IC153: Calculus 1
(Lecture 11)

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Series of real numbers

Definition

Let (a_n) be a sequence of real numbers. Then

- $\sum_{n=1}^{\infty} a_n := a_1 + a_2 + a_3 + \dots$ is called a series.
- $S_n = a_1 + a_2 + \cdots + a_n$ is called the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$.
- sequence (S_n) is called the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$.

Example

- $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots.$
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$

Convergence/divergence of a series

Definition

If $S_n \longrightarrow S$ for some $S \in \mathbb{R}$, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and denote is by $\sum_{n=1}^{\infty} a_n = S$.

If (S_n) does not converge, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

2. $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ diverges.

$$S_n = \log\frac{2}{1} + \log\frac{3}{2} + \cdots + \log\frac{n}{n-1} + \log\frac{n+1}{n}$$

$$= \cancel{\log 2} - \log 1 + \cancel{\log 3} - \cancel{\log 2} + \cancel{\cdots} + \cancel{\log n} - \cancel{\log(n-1)} + \log(n+1) - \cancel{\log n}$$

$$= \log(n+1) - 0 = \log(n+1) \longrightarrow \infty$$

Necessary condition for convergence

Theorem

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \longrightarrow 0$.

Proof: $S_n \longrightarrow s$, $S_{n-1} \longrightarrow s$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0$$

Example

1. $\sum_{n=1}^{\infty} 2^n$ diverges. $2^n \longrightarrow \infty$ as $n \to \infty$
2. $\sum_{n=1}^{\infty} x^n$ diverges if $|x| \geq 1$.

Note: The converse of the above result is not true. Example: $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$. $\log \frac{n+1}{n} = \log\left(1+\frac{1}{n}\right) = \log 1 = 0$

Necessary and sufficient condition for convergence

Theorem

Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded above.

Proof: $(\Leftarrow)$ $S_n \leq S_{n+1}$ $\forall n \geq 1 \Rightarrow (S_n)$ is $\uparrow$

if (S_n) is bounded above $\Rightarrow (S_n)$ is convergent.

$(\Rightarrow)$ Let $\sum_{n=1}^{\infty} a_n = l \Leftrightarrow (S_n) \to l \Rightarrow (S_n)$ is bounded

$\Rightarrow (S_n)$ is bounded above.

Example

- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$S_{2^k} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^k} \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$

$\cdots + 2^{k-1} \cdot \frac{1}{2^k}$

$\geq 1 + \frac{k}{2} \Rightarrow (S_n)$ is not bounded above

$\Rightarrow \sum \frac{1}{n}$ is not convergent.

Theorem

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: $S_n = a_1 + a_2 + \cdots + a_n$

$T_n = |a_1| + |a_2| + \cdots + |a_n|$

Recall: (S_n) is Cauchy if $\forall\, \epsilon > 0\ \exists\, N = N_1$ s.t. $n > m$

$\underbrace{|S_n - S_m|}_{} < \epsilon \quad \forall\ n, m > N_1$

$$\left|\sum_{i=m+1}^{n} a_i\right| \leq \sum_{i=m+1}^{n} |a_i| = T_n - T_m < \epsilon$$

$\forall\ n, m \geq N_1$

Note: We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if series $\sum_{n=1}^{\infty} |a_n|$ converges. Therefore, the above result implies the following.

$$\text{Absolutely convergent} \implies \text{convergent}$$

$\not\Leftarrow$

Example

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely.

Proof: $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is not convergent

Claim: S_{2n} is a convergent sequence

Pf: $S_2 = 1 - \frac{1}{2} > 0$ $\quad S_4 = S_2 + \frac{1}{3} - \frac{1}{4} > S_2$

$S_6 = S_4 + \frac{1}{5} - \frac{1}{6} > S_4 \; \cdots \Rightarrow S_{2n}$ is increasing.

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots \leq 1$$

$\Rightarrow S_{2n} \longrightarrow s$ —— (1)

$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}\left(S_{2n} + \frac{1}{2n+1}\right) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} = s$ —— (2)

$\Rightarrow S_n \longrightarrow s$

Questions?

Alternating series test

Theorem

Let (b_n) be a decreasing sequence of non-negative real numbers such that $\lim_{n\to\infty} b_n = 0$. Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ is convergent.

Proof:

Claim: S_{2n} is a convergent sequence.

$$\left.\begin{aligned} S_2 &= b_1 - b_2 \geq 0 \\ S_4 &= S_2 + \underbrace{b_3 - b_4}_{\geq 0} \geq S_2 \\ &\vdots \\ S_{2k+2} &\geq S_{2k} \end{aligned}\right\} \Rightarrow \quad S_{2n} \text{ is } \uparrow.$$

$$\begin{aligned} S_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - \cdots \\ &= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots \leq b_1 \end{aligned}$$

Alternating series test

Proof (Cont.): $\underbrace{S_{2n} \text{ is } \uparrow + S_{2n} \text{ is bounded above}}$

$$\Downarrow$$

$$S_{2n} \longrightarrow l$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \underbrace{\lim_{n\to\infty} b_{2n+1}}_{=0}$$

$$= \lim_{n\to\infty} S_{2n} = l$$

$$S_{2n} \to l \;\; \& \;\; S_{2n+1} \longrightarrow l \implies S_n \longrightarrow l.$$

Problem: Show that the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ is convergent.

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$\frac{1}{\sqrt{n}}$ is $\downarrow$ &

$\frac{1}{\sqrt{n}} \longrightarrow 0$