Lecture #13 (IC152)

Recall:

Matrix of an innerproduct

- $\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ ordered basis for ips V
- $M = [\langle \cdot, \cdot \rangle]_{\mathcal{B}} = (M_{jk})_{j,k}$, $M_{jk} = \langle \alpha_k, \alpha_j \rangle$

Observations. M is Hermitian & positive definite

Conversely if G, is Hermitian & positive definite then G defines an inner product on V

$$\langle x, y \rangle_G = Y^* G X$$

where X & Y are co-ordinates of x & y relative to any ordered basis $\underline{\mathcal{B}}$ of V.

Orthogonal vectors

Let V be an ips, then

$$\langle \alpha, \beta \rangle = 0 \iff \alpha \ \& \ \beta \text{ are orthogo}$$

* $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

then S is orthogonal subset of V if

$$\langle \alpha_i, \alpha_j \rangle = 0 \quad \text{if } i \neq j$$

* Orthonormal if

$$\langle \alpha_i, \alpha_j \rangle = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

Theorem: Any orthogonal set of nonzero vectors is linearly independent.

Corollary: Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be an orthogonal set & $\beta \in \text{Span}(S)$

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots c_n\alpha_n \quad (\text{uniquely})$$

$$\langle \beta, \alpha_i \rangle = c_i \langle \alpha_i, \alpha_i \rangle$$

$$\Rightarrow \quad c_i = \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{\langle \beta, \alpha_i \rangle}{\|\alpha_i\|^2}$$

$$\Rightarrow \quad \boxed{\beta = \sum_{i=1}^{n} \frac{\langle \beta, \alpha_i \rangle}{\|\alpha_i\|^2}\, \alpha_i} \checkmark$$

<u>Corollary</u>: If $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is orthonormal then

$$\boxed{\beta = \sum_{i=1}^{n} \underline{\underline{\langle \beta, \alpha_i \rangle}}\, \alpha_i}$$

<u>Example</u>: $S = \{(2,0,0), (0,5,0)\} \subset \mathbb{R}^3$, standard ip's.

i) Is S an orthogonal set?

ii) Is S orthonormal?

iii) Write $(4, 10, 0)$ in the linear combinations of vectors in S.

$\langle (2,0,0), (0,5,0) \rangle = 0$

Answer:

i) $\langle (2,0,0),(0,5,0)\rangle = 0$

ii) $\|(2,0,0)\|^2 = \langle (2,0,0),(2,0,0)\rangle$

$= 4$

$\|(2,0,0)\| = 2$

||^{ly} $\|(0,5,0)\| = 5$

Can you make S to be orthonormal

Yes !!

$$S' = \left\{ \frac{1}{2}(2,0,0), \frac{1}{5}(0,5,0) \right\}$$

iii) $$(4,10,0) = \frac{\langle (4,10,0),(2,0,0)\rangle}{4}(2,0,0) + \frac{\langle (4,10,0),(0,5,0)\rangle}{25}(0,5,0)$$

$$= 2(2,0,0) + 2(0,5,0)$$

Question: Can you make a linearly independent subset of an ips, orthogonal.

Example: Let $\{\alpha, \beta\}$ is a linearly indep. subset of V.

$\{u, v\}$ which is orthogonal in V.

Choose $u = \alpha$, $v = \beta - c\alpha$

such that $\langle u, v \rangle = 0$

$$\Rightarrow \langle \alpha, \beta - c\alpha \rangle = \langle \alpha, \beta \rangle - \bar{c}\,\|\alpha\|^2$$

$$\Rightarrow \quad \bar{c} = \frac{\langle \alpha, \beta \rangle}{\|\alpha\|^2}$$

$$\Rightarrow \quad c = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}$$

$$\left\{ u = \alpha, \quad v = \boxed{\beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha} \right\} \text{ orthogonal set..}$$

Note $\left\{ \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\}$ is an orthonormal set.

$\langle u, v \rangle = 0$

$\langle c_1 u, c_2 v \rangle$

$= c_1 \bar{c}_2 \langle u, v \rangle$

Theorem (Gram-Schmidt Process)

... inner product space &

Let V be an inner product ...

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a linearly independent subset of V. Then

$S' = \{\beta_1, \beta_2, \ldots \beta_n\}$ is an orthogonal subset of V such that $\operatorname{span}(S') = \operatorname{span}(S)$.

Moreover

$$(*) \qquad \beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{\langle \alpha_k, \beta_j \rangle}{\|\beta_j\|^2} \beta_j \quad \forall\, 2 \le k \le n.$$

$\ldots \langle \alpha_1, \alpha_2 \rangle \ldots$
$= 0$

Proof: We prove using mathematical induction. For $n=1$, choose $\beta_1 = \alpha_1 \Rightarrow S' = S \Rightarrow \operatorname{span}(S') = \operatorname{span}(S)$.

Now assume that for $n = k-1$, theorem holds good.

$\Big\{ S_{k-1} = \{\alpha_1, \alpha_2, \ldots \alpha_{k-1}\}, \Rightarrow S'_{k-1} = \{\beta_1, \beta_2 \ldots \beta_{k-1}\}$ such that S'_{k-1} is orthogonal & $\operatorname{span}(S_{k-1}) = \operatorname{span}(S'_{k-1})$

Moreover β_j are obtained by $(*)$ $\Big\}$

Now we show that result is true for $x=k$ also.

Objective: If β_k is obtained using (*) then we have to show

i) $S_k' = \{\beta_1, \beta_2, \ldots \beta_{k-1}, \beta_k\}$ is orthogonal

ii) span $S_k' =$ span S_k

i) $$\beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{\langle \alpha_k, \beta_j \rangle}{\|\beta_j\|^2} \beta_j.$$

$$\langle \beta_k, \beta_i \rangle = \langle \alpha_k, \beta_i \rangle - \sum_{j=1}^{k-1} \frac{\langle \alpha_k, \beta_j \rangle}{\|\beta_j\|^2} \langle \beta_j, \beta_i \rangle$$

$$= \langle \alpha_k, \beta_i \rangle - \frac{\langle \alpha_k, \beta_i \rangle}{\|\beta_i\|^2} \langle \beta_i, \beta_i \rangle$$

$$= 0 \quad \forall\ i = 1, 2, \ldots k-1$$

$\Rightarrow$ S_k' is orthogonal.

ii) $\beta_k \neq 0$. If $\beta_k = 0$

$\sum^{k-1} \langle \alpha \ldots \beta \ldots \rangle$

$$\alpha_k = \sum_{j=1} \frac{(\alpha_k, \beta_j)}{\|\beta_j\|^2} \beta_j$$

As span $S'_{k-1} = $ span S_{k-1}

$\Rightarrow \alpha_k \in$ span S_{k-1}

$\Rightarrow S_k$ is not linearly independent which is a contradiction.

Hence $\beta_k \neq 0$ ✓

Now as any ~~o~~ orthogonal set of non zero vectors is linearly independent,

$\dim S'_k = \dim S_k \Rightarrow$ span $S'_k =$ span S_k.

<u>Example</u>: $\alpha_1 = (1, 0, 1, 0)$ $\alpha_2 = (1, 1, 1, 1)$

$\alpha_3 = \{0, 1, 2, 1\}$

$S = \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}^4$ (standard ip,

$S' = \{\beta_1, \beta_2, \beta_3\}$

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1$$

$$= (1,1,1,1) - \frac{1+0+1+0}{(\sqrt{2})^2} (1,0,1,0)$$

$$= (1,1,1,1) - (1,0,1,0)$$

$$\beta_2 = (0,1,0,1)$$

$$\beta_3 = \alpha_3 - \sum_{j=1}^{2} \frac{\langle \alpha_3, \beta_j \rangle}{\|\beta_j\|^2} \beta_j$$

$$= \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2$$

$$\langle \alpha_3, \beta_1 \rangle = 2, \quad \langle \alpha_3, \beta_2 \rangle = 2$$

$$\beta \quad (0,1,2,1) \quad 2 \, (1,0,1,0) \; - \; 2 \, (0,1,0,1)$$

$$\gamma_3 = (0,1,2,1) - \frac{\;}{2}(1,0,1,0) - \frac{\;}{2}(0,1,0,1)$$

$$= (0,1,2,1) - (1,0,1,0) - (0,1,0,1)$$

$$\beta_3 = (-1,0,1,0)$$

$S' = \{\beta_1, \beta_2, \beta_3\}$ is orthogonal

$S'_N = \left\{\frac{1}{\sqrt{2}}\beta_1, \frac{1}{\sqrt{2}}\beta_2, \frac{1}{\sqrt{2}}\beta_3\right\}$ is orthonormal.

$$\text{span } S = \text{span } S'$$

$$\begin{cases} \{\beta_1, \beta_2, \beta_3\} \\ \checkmark \{c_1\beta_1, c_2\beta_2, c_3\beta_3\} \\ \quad \subset \text{span}\{\beta_1, \beta_2, \beta_3\} \end{cases}$$