IC153: Calculus 1
(Lecture 14)

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Recap of the previous lecture

- Riemann Integration: Motivation
- Partition of interval and their refinement
- Lower sum and upper sum of a function
- Riemann integrable functions
- Example of bounded non-integrable function

$$\sup_P L(P,f) = \inf_P U(P,f)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$$

Riemann's criterion for integrability

Theorem

Let f be a bounded function on $[a, b]$. Then, f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P, f) - L(P, f) < \epsilon$.

Proof: $(\Longleftarrow)$

For any partition P of $[a,b]$

$$L(P,f) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq U(P,f) \quad \text{—} \; 1$$

To prove: $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$

Let $\epsilon > 0$. Hypothesis implies that $\exists$ P s.t.

$$U(P,f) - L(P,f) < \epsilon \quad \text{———} \; 2$$

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq U(P,f) - L(P,f) < \epsilon$$

$(\Rightarrow)$. f is integrable. Let $\epsilon > 0$

By definition $\int_a^b f\,dx = \underline{\int_a^b} f\,dx = \sup_P L(P, f)$

$\Rightarrow \exists\ P_1$ s.t. $\int_a^b f\,dx - L(P_1, f) < \epsilon/2$

Similarly $\exists\ P_2$ s.t.

$U(P_2, f) - \int_a^b f\,dx < \epsilon/2$

Take $P = P_1 \cup P_2$.

$$U(P, f) - L(P, f) \le U(P_2, f) - L(P_1, f)$$
$$= U(P_2, f) - \int_a^b f\,dx + \int_a^b f\,dx - L(P_1, f)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Recall: If Q_1 is refinement of Q_2 then $U(Q_1, f) \le U(Q_2, f)$
$L(Q_1, f) \ge L(Q_2, f)$

Corollary

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. If there is a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \longrightarrow 0$, then f is integrable.

Proof: $U(P_n, f) - L(P_n, f) \longrightarrow 0$

For any $\epsilon > 0$ $\exists$ n_0 s.t.

$$U(P_n, f) - L(P_n, f) < \epsilon \quad \forall \; n \geq n_0$$

In particular,

$$U(P_{n_0}, f) - L(P_{n_0}, f) < \epsilon$$

$\xrightarrow{\text{Riemann Cri}}$ f is integrable.

Uniform continuity

Recall: Let $f : D \longrightarrow \mathbb{R}$ be a continuous function. Then for each $x_0 \in D$ and for given $\epsilon > 0$ there exists $\delta(x_0, \epsilon) > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon.$$

Definition

A function $f : D \longrightarrow \mathbb{R}$ is said to be uniform continuous on D if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon \text{ for all } x, y \in D.$$

Theorem

Uniform continuity $\Longrightarrow$ continuity.

$\not\Longleftarrow$

Example

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$.
For $\epsilon = 2$ and $x_0 = 1$, $\delta = \frac{1}{2}$ does the job. However $\delta = \frac{1}{2}$ does not work for $\epsilon = 2$ and $x_0 = 10$.

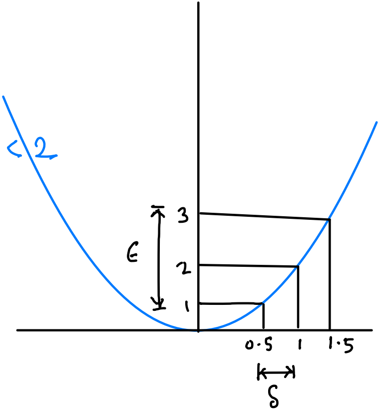
$|x-1| < \frac{1}{2}$, then $\frac{1}{2} < x < \frac{3}{2}$.

$\Rightarrow \quad -\frac{3}{4} < x^2 - 1 < \frac{5}{4} \Rightarrow |f(x) - f(x_0)| < 2$

Therefore

$|x-1| < \frac{1}{2} \Rightarrow |f(x) - f(1)| < 2$

$\Rightarrow \delta = \frac{1}{2}$ works.



Now take $\epsilon = 2$, $x_0 = 10$ & $\delta = \frac{1}{2}$.

We have $f(x) - f(x_0) = x^2 - 100$.

If $x = x_0 + \frac{1}{4}$ then $|x - x_0| < \frac{1}{2} \Rightarrow x \in \left(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right)$

but $|f(x) - f(x_0)| = \left| f\left(10 + \frac{1}{4}\right) - f(10) \right|$

$$= \left| \left(10 + \frac{1}{4}\right)^2 - 10^2 \right| = \left| 5 + \frac{1}{16} \right| > 2 = \epsilon$$

$\Rightarrow$ for $\epsilon = 2$, $\delta = \frac{1}{2}$ works for $x_0 = 1$ but not for $x_0 = 10$.

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then f is uniformly continuous of $[a, b]$.

Proof: Suppose f is not uniform continuous.

$\Rightarrow \exists\ \epsilon_0 > 0$ s.t. $\underbrace{\text{no } \delta \text{ works}}_{\Downarrow}$

for any $\delta > 0$ $\exists$ $x, y \in [a, b]$ s.t.

$|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$

$\Rightarrow$ In particular $\exists$ $x_n, y_n \in [a, b]$ s.t.

$|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$ —— (1)

Since $(x_n) \subseteq [a, b]$, (x_n) is bounded

$\Rightarrow$ (x_n) has a convergent subseq.

Let $x_{n_k} \longrightarrow x_0$.

$$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \longrightarrow 0$$

$$\Rightarrow \quad y_{n_k} \longrightarrow x_0$$

Since f is cts, $x_{n_k} \longrightarrow x_0$, $y_{n_k} \longrightarrow x_0$

$$f(x_{n_k}) \longrightarrow f(x_0), \qquad f(y_{n_k}) \longrightarrow f(x_0)$$

$$\Rightarrow \quad |f(x_{n_k}) - f(y_{n_k})| \longrightarrow 0$$

This is a contradiction to eq^n (1).

Applications of Riemann's criterion for integrability

Theorem

If f is continuous on $[a, b]$ then f is integrable.

Proof: $f : [a,b] \longrightarrow \mathbb{R}$ cts. $\Rightarrow f$ is unif. cts.

$\Rightarrow \forall \epsilon > 0, \exists \delta > 0$ s.t.

$$|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon$$

Recall: (Riemann's criterion) If for all $\epsilon > 0$ $\exists$ a partition P s.t. $U(P,f) - L(P,f) < \epsilon$, then f is integrable.

Let $\epsilon > 0$.

Let P be a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$

s.t. $x_i - x_{i-1} < \delta \quad \forall \ i$

$$\Rightarrow \quad |f(x) - f(y)| < \epsilon \quad \forall \quad x, y \in [x_{i-1}, x_i] \quad \forall \ i.$$

$$\Rightarrow \quad \underbrace{M_i}_{\substack{\| \\ \sup\limits_{x \in [x_{i-1}, x_i]} f(x)}} - \underbrace{m_i}_{\substack{\| \\ \inf\limits_{x \in [x_{i-1}, x_i]} f(x)}} < \epsilon \quad \forall \quad i = 1, 2, \ldots, n$$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \epsilon \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \epsilon (b - a)$$

Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is a monotone function then f is integrable.

Proof: Let f be monotonically increasing.
Choose partition P_n of $[a,b]$ s.t.

$$x_i - x_{i-1} = \frac{b-a}{n}$$

$$U(P_n, f) - L(P_n, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$$

$$= \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)(x_i - x_{i-1})$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)$$

$$= \frac{b-a}{n} \big(f(b) - f(a)\big) \longrightarrow 0$$

$\Rightarrow$ f is integrable.

Questions?

Additional discussion: Limit and integration

For $n \geq 1$, define $g_n : (0, 1] \longrightarrow \mathbb{R}$ as follows.

$$g_n(x) = \begin{cases} n & \text{if } x \in (0, 1/n], \\ 0 & \text{if } x \in (1/n, 1]. \end{cases}$$

$$\lim_{n\to\infty} \int g_n(x)\,dx = 1$$

$$\int \lim_{n\to\infty} g_n(x)\,dx = 0$$

$g_2 = 0$ on $(\tfrac{1}{2}, 1]$

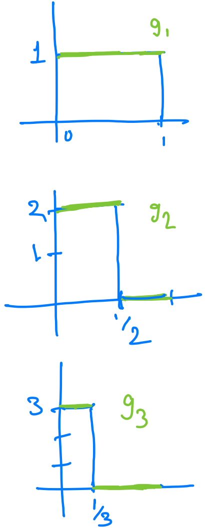
$g_3 = 0$ on $(\tfrac{1}{3}, 1]$

$g_4 = 0$ on $(\tfrac{1}{4}, 1]$

$g_5 = 0$ on $(\tfrac{1}{5}, 1]$

$\vdots$

$(\lim g_n) = 0$ on $(0, 1]$



Uniform continuous function preserves Cauchy sequences

Theorem

Let $f : D \longrightarrow \mathbb{R}$ be a uniform continuous function on D. If (a_n) is Cauchy sequence in D, then $(f(a_n))$ is a Cauchy sequence in $\mathbb{R}$.

Try to prove this at home.