Consider the following problem

$$\text{Min } Z = 3x_1 + 4x_2$$

subject to $\quad 2x_1 + 3x_2 \geq 8$

$5x_1 + 2x_2 \geq 12, \quad x_1, x_2 \geq 0$

Sol^n The first step is to convert the inequalities into equations. We add two surplus variable x_3 and x_4 that have to subtracted from the left hand side values to equate to the RHS values. The problem is now

$$\min Z = 3x_1 + 4x_2$$

s.t

$$\left.\begin{aligned} 2x_1 + 3x_2 - x_3 &= 8 \\ 5x_1 + 2x_2 - x_4 &= 12, \quad x_i \geq 0 \end{aligned}\right\}$$

We need a starting B.F.S to start the simplex algorithm. The easiest among them is to fix surplus variable as basic variable and solve them. Unfortunately this solution is $x_3 = -8$ and $x_4 = -12$ which is infeasible. Hence this is not a B.F.S and we cannot use this as initial B.F.S to start the Simplex algorithm.

Thus we have to identify a starting B.F.S which has some other set of basic variables

One way to do that is to follow the algebraic method till we get a B.F.S. and start with this as the first iteration for simplex method. We don't follow this approach because it may take many trials in an algebraic method before we get a first B.F.S.

we normally use an indirect approach to get starting solution for the simplex table. If we introduced two new variables a_1 and a_2 such that

$$2x_1 + 3x_2 - x_3 + a_1 = 8$$

$$5x_1 + 2x_2 - x_4 + a_2 = 12$$

Then a_1 and a_2 will automatically provide the starting B.F.S where $a_1 = 8$, $a_2 = 12$ are basic variables and then we can proceed to the simplex table.

We should understand that a_1 and a_2 are not part of our original problem and have been introduced to get a B.F.S. These are called artificial variables. Since they are not part of the original problem, we have to ensure that they should not be in the optimal solution (when we find the optimum). We ensure this by giving a very large and positive value to the objective function.

The problem is now

$$\min Z = 3x_1 + 4x_2 + 0.x_3 + 0.x_4 + Ma_1 + Ma_2$$

where $M > 0$ and large

$$\text{s.t} \quad 2x_1 + 3x_2 - x_3 + a_1 = 8$$

$$5x_1 + 2x_2 - x_4 + a_2 = 12$$

$$x_i \geq 0, a_i \geq 0$$

Note 1

If the original problem has an optimal soln, it will not involve a_1 or a_2 because we added a large positive co-efficient in the objective function for a_1 and a_2.

Note

$$\min Z = -\max(-Z)$$

Since our standard form has a maximum objective function, we multiply the co-efficient of the objective function by -1 to convert it to a maximization problem.

$$-3x_1 - 4x_2 - 0.x_3 - 0.x_4 - Ma_1 - Ma_2$$

C_J		-3	-4	0	0	$-M$	$-M$		
C_B	x_B	x_1	x_2	x_3	x_4	a_1	a_2	R.H.S	θ
$-M$	a_1	2	3	-1	0	1	0	8	4
$-M$	a_2	5	2	0	-1	0	1	12	$\frac{12}{5}$ →
$C_J - Z_J$		$7M-3$ ↑	$5M-4$	$-M$	$-M$	0	0		
$-M$	a_1	0	$\frac{11}{5}$	-1	$\frac{2}{5}$	1	$-\frac{2}{5}$	$\frac{16}{5}$	$16/11$ →
-3	x_1	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{12}{5}$	6
$C_J - Z_J$		0	$\frac{11M-14}{5}$ ↑	$-M$	$\frac{2M-3}{5}$	0	$\frac{-7M+3}{5}$		$R_1' = R_1 - 2R_2'$
-4	x_2	0	1	$-\frac{5}{11}$	$\frac{2}{11}$	$\frac{5}{11}$	$-\frac{2}{11}$	$\frac{16}{11}$	
-3	x_1	1	0	$\frac{2}{11}$	$-\frac{3}{11}$	$-\frac{2}{11}$	$\frac{3}{11}$	$\frac{20}{11}$	
$C_J - Z_J$		0	0	$-\frac{14}{11}$	$-\frac{1}{11}$	$-M+\cdots$ (−ve)	$-M+\cdots$ (−ve)		

Here the algorithm terminates as all the non-basic variables x_3, x_4, a_1, a_2 have negative values of $C_J - Z_J$. The optimal soln is

$x_1 = \frac{20}{11}$, $x_2 = \frac{16}{11}$ and optimal value $= -\left(-3\cdot\frac{20}{11} - 4\cdot\frac{16}{11}\right)$

$$= \frac{124}{11}$$

The simplex method involving big M is called the Big M method.

Prob 2 Solve the problem

$$\text{Max } Z = 7x_1 + 5x_2$$

$$\text{s.t} \quad 2x_1 + 3x_2 = 7$$

$$5x_1 + 2x_2 \geq 11 \quad , \quad x_1, x_2 \geq 0$$

we convert the second inequality as equation by adding a surplus variable x_3.

$$\text{Max } Z = 7x_1 + 5x_2$$

$$\text{s.t} \quad 2x_1 + 3x_2 = 7$$

$$5x_1 + 2x_2 - x_3 = 11$$

Since we are not getting the initial B.F.S by taking x_3 as basic variable, we add two artificial variable a_1 and a_2. The problem transformed to

$$\text{Max } Z = 7x_1 + 5x_2 + 0 \cdot x_3 - Ma_1 - Ma_2$$

S.t $\quad 2x_1 + 3x_2 + a_1 = 7$

$\quad 5x_1 + 2x_2 - x_3 + a_2 = 11$

	C_j	7	5	0	$-M$	$-M$		
C_B	X_B	x_1	x_2	x_3	a_1	a_2	R.H.S	θ
$-M$	a_1	2	3	0	1	0	7	$\frac{7}{2}$
$-M$	a_2	5	2	-1	0	1	11	$\frac{11}{5}$ $\rightarrow$
	$C_j - Z_j$	$7+7M$ $\uparrow$	$5+5M$	$-M$	0	0		
$-M$	a_1							
7	x_1							