Lecture #10 (IC152)

Def^n: (Inner Product)

Let V be a vector space over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). An inner product on V is a function $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$, $(\alpha, \beta) \in V \times V \longmapsto \langle \alpha, \beta \rangle \in \mathbb{F}$ satisfying the following properties

1) $\langle \alpha, \alpha \rangle > 0$ if $\alpha \neq 0$ $\forall \alpha$ in V
   & $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0$

2) $\langle \alpha + \beta, \gamma \rangle = \langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$
   $\forall \alpha, \beta, \gamma$ in V.

3) $\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$ ✓

4) ✓ $\langle c\alpha, \beta \rangle = c \langle \alpha, \beta \rangle$

$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$V = \mathbb{R}^3$ ✓

$\langle \alpha, \beta \rangle$

$\alpha \cdot \beta$

(under complex field)

Remark : a) In property 3) (...) complex conjugate is necessary.

$\alpha \neq 0$, $\langle \alpha, \alpha \rangle > 0$, $\langle i\alpha, i\alpha \rangle \underset{(4)}{=} i \langle \alpha, i\alpha \rangle$

$\underset{(3)}{=} i \langle i\alpha, \alpha \rangle$ ⟶ without complex conjugate

$\underset{(4)}{=} i \cdot i \langle \alpha, \alpha \rangle$

$= - \langle \alpha, \alpha \rangle$

< 0

which is a contradiction of property 1) for $\beta = i\alpha$.

b) Observe that $\langle \alpha, c\beta \rangle \underset{(3)}{=} \overline{\langle c\beta, \alpha \rangle} \underset{(4)}{=} \overline{c \langle \beta, \alpha \rangle}$

$= \bar{c}\, \overline{\langle \beta, \alpha \rangle} = \bar{c} \langle \alpha, \beta \rangle$

Example :- Let $V = \mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

$x = (x_1, x_2, \ldots x_n)$, $x_i \in \mathbb{F}$

$y = (y_1, y_2, \ldots y_n)$

then $\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \ldots x_n \bar{y}_n = \sum_{j=1}^{n} x_j \bar{y}_j$

... inner product on $\mathbb{F}^n$

Claim: $\langle \cdot , \cdot \rangle$ defines an ... product on $\mathbb{F}$

& is known as "Standard inner product"

Let us verify

* $\langle x, x \rangle = \sum_{j=1}^{n} |x_j|^2 \geq 0 \Rightarrow \langle x, x \rangle = 0$ iff $x = 0$

* $\langle x+y, z \rangle = \langle (x_1+y_1, x_2+y_2, \dots x_n+y_n), (z_1, z_2, \dots z_n) \rangle$

$$= \sum_{j=1}^{n} (x_j + y_j)\overline{z}_j = \sum_{j=1}^{n} (x_j \overline{z}_j + y_j \overline{z}_j)$$

$$= \sum_{j=1}^{n} x_j \overline{z}_j + \sum_{j=1}^{n} y_j \overline{z}_j$$

$$= \langle x, z \rangle + \langle y, z \rangle$$

* $\langle y, x \rangle = \overline{\langle x, y \rangle}$

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y}_j$$

$$\overline{\langle x, y \rangle} = \overline{\sum_{j=1}^{n} x_j \overline{y}_j} = \sum_{j=1}^{n} \overline{x_j \overline{y}_j} = \sum_{j=1}^{n} \overline{x}_j y_j$$

$$= \sum_{j=1}^{n} y_j \overline{x}_j$$

$$= \langle y, x \rangle$$

$$* \quad \langle cx, y \rangle = \sum_{j=1}^{n} c x_j \bar{y}_j = c \sum_{j=1}^{n} x_j \bar{y}_j = c \langle x, y \rangle$$

$c(x_1, x_2, \ldots x_n)$
$= (cx_1\ cx_2 \ldots cx_n)$

<u>Note</u>: $\langle x, cy \rangle = \sum_{j=1}^{n} x_j \overline{c y_j} = \sum_{j=1}^{n} \bar{c} x_j \bar{y}_j$
$= \bar{c} \langle x, y \rangle$

Hence all the properties are satisfied.

<u>Example</u>: $V = M_{n\times n}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$1) \text{ — } \langle A, B \rangle = \sum_{j,k=1}^{n} a_{jk} \bar{b}_{jk}$$

$A = (a_{jk})_{j,k=1,\ldots n}$. $B = (b_{jk})$

<u>Claim</u> 1) defines an innerproduct on V.

Excercise :

Example: $V = \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

Claim: It is an inner product.

$$\langle (x_1, x_2), (x_1, x_2) \rangle = x_1^2 - x_1 x_2 - x_1 x_2 + 4 x_2^2$$

$$= x_1^2 - \underline{\underline{2 x_1 x_2}} + 4 x_2^2$$

$$= (x_1 - x_2)^2 + 3 x^2$$

$$\geq 0$$

0 only if $(x_1 - x_2) = 0$ & $3 x_2^2 = 0$

$x_1 = x_2$ & $x_2 = 0$

$\Rightarrow x_1 = x_2 = 0$

$x = 0$

Example:

Let $\langle x, y \rangle$ is an inner product.

Claim $\langle x, y \rangle_\lambda = \lambda \langle x, y \rangle$ $\mathbb{R} \ni \lambda > 0$

$\langle x, y + z \rangle \quad \lambda \langle x, z \rangle$

$$\checkmark\quad \langle x+y, z\rangle_\lambda = \lambda\langle x+y, z\rangle = \lambda\langle x, z\rangle + \lambda\langle y, z\rangle$$
$$= \langle x, z\rangle_\lambda + \langle y, z\rangle_\lambda$$

$$\langle x, y\rangle_\lambda = \overline{\langle y, x\rangle_\lambda} \quad \checkmark$$

$$\langle cx, y\rangle_\lambda = \lambda\langle cx, y\rangle = \lambda c\langle x, y\rangle = c\langle x, y\rangle_\lambda$$

$$\overline{\langle x, y\rangle_\lambda} = \overline{\lambda\langle x, y\rangle} = \lambda\,\overline{\langle x, y\rangle} = \lambda\langle y, x\rangle$$
$$= \langle y, x\rangle_\lambda$$

Remark: V, W be vector spaces over a field $\mathbb{F}$

$T: V \to W$ non-singular linear transformation

Let $\langle \cdot, \cdot\rangle$ be an inner product on W

Then $\langle \alpha, \beta\rangle_T = \langle T\alpha, T\beta\rangle$ ✓

Verify if $\langle \alpha, \beta\rangle_T$ defines an inner product on V

$\Rightarrow$ What if $W = V$ then from a given inner product on V we can define another inner product on V ... linear operator via a non-singular ...

using $\langle \alpha, \beta \rangle_T = \langle T\alpha, T\beta \rangle$ ✓

✓ $\langle \, , \, \rangle$ is an innerproduct on V (given)

Let us try to prove the claim !!

* $\langle \alpha, \alpha \rangle_T = \langle T\alpha, T\alpha \rangle \geq 0$ & 0 iff $T\alpha = 0$

$\Downarrow$

$\alpha = 0$

(as T is non-singular)

* $\langle \alpha+\beta, \gamma \rangle_T = \langle T(\alpha+\beta), T\gamma \rangle$

$= \langle T\alpha + T\beta, T\gamma \rangle$

$= \langle T\alpha, T\gamma \rangle + \langle T\beta, T\gamma \rangle$

$= \langle \alpha, \gamma \rangle_T + \langle \beta, \gamma \rangle_T$

* Rest of the properties also hold true !!

(Exercise)