Lagrange multiplier method

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function. Lagrange multiplier method is used for maximizing or minimizing a general objective function subject to a constraint of the form $g(x,y) = k$.

The following steps consitutes the method of Lagrange multiplier.

1. Find ∇f and ∇g in terms of x and y. Then set up the equations

$$\boxed{\begin{aligned} \nabla f &= \lambda \nabla g \\ \text{and} \quad g(x,y) &= k \end{aligned}}$$

where λ is a scalar which is known as Lagrange multiplier

This will give you a system of equations based on the components of the gradients.

2. Solve this system of equations to get x, y, λ.

3. For each of this solutions, find $f(x,y)$ and compare the values. Every solution that gives a maximum value is a maximum point and every sol^n that gives a minimum value is a minimum point.

Prob 1 Use Lagrange multiplier method to find the maximum and minimum values of the function $f(x,y) = 3x + y$ subject to the given constraints $x^2 + y^2 = 10$.

Soln For this problem

$$f(x,y) = 3x + y$$

$$g(x,y) = k \Rightarrow x^2 + y^2 = 10$$

Let us go through the steps

$$\nabla f = (3, 1) \quad \text{and} \quad \nabla g = (2x, 2y)$$

Now $\nabla f = \lambda \nabla g$ and

$g(x,y) = 10$ gives us

$$\Rightarrow \quad (3, 1) = \lambda (2x, 2y)$$

and $x^2 + y^2 = 10$

$$\Rightarrow \quad 2x\lambda = 3, \quad 2y\lambda = 1$$

and $x^2 + y^2 = 10$

$$\Rightarrow \quad x = \left(\frac{3}{2\lambda}\right) \quad \text{and} \quad y = \left(\frac{1}{2\lambda}\right)$$

now substituting this in $x^2 + y^2 = 10$, we get

$$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$$

$$\Rightarrow \quad (9+1) = 10 \times 4\lambda^2$$

$$\Rightarrow \quad \lambda = \pm \frac{1}{2}$$

$\underline{\lambda = \frac{1}{2}}$, $\quad x = 3$ and $y = 1$

$\underline{\lambda = -\frac{1}{2}}$, $\quad x = -3$, $y = -1$

we get the following extreme points $(3, 1)$ and $(-3, -1)$ of the given optimization problem.

we can classify them by simply finding this value.

$$\boxed{\begin{aligned} f(3,1) &= 10 \\ f(-3,-1) &= -10 \end{aligned}}$$

$\therefore$ $(3,1)$ is a maximum point and $(-3,-1)$ is minimum point

Prob 2 Use Lagrange multiplier method to find the maximum and minimum of $f(x,y) = xy$ subject to $x^2+y^2 = 8$.

Soln maximum value is 4 and it is occuring at $(2,2)$ and $(-2,-2)$.

minimum value is -4 and it is occuring at $(-2,2)$ and $(2,-2)$.

□ For functions of three variables $f(x,y,z)$ subject to two constraints $g(x,y,z) = k$ and $h(x,y,z) = c$.

Then we can consider

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

where λ and μ are Lagrange multipliers.

The following steps is used to find maximum or minimum values for $f(x,y,z)$ subject to $g(x,y,z) = k$ and $h(x,y,z) = c$

1. Find ∇f, ∇g and ∇h. Then set up the equations

$$\boxed{\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)}$$

$$\boxed{g(x,y,z) = k}$$

and $$\boxed{h(x,y,z) = c}$$

Note that we have now five unknowns. If we break the first equation, we will get three equation and hence we will have five equations.

Prob 3 Use Lagrange multiplier method to find the maximum and minimum values of the function $f(x,y,z) = 3x - y - 3z$ subject to $x+y-z=0$ and $x^2+2z^2=1$.

Sol^n Here, $f(x,y,z) = 3x - y - 3z$

$g(x,y,z) = k \Rightarrow x+y-z=0$

$h(x,y,z) = c \Rightarrow x^2+2z^2=1$

$\nabla f = (3, -1, -3)$

$$\nabla g = (1, 1, -1) \quad \text{and} \quad \nabla h = (2x, 0, 4z)$$

Then the set of equations

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$g(x, y, z) = K$$

$$\text{and} \quad h(x, y, z) = C$$

This implies that

$$(3, -1, -3) = \lambda(1, 1, -1) + \mu(2x, 0, 4z)$$

$$x + y - z = 0$$

$$x^2 + 2z^2 = 1$$

$$\Rightarrow \quad \lambda + 2\mu x = 3 \quad \text{—— (I)}$$

$$\lambda + 0 = -1 \quad \text{—— (II)}$$

$$-\lambda + 4z\mu = -3 \quad \text{—— (III)}$$

$$x + y - z = 0 \quad \text{—— (IV)}$$

$$x^2 + 2z^2 = 1 \quad \text{—— (V)}$$

By (2), $\lambda = -1$,

From (I) & (III), we get $z = -\frac{1}{\mu}$ and $x = \frac{2}{\mu}$

Substitute these value of x and z in (v),

we get

$$\frac{4}{\mu^2} + \frac{2}{\mu^2} = 1$$

$$\Rightarrow \quad 6 = \mu^2 \Rightarrow \mu = \pm\sqrt{6}$$

$\underline{\mu = \sqrt{6}}$, $\quad x = \frac{2}{\sqrt{6}}$ and $z = -\frac{1}{\sqrt{6}}$

and $\quad y = z - x = -\frac{3}{\sqrt{6}}$

$\underline{\mu = -\sqrt{6}}$ $\quad x = -\frac{2}{\sqrt{6}}$ and $z = \frac{1}{\sqrt{6}}$

and $\quad y = \frac{3}{\sqrt{6}}$

$\therefore$ The two extreme points to the original optimization problem are

$$\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \quad \text{and}$$

$$\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$\boxed{\text{Prob 2}}$ A rectangular box without a lid is to be made from 12 m2 of cardboard. Find the maximum volume of such a box.

Soln Let x, y and z be the length, width and height, of the box in meters.

$z = 2$

Then we have to

$$\max \; xyz$$

$$\text{Subject to} \quad xy + 2yz + 2zx = 12$$

$\therefore$ The maximum volume is 4 with $x = 2, y = 2$ and $z = 1$.