Tutorial - 3 : Calculus II

1. Let $D = [0,1] \times [0,1] \subset \mathbb{R}^2$ and $P = \{(x_i, y_j) : i = 0, 1, \dots, n;\ j = 0, 1, \dots, k\}$ be a partition of D where $x_i = \frac{i}{n}$ and $y_j = \frac{j}{k}$. Also let $f(x,y) = xy$, $(x,y) \in D$.

   i) Calculate $L(P,f)$ & $U(P,f)$

   ii) Calculate $\iint_D f(x,y)\, d(x,y)$ using $U(P,f)$ and $L(P,f)$.

   iii) Calculate $\iint_D f(x,y)\, d(x,y)$ using Fubini's theorem and verify the value.

2. Let $\phi : [a,b] \to \mathbb{R}$ be a bounded function of one variable. Also let $f : [a,b] \times [c,d] \to \mathbb{R}$ be defined as $f(x,y) = \phi(x)$ $\forall (x,y) \in [a,b] \times [c,d]$. Prove that

   f is double integrable on $[a,b] \times [c,d] \iff \phi$ is Riemann integrable on $[a,b]$.

3. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2}, & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2}, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

   i) Show that f is not integrable

   ii) Calculate the iterated integrals.

④ Change the order of the following integrals & write down the iterated integral.

i) $\int_0^1 \left[\int_1^{e^x} dy \right] dx$ ii) $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx \right] dy$

⑤ Sketch the region and evaluate the integrals

i) $\int_0^{\pi} \left[\int_x^{\pi} \frac{\sin y}{y}\, dy \right] dx$

ii) $\int_0^1 \left[\int_y^1 x^2 e^{xy}\, dx \right] dy$

iii) $\int_1^4 \left[\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy \right] dx$

⑥ Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$ and $y = x$.

⑦ Find the area of the region bounded by the following curves in the first quadrant of xy plane.

i) $y = x^2$ and $y = x$.

ii) $x = e^y$ and $x = 2$.

iii) $y = \ln x$, $y = 2\ln x$, $x = e$.

(8) i) Sketch the domain $D=\{(x,y)\in\mathbb{R}^2 : y=x^2,\ y=1,\ x=2\}$ in the xy-plane.

ii) Express D in the form of elementary region of type-1 and type-2 both.

iii) Evalute the integral $\iint_D (x^2+y^2)\, d(x,y)$ using Fubini's Theorem for elementary region of type1 and type-2 both and deduce that the value is same in both the cases.

(9) Let D be the 3 dimensional region bounded by the plane $x+y+z=a$ $(a>0)$, $x=0$, $y=0$, $z=0$. Then evaluate

$$\iiint_D (x^2+y^2+z^2)\, d(x,y,z).$$

(10) Let S be the sphere of radius 5 centered at $(0,0,0)$ and D be the region under the sphere that lies above the plane $z=3$. Set the limit of integration for evaluating the triple integral $\iiint_D F(x,y,z)\, d(x,y,z)$ for any fun F.

(11) In each of the following region D evaluate $\iiint_D f(x,y,z)\, d(x,y,z)$ where $f(x,y,z)=1$ $\forall (x,y,z)\in D$.

i) D = The region between the cylinder $z=y^2$ and xy-Plane that is bounded by $x=0$, $x=1$, $y=-1$, $y=1$

ii) The wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$.