Lecture #15 (IC152)

Find out orthogonal basis from given set of linearly independent vectors in an ips.

$\alpha_1 = (1, 0, 1)$ $\alpha_2 = (1, 0, -1)$ $\alpha_3 = (0, 3, 4)$

Orthogonal vectors

$\beta_1 = (1, 0, 1)$ & $\beta_2 = (1, 0, -1)$ & $\beta_3 = (0, 3, 0)$

$\langle \alpha_1, \alpha_2 \rangle = 1 + 0 + (-1) = 0$

Lemma :- If $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ is an orthogonal set of non zero vectors, then the vectors $\{\beta_1, \beta_2, \dots \beta_n\}$ obtained from Gram-Schmidt process satisfy

$$\beta_i = \alpha_i \quad \forall\, i = 1, 2, \dots n.$$

Proof: Follows from induction

for $n = 1$ it is true, $\beta_1 = \alpha_1$.

Let it is true ~~for~~ $n = k$, then we have to

Show that it's true for ...

It means you have $\{\beta_1, \beta_2, \dots \beta_k\}$ orthogonal

& $\beta_i = \alpha_i \ \forall \ i = 1, 2, \dots k$

To show: $\{\beta_1, \beta_2, \dots \beta_k, \beta_{k+1}\}$ is orthogonal (holds if

$$\beta_{k+1} = \alpha_{k+1} - \sum_{j=1}^{k} \frac{\langle \alpha_{k+1}, \beta_j \rangle}{\|\beta_j\|^2} \beta_j \Big) \ \& \ \beta_{k+1} = \alpha_{k+1}!!$$

As $\beta_j = \alpha_j \ \forall \ j = 1, 2, \dots k$

$$\Rightarrow \beta_{k+1} = \alpha_{k+1} - \sum_{j=1}^{k} \frac{\overbrace{\langle \alpha_{k+1}, \alpha_j \rangle}^{=0}}{\|\alpha_j\|^2} \alpha_j$$

$$\Rightarrow \beta_{k+1} = \alpha_{k+1}.$$

Theorem :- Suppose $S = \{v_1, v_2, \dots v_k\}$ is an orthonormal set in an n-dimensional ips V.

Then

i) S can be extended to an orthonormal basis $S' = \{\underline{v_1, v_2, \dots v_k}, \underline{v_{k+1}, \dots v_n}\}$ for V.

ii) $W = \text{span}(S)$ then $S_1 = \{\underline{v_{k+1}, v_{k+2}, \dots v_n}\}$ is an orthonormal basis for $W^{\perp}$. $n-k$

iii) If W is any subspace of V then
$$\dim V = \dim W + \dim W^{\perp}$$

<u>Proof</u> :- By LA-I, S can be extended to a basis of V say $\{v_1, v_2, \dots v_k, \underline{\alpha_{k+1}, \dots \alpha_n}\}$

Now apply Gram-Schmidt process to get an orthogonal set

$$\longrightarrow \tilde{S} = \{\underline{v_1, v_2, \dots v_k}, \underline{\beta_{k+1}, \beta_{k+2}, \dots \beta_n}\}$$

(Use Lemma as above)

Now normalize the set $\tilde{S}$ to get

$$S' = \{\underline{\check{v}_1, \check{v}_2, \dots \check{v}_k}, v_{k+1}, v_{k+2}, \dots v_n\}$$

where $v_{k+1} =: \dfrac{\beta_{k+1}}{\|\beta_{k+1}\|}$

ii) We have to show two things

a) S_1 is linearly independent (which is true as S_1 is a set of nonzero orthogonal vectors)

b) S_1 spans $W^\perp$

$x \in V$

$x = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$

$\langle x, v_i \rangle = c_i \langle v_i, v_i \rangle$

$= c_i$

$\Rightarrow x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$

Let $x \in W^\perp \Rightarrow x \in V$

$\Rightarrow x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$ ✓

$= \underline{\sum_{i=k+1}^{n} \langle x, v_i \rangle v_i}$ As $v_1, v_2, \ldots v_k \in W$ & $\langle x, v_i \rangle = 0$ as $x \in W^\perp$

which is a linear combination of vectors in S_1

$\Rightarrow S_1$ spans $W^\perp$.

iii) $n = k + (n-k)$

$\dim V = \dim W + \dim W^\perp$

<u>Popular identities/inequalities</u>

Parseval's Identity: Let V be a finite dimensional inner product space over $\mathbb{F}$. Let $\{v_1, v_2, \ldots v_n\}$ be an orthonormal basis for V. Then for any $x, y \in V$, we have

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

Proof:-

$$x = \sum_{i=1}^{n} \underline{\underline{\langle x, v_i \rangle}} v_i$$

$$y = \sum_{j=1}^{n} \langle y, v_j \rangle v_j$$

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i , \sum_{j=1}^{n} \underline{\underline{\langle y, v_j \rangle}} v_j \right\rangle$$

$$= \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

Bessel's Inequality ($\dim V < \infty$)

Let V be an ips & $S = \{v_1, v_2, \ldots v_n\}$ be an orthonormal subset of V. Then for any $x \in V$

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$$

Note: If S is a basis of V (ie $\dim V = n$) then by Parseval's identity

$$\langle x, x \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_i \rangle}$$

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$$

If not, can we extend S upto a basis of V? Yes for V finite dimensional vector spaces (as per LA-I)

If V is finite dimensional, we can extend S upto a basis (orthonormal) of V, say $\{v_1, v_2, \ldots v_n, v_{n+1}, \ldots v_m\}$ ($\dim V = m$)

$S_1 = \{v_1, v_2, \ldots v_n,$

then $\langle x, x\rangle = \sum_{i=1}^{m} \langle x, v_i\rangle \overline{\langle x, v_i\rangle}$

$$= \sum_{i=1}^{m} |\langle x, v_i\rangle|^2$$

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$$

Recall on IPS

- Definition of Inner Product, $F = \mathbb{R}/\mathbb{C}$.
- Inner product spaces $(V, \langle \cdot, \cdot \rangle)$
- Matrix of inner product: It is Hermitian & positive definite
- If a matrix is Hermitian & positive definite then it give rise to an ip.
- length/norm of ips.

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha\rangle}.$$

$\langle \cdot, \cdot \rangle$

$\langle \alpha, \beta\rangle_T = \langle T\alpha, T\beta\rangle$

- Polarization identities.
- Orthogonality.
- Pythagoras Theorem. If $\langle \alpha, \beta \rangle = 0 \Rightarrow \|\alpha\|^2 + \|\beta\|^2 = \|\alpha+\beta\|^2$
- Gram-Schmidt Process
- Orthogonal compliment of a set : a vector subspace
- Any vector y in V can be written as (uniquely)

  $y = u + v$, where $u \in W$, $v \in W^\perp$

  for any subspace W of V ($\dim V < \infty$)

  This vector 'u' is called orthogonal projection of y in W.

  $\longrightarrow$ $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$, where

  $\{v_1, v_2, \ldots v_n\}$ is an orthonormal basis for W.

- Any orthonormal set of an ips can be extended to a basis of ips.

- $V = W \oplus W^{\perp}$ (if y in V can be written uniquely as $y = u + v$, $u \in W$, $v \in W^{\perp}$ & $W \cap W^{\perp} = \{0\}$.)

Make up quiz on thursday, 9:30 AM.
