② $$\alpha_k = -\frac{g^k d^k}{(d^k)^T Q d^k}$$

③ $$x^{k+1} = x^k + \alpha_k d^k$$

<u>Theorem</u> For any starting point $x^{(0)}$, the conjugate direction algorithm converges to unique x^* (that solves $Qx=b$) in n steps; that is $x(n) = x^*$.

<u>Proof</u> Since $\{d^{(0)}, \ldots d^{n-1}\}$ are l.i, they must span the whole space $\mathbb{R}^n$. So for the vector $x^* - x^0 \in \mathbb{R}^n$, $\exists$ $\beta_0, \ldots \beta_{n-1}$ s.t

$$x^* - x^0 = \beta_0 d^0 + \beta_1 d^1 + \cdots + \beta_{n-1} d^{n-1}$$

So now premultiplying this equation by $(d^k)^T Q$ for $k = 0, \ldots n-1$, we get

$$(d^k)^T Q (x^* - x^0) = \beta_k (d^k)^T Q d^k$$

$$\Rightarrow \beta_k = \frac{(d^k)^T Q (x^* - x^0)}{(d^k)^T Q d^k} \quad \text{— (A)}$$

Again $x^{(k)} = x^0 + \alpha_0 d^0 + \cdots + \alpha_{k-1} d^{k-1}$

$0 \le k < n-1$

$\Rightarrow x^k - x^0 = \alpha_0 d^{(0)} + \cdots + \alpha_{k-1} d^{k-1}$

Now $x^* - x^{(0)} = (x^* - x^{(k)}) + (x^k - x^{(0)})$

premultiplying $(d^k)^T Q$ we get

$$(d^k)^T Q (x^* - x^0) = (d^k)^T Q (x^* - x^k) + (d^k)^T Q (x^k - x^0)$$

$ = 0$

$$= (d^k)^T Q (x^* - x^k)$$

$$= (d^k)^T [b - Q x^k] = -(d^k)^T g^k$$

$[\because Q(x^*) = b$ and $g(k) = \nabla f(x^k) = Q x^k - b]$

By (A) & (B),

$$\beta_k = \frac{-(d^k)^T g^k}{(d^k)^T Q d^k}$$

$\therefore \alpha_k = \beta_k \quad \forall k$

$$x^* = x_0 + \alpha_0 d^0 + \cdots + \alpha_{n-1} d^{n-1}$$

$$= x_n$$