IC105: Probability and Statistics **2021-22-M**

Lecture 4: Statistically Independent Events

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Example 4.1. *A bowl contains 3 red and 5 blue chips. All chips that are of the same colour are identical. Two chips are drawn successively at random and without replacement. Define events*

$$A : \textit{first draw resulted in a red chip},$$
$$B : \textit{second draw resulted in a blue chip}.$$

Find $P(A \cap B)$, $P(A)$ and $P(B)$.

Solution: $P(A) = \dfrac{3}{8},\ P(B|A) = \dfrac{5}{7}$ and

$$P(B) = P(A \cap B) + P(A^c \cap B) = P(B|A)P(A) + P(B|A^c)P(A^c) = \frac{5}{7} \times \frac{3}{8} + \frac{4}{7} \times \frac{5}{8} = \frac{35}{56}.$$

Note that here the outcomes of second draw is dependent on outcome of first draw ($P(B|A) \neq P(B)$). Also,

$$P(A \cap B) = P(A)P(B|A) = \frac{3}{8} \times \frac{5}{7} = 0.2679.$$

Theorem 4.2 (Theorem of Total Probability)**.** *For a countable set Δ (that is elements of Δ can either be put in 1-1 correspondence with $\mathbb{N} = \{1, 2, \dots\}$ or with $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$), let $\{E_\alpha : \alpha \in \Delta\}$ be a countable collection of mutually exclusive (i.e., $E_\alpha \cap E_\beta = \phi,\ \forall\, \alpha \neq \beta$) and exhaustive (i.e., $P\left(\bigcup_{\alpha \in \Delta} E_\alpha\right) = 1$) events. Then, for any $E \in \mathfrak{F}$,*

$$P(E) = \sum_{\alpha \in \Delta} P(E \cap E_\alpha) = \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E|E_\alpha)P(E_\alpha).$$

Proof. Since $P\left(\bigcup_{\alpha \in \Delta} E_\alpha\right) = 1$, we have

$$\begin{aligned}
P(E) &= P\left(E \bigcap \left(\bigcup_{\alpha \in \Delta} E_\alpha\right)\right) = P\left(\bigcup_{\alpha \in \Delta} (E \cap E_\alpha)\right) \\
&= \sum_{\alpha \in \Delta} P(E \cap E_\alpha), \ \ (E_\alpha\text{'s are disjoint} \implies \text{their subsets } (E \cap E_\alpha)\text{'s are disjoint}) \\
&= \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E \cap E_\alpha), \ \ (P(E_\alpha) = 0 \implies P(E \cap E_\alpha) = 0,\ \alpha \in \Delta) \\
&= \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E|E_\alpha)P(E_\alpha).
\end{aligned}$$

This completes the proof. □

Example 4.3. *A population comprises of 40% female and 60% male. Suppose that 15% of female and 30% of male in the population smoke. A person is selected at random from the population.*

(a) Find the probability that he/she is a smoker.

(b) Given that the selected person is smoker, find the probability that he is male.

Solution: Define the events

$$\begin{aligned} M &: \text{selected person is a male,} \\ F = M^c &: \text{selected person is a female,} \\ S &: \text{selected person is a smoker,} \\ T = S^c &: \text{selected person is a non-smoker.} \end{aligned}$$

We have $P(F) = 0.4,\ P(M) = 0.6,\ P(F \cup M) = P(F) + P(M) = 1,\ P(S|F) = 0.15,\ P(T|F) = 0.85,\ P(S|M) = 0.30,\ P(T|M) = 0.70$.

(a) By using Theorem of total probability, we get

$$P(S) = P(S \cap F) + P(S \cap M) = P(S|F)P(F) + P(S|M)P(M) = 0.15 \times 0.4 + 0.30 \times 0.6 = 0.24.$$

(b)

$$P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{P(S|M)P(M)}{P(S)} = \frac{0.30 \times 0.60}{0.24} = \frac{3}{4}.$$

Theorem 4.4 (Bayes' Theorem)**.** *Let $\{E_\alpha : \alpha \in \Delta\}$ be a countable collection of mutually exclusive and exhaustive events and let E be any event $P(E) > 0$. Then, for $j \in \Delta$ with $P(E_j) > 0$,*

$$P(E_j|E) = \frac{P(E|E_j)P(E_j)}{\sum\limits_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E|E_\alpha)P(E_\alpha)}.$$

Proof. For $j \in \Delta$,

$$P(E_j|E) = \frac{P(E_j \cap E)}{P(E)} = \frac{P(E|E_j)P(E_j)}{\sum\limits_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E|E_\alpha)P(E_\alpha)}, \quad \text{(using Theorem of total probability).}$$

This completes the proof. $\square$

Remark 4.5. *(a) Suppose that occurrence of any of the mutually exclusive and exhaustive events $\{E_\alpha : \alpha \in \Delta\}$ (where Δ is a countable set) may cause the occurrence of an event E. Given that the event E has occurred (i.e., given the effect), Bayes' Theorem provides the conditional probability that the event E (effect) is caused by occurrence of event $E_j,\ j \in \Delta$.*

(b) In Bayes' Theorem $\{P(E_j) : j \in \Delta\}$ are called prior probabilities and $\{P(E_j|E) : j \in \Delta\}$ are called posterior probabilities.

Example 4.6. *Bowl C_1 contains 3 red and 7 blue chips. Bowl C_2 contains 8 red and 2 blue chips. Bowl C_3 contains 5 red and 5 blue chips. All chips of the same colour are identical.*

A die is cast and a bowl is selected as per the following schemes:

Bowl C_1 is selected if 5 or 6 spots show on the upper side,
Bowl C_2 is selected if 2,3 or 4 spots show on the upper side,
Bowl C_3 is selected if 6 spots show on the upper side.

The selected bowl is handed over to another person who drawns two chips at random from this bowl. Find the probability that:

(a) Two red chips are drawn.

(b) Given that drawn chips are both red, find the probability that it came from bowl C_3.

Solution: Define the events

A_i: selected bowl is C_i, $i = 1, 2, 3$, and R: the chips drawn from the selected bowl are both red.

Then $P(A_1) = \frac{2}{6} = \frac{1}{3}$, $P(A_2) = \frac{3}{6} = \frac{1}{2}$, $P(A_3) = \frac{1}{6}$. Note that $\{A_1, A_2, A_3\}$ are mutually exclusive and exhaustive.

(a)

$$P(R) = P(R|A_1)P(A_1) + P(R|A_2)P(A_2) + P(R|A_3)P(A_3) = \frac{\binom{3}{2}}{\binom{10}{2}} \times \frac{1}{3} + \frac{\binom{8}{2}}{\binom{10}{2}} \times \frac{1}{2} + \frac{\binom{5}{2}}{\binom{10}{2}} \times \frac{1}{6} = \frac{10}{27}.$$

(b)

$$P(A_3|R) = \frac{P(R|A_3)P(A_3)}{P(R)} = \frac{\frac{\binom{5}{2}}{\binom{10}{2}} \times \frac{1}{6}}{\frac{10}{27}} = \frac{1}{10}.$$

Remark 4.7. *In the above example,*

$$P(A_1|R) = \frac{P(R|A_1)P(A_1)}{P(R)} = \frac{\frac{\binom{3}{2}}{\binom{10}{2}} \times \frac{1}{3}}{\frac{10}{27}} = \frac{3}{50},$$

$$P(A_2|R) = \frac{P(R|A_2)P(A_2)}{P(R)} = \frac{\frac{\binom{8}{2}}{\binom{10}{2}} \times \frac{1}{2}}{\frac{10}{27}} = \frac{21}{25},$$

$$P(A_1|R) = \frac{3}{50} < \frac{1}{3} = P(A_1) \iff P(A_1 \cap R) < P(A_1)P(R) \longleftrightarrow R \text{ has negative information about } A_1,$$

$$P(A_2|R) = \frac{21}{25} > \frac{1}{2} = P(A_2) \iff P(A_2 \cap R) > P(A_2)P(R) \longleftrightarrow R \text{ has positive information about } A_2,$$

$$P(A_3|R) = \frac{1}{10} < \frac{1}{6} = P(A_3) \iff P(A_3 \cap R) < P(A_3)P(R) \longleftrightarrow R \text{ has negative information about } A_3.$$

Note that proportion of red chips in C_2 > proportion of red chips in C_i, $i = 1, 3$.

Independent Events:

Definition 4.8. *Let $\{E_j : j \in \Delta\}$ be a collection of events.*

(i) Events $\{E_j : j \in \Delta\}$ are said to be pairwise independent if for any pair of events E_α and E_β ($\alpha, \beta \in \Delta, \alpha \neq \beta$) in the collection $\{E_j : j \in \Delta\}$, we have

$$P(E_\alpha \cap E_\beta) = P(E_\alpha)P(E_\beta).$$

(ii) Events $\{E_1, E_2, \ldots, E_n\}$ are said to be independent if for any subcollection $\{E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_k}\}$ of $\{E_1, E_2, \ldots, E_n\}$ ($k = 1, 2, \ldots, n$), we have

$$P\left(\bigcap_{j=1}^{k} E_{\alpha_j}\right) = \prod_{j=1}^{k} P(E_{\alpha_j}).$$

(iii) Let $\Delta \subseteq \mathbb{R}$ be an arbitrary index set so that $\{E_\alpha : \alpha \in \Delta\}$ is an arbitrary collection of events. Events $\{E_\alpha : \alpha \in \Delta\}$ are said to be independent if any finite subcollection of events in $\{E_\alpha : \alpha \in \Delta\}$ forms a collection of independent events.

Theorem 4.9. *Let $E_1, E_2, \ldots$ be collection of independent events. Then*

$$P\left(\bigcap_{k=1}^{\infty} E_k\right) = \prod_{k=1}^{\infty} P(E_k).$$

Proof. Let $B_n = \bigcap_{k=1}^{n} E_k$, $n = 1, 2, \ldots$. Then $B_n \downarrow$ and $P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$. But $\bigcap_{n=1}^{\infty} B_n = \bigcap_{k=1}^{\infty} E_k$ and $P(B_n) = P\left(\bigcap_{k=1}^{n} E_k\right) = \prod_{k=1}^{n} P(E_k)$. Then,

$$P\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} \prod_{k=1}^{n} P(E_k) = \prod_{k=1}^{\infty} P(E_k).$$

This completes the proof. $\square$

Remark 4.10. *(i) To verify that n events $E_1, E_2, \ldots, E_n$ are independent one must verify*

$$\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1 \quad \textit{conditions}.$$

For an example to conclude that three events E_1, E_2, E_3 are independent, the following four (as $2^3 - 3 - 1 = 4$) conditions must be verified:

$$P(E_1 \cap E_2) = P(E_1)P(E_2), \;\; P(E_1 \cap E_3) = P(E_1)P(E_3), \;\; P(E_2 \cap E_3) = P(E_2)P(E_3),$$

and

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3).$$

(ii) Any subcollection of independent events is independent. In particular, the independence of a collection of events implies their pairwise independence.

(iii) If E_1 and E_2 are independent events $(P(E_1) > 0,\; P(E_2) > 0)$, then

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1)P(E_2)}{P(E_2)} = P(E_1),$$

that is, conditional probability of E_1 given E_2 is the same as unconditional probability of E_1.

Similarly, if E_1, E_2 and E_3 are independent events then $P(E_1|E_2 \cap E_3) = P(E_1)$.

Example 4.11. *Consider the probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{1, 2, 3, 4\}$ and $P(\{i\}) = 1/4,\;\; i = 1, 2, 3, 4$. Let $A = \{1, 4\},\; B = \{2, 4\},\; C = \{3, 4\}$. Then, show that A, B and C are pairwise independent but not independent.*

Solution: We have $P(A) = P(B) = P(C) = 1/2$. Also, $P(A \cap B) = P(A \cap C) = P(B \cap C) = P(\{4\}) = 1/4$. Thus,

$$P(A \cap B) = P(A)P(B), \;\; P(A \cap C) = P(A)P(C), \;\; P(B \cap C) = P(B)P(C),$$

which implies that A, B and C are pairwise independent. However,

$$P(A \cap B \cap C) = P(\{4\}) = 1/4 \neq 1/8 = P(A)P(B)P(C),$$

which implies that A, B and C are not independent although they are pairwise independent.

Example 4.12. *Let $E_1, E_2, \ldots, E_n$ be a collection of independent events. Show that,*

(a) for any permutation $(\alpha_1, \ldots, \alpha_n)$ of $(1, \ldots, n)$, $E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_n}$ are independent;

(b) $E_1, E_2, \ldots, E_k, E_{k+1}^c, \ldots, E_n^c$ are independent for any $k \in \{0, 1, \ldots, n-1\}$;

(c) E_1^c and $E_2 \cup E_3^c \cup E_5$ are independent.

(d) $E_1 \cup E_2^c$, E_3^c and $E_4 \cap E_5^c$ are independent.

Remark 4.13. *When we say that the two random experiments are performed independently, it means that the events associated with two random experiments are independent.*