IC105

Tutorial #4 (Solutions)

1. (a) The number of trains arriving on days 1, 2 and 3 is independent of the number of trains arriving on day 0.

Let X_i be the # of trains that arrive on i^{th} day $i = 1, 2, 3$.

Given $X_i \sim Po(3)$. (as $\lambda = 3$)

Let $N = \sum_{i=1}^{3} X_i$ be the total no. of trains that arrive on days 1, 2, 3.

$\Rightarrow N \sim Po(3\lambda) = Po(9)$.

$\therefore$ P(no trains on days 1, 2, 3 | One train on Day 0)

= P(no train on days 1, 2, 3) (because of independence).

$= P(N=0) = \dfrac{e^{-9}9^0}{0!} = e^{-9}$.

(b) P(second arrival is after Day 3 | one train arrival on Day 0)

= P(no train on Day 1, 2, 3 | one train arrival on Day 0)

$= e^{-9}$. (by part (a)).

(c) P(0 trains in first 2 days and 4 trains on Day 4)

= P(0 trains in first 2 days) P(4 trains on Day 4) (independence)

$= e^{-2\lambda}\dfrac{(2\lambda)^0}{0!} \cdot e^{-\lambda}\dfrac{\lambda^4}{4!} = e^{-9}\dfrac{3^4}{4!}$.

(d) P(5^{th} train arrival in more than 2 days) = P(at most 4 trains on first 2 days)

$= \sum_{k=0}^{4}$ P(exactly k arrival in first 2 days)

$= \sum_{k=0}^{4} e^{-2\lambda}\dfrac{(2\lambda)^k}{k!} = e^{-6}(1+6+18+36+54) = 115e^{-6}$.

(2) $n = 10$, Success probability $= p$. (heads).

(a) Note that $P(A) = \binom{8}{6} p^6 (1-p)^{8-6} > 0$.

$$\therefore \quad P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\binom{8}{6} p^6 (1-p)^2 p}{\binom{8}{6} p^6 (1-p)^2} = p = P(B)$$

$\Rightarrow$ A and B are independent.

(b) let F: 3 heads in first 4 tosses, L: 2 heads in the last 3 tosses.

$$P(F \cap L) = P(F)P(L) \quad \text{(independence)}$$
$$= \binom{4}{3} p^3 (1-p) \cdot \binom{3}{2} p^2 (1-p) = 12 p^5 (1-p)^2.$$

(c) let G: 4 heads in the first 7 tosses, H: 2nd head on 4th toss.

$$P(H|G) = \frac{P(H \cap G)}{P(G)} = \frac{\binom{3}{1} p(1-p)^2 \cdot p \cdot \binom{3}{2} p^2 (1-p)}{\binom{7}{4} p^4 (1-p)^3} = \frac{9}{35}.$$

(d) let I: 5 heads in the first 8 tosses, J: 3 heads in the last 5 tosses.

$P(I \cap J) = P(I \cap J \mid \text{one head in tosses 6-8})\, P(\text{one head in tosses 6-8})$
$+ P(I \cap J \mid \text{two " " " "})\, P(\text{two " " " "})$
$+ P(I \cap J \mid \text{three " " " "})\, P(\text{three " " " "})$

$$= \binom{5}{4} p^4 (1-p) \cdot p^2 \cdot \binom{3}{1} p(1-p)^2$$
$$+ \binom{5}{3} p^3 (1-p)^2 \cdot \binom{2}{1} p(1-p) \cdot \binom{3}{2} p^2 (1-p)$$
$$+ \binom{5}{2} p^2 (1-p)^3 \cdot (1-p)^2 \cdot p^3$$
$$= 15 p^7 (1-p)^3 + 60 p^6 (1-p)^4 + 10 p^5 (1-p)^5$$

③ Let X denote the time (in minutes) past 7:00 am that the passenger arrive at the bus stop. Then, $X \sim U(0,30)$.

ⓐ $P(\text{passenger waits} < 5 \text{ mins for a bus}) = P(10 < X < 15) + P(25 < X < 30)$

$$= \frac{5}{30} + \frac{5}{30} = \frac{1}{3}.$$

ⓑ $P(\text{passenger waits} > 12 \text{ mins for a bus}) = P(0 < X < 3) + P(15 < X < 18)$

$$= \frac{3}{30} + \frac{3}{30} = 1/5.$$

④

Let O be the cut point.

Let X be the length AO $\Rightarrow X \sim U(0,2)$.

Required probability $= P(\max\{X, 2-X\} \geq 2\min\{X, 2-X\})$

i) If $X > 2-X \Rightarrow X \geq 2(2-X) \Rightarrow X \geq \frac{4}{3}$.

ii) If $X < 2-X \Rightarrow 2-X \geq 2X \Rightarrow X \leq \frac{2}{3}$.

$\therefore$

$P(\max\{X, 2-X\} \geq 2\min\{X, 2-X\})$

$= P(X \leq \frac{2}{3}) + P(\frac{4}{3} < X < 2)$

$= \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3} = 2/3.$

⑤ Let X denote the life of a bulb in hours.

Given $X \sim Exp(\theta)$ with $E(X) = 1/\theta = 50$ $\Rightarrow \theta = 1/50$.

So,

$$f_X(x) = \begin{cases} \frac{1}{50} e^{-x/50} & , x > 0 \\ 0 & , o/w \end{cases}$$

$$P(\text{A bulb works after 100 hours}) = P(X > 100) = \frac{1}{50}\int_{100}^{\infty} e^{-x/50}\,dx$$

$$= e^{-2} = p \text{ (say)}$$

Let Y denote the number of bulb working after 100 hrs.

Then, $Y \sim Bin(10, e^{-2})$

$$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$$

$$= 1 - (1-e^{-2})^{10} - 10(e^{-2})(1-e^{-2})^{9}$$