IC 105

Tutorial #7 (Solutions)

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① $P(N_1=k_1, N_2=k_2, \ldots, N_n=k_n \mid N_I=k) = \dfrac{P(N_1=k_1, N_2=k_2, \ldots, N_n=k_n, N_I=k)}{P(N_I=k)}$

$$= \frac{P(N_1=k_1, N_2=k_2, \ldots, N_n=k_n)}{P(N_I=k)}$$

$$= \frac{1}{P(N_I=k)} \prod_{i=1}^{n} P(N_i=k_i) \quad \text{(independence)}$$

$$= \frac{\prod_{i=1}^{n} \frac{(c_i\lambda)^{k_i} e^{-c_i\lambda}}{k_i!}}{\frac{(c\lambda)^{k} e^{-c\lambda}}{k!}} \quad (\text{if } N_i \sim P(c_i\lambda))$$

$$= \frac{k!}{k_1!\,k_2!\cdots k_n!} \left(\frac{c_1}{c}\right)^{k_1}\left(\frac{c_2}{c}\right)^{k_2}\cdots\left(\frac{c_n}{c}\right)^{k_n}$$

$$= \binom{k}{k_1\ k_2\ \ldots k_n}\left(\frac{c_1}{c}\right)^{k_1}\left(\frac{c_2}{c}\right)^{k_2}\cdots\left(\frac{c_n}{c}\right)^{k_n}$$

(†) pmf of multinomial distn.

* Multinomial Distribution (Throw an n-sided die k times, whose side i comes up with probability $p_i = c_i/c$. (†) denotes the probability that side i comes up k_i times $i=1,2,\ldots,n$.

② Given $X \sim U(0,1)$ and $Y \sim U(0,1)$; $X \perp\!\!\!\perp Y$.

The joint density of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1, \\ 0, & \text{o/w}. \end{cases}$$

Take $U = XY$, $V = X$ $\qquad h_1(x,y) = xy$, $h_2(x,y) = x$.

$h_1^{-1}(u,v) = v$, $h_2^{-1}(u,v) = u/v$.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}.$$

As $x \in (0,1)$ and $y \in (0,1)$ $\Rightarrow$ $u \in (0,1)$, $v \in (0,1)$.

So, the joint of U,V is

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{v}, & 0<v<1,\ 0<u<v, \ (\underline{\text{Same as } 0<u<v<1}). \\ 0, & o/w. \end{cases}$$

Thus, the marginal of U is given by

$$f_U(u) = \int_u^1 \frac{1}{v}\, dv, \quad 0<u<1$$

$$= -\ln u, \quad 0<u<1.$$

So,

$$f_U(u) = \begin{cases} -\ln u, & 0<u<1, \\ 0, & o/w. \end{cases}$$

(b) Take $U = \frac{X}{Y}$, $V = X$. So, $h_1(x,y) = \frac{x}{y}$, $h_2(x,y) = x$.

$\Rightarrow$ $h_1^{-1}(u,v) = v$, $h_2^{-1}(u,v) = \frac{v}{u}$. $J = \begin{vmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u^2}$.

The joint of U,V is

$$f_{U,V}(u,v) = \begin{cases} \frac{v}{u^2}, & 0<v<u,\ 0<v<1, \\ 0, & o/w. \end{cases}$$

So, the marginal of U is

$$f_U(u) = \int_0^u \frac{v}{u^2}\, dv, \quad 0<u<1 \qquad \text{and} \qquad f_U(u) = \int_0^1 \frac{v}{u^2}\, dv, \quad u>1$$

$$= \frac{1}{2} \qquad\qquad\qquad\qquad = \frac{1}{2u^2}.$$

Hence,

$$f_U(u) = \begin{cases} \frac{1}{2}, & 0<u<1, \\ \frac{1}{2u^2}, & u>1, \\ 0, & o/w. \end{cases}$$

(3) The joint p.m.f. of (X,Y) is given as

$f_{X,Y}(x,y)$

$Y \backslash X$	-1	0	1	$f_Y(y)$
-2	$1/6$	$1/12$	$1/6$	$5/12$
1	$1/6$	$1/12$	$1/6$	$5/12$
2	$1/12$	0	$1/12$	$2/12$
$f_X(x)$	$5/12$	$2/12$	$5/12$	1

Are X,Y independent? (No, Why?)

let $U = |X|$, $V = Y^2$: $S_U = \{0,1\}$, $S_V = \{1,4\}$

The joint p.m.f. of (U,V) is

$f_{U,V}(u,v)$

$V \backslash U$	0	1	$f_V(v)$
1	$1/12$	$2/6$	$5/12$
4	$1/12$	$6/12$	$7/12$
$f_U(u)$	$2/12$	$10/12$	1

Are U,V independent? (No, Why?)

(4) Given X_1, X_2 iid $\text{Exp}(\lambda)$. Then the joint density of X_1 and X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda^2 e^{-\lambda(x_1+x_2)} & , x_1>0, x_2>0, \\ 0 & , \text{o/w} \end{cases}$$

let $Y_1 = X_1$, $Y_2 = X_1 + X_2$. So, $h_1(x_1,x_2) = x_1$, $h_2(x_1,x_2) = x_1+x_2$

So, $h_1^{-1}(y_1,y_2) = y_1$, $h_2^{-1}(y_1,y_2) = y_2 - y_1$

$y_1 > 0$ and $y_2 - y_1 > 0$. $J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

The joint p.d.f. of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \lambda^2 e^{-\lambda y_2} & , y_1>0, y_2>y_1 \quad (\text{or } y_2>y_1>0) \\ 0 & , \text{o/w}. \end{cases}$$

The marginal p.d.f of Y_2 is

$$f_{Y_2}(y_2) = \int_0^{y_2} f_{Y_1,Y_2}(y_1,y_2)\,dy_1 \quad , y_2>0.$$

$$= \int_0^{y_2} \lambda^2 e^{-\lambda y_2}\,dy_1 = \lambda^2 y_2 e^{-\lambda y_2} \quad , y_2>0.$$

So,

$$f_{Y_2}(y_2) = \begin{cases} \lambda^2 y_2 e^{-\lambda y_2} & , y_2>0, \\ 0 & , \text{o/w}. \end{cases}$$

$\Rightarrow Y_2 \sim GAM(2,\lambda)$ (a gamma rv).

The conditional pdf $f_{Y_1|Y_2}$ is

$$f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} & , 0<y_1<y_2, \\ 0 & , \text{o/w}. \end{cases}$$

$$= \begin{cases} \frac{1}{y_2} & , 0<y_1<y_2, \\ 0 & , \text{o/w}. \end{cases}$$

$\therefore f_{Y_1|Y_2=1}(y_1|y_2=1) = 1.$

(5) (a) Given Asset's Gain $R \sim N(7, 10^2)$

The bank becomes insolvent if $R \leq -5$.

$$P(R \leq -5) = P\left(\frac{R-7}{10} \leq \frac{-5-7}{10} \right) = \Phi(-1.2)$$
$$= 1 - \Phi(1.2) \simeq 0.115$$

Thus, by investing in just this one asset, the bank has a 11.5% chance of becoming insolvent.

(b) Given, ith Asset's gain $R_i \sim N(7, 10^2)$ and R_i's are independent $i = 1, 2, \ldots, 20$.

$$R = \frac{\sum_{i=1}^{20} R_i}{20} \quad \text{(Sample mean)}.$$

$$\Rightarrow R \sim N\left(7, \frac{100}{20}\right) = N(7, 5).$$

$$P(R \leq -5) = P\left(\frac{R-7}{\sqrt{5}} \leq \frac{-5-7}{\sqrt{5}} \right) = \Phi(-5.367)$$
$$= 1 - \Phi(5.367)$$
$$\simeq 0.0000000439$$
$$(= 4.39 \times 10^{-8})$$

Thus, by diversifying and assuming that the 20 assets have independent gains, the bank has seemingly decreased its probability of becoming insolvent to a palatable value.

(c) Given, $R_i \sim N(7, 10^2)$ and $\rho(R_i, R_j) = 1/2$ if $i \neq j$ $i, j \in \{1, 2, \ldots, 20\}$.

$i = 1, 2, \ldots, 20$,

for $i \neq j$,

$$Cov(R_i, R_j) = \rho(R_i, R_j) \sqrt{Var(R_i) Var(R_j)}$$
$$= \frac{1}{2} \times \sqrt{10^2 \times 10^2} = 50.$$

Given $R = \frac{\sum_{i=1}^{20} R_i}{20} \sim N(7, Var(R))$

Also,

$$Var(R) = Var\left(\frac{R_1 + R_2 + \cdots + R_{20}}{20}\right)$$

$$= \frac{1}{400} Var\left(\sum_{i=1}^{20} R_i\right)$$

$$= \frac{1}{400}\left(\sum_{i=1}^{20} Var(R_i) + 2\sum_{1 \le i < j \le 20} Cov(R_i, R_j)\right)$$

$$= \frac{1}{400}\left[20 \times 100 + \binom{20}{2} \cdot 50\right] = 52.5 .$$

$\therefore$ $R \sim N(7, 52.5)$.

$$P(R \le -5) = P\left(\frac{R-7}{\sqrt{52.5}} \le \frac{-5-7}{\sqrt{52.5}}\right) = \Phi(-1.656)$$

$$= 1 - \Phi(1.656)$$

$$\approx 0.0488 .$$

Thus, by taking into acount the positive correlation between the assets' gains, we are no longer as comfortable with the probability of insolvency as we thought we were in part (b).