MA202: Calculus II

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Lecture Notes

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Module 3

Lecture 6

- Let D be a bounded subset of $\mathbb{R}^2$ and consider the identity function $id : D \to \mathbb{R}$ defined as

$$id(x, y) = 1, \ \forall \ (x, y) \in D.$$

- We say that D has an area if the function id is integrable over D and in this case the area can be obtained by

$$\text{Area of } D \quad = \iint_D id(x, y) d(x, y) = \iint_D d(x, y).$$

$D = \left\{ \left(\frac{1}{n}, \frac{1}{k} \right) : n, k \in \mathbb{N} \right\}$. $\text{Area}(D) = 0 \Leftrightarrow \partial D$ is of content zero set

Application of Double Integral

Important Remark

- The function $id : D \to \mathbb{R}$ is continuous (constant function).
- If the boundary ∂D of D is of content zero, then the continuous function $id : D \to \mathbb{R}$ is integrable on D.
- The converse also holds, that is, if the function id is integrable on D, then ∂D is of content zero.

Basically we have the following theorem.

Theorem

Let D be a bounded subset of $\mathbb{R}^2$. Then

1. D has an area $\Leftrightarrow \partial D$ is of content zero.
2. D has an area and area $(D) = 0 \Leftrightarrow D$ is of content zero.

Examples:

1. If $D_1 = R = [a, b] \times [c, d]$ then ∂D_1 is of content zero. Hence *id* over D_1 is integrable and the double integral is equal to $(b - a)(d - c)$.
2. If $D_2 = \{(x, y) \in R : x, y \in \mathbb{Q}\}$. It is easy to check that the function *id* is not integrable over D_2 (by extending the function). Another approach is that, $\partial D_2 = R$ and it is not of content zero. Hence *id* is not integrable.

Examples:

$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous. Then ∂D is of content zero, and by the Fubini theorem,

$$\text{Area}\,(D) = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 1\, dy \right) dx = \boxed{\int_a^b \big(\phi_2(x) - \phi_1(x)\big)\, dx,}$$

which was our definition of the **area between the curves** $y = \phi_1(x)$ and $y = \phi_2(x)$, $x \in [a, b]$.

The following relation is important to estimate a double integral.

- Let D be a bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is an integrable function. Also let $|f| \leq c$ on D (i.e., $|f(x, y)| \leq c$ for all $(x, y) \in D$), the

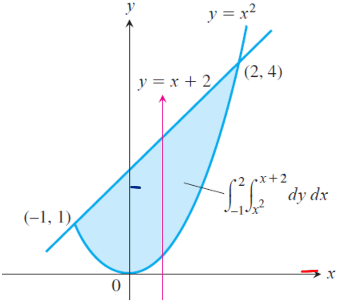
$$\left| \iint_D f(x, y) d(x, y) \right| \leq \iint_D |f(x, y)| d(x, y) \leq c \iint_D d(x, y) = c.A(D).$$

where $A(D)$ denotes the area of D.

Application of Double Integral

Find the area of the region D enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

- Check that the above region is a elementary region and the conditions of Fubini Theorem satisfies.

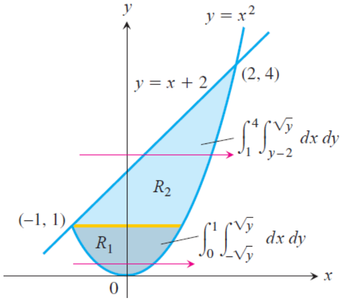


Find the area of the region D enclosed by the parabola and the line $y = x + 2$.

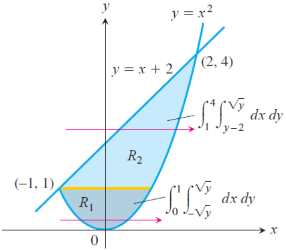
- The area A of the region can be calculated by using iterated integral as follows

$$\iint_D d(x,y) = \int_{x=-1}^{x=2} \left(\int_{y=x^2}^{y=x+2} dy \right) dx = \frac{9}{2}.$$

Important Remark: If the integral is evaluated with the help of another iterated integral then check that when we try to move the horizontal line on the whole region then the x-limit is changing. Look into the following picture.



Check that $R_1 \cap R_2$ is of content zero set



Therefore we need to divide the whole region into two subregions D_1 and D_2 and evaluate the integral as follows.

$$\iint_D d(x,y) = \iint_{D_1} d(x,y) + \iint_{D_2} d(x,y)$$

$$= \int_{y=0}^{y=1} \left(\int_{x=-\sqrt{y}}^{x=\sqrt{y}} dx \right) dy + \int_{y=1}^{y=4} \left(\int_{x=y-2}^{x=\sqrt{y}} dx \right) dy$$

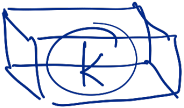
Triple Integrals

Over a cuboid:

- The double integrals of functions of two variables can be directly extended to the triple integral of functions of three variables. No new concept is needed in this case.
- Like double integral, first we define the triple integral over a cuboid $K = [a, b] \times [c, d] \times [p, q]$ where $a < b,\ c < d,\ p < q$.
- Also let $f : K \to \mathbb{R}$ be a bounded function.
- First we take a partition P of the cuboid K where

$$P = \{(x_i, y_j, z_l) : i = 0, \cdots, n,\ j = 0, \cdots, k,\ l = 0, \cdots, r\}$$

- $a = x_0 < x_1 < \cdots < x_n = b,\ c = y_0 < y_1 < \cdots < y_k = d,$
  $p = z_0 < z_1 < \cdots < z_r = q.$



- Define the quantities m_{ijl} and M_{ijl} as usual and the upper/lower triple sum $U(P, f)$ and $L(P, f)$ as follows

$$\begin{aligned} U(P, f) &= \sum_i \sum_j \sum_l M_{ijl}(x_i - x_{i-1})(y_j - y_{j-1})(z_l - z_{l-1}), \\ L(P, f) &= \sum_i \sum_j \sum_l m_{ijl}(x_i - x_{i-1})(y_j - y_{j-1})(z_l - z_{l-1}), \end{aligned}$$

- Define the lower triple integral $L(f)$ and upper triple integral $U(f)$ as the supremum and infimum value (taken over all the partitions of K) of $L(P, f)$ and $U(P, f)$ respectively.
- $f : K \to \mathbb{R}$ is said to be integrable if $L(f) = U(f)$ and the value is called triple integral and denoted by

$$L(f) = U(f) = \iiint_K f(x, y, z)d(x, y, z) = \iiint_K f.$$

Examples:

- If $f = id$ (i.e, $f(x,y,z) = 1$ over K) then the triple integral of f over K is the volume of the cuboid.
- If f be the trivariate Dirichlet function on K (i,e, $f(x,y,z) = 1$ when all x, y, z are rational and 0 otherwise) then similarly as double integral it can be deduced that f is not integrable.
  $(L(P,f) = L(f) = 0$ and $U(P,f) = U(f) = 1)$

Fubini Theorem on cuboids:

Let f be integrable on K, and let I denote its triple integral. Suppose for each fixed $x \in [a, b]$, the double integral $\iint_{[c,d]\times[p,q]} f(x, y, z)d(y, z)$ exists. Then the **iterated integral** $\int_a^b (\iint_{[c,d]\times[p,q]} f(x, y, z)d(y, z))dx$ exists and equals I. Further, if for each fixed $(x, y) \in [a, b] \times [c, d]$, the Riemann integral $\int_p^q f(x, y, z)dz$ exists, then the **iterated integral** $\int_a^b \left[\int_c^d \left(\int_p^q f(x, y, z)dz\right)dy\right]dx$ exists and equals I.

* Similar statement holds when the variables are interchanged.

Triple Integrals over bounded set:

Let D be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a cuboid $K := [a, b] \times [c, d] \times [p, q]$ such that $D \subset K$, and define $f^* : K \to \mathbb{R}$ by

$$f^*(x, y, z) := \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is **integrable** over D if f^* is integrable on K, and in this case, the **triple integral** of f (over D) is defined to be the triple integral of f^* (on K), that is,

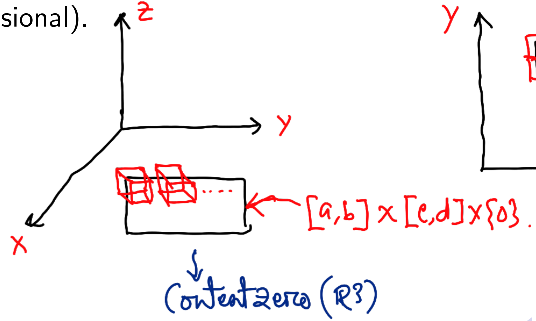
$$\iiint_D f(x, y, z) d(x, y, z) := \iiint_K f^*(x, y, z) d(x, y, z).$$

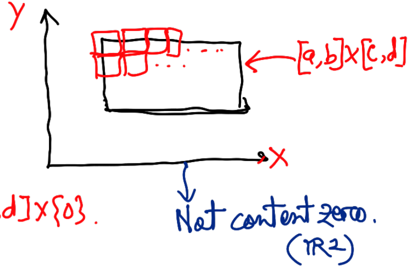
We may also denote the triple integral by $\iiint_D f$.

Set of content zero

A bounded subset E of $\mathbb{R}^3$ is of (three-dimensional) content zero if for every $\epsilon > 0$ there are finitely many cuboids whose union contains E and the sum of whose volumes is less than ϵ

Check that the subset $[a, b] \times [c, d] \times \{0\}$ of $\mathbb{R}^3$ is of content zero (three dimensional) but the subset $[a, b] \times [c, d]$ is not of content zero (two dimensional).





Which functions are triple integrable? The answer is similar as double integral.

Theorem

Let D be a bounded subset of $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be a bounded function. Suppose (i) the set of discontinuities of f in D is of (three-dimensional) content zero and (ii) the boundary ∂D of D is of (three-dimensional) content zero. Then f is integrable over D.

* Sufficient condn

Triple Integrals

Elementary Regions:

Suppose D_0 is a subset of $\mathbb{R}^2$ having an area, that is, ∂D_0 is of two-dimensional content zero. Let $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ be continuous, and let $\psi_1 \leq \psi_2$. Consider an **elementary region**

$$D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0 \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y)\}.$$

Then ∂D is of three-dimensional content zero. Hence if a function is continuous on $\bar{D}$, then it is integrable over D.

Cavalieri Principle:

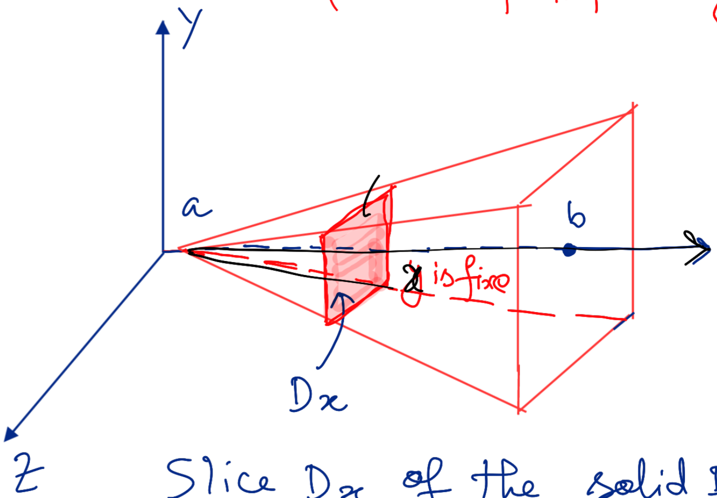
Let D be a bounded subset of $\mathbb{R}^3$ and $f : D \to \mathbb{R}$ be an integrable function and I denotes the triple integral of f over D.

(a) Suppose $D = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, \text{ and } (y, z) \in D_x\}$ where D_x is a subset of $\mathbb{R}^2$ whose boundary is of content zero (two dimensional) and for each fixed $x \in [a, b]$ the double integral $\iint_{D_x} f(x, y, z)d(y, z)$ exists. Then the iterated integral

$$\int_a^b \left(\iint_{D_x} f(x, y, z)d(y, z) \right) dx$$

exists and equal to the triple integral I.

Cavalieri Principle (contd.): (Domain of triple integration)



Slice D_x of the solid D.

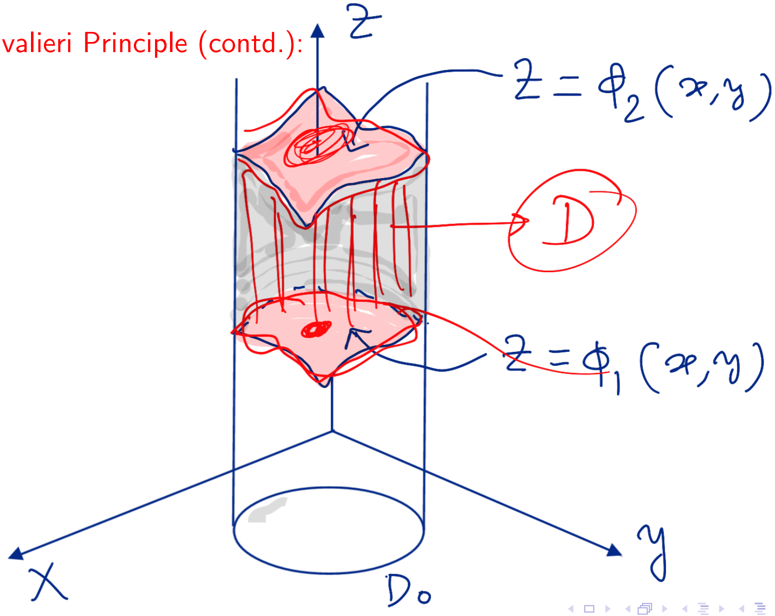
Cavalieri Principle (contd.):

(b) Suppose $D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0 \text{ and } \phi_1(x, y) \leq z \leq \phi_2(x, y)\}$ where D_0 is a subset of $\mathbb{R}^2$ whose boundary is of content zero (two dimensional), $\phi_1, \phi_2 : D_0 \to \mathbb{R}$ are integrable function such that $\phi_1 \leq \phi_2$ and for each fixed $(x, y) \in D_0$ the Riemann integral $\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)dz$ exists. Then the iterated integral

$$\iint_{D_0} \left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x, y, z)dz \right) d(x, y)$$

exists and equal to the triple integral I.

Cavalieri Principle (contd.):



Cavalieri Principle (contd.):

- In the statement (a) of Cavalieri Principle if D_x is an elementary region in the yz-plane for every fixed $x \in [a, b]$ then the Fubini Theorem for elementary region (double integral) can be applied to $\iint_{D_x} f(x, y, z) d(y, z)$ provided the function $f(x, y, z)$ (here x is fixed) satisfies the required hypothesis of Fubini Theorem.

In particular, if D_0 is an elementary region in $\mathbb{R}^2$ given by $D_0 := \{(x, y) : \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \leq \phi_2$, then

$$\iiint_D f =' \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right) dy \right) dx.$$

Important Remark:

- The above two observations lead us to an important feature to calculate of triple integral.
- The evaluation of a triple integral can be reduced to the evaluation of several Riemann integrals
- For example, if $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and $f : D \to \mathbb{R}$ is a continuous function. Then we have

$$\iiint_D f(x,y,z)d(x,y,z) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdydx.$$

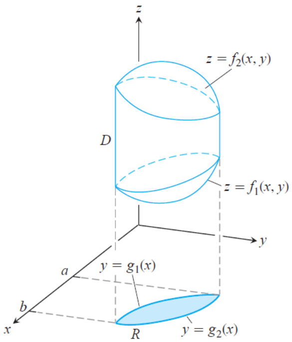
Method to solve problems

Use Cavalieri Principal + Fubini Theorem for elementary region (for the double integration)

Triple Integrals: Finding limit of Integration

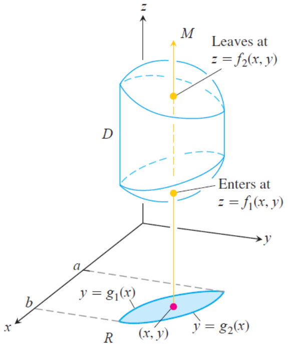
- As shown above the evaluation of triple integral can be made to evaluation of several Riemann integral under certain conditions.
- Similar as double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.
- Next we mention the procedure to find the limits of $x, \ y, \ z$ when the integral $\iiint_D F(x,y,z)d(x,y,z)$ over the region D will be evaluated with the help of the iterated integral where we integrate first with respect to z, then with respect to y, finally with x.

- Sketch the region D along with its vertical projection R in the xy-plane (The shadow of D on the xy plane). Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



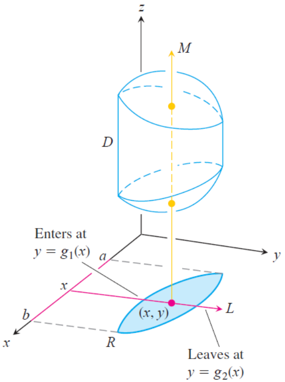
Triple Integrals: Finding limit of Integration

- Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limit of integration.



Triple Integrals: Finding limit of Integration

- Draw a line L passing through a typical point (x, y) in R parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limit of integration.



- Choose x-limits that include all lines through R parallel to the y-axis.
- The integral is

$$\boxed{\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)dz\ dy\ dx.}$$

- A similar process can be followed if the order of integration is changed.

* The above steps are same as expressing the domain D in the form of elementary region.

Triple Integrals Applications

- Let D be a bounded subset of $\mathbb{R}^3$ and consider the identity function $id : D \to \mathbb{R}$ defined as

$$id(x, y, z) = 1, \ \forall \ (x, y, z) \in D.$$

- We say that D has a volume if the function id is integrable over D and in this case the volume can be obtained by

$$\text{Volume of } D \quad = \iiint_D id(x, y, z) d(x, y, z) = \boxed{\iiint_D d(x, y, z).}$$

Important Remark

- The function $id : D \to \mathbb{R}$ is continuous (constant function).
- If the boundary ∂D of D is of content zero (three dimensional), then the continuous function $id : D \to \mathbb{R}$ is integrable (triple) on D.
- The converse also holds, that is, if the function id is integrable on D, then ∂D is of content zero.

Basically we have the following theorem.

Theorem

Let D be a bounded subset of $\mathbb{R}^3$. Then

1. D has a volume $\Leftrightarrow$ ∂D is of content zero.
2. D has a volume and volume $(D) = 0$ $\Leftrightarrow$ D is of content zero.

Triple Integrals Applications

Examples:

1. If $D_1 = K = [a,b] \times [c,d] \times [p,q]$ then ∂D_1 is of content zero. Hence *id* over D_1 is integrable and the double integral is equal to $(b-a)(d-c)(q-p)$. This is equal to the volume of the cuboid.
2. If $D_2 = \{(x,y,z) \in K : x,y,z \in \mathbb{Q}\}$. It is easy to check that the function *id* is not integrable over D_2 (by extending the function). Another approach is that, $\partial D_2 = K$ and it is not of content zero. Hence *id* is not integrable.

Triple Integrals : Example

$z=x^2+y^2$

Find the volume of the region D formed by the paraboloid and the plane $z=2$ in the first octant.

Our main approach is to express the region D as an elementary region. Here z varies from x^2+y^2 to 2, y varies from $y=0$ to $y=\sqrt{2-x^2}$ and x varies from 0 to $\sqrt{2}$. Hence the region can be written as

$$D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq \sqrt{2},\ 0 \leq y \leq \sqrt{2-x^2},\ x^2+y^2 \leq z \leq 2\}.$$

This proves that the region D is an elementary region and the required volume is given by

$$V = \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} \int_{z=x^2+y^2}^{z=2} dz\ dy\ dx$$