Duality and Sensitivity Analysis

Let us consider a linear programming problem

$$\text{Max } Z = 6x_1 + 5x_2$$

$$\text{s.t} \quad x_1 + x_2 \le 5$$

$$3x_1 + 2x_2 \le 12, \quad x_1, x_2 \ge 0$$

Let us assume that we do not know the optimum solution. Let Z^* be the optimum value of Z. Let us try to find a value higher than Z^* without solving the problem. We wish to find as small a value as possible but it should be more than Z^*.

1. The obvious value is infinity.

2. Let us multiply the second constraints by 3, then we have

$$9x_1 + 6x_2 \le 36$$

This implies $Z = 6x_1 + 5x_2 \le 9x_1 + 6x_2 \le 36.$

$x_1 \ge 0, x_2 \ge 0$

3. Multiply first constraint by 6

$$Z = 6x_1 + 5x_2 \le 6x_1 + 6x_2 \le 30$$

4. Multiply first by 1 and second by 2

$$z \le 7x_1 + 5x_2 \le 29$$

The lowest value that can be achieved will have a certain constant y_1 and y_2 multiplied to the first and second constraints group.

In order to get the upper bound for z^*

$$y_1 + 3y_2 \ge 6$$
$$y_1 + 2y_2 \ge 5 \quad , \; y_1, y_2 \ge 0$$

Then $5y_1 + 12y_2$ is an upper bound for z^*. The lowest or minimum value of $5y_1 + 12y_2$ for such y_1, y_2 satisfying

$$y_1 + 3y_2 \ge 6$$
$$y_1 + 2y_2 \ge 5$$

is a best upper estimate for z^*.

The above problem is called the dual of the given problem.

Dual of the primal problem

If the primal problem is

$$\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t $\quad Ax \le b \quad$ where $A = [a_{ij}]_{i=1,\, j=1}^{m\quad n}$

$b = [b_1, \ldots b_m]^T$

$x = [x_1, \ldots x_n]^T$

Then its dual problem is

$$\text{Min } W = by$$

s.t $\quad A^T y \ge c \quad$ where $c = [c_1, c_2 \ldots c_n]^T$

$y = [y_1, \ldots y_m]^T \ge 0$

Prob 1 Write the dual of the problem

$$\text{Max } Z = 3x_1 + 4x_2$$

s.t
$$x_1 + x_2 \le 12$$
$$2x_1 + 3x_2 \le 30$$
$$x_1 + 4x_2 \le 36 \quad , x_1, x_2 \ge 0$$

Prob 2

$$\text{Max } Z = -x_1 + 5x_2$$

s.t $\quad 2x_1 - 3x_2 \ge 1$

$x_1 + x_2 \le 3, \quad x_1, x_2 \ge 0$

Relationship between the primal and dual problem

Primal	Dual
Maximization	Minimization
Minimization	Maximization
Number of variable (n)	Number of constrains (n)
Number of constraints (m)	Number of variable (m)
RHS (b)	Objective function co-efficient
Objective function co-efficients (c)	RHS
Constraint co-efficient (A)	Constraint co-efficient (A^T)

Weak duality theorem

For a maximization primal, every feasible solution to the dual has a objective function value greater than or equal to every feasible solution to the primal.

Proof Let $[x_1, x_2, \ldots x_n]^T$ be a feasible solution to the primal problem and $[y_1, y_2, \ldots y_m]^T$ be a feasible solution to the dual problem.

By the construction of the dual problem we get

$$b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$$

is an upper bound for Z^*.

$$\therefore \quad Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \le Z^* \le b_1 y_1 + \ldots + b_m y_m$$

Note 1 If the primal is unbounded, what can you say about dual?

Let $[y_1, y_2 \ldots y_m]$ be a feasible solution to the dual.

Let $W = b_1 y_1 + \ldots + b_m y_m$

$$Z \le Z^* \le W = b_1 y_1 + \ldots + b_m y_m$$

$\therefore$ the primal becomes bounded which lead us to a contradiction.

$\therefore$ The dual does not have a feasible soln.

Note 2 what happens if the primal is itsself infeasible? can you say dual is unbounded. The answer is no. The dual is unbounded or infeasible.