INDIAN INSTITUTE OF TECHNOLOGY BHILAI

CS203: Theory of Computation I

Tutorial Sheet 1

- *Solve the following problems before the Tutorial.*

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1. For strings x and y, prove $(xy)^R = y^R x^R$.

> **Solution:** Prove by induction on the length of string x.
>
> **Base Case:** Let $|x| = 0$. Therefore $x = \epsilon$. Then,
> $(xy)^R = (\epsilon y)^R = y^R = y^R\epsilon = y^R\epsilon^R = y^R x^R$.
>
> **Inductive Hypothesis:** Assume true for all strings x of length $\leq n$.
>
> We show that property then holds for x of length $= n + 1$.
>
> We know $x = av$, where $|v| = n$ and $a \in \sum$ (a is a single character). Then
>
> $$\begin{aligned}(xy)^R &= ((av)y)^R \\ &= (a(vy))^R && \text{(associativity of concatenation)} \\ &= (vy)^R a && \text{(definition of reverse)} \\ &= (y^R v^R)a && \text{(inductive hypothesis)} \\ &= y^R(v^R a) && \text{(associativity of concatenation)} \\ &= y^R(av)^R && \text{(definition of reverse)} \\ &= y^R x^R && \text{(since } x = av\text{)}\end{aligned}$$

2. For language L_1 and L_2, prove $(L_1L_2)^R = L_2^R L_1^R$ and $(L_1 \cap L_2)^R = L_1^R \cap L_2^R$.

> **Solution:** a) Let $\sigma \in (\mathrm{L}_1\ \mathrm{L}_2)^R$.
> Hence $\sigma^R \in \mathrm{L}_1\ \mathrm{L}_2$.
> Let $\sigma^R = \mathrm{xy}$ such that $\mathrm{x} \in \mathrm{L}_1$ and $\mathrm{y} \in \mathrm{L}_2$.
> Now $\sigma = (\mathrm{xy})^R = \mathrm{y}^R\ \mathrm{x}^R \in \mathrm{L}_2^R\ \mathrm{L}_1^R$.
>
> Let $\sigma \in \mathrm{L}_2^R\ \mathrm{L}_1^R$.
> Hence $\sigma = \mathrm{xy}$ such that $\mathrm{x} \in \mathrm{L}_2^R$ and $\mathrm{y} \in \mathrm{L}_1^R$.
> Then $\mathrm{x}^R \in \mathrm{L}_2$ and $\mathrm{y}^R \in \mathrm{L}_1$.
> Hence $\sigma^R = \mathrm{y}^R\ \mathrm{x}^R \in \mathrm{L}_1\ \mathrm{L}_2$.
> So, $\sigma \in (\mathrm{L}_1\ \mathrm{L}_2\ )^R$. Thus Proved.
>
> b) Prove: $(L1 \cap L2)^R = L_1^R \cap L_2^R$.
> Let $x \in (L_1 \cap L_2) \implies x \in L_1$ and $x \in L_2$.
> If $x \in L_1$, then $x^R \in L_1^R$.
> If $x \in L_2$, then $x^R \in L_2^R$.
> If $x \in (L_1 \cap L_2)$, then $x^R \in (L_1 \cap L_2)^R \implies x^R \in L_1^R$ and $x^R \in L_2^R$.

3. For language L, prove $L^+ = L^*$ if and only if $\epsilon \in L$.

4. Let $L = \{ab, aa, baa\}$. Which of the following strings are in L^*: *abaabaaabaa*, *aaaabaaaa*, *baaaaabaaaab*, *baaaaabaa*? Which strings are in L^4?

Solution: The strings in L* are: abaabaaabaa, aaaabaaaa, baaaaabaa.
L^4 = { abababab, abababaa, abababbaa, aaaaaaaa, aaaaaaab, aaaaaabaa, baabaabaaab, baabaabaaaa, baabaabaabaa, ... }

string	L^*	L^4	Reason
abaabaaabaa	Yes	No	combination of (ab,aa,baa,ab,aa).
aaaabaaaa	Yes	Yes	combination of (aa,aa,baa,aa).
baaaaabaaaab	No	No	combination of (baa,aa,ab,aa,aa,b).
baaaaabaa	Yes	Yes	combination of (baa,aa,ab,aa).

5. Let $\sum = \{a, b\}$ and $L = \{aa, bb\}$. Use set notation to describe $\overline{L}$, complement of L.

Solution: $\bar{L} = \sum^*$ - L $= \{w \in \sum^* : w \neq aa, w \neq bb\}$

6. Let $L_1 = \{\varepsilon, a\}$ and $L_2 = \{a, b\}$. List the elements of the following sets.
   (i) L_1^2
   (ii) L_2^3
   (iii) L_1L_2
   (iv) L_1^+
   (v) L_2^*

Solution:

(i) $L_1^2 = \{\epsilon$, a ,aa }

(ii) L_2^3 = { aaa, aab, aba, abb, baa, bab, bba, bbb }

(iii) L_1 L_2 = {, a, b, aa, ab }

(iv) $L_1^+ = \{\epsilon$, a, aa, aaa, $\cdots\}$

(v) $L_2^* = \{\epsilon$, a, b, aa, ab, ba, bb, $\cdots\}$

7. Find Kleene star (L^*) of the language $L = \{\varepsilon, 0, 01\}$.

Solution: L = $\{\epsilon$, 0, 01, 00, 001, 010, 0101, ...}
i.e. L^* is set of all strings starting with 0's and no two consecutive 1's, including empty string.

8. Prove distributive properties for the languages L_1, L_2, L_3
   (i) $(L_1 \cup L_2)L_3 = L_1L_3 \cup L_2L_3$

Solution: Suppose x $\in$ ($L_1 \cup L_2$) L_3
$\Rightarrow$ x = x_1 x_2, for some $x_1 \in (L_1 \cup L_2)$ and some $x_2 \in L_3$
$\Rightarrow$ x = x_1 x_2, for some $x_1 \in L_1$ or $x_1 \in L_2$ and some $x_2 \in L_3$
$\Rightarrow$ x = x_1 x_2, for some $x_1 \in L_1$ and $x_2 \in L_3$ or some $x_1 \in L_2$ and $x_2 \in L_3$
$\Rightarrow$ x $\in L_1$ L_3 or L_2 L_3
$\Rightarrow$ x $\in L_1$ $L_3 \cup L_2$ L_3

Conversely, suppose x $\in L_1$ $L_3 \cup L_2$ L_3 .
Without loss of generality, assume x $\notin L_1$ L_3. Then x $\in L_2$ L_3.

$\Rightarrow$ x = x_3 x_4, for some $x_3 \in L_2$ and $x_4 \in L_3$

$\Rightarrow$ x = x_3 x_4, for some $x_3 \in (L_1 \cup L_2)$ and $x_4 \in L_3$

$\Rightarrow$ x $\in$ (($L_1 \cup L_2$) L_3).

Hence, ($L_1 \cup L_2$) L_3 = L_1 $L_3 \cup L_2$ L_3

(ii) $L_1(L_2 \cup L_3) = L_1L_3 \cup L_1L_3$

Solution: L_1 ($L_2 \cup L_3$) = L_1 $L_3 \cup L_1$ L_3.
Suppose that $L_1(L_2 \cup L_3)$
$\implies x = x_1x_2$, for some $x_1 \in L_1$, and some $x_2 \in (L_2 \cup L_3)$
$\implies x = x_1x_2$, for some $x_1 \in L_1$, and $x_2 \in L_2$ or $x_2 \in L_3$
$\implies x = x_1x_2$, for some $x_1 \in L_1$ and $x_2 \in L_2$, or $x_1 \in L_1$ and $x_2 \in L_3$
$\implies x \in L_1L_2$ or $x \in L_1L_3$
$\implies x \in L_1L_2 \cup x \in L_1L_3$.
Conversely, suppose $x \in L_1L_2 \cup x \in L_1L_3 \implies x \in L_1L_2$ or $x \in L_1L_3$.
Without loss of generality, assume $x \notin L_1L_2$. Then $x \in L_1L_3$.
$\implies x = x_3x_4$, for some $x_3 \in L_1$ and $x_4 \in L_3$
$\implies x = x_3x_4$, for some $x_3 \in L_1$ and $x_4 \in (L_2 \cup L_3)$
$\implies x \in L_1(L_2 \cup L_3)$
Hence, $L_1(L_2 \cup L_3) = L_1L_2 \cup L_1L_3$.

9. Prove $L^*L = LL^* = L^+$.

Solution: As we know that, L^+ is defined to be L + LL + LLL + ...
Also, $L^* = \epsilon$ + L + LL + LLL + ...
Thus, LL^* = $L\epsilon$ + LL + LLL + . . .
When we remember that $L\epsilon$ = L, we see that the infinite expressions for LL^* and L^+ are the same.
That proves $L^+ = LL^*$.
The proof $L^+ = L^*L$ is similar.

10. Write the regular expressions corresponding to the following languages:

(i) The set of all strings over some alphabet $\sum = \{0, 1\}$ with even number of $0'$s.

Solution: $1^*(01^*01^*)^*$ or $(1+01^*0)^*$

(ii) The set of all strings over some alphabet $\sum$ that have an a in the 5th position from the right.

Solution: $(a+b)^*a(a+b)(a+b)(a+b)(a+b)$

(iii) The set of all strings over some alphabet $\sum$ with no consecutive a's.

Solution: (b+ab)*(ϵ+a)

(iv) The set of all strings over $\{a, b\}$ in which every occurrence of b is not before an occurrence of a.

Solution: a*b*