MA202: Calculus II

Instructor: Dr. Arnab Patra

Lecture Notes

Department of Mathematics
Indian Institute of Technology Bhilai

Module 3

Lecture ~~6~~ 7

For Module-3 refer to the following book.

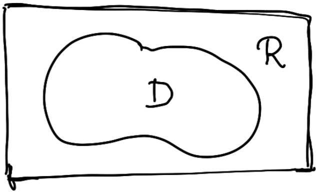
* A Course in Multivariable calculus and Analysis — S.R. Ghorpade, B.V. Limaye. SPRINGER

Double Integral on a Bounded set

- Let D be a bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a bounded function.
- Consider a rectangle R containing D.
- Define a function $f^* : R \to \mathbb{R}$ such that

$$f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{otherwise} \end{cases}$$

- Check that f^* is a bounded function on R as f is bounded on D.



Double Integral on a Bounded set

Definition

We say that $f : D \to \mathbb{R}$ integrable over D if $f^* : R \to \mathbb{R}$ is integrable over R. In this case we define

$$\iint_D f(x,y)d(x,y) = \iint_R f^*(x,y)d(x,y).$$

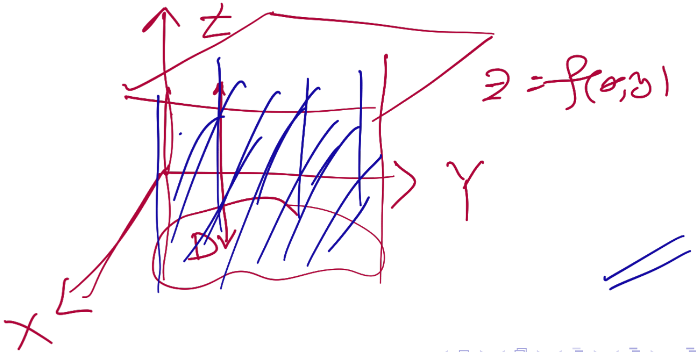
This is called the double integral of f over D.

Note that the integrability of f over D and the value of the double integral is independent of the choice of the rectangle R containing D.

Double Integral on a Bounded set

- Geometrical Interpretation: Let $f : D \to \mathbb{R}$ be integrable on D and non-negative. The double integral of f on D gives the volume of the solid formed under the surface $z = f(x, y)$ and above the set D. In other words it gives the volume of the following set

$$\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, \ \& \ 0 \leq z \leq f(x, y)\}.$$



Double Integral on a Bounded set

- Let $\phi_1,\ \phi_2 : [a, b] \to \mathbb{R}$ are continuous functions on $[a, b]$ such that $\phi_1 \leq \phi_2$ and let

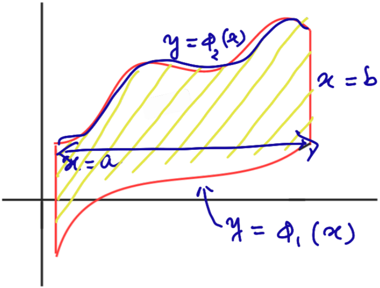
$$D_1 = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ \&\ \phi_1(x) \leq y \leq \phi_2(x)\}.$$

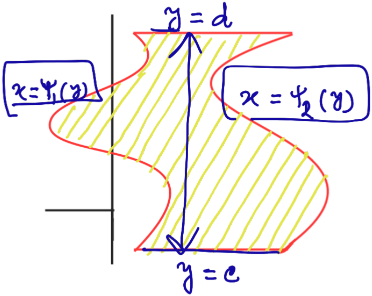
- Let $\psi_1,\ \psi_2 : [c, d] \to \mathbb{R}$ are continuous functions on $[c, d]$ such that $\psi_1 \leq \psi_2$ and let

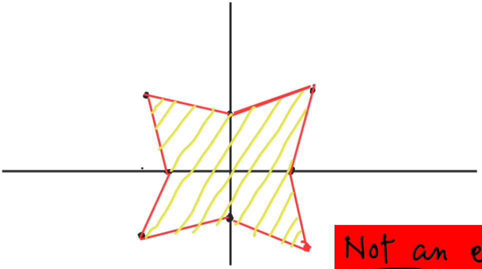
$$D_2 = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d,\ \&\ \psi_1(y) \leq x \leq \psi_2(y)\}.$$

* Instead of continuity we can assume that $\underline{\phi_1, \phi_2, \psi_1,}$ ψ_2 are Riemann-integrable.

Double Integral on a Bounded set







Not an elementary region

Double Integral on a Bounded set

Elementary region

In both the cases mentioned above, D_1 and D_2 are called elementary region in $\mathbb{R}^2$. In first case it is called elementary region of type I, and in the latter case, it called elementary region of type II.

- A rectangle in $\mathbb{R}^2$ is an elementary region of both type I and type II,
- The disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$ is an elementary region in $\mathbb{R}^2$ as it can be written as

$$D = \{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a, \;\&\; -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}\}.$$

Fubini Theorem over Elementary region

Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous, then the iterated integral $\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

(ii) If $D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)\}$, where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are continuous, then the iterated integral $\int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \right) dy$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

Proof (Sketch):

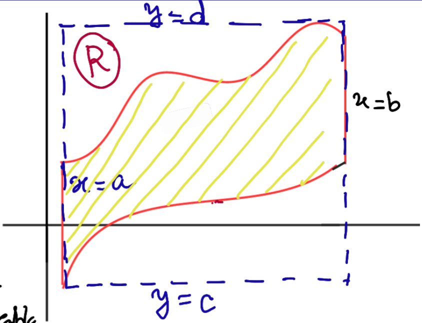
(i) Let $c = \inf\{\phi_1(x) : x \in [a,b]\}$

$d = \sup \{\phi_2(x) : x \in [a,b]\}$

Then $D \subset R = \boxed{[a,b] \times [c,d]}$

Since f is integrable on D, the extended function f^* is integrable on R. Now use the Fubini Theorem for rectangle on the function f^*.

(ii) Similar.



$D \subset R = [a,b] \times [c,d]$

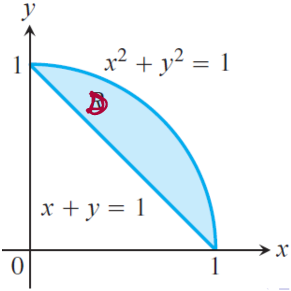
$f^* : R \to \mathbb{R}$ by

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & \text{otherwise} \end{cases}$$

Procedure to solve problems

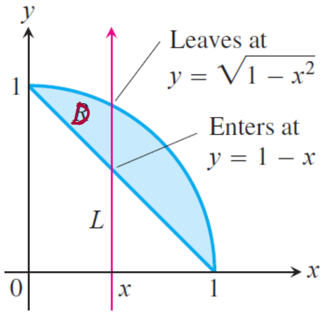
To apply Fubini Theorem in elementary regions we need to know how to find the limits of iterated integral when the regions are complicated. The procedure is mentioned below:

- When evaluating $\iint_D f(x,y)d(x,y)$ with integrating first with respect to y and then with respect to x, do the following:
- Sketch the region of integration and label the bounding curves. As an example let D be the region bounded by the line $x+y=1$ and the circle $x^2+y^2=1$.

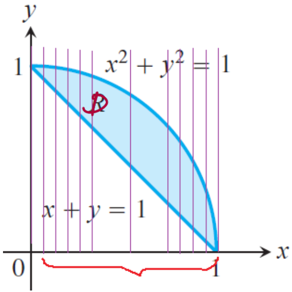


Procedure to solve problems

- Find the y-limits: Imagine a vertical line L cutting through D in the direction of increasing y. Mark the y-values where L enters and leaves the region D. These are the y-limits of integration and are usually functions of x (may be constants also).



- Find the x-limits: Choose x-limits that include all the vertical lines which are needed to cover the region D.



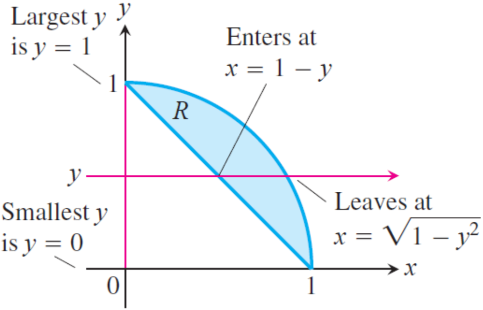
Procedure to solve problems

- In this case the integral will be

$$\iint_D f(x,y)d(x,y) = \int_{x=0}^{x=1} \left(\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)dy \right) dx$$

- Similarly to evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in the previous steps.

Procedure to solve problems



- In this case the integral will be

$$\iint_D f(x,y)d(x,y) = \int_{y=0}^{y=1} \left(\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)dx \right) dy$$

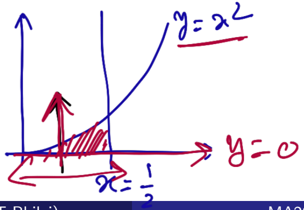
Fubini Theorem over Elementary region

(i) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1/2 \text{ and } 0 \leq y \leq x^2\}$ and $f(x, y) := x + y$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$I := \iint_D (x+y)\,d(x,y) = \int_0^{1/2} \left(\int_0^{x^2} (x+y)\,dy \right) dx,$$

which is equal to

$$\int_0^{1/2} \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x^2} dx \;=\; \int_0^{1/2} \left(x^3 + \frac{x^4}{2} \right) dx \;=\; \frac{3}{160}.$$



(ii) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 2x\}$ and $f(x, y) := e^{x^2}$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$\iint_D f = \int_0^1 \left(\int_0^{2x} e^{x^2}\, dy \right) dx = \int_0^1 2x\, e^{x^2}\, dx = \boxed{e - 1.}$$

$$\boxed{\text{Calculate the other iterated integral}}$$

Double Integral on a Bounded set

- Let D be a bounded subset of $\mathbb{R}^2$. Which functions $f : D \to \mathbb{R}$ are integrable?
- Consider the following example. Let $R = [a, b] \times [c, d]$ and $D = \{(x, y) \in R : x,\ y\ \in \mathbb{Q}\} = R \cap \mathbb{Q}^2$ and let $f : D \to \mathbb{R}$ defined as $f(x, y) = 1$ for all $(x, y) \in D$. Since f is defined on D so f is continuous over D but the function $f^* : R \to \mathbb{R}$ is the bivariate Dirichlet function [Here $\partial D = R$ which is not a content zero set]

$$f^*(x, y) = \begin{cases} 1, & \text{if } (x, y) \in D \\ 0, & \text{otherwise} \end{cases}$$

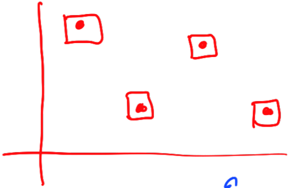
  and it already shown that f^* is not integrable over R. Hence by definition, f is also not integrable over D.
- This shows that even the continuous functions may not be double integrable!

Double Integral on a Bounded set

- In order to seek conditions under which a bounded function f defined on a bounded subset D of $\mathbb{R}^2$ is integrable over D, we introduce a new concept.

Content zero set

A bounded subset E of $\mathbb{R}^2$ is said to be of (two-dimensional) content zero if for every $\epsilon > 0$ there are finitely many rectangles whose union contains E and sum of whose area is less than ϵ.



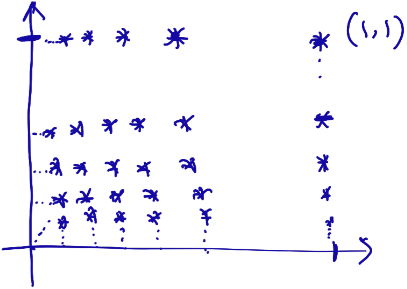
Fix $\varepsilon > 0$. Let there are hundred points. Choose 100 rectangle containing those 100 points with length & width both same as $\frac{\sqrt{\varepsilon}}{100}$. Then area of each rectangle is $\frac{\varepsilon}{100^2}$ and sum of their area is

$$\frac{\varepsilon}{100^2} + \frac{\varepsilon}{100^2} + \cdots + \frac{\varepsilon}{100^2} = \frac{\varepsilon}{100} < \varepsilon$$

Double Integral on a Bounded set

Examples:

1. Every finite subset of $\mathbb{R}^2$ is of content zero,
2. The infinite set $\{(\frac{1}{n}, \frac{1}{k}) : n, k \in \mathbb{N}\}$ is content zero in $\mathbb{R}^2$.
3. The subset $\{(x, y) \in [0, 1] \times [0, 1] : x, y \in \mathbb{Q}\}$ of $\mathbb{R}^2$ is not of content zero.



(iv) Let $\varphi : [a, b] \to \mathbb{R}$ be an integrable function. Then its graph $E := \{(x, \varphi(x)) : x \in [a, b]\}$ is of content zero. To see this, let $\epsilon > 0$. By the Riemann condition, there is a partition $P := \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that $U(P, \varphi) - L(P, \varphi) < \epsilon$. Then $E \subset \bigcup_{i=1}^{n} R_i$, where $R_i := [x_{i-1}, x_i] \times [m_i(\varphi), M_i(\varphi)]$ and

$$\text{Area}\,(R_1) + \cdots + \text{Area}\,(R_n) = U(P, \varphi) - L(P, \varphi) < \epsilon.$$

Similarly, if $\psi : [c, d] \to \mathbb{R}$ is an integrable function, then the set $E := \{(\psi(y), y) : y \in [c, d]\}$ is of content zero.

Double Integral on a Bounded set

Theorem

Let D be a bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be a bounded function. If the boundary ∂D of D is of (two-dimensional) content zero and if the set of discontinuities of f in D is also of (two-dimensional) content zero, then f is integrable over D.

- The above conditions are sufficient conditions. Integrability of f does not necessarily imply the conditions.

$\boxed{\partial \mathbb{Q} = \mathbb{R}}$,

Double Integral on a Bounded set

Fubini's Theorem revisited: Fubini's Theorem of elementary regions can be weakened with the following assumptions

- $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are bounded functions and the point of discontinuities of both ϕ_1 and ϕ_2 are of content zero (in one dimension).
- $f : D \to \mathbb{R}$ is a bounded function on D whose set of discontinuities is of content zero (in two dimension).

The following result gives the concept of domain additivity of double integral on bounded subsets.

Domain additivity

Let D be a bounded subset of $\mathbb{R}^2$ and let D_1 and D_2 are subsets of D such that

- $D_1 \cup D_2 = D$,
- $D_1 \cap D_2$ is of content zero.

If $f : D \to \mathbb{R}$ is a bounded function such that f is integrable over D_1 and D_2 then f is integrable over D and

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$