Lect-4 (IE-200)

Que 1: Prove the virial theorem for elliptical orbit.
(Hint: You will have to calculate K.E & P.E averaged over time period)

Sol^n: The orbital eq^n is

$$\frac{1}{r} = \frac{GMm^2}{L^2}(1+\varepsilon\cos\theta)$$

For elliptical orbit $\quad \frac{GMm^2}{L^2} = \frac{1}{a(1-\varepsilon^2)}$ where 'a' - semi-major axis, $\varepsilon \rightarrow$ eccentricity

with $\quad \varepsilon = \sqrt{1+\frac{2EL^2}{G^2M^2m^3}} \quad$ —(1)

$$\Rightarrow \quad \frac{1}{r} = [a(1-\varepsilon^2)]^{-1}(1+\varepsilon\cos\theta)$$

from (1), $\quad E = \frac{(\varepsilon^2-1)G^2M^2m^3}{2L^2} = \frac{(\varepsilon^2-1)GMm}{2}\cdot\frac{GMm^2}{L^2}$

$$= \frac{\cancel{(\varepsilon^2-1)}GMm}{2}\,\frac{1}{a\cancel{(1-\varepsilon^2)}}$$

$$\boxed{E = -\frac{GMm}{2a}} \quad \text{——— (2)}$$

$P.E = V = -\frac{GMm}{r}$, clearly r is variable for elliptical orbit.

But for P.E averaged over time period (T)

$$\langle V\rangle = \left\langle -\frac{GMm}{r}\right\rangle = -GMm\left\langle\frac{1}{r}\right\rangle = -\frac{GMm\int_0^T \frac{1}{r}\,dt}{T}$$

Now, $\quad \left\langle\frac{1}{r}\right\rangle = \frac{1}{T}\int_0^{2\pi}\frac{m}{L}\frac{r^2}{r}\,d\theta \qquad \left\{\text{Using } L = mr^2\frac{d\theta}{dt}\right\}$

$$= \frac{m}{TL}\int_0^{2\pi} r\,d\theta$$

$$= \frac{m}{TL}\,\frac{L^2}{GMm^2}\int_0^{2\pi}\frac{1}{1+\varepsilon\cos\theta}\,d\theta$$

$$= \frac{L}{TGMm} \frac{2\pi}{\sqrt{1-\varepsilon^2}}$$

Now, $T^2 = \frac{4\pi^2 a^3}{GM}$, so

$$\Rightarrow \langle \frac{1}{r} \rangle = \frac{L}{GMm} \frac{\sqrt{GM}}{a^{3/2}\cancel{(2\pi)}} \frac{\cancel{(2\pi)}}{\sqrt{1-\varepsilon^2}}$$

$$= \cancel{\sqrt{\frac{L^2}{GMm^2}}} \ \cancel{\frac{1}{\sqrt{a(1-\varepsilon^2)}}} \ \frac{1}{a} \Rightarrow \langle \frac{1}{r} \rangle = \frac{1}{a}$$

So, P.E averaged over one time period

$$T = \langle V \rangle = -\frac{GMm}{a} \quad \text{— (3)}$$

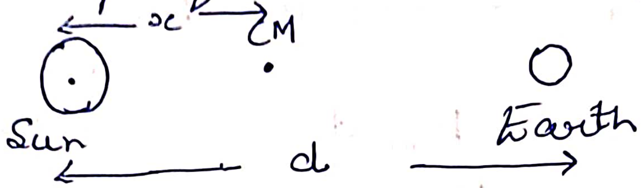
from (1) & (3) $\boxed{E = \frac{\langle V \rangle}{2}}$ (Verial theorem)

IC-200 (Lec-3)

Que 1 Calculate the positions of centre of mass in:
(a) Sun - Earth System (b) Earth - Moon System

Soln Sun - Earth System

Let us shift the frame of reference to centre of sun.



The distance between sun and earth is taken as d and distance of COM from centre of sun be x (say).

We know $M_{sun} \simeq 1.9891 \times 10^{30}$ kg

$M_{earth} \simeq 5.972 \times 10^{24}$ kg

$d = 147.11 \times 10^{6}$ km

The centre of mass is given by

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

As we have shifted origin to centre of Sun $\Rightarrow x_1 = 0$ and $x_2 = d$

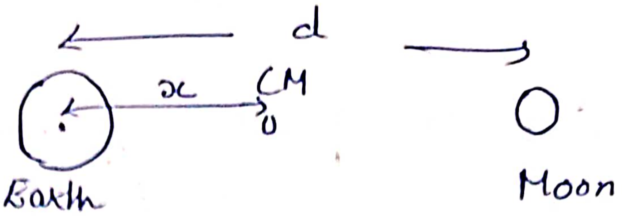
Thus
$$x = \frac{(M_{earth})\, d}{M_{sun} + M_{earth}}$$

$$= \frac{(5.972 \times 10^{24})(147.11 \times 10^{6})}{(1.9891 \times 10^{30}) + (5.972 \times 10^{24})}$$

$$= 441.676 \text{ km}$$

Thus COM is a distance of 441.676 km from centre of Sun.

(b) Earth-Moon System



$$M_{earth} = 5.972 \times 10^{24} \text{ kg}$$
$$M_{moon} = 7.34767 \times 10^{22} \text{ kg}$$
$$d = 384,400 \text{ km}$$

Thus $$x = \frac{M_m d}{M_{earth} + M_{moon}} = \frac{(7.34767 \times 10^{22}) \times (384,400)}{(5.972 \times 10^{24}) + (7.34767 \times 10^{22})}$$

$$\simeq 4671.952 \text{ km}$$

Thus, COM lies at a distance 4671.952 km from centre of Earth.

Que 2 In Bohr model of hydrogen atom, calculate velocity of electron in innermost orbit and compare it with speed of light.

Solⁿ We know the attractive force between electron and proton of hydrogen atom is providing the centripetal force required for motion of e^- around nucleus (for H-atom it is just a proton).

Thus $$\frac{(Ze)(e)}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r}$$

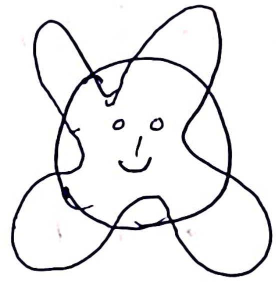
where r is orbital radius, v is velocity

Also $Z=1$ for H-atom.

Thus $$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r}$$

$$v^2 = \frac{r e^2}{4\pi\varepsilon_0 m r^2} \quad (1) \Rightarrow \boxed{v^2 = \frac{e^2}{4\pi\varepsilon_0 m r}}$$

Also, from the wave nature we know for a wave to be completly in phase, the circumference of orbit should be equal to integral multiple of wave length(λ)



ie $\quad 2\pi r = n\lambda$

n = an integer, r = radius of orbit

also $\quad \lambda = \frac{h}{mv}$

$$\Rightarrow \quad 2\pi r = \frac{nh}{mv}$$

$$\Rightarrow \boxed{mvr = n\frac{h}{2\pi}} \quad \Rightarrow \quad r = \left(\frac{nh}{2\pi}\right)\frac{1}{mv}$$

Substituting in (1)

$$\cancel{v^2 = \left(\frac{nh}{2\pi}\right)} \qquad v^2 = \frac{e^2 (2\cancel{\pi})\cancel{m}v}{\underset{2}{\cancel{4\pi}}\varepsilon_0 \cancel{m}(nh)}$$

$$\Rightarrow \boxed{v = \frac{e^2}{2nh\varepsilon_0}}$$

for innermost orbit $n=1$

$$v = \frac{e^2}{2h\varepsilon_0} = \frac{(1.6\times10^{-19})^2}{2\times(6.626\times10^{-34})\times(8.85\times10^{-12})}$$

$$\simeq 1.37\times10^{6}\ m/s$$

In comparison with speed of light $c = 3\times10^8\ m/s$

$$\frac{v}{c} = \frac{1.37\times10^6\ m/s}{3\times10^8\ m/s} \simeq 0.00457$$

Que 4 Calculate the wavelength of photon emitted when e^- jumps from $n=2$ to $n=1$ in H-atom; then repeat the same calculations for Deuterium atom.

What is the difference in wavelength that you calculated in hydrogen and deuterium.

Sol^n The wavelength is given by

$$\frac{1}{\lambda} = R Z^2 \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

n_f = final state
n_i = initial state

for $(n=2) \longrightarrow (n=1)$, $n_f = 1$, $n_i = 2$, (for H-atom, $Z=1$)

$$\Rightarrow \frac{1}{\lambda_H} = R_H \left[\frac{1}{1} - \frac{1}{4} \right] = \frac{3}{4} R_H$$

where R_H is Rydberg constant for Hydrogen atom

The Rydberg constant for hydrogen atom is

$$R_H = \frac{\mu}{m_e} R_\infty = \frac{R_\infty}{\left(1 + \frac{m_e}{M_p}\right)}$$

{ $\mu \rightarrow$ reduced mass $= \frac{mM}{m+M}$]

where $R_\infty = 109737 \ cm^{-1}$

$$\text{Thus} \quad R_H = \frac{109737}{\left(1 + \frac{1}{1836}\right)} \simeq 109678 \ cm^{-1}$$

* $$\frac{1}{\lambda_H} = \frac{3}{4}(109678) \Rightarrow \lambda_H = 1.21568 \times 10^{-5}\ \text{cm}$$
$$\simeq 1215.68\ \text{Å}$$

Deuterium atom is H-atom with neutron in nucleus.
The Rydberg constant for Deuterium atom is

$$R_D = \frac{R_\infty}{\left(1 + \frac{m_e}{M_p + M_n}\right)} = \frac{109737\ \text{cm}^{-1}}{\left(1 + \frac{1}{2 \times 1836}\right)}$$

$$R_D \simeq 109707\ \text{cm}^{-1}$$

Thus, wavelength is given by

$$\frac{1}{\lambda_D} = \frac{3}{4} R_D = \frac{3}{4}(109707)$$

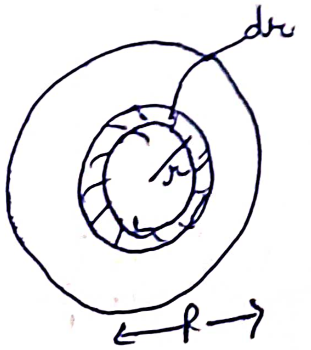
$$\Rightarrow \lambda_D = 1215.35\ \text{Å}$$

The Shift in wavelength is $\simeq$ 0.33 Å

2) Prove that total gravitational potential energy of sun is $-\frac{3}{5}\frac{GM^2}{R}$

Sol^n: P.E of sphere (or Sun if it has uniform density)

$$= -\int_0^R \underbrace{\frac{GM(r)}{r}}_{\text{Mass interior to } r} \underbrace{\rho\, 4\pi r^2\, dr}_{\text{shell}}$$



$$= -\int_0^R \frac{G}{r}\left(\frac{4}{3}\pi r^3 \rho\right) 4\pi r^2 \rho\, dr$$

for uniform density ρ

$$= \frac{-16\pi^2 G\rho^2}{3}\int_0^R r^4\, dr$$

$$= \frac{-16\pi^2 G\rho^2 R^5}{15}$$

But $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ or $\rho^2 = \frac{9M^2}{16\pi^2 R^6}$

$\Rightarrow$ $P.E = -\frac{16\pi^2 G (9M^2) R^5}{15 (16\pi^2 R^6)} = -\frac{3}{5}\frac{GM^2}{R}$

$\Rightarrow \boxed{P.E = -\frac{3}{5}\frac{GM^2}{R}}$ { more realistic case would be when ρ is varying with r }

3) Use the virial theorem argument to calculate the temp of sun if
(a) it expands to 10 times of its current size.
(b) it shrinks to a radius of 10 km.

Sol^n: If virial theorem holds true, then

$$K.E = -\frac{1}{2} P.E$$

from thermodynamics, $K.E = N k_B T$
and $P.E = -\frac{3}{5}\frac{GM^2}{R}$

$\Rightarrow$ $T = \frac{3}{10}\frac{GM^2}{RNk_B}$, here $k_B = 1.38 \times 10^{-23}$

Also $N = \frac{2M}{m_p}$ where $M = 2 \times 10^{30}$ kg
$M_p = 1.67 \times 10^{-27}$ kg

$\Rightarrow$ $T = \frac{3}{10}\frac{GM^2 m_p}{R(2M)k_B}$

$$\boxed{T = \frac{3}{20}\frac{GMm_p}{k_B R}}$$

(a). for $R = 10 R_0$

$$T = \frac{3}{20} \times \frac{6.75 \times 10^{-11} \times 2 \times 10^{30} \times 1.67 \times 10^{-27}}{1.38 \times 10^{-23} \times 10 \times 7 \times 10^{8}}$$

$$\boxed{T \simeq 3.5 \times 10^{5} \text{ K}}$$

(b) $R = 10 \text{ km} = 10^{4} \text{ m}$

$$T = \frac{3 \times 6.75 \times 10^{-11} \times 2 \times 10^{30} \times 1.67 \times 10^{-27}}{20 \times 1.38 \times 10^{-23} \times 10^{4}}$$

$$\Rightarrow \quad \boxed{T \simeq 2.45 \times 10^{11} \text{ K}}$$

4) There must be temp. gradient in Sun's successive shells. Using this information, try to explain Limb darkening in words.

Solⁿ: Limb darkening is an optical effect seen in stars (including Sun) where central part of disk appears brighter than edge or limb. Its understanding offered early solar astronomers an opportunity to construct models with such gradients. This encouraged the development of theory of radiative transfer.

Que2 : The Sun has total luminosity of $L \simeq 4 \times 10^{26}$ J/s. Describe the specific intensity I_ν for it.

Sol^n: Blackbody Planck's function is

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_BT} - 1}$$

The units of this quantity are $Sr^{-1} m^{-2} Hz^{-1}$

Here $B(\nu, T)$ is nothing but specific intensity I_ν emitted by blackbody.

To obtain the luminosity from it, we must integrate

(a) over solid angle
(b) over frequency
(c) over the area

For Sun, $$L = \iiint \frac{2\pi\nu^3}{c^2} \frac{1}{e^{h\nu/k_BT} - 1} d\nu \, d\Omega \, dA$$

This integral after solving comes out to be

$L = 4\pi R^2 \sigma T^4$, where σ = Stefan's constant

Therefore, $L = 4 \times 10^{26}$ J/s ; intensity is $B(\nu, T)$ as written above for $T \simeq 5000$ K.

IC-200 (Lect-5)

Que 1: A glass slab of thickness 0.2m absorbs 50% of light passing through it. How thick a slab of similar glass be required

(a) in order to absorb 90% of light passing through it.

(b) in order to absorb 99% of light passing through it.

(c) in order to absorb 99.9% of light passing through it.

(Assume emission coefficient $S_\nu = 0$).

Solⁿ:

$$I_\nu = I_\nu(0)e^{-\tau} + S_\nu(1-e^{-\tau})$$

where I_ν = specific intensity, τ = optical depth.

Given $I_\nu = \frac{I_\nu(0)}{2}$ when $S_\nu = 0$; $\tau = \int_0^z \alpha\, dz$ {z = depth (in metre)}

$$\Rightarrow \frac{1}{2} = (1).e^{-\int_0^{0.2} \alpha\, dz}$$

$$\Rightarrow \frac{1}{2} = e^{-0.2\alpha} \Rightarrow 2 = e^{0.2\alpha}$$

$$\Rightarrow \alpha = \frac{\ln 2}{0.2} = \frac{0.693147}{0.2}$$

$$\boxed{\alpha = 3.46574}$$

(a)

$$I_\nu(0) \times \frac{10}{100} = I_\nu(0)\, e^{-\alpha\int_0^x dz}$$ {90% absorb ≈ 10% ...}

$$\frac{1}{10} = e^{-3.4674[x]}$$

$$\Rightarrow x = \frac{\ln 10}{3.46574} \Rightarrow \boxed{x = 0.664386m}$$

(b)

$$I_\nu(0) \times \frac{1}{100} = I_\nu(0)\, e^{-\alpha\int_0^x dz}$$

$$\Rightarrow x = \frac{\ln(100)}{3.46574} \Rightarrow \boxed{x = 1.32877\ m}$$

(c) $\frac{0.1}{100} I_\nu(0) = I_\nu(0)\, e^{-\alpha x}$

$\Rightarrow x = \frac{\ln(1000)}{3.46574} \Rightarrow \boxed{x = 1.99316\ m}$