Tutorial-5

(1) X denotes the time (in hours) needed to locate and rectify a prob. $X \sim N(10, 9)$

The required prob is $P(X \leq 15)$

$$= P\left(\frac{X-10}{3} \leq \frac{15-10}{3}\right)$$

$$= P(Z \leq 5/3) = \Phi(5/3) = 0.9525$$

(2) Prob. of giving opinion in favor is $p = \frac{1}{2}$

so $q = 1/2$, $n = 100$, $np = 50$.

$npq = 25$

$X \to$ no of adults in favour of the project

$X \sim Bin(100, \frac{1}{2})$

$$P(X \geq 60) = P\left(\frac{X - np}{\sqrt{npq}} \geq \frac{60 - np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{X-50}{\sqrt{25}} \geq \frac{60-50}{\sqrt{25}}\right) \approx P(Z \geq 2)$$

$$= 1 - \Phi(2) = 1 - 0.9772 = 0.0228.$$

(3) X denote the length of dimeter

$X \sim N(3, 0.005^2)$

Required prob. P(ball bearing is scarpped)

$= 1 - P(2.99 < X < 3.01)$

$= 1 - P(-2 < Z < 2)$

$= 2\Phi(-2)$ $\quad [\because \Phi(2) + \Phi(-2) = 1]$

$= 2 \times 0.0228 = 0.0456.$

So 4.56% of of balls will be scrapped.

(4) X be the hight which will cleave by the high jumper

$X \sim N(200, 100)$

Let a be such that $P(X > C) = 0.95$

$\Rightarrow P\left(\frac{X-200}{10} > \frac{C-200}{10}\right) = 0.95$

$\Rightarrow P\left(Z > \frac{C-200}{10}\right) = 0.95$

$\Rightarrow P\left(Z \le \frac{C-200}{100}\right) = 0.05$

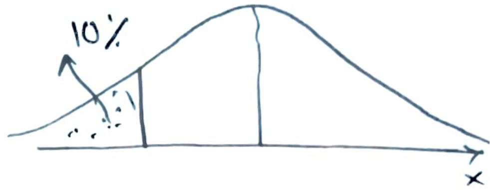
$\frac{C-200}{100} = -1.645 \Rightarrow \frac{200-C}{100} = 1.645$

$\Rightarrow C = 183.55$ cm.

Further d is s.t.

$$P(x>d) = 0.1 \Rightarrow \frac{200-d}{10} = -1.28 \Rightarrow d = 212.80 \text{ cm}$$

(3) X denote the marks. $X \sim N(74 \cdot 62.41)$



(a) $P(X<c) = 0.1$

$$P\left(Z < \frac{c-74}{\sqrt{62.41}}\right) = 0.1 \Rightarrow \frac{c-74}{\sqrt{62.41}} = -1.28$$

$$\Rightarrow c \approx 64.$$

lowest passing marks $= 64$.

(b)

$$P(X>d) = 0.05$$

$$P\left(Z \leq \frac{d-74}{\sqrt{62.41}}\right) = 0.95$$

$$\frac{d-74}{\sqrt{62.41}} = 1.645 \Rightarrow d \approx 86.99.$$

So highest of B is 86.

6) $X = 0, 1, 2, 3$ for white, red, black and blue balls respectively

Y = number in the balls $= 0, 1, 2, 3, 4$.

$f_{X,Y}(i,j) = P(X=i, Y=j)$

marginal of Y.

Y \ X	0	1	2	3	$f_Y(y)$
0	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{4}{14}$
1	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{4}{14}$
2	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	0	$\frac{3}{14}$
3	$\frac{1}{14}$	$\frac{1}{14}$	0	0	$\frac{2}{14}$
4	$\frac{1}{14}$	0	0	0	$\frac{1}{14}$
marginal of X ← $f_X(x)$	$5/14$	$4/14$	$3/14$	$2/14$	1

The marginal of X and Y are:

$$f_X(x) = \begin{cases} 5/14, & x=0 \\ 4/14 & x=1 \\ 3/14, & x=2 \\ 2/14, & x=3 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = \begin{cases} 4/14, & y=0 \\ 4/14, & y=1 \\ 3/14, & y=2 \\ 2/14, & y=3 \\ 1/14, & y=4 \end{cases}$$

(7) X - denote the time of arrival of boy

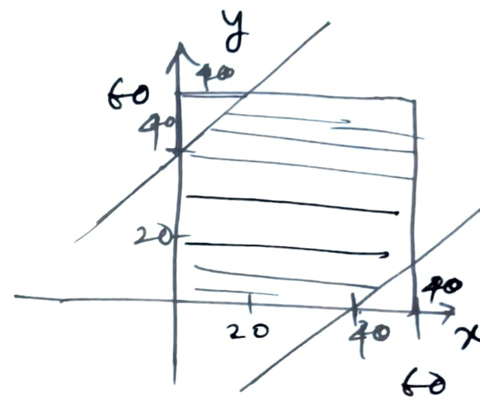
$Y \rightarrow$ denote the time of arrival of girl.

Then $X \sim U(0,60)$, $Y \sim U(0,60)$ and they are independent

So $f_{X,Y}(x,y) = \begin{cases} \frac{1}{3600}, & 0<x<60,\ 0<y<60 \\ 0, & o/w \end{cases}$

Then the required prob is

$P(|X-Y|<20) = P(-20 < X-Y < 20)$



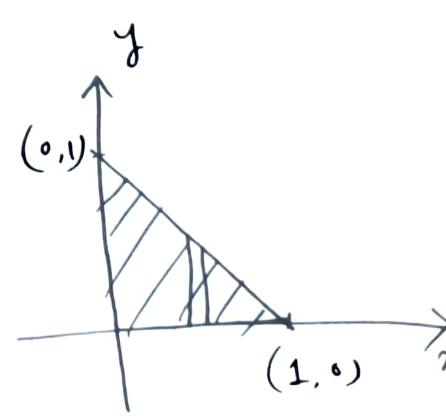
$= \frac{60\times 60 - 2\times\frac{1}{2}\times 40\times 40}{3600}$

$= 5/9$.

So the required probability is $5/9$.

(8) The joint density is given as

$f_{X,Y}(x,y) = \begin{cases} K(1-x-y), & x>0,\ y>0,\ x+y<1 \\ 0, & o/w. \end{cases}$



$\int_{x=0}^{1}\int_{y=0}^{1-x} f_{X,Y}(x,y)\,dy\,dx = 1$

$= \int_0^1 \left[\int_0^{1-x} K(1-x-y)\,dy\right] dx = 1 \quad \Rightarrow K = 6.$

The marginal pdf of x is

$$f_X(x) = \int_{y=0}^{1-x} 6(1-x-y)\,dy \qquad 0 < x < 1$$

$$= 3(1-x)^2, \quad 0 < x < 1$$

$$f_X(x) = \begin{cases} 3(1-x)^2, & 0 < x < 1 \\ 0, & o/w. \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{x=0}^{1-y} 6(1-x-y)\,dx, & 0 < y < 1 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} 3(1-y)^2, & 0 < y < 1 \\ 0, & o/w. \end{cases}$$

$$E(X) = \int_0^1 3x(1-x)^2\,dx = \frac{1}{4}.$$

$$E(Y) = \int_0^1 3y(1-y)^2\,dy = 1/4.$$

$$E(XY) = \int_0^0 \int_0^{1-x} 6(1-x-y)\cdot xy\,dy\,dx = \frac{1}{20}$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{20} - \frac{1}{16} = \frac{-5+4}{80} = -\frac{1}{80}$$

$$\sigma_X^2 = E(X^2) - E(X)^2 = \frac{1}{10} - \frac{1}{16} = \frac{16-10}{160} = \frac{6}{160} = \frac{3}{80}$$

$$\sigma_Y^2 = \frac{3}{80}$$

$$\rho_{X,Y} = \frac{-\frac{1}{80}}{\sqrt{\frac{3}{80} \times \frac{3}{80}}} = -\frac{1}{\cancel{80}} \times \frac{\cancel{80}}{3} = -\frac{1}{3}.$$

(9) (a) We have $c \sum_{x=0}^{3} \sum_{y=1}^{4} (3x+4y) = 1 \Rightarrow c = \frac{1}{232}$

(b) The marginal p.m.f of X

$$f_X(x) = \sum_{y=1}^{4} \frac{1}{232}(3x+4y), \quad x = 0,1,2,3$$

$$= \frac{1}{232}\left[12x + 4(1+2+3+4)\right]$$

$$= \frac{1}{232}(12x+10) = \frac{1}{58}(3x+10)$$

$$f_X(x) = \begin{cases} \frac{1}{58}(3x+10), & x = 0,1,2,3 \\ 0 & \text{o/w} \end{cases}$$

Illy

$$f_Y(y) = \begin{cases} \sum_{x=0}^{3} \frac{1}{232}(3x+4y), & y = 1,2,3,4 \\ 0 & \text{o/w} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{116}(9+8y), & y = 1,2,3,4 \\ 0, & o/w. \end{cases}$$

(c) $$P(X \geq 2 \mid Y \leq 3) = \frac{P(X \geq 2, Y \leq 3)}{P(Y \leq 3)}$$

$$= \frac{\frac{1}{232}\sum_{x=2}^{3}\sum_{y=1}^{3}(3x+4y)}{\sum_{y=1}^{3}\frac{1}{116}(9+8y)} =$$

$$P(Y=2 \mid X=3) = \frac{P(Y=2, X=3)}{P(X=3)} = \frac{P(X=3, Y=2)}{P(X=3)}$$

$$= \frac{\frac{1}{232}(9+8)}{\frac{1}{58}(9+10)}$$

(10) (a) The joint p.m.f is given as

X \ Y	-1	0	1	$f_X(x)$
0	0	1/3	0	1/3
1	1/3	0	1/3	2/3
$f_Y(y)$	1/3	1/3	1/3	1

$$f_X(x) = \begin{cases} \frac{1}{3}, & x = 0 \\ 2/3 & x = 1 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{3}, & y = -1,0,1 \\ 0, & o/w. \end{cases}$$

(b) $E(X) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = 2/3$

$E(Y) = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$

$$E(XY) = \sum_{x=0,1} \sum_{y=-1,0,1} xy \, f_{X,Y}(x,y)$$

$$= 0 \cdot (-1) \cdot 0 + 0 \cdot 0 \cdot \frac{1}{3} + 0 \cdot (1) \cdot 0 + 1 \cdot (-1) \frac{1}{3} + 1 \cdot 0 \cdot \frac{1}{3}$$
$$+ 1 \cdot 1 \cdot \frac{1}{3} = 0$$

So $\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$

$= 0 - 0 = 0.$

$\rho_{xy} = 0.$

(c) We have $f_X(0) = \frac{1}{3}$, $f_Y(0) = \frac{1}{3}$, $f_{X,Y}(0,0) = \frac{1}{3}$

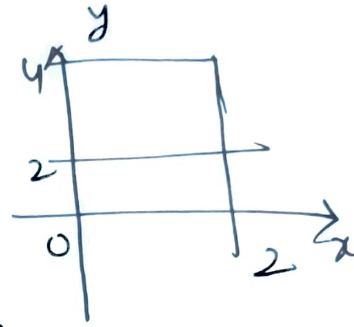
$\neq f_X(0)\, f_Y(0).$

So X and Y aree not independent.

(11) Same as 8. Do yourself.

(12)

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0, & o/w. \end{cases}$$



(a) The marginal density of X is

$$f_X(x) = \begin{cases} \displaystyle\int_2^4 \left(\frac{6-x-y}{8}\right) dy, & 0<x<2 \\ 0, & o/w. \end{cases}$$

$$= \begin{cases} \dfrac{6-2x}{8}, & 0<x<2 \\ 0, & o/w. \end{cases}$$

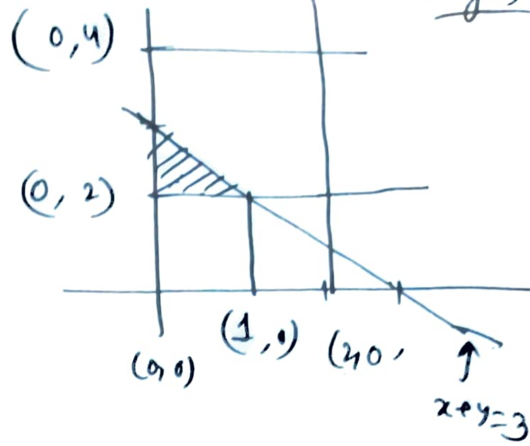
$||^y$

$$f_Y(y) = \begin{cases} \displaystyle\int_0^2 \frac{6-x-y}{8}\, dx, & 2<y<4 \\ 0, & o/w \end{cases} = \begin{cases} \dfrac{5-y}{4}, & 2<y<4 \\ 0, & o/w. \end{cases}$$

(b) $$P(X<1, Y<3) = \int_{x=0}^{1}\int_{y=2}^{3} \frac{6-x-y}{8}$$

$$= \frac{1}{16}\int_0^1 (7-2x)\, dx = \frac{3}{8}.$$

$$P(X+Y<3) = \int_{x=0}^{1} \int_{y=2}^{3-x} \left(\frac{6-x-y}{8}\right) dy\, dx$$



$$= \frac{1}{16} \int_{0}^{1} (x^2 - 8x + 7)\, dx = 5/24.$$

The conditional p.d.f of X given $Y=y$ is $(2<y<4)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)}, \quad 0<x<2$$

So for $2<y<4$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{6-x-y}{2(5-y)}, & 0<x<2 \\ 0, & o/w. \end{cases}$$

So $$P(X<1 \mid Y=3) = \int_0^1 f_{X|Y}(x|3)\, dx = \int_0^1 \frac{6-x-3}{2(5-3)}\, dx$$

$$= \frac{1}{4} \int_0^1 (3-x)\, dx = 5/8.$$

Now $$P(X<1 \mid Y<3) = \frac{P(X<1,\ Y<3)}{P(Y<3)}$$

$$P(Y<3) = \int_2^3 \frac{5-y}{4}\,dy = 5/8$$

So $$P(X<1 \mid Y<3) = \frac{3}{8} \Big/ 5/8 = 3/5$$