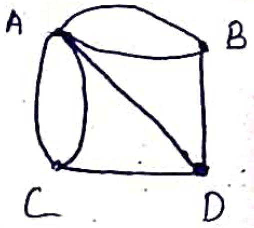
Discrete ~~maths~~ structures - 2

Quiz - 20
Assignment - 20
tierce - 60

1) Graph theory.

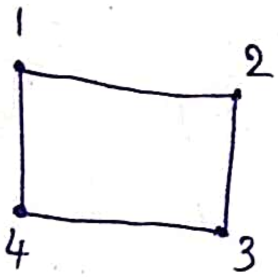
A Graph is a tuple (V, E) where
V is a set called Vertex set,
E is $\subseteq (V \times V)$

Konigs bridge prob

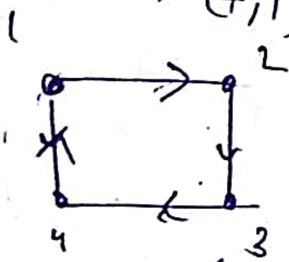


Eg:- $V = \{1, 2, 3, 4\}$

$E = \{(1,2), (2,3), (3,4), (4,1)\}$



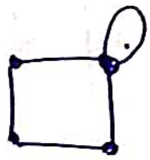
Undirected graphs



directed graphs

Any map can be colo by 4 colour

Simple graph :- which doesn't have edg b/w same nodes.
(x, x),



(atmost 1 vertex)

Path :- Traversing from one edge to other by using other nodes.

A Sequence of Vertices $\{v_1, v_2, \ldots v_k\}$ such that $(v_i, v_{i+1}) \in E$ for $i = 1$ to $K-1$.

Path is a sequence of edges $\{e_1, e_2, \ldots e_k\}$ s.t $e_i = (a,b)$ then $e_{i+1} = (b,c)$ for some $c \in V$.

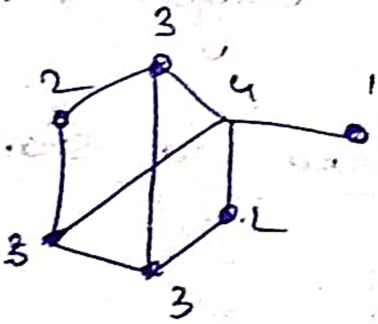
~~For undirected :-~~

two vertices a, b are called adjacent if $(a,b) \in E$.

An edge e is said to be incident to a vertex v if $e = (v,u)$ for some u.

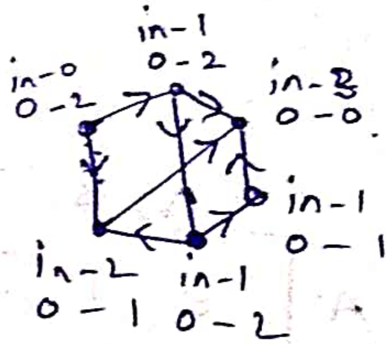
<u>Degree of a node</u> :– (For undirected)

How many edges are incident to it.



For directed — in degree / out degree

$\boxed{\text{Sum of all degrees} = 2 * \text{no. of edges}}$



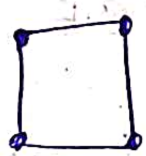
Can there be a graph where no. of odd degree nodes is odd.

No.

* odd degree vertices are always even.

Cycle :-

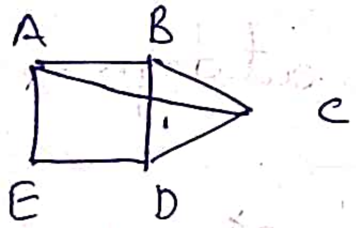
A path for which starting vertex and end vertex is same then it is a cycle. It is a closed path.



Connectedness:-

A graph is called connected, if there exists a path between every two ~~edges~~ vertices.

How to store a graph



$$
\begin{array}{c|ccccc}
 & A & B & C & D & E \\
\hline
A & 0 & 1 & 1 & 0 & 1 \\
B & 1 & 0 & 1 & 1 & 0 \\
C & 1 & 1 & 0 & 1 & 0 \\
D & 0 & 1 & 1 & 0 & 1 \\
E & 1 & 0 & 0 & 1 & 0
\end{array}
$$

"Adjacency list — Can stroe graphs.

Problem:-

Can we find a graph where every vertex have unique degree??

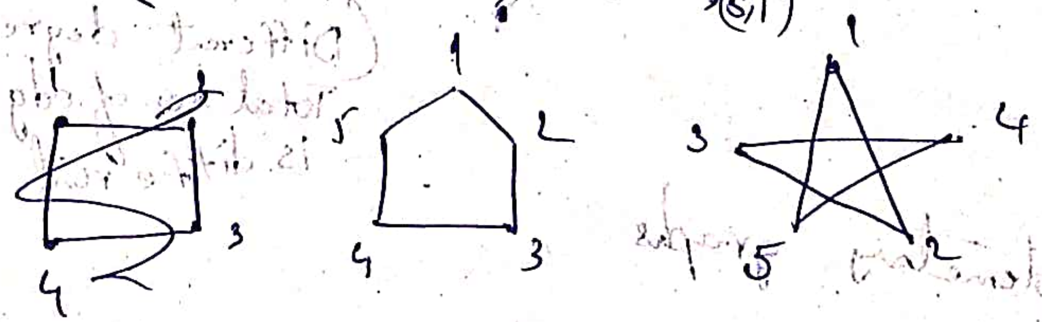
Not possible.

if let the degrees be 1 to N.

then for the node with N degree cannot have degree N.

Q) V: {1, 2, 3, 4, 5}

E = {(1,2), (2,3), (3,4), (4,5), (5,1)}



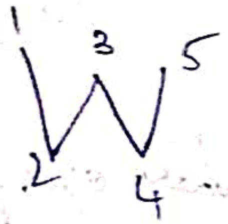
isomorphism :-

Two graphs G, H are called isomorphic if there exists a bijection from $f : V(G) \to V(H)$

Such that $(U,V) \in E(G)$, iff

$(f(U), f(V)) \in E(H)$.

Problem :-



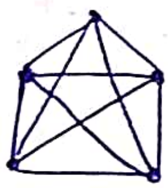
(A - 1) (or) (D - 1)

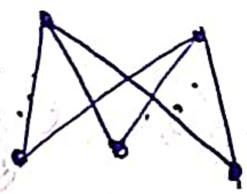
(E - 2) (or) (B - 2)

(E - 3) (or) (E - 3)

(B - 4) (or) (C - 4)

(D - 5) (or) (A - 5)



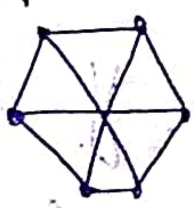


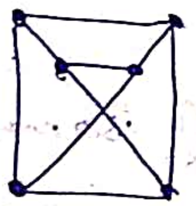
Not isomorphic

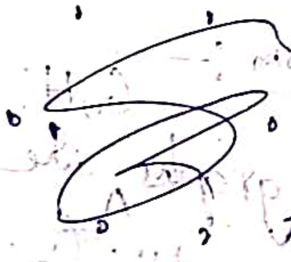
(Different degrees)

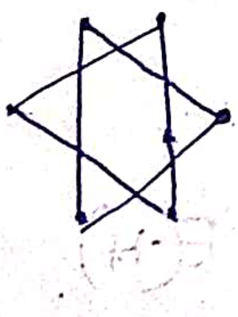
Total no. of edges is different

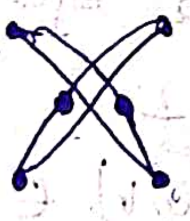
Complemetary graphs



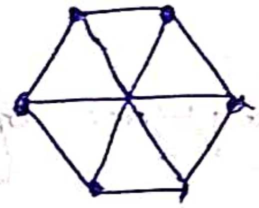


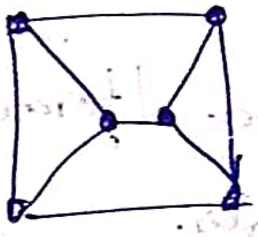


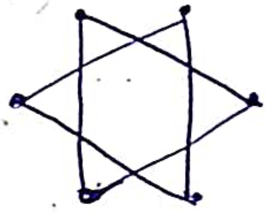




two cycle of 3

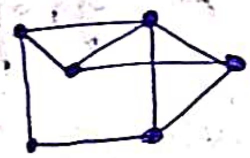






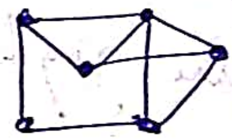
Subgraph :—

A graph $H = (V_1, E_1)$ is said to be subgraph of a graph $G = (V_2, E_2)$, if $V_1 \subseteq V_2$, $E_1 \subseteq E_2$.



induced subgraph :—

For a given graph $G = (V, E)$ an induced subgraph induced by a subset $V' \subseteq V$ is the subgraph (V', E') such that for $(a,b) \in E$, if $a, b \in V'$ and $(a,b) \in E'$



(Graph formed by selecting some nodes and drawing Edges b/w nodes present in Graph)

Walk :- ~~(atmost~~

A walk is ~~an seq~~ alterenating sequence of vertices and edges.

$(v_1, e_1, v_2, e_2, \ldots e_{n-1}, v_n)$ such that $e_i = (v_i, v_{i+1})$

Trail :- (atmost 1 edge)

A trail is a walk where no edge is repeated. (closed trail) $\to$ end point = start point.

Eulerian graph :-

A graph is called eulerian if it has a closed trail that visits every edge of the graph.

Fact 1 :- In a eulerian graph, every vertex have even degree.

$v_1, e_1, v_2, e_2 \ldots v_1$

No. of entering edges = No. of exiting edges.

Fact 2 :- If all the vertices are even degree in a graph, then the graph is eulerian.

Fact 3 :- If there is a graph where every vertex have degree ≥ 2, Then this graph must have a cycle.

Proof :-

(longest path)

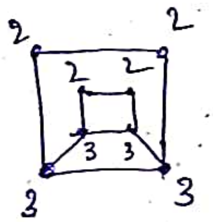
Fact 2 :-

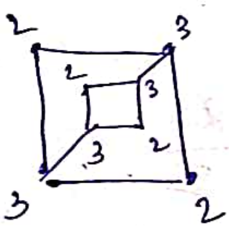
Proof :-

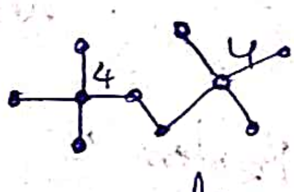
Since, every edge has even degree, there will be a cycle.

Fact 4 :- Every eulerian graph can be written as union of edge-disjoint cycles.

Problem :-

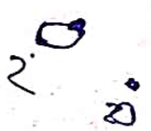






2° not Connected

4 & 4° are not Connected.



Problem 3

$G = (V, E)$

$|V| = 20$ Every vertex either. 3(or) 7.

$|E| = 62$

$$3x + 7(20 - x) = 124$$

$$-4x + 140 = 124$$

$$-4x = \cancel{20}\ -16$$

$$x = 4$$

$$\begin{pmatrix} 3^\circ & 7^\circ \\ 4, & 16 \end{pmatrix}$$

Problem 4

Does there exist a graph with ~~10~~ 7 vertices where every node has degree 3?

Sol

Total edges $= \frac{7 \times 3}{2} = \frac{21}{2}$

Not possible.

~~Problem 5~~

diameter

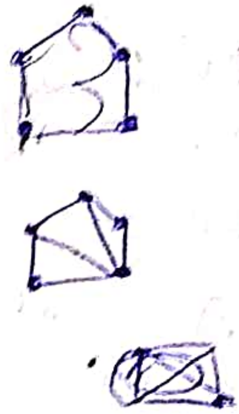
For any graph, distance b/w two vertices is the shortest length path b/w these two vertices

~~Diameter is max(u,v ∈ V)~~

Diameter is $\max(\text{dist}(v,v))$ $v,v \in V$.

Problem 5:- 5 vertex, 7 edges, Simple graph has diameter ≤ 2

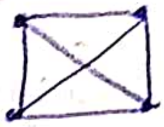
Sol degree of every vertex is atleast $= 2$. P $\left[\frac{14}{5}\right]$



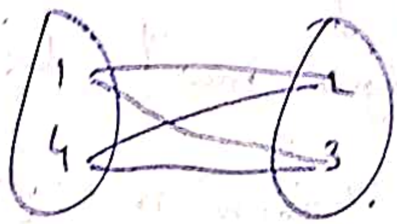
Some special type of graphs :-

(1) Complete graphs

no. of edges $= \frac{n(n-1)}{2}$



(2) Bypartite graph :-



Fact :-

All cycles are of even length.

Fact 2 :-

If for a graph all simple cycles are of even length then the graph is bipartite.

Tree :-

A connected graph without any cycle

Def 2 A collection of trees is a forest.

Property :- Between any two vertices in a tree there is a unique path

Property 2 :- There is exactly $n-1$ edges in an n node ~~edge~~ tree (Proof:- $n_1 + n_2$

$(n_1-1)+(n_2-1)+1$)

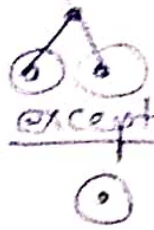
Property 3 :- A connected graph with n nodes and $n-1$ edges must be a tree

(Proof :- if there is a Cycle. Remove a edge in Cycle. Then there would be $(n-2)$ edges, which violate the previous property

Property :- For any tree $n \geq 2$ nodes there is atleast two degree '1' vertices.

Proof ~~Sum of degrees $\geq$ 2n-1~~

exception:



Case 1) No node of degree '1'
~~→ every node~~
Case 2) One node of degree '1'

Case 1) Every node has degrees ≥ 2

$$\text{Sum of degrees} \geq 2n > 2(n-1) = 2|E|.$$

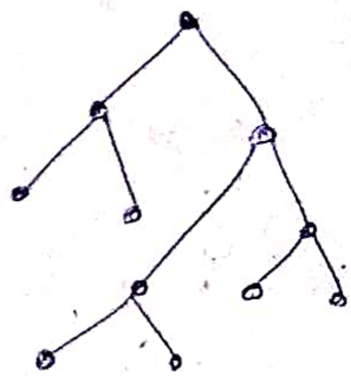
Case 2) $$\sum_{v \in t} deg(v) \geq 1 + 2(n-1) = 2n-1 > (2(n-1)) = 2|E|$$

* If there is a forest of 5 trees, and total number of nodes = n, then no. of edges $= n-1-4 = n-5$

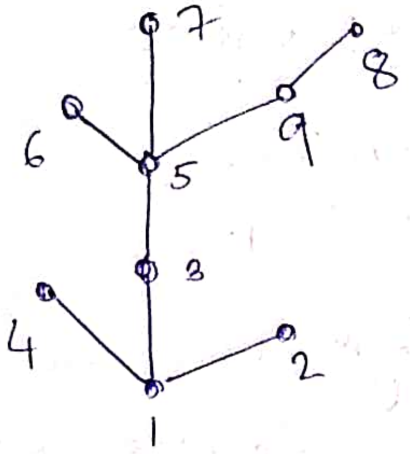
How to store a tree :-

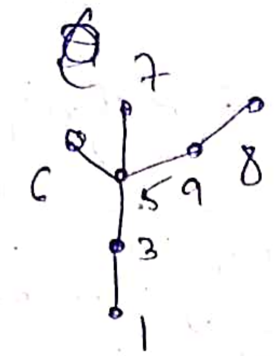
Use adjacency matrices $O(n^2)$ space.

1101001110100 1101000



1 → go down
0 → go up

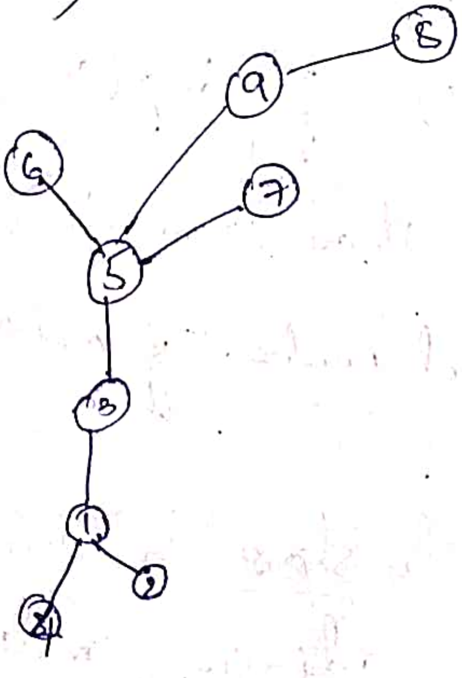




9 8

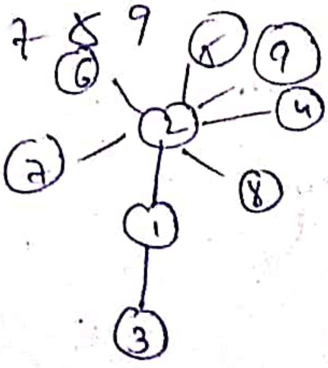
(1, 1, 3, 5, 5, 5, 9) Length is $n-2$. Then there are n nodes

1, 2, 3, 4, 5, 6, 7, 8, 9



(1 2 2 2 2 2 2)

1 2 3 4 6 6 7 8 9



No. of ~~level~~ label trees $= n^{n-2}$

Spanning tree :-

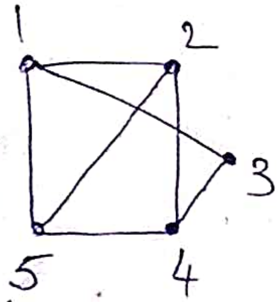
A subgraph of a graph is called a spanning tree if it is a tree and it covers all the vertices of the graph.

Weighted graph

$W : E \rightarrow \mathbb{R}.$

Cut of a Graph :-

A partition of a vertex

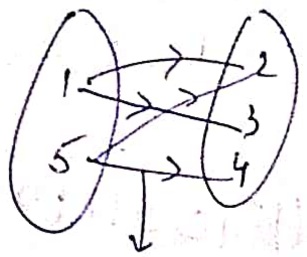


(1,5) (2,3,4)

Cut property :-

~~In ar weight~~

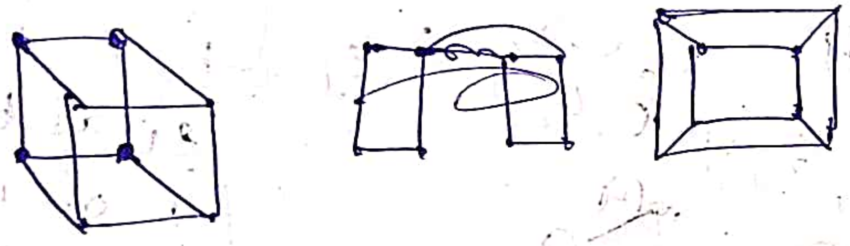
For any cut $(S, \underline{V-S})$ the minimum weighted edge that ~~Crv Crvg~~s ~~connects the~~ ~~Corresponding the cut~~ ~~Crosses Corresponds~~ the cut crosses must be in the MST.

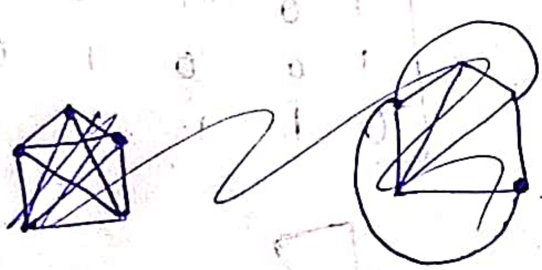


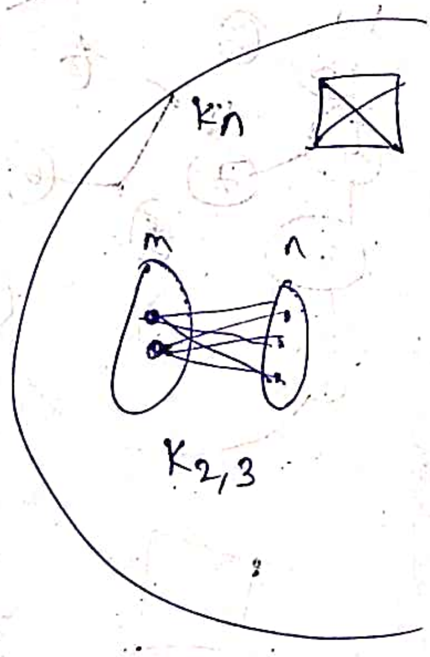
Cut edges

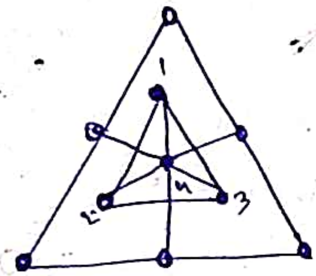
Plannar graphs :—

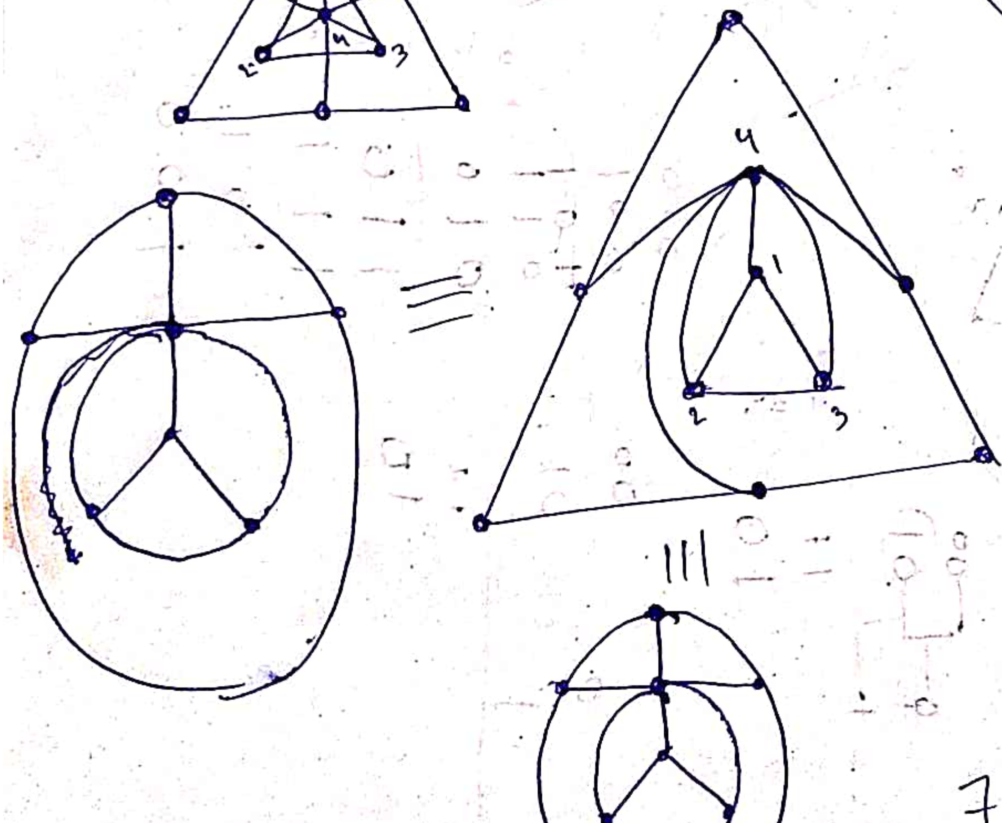
~~Connected~~ A graph is called plannar, if it can be drawn on a plane such a way that no edges are crossed.







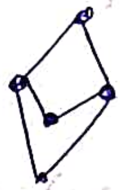




~~$7+8 = 10+2$~~

$7+7 = 12+2$

Def :- A connected planar graph partitions the plane into several regions one of which is infinite, These regions are called faces.



Euler's formula :-

No. of vertices + No. of Faces = No. of edges +2

$$|V| + |F| = |E| + 2$$

$5+3 \neq 6+2$

Proof by induction :-

Base Case :- base case 1 vertex

Induction hypothesis :-

Suppose for any n vertices m edges the equation holds.

* If $|V| \geq 3$, then $|E| \leq 3|V| - 6$.

If $V \geq 3$

$2E \geq 3|F|$

* $\sum l(F_i) = 2E \geq 3|F|$

$2|E| \geq 3|F|$

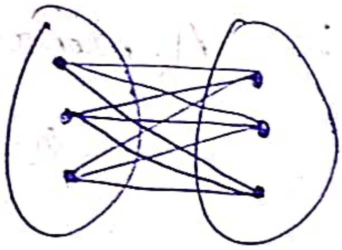
$2|E| \geq 3(|E| - |V| + 2)$

$|E| \leq 3|V| - 6$.

$2(V+F-2) \geq 3F$

$2V-4 \geq F$

$\boxed{2|V| - 4 \geq |F|}$



Problem :—

G is planar
→ atleast one vertex has degree 8. → (9 nodes)
→ Every vertex have degree atleast 5.
Show that G has atleast 15 vertices.

$\frac{45}{\cancel{2}} =$

sum of degrees ≥ 45

$E \geq \frac{45}{2} \geq 22$

$n =$ no. of nodes of graph.

$5(n-1) + 8 \leq \Sigma d_i \leq 8n$

$5n + 3 \leq 2|E| \cancel{\leq 8n}$

$\frac{5n+3}{2} \leq |E| \cancel{\leq 4n}$

$|E| \leq 3n - 6$

$\frac{5n+3}{2} \leq 3n - 6$

$5n + 3 \leq 6n - 12 \Rightarrow \boxed{n \geq 15}$

$\cancel{9 \times 5}$

$|V| + |F| = |$

$|E| \leq 3|V| -$

$22 + 6 \leq 3|V|$

$\cancel{\frac{28}{3} \leq |V|}$

$\frac{45}{2} \leq n$

$E = 22 +$

$E \geq 22$

$|E| \leq 3n - 6$

$3n \geq 22 +$

$3n \geq 28 + 9$

$n \geq \frac{28+9}{3}$

Problem :- If you have a planar graph with n vertices $n \geq 3$, atmost $2n-4$ faces.

$E \leq 3V-6$

Sol

$\cancel{|E| + |F| \geq |V| + 2}$

$V + F \geq E + 2$

$\cancel{|E| + |F| \geq n+2}$

$n + F \leq 3n - 6 + 2$

$\cancel{3n - 6 + F \geq n+2}$

$F \leq 2n - 4$

$\cancel{F \geq}$

$V + F - 2 \leq 3V - 6$

$2V \geq F + 4$

$2V - 4 \geq F$

Graph Colouring.

Minimum colouring problem :-

$f : V \rightarrow C$

st $(u, v) \in E, f(u) \neq f(v)$

Fact :- Any planar graph is 4-colourable

Thm :- Any plannar graph is 5 colorable.

Proof :- For graphs with $n \leq 5$ trivialy true.

Consider graphs with $n > 5$.

Claim :- One node of degree ≤ 5 exists.

$\cancel{2|E| \geq 6|V|}$

$\cancel{|E| \geq 3|V|}$

Problems on graph:-

Q1 What is the largest possible number of vertices in a graph with 19 edges and all vertices having degree atleast 3.

Sol

$$2 \times 19 = \cancel{3\times v}\ 3\times(V-2) + 8 .$$

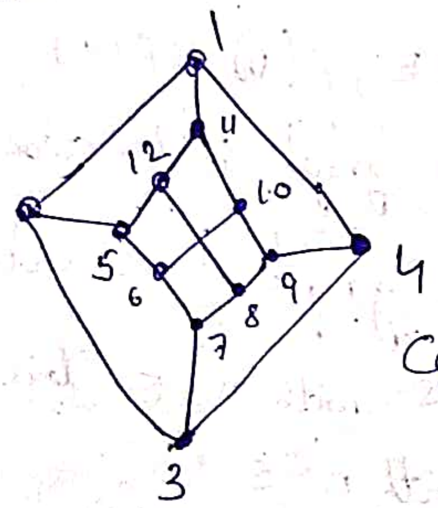
38

$$38 = 3v + 2$$

$$V = 12$$

Q2 Verify whether the following graph is bipartite

(a)



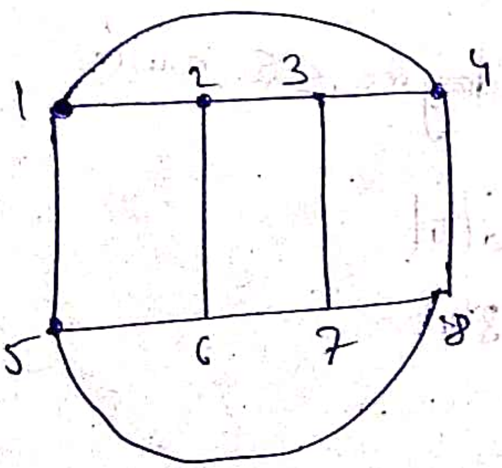
1 10 12

2 4 11

Cycle of odd length

Not bipartite

(b)

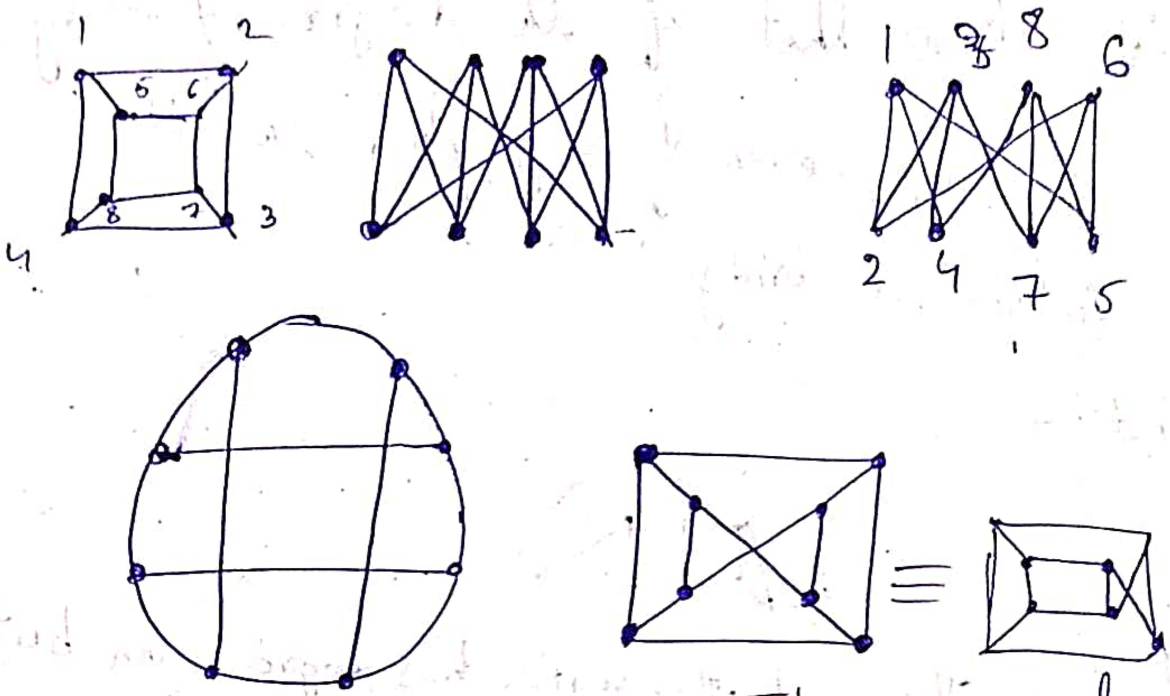


1 6 3 8

2 4 5 7 ~~9~~

bipartite .

Q1



There is a cycle of length 5 (Not Biparti-

Q2 How many simple directed graph exists with n vertices.

Sol

$4^{{}^nC_2}$ $\qquad n-1 \le V \le \frac{n(n-1)}{2}$

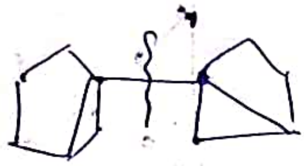
Q3 Let G be a graph where every vertex have degree atleast d. Prove that G contains a path of length atleast d.

Proof If there exists a longest path P.

There d-1 edges of last vertex of longest path. degree is atleast d.

As this P is longest, so these d-1 must be all nodes present in path, Q2: Prove that if the degree of every vertex is even of a graph, then it contains no bridge.

Proof: -



Then all other vertex has degree even but the disconnected vertex becoms odd.
As there is no graph with with odd number of odd vertices.

$$ecentricity(v) = \max_{u \in V} dist(u,v)$$

Radius (G) = minimum ecentricity of a vertex in G.

$$radius(G) = \min_{v \in V} ecentricity(v)$$

Proove that for every graph

$$radius(G) \le diam(G) \le 2 \cdot radius(G)$$

Proof: - $\min_{v \in V} \left(ecentricity(v)\right)$

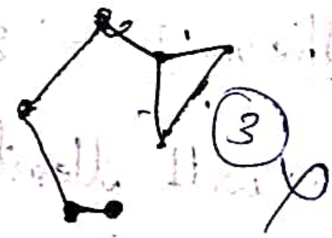
Q Let G be a graph with no induced subgraph P_4 or C_3. Prove that G is bipartite.

Q A tree one vertex of degree

K. Then the tree must contain k nodes of degree 1.

$P_n \to$ Path of n vertices

$C_n \to$ Cycle of n vertices.



Even cycle

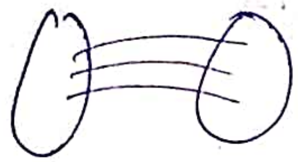
~~Prog~~ Probabilistic methods in Combinatrics

→ Use probability to certainity :

Probability Space :— discreate space

expectation.

Cut of a graph :—



Objective :— Find a cut with maximum number of cut edges.

Theorm1 :— There exists a cut with atleast $|E|/2$ edges

$\Updownarrow$

Theorm 2 :— Every graph has a bipartite subgraphs with atleast $|E|/2$ edges.

Proof :— Let G be a graph

For every vertex

→ Put it in S with probability $1/2$

Let $X_i = 1$ if the edge e_i is a cut edge.

$= 0$ otherwise

$$X = \sum_{e \in E} X_e$$

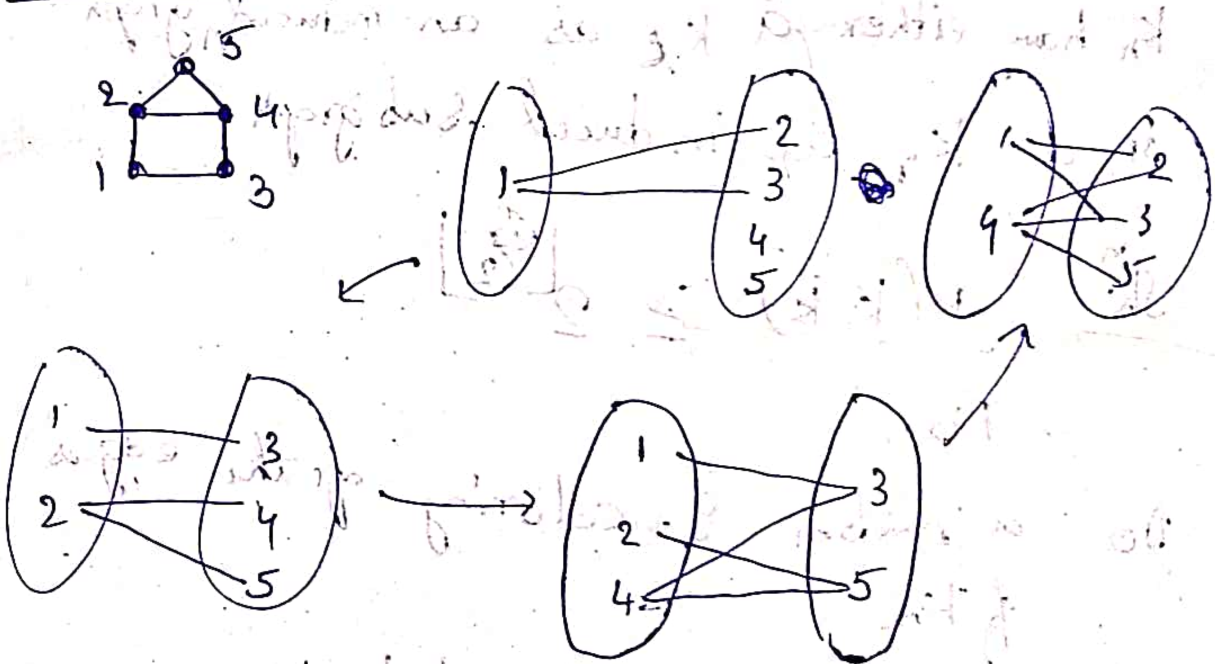
$$E(x) = \sum_{e \in E} E(x_e)$$

$$= \sum_{e \in E} 1 \cdot P(x_e = 1) + \sum_{e \in E} 0 \cdot P(x_e = 0)$$

$$= \frac{|E|}{2}.$$

$\Rightarrow$ There exist a cut with $\geq \frac{|E|}{2}$ edges.

Example :—



$$\deg(v) = d_{out}(v) + d_{in}(v)$$

$$\forall v \in S \quad d_{out}(v) \geq d_{in}(v).$$

$$\Rightarrow d_{out}(v) \geq \frac{\deg(v)}{2}.$$

$$\sum_{v \in V} d_{out}(v) = 2 \times (\text{cutedge}(G)).$$

$$\sum_{v \in V} d_{out}(v) \geq |E|$$

$$2 \times \text{Cutedges} \geq E$$

$$\text{Cutedges} \geq \frac{|E|}{2}$$

<u>ramsey</u> <u>number</u> :— $R(l, r)$

what is the minimum number n,

Such that ~~K_n~~ any ~~four~~ two colouring of the edges of K_n have either a K_ℓ as an induced subgraph or a K_r as induced sub graph.

<u>Th^m</u> $R(k, k) \geq 2^{\lfloor k/2 \rfloor}$.

Fix n

Do a random 2-coloring of the edges of K_n.

Let R be any subset of V, $|R| = k$.

A_R = Induced subgraph on A_R has all the edges of same color.

$$P(A_R) = \frac{2}{2^{\binom{k}{2}}}$$

The probability that atleast one such R exists is

$$P\left(\bigvee_{R \subset V} A_R\right) \leq \sum_{R \subset V} P(A_R)$$

$$\leq \binom{n}{k} \frac{2}{2^{\binom{k}{2}}}$$

~~Probability of no such subset exist > 0~~

$$\binom{2^{\binom{k}{2}}}{k} \frac{2}{2^{\binom{k}{2}}}$$

$$\frac{2^{\binom{k}{2}}!}{\left(2^{\binom{k}{2}}-k\right)!\,k!} \times \frac{2}{2^{\binom{k}{2}}} \qquad \frac{2^{k/2}!}{\left(2^{k/2}-k\right)!\,k!} \times \frac{2}{2^{\frac{k(k-1)}{2}}}$$

$$= \frac{2^{\frac{(k-2)}{2}}! \times 2}{\left(2^{k/2}-k\right)!\,k!}$$

$$=$$

Q 1600 Students

16000 Teams are formed

Each Team has 80 students. There exists ~~att~~ atleast two teams where 4 members are common.

Pick 2 Committes at random.

$X_i = 1$, if the ith student is in both the comities'

$= 0$ otherwise.

$X = X_1 + X_2 + \cdots X_{1600}$

Claim $E(X) \geq 4$.

4

$$\begin{array}{r} 16000 \\ \times 80 \\ \hline 1280000 \end{array}$$

$$E(X) = \sum_{i=1}^{1600} E(X_i)$$

Let n_i be the ~~number of comities~~ team where i-th student belong

$$E(X_i) = \frac{\binom{n_i}{2}}{\binom{16000}{2}}$$

$$\sum_{i=1}^{1600} n_i = 16000 \times 80$$

$$\bar{n} = \frac{16000 \times 80}{1600} = 800$$

$$E(x) \geq \frac{1600 \times \binom{\bar{n}}{2}}{\binom{16000}{2}}$$

$$\geq \frac{\cancel{1600} \times 800 \times 799}{\cancel{16000}_{10} \times 15999}$$

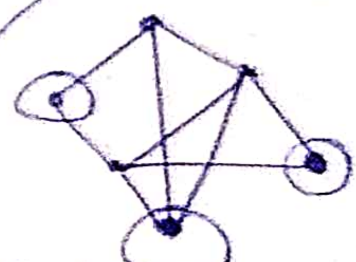
$$\geq \frac{80 \times 799}{15999}$$

$$E(x) \geq 3.995$$

$$E(x) \geq 4$$

Independent set :-

Simple graph G, A subset $V' \subset V$ is said to be independent set if $\forall u,v \in V'$ $(u,v) \notin E$



Maximum independent set problem is NP-hard :-

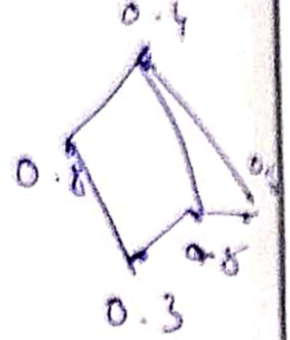
Theorm :- Every graph has an independent set of size atleast $k = \sum_{v \in V} \frac{1}{deg(v)+1}$

$$\geq \frac{1}{3} \times 4 = \frac{4}{3}$$

For every node $v \in V$.

assign some weight to v uniformly randomly in $[0, 1]$

Call vertex as a local minimum of



Claim :- $c(v) \leq c(u) \; \forall \; v, u \in E$

Collection of local minimum ~~from~~ form an independent set

Theorem :- Let $X_i = 1$ if v_i is a local minimum.

$= 0$ otherwise,

0.4 vertex is local minimum

$$X = \sum_{v \in V} X_i$$

$$E(X) = \sum_{v_i \in V} E(X_i)$$

$$= \sum_{v_i \in V} P(X_i = 1)$$

$$= \sum_{v_i \in V} \frac{1}{\deg(v_i) + 1}$$

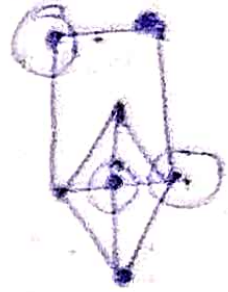
This implies that there exist a Independent set of size atleast $\sum_{v \in V} \frac{1}{\deg(v) + 1}$

Dominating Set :-

A ~~set~~ subset $V' \subseteq V$ is said to be a dominating set if for every vertex

$v \in V - V'$, $\exists$ a vertex $u \in V'$

s.t $(u, v) \in E$



Minimum dominating set problems

Every graph with minimum degree δ, has a dominating set of size $\leq \dfrac{n \cdot (1 + \log(1+\delta))}{1+\delta}$

Proof :- Let $D = \phi$

for every vertex v, put v in D with probability p.

Let x be the set of nodes who do not have neighbour in D.

include x in D.

$\boxed{\text{return } D \cup X}$.

$X_i = 1$ if v_i is in $D \cup X$

$= 0$ otherwise.

$$X = \sum_{i \in V} X_i$$

$$= \sum_{i \in D} X_i + \sum_{i \in X} X_i$$

$\forall i \in D \quad E(x_i) = p$

$$\sum E(x_i) = np$$

$E(Y_i)$ = The probability that the node V_i is not picked and non of its negiblours are also not picked.

$$= (1-p)^{\deg(v_i)+1}$$

$$E(X) \le np + n(1-p)^{\delta+1}$$

$$(1-p)^{\delta+1} \le e^{-p(\delta+1)},$$

$$1-p \le e^{-p}$$

$$\Rightarrow E(x) = np + ne^{-p(1+\delta)}$$

minimum at $p = \frac{\log(1+\delta)}{1+\delta}$

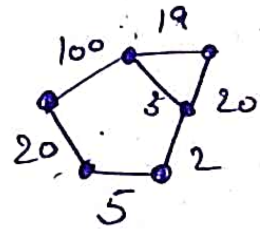
$$E(x) = \frac{n\log(1+\delta)}{1+\delta} + ne^{-\log(1+\delta)}$$

$$= n\left(\frac{\log(1+\delta)+1}{1+\delta}\right)$$

min Cut :-

Partition V into V_1, V_2

s.t $\sum_{e \in (E(V_1,V_2))} W(e)$ is minimum



st Cut Problem :-