y be a r.v. of continuous type with d.f. F such that $F(0) = 0$. Then Y has LoM property (i.e. $\bar{F}(s + t) = \bar{F}(s)\bar{F}(t)\ \forall\, s, t > 0$, where $\bar{F} = 1 - F$) iff $Y \sim Exp(\theta)$, for some $\theta > 0$.*

Proof. Let $Y \sim Exp(\theta)$, $\theta > 0$. Then Y has LoM property (already discussed). Now suppose that $F(0) = 0$ and Y has LoM property. Then

$$\bar{F}(s + t) = \bar{F}(s)\bar{F}(t)\ \forall\, s, t > 0,$$
$$\implies \bar{F}(s_1 + s_2 + \cdots + s_m) = \bar{F}(s_1)\bar{F}(s_2)\cdots\bar{F}(s_m), \quad s_i > 0, \ i = 1, 2, \ldots, m,$$
$$\implies \bar{F}\left(\frac{m}{n}\right) = \bar{F}\left(\underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}\right) = \left[\bar{F}\left(\frac{1}{n}\right)\right]^m \ \forall\, m, n \in \mathbb{N}, \tag{14.1}$$
$$\implies \bar{F}(1) = \left[\bar{F}\left(\frac{1}{n}\right)\right]^n \ \forall\, n \in \mathbb{N}, \tag{14.2}$$
$$\implies \bar{F}\left(\frac{m}{n}\right) = \left[\bar{F}(1)\right]^{m/n} \ \forall\, m, n \in \mathbb{N}. \tag{14.3}$$

Let $\lambda = \bar{F}(1)$ so that $0 \leq \lambda \leq 1$.

$$\lambda = 0 \implies \bar{F}\left(\frac{1}{n}\right) = 0\ \forall\, n \in \mathbb{N} \text{ (using 14.2)} \implies \bar{F}(0) = 0 \implies F(0) = 1 \ \text{(contradiction, since } F(0) = 0)$$

$$\lambda = 1 \implies \bar{F}(m) = \left[\bar{F}(1)\right]^m = 1\ \forall\, m \in \mathbb{N} \implies \lim_{m\to\infty} \bar{F}(m) = 1 \implies \lim_{m\to\infty} F(m) = 0 \ \to \text{contradiction.}$$

Thus $\lambda \in (0, 1)$. Let $\lambda = e^{-1/\theta}$, $\theta > 0$ ($\theta = -1/\ln\lambda$). Then using (14.3), $\bar{F}(r) = e^{-r/\theta}\ \forall\, r \in IQ \cap (0, \infty)$. Let $x \in IQ \cap (0, \infty)$. Then there exists a sequence $\{r_n\}_{n\geq 1}$ in $IQ \cap (0, \infty)$ such that $r_n \to x$. Then

$$\bar{F}(x) = \bar{F}\left(\lim_{n\to\infty} r_n\right) = \lim_{n\to\infty} \bar{F}(r_n) = \lim_{n\to\infty} e^{-r_n/\theta} = e^{-x/\theta},$$
$$\implies F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-x/\theta}, & x \geq 0, \end{cases} \implies Y \sim Exp(\theta).$$

This completes the proof. $\square$

Example 14.7. *X : Waiting time for occurrence of an event E. Suppose that $X \sim Exp(3)$. Then the conditional probability that the waiting time for occurrences of E is atleast 5 hrs given that it has not occurred in first two hrs* $=P(X > 5|X > 2) = P(X > 3) = e^{-1}$.

Chi-squared Distribution: Let $n \in \mathbb{N}$. Then $GAM\left(\frac{n}{2}, 2\right)$ distribution is called Chi-squared distribution with n degrees of freedom (denoted by χ^2_n). Let $X \sim \chi^2_n$. The p.d.f. of X is

$$f_X(x) = \begin{cases} \dfrac{e^{-x/2}x^{n/2-1}}{2^{n/2}\Gamma(n/2)}, & x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Mean = $E(X) = n$, $\text{Var}(X) = \mu_2 = \sigma^2 = 2n$, coefficient of skewness = $\beta_1 = 2\sqrt{\frac{2}{n}}$, Kurtosis = $\nu_1 = 3 + \frac{12}{n}$, m.g.f. $M_X(t) = (1-2t)^{-n/2}$, $t < \frac{1}{2}$.

Theorem 14.8. *Let $X_1, X_2, \ldots, X_k$ be independent with $X_i \sim \chi^2_{n_i}$, $n_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$. Then $\sum_{i=1}^k X_i \sim \chi^2_n$, where $n = \sum_{i=1}^k n_i$.*

For various values of $n \in \mathbb{N}$ and $\alpha \in (0,1)$, tables for $(1-\alpha)$th quantile of χ^2_n distribution (*i.e.* $\tau_{n,\alpha}$ satisfying $P(\chi^2_n \leq \tau_{n,\alpha}) = 1-\alpha$) are available in various textbook.

14.0.2. Beta Distribution

For $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty e^{-(s+t)} s^{\alpha-1} t^{\beta-1} \mathrm{d}s\mathrm{d}t \\
&= \int_0^1 \int_0^\infty e^{-v} (uv)^{\alpha-1} ((1-u)v)^{\beta-1} |v| \mathrm{d}v\mathrm{d}u, \\
&\qquad\qquad \text{making transformation: } s = uv,\ t = (1-u)v, \text{ Jacobian : } J = v \\
&= \Gamma(\alpha+\beta) \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} \mathrm{d}u
\end{aligned}$$

$$\implies \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u \to \text{Beta function (function of } (\alpha,\beta),\ \alpha > 0,\ \beta > 0).$$

Note that $B(\alpha, \beta) = B(\beta, \alpha)$, $\forall\ \alpha,\ \beta > 0$.

Definition 14.9. *For given $\alpha > 0$ and $\beta > 0$, a r.v. X is said to have the beta distribution with parameter (α, β) (written as $X \sim Be(\alpha, \beta)$) if its p.d.f. is given by*

$$f(x|\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & 0 < x < 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Suppose that $X \sim Be(\alpha, \beta))$, for some $\alpha > 0$ and $\beta > 0$. Then

$$\begin{aligned}
E(X^r) &= \frac{B(\alpha+r,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+r)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+r)}, \quad r > -\alpha, \\
\text{Mean} &= \mu_1' = E(X) = \frac{\alpha}{\alpha+\beta}, \quad \mu_2' = E(X^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}, \\
\mu_2 &= \sigma^2 = \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}, \\
\text{Mode} &= M_0 = \frac{\alpha-1}{\alpha+\beta-2}, \quad \text{if } \alpha > 1 \text{ and } \alpha+\beta > 2, \\
\text{Skewness} &= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{\sqrt{\alpha\beta}(\alpha+\beta+2)}, \\
\text{Kurtosis} &= \nu_1 = \frac{\mu_4}{\mu_2^2} = \frac{6[(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)} + 3 \\
&\qquad = \frac{6[\alpha^3+\alpha^2(1-2\beta)+\beta^2(1+\beta)-2\alpha\beta(2+\beta)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}.
\end{aligned}$$

Let $X \sim Be(\alpha, \alpha)$, $\alpha > 0$. Then

$$f(x|\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\alpha-1}}{B(\alpha,\alpha)}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $X \stackrel{d}{=} 1 - X \implies X - \frac{1}{2} \stackrel{d}{=} \frac{1}{2} - X$. Thus, if $X \sim Be(\alpha, \alpha)$. Then the distribuion of X is symmetric about $1/2$.

Theorem 14.10 (Relationship between Beta and Binomial Distribution)**.** *For $m, n \in \mathbb{N}$ and $x \in (0, 1)$, let $X \sim Be(m, n)$ and $Y \sim Bin(m + n - 1, x)$. Then $P(X \le x) = P(Y \ge m)$, i.e.*

$$\frac{1}{B(m,n)} \int_0^x t^{m-1}(1-t)^{n-1}\mathrm{d}t = \sum_{j=m}^{m+n-1} \binom{m+n-1}{j} x^j (1-x)^{m+n-1-j}.$$

Proof. Fix $m, n \in \mathbb{N}$ and $x \in (0, 1)$. Let

$$\begin{aligned}
I_{m,n} = LHS &= \frac{(m+n-1)!}{(m-1)!(n-1)!} \int_0^x t^{m-1}(1-t)^{n-1}\mathrm{d}t \\
&= \binom{m+n-1}{m} x^m (1-x)^{n-1} + \frac{(m+n-1)!}{m!(n-2)!} \int_0^x t^m (1-t)^{n-2}\mathrm{d}t \\
&= \binom{m+n-1}{m} x^m (1-x)^{n-1} + I_{m+1,n-1}.
\end{aligned}$$

Proceeding recursively give the result. □

$$\begin{aligned}
\text{m.g.f. } M_X(t) = E(e^{tX}) &= \frac{1}{B(\alpha,\beta)}\int_0^1 e^{tx}x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{1}{B(\alpha,\beta)}\int_0^1 \left(\sum_{j=0}^{\infty}\frac{t^j x^j}{j!}\right) x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x \\
&= \frac{1}{B(\alpha,\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^1 x^{j+\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{1}{B(\alpha,\beta)}\sum_{j=0}^{\infty}\frac{B(\alpha+j,\beta)t^j}{j!}, \quad t\in\mathbb{R}.
\end{aligned}$$

Example 14.11. *Time (in hours) to finish a job follows beta distribution with mean $\frac{1}{3}$ hrs. and variance $\frac{2}{63}$ hrs. Find the probability that the job will be finished in 30 minutes.*

Solution: Define $X =$ time to finish job (in hours) $\sim Be(\alpha,\beta)$, say.

$E(X) = \frac{1}{3} \implies \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$, $\mathrm{Var}(X) = \frac{2}{63} \implies \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2}{63}$. This implies $\alpha = 2$ and $\beta = 4$. Thus, $X \sim Be(2,4)$. Required probability

$$P(X < \frac{1}{2}) = \frac{1}{B(2,4)}\int_0^{1/2} x(1-x)^3\mathrm{d}x = \frac{13}{16}.$$

14.0.3. Normal Distribution

Recall that

$$\begin{aligned}
\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-t}t^{-1/2}\mathrm{d}t &= 2\int_0^{\infty} e^{-x^2}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\mathrm{d}t \\
\implies \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\mathrm{d}t &= 1 \\
\implies \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t-\mu)^2/2\sigma^2}\mathrm{d}t &= 1 \ \forall\, \mu\in\mathbb{R} \text{ and } \sigma > 0.
\end{aligned}$$

Definition 14.12. *Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be given constants. An absolutely continuous type r.v. is said to follow a normal distribution with patrameters $\mu \in \mathbb{R}$ and $\sigma > 0$ (written as $X \sim N(\mu,\sigma^2)$) if its p.d.f. is given by*

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \ -\infty < x < \infty.$$

The $N(0,1)$ distribution is called standard normal distribution. The p.d.f. and d.f. of a standard normal distribution are denoted by $\phi(z)$ and $\Phi(z)$, respectively, so that

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, \ -\infty < z < \infty, \ \Phi(z) = \int_{-\infty}^{z}\phi(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\mathrm{d}t, \ z\in\mathbb{R}.$$

$$X \sim N(\mu,\sigma^2) \implies f(\mu-x|\mu,\sigma) = f(\mu+x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}, \ x\in\mathbb{R}.$$

This implies $X-\mu \overset{d}{=} \mu - X$ (distribution of X is symmetric about μ) $\implies$ $E(X)=\mu$ and $F(\mu|\mu,\sigma)=\frac{1}{2}$. Moreover,

$$P(X-\mu \le x) = P(\mu - X \le x) \implies F(\mu+x|\mu,\sigma) = 1 - F(\mu - x|\mu,\sigma)\ \forall\, x \in \mathbb{R}.$$

In particular,

$$\Phi(0) = \frac{1}{2} \text{ and } \Phi(-z) + \Phi(z) = 1\ \forall\, z \in \mathbb{R}.$$

The p.d.f. $f(x|\mu,\sigma) \uparrow$ in $(-\infty,\mu)$ and $\downarrow$ in (μ,∞) $\implies$ mode = $m_0 = \mu$. Thus mean = median = mode = μ.

Let $X \sim N(\mu,\sigma^2)$. Then m.g.f. of X is

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x, \text{ take } \frac{x-\mu}{\sigma} = z,\ \ x = (\mu+\sigma z) \\ &= \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2\sigma t z + \sigma^2 t^2)} \mathrm{d}z \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} \mathrm{d}z = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad t \in \mathbb{R}. \end{aligned}$$

· Let $X \sim N(\mu,\sigma^2)$. Then $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$.

$$M_Z(t) = E(e^{t\frac{(X-\mu)}{\sigma}}) = e^{-\mu t/\sigma} M_X(t/\sigma) = e^{-\mu t/\sigma} e^{\mu t/\sigma + t^2/2} = e^{t^2/2}\ \forall\, t \in \mathbb{R} \rightarrow \text{m.g.f. of } N(0,1)$$
$$\implies Z \sim N(0,1).$$

Theorem 14.13. *Let $X \sim N(\mu,\sigma^2)$ and $Z \sim N(0,1)$.*

(a) For $a \neq 0$, $b \in \mathbb{R}$, $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$.

(b) $Z \overset{d}{=} \frac{X-\mu}{\sigma} \sim N(0,1)$.

(c)

$$E(Z^r) = \begin{cases} 0, & \text{if } r = 1,3,5,\ldots, \\ \frac{r!}{2^{r/2}(r/2)!}, & \text{if } r = 2,4,6,\ldots. \end{cases}$$

(d) Mean = $\mu_1' = E(X) = \mu$; Variance = $\mu_2 = \sigma^2$; coefficient of skewness = $\beta_1 = 0$; kurtosis = $\nu_1 = 3$.

(e) $Z^2 \sim \chi_1^2$.

Proof. (a) Note that

$$\begin{aligned} M_Y(t) = E(e^{tY}) &= E\left(e^{t(aX+b)}\right) \\ &= e^{bt} E\left(e^{(ta)X}\right) = e^{bt} M_X(at) \\ &= e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2 (at)^2} \\ &= e^{(a\mu+b)t + \frac{1}{2}(a^2\sigma^2)t^2}, \ \ t \in \mathbb{R} \implies Y \sim N(a\mu+b, a^2\sigma^2). \end{aligned}$$

(b) Follows from (a) by taking $a = \frac{1}{\sigma}$ and $b = -\frac{\mu}{\sigma}$.

(c) $M_Z(t) = e^{t^2/2} = \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k k!}$, $t \in \mathbb{R}$.

$$E(Z^r) = \text{ Coefficient of } \frac{t^r}{r!} \text{ in the expansion of } M_Z(t) = \begin{cases} 0, & \text{if } r = 1, 3, 5, \ldots, \\ \frac{r!}{2^{r/2}(r/2)!}, & \text{if } r = 2, 4, 6, \ldots. \end{cases}$$

(d) $Z \stackrel{d}{=} \frac{X-\mu}{\sigma}$.

$$E\left(\frac{X-\mu}{\sigma}\right) = E(Z) = 0 \implies \mu_1' = E(X) = \mu,$$
$$E\left(\left(\frac{X-\mu}{\sigma}\right)^2\right) = E(Z^2) = 1 \implies \mu_2 = E((X-\mu)^2) = \sigma^2,$$
$$E\left(\left(\frac{X-\mu}{\sigma}\right)^3\right) = E(Z^3) = 0 \implies \mu_3 = E((X-\mu)^3) = 0,$$
$$E\left(\left(\frac{X-\mu}{\sigma}\right)^4\right) = E(Z^4) = 3 \implies \mu_4 = 3\sigma^4 = 3,$$
$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3}{\mu_2^2} = 0, \text{ kurtosis} = \frac{\mu_4}{\mu_2^2} = 3.$$

(e) Let $Y = Z^2$. Then

$$M_Y(t) = E(e^{tZ^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-z^2/2} \mathrm{d}z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(1-2t)}{2} z^2} \mathrm{d}z = (1-2t)^{-1/2}, \ t < \frac{1}{2} \implies Z^2 \sim \chi_1^2.$$

This completes the proof. □

Corollary 14.14. *Let $X_1, X_2, \ldots, X_k$ be independent and let $X_i \sim N(\mu_i, \sigma_i^2)$, $-\infty < \mu_i < \infty$, $\sigma_i > 0$, $i = 1, 2, \ldots, k$. Then $\sum_{i=1}^{k} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi_k^2$.*

Remark 14.15. *(i) In $N(\mu, \sigma^2)$ distribution the parameters $\mu \in (-\infty, \infty)$ and $\sigma^2 > 0$ are respectively, the mean and variance of the distribution.*

(ii) If $X \sim N(\mu, \sigma^2)$, then

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right), \ x \in \mathbb{R}.$$

Let τ_α be the $(1-\alpha)$th quantile of Φ then $\Phi(-\tau_\alpha) = 1 - \Phi(\tau_\alpha) = \alpha$. Tables for values of $\Phi(x)$ for different values of x are available in various text books.

Example 14.16. *Let $X \sim N(2, 4)$. Find $P(X \leq 0)$, $P(|X| \geq 2)$, $P(1 < X \leq 3)$ and $P(X \leq 3 | X > 1)$.*

Solution: $P(X \leq 0) = \Phi\left(\frac{0-2}{2}\right) = \Phi(-1) = 1 - \Phi(1) = .1587$,

$$P(|X| \geq 2) = P(X \leq -2) + P(X \geq 2) = \Phi\left(\frac{-2-2}{2}\right) + 1 - \Phi\left(\frac{2-2}{2}\right)$$
$$= \Phi(-2) + 1 - \Phi(0) = 0.0228 + 0.5 = 0.5228,$$

$$P(1 < X \leq 3) = P(X \leq 3) - P(X \geq 1) = \Phi\left(\frac{3-2}{2}\right) + 1 - \Phi\left(\frac{1-2}{2}\right) = 2\Phi(0.5) - 1 = 2 \times 0.6915 - 1 = 0.383,$$

$$P(X \leq 3 | X > 1) = \frac{P(1 < X \leq 3)}{P(X > 1)} = \frac{.383}{1 - \Phi\left(\frac{1-2}{2}\right)} = \frac{0.383}{\Phi(0.5)} = 0.55599.$$

Theorem 14.17. *Let $X_1, X_2, \ldots, X_k$ be independent r.v.'s and let $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, \ldots, k$. Let $a_1, a_2, \ldots, a_k$ be real constants such that $\sum_{i=1}^k a_i^2 > 0$. Then $Y = \sum_{i=1}^k a_i X_i \sim N(\sum_{i=1}^k a_i \mu_i, \sum_{i=1}^k a_i^2 \sigma_i^2)$.*

Proof. Note that

$$
\begin{aligned}
M_Y(t) = E(e^{t\sum_{i=1}^k a_i X_i}) = E\left(\prod_{i=1}^k e^{t a_i X_i}\right) &= \prod_{i=1}^k E(e^{t a_i X_i}), \quad \text{(independent of } X_i\text{'s)} \\
&= \prod_{i=1}^k M_{X_i}(t a_i) = \prod_{i=1}^k e^{\mu_i t a_i + \frac{1}{2}\sigma_i^2 t^2 a_i^2} = e^{(\sum_{i=1}^k a_i \mu_i)t + \frac{(\sum_{i=1}^k a_i^2 \sigma_i^2)t}{2}} \\
&\to \text{m.g.f. of } N\left(\sum_{i=1}^k a_i \mu_i, \sum_{i=1}^k a_i^2 \sigma_i^2\right).
\end{aligned}
$$

By uniqueness of m.g.f.'s $Y \sim N(\sum_{i=1}^k a_i \mu_i, \sum_{i=1}^k a_i^2 \sigma_i^2)$. $\square$