IC105: Probability and Statistics **2021-22-M**

Lecture 5: Random Variables and their Distribution Functions

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Let $(\Omega, \mathcal{F}, P)$ be a given probability space. In some situations we may not be directly interested in the sample space Ω; rather we may be interested in some numerical aspect of Ω.

Example 5.1. *A fair coin (head and tail are equally likely) is tossed three times independently. Then,*

$$\Omega = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$$

and $P(\{\omega\}) = 1/8$ for all $\omega \in \Omega$. Suppose that we are interested in number of heads in three tosses, i.e., we are interested in the function $X : \Omega \to \mathbb{R}$ defined as

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = TTT, \\ 1, & \text{if } \omega \in \{HTT, THT, TTH\}, \\ 2, & \text{if } \omega \in \{HHT, HTH, THH\}, \\ 3, & \text{if } \omega = HHH. \end{cases}$$

Clearly the values assumed by X are random with

$$P(X = 0) = P(X = 3) = 1/8 \text{ and } P(X = 1) = P(X = 2) = 3/8.$$

Hence $P(X \in \{0, 1, 2, 3\}) = 1$.

Definition 5.2. *Let $(\Omega, \mathcal{F}, P)$ be a given probability space. A real valued* **measurable** *function $X : \Omega \to \mathbb{R}$ (defined on sample space Ω) is called a random variable (r.v.).*

Note: From rigorous mathematical point of view a random variable is a real valued function with some technical condition. In this course we are ignoring these technical details. For all practical purpose r.v. is a real valued function defined on Ω.

For a function $Y : \Omega \to \mathbb{R}$ and $A \subseteq \mathbb{R}$, define

$$Y^{-1}(A) = \{\omega \in \Omega : Y(\omega) \in A\}.$$

Then it is straightforward to prove the following result:

Proposition 5.3. *Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ and $A_\alpha \subseteq \mathbb{R}$, $\alpha \in \Lambda$, where Λ is an arbitrary index set. Let $Y : \Omega \to \mathbb{R}$ be a given function. Then*

(a) If $A \cap B = \phi$, then $Y^{-1}(A) \cap Y^{-1}(B) = \phi$;

(b) $Y^{-1}(A^c) = (Y^{-1}(A))^c$ (that is, $Y^{-1}(\mathbb{R} - A) = Y^{-1}(\mathbb{R}) - Y^{-1}(A) = \Omega - Y^{-1}(A)$);

(c) $Y^{-1}\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) = \bigcup_{\alpha\in\Lambda} Y^{-1}(A_\alpha)$;

(d) $Y^{-1}\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right) = \bigcap_{\alpha\in\Lambda} Y^{-1}(A_\alpha)$.

For a probability space $(\Omega, \mathcal{F}, P)$ and a r.v. $X : \Omega \to \mathbb{R}$, note that $\forall$ $B \subseteq \mathscr{B}$

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

Thus, one can define a set function $P_X : \mathscr{B} \to [0, 1]$ by

$$P_X(B) = P(X^{-1}(B)) = P\left(\{\omega \in \Omega : X(\omega) \in B\}\right), \;\; B \in \mathscr{B},$$

where $\mathscr{B}$ is some class of subsets of $\mathbb{R}$. Here, also for all practical purpose we will take $\mathscr{B}$ to be a sigma algebra formed by open subsets of $\mathbb{R}$.

We simply write

$$P_X(B) = P\left(\{\omega \in \Omega : X(\omega) \in B\}\right) = P(X \in B), \;\; B \in \mathscr{B}.$$

We have the following scenario $(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathscr{B}, P_X)$.

Theorem 5.4 (Induced probability space / measures)**.** *$(\mathbb{R}, \mathscr{B}, P_X)$ (as defined above) is a probability space, i.e., $P_X(\cdot)$ is a probability function defined on $\mathscr{B}$.*

Proof. (i) $P_X(\mathbb{R}) = P(X \in \mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$.

(ii) For any $B \in \mathscr{B}$, $P_X(B) = P(X^{-1}(B)) \geq 0$.

(iii) Let $\{B_n\}$ be a collection of mutually exclusive events in $\mathscr{B}$. Then,

$$\begin{aligned}
P_X\left(\bigcup_{n=1}^{\infty} B_n\right) &= P\left(X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right)\right) \\
&= P\left(\bigcup_{n=1}^{\infty} X^{-1}(B_n)\right), \quad \text{(Proposition } 5.3(c)) \\
&= \sum_{n=1}^{\infty} P(X^{-1}(B_n)), \;\; (P \text{ is a probability measure and using Proposition } 5.3(a)) \\
&= \sum_{n=1}^{\infty} P_X(B_n).
\end{aligned}$$

This completes the proof. □

Definition 5.5. *The probability function P_X defined above is called the probability function/ measure induced by r.v. X and $(\mathbb{R}, \mathscr{B}, P_X)$ is called the probability space induced by r.v. X.*

The induced probability measure P_X describes the random behaviour of X.

Example 5.6. *Toss a coin three times independently. Then,*

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\} \text{ and } P(\{\omega\}) = 1/8, \;\; \forall\, \omega \in \Omega$$

and $X : \Omega \to \mathbb{R}$ (number of heads in three tosses) is defined by

$$X(\omega) = \begin{cases} 0, & \text{if } \omega \in \{TTT\}, \\ 1, & \text{if } \omega \in \{HTT, THT, TTH\}, \\ 2, & \text{if } \omega \in \{HHT, HTH, THH\}, \\ 3, & \text{if } \omega \in \{HHH\}. \end{cases}$$

Obviously, $X : \Omega \to \mathbb{R}$ *is r.v. with induced probability space given by* $(\mathbb{R}, \mathscr{B}, P_X)$*, where*

$$\begin{aligned} P_X(\{0\}) &= P(\{TTT\}) = 1/8, \\ P_X(\{1\}) &= P(\{HTT, THT, TTH\}) = 3/8, \\ P_X(\{2\}) &= P(\{HHT, HTH, THH\}) = 3/8, \\ P_X(\{3\}) &= P(\{HHH\}) = 1/8. \end{aligned}$$

Now for any $B \in \mathscr{B}$*,*

$$P_X(B) = P(X^{-1}(B)) = P\left(\{\omega \in \Omega : X(\omega) \in B\}\right) = \sum_{i \in B \cap \{0,1,2,3\}} P_X(\{i\}).$$

Definition 5.7. *Let* X *be a r.v. defined on probability space* $(\Omega, \mathcal{F}, P)$ *and let* $(\mathbb{R}, \mathscr{B}, P_X)$ *denote the probability space induced by* X*. Define the function* $F_X : \mathbb{R} \to \mathbb{R}$ *by*

$$F_X(x) = P(X \leq x) = P(X^{-1}(-\infty, x]) = P_X((-\infty, x]), \quad x \in \mathbb{R}.$$

The function F_X *is called the cumulative distribution function (c.d.f.) or simply the distribution function (d.f.) of r.v.* X*.*

Note: Whenever there is no ambiguity we will drop subscript X in F_X to represent d.f. of a r.v. by F. It can be shown (in advanced courses) that the c.d.f. $F_X(\cdot)$ of a r.v. X determines the induced probability measure $P_X(\cdot)$ uniquely. Thus to study the random behaviour of r.v. X it suffices to study its d.f. F.

Example 5.8. *In the previous example*

$$P(X = 0) = P_X(\{0\}) = 1/8,\ P(X = 1) = P_X(\{1\}) = 3/8 = P(X = 2) = P_X(\{2\})$$

and $P(X = 3) = P_X(\{3\}) = 1/8$*. Then, the d.f. of* X *is obtained as*

$$F_X(x) = P(X \leq x) = P\left(\{\omega : X(\omega) \leq x\}\right) = \sum_{\substack{i \in \{0,1,2,3\} \\ i \leq x}} P_X(\{i\}) = \begin{cases} 0, & x < 0, \\ 1/8, & 0 \leq x < 1, \\ 1/8 + 3/8 = 1/2, & 1 \leq x < 2, \\ 7/8, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

Theorem 5.9. *Let* $F(\cdot)$ *be the c.d.f. of a r.v.* X *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and let* $(\mathbb{R}, \mathscr{B}, P_X)$ *be the probability space induced by* X*. Then*

(i) F *is non-decreasing,*

(ii) $F(x)$ *is right continuous,*

(iii) $F(-\infty) = \lim_{n \uparrow \infty} F(-n) = 0$ *and* $F(\infty) = \lim_{n \uparrow \infty} F(n) = 1$*.*

Conversely, any function $G(\cdot)$ *satisfying properties (i)-(iii) is a d.f. of some r.v.* Y *defined on a probability space* $(\Omega^*, \mathcal{F}^*, P^*)$*.*

Proof. (i) Let $-\infty < x < y < \infty$. Then $(-\infty, x] \subseteq (-\infty, y] \implies P_X((-\infty, x]) \leq P_X((-\infty, y])$. This implies that $F(x) \leq F(y)$.

(ii) Since F is monotone and bounded below (by 0), $\lim_{h \downarrow 0} F(x + h) = F(x+)$ exists $\forall\, x \in \mathbb{R}$. Therefore,

$$F(x+) = \lim_{h \downarrow 0} F(x + h) = \lim_{n \to \infty} F\left(x + \frac{1}{n}\right) = \lim_{n \to \infty} P_X\left((-\infty, x + 1/n]\right).$$

Let $A_n = (-\infty, x + 1/n]$, $n = 1, 2, \ldots$. Then $A_n \downarrow$ and $\bigcap_{n=1}^{\infty}(-\infty, x+1/n] = (-\infty, x]$. Thus,

$$F(x+) = P_X\left(\bigcap_{n=1}^{\infty}(-\infty, x+1/n]\right) = P_X((-\infty, x]) = F(x).$$

(iii) Note that

$$\begin{aligned}
F(-\infty) = \lim_{n\to\infty} F(-n) = \lim_{n\to\infty} P_X((-\infty,-n]) &= P_X\left(\bigcap_{n=1}^{\infty}(-\infty,-n]\right), \quad ((-\infty,-n]\downarrow) \\
&= P_X(\phi), \quad \left(\bigcap_{n=1}^{\infty}(-\infty,-n] = \phi\right) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
F(+\infty) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P_X((-\infty,n]) &= P_X\left(\bigcup_{n=1}^{\infty}(-\infty,n]\right), \quad ((-\infty,n]\uparrow) \\
&= P_X(\mathbb{R}), \quad \left(\bigcup_{n=1}^{\infty}(-\infty,n] = \mathbb{R}\right) \\
&= 1.
\end{aligned}$$

This completes the proof. □

Remark 5.10. *(i) Since any distribution function is monotone and bounded above (by 1),* $\lim_{h\downarrow 0} F(x-h) = F(x-)$ *exists* $\forall x \in \mathbb{R}$*. Moreover,*

$$\begin{aligned}
F(x-) = \lim_{h\downarrow 0} F(x-h) = \lim_{n\to\infty} F(x-1/n) = \lim_{n\to\infty} P_X((-\infty, x-1/n]) \\
&= P_X\left(\bigcup_{n=1}^{\infty}(-\infty, x-1/n]\right), \quad ((-\infty, x-1/n]\uparrow) \\
&= P_X((-\infty, x)) = P(X < x).
\end{aligned}$$

(ii) From the calculus we know that any monotone function is either continuous on $\mathbb{R}$ *or it has atmost countable number of discontinuities. Thus any c.d.f* $F(x)$ *is either continuous on* $\mathbb{R}$ *or has atmost countable number of discontinuities. Since, for any* $x \in \mathbb{R}$*,* $F(x+)$ *and* $F(x-)$ *exist,* F *has only jump discontinuities* $(F(x) = F(x+) > F(x-))$.

(iii) A distribution function F *is continuous at* $a \in \mathbb{R}$ *iff* $F(a) = F(a-)$.

(iv) For any $a \in \mathbb{R}$*,* $P(X = a) = P(X \le a) - P(X < a) = F(a) - F(a-)$*. Thus, a d.f.* F *is continuous at* $a \in \mathbb{R}$ *iff* $P(X = a) = F(a) - F(a-) = 0$.

(v) For $-\infty < a < b < \infty$*,* $P(X \le b) = P(X \le a) + P(a < X \le b)$.

$$P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a).$$

Similarly, for $-\infty < a < b < \infty$*,*

$$\begin{aligned}
P(a < X < b) &= P(X < b) - P(X \leq a) = F(b-) - F(a),\\
P(a \leq X \leq b) &= P(X \leq b) - P(X < a) = F(b) - F(a-),\\
P(a \leq X < b) &= P(X < b) - P(X < a) = F(b-) - F(a-),\\
P(a < X < b) &= P(X < b) - P(X \leq a) = F(b-) - F(a),\\
P(X > a) &= 1 - P(X \leq a) = 1 - F(a),\\
P(X \geq a) &= 1 - P(X < a) = 1 - F(a-).
\end{aligned}$$

Example 5.11. *Consider the function* $G : \mathbb{R} \to \mathbb{R}$ *defined by*

$$G(x) = \begin{cases} 0, & \text{if } x < 0,\\ \frac{x}{3}, & \text{if } 0 \leq x < 1,\\ \frac{1}{2}, & \text{if } 1 \leq x < 2,\\ \frac{2}{3}, & \text{if } 2 \leq x < 3,\\ 1, & \text{if } x \geq 3. \end{cases}$$

(a) Show that G *is d.f. of some r.v.* X*,*

(b) Find $P(X = a)$ *for various values of* $a \in \mathbb{R}$*,*

(c) Find $P(X < 3),\ P\left(X \geq \frac{1}{2}\right),\ P(2 < X \leq 4),\ P(1 \leq X < 2), P(2 \leq X \leq 3)$ *and* $P\left(\frac{1}{2} < X < 3\right)$.

Solution: (a) Clearly G is non-decreasing in $(-\infty, 0),\ (0,1),\ (1,2),\ (2,3)$ and $(3,\infty)$. Moreover,

$$\begin{aligned}
&G(0) - G(0-) = 0 \geq 0,\ \ G(1) - G(1-) = \frac{1}{2} - \frac{1}{3} > 0,\\
&G(2) - G(2-) = \frac{2}{3} - \frac{1}{2} > 0,\ \ G(3) - G(3-) = 1 - \frac{2}{3} > 0.
\end{aligned}$$

It follows that G is non-decreasing.

Clearly G is continuous (and hence right continuous) on $(-\infty, 0),\ (0,1),\ (1,2),\ (2,3)$ and $(3,\infty)$. Moreover,

$$\left.\begin{array}{lll}
G(0+) - G(0) & = 0 - 0 & = 0\\
G(1+) - G(1) & = 1/2 - 1/2 & = 0\\
G(2+) - G(2) & = 2/3 - 2/3 & = 0\\
G(3+) - G(3) & = 1 - 1 & = 0
\end{array}\right\} \implies G \text{ is right continuous on } \mathbb{R}.$$

Also, $G(+\infty) = \lim\limits_{x\to\infty} G(x) = 1\ \ \&\ G(-\infty) = \lim\limits_{x\to\infty} G(-x) = 0$. Thus, G is a d.f. of some random variable X.

(b) The set of discontinuity points of F is $D = \{1,2,3\}$. Thus,

$$\begin{aligned}
P(X = a) &= G(a) - G(a-) = 0,\ \ \forall\, a \neq 1,2,3,\\
P(X = 1) &= G(1) - G(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},\\
P(X = 2) &= G(2) - G(2-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},\\
P(X = 3) &= G(3) - G(3-) = 1 - \frac{2}{3} = \frac{1}{3}.
\end{aligned}$$

(c) Note that

$$
\begin{aligned}
P(X < 3) &= G(3-) = \frac{2}{3}, \\
P\left(X \geq \frac{1}{2}\right) &= 1 - G\left(\frac{1}{2}-\right) = 1 - \frac{1}{6} = \frac{5}{6}, \\
P(2 < X \leq 4) &= G(4) - G(2) = 1 - \frac{2}{3} = \frac{1}{3}, \\
P(1 \leq X < 2) &= G(2-) - G(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \\
P(2 \leq X \leq 3) &= G(3) - G(2-) = 1 - \frac{1}{2} = \frac{1}{2}, \\
P\left(\frac{1}{2} < X < 3\right) &= G(3-) - G\left(\frac{1}{2}\right) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}.
\end{aligned}
$$