Q.1. Find the matrix of the following inner products relative to given ordered basis $\mathcal{B}$ -

i) $V = P_2(\mathbb{R})$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$, $\mathcal{B} = \{1, x, x^2\}$.

Sol→ Let M be the matrix of inner product then we know that $M_{ij} = \langle \alpha_j, \alpha_i \rangle$ where $\alpha_1 = 1$, $\alpha_2 = x$, $\alpha_3 = x^2$. Thus,

$$M = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/5 \end{pmatrix}.$$

ii) $V = \mathbb{C}^3(\mathbb{C})$ with $\langle \alpha, \beta \rangle = \sum_{j=1}^{3} \alpha_j \overline{\beta_j}$ with any ordered basis $\mathcal{B}$.

Sol→ Suppose $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ then,

$\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$.

$$M = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_3, \alpha_1 \rangle \\ \langle \alpha_1, \alpha_2 \rangle & \langle \alpha_2, \alpha_2 \rangle & \langle \alpha_3, \alpha_2 \rangle \\ \langle \alpha_1, \alpha_3 \rangle & \langle \alpha_2, \alpha_3 \rangle & \langle \alpha_3, \alpha_3 \rangle \end{pmatrix}$$

where $\alpha_1 = 1$, $\alpha_2 = x$, $\alpha_3 = x^2$. Thus,

$$M = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/3 \\ 1/2 & 1/3 & 1/5 \end{pmatrix}.$$

ii) $V = \mathbb{C}^3(\mathbb{C})$ with $\langle \alpha, \beta \rangle = \sum_{j=1}^{3} \alpha_j \overline{\beta_j}$ with any ordered basis $\mathcal{B}$.

Sol→ Suppose $\mathcal{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ then,

$\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (0,0,1)$. ✓

$$M = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_2, \alpha_1 \rangle & \langle \alpha_3, \alpha_1 \rangle \\ \langle \alpha_1, \alpha_2 \rangle & \langle \alpha_2, \alpha_2 \rangle & \langle \alpha_3, \alpha_2 \rangle \\ \langle \alpha_1, \alpha_3 \rangle & \langle \alpha_2, \alpha_3 \rangle & \langle \alpha_3, \alpha_3 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q. Apply Gram Schmidt process to the following subset S of inner product space V to get an orthonormal basis for span of S.

i) $V = \mathbb{R}^3$ with standard inner product.

$S = \{(1,0,1), (0,1,1), (1,3,3)\}$.

Sol→ We know that the Gram-Schmidt formula is,

$$\beta_k' = \alpha_k - \sum_{j=1}^{k-1} \frac{\langle \alpha_k, \beta_j' \rangle}{\|\beta_j'\|^2} \beta_j', \quad k = 2, 3, \ldots n.$$

$$\therefore \beta_1' = \alpha_1 = (1,0,1), \quad \therefore \beta_1 = \frac{\beta_1'}{\|\beta_1'\|}$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1' \rangle}{\|\beta_1'\|^2} \beta_1', \quad \therefore \beta_2 = \frac{\beta_2'}{\|\beta_2'\|}$$

$$= (0,1,1) - \frac{\langle (0,1,1), (1,0,1) \rangle}{2} (1,0,1)$$

$$= \frac{1}{2}(-1, 2, 1)$$

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ii) $V = P_2(\mathbb{R})$ with inner product $\langle f, g \rangle$
$= \int_0^1 f(x)\, g(x)\, dx$, $S = \{1, x, x^2\}$.

Sol → Here, $\alpha_1 = 1$, $\alpha_2 = x$ & $\alpha_3 = x^2$.

Thus,

$$\beta_1' = \alpha_1 = 1.$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1.$$

$$= x - \frac{\langle x, 1 \rangle}{\|1\|^2} . 1.$$

$$= x - 1 \qquad \beta_2$$

$$\beta_3' = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2.$$

$$= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} - \frac{\langle x^2, x-1 \rangle}{\|x-1\|^2} (x-1)$$

$$= x^2 - \frac{1}{2} + \frac{1}{4}(x-1)$$

$$= x^2 + \frac{1}{4}x - \frac{3}{4}$$

Q.4. $S^\perp$ for $S = \{(1,0,i),(1,2,1)\}$

$A = (a,b,c) \in S^\perp$ & if A solves the following system of eq,

$$\langle A, (1,0,i) \rangle = 0$$

$$\Rightarrow a - ic = 0$$

$$\langle A, (1,2,1) \rangle = 0$$

$$a + 2b + c = 0$$

$$S^\perp = \left\{ \left(i, \frac{-i-1}{2}, 1 \right) \right\}$$

Since,

$$\dim V = \dim S + \dim S^\perp$$

$$\therefore \dim S^\perp = 1$$

& A will span $S^\perp$.

$\therefore \dim S^{\perp} = 1$

& A will span $S^{\perp}$.

Q.5. $\alpha = (2,1,3)$

$W = \{(x,y,z) : x+3y-2z=0\}$.

Projection of α over $W = ?$

Sol→ If $B = \{v_1, v_2 \ldots v_k\}$ is orthonormal basis for W then orthogonal projection of α

on W is,

$$\underline{u} = \underline{\sum_{i=1}^{k} \langle \alpha, v_i \rangle v_i}$$

So we have to find $B = ?$

$\dim W = 2$

Because,

$$\underline{x+3y = 2z}.$$

$B = \{(2,0,1), (-3,1,0)\}$

on W is,

$$\underline{u} = \underline{\sum_{i=1}^{k} \langle \underline{a}, \underline{v_i} \rangle \underline{v_i}}$$

So we have to find $k = ?$

$\dim W = 2$

Because,

$$\underline{x + 3y = 2z}.$$

$$\bar{B} = \underline{\{(2, 0, 1), (-3, 1, 0)\}}$$

$$B = \left\{ \frac{1}{\sqrt{5}}(2, 0, 1), \frac{1}{\sqrt{70}}(-3, 1, 0) \right\}$$

$$u = \frac{1}{\sqrt{5}} \left\langle (2, 1, 3), \frac{1}{\sqrt{5}}(2, 0, 1) \right\rangle (2, 0, 1)$$

$$\underline{\frac{1}{\sqrt{70}} \left\langle (2, 1, 3), \frac{1}{\sqrt{70}}(-3, 1, 0) \right\rangle} (-3, 1, 0)$$

$$= \frac{1}{14}(29, 17, 40).$$

Sol → Here, $\alpha_1 = 1, \alpha_2 = x$ & $\alpha_3 = x^2$.

Thus,

$$\beta_1' = \alpha_1 = 1.$$

$$\beta_2' = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1.$$

$$= x - \frac{\langle x, 1 \rangle}{\|1\|^2} . 1.$$

$$= x - 1$$

$$\beta_2 = \frac{\beta_2'}{\|\beta_2'\|}$$

$$\beta_3' = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2.$$

$$= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} - \frac{\langle x^2, x-1 \rangle}{\|x-1\|^2} (x-1)$$

$$= x^2 - \frac{1}{3} + \frac{1}{4}(x-1)$$

$$= x^2 + \frac{1}{4}x - \frac{3}{4}$$

$V = \mathbb{C}^4(\mathbb{C})$ with standard inner product,

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$$\frac{1}{\sqrt{70}}\left\langle (2,1,3), \frac{1}{\sqrt{70}}(-3,1,0)\right\rangle (-3,1,0)$$

$$= \frac{1}{14}(29, 17, 40).$$

ii) $V = P(\mathbb{R})$ $\langle f,g\rangle = \int_0^1 f(x)\,g(x)\,dx$

$h(x) = 4+3x-2x^2$, $W = P_1(\mathbb{R})$.

$\bar{B} = \{1, x\}$

$B = 1,\ 3\left(x-\frac{1}{2}\right)$

$\left\langle x-\frac{1}{2}, x-\frac{1}{2}\right\rangle$

$x^2 - x + \frac{1}{4}$

$\frac{1}{3} - \frac{1}{2} + \frac{1}{4}$

$\mathcal{B} = \left(1, 2\sqrt{3}\left(x-\frac{1}{2}\right)\right)$ ✓

$u = \langle 4+3x-2x^2, 1\rangle \cdot 1$

$+ \langle 4+3x-2x^2, 2\sqrt{3}(x-\frac{1}{2})\rangle \; 2\sqrt{3}\left(x-\frac{1}{2}\right)$

$u = ?$

<u>Q.6</u> $S_0 \subseteq S$ implies $S^\perp \subseteq S_0^\perp$.

For any $x \in S^\perp$, $\langle x, y\rangle = 0 \;\; \forall\, y \in S$.

Now $S_0 \subset S$, that implies $\langle x, y\rangle = 0 \;\forall$ $y \in S_0$.

$\Rightarrow x \in S_0^\perp$

ii) $S \subseteq (S^\perp)^\perp$ implies span $(S) \subset (S^\perp)^\perp$

Now $S_0 \subset S$, that implies $\langle x, y \rangle = 0$

$y \in S_0$.

$\Rightarrow x \in S_0^{\perp}$

ii) $S \subseteq (S^{\perp})^{\perp}$ implies $\operatorname{span}(S) \subset (S^{\perp})^{\perp}$

Take $x \in S$ then for any $y \in S^{\perp}$

$\langle x, y \rangle = 0 \quad \forall y \in S^{\perp}$.

$x \in (S^{\perp})^{\perp}$.

$S \subseteq (S^{\perp})^{\perp}$

Since $(S^{\perp})^{\perp}$ is a subspace &

* if $A \subset W$ where W is a subspace then $\operatorname{span} A \subset W$.

$\operatorname{span} S \subset (S^{\perp})^{\perp}$.

Since $(S^{\perp})^{\perp}$ is a subspace &

* if $A \subset W$ where W is a subspace

then span $A \subset W$.

$\Rightarrow$ $\boxed{\text{span } S \subset (S^{\perp})^{\perp}}$ — 0

If $x, y \in (S^{\perp})^{\perp}$

then $x+y \in (S^{\perp})^{\perp}$.

then $\langle x, z \rangle = 0$ for any $z \in S^{\perp}$

& $\langle y, z \rangle = 0$ for any $z \in S^{\perp}$

$\langle x+y, z \rangle = 0$ for any $z \in S^{\perp}$.

$x+y \in (S^{\perp})^{\perp}$.

If $A \subset W$

subspace

$\downarrow$

$x+y \in (S^{\perp})^{\perp}$.

If $A \subset W$
$\downarrow$
subspace

Claim — span $A \subset W$.

because let $a, b \in A \subset W \to$ subspace then

$\alpha a + \beta b \in$ span A for any $\alpha, \beta \in F$.

span $A \subset W$.

c) $W = (W^{\perp})^{\perp}$

From (ii), $W \subset (W^{\perp})^{\perp}$

For proving $(W^{\perp})^{\perp} \subset W$.

Let $x \in (W^{\perp})^{\perp}$ such that $x \in$

If $A \subset W$
$\downarrow$
subspace

Claim — span $A \subset W$.

because let $a, b \in A \subset W$ then $\rightarrow$ subspace

$\alpha a + \beta b \in$ span A for any $\alpha, \beta \in F$.

span $A \subset W$.

c) $W = (W^{\perp})^{\perp}$

from ii), $W \subset (W^{\perp})^{\perp}$

For proving $(W^{\perp})^{\perp} \subset W$.

Let $x \in (W^{\perp})^{\perp}$ such that $x \notin W$.

then there will be some $y \in W^{\perp}$ s.t.

$\langle x, y \rangle \neq 0$. but this is contradiction

as $x \in (W^{\perp})^{\perp}$ then $\langle x, y \rangle = 0 \ \forall y \in W^{\perp}$.

Let $x \in (W^\perp)^\perp$ such that $x \notin W$.

then there will be some $y \in W^\perp$ s.t.

$\langle x, y \rangle \neq 0$. but this is contradiction

as $x \in (W^\perp)^\perp$ then $\langle x, y \rangle = 0 \ \forall y \in W^\perp$.

d) $V = W \oplus W^\perp$.

For any $y \in V$ we have $y = u + v$

where $\underline{u} \in W$ & $v \in W^\perp$ & also

$\underline{W \cap W^\perp = \{0\}}$ because,

if $x \in W$ & $x \in W^\perp$ s.t. $x \neq 0$ then

$\underline{\langle x, x \rangle = 0}$

$\underline{\text{but } x \neq 0.}$

This ~~is~~ contradicts def^n of inner

product. Hence $\{0\} = W \cap W^\perp$.

where $u \in W$ & $v \in W^{\perp}$ & also

$W \cap W^{\perp} = \{0\}$ because,

if $x \in W$ & $x \in W^{\perp}$ s.t. $x \neq 0$ then

$\langle x, x \rangle = 0$

but $x \neq 0$.

This ~~is~~ contradicts def^n of inner product. Hence $\{0\} = W \cap W^{\perp}$.

$\therefore V = W \oplus W^{\perp}$.

Q.7. $V \to$ I.P.S.

W is subspace of V.

Iff $x \notin W$, P.T. $y \in V$ s.t. $y \in W^{\perp}$ but $\langle x, y \rangle \neq 0$.

y. T. V — I.P.S.

W is subspace of V.

If $x \notin W$, P.T. $y \in V$ s.t. $y \in W^{\perp}$ but

$$\langle x, y \rangle \neq 0.$$

Sol → As $V = W \oplus W^{\perp}$, there ~~is~~ are

unique vectors u & v s.t.;

$$x = u + v$$

Now we know that $x \notin W$

$\therefore$ $v \neq 0$.

Because if $v = 0$ then $x = u \in W$.

Now suppose $y = v$ then,

$$\langle x, y \rangle = \langle u+v, y \rangle = \langle u+v, v \rangle$$

$$= \langle u, v \rangle + \langle v, v \rangle$$

$$= 0 + \langle v, v \rangle \neq 0$$

Because $v \neq 0$.

$= \langle u, v \rangle + \langle v, v \rangle$

$= 0 + \underline{\langle v, v \rangle} \neq 0$

Because $v \neq 0$.

Q.8 $V = C([-1,1], \mathbb{R})$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Suppose W_e & W_o denotes the subspaces of V

$W_e \downarrow$ even funct

$W_o \downarrow$ odd funct.

P.T. $W_e^{\perp} = W_o$.

$$\underset{\text{even}}{\langle f}, \underset{\text{odd}}{g \rangle} = \int_{-1}^{1} \underbrace{f(x)}_{\text{even}}\, \underbrace{g(x)}_{\text{odd}}\, dx.$$

$$\int_{-a}^{a} f(x)\, dx = 0$$

if f is odd.

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$$\langle f, g\rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$$

Suppose $\underset{\substack{\downarrow \\ \text{even} \\ \text{funct}}}{W_e}$ & $\underset{\substack{\downarrow \\ \text{odd} \\ \text{funct.}}}{W_o}$ denotes the subspaces of V

P.T. $W_e^{\perp} = W_o$.

$$\underset{\substack{\downarrow \\ \text{even}}}{\langle f}, \underset{\substack{\downarrow \\ \text{odd}}}{g\rangle} = \int_{-1}^{1} \underbrace{f(x)}_{\text{even}}\,\underbrace{g(x)}_{\text{odd}}\,dx.$$

$$= \int_{-1}^{1} \underbrace{h(x)}_{\text{odd}}\,dx = 0.$$

* $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd.

$$\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x) + \int_{0}^{a} f(x)$$

$$= \int_{0}^{a} f(-x) + \int_{0}^{a} f(x)$$

$$= -\cancel{\int_{0}^{a} f(x)} + \cancel{\int_{0}^{a} f(x)}$$

$$= 0.$$

$= 0$

As we know that product of odd functn & even functn is odd therefore if $f \in W_e$, $g \in W_o$ then $\langle f, g \rangle = 0$. Thus $W_o \subset W_e^{\perp}$.

$W_e^{\perp} \subset W_o$

* Now if possible let us take some $h \in W_e^{\perp}$ but $h \notin W_o$. Then we know that $h = \psi + \phi$ where ϕ is

is odd & ψ is even.

Now,

$$\langle \underset{\in W_e^{\perp}}{h}, \underset{\text{even}}{\psi} \rangle = \langle \psi, \psi \rangle + \langle \phi, \psi \rangle$$

$$= \langle \psi, \psi \rangle + 0 \neq 0.$$

This is contradictn.

$\therefore h \notin W_o$