Tutorial-4 : Solution

① a) The polar transformation is $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian is $J = r$. The region $D = \{(x,y) : x^2+y^2 \le 1\}$ is transformed to $E = \{(r,\theta) : 0 \le r \le 1 \;\&\; -\pi \le \theta \le \pi\}$.

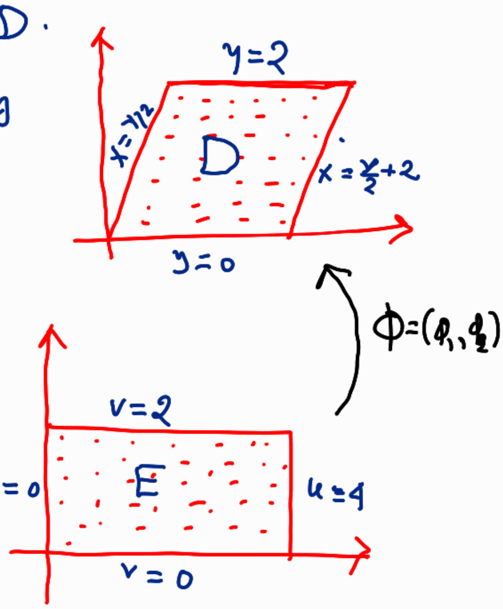
Then we have

$$\iint_D e^{x^2+y^2}\, d(x,y) = \iint_E e^{r^2}\, r\, d(r,\theta) = \int_{-\pi}^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta$$

$$= \pi(e-1).$$

b) Given $D = \left\{(x,y) : 0 \le y \le 2,\ \frac{y}{2} \le x \le \frac{y+4}{2}\right\}$ and

$f(x,y) = y^3(2x-y)\, e^{(2x-y)^2}$, $(x,y) \in D$.



$\boxed{\Phi(E)=D}$

Given transformations are $u = 2x-y$ & $v = y$

$\Rightarrow\ x = \phi_1(u,v) = \dfrac{u+v}{2}$, $y = \phi_2(u,v) = v$.

The new region E in uv plane is

$E = \{(u,v) : 0 \le u \le 4 \;\&\; 0 \le v \le 2\}$

The Jacobian is $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$

Then we have

$$\iint_D f(x,y)\, d(x,y) = \iint_E v^3 u\, e^{u^2}\, \frac{1}{2}\, d(u,v)$$

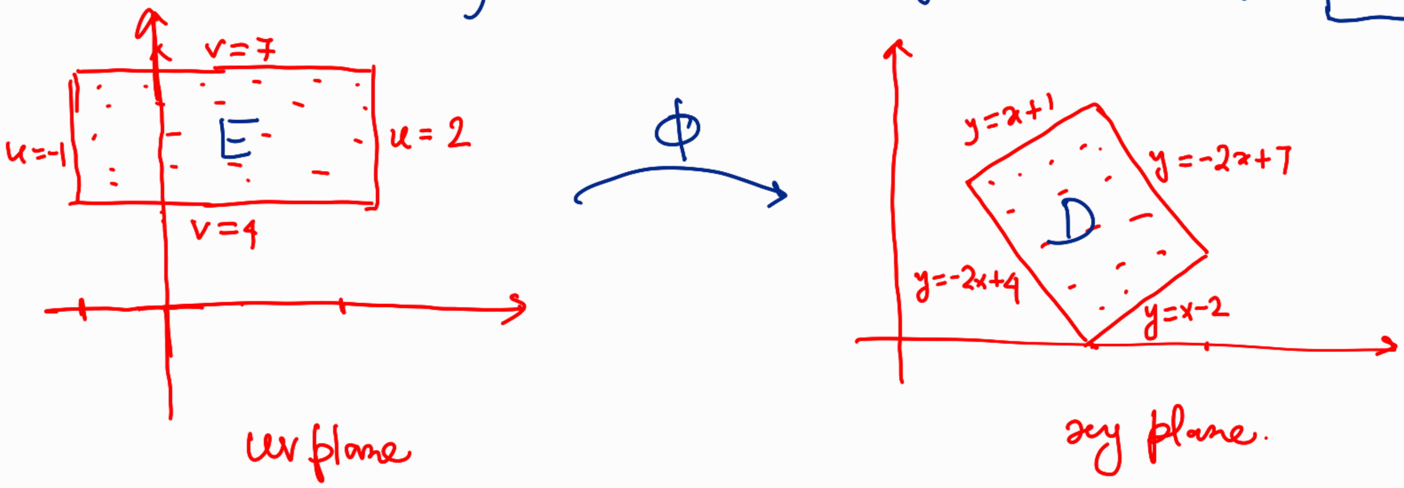
$$= e^{16} - 1$$

c) Given transformation $x = ar\cos\theta$, $y = br\sin\theta$.

The new region E in (r,θ) co-ordinate system is the closed unit disk. Also the Jacobian $J = abr$. Hence

$$\iint\limits_D f(x,y)\,d(x,y) = \iint\limits_E (br\sin\theta)^2 abr\,d(r,\theta) = ab^2\frac{\pi}{4}.$$

d) Given transformation is $u = x-y$ and $v = 2x+y$. $\boxed{J = \frac{1}{3}}$



$$E = \{(u,v) : -1 \le u \le 2 \ \& \ 4 \le u \le 7\}.$$

$$\therefore \iint\limits_D f(x,y)\,d(x,y) = \iint\limits_E uv\,|J|\,d(u,v) = \frac{1}{3}\iint\limits_E uv\,d(u,v) = \frac{33}{4}.$$

(2) Give $x = au$, $y = bv$. The Jacobian $J = ab$. The required region in xy plane is $D = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$. The new region in uv plane is $E = \left\{ (u,v) : u^2+v^2 \leq 1 \right\}$. Hence

$$\text{Area}(D) = \iint_D d(x,y) = \iint_E ab\, d(u,v) = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv\, du$$

$$= ab\pi.$$

(3) a) $u = x+y+z$, $\quad v = \dfrac{y+z}{x+y+z}$, $\quad w = \dfrac{z}{y+z}$

$\Rightarrow x = u(1-v)$, $\quad y = uv(1-w)$, $\quad z = uvw$

<u>Jacobian = $u^2 v$</u>

$D = \left\{ (x,y,z) : x,y,z \geq 0,\ x \leq y+z,\ 1 \leq 2(x+y+z) \leq 2 \right\}$

E = The cuboid $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times [0,1]$ (in uvw-plane).

$$\iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E f\big(u(1-v), uv(1-w), uvw\big)\, |u^2 v|\, d(u,v,w)$$

$$= \iiint_E u^2 v w\, d(u,v,w) = \frac{7}{128}.$$

b) Cylindrical co-ordinate transformation is
$x = r\cos\theta$, $\quad y = r\sin\theta$ and $z = z$. and $J = r$.

The new region E in uv plane is

$$E = \{(r, \theta, z) : 0 \le r \le 1,\ -\pi \le \theta \le \pi,\ 0 \le z \le 1\}$$

$$= [0,1] \times [-\pi, \pi] \times [0,1]$$

$$\therefore \iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E z\sqrt{1-r^2}\, .\, r\, d(r,\theta,z)$$

$$= \frac{\pi}{3}.$$

ⓒ $(x,y,z) \longrightarrow (r, \phi, \theta)$

$x = r\sin\phi\cos\theta,\quad y = r\sin\phi\sin\theta,\quad z = r\cos\phi;\quad J = r^2\sin\phi.$

$D = \{(x,y,z) : x^2+y^2+z^2 \le a^2\}$. The new region E is

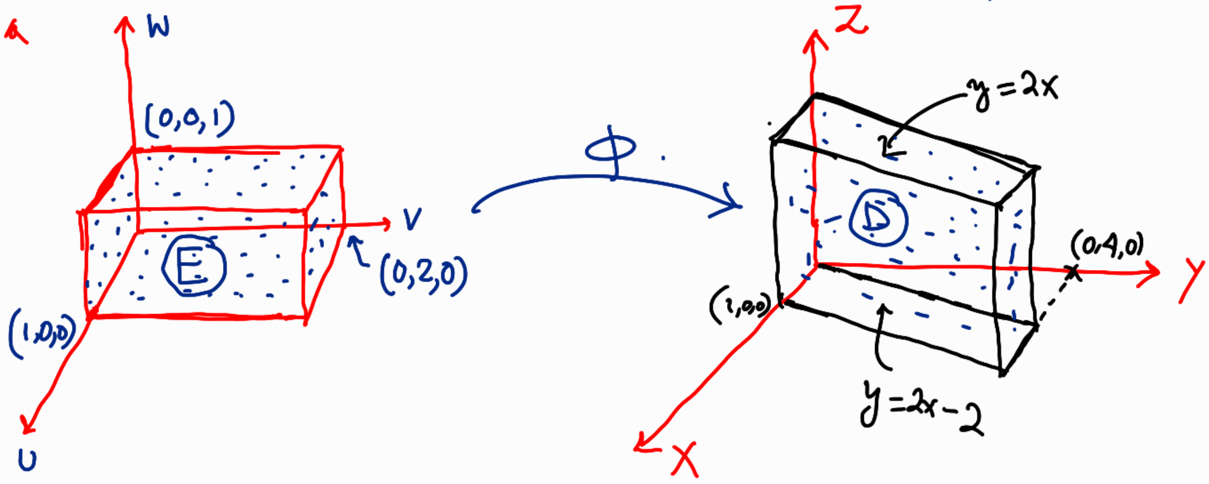
$$E = \{(r, \phi, \theta) : 0 \le r \le a,\ 0 \le \phi \le \pi,\ -\pi \le \theta \le \pi\}$$

$$= [0,a] \times [0,\pi] \times [-\pi,\pi]$$

$$\therefore \iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E \left(r^2\cos^2\phi \cdot r^2\sin\phi\right) d(r,\phi,\theta)$$

$$= \frac{4\pi a^5}{15}.$$

d) Given $u = 2x-y$, $v = y/2$, $w = z/3 \Rightarrow x = u+v$, $y = 2v$, $z = 3w$ & $J = 6$.



$D = \{(x,y,z) : 0 \le z \le 3,\ 0 \le y \le 4,\ 2x = y \text{ and } y + 2 = 2x\}$

The new region in uvw plane is the cuboid

$$E = [0,1] \times [0,2] \times [0,1].$$

$$\iiint_D f = \iiint_E 6(u+w)\, d(u,v,w) = 12.$$

(4) Simple calculations. Use the fact that $f_{xy} = f_{yx}$ for all the functions & all the variables.

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