Lecture #12 (IC152)

Matrix representation of an inner product

Let V be n-dimensional v. space & $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be an ordered basis for V.

Let us consider

$$\langle \alpha, \beta \rangle = \Big\langle \sum_{k=1}^{n} x_k \alpha_k , \underbrace{\sum_{j=1}^{n} y_j \alpha_j}_{\beta} \Big\rangle$$

$$= \sum_{k=1}^{n} x_k \langle \alpha_k, \beta \rangle$$

$$= \sum_{k=1}^{n} x_k \Big\langle \alpha_k, \sum_{j=1}^{n} y_j \alpha_j \Big\rangle$$

$$= \sum_{k=1}^{n} \underline{x_k} \sum_{j=1}^{n} \underline{\overline{y_j}}\, \underline{\langle \alpha_k, \alpha_j \rangle}$$

$$= \sum^{n} \overline{y_j} M_{jk} x_k$$

Recall

1. Inner product
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$
$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

2. Inner product space

3. length $(V, \langle \cdot,\cdot \rangle)$
$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$

4. $\langle \alpha, \beta \rangle = \frac{1}{4}\|\alpha+\beta\|^2 - \frac{1}{4}\|\alpha-\beta\|^2$

$\langle \alpha_1, y_1\alpha_1 + y_2\alpha_2 \rangle$
$\overline{y_1}\langle \alpha_1, \alpha_1 \rangle$
$+ \overline{y_2}\langle \alpha_1, \alpha_2 \rangle$

$\langle \alpha, c\beta \rangle$
$= \overline{c}\langle \alpha, \beta \rangle$

$j,k=1$

where $\quad M_{jk} = \langle \alpha_k, \alpha_j \rangle$

$1\times n \times n \times n \times n \times 1 \in \mathbb{F}$

$$= Y^* M X$$

where X & Y are co-ordinates of α, β w.r.t. ordered basis $\mathcal{B}$.

Here M is known as matrix of innerproduct $\langle \cdot , \cdot \rangle$ relative to ordered basis $\mathcal{B} = \{\alpha_1, \ldots \alpha_n\}$

$$\left\{ \boxed{M_{jk} = \langle \alpha_k, \alpha_j \rangle} \right\}$$

<u>Example</u>: $V = \mathbb{R}^2, F = \mathbb{R} \quad \langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2$

$\mathcal{B} = \{(1,0), (0,1)\}$

$M_{11} = \langle (1,0), (1,0) \rangle\, 1 \qquad M_{1,2} = \langle (0,1), (1,0) \rangle = 0$

$M_{21} = \langle (1,0)\ (0,1) \rangle = 0 \qquad M_{22} = \langle (0,1)(0,1) \rangle = 1$

$M = \begin{bmatrix} M_{11} & M_{12} \\ \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ \end{bmatrix}$

$$M = \begin{bmatrix} M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Observations

1) Matrix M is Hermitian !! $\left(M_{jk} = \langle \alpha_k, \alpha_j \rangle = \overline{\langle \alpha_j, \alpha_k \rangle} = \overline{M_{kj}} \right)$

2) $X^* M X > 0$ if $X \neq 0$ $\left(\text{if } \beta = \alpha, \Rightarrow Y^* = X^* \text{ \& hence } \langle \alpha, \alpha \rangle = X^* M X > 0 \text{ if } \alpha \neq 0 \text{ if } X \neq 0 \right)$

Remark :-

For a given Hermitian matrix M, satisfying

✓ ⟶ ✓ $X^* M X > 0 \ \forall \ X \neq 0$, (positive definite)

give rise to an inner product defined as

$$\langle \alpha, \beta \rangle = Y^* M X.$$

A positive definite matrix is invertible. If not, $\exists X \neq 0$ s.t. $MX = 0$, $X^* M X = 0$

Example :- $M = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$,

Take any ordered basis $\mathcal{B}$ and find $[\alpha]_{\mathcal{B}} = X$, $[\beta]_{\mathcal{B}} = Y$

then $\langle \alpha, \beta \rangle = Y^{*} M X$ defines an inner product on 2-dim vector space

Parallelogram Law

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space then

$$\boxed{\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\left(\|\alpha\|^2 + \|\beta\|^2\right)}$$

Proof $\Rightarrow$

$$\|\alpha+\beta\|^2 = \langle \alpha+\beta, \alpha+\beta \rangle$$

$$= \|\alpha\|^2 + 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \|\beta\|^2$$

$$\|\alpha-\beta\|^2 = \|\alpha\|^2 - 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \|\beta\|^2$$

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\left(\|\alpha\|^2 + \|\beta\|^2\right)$$

Excercise.

$$\|\alpha\| = 3 \quad \|\alpha+\beta\| = 4 \quad \|\alpha-\beta\| = 5$$

$$\|\beta\| = ?$$

$$\Rightarrow 2\|\beta\|^2 = 16 + 25 - 18 = 23$$

$$\|\beta\| = \sqrt{\frac{23}{2}}$$

Definition (Orthogonal vectors)

Let $(V, \langle \cdot , \cdot \rangle)$ be an ips then α & $\beta \in V$ are called orthogonal if $\langle \alpha , \beta \rangle = 0$.

Let $S \subset V$, then S is an orthogonal subset of V if any arbitrary pair of vectors of S is orthogonal.

$$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$$

then $\langle \alpha_i, \alpha_j \rangle = 0 \;\forall\; i, j = 1, 2, \ldots, n$, $i \neq j$

Remark: A set S is called orthonormal if it is orthogonal & length of each vector in S is 1.

i.e. if $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$

then $\langle \alpha_i, \alpha_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & i = j \end{cases}$

Ex. Let $V = \mathbb{R}^2$ & standard inner product

$\alpha = (1, -1)$ $\beta = (2, 2)$

$\langle \alpha, \beta \rangle = 1 \cdot 2 + (-1) \cdot 2 = 0$

$\Rightarrow$ α, β are orthogonal to each other.

$\langle \alpha, \alpha \rangle$

$= 1^2 + 1^2 = 2$

$\|\alpha\| = \sqrt{2}$

$\alpha' = \dfrac{\alpha}{\|\alpha\|} = \dfrac{1}{\sqrt{2}}(1, -1) = \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

$\beta' = \dfrac{1}{\ldots}(2, 2) = \left(\dfrac{1}{\ldots}, \dfrac{1}{\ldots}\right)$

$\Gamma = \frac{}{2\sqrt{2}}(\ , \ ) \quad \frac{}{\sqrt{2}} \ \frac{}{\sqrt{2}}$

$\langle \alpha', \beta' \rangle = 0$ & $\|\alpha'\| = 1$, $\|\beta'\| = 1$

α', β' are orthonormal.

Theorem :- An orthogonal set of nonzero vectors is linearly independent.

Proof :- $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ orthogonal set

Assume $c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n = 0$

$$c_1\langle \alpha_1, \alpha_i \rangle + \underline{c_2\langle \alpha_2, \alpha_i \rangle + \ldots \quad c_n\langle \alpha_n, \alpha_i \rangle}$$

$$= \langle 0, \alpha_i \rangle = 0$$

$$c_i \|\alpha_i\|^2 = 0$$

$$c_i = 0 \quad \forall \ i = 1, 2, \ldots n$$

$\Rightarrow$ S is linearly independent.

Recall :- $\langle \alpha, \beta \rangle = 0 \quad \forall \ \beta \in V,$

$[\langle 0, \beta \rangle = 0$

$\exists\, \alpha = 0$!!

$\forall \beta \in V$