Lecture #7 (IC152)

Recall $L(V,V) = \{T : V \to V,\ T \text{ is linear}\}$ vector space

$\dim L(V,V) = n^2$, if $\dim V = n$.

Take V, vector space of dimension 'n' over a field F

and $T : V \longrightarrow V$ be a linear operator.

Let $p \in P(F)$ (vector space of polynomials over a field F)

then $p(T) : V \longrightarrow V$ is also a linear operator?

What about $p(0)$, or $\underline{0(T)}$?

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$

$p(T) = \underline{a_n T^n + a_{n-1} T^{n-1} + \cdots a_1 T + a_0 I}$

$T^2 = T \circ T$

0 polynomial is an example (trivial) of a polynomial which sends every linear operator to zero linear operator.

What about existence of nontrivial polynomials having the same behaviour (sending $T \to 0$)

polynomials ...

Definition (Annihilating polynomial)
A polynomial $p \in P(\mathbb{F})$ is called an annihilating polynomial for $T: V(\mathbb{F}) \to V(\mathbb{F})$ (linear operator on a vector space V over a field $\mathbb{F}$) if

$$p(T) = 0 \text{ (operator)}$$

If p & q are annihilating polynomial for T,
then $(p+q)(T) = p(T) + q(T) = 0 + 0 = 0$

$\Rightarrow$ $p+q$ is also an annihilating polynomial for T.

Similarly $(pq)(T) =: p(T)\, q(T)$

Does every linear operator on a finite dimensional v. space has an annihilating polynomial?

Answer: Yes

$\dim(L(V,V)) = n^2, \quad n = \dim V)$

Then $\{I, T, T^2, \dots T^{n^2}\}$ must be linearly dependent

$$\Rightarrow \quad c_0 I + c_1 T + c_2 T^2 + \dots c_{n^2} T^{n^2} = 0$$ and not all of $c_i = 0$.

$\Rightarrow$ there exist a polynomial

$$p(x) = c_{n^2} x^{n^2} + c_{n^2-1} x^{n^2-1} + \dots c_1 x + c_0$$

(nontrivial)

which annihilates 'T' $(p(T) = 0)$.

$\Longrightarrow$ $\exists$ a nontrivial annihilating polynomial for a given linear operator on a finite dimensional vector space of degree not more that n^2, $(\dim V = n)$.

Objective: To look for a least degree annihilating polynomial for a given linear operator on a finite dimensional vector space.

Definition (Minimal polynomial)

A polynomial $p \in P(\mathbb{F})$ is called minimal polynomial for a linear operator $T: V(\mathbb{F}) \to V(\mathbb{F})$, $(\dim V < \infty)$ if

a) p is monic polynomial

b) $p(T) = 0$

c) If there is another polynomial $q \in P(\mathbb{F})$ such that $q(T) = 0$, then degree of $q \geq$ degree of p.

Remark:- A minimal polynomial is unique !!

If NOT, let p & q are two minimal polynomial for T. Then $(p-q)(T) = 0$ as $p(T) = 0$, $q(T) = 0$.

Moreover degree of $(p-q) <$ degree of p or q which contradicts p & q are minimal polynomial as $p-q$ is an annihilating polynomial with lesser degree.

Note that degree of minimal polynomial is $\leq n^2$

Note that

if $\dim V = n$.

We need to refine this estimate.