Tutorial-2 Solution

1) (i) $F(\frac{3}{2}) = \frac{3}{4} \neq F(3/2+) = 1 \Rightarrow F(x)$ is not right continuous at $\frac{3}{2}$. So $F(x)$ is not a cdf

(ii) $F(1-) = F(1) = F(1+) = 1-1 = 0$.

So $F(x)$ is continuous everywhere hence right continuous.

$F(x)$ is non-decreasing

$\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$.

$\Rightarrow F(x)$ is a c.d.f.

(iii) $F(0) = 0$, $F(0+) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

So $F(0) \neq F(0+) \Rightarrow$ not right continuous

So F is not a cdf.

(iv) $F(0) = 0$, $F(0+) = 0 \Rightarrow F(0) = F(0+)$

$F(1) = \frac{1+1}{8} = \frac{1}{4} = F(1+)$

$F(2) = \frac{4+1}{8} = 5/8 = F(2+)$

$F(3) = F(3+) = 1$

So F is right continuous at $0, 1, 2, 3$

Also $F(x)$ is continuous on $(-\infty, 1)$, $(1,2)$, $(2,3)$ $(3,\infty)$.

So F is right continuous on $\mathbb{R}$.

F is non-decreasing in $(-\infty, 0)$, $(0,1)$, $(1,2)$, $(2,3)$ & $(3,\infty)$. Also

$$F(0) - F(0-) = 0 \geq 0.$$

$$F(1) - F(1-) = \frac{1}{4} - \frac{1}{8} > 0$$

$$F(2) - F(2-) = \frac{5}{8} - \frac{1}{4} > 0$$

$$F(3) - F(3-) = 1 - 5/8 > 0$$

So $F(x)$ is non-decreasing in $\mathbb{R}$.

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$$

$\Rightarrow F(x)$ is a c.d.f.

(2) $$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}$$

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1$$

Since $F(x)$ is continuous everywhere so right continuous

$F(x)$ is also non-decreasing

$\Rightarrow F(x)$ is a c.d.f.

$P(2 < X \leq 3) = F(3) - F(2) = e^{-2} - e^{-3}$

$P(-2 < X \leq 3) = 1 - e^{-3}$

$P(1 \leq X < 4) = F(4) - F(1) = e^{-1} - e^{-4}$

$P(5 \leq X < 8) = F(8) - F(5) = e^{-5} - e^{-8}$

(3) Since F is right continuous so

$$F(20) = F(20+)$$

$$\Rightarrow \quad 16K^2 - 16K + 3 = 0 \quad \Rightarrow K = \frac{1}{4} \text{ or } K = \frac{3}{4} \quad (1)$$

Also F is non-decreasing

$$F(5-) \leq F(5) \Rightarrow \frac{2}{3} \leq \frac{7}{6} - K$$

$$\Rightarrow K \leq \frac{7}{6} - \frac{2}{3} = \frac{1}{2} \quad (2)$$

From (1) & (2) we have $K = \frac{1}{4}$.

$$F(x) = \begin{cases} 0, & x < 2 \\ 2/3 & 2 \leq x < 5 \\ \frac{11}{12}, & 5 \leq x < 9 \\ \frac{91}{96}, & 9 \leq x < 14 \\ 1, & x \geq 14 \end{cases}$$

The set of discontinuity points $D = \{2, 5, 9, 14\}$

Moreover

$P(X=2) = F(2) - F(2-) = 2/3$

$P(X=5) = F(5) - F(5-) = \frac{1}{4}$

$P(X=9) = F(9) - F(9-) = \frac{1}{32}$

$P(X=14) = F(14) - F(14-) = 5/96$

$P(X \in D) = P(X=2) = P(X=5) + P(X=9) + P(X=14)$

$= 1$

so X is a discrete r.v. with support

$S = \{2, 5, 9, 14\}$

p.m.f of X given as

$$f_X(x) = P(X=x) = \begin{cases} 2/3, & x=2 \\ 1/4, & x=5 \\ 1/32, & x=9 \\ \frac{5}{96}, & x=14 \\ 0, & o/w. \end{cases}$$

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(4) The set of discontinuity points of $F(x)$ are

$D = \{1, 2, 5/2\}$. Since $D \neq \phi$ so the r.v. is not of continuous type.

$P(X \in D) = P(X=1) + P(X=2) + P(X=5/2)$

$= F(1) - F(1-) + F(2) - F(2) + F(5/2) - F(5/2-)$

$= \frac{11}{48} < 1 \Rightarrow X$ is not a discrete r.v.

(b)

$P(1 < X \leq 5/2) = F(5/2) - F(1) = 1 - \frac{1}{3} = 2/3$

$P(1 < X < 5/2) = F(5/2-) - F(1) = \frac{15}{16} - \frac{1}{3} = \frac{29}{48}$

$P(1 \leq X < 5/2) = F(5/2-) - F(1-) = \frac{15}{16} - \frac{1}{4} = \frac{11}{16}$

$P(-2 \leq X < 1) = F(1-) - F(-2-) = \frac{1}{4} - 0 = \frac{1}{4}$

$P(X \geq 2) = 1 - F(2-) = 1 - \frac{1}{3} = \frac{1}{3}$

$P(X > 2) = 1 - F(2) = 1 - \frac{3}{4} = \frac{1}{4}$

(5) (a) Since X is continuous type with p.d.f $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-1/2}^{1/2} (k - |x|)\,dx = 1$$

$$\Rightarrow \int_{-1/2}^{0} (k+x)\,dx + \int_{0}^{1/2} (k-x)\,dx = 1 \Rightarrow k = 5/4$$

Also for $k = 5/4$, $f(x) \geq 0 \quad \forall x \in \mathbb{R}$.

(b) $$P(X < 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-1/2}^{0} \left(\frac{5}{4} + x\right) dx = \frac{1}{2}$$

$= P(X \leq 0)$.

$||^{ly}$ do others

(c) for $x < \frac{1}{2}$, $F_X(x) = 0$.

$-\frac{1}{2} \le x < 0$, $F_X(x) = \int_{-1/2}^{x} \left(\frac{5}{4} + t\right) dt = \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}$

$0 \le x < \frac{1}{2}$, $F_X(x) = \int_{-1/2}^{0} \left(\frac{5}{4} + t\right) dt + \int_{0}^{x} \left(\frac{5}{4} - t\right) dt$

$= -\frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}$

$x \geq \frac{1}{2}$, $F_X(x) = \int_{-1/2}^{0} \left(\frac{5}{4} + t\right) dt + \int_{0}^{1/2} \left(\frac{5}{4} - t\right) dt = 1$.

$$F_X(x) = \begin{cases} 0, & x < -1/2 \\ \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ -\frac{x^2}{2} + \frac{5x}{4} + \frac{1}{2}, & 0 \le x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

$$= \begin{cases} 0, & x < -1/2 \\ -\frac{x|x|}{2} + \frac{5}{4}x + \frac{1}{2}, & -\frac{1}{2} \le x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

(6) Similare to (5)

(7) $f(x) = \begin{cases} \frac{1}{\beta}\left(1 - \frac{|x-\alpha|}{\beta}\right), & \alpha-\beta < x < \alpha+\beta \\ 0, & o/w \end{cases}, \quad \alpha \in \mathbb{R}, \beta > 0$

$$\int_{\alpha-\beta}^{\alpha+\beta} \frac{1}{\beta}\left\{1 - \frac{|x-\alpha|}{\beta}\right\} dx, \quad y = \frac{x-\alpha}{\beta}$$

$$= \int_{-1}^{1}\left(1-|y|\right) dy = 2\int_{0}^{1}(1-y)\, dy = 1$$

Also $f(x) \geq 0 \quad \forall\, x \in \mathbb{R}$.

$\Rightarrow$ $f(x)$ is a pdf.

$$F(x) = \int_{\alpha-\beta}^{x} \frac{1}{\beta}\left(1 - \frac{|t-\alpha|}{\beta}\right), \quad \alpha-\beta < t < \alpha+\beta$$

$$= \int_{-1}^{\frac{x-\alpha}{\beta}} \left(1-|y|\right) dy, \quad y = \frac{t-\alpha}{\beta}$$

If $\frac{x-\alpha}{\beta} < 0$ i.e. $x < \alpha$, we have

$$F(x) = \int_{-1}^{\frac{x-\alpha}{\beta}} (1+y)\, dy = \left.\frac{(1+y)^2}{2}\right|_{-1}^{\frac{x-\alpha}{\beta}}$$

$$= \frac{1}{2}\left[1 + \frac{x-\alpha}{\beta}\right]^2, \quad \alpha - \beta < x \le \alpha$$

Now if $\frac{x-\alpha}{\beta} > 0$, so for $\alpha \le x < \alpha + \beta$

$$F(x) = \int_{-1}^{0} (1 - |y|)\, dy + \int_{0}^{\frac{x-\alpha}{\beta}} (1 - |y|)\, dy$$

$$= \frac{1}{2} + \int_{0}^{\frac{x-\alpha}{\beta}} (1-y)\, dy = 1 - \frac{1}{2}\left(1 - \frac{x-\alpha}{\beta}\right)^2$$

$$F(x) = \begin{cases} 0, & x \le \alpha - \beta \\ \frac{1}{2}\left[1 + \left(\frac{x-\alpha}{\beta}\right)\right]^2, & \alpha - \beta < x \le \alpha \\ 1 - \frac{1}{2}\left(1 - \frac{x-\alpha}{\beta}\right)^2, & \alpha < x \le \alpha + \beta \\ 1, & x > \alpha + \beta \end{cases}$$

$E(X) = \alpha.$

$$Var(X) = E(X-\alpha)^2 = \int_{\alpha-\beta}^{\alpha+\beta} \frac{(x-\alpha)^2}{\beta}\left\{1 - \frac{|x-\alpha|}{\beta}\right\} dx$$

$$= \beta^2 \int_{-1}^{1} y^2 (1 - |y|)\, dy = 2\beta^2 \int_{0}^{1} y^2 (1-y)\, dy$$

$$= \beta^2/6$$