Operations Research (MA 605)

Reference: (a) Operations Research An Introduction by Hamdy A. Taha

(b) An Introduction to optimization Edwin K. P. Chong & S. H. Zak

Introduction The first formal activities of Operations Research (OR) were initiated in England during world war II, when a team of British scientists set out to make scientifically based decisions regarding the best utilization of war material. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in civilization sector.

The principal phases for implementing OR in practice include the following stages:

1. construction of the model / formulation of the problem
2. solution of the model
3. Implementation

Construction of the model

① Consider forming a maximum area rectangle out of a piece of wire of length L inches. What should be the best width and height of the rectangle?

Let w be the width of the rectangle and h be the height of the rectangle in inches.

Based on the definition, the restrictions of the situation can be expressed as

ⓐ width of the rectangle + height of the rectangle = half the length of the wire

ⓑ width and height cannot be negative

These restrictions are transformed algebrically to

ⓐ $2(w+h) = L$

ⓑ $w, h \geq 0$

The only remaining component of the modeling is now the objective of the problem, namely maximization of the area of the rectangle. Let z be the area of the rectangle.

Then the complete model becomes

maximize $z = wh$

subject to $2(w+h) = L, \; w, h \geq 0$

Based on the preceding example, the general OR model can be organized in the following general format :

minimize or maximize objective function

subject to constraints

A solution of the model is feasible if it satisfies all the constraints. It is optimal if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function. In rectangle problem, a feasible alternative must satisfy the conditions $w+h = \frac{L}{2}$ where w and h are nonnegative variables. This definition leads to an infinite number of feasible solutions and the optimal solution is $w = h = \frac{L}{4}$.

Solving the OR model

The most prominent OR technique is linear programming. It is designed for models with linear objective and constraint functions. Other techniques include integer programming, dynamic programming, network program and nonlinear programming.

Linear Programming Problems

Let us begin our discussion of Linear Programming with the following example.

Product mix problem

Consider a small manufacturer making two products A and B. Two resources R_1 and R_2 are required to make these products. Each unit of product A requires 1 unit of R_1 and 3 units of R_2. Each unit of product B requires 1 unit of R_1 and 2 units of R_2. The manufacturer has 5 units of R_1 and 12 units of R_2 available. The manufacturer also makes a profit of Rs 6 per unit of product A sold and Rs 5 per unit of product B sold. The manufacturer wants to determine the number of units of product A and B to produce that maximizes the total profit.

The mathematical model for the problem

Let x be number of units of product A to be produced and y be the no. of units of prod. B to be produced.

Let Z represents the total profit of the company. Then the objective of the model is

$$\max Z = 6x + 5y$$

that is manufacturer would determine x and y such that this function Z has a maximum value.

The requirement of the resources R_1 and R_2 are $x + y$ and $3x + 2y$ resp. and the manufacturer has to ensure that these are available. Thus

$$x + y \leq 5 \quad \text{and} \quad 3x + 2y \leq 12$$

Also, it is necessary that $x, y \geq 0$

Thus the model is

$$\max Z = 6x + 5y$$

$$\text{s.t} \quad x + y \leq 5 \qquad (?)$$

$$3x + 2y \leq 12, \quad x, y \geq 0$$

Any values of x and y that satisfy the constraints constitute a feasible solution. Otherwise the solution is infeasible. For instance, $x = 1, y = 1$ — feasible

$x = 5, y = 1$ — infeasible.

The best feasible solution that maximized the total profit is called optimal solution.

Terminology The problem variables x and y are called decision variables and they represent the solution or the output decision from the problem. The profit function that the manufacturer wishes to increase or decrease is called objective function. The conditions matching the resources availability and resources requirement are called constraints. We have also explicitely state that the decision variable should take non negative values. This is true for all LP problem. This is called non-negative restriction.

The problem that we have written down in algebraic form represents the mathematical model is called formulation of the problem. The problem formulation has the following steps :

(a) Identifying the decision variables

(b) writing the objective function

(c) writing the constraints

(4) writing the non-negative restriction.

A linear programming problem has a linear objective functions, linear constraints and the non-negative restrictions.

Prob 1 Reddy Mikks produces both interior and exterior paints from two raw materials M_1 and M_2. The following table provides you the basic data of the problem:

Tons of raw material per ton of

	Exterior paint	Interior paint	maximum availability
M_1	6	4	24
M_2	1	2	6
Profit per ton	5	4	

A survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons. The company wants to determine the optimum product mix of interior and exterior paint that maximizes the benefit.

<u>model</u>

x — exterior

y — interior

$$\max \quad 5x + 4y$$

$$\begin{aligned} \text{s.t} \quad 6x + 4y &\leq 24 \\ x + 2y &\leq 6 \\ y - x &\leq 1 \\ y &\leq 2 \end{aligned} \quad , \; x, y \geq 0$$

Standard form of a L.P.P

The linear function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is known as objective function, which has to optimize.

subjects to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq = \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq = \geq b_2$$
$$\cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq = \geq b_m$$

and $x_1, x_2, \ldots x_n \geq 0$

Let $C = [c_1, c_2 \ldots c_n]$ — price vector

$x = [x_1, \ldots x_n]$ — decision variable

$b = [b_1 \ldots b_m]$ — requirement

If all the constraint of the above problem are equations, then LPP reduced to

$$\text{Optimize } Z = cx$$

$$\text{s.t. } Ax = b, \quad x \geq 0$$

where $A = [a_{ij}]_{i=1\ j=1}^{m\ \ n}$ is a $m \times n$ matrix.

This is the standard form of an L.P.P.

There are various methods of finding the optimal solution of a L.P.P.

1. Geometrical method or graphical method

2. Algebraic method (simplex method)

<u>Feasible solution to a L.P.P</u>

A set of values of the variables which satisfy all constraints and all non negative restriction of the variables is known as feasible solution to the L.P.P.