Floating Point Numbers and Excess-k Format for Signed Integers

Floating Point Number Representation

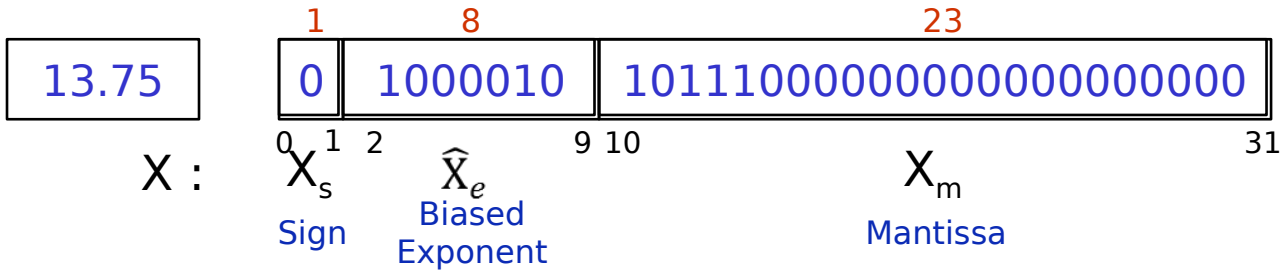
- Example: -6.3245×10^{-2}

  $13.75 = 1101.11 \times 2^{0}$

  $= 1.10111 \times 2^{3}$

  Normalized form

- IEEE Standard 754
- 32-bit single precision



  - Exponent is represented in Excess-k representation
  - bias, $k=2^{(8-1)}-1=2^{7}-1 = 127$
  - Excess-127
  - Example: True exponent=3,

    Biased exponent= 3+127 = 130

Excess-k Representation for Signed Integers

- Signed integers can also be represented using Excess-k format
- Integers obtained after representing the signed integers in excess-k format are called as biased integers
- Biased integer = true integer + k
  - k is called as bias
  - For any n-bit integers, bias, $k=2^{(n-1)}-1$
  - True integer: The actual value of an integer. It can be positive or negative value
  - Biased integer: The positive integer value obtained by adding bias to the actual integer
- This representation is typically used in representing the exponent part of the floating point number

Illustration of Excess-7 Format for 4-bit Signed Integers

Biased integer= True integer+k

x	$\hat{x}$	$\hat{x}$ *in binary*
-7	0	0000
-6	1	0001
-5	2	0010
-4	3	0011
-3	4	0100
-2	5	0101
-1	6	0110
0	7	0111
1	8	1000
2	9	1001
3	10	1010
4	11	1011
5	12	1100
6	13	1101
7	14	1110
8	15	1111

32-bit Single Precision

Range of numbers

32-bit Single Precision

- Exponent field is 8-bit in length
- Exponent is represented in Excess-k format
- Biased exponent is in the range: $0 \leq \hat{X}_e \leq 255$
- The biased exponent value 0 and 255 is used to represent special values
- Actual biased exponent takes the values from 1 to 254
  - Hence, true exponent is in the range: $-126 \leq X_e \leq +127$

X_e	$\hat{X}_e$	X_m	*Remark*
-	0	0	The value exact 0 is represented

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-	0	≠0	Denormalized value

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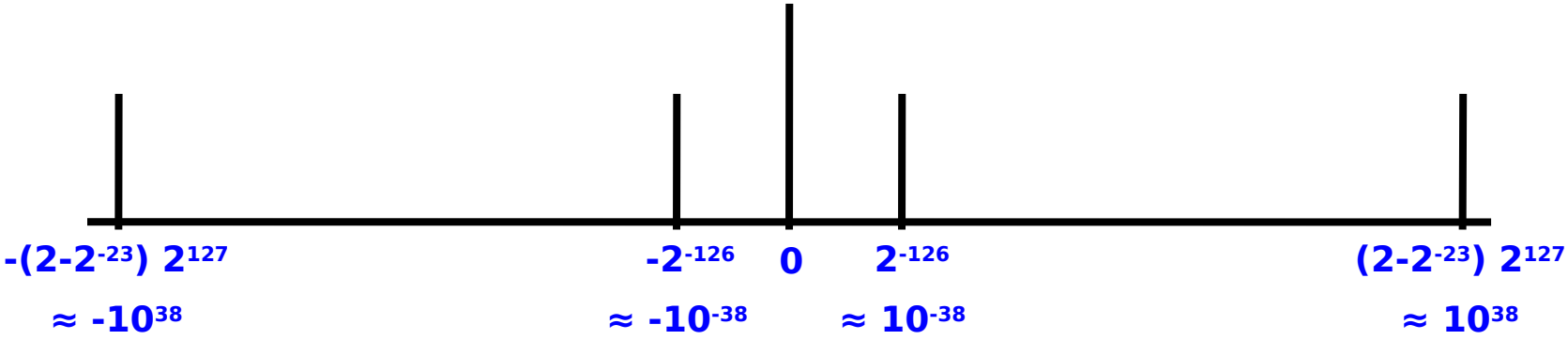
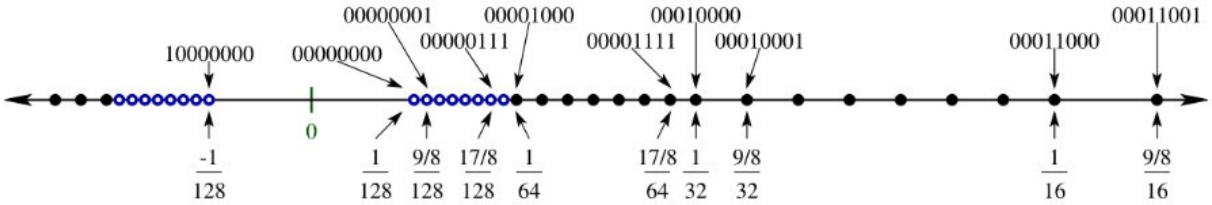
-	0	0	The value exact 0 is represented
-	255	0	The value ∞ is represented
-	0	≠0	Denormalized value
-	255	≠0	Not a number (NaN)
-126 to 127	1 to 254	0 or ≠0	Normalized value

Range and Resolution in 32-bit Single Precision

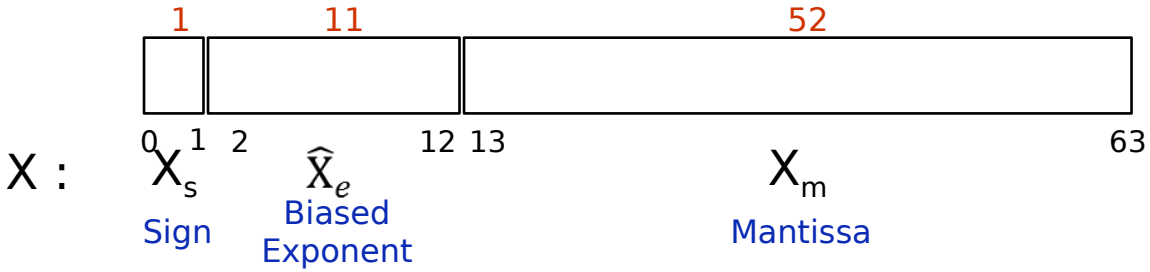
- Range:



- Resolution:
  - Different exponent will have different resolution



64-bit Double Precision



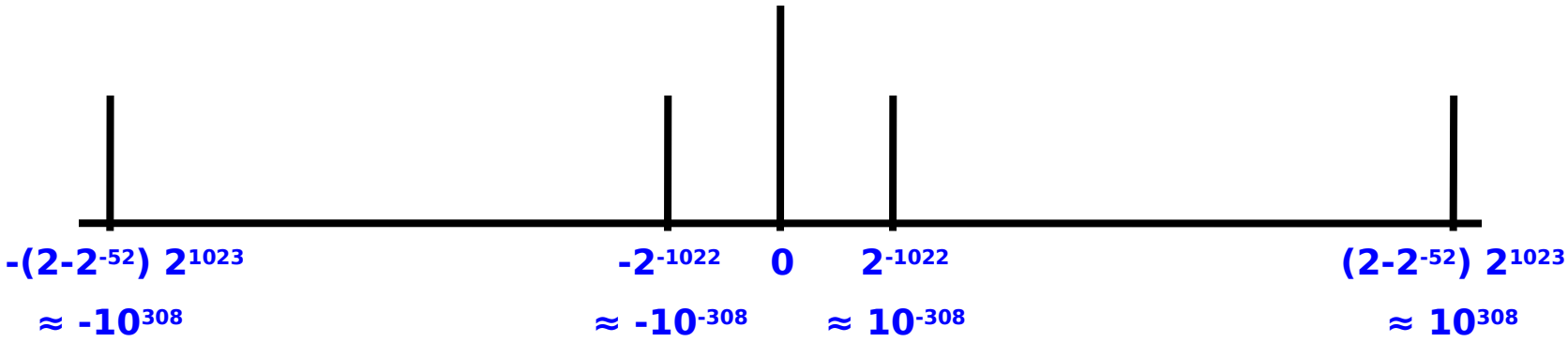
- Exponent field is 11-bit in length
- Exponent is represented in Excess-1023 format
- Biased exponent is in the range: $0 \leq \hat{X}_e \leq 2047$
- The biased exponent value 0 and 2047 is used to represent special values
- Actual biased exponent takes the values from 1 to 2046
  - Hence, true exponent is in the range:

$$-1022 \leq X_e \leq +1023$$

- Resolution: $2^{-52+\text{true exponent}}$

Range and Resolution in 64-bit Double Precision

- Range:



- Resolution:
  - Different exponent will have different resolution
  - $2^{-52+\text{true exponent}}$

Arithmetic Operations on Floating Point Numbers

Floating Point Addition/Subtraction

- X: X_s $\hat{X}_e$ X_m
- Y: Y_s $\hat{Y}_e$ Y_m
- $Z = X + Y$ or $Z = X - Y$
- Resultant Z: Z_s $\hat{Z}_e$ Z_m
- Focus: 32-bit single precision floating point numbers
- Addition Subtraction Rule:
  1. Choose the number with smallest exponent
     - Shift its mantissa right a number of steps equal to the difference of exponent
  2. Set the exponent of the result equal to the larger exponent
  3. Perform addition/subtraction on the mantissas and determine the sign of the result
  4. Normalize the resulting value, if necessary

Floating Point Addition/Subtraction: Example 1

- X: $X_s\ \hat{X}_e\ X_m$ X: $1.00000...00 \times 2^0$
- Y: $Y_s\ \hat{Y}_e\ Y_m$ Y: $1.11110...00 \times 2^{-5}$
- $Z = X + Y$
- Addition Subtraction Rule:
  1. Choose the number with smallest exponent and let that be Y

     Y: $1.11110...00 \times 2^{-5}$

     Shift its mantissa right a number of steps equal to the difference of exponents

     difference = |0+5| = 5

     Y: $0.0000111110...00 \times 2^0$
  2. Perform addition/subtraction on the mantissas and determine the sign of the result

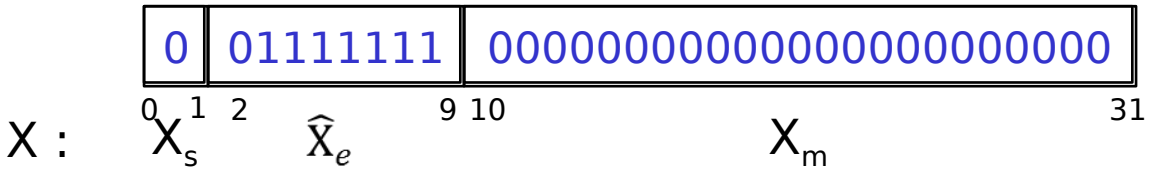
     X: $1.0000000000...00 \times 2^0$

     Y: $0.0000111110...00 \times 2^0$

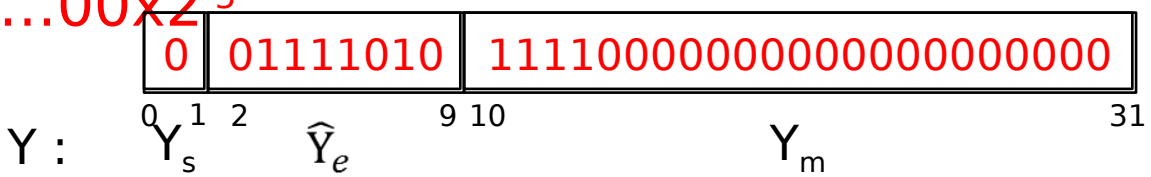
     Z: $1.0000111110...00 \times 2^0$
  3. Normalize the resulting value, if necessary

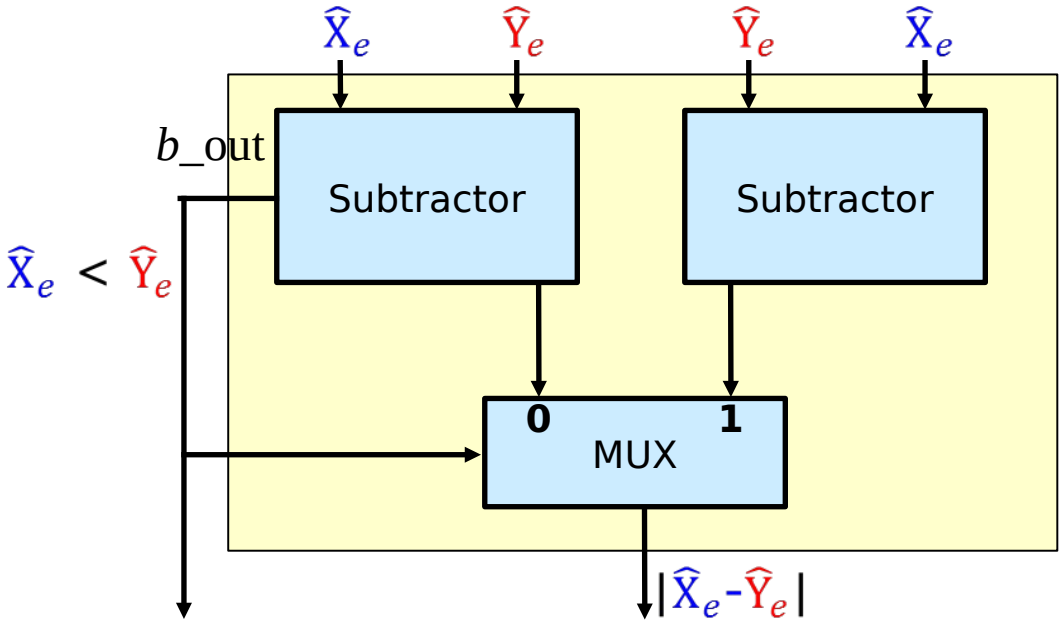
Exponent Comparator

X: $1.00000\ldots00 \times 2^{0}$



Y: $1.11110\ldots00 \times 2^{-5}$





X	Excess-3	2s complement
-7	0000	1001
-6	0001	1010
-5	0010	1011
-4	0011	1100
-3	0100	1101
-2	0101	1110
-1	0110	1111

Floating Point Multiplication and Division

32-bit single precision

Multiply rule:

Add the exponent and subtract 127 (i.e. bias)

Multiply the mantissas and determine the sign of the result

Normalize the resulting value, if necessary

Ariane 5

- Exploded 37 seconds after liftoff
- Cargo worth $500 million

Why

- Computed horizontal velocity as floating-point number
- data conversion from 64-bit floating point to 16-bit signed integer value
- Worked OK for Ariane 4
- Overflowed for Ariane 5
  - Used same software