IC105: Probability and Statistics **2021-22-M**

Lecture 7: Functions of a Random Variable

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We state the following theorem without proof.

Theorem 7.1. *Let X be a r.v. with d.f. F. Suppose that F is differentiable everywhere except (possibly) on a countable set E. Further suppose that $\int_{-\infty}^{\infty} F'(t)\mathrm{d}t = 1$. Then, X is a continuous r.v with p.d.f.*

$$f(x) = \begin{cases} F'(x), & x \in E^c, \\ 0, & x \in E. \end{cases}$$

Remark 7.2. *(i) The p.d.f. determines the d.f. uniquely. Converse is not true. However, the d.f. determines the p.d.f. almost uniquely (they may vary on sets that have no length (or have zero content)). Thus it is enough to study the p.d.f. of a continuous r.v.*

(ii) Let X be continuous r.v. with p.d.f $f(x)$. Then,

$$(a) \;\; f(x) \geq 0, \;\; \forall\, x \in \mathbb{R} \;\; \textit{and} \;\; (b) \;\; \int_{-\infty}^{\infty} f(t)\mathrm{d}t = 1.$$

Conversely, suppose that $g : \mathbb{R} \to \mathbb{R}$ is a function such that

$$(a) \;\; g(x) \geq 0, \;\; \forall\, x \in \mathbb{R}, \quad (b) \;\; \int_{-\infty}^{\infty} g(t)\mathrm{d}t = 1.$$

Then, $g(\cdot)$ is the p.d.f. of some continuous r.v. having support $T = \left\{ x \in \mathbb{R} : \int_{x-h}^{x+h} g(t)\mathrm{d}t > 0, \;\; \forall\, h > 0 \right\}$.

Example 7.3. *Let X be a r.v. with d.f.*

$$F(x) = \begin{cases} 0, & \textit{if } x < 0, \\ x/4, & \textit{if } 0 \leq x < 1, \\ x/3, & \textit{if } 1 \leq x < 2, \\ 3x/8, & \textit{if } 2 \leq x < 5/2, \\ 1, & \textit{if } x \geq 5/2. \end{cases}$$

Examine whether X is a continuous r.v. or a discrete r.v. or none?

Solution: Let D be the set of discontinuity points of F. Then $D = \{1, 2, 5/2\}$. So, $D \neq \phi \implies X$ is not a continuous r.v. So

$$\sum_{x \in D} [F(x) - F(x-)] = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + \left(1 - \frac{15}{16}\right) = \frac{11}{48} < 1 \implies X \text{is not a discrete r.v.}$$

Thus, X is neither a discrete nor a continuous r.v.

Example 7.4. *Let X be a r.v with d.f.*

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2/2, & \text{if } 0 \le x < 1, \\ x/2, & \text{if } 1 \le x < 2, \\ 1, & \text{if } x \ge 2. \end{cases}$$

Show that X is a continuous r.v. Find the p.d.f. of X and support of X.

Solution: Clearly F is continuous everywhere. Moreover, F is differentiable everywhere except at three (countable) points $0, 1, 2$, and

$$F'(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 < x < 1, \\ 1/2, & \text{if } 1 < x < 2, \\ 0, & \text{if } x \ge 2. \end{cases}$$

Also, $\displaystyle\int_{-\infty}^{\infty} F'(x)\mathrm{d}x = \int_0^1 x\mathrm{d}x + \int_1^2 \frac{1}{2}\mathrm{d}x = 1 \implies$ X is continuous r.v. with p.d.f.

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1, \\ 1/2, & \text{if } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The support of X is

$$S = \{x \in \mathbb{R} : F(x+h) - F(x-h) > 0, \ \forall\, h > 0\} = \left\{ x \in \mathbb{R} : \int_{x-h}^{x+h} f(t)\mathrm{d}t > 0, \ \ \forall\, h > 0 \right\} = [0, 2].$$

Example 7.5. *Let X be a continuous r.v. with p.d.f.*

$$f(x) = \begin{cases} x^2, & \text{if } 0 < x < 1, \\ ce^{-x}, & \text{if } x \ge 1, \quad \text{where } c \ge 0 \text{ is a constant}, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the value of c,
(b) Find $P(1/2 \le X \le 2)$,
(c) Find the support of X,
(d) Find the d.f. of X.

Solution: (a) We have

$$\int_a^b f(x)\mathrm{d}x = 1 \implies \int_0^1 x^2\mathrm{d}x + \int_1^\infty ce^{-x}\mathrm{d}x = 1 \implies 1/3 + ce^{-1} = 1 \implies c = \frac{2e}{3}.$$

(b) Observe that,

$$\begin{aligned} P(1/2 \le X \le 2) = \int_{1/2}^2 f(x)\mathrm{d}x &= \int_{1/2}^1 x^2\mathrm{d}x + c\int_1^2 e^{-x}\mathrm{d}x \\ &= \frac{1}{3}(1 - 1/8) + c(e^{-1} - e^{-2}) = \frac{7}{24} + \frac{2}{3}(1 - e^{-1}). \end{aligned}$$

(c) The support of X is $S = \left\{ x \in \mathbb{R} : \int_{x-h}^{x+h} f(t)\mathrm{d}t > 0, \ \forall\, h > 0 \right\} = [0, \infty)$.

(d) The d.f. of X is $F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t$. For $x < 0$, clearly $F(x) = 0$. For $0 \le x < 1$,

$$F(x) = \int_0^x t^2 \mathrm{d}t = x^3/3.$$

For $x \ge 1$,

$$F(x) = \int_0^1 t^2 \mathrm{d}t + c\int_1^x e^{-t}\mathrm{d}t = \frac{1}{3} + \frac{2}{3}(1 - e^{-(x-1)}).$$

Thus,

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x^3}{3}, & \text{if } 0 \le x < 1, \\ \frac{1}{3} + \frac{2}{3}(1 - e^{-(x-1)}), & \text{if } x \ge 1. \end{cases}$$

Remark 7.6. *Let X be a continuous r.v. with p.d.f. $f(\cdot)$. If f is continuous at $x \in \mathbb{R}$, then*

$$f(x) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{x-\delta/2}^{x+\delta/2} f(t)\mathrm{d}t \implies P(x - \delta/2 \le X \le x + \delta/2) \approx \delta f(x), \text{ for small } \delta > 0,$$

that is, $P(x - \mathrm{d}x \le X \le x + \mathrm{d}x) \approx f(x)\mathrm{d}x$.

7.1. Probability Distribution of a Function of Discrete Random Variable

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a r.v. with d.f. F, p.m.f. f and support S. Let $h : \mathbb{R} \to \mathbb{R}$ be a given function. Define $Z : \Omega \to \mathbb{R}$ as

$$Z(\omega) = h(X(\omega)), \ \ \omega \in \Omega.$$

Then Z is a r.v. and it is a function of r.v. X. Since we are only interested in values of random variables X and Z and not in the original probability space $(\Omega, \mathcal{F}, P)$, we simply write $X(\omega), \omega \in \Omega$ as X and $Z(\omega), \omega \in \Omega$ as Z.

We have $F(x) = P(X \le x)$, $f(x) = P(X = x)$, $x \in \mathbb{R}$, $P(X \in S) = 1$ and $P(X = x) > 0$ for all $x \in S$.

Define $T = h(S) = \{h(x) : x \in S\}$. For any set $A \subseteq \mathbb{R}$, define

$$h^{-1}(A) = \{x \in S : h(x) \in A\}.$$

Then T is a countable set. Also, $P(Z = z) > 0, \ \forall\, z \in T$ (since $P(X = x) > 0, \forall x \in S$) and $P(Z \in T) = 1$ (since $P(X \in S) = 1$). It follows that Z is a discrete r.v. Moreover, for $z \in T$,

$$P(Z = z) = P(h(X) = z) = \sum_{\{x \in S : h(x) = z\}} P(X = x) = \sum_{x \in h^{-1}(\{z\})} P(X = x) = \sum_{x \in h^{-1}(z)} f(x),$$

and for any $z \notin T$, $P(Z = z) = 0$. Thus, we have the following theorem:

Theorem 7.7. *Let X be a discrete r.v. with support S, d.f. F and p.m.f. f. Let $h : \mathbb{R} \to \mathbb{R}$ be a given function. Then, $Z = h(X)$ is a discrete r.v. with support $T = \{h(x) : x \in S\}$ and p.m.f.*

$$g(z) = \begin{cases} \displaystyle\sum_{x \in h^{-1}(\{z\})} f(x), & \textit{if } z \in T, \\ 0, & \textit{otherwise}, \end{cases}$$

and d.f.

$$G(z) = P(Z \leq z) = \sum_{\{t \in T : t \leq z\}} g(t) = \sum_{\{x \in S : h(x) \leq z\}} f(x) = \sum_{x \in h^{-1}((-\infty, z]) \cap S} f(x).$$

In particular, if $h : S \to \mathbb{R}$ is one-one then

$$g(z) = \begin{cases} f(h^{-1}(z)), & \textit{if } z \in T, \\ 0, & \textit{otherwise}. \end{cases}$$

Example 7.8. *Let X be a discrete r.v. with p.m.f.*

$$f(x) = \begin{cases} 1/7, & \textit{if } x \in \{-2, -1, 0, 1\}, \\ 3/14, & \textit{if } x \in \{2, 3\}, \\ 0, & \textit{otherwise}. \end{cases}$$

Find the p.m.f. and d.f. of $Y = X^2$.

Solution: Here, the support of X is $S = \{-2, -1, 0, 1, 2, 3\}$. By Theorem 7.7, $Y = X^2$ is discrete r.v. with support $T = \{0, 1, 4, 9\}$ and p.m.f.

$$g(z) = P(X^2 = z) = \begin{cases} P(X = 0), & \text{if } z = 0, \\ P(X = -1) + P(X = 1), & \text{if } z = 1, \\ P(X = -2) + P(X = 2), & \text{if } z = 4, \\ P(X = -3) + P(X = 3), & \text{if } z = 9, \\ 0, & \text{otherwise}. \end{cases} = \begin{cases} 1/7, & \text{if } z = 0, \\ 2/7, & \text{if } z = 1, \\ 5/14, & \text{if } z = 4, \\ 3/14, & \text{if } z = 9, \\ 0, & \text{otherwise}. \end{cases}$$

The d.f. of Y is

$$G(z) = P(Y \leq z) = \begin{cases} 0, & \text{if } z < 0 \\ 1/7, & \text{if } 0 \leq z < 1 \\ 3/7, & \text{if } 1 \leq z < 4 \\ 11/14, & \text{if } 4 \leq z < 9 \\ 1, & \text{if } z \geq 9. \end{cases}$$

Example 7.9. *In Example 7.8, directly find the d.f. of $Y = X^2$ (i.e. find d.f. of Y before finding the p.m.f. of Y). Hence find the p.m.f. of Y.*

Solution: By Theorem 7.7, Y is a discrete r.v. with support $T = \{0, 1, 4, 9\}$. Thus the d.f. of Y is

$$G(z) = P(Y \leq z) = P(X^2 \leq z) = \begin{cases} 0, & z < 0, \\ P(X^2 = 0), & 0 \leq z < 1, \\ P(X^2 = 0) + P(X^2 = 1), & 1 \leq z < 4, \\ P(X^2 = 0) + P(X^2 = 1) + P(X^2 = 4), & 4 \leq z < 9, \\ 1, & z \geq 9. \end{cases}$$

$$= \begin{cases} 0, & z < 0, \\ \frac{1}{7}, & 0 \leq z < 1, \\ \frac{1}{7} + \frac{1}{7} + \frac{1}{7}, & 1 \leq z < 4, \\ \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{3}{14}, & 4 \leq z < 9, \\ 1, & z \geq 9. \end{cases} = \begin{cases} 0, & z < 0, \\ 1/7, & 0 \leq z < 1, \\ 3/7, & 1 \leq z < 4, \\ 11/14, & 4 \leq z < 9, \\ 1, & z \geq 9. \end{cases}$$

The p.m.f. of Y is

$$g(z) = \begin{cases} G(z) - G(z-), & \text{if } z \in T, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 1/7, & \text{if } z = 0, \\ 2/7, & \text{if } z = 1, \\ 5/14, & \text{if } z = 4, \\ 3/14, & \text{if } z = 9, \\ 0, & \text{otherwise.} \end{cases}$$

7.2. Probability Distribution of a Function of Continuous Random Variable

Let X be a continuous r.v. with d.f. F, p.d.f. $f(\cdot)$ and support $S = \{x \in \mathbb{R} : F(x+h) - F(x-h) = \int_{x-h}^{x+h} f(t)\mathrm{d}t > 0,\ \forall\, h > 0\}$. For convenience assume that $S = [a, b]$ and $\{x \in \mathbb{R} : f(x) > 0\} = (a, b)$, for some $-\infty \leq a < b \leq \infty$ (with the convention that $[-\infty, b] \equiv (-\infty, b)$, $\forall\, b \in \mathbb{R}$, $[a, \infty] \equiv (a, \infty)$, $\forall\, a \in \mathbb{R}$ and $[-\infty, \infty] \equiv (-\infty, \infty)$).

Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that h is strictly monotone and differentiable function on S. Then $Z = h(X)$ is a r.v. with d.f. $G(z) = P(Z \leq z) = P(h(X) \leq z)$, $z \in \mathbb{R}$.

For any sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$, define $h(A) = \{h(x) : x \in A\}$ and $h^{-1}(B) = \{x \in \mathbb{R} : h(x) \in B\}$. Clearly $P(X \in (a, b)) = 1$ and therefore $P(h(X) \in h((a, b)) = 1)$. Consider the following cases:

Case I: $h(\cdot)$ is strictly increasing on S

We have $P(h(a) < Z < h(b)) = 1$. Therefore, for $z < h(a)$, $P(Z \leq z) = 0$ and for $z \geq h(b)$, $P(Z \leq z) = 1$. For $h(a) < z < h(b)$,

$$G(z) = P(h(X) \leq z) = P(X \leq h^{-1}(z)) = \int_{-\infty}^{h^{-1}(z)} f(t)\mathrm{d}t = \int_{a}^{h^{-1}(z)} f(t)\mathrm{d}t = \int_{h(a)}^{z} f(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y) \right| \mathrm{d}y.$$

Thus,

$$G(z) = \begin{cases} 0, & \text{if } z < h(a), \\ \displaystyle\int_{h(a)}^{z} f(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y) \right| \mathrm{d}y, & \text{if } h(a) \leq z < h(b), \\ 1, & \text{if } z \geq h(b). \end{cases}$$

Since f is continuous on (a,b) it follows that $G(z)$ is differentiable everywhere except possibly at $z = h(a)$ and $z = h(b)$. Moreover,

$$G'(z) = \begin{cases} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right|, & \text{if } h(a) < z < h(b), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\int_{-\infty}^{\infty} G'(z)\mathrm{d}z = \int_{h(a)}^{h(b)} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right| \mathrm{d}z = \int_a^b f(t)\mathrm{d}t = 1.$$

It follows that Z is a continuous r.v. with p.d.f.

$$g(z) = \begin{cases} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right|, & \text{if } h(a) < z < h(b), \\ 0, & \text{otherwise} \end{cases}$$

and support $S = [h(a), h(b)]$.

Case II: $h(\cdot)$ is strictly decreasing on S

Here $P(h(b) < h(X) < h(a)) = 1$ and $G(z) = P(h(X) \leq z)$, $z \in \mathbb{R}$. Clearly, for $z < h(b)$, $G(z) = 0$ and for $z \geq h(a)$, $G(z) = 1$. For $h(b) < z < h(a)$,

$$G(z) = P(X \geq h^{-1}(z)) = \int_{h^{-1}(z)}^{\infty} f(t)\mathrm{d}t = \int_{h^{-1}(z)}^{b} f(t)\mathrm{d}t = \int_{h(b)}^{z} f(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y) \right| \mathrm{d}y.$$

Thus,

$$G(z) = \begin{cases} 0, & \text{if } z < h(b), \\ \int_{h(b)}^{z} f(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y) \right| \mathrm{d}y, & \text{if } h(b) \leq z < h(a), \\ 1, & \text{if } z \geq h(a). \end{cases}$$

Since f is continuous on (a,b), it follows that $G(\cdot)$ is differentiable everywhere except possibly at $h(a)$ and $h(b)$. Moreover,

$$G'(z) = \begin{cases} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right|, & \text{if } h(b) < z < h(a), \\ 0, & \text{otherwise} \end{cases}$$

and

$$\int_{-\infty}^{\infty} G'(z)\mathrm{d}z = \int_{h(b)}^{h(a)} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right| \mathrm{d}z = \int_a^b f(t)\mathrm{d}t = 1.$$

Consequently, Z is a continuous r.v. with p.d.f.

$$g(z) = \begin{cases} f(h^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right|, & \text{if } h(b) < z < h(a), \\ 0, & \text{otherwise} \end{cases}$$

and support $S = [h(b), h(a)]$.

Combining Case I and Case II, we get the following result:

Theorem 7.10. *Let X be a continuous r.v. with p.d.f. $f(\cdot)$ and support $S = [a,b]$ for some $-\infty \leq a < b \leq \infty$. Suppose that $\{x \in \mathbb{R} : f(x) > 0\} = (a,b)$ and that f is continuous on (a,b). Let $h : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable and strictly monotone on (a,b). Then, $Z = h(X)$ is a continuous r.v. with p.d.f*

$$\begin{cases} f(h^{-1}(z)) \left| \dfrac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right|, & \textit{if } z \in h((a,b)), \\ 0, & \textit{otherwise}, \end{cases}$$

and support $S = [\min\{h(a), h(b)\}, \max\{h(a), h(b)\}]$.

The following theorem is a generalization of the above result and can be proved on similar lines.

Theorem 7.11. *Let X be a continuous r.v. with p.d.f. $f(\cdot)$ and support $S = \bigcup_{i \in \Lambda} [a_i, b_i]$, where Λ is a countable set and $[a_i, b_i]$'s are disjoint intervals. Suppose that $\{x \in \mathbb{R} : f(x) > 0\} = \bigcup_{i \in \Lambda} (a_i, b_i)$ and that f is continuous in each (a_i, b_i), $i \in \Lambda$. Let $h : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable and strictly monotone in each (a_i, b_i), $i \in \Lambda$ (h may be monotonic increasing in some (a_i, b_i) and monotonic decreasing in some (a_i, b_i)). Let $h_i^{-1}(\cdot)$ be the inverse function of h_i on (a_i, b_i), $i \in \Lambda$. Then, $Z = h(X)$ is a continuous r.v. with p.d.f.*

$$g(z) = \sum_{j \in \Lambda} f(h_j^{-1}(z)) \left| \frac{\mathrm{d}}{\mathrm{d}z} h_j^{-1}(z) \right| I_{h_j((a_j,b_j))}(z), \ \ \textit{where} \ \ I_{h_j((a_j,b_j))}(z) = \begin{cases} 1, & z \in h_j((a_j,b_j)), \\ 0, & \textit{otherwise}. \end{cases}.$$

Remark 7.12. *Theorem 7.10 and Theorem 7.11 hold even in situations where the function h is differentiable everywhere except possibly at a finite number of points in S.*