# Lecture #2 (IC152)

Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$ and $T: V \to V$, then $\exists$ $n \times n$ matrix $([T]_{\mathcal{B}})$ is a matrix representation of $T$ relative to an ordered basis $\mathcal{B}$ of $V$.

Let $\mathcal{B}'$ be another basis of $V$, then

$[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ are similar.

Definition: $\det(T) := \det [T]_{\mathcal{B}}$

& $tr(T) := tr[T]_{\mathcal{B}}$

" Objective is to look for an ordered basis of $V$ $(\dim V = n)$ of $V$ such that $[T]_{\mathcal{B}} = diag(d_1, d_2, \ldots d_n)$."

iff $T\alpha_k = d_k\alpha_k$. what is the null space of $[T]_{\mathcal{B}}$, $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

$$Null(T) = \{\alpha_k\text{'s} : T\alpha_k = d_k\alpha_k \text{ with } d_k = 0\}$$

$T\alpha_1 = d_1\alpha_1 = d_1\alpha_1 + 0\alpha_2 + \cdots 0\alpha_n$ $\quad$ $Range(T) = \{\alpha_k\text{'s} : T\alpha_k = d_k\alpha_k, d_k \neq 0\}$

$$T\alpha_2 = d_2\alpha_2 = 0\alpha_1 + d_2\alpha_2 + \cdots 0\alpha_n$$

$$\vdots$$

$$T\alpha_n = d_n\alpha_n = 0\alpha_1 + 0\alpha_2 + \cdots d_n\alpha_n$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ 0 & 0 & d_3 & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

**Definition** (Characteristic roots / eigen values)

Let $V$ be a vector space over a field $\mathbb{F}$ & $T: V \to V$ be a linear operator on $V$, then a scalar $c \in \mathbb{F}$ is called an eigen value of $T$ if $\exists\ \alpha \neq 0$ in $V$ such that

$$T\alpha = c\alpha.$$

The vector $\alpha$ is called eigenvector / characteristic vector associated with eigenvalue $c$.

**Example:** $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x, y) = (x + y, y)$$

Claim: $1 \in \mathbb{R}$, is an eigenvalue of $T$.

We have to find a nonzero vector $(x_0, y_0)$ in $\mathbb{R}^2$ ($x_0 \neq 0$ or $y_0 \neq 0$) such that $T(x_0, y_0) = 1.(x_0, y_0)$

One such choice is $\underline{\underline{(1, 0)}}$ for $(x_0, y_0)$.

Is 1 the unique eigenvalue for T ?

If not, then

$$T(x_0, y_0) = c(x_0, y_0)$$

$$\Rightarrow (x_0 + y_0, y_0) = c(x_0, y_0)$$

$$\Rightarrow x_0 + y_0 = cx_0 \ \& \ y_0 = cy_0$$

$$\downarrow \qquad\qquad \downarrow$$

$$c = 1 \qquad\qquad y_0 = 0$$

Example :- $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (5x, 3y)$ ✓

then $c = 5, 3$ are eigen values for T.

When $c = 5$, $\alpha = (1, 0)$ is an eigenvector

$$T(1, 0) = (5, 0) = 5(1, 0)$$

When $c = 3$, $\alpha = (0, 1)$ is an eigenvector.

$$T(0, 1) = (0, 3) = 3(0, 1)$$

**Remark**: If $c \in \mathbb{F}$ is an eigenvalue of T then $\exists\, \alpha \neq 0$ in V s.t.

$I : V \to V$
$I\alpha = \alpha$.

$$T\alpha = c\alpha$$

$$\Rightarrow (cI - T)\alpha = 0 \text{ for some } \alpha \neq 0$$

If $\text{null}(cI - T) \neq \{0\}$ then c is an eigenvalue of T.

Thus if c is an eigenvalue of T, then $(cI - T)$ is not invertible. Equivalently

$$\det(cI - T) := \det(cI - [T]_{\mathcal{B}}) = 0$$

**Definition**: The polynomial $\det(xI - [T]_{\mathcal{B}})$ is called as **characteristic polynomial** of $T : V \to V$, ($\dim V < \infty$) $\mathcal{B}$ is an ordered basis of V.

**Remark**: 1. characteristic polynomial is of degree $\dim V$ and is monic. (coefficient of highest degree term is 1)

2. roots of characteristic polynomial are eigenvalues of T

are eigenvalues.

3. If $\dim V = n$, then $T$ has atmost $n$-eigenvalues.

**Example:** $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (-y, x)$

$T(1,0) = (0,1) = 0e_1 + e_2$

$T(0,1) = (-1,0) = -e_1 + 0e_2$

$\mathcal{B} = \{e_1, e_2\}$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det\left( x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \det\left( \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \right) = x^2 + 1$$

which has no roots. Hence no-eigenvalues.

**Example:** $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (x+y,\ y+z,\ z+x)$

$\mathcal{B} = \{e_1, e_2, e_3\}$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial for T,

$$\det\left( \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - [T]_{\mathcal{B}} \right)$$

$$= \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & -1 \\ -1 & 0 & x-1 \end{vmatrix}$$

$$= (x-1)^3 - 1 = x^3 - 3x^2 + 3x - 2$$

Thus $f(x) = (x-2)(x^2-x+1)$ is the characteristic polynomial of T.

Observe that

$x = 2, \ \dfrac{1 \pm \sqrt{3}\, i}{2}$ are the roots of

$f(x)$ but $\frac{1 \pm \sqrt{3}i}{2} \notin \mathbb{R}$ hence $x = 2$ is the only eigenvalue of $T$.

Let us see the following definition

**Definition:** The set $E_c = \{\alpha \in V : T\alpha = c\alpha\}$ is the eigenspace corresponding to an eigenvalue $c$ of $T : V \to V$.

**Remark:** $E_c$ is a vector subspace of $V$.

**Proof:** $d \in F$, $\alpha, \beta \in E_c$

To show, $d\alpha + \beta \in E_c$

i.e. $T(d\alpha + \beta) = c(d\alpha + \beta)$

L.H.S.

$$T(d\alpha + \beta) = d\,T\alpha + T\beta = d\,c\alpha + c\beta$$
$$= c(d\alpha + \beta)$$

$\Rightarrow$ $d\alpha + \beta \in E_c$ and hence $E_c$ is a vector subspace of $V$.

To find out eigenvector for $c = 2$ in last example we need to solve the following system of equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y = 0 \Rightarrow x = y$$

$$y = z$$

$$x = z$$

$$\Rightarrow x = y = z$$

$\Rightarrow$ solution space in spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Excercise :

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Find out the eigen values & corresponding eigen spaces.!!