

1.3 Dominating Sets

Definition 7. Let $G = (V, E)$ be a graph. A set of vertices $D \subseteq V$ is called **dominating** with respect to G if every vertex in $V \setminus D$ is adjacent to a vertex in D .

Theorem 8. Suppose that $G = (V, E)$ is a graph with n vertices, and that $\delta(G) = \delta$, the minimum degree amongst G 's vertices, is strictly positive. Then G contains a dominating set of size less than or equal to

$$\frac{n \cdot (1 + \log(1 + \delta))}{1 + \delta}$$

Proof. Create a subset of G 's vertices by choosing each $v \in V$ independently with probability p ; call this subset X . Let Y be the collection of vertices in $V \setminus X$ without any neighbors in X ; then, by definition, $X \cup Y$ is a dominating set for G .

What is the expected size of $|X \cup Y|$? Well; because they are disjoint subsets, we can calculate $|X \cup Y|$ by simply adding $|X|$ to $|Y|$:

$$\begin{aligned} \mathbb{E}(|X|) &= \sum_{v \in V} \mathbb{E}(\mathbb{1}_{\{v \text{ is chosen}\}}) \\ &= p \cdot n, \text{ while} \\ \mathbb{E}(|Y|) &= \sum_{v \in V} \mathbb{E}(\mathbb{1}_{\{v \text{ is in } Y\}}) \\ &= \sum_{v \in V} \mathbb{E}(\mathbb{1}_{\{v \text{ isn't in } X, \text{ nor are any of its neighbors}\}}) \\ &= \sum_{v \in V} \mathbb{E}(1 - p)^{\deg(v)+1}, \text{ (b/c we've made } \deg(v) + 1 \text{ choices independently)} \\ &\leq \sum_{v \in V} (1 - p)^{\delta+1} \\ &= n(1 - p)^{\delta+1}. \end{aligned}$$

This tells us that

$$\begin{aligned} \mathbb{E}(|X \cup Y|) &\leq np + n(1 - p)^{\delta+1} \\ &\leq np + ne^{-p(\delta+1)}, \end{aligned}$$

which has a minimum at

$$p = \frac{\log(1 + \delta)}{1 + \delta}.$$

Thus, for such p , we can find a dominating set of size at most

$$\frac{n \cdot (1 + \log(1 + \delta))}{1 + \delta},$$

as claimed. □

Independent set

In a graph G , a subset of vertices $S \subset V$ is said to be an *independent set* if no two vertices $u, v \in S$ are adjacent in G . The problem of determining the size of the largest independent set in G is NP-hard. However, we can once again apply the probabilistic method to establish a good lower bound on the size of the maximum independent set for any graph.

Theorem 2.6. *Any graph $G = (V, E)$ contains an independent set $S \subset V$ such that $|S| \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}$ where $\deg(v)$ is the number of vertices adjacent to v in G .*

Proof. Assign a random weight w_v to each vertex $v \in V$, choosing the weights independently and uniform from the interval $[0, 1]$. Call $v \in V$ a local minimum if $w_v < w_u$ for each vertex u adjacent to v . Since no two adjacent vertices can be local minima, the set of local minima form an independent set. Additionally, any vertex among v and its neighbors are equally likely to have minimum weight. Therefore, for each v , the probability that v is a local minimum is $\frac{1}{\deg(v) + 1}$.

Now, we again use indicator variables for each vertex. Let X be the number of local minima, and X_v be an indicator random variable for the event that v is a local minimum. By linearity of expectation,

$$\mathbb{E}[X] = \sum_{v \in V} \mathbb{E}[X_v] = \sum_{v \in V} \frac{1}{\deg(v) + 1}.$$

Therefore, there must be at least one independent set of that size. □

