Lecture # 3 (IC 152)

Theorem :- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof :- Let α, β be two eigenvectors corresponding to distinct eigenvalues c & d. i.e. $\alpha \neq 0, \beta \neq 0$

& $T\alpha = c\alpha$, $T\beta = d\beta$. ✓

For,
$$c_1 \alpha + c_2 \beta = 0 \quad \text{——— (1)}$$
$$T(c_1\alpha + c_2\beta) = T(0) = 0$$
$$c_1 T\alpha + c_2 T\beta = 0$$
$$c_1 c\alpha + c_2 d\beta = 0 \quad \text{—— (2)}$$
$$-c\, c_2 \beta + c_2 d \beta = 0$$
$$c_2(-c+d)\beta = 0$$

As $\beta \neq 0$, & $c \neq d \Rightarrow c_2 = 0$

Recall

We need to find $0 \neq \alpha \in V$ s.t.
$T\alpha = c\alpha$ if
c is an eigenvalu

$(T - cI)\alpha = 0$

$\Rightarrow \alpha \in \text{Null}(T-cI$

$\Rightarrow \text{Null}(T-cI) \not\equiv \{0$

$\Rightarrow (T-cI)$ is not invertible

$\Rightarrow \det(T-cI) = 0$

$\det(cI - T) \rightarrow$ Characteristic polynomial

Similarly $c_1 = 0$

Thus α & β are linearly independent.

Mathematical Induction: Let $\{\alpha_1, \alpha_2, \dots \alpha_j\}$ are linearly independent eigenvectors corresponding to distinct eigenvalues $c_1, c_2, \dots c_j$

Let if possible

$$\sum_{k=1}^{j} d_k \alpha_k = \alpha_{j+1},$$ where α_{j+1} is an eigenvector corresponding to eigen value c_{j+1}.

$$\sum_{k=1}^{j} d_k T\alpha_k = T\alpha_{j+1}$$

$$\Rightarrow \sum_{k=1}^{j} d_k c_k \alpha_k = c_{j+1} \alpha_{j+1}$$

$$\sum_{k=1}^{j} d_k c_k \alpha_k = c_{j+1} \sum_{k=1}^{j} d_k \alpha_k$$

$$\sum_{k=1}^{j} d_k (c_k - c_{j+1}) \alpha_k = 0$$

As α_k's are linearly independent

$\Rightarrow d_k(c_k - c_{j+1}) = 0 \;\forall\; k = 1, 2, \dots j$

Observe that $d_{k_0} \neq 0$ for some $k_0 \in \{1, 2, \dots j\}$

a contradiction as

$\Rightarrow c_{k_0} = c_{j+1}$ ...

$c_1, c_2, \dots c_{j+1}$ are distinct.

- Let $T: V \to V$, $\dim V < \infty$. Let V has a basis (ordered) of eigenvectors of T. Then, $\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$

$$T\alpha_k = c_k \alpha_k = 0\alpha_1 + 0\alpha_2 + \dots c_k\alpha_k + 0\alpha_{k+1} \dots 0\alpha_n \quad \forall\, k = 1, 2, \dots n$$

$$\Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & c_n \end{bmatrix} \checkmark$$

Note that c_k's need not be distinct!!

- If $[T]_{\mathcal{B}} = \mathrm{diag}(c_1, c_2, \dots c_n)$ then it is necessary for $\mathcal{B}$ to be formed by eigenvectors of T.

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$

Then $T\alpha_k = c_k \alpha_k \Rightarrow \mathcal{B}$ consists of eigenvectors of T.

Definition :- Let $T: V \to V$, $(\dim V < \infty)$ be a linear

Definition :- ... operator. Then T is said to be diagonalizable if there exists a basis $\mathcal{B}$ of V consisting of eigenvectors of T.

Let us see matrix analog of diagonalization.

Definition :- A matrix A is said to be diagonalizable if corresponding linear operator is diagonalizable.

Lemma :- A matrix A is diagonalizable iff it is similar to a diagonal matrix i.e. $\exists$ a invertible matrix Q. such that

$$A = Q D Q^{-1}$$, where D is a diagonal matrix.

If $A \in M_{n\times n}(\mathbb{R})$

$\exists$ $T : \mathbb{R}^n \to \mathbb{R}^n$

$Te_k = Ae_k$

$[T]_{\mathcal{B}} = A$

$\mathcal{B} = \{e_1, e_2, \dots e_n\}$

Proof :- By definition

$A = [T_A]_{\mathcal{B}}$, $\mathcal{B}$ = standard ordered basis of $\mathbb{R}^n$ $= \{e_1, e_2, \dots e_n\}$ $A \in M_{n\times n}(\mathbb{R})$

As A is diagonalizable $\quad T_A e_k = Ae_k$

$\Rightarrow$ T_A is diagonalizable

$\Rightarrow \exists\ \mathcal{B}' \subset V$ s.t.

$$[T_A]_{\mathcal{B}'} = D = \text{diagonal matrix}$$

As $[T_A]_{\mathcal{B}}$ & $[T_A]_{\mathcal{B}'}$ are similar i.e. $\exists\ Q$ (invertible) in $M_{n\times n}(\mathbb{R})$ such that

$$[T_A]_{\mathcal{B}} = Q\,[T_A]_{\mathcal{B}'}\,Q^{-1}$$

$$A = Q D Q^{-1}$$

Conversely if $A = QDQ^{-1}$, then A is diagonalizable.

Let $D = \text{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$

$\Rightarrow De_j = \lambda_j e_j$ ✓

Claim if Qe_j is an eigenvector of A

$$A(Qe_j) = (QDQ^{-1})(Qe_j)$$
$$= QD(Q^{-1}Qe_j)$$
$$= QDe_j = Q\lambda_j e_j = \lambda_j (Qe_j)$$

$\Rightarrow \{Qe_1, Qe_2, \dots Qe_n\}$ is a basis of V

consisting of eigenvectors of A.

$\implies$ A diagonalizable.

* Q consists of columns of eigenvectors of T or A.

Theorem:- Let $T: V \to V$, ($\dim V < \infty$), $c_1, c_2, \ldots c_k$ are distinct eigenvalues of T. & E_i be the eigenspace corresponding to eigenvalue c_i. Then

T is diagonalizable

$$\Updownarrow$$

$$f = (x-c_1)^{d_1}(x-c_2)^{d_2}\cdots(x-c_k)^{d_k}$$

with $\dim E_i = d_i$ & $\boxed{d_1+d_2+\cdots d_k = n}$.

Once T is diagonalizable, i.e. $\exists\ \mathcal{B}$ consisting of eigen vectors of T s.t.

$$[T]_{\mathcal{B}} = \begin{bmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{bmatrix}$$

The characteristic polynomial for T

$$f(x) = (x-c_1)(x-c_2)\cdots(x-c_n)$$

... is repeated

but c_i's need not be distinct. Let c_i is ...

d_i times then

$$f(x) = \underline{(x-c_1)}^{d_1}(x-c_2)^{d_2}\cdots(x-c_k)^{d_k}$$

& $d_1+d_2+\cdots d_k = n$

Observe that $\dim E_i = \dim\left(\text{null}\left(\underline{c_i I - [T]_{\mathcal{B}}}\right)\right)$

$= d_i$

<u>Remark:</u>

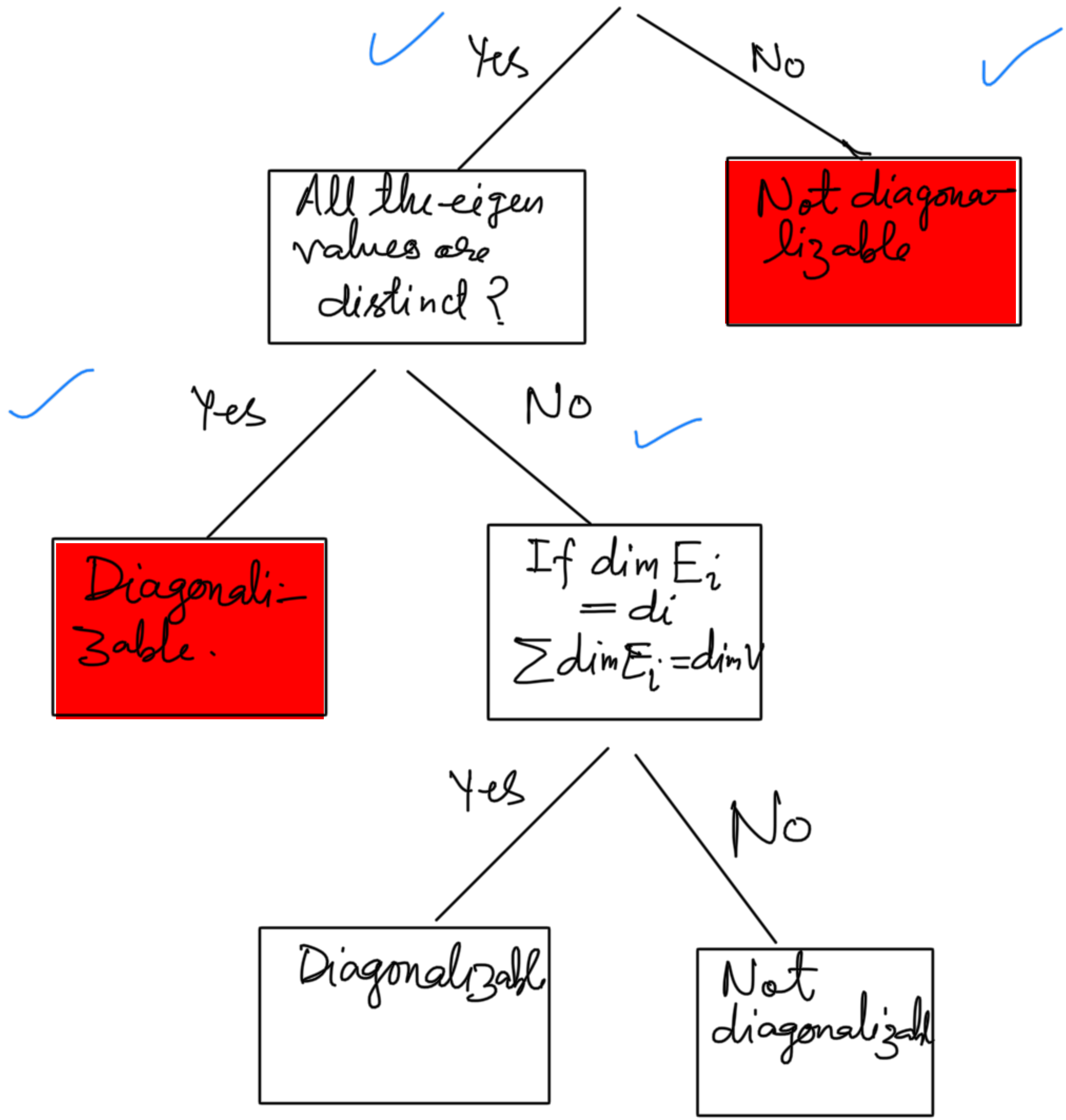
Algebraic multiplicity of an eigen value = no. of times an eigen value is repeated.

Geometric multiplicity of an eigen value = dim of corresponding eigenspace

Geometric multiplicity $\leq$ Algebraic multiplicity

<u>Diagonalizability Test</u>

$\boxed{\text{Is Char poly completely factorized in linear factors}}$


Yes
No
All the eigen values are distinct?
Not diagonalizable
Yes
No
Diagonalizable.
If $\dim E_i = d_i$
$\sum \dim E_i = \dim V$
Yes
No
Diagonalizable
Not diagonalizable

Example

Example :

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$Te_j = Ae_j$

Characteristic polynomial

$= (x+1)^2 (x-3)$

Eigenvalues $= -1, 3$

-1 is repeated twice.

$E_{-1} =$ null space of $(A+I)$

$$= \left\langle \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\rangle$$

$E_3 = \text{null}\ (A - 3I)$

$E_3 = \text{null}(A - 3I)$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$2x_2 = x_3$$

$$E_3 = \left\langle \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\rangle$$

Thus $\dim E_{-1} = 2$ $\quad \dim E_3 = 1$

$$\dim E_{-1} + \dim E_3 = 3 = \dim V.$$

Hence A is diagonalizable.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[T]_{\mathcal{B}'} = D, \quad \text{where } \mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$