Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Tutorial Sheet 2

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1. Test the diagonalizability of the following linear operators

   (i) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined as $Tf = f'$.

   (ii) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined as $(Tf)(x) = f'(x) + f''(x)$.

   (iii) $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined as $T(x, y, z) = (4x + y, 2x + 3y + 2z, x + 4z)$

   (iv) $T : \mathbb{C}^2(\mathbb{C}) \to \mathbb{C}^2(\mathbb{C})$ defined as $T(w, z) = (w + iz, z + iw)$.

2. Let λ be an eigenvalue of a linear operator T on V, then show that λ^k is an eigenvalue of T^k. Can we generalize the above result, i.e., if λ is an eigenvalue of T and μ is an eigenvalue for S, then $\lambda\mu$ is an eigenvalue for TS?
   Let v be an eigenvector of T corresponding to eigenvalue λ then v is also an eigenvector of T^k corresponding to eigenvalue λ^k as $T^k v = T^{k-1}(Tv) = T^{k-1}(\lambda v) = \lambda T^{k-2}(Tv) = \lambda T^{k-2}(\lambda v) = \lambda^2 T^{k-2} v = \cdots = \lambda^{k-1} Tv = \lambda^k v$. The second part is not always true. We can give the following counter example: $\lambda = 2$ is an eigenvalue of $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $\mu = 3$ is an eigenvalue of $B = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ but $\lambda\mu = 6$ is not an eigenvalue of $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. Without finding roots of characteristic polynomial, figure out the eigenvalues of the following matrix.
$$A = \begin{bmatrix} -2 & 10 & -6 \\ 5 & -18 & 15 \\ 3 & -10 & 9 \end{bmatrix}$$
   Observe that first and third column are linearly dependent and hence 0 is one of the eigenvalues. Moreover, row sum of each row is 2 implies 2 is also an eigenvalue. As trace is -11, third eigen value will be -13. You can prove that "if row sum of each row of a matrix is λ then λ is an eigenvalue of the matrix" by showing that $v = (1, 1, 1)^t$ satisfies $Av = \lambda v$.

4. Show that a diagonalizable linear transformation on a finite dimensional vector space having only one eigenvalue is a scalar multiple of identity operator.
   Let T be a diagonalizable linear operator on a $n-$dimensional vector space V with λ as its only eigenvalue (ofcourse repeated n times). As T is diagonalizable, V has a basis

of eigenvectors of T, say $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ such that $T\alpha_i = \lambda\alpha_i$ for all $i = 1, 2, \cdots, n$. Thus for any $\alpha \in V$,

$$T\alpha = T\left(\sum_{i=1}^{n} c_i\alpha_i\right) = \sum_{i=1}^{n} c_i T(\alpha_i) = \sum_{i=1}^{n} c_i\lambda\alpha_i = \lambda\sum_{i=1}^{n} c_i\alpha_i = \lambda\alpha.$$

5. Let trace of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be α and λ be one of the eignevalues of T. If the eigenspace corresponding to the eigenvalue $\lambda \in \mathbb{R}$ of T is 2- dimensional. Then find all the choices of eigenvalues of T. Is T diagonalizable for your choices of eigenvalues?
Using the result, $1 \leq GM \leq AM$, we ensure that λ is repated atleast twice. As sum of eigenvalues is equal to the trace, third eigenvalue (say β) will be $\beta = \alpha - 2\lambda$. Now if β is different from λ, i.e., $\alpha \neq 3\lambda$, T will be diagonalizable. If $\beta = \lambda$, i.e., $\alpha = 3\lambda$, T will not be diagonalizable.

6. Let n be a positive integer. Find A^n for the following matrix

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

Show that the matrix A is diagonalizable and hence there exist inveritble matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. Now we claim that $A^n = QD^nQ^{-1}$. We prove our claim by induction. First show that $A^2 = A.A = QDQ^{-1}QDQ^{-1} = QD^2Q^{-1}$. Assume that for $k > 2$, $A^k = QD^kQ^{-1}$. Now $A^{k+1} = AA^k = QDQ^{-1}QD^kQ^{-1} = QD^{k+1}Q^{-1}$.
As D^n is easy to compute (raise power n to each diagonal entry) and Q is known, A^n will be easily computed.

7. Check if the matrices $A \in M_{n\times n}(\mathbb{R})$ given below are diagonalizable. Also find an invertible matrix Q and diagonal matrix D such that $A = QDQ^{-1}$.

(i) $\begin{bmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{bmatrix}$

8. As an application of diagonalizability: Find a general solution of the following system of differential equations

$$x' = x + y$$
$$y' = 4x + y$$

where $x = x(r)$ and $y = y(r)$ are real valued functions of $r \in \mathbb{R}$.
Let us write the above system as $X' = AX$, where $X(r) = (x(r), y(r))^t \in \mathbb{R}^2$ with $X'(r) = (x'(r), y'(r))^t$ and $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. Observe that $3, -1$ are the eigenvalues of A. We obtain the eigenspaces corresponding to these eigenvalues as $E_3 =< (1,2)^t >$ and $E_{-1} =< (1,-2)^t >$. Thus we can find invertible matrix Q and a diagonal matrix D

$$Q = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

such that $A = QDQ^{-1}$. Now we substitute this into $X' = AX$, to get $X' = QDQ^{-1}X$ or $Q^{-1}X' = DQ^{-1}X$. Now we can make use of $(Q^{-1}X)' = Q^{-1}X'$ to obtain $Y' = DY$, where $Y = Q^{-1}X$. As D is diagonal, the system $Y' = DY$ is not mutually dependent and gives rise to the solutions $Y(r) = (c_1e^{3r}, c_2e^{-r})^t$, where c_1, c_2 are arbitrary constants. Thus we get, $X(r) = QY(r)$ as the solution of the given system.