# Lecture #5 (IC152)

Recall,

Real eigenvalues — Hermitian matrix : $A = A^* =$ transpose of complex conjugate of A ✓

Purely imaginary or zero ← Skew-Hermitian : $A^* = -A$

$|x| = 1$ ← Unitary : $A^*A = AA^* = I$ ✓

<u>Normal matrix</u> : A matrix (square) A satisfying

$$AA^* = A^*A.$$

<u>Theorem</u> (Spectral Theorem) ✓

Given any normal matrix A ∃ a unitary matrix C such that $C^*AC$ is a diagonal matrix. i.e.

$$C^*AC = D \quad (\Rightarrow A = CDC^*)$$

For Hermitian, $AA^* = AA = A^2$ ⇒ Hermitian matrix is a normal matrix.

$$A^*A = A \cdot A = A^2$$

... normal matrices

Skew-Hermitian are also normal matrices.

**Example 1:** $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = A$

$\Rightarrow$ A is Hermitian.

Char polynomial for A is

$$f(x) = \begin{vmatrix} x & -i \\ i & x \end{vmatrix} = x^2 - 1$$

Eigenvalues are $\pm 1$

Eigenspace, $E_1$ = null space of $(I - A)$

or solution space of

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{aligned} z - iw &= 0 \\ iz + w &= 0 \end{aligned} \right\} \Rightarrow z = iw$$

$$E_1 = \text{span of } \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle$$

Similarly $E_{-1}$ = null space of $(-I - A)$

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$$= \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle \quad \text{check it !!}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \text{ of } \mathbb{C}^2(\mathbb{C}). \qquad \mathbb{C}^2(\mathbb{R})$$

Example 2.

$$A = \begin{bmatrix} -i & 2+i \\ -2+i & 0 \end{bmatrix}$$

$$A^* = \begin{bmatrix} i & -2-i \\ 2-i & 0 \end{bmatrix} = - \begin{bmatrix} -i & 2+i \\ -2+i & 0 \end{bmatrix} = -A$$

$\Rightarrow$ $A$ is Skew-Hermitian.

Characteristic polynomial.

$$f(x) = \begin{vmatrix} x+i & -2-i \\ 2-i & x \end{vmatrix} = x^2 + ix + 5$$

Eigenvalues will be : $\dfrac{-i \pm \sqrt{-1-20}}{2}$

$$= \frac{-i \pm \sqrt{21}\, i}{2} \Rightarrow \frac{-(1+\sqrt{21})\, i}{2},$$

$$\text{and } \frac{(-1+\sqrt{21})\, i}{2}$$

To find out basis for $\mathbb{C}^2(\mathbb{C})$ consisting of eigenvectors of A. !!

Example

$$A = \frac{1}{2} \begin{bmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{bmatrix}$$

Check!! $AA^* = A^*A = I$, A is unitary.

$$A^* = \frac{1}{2} \begin{bmatrix} 1 & -i & 1-i \\ i & 1 & -1-i \\ -1-i & 1-i & 0 \end{bmatrix}$$

$(-1+i)(1-i)$

$-\cancel{1}+i+i+\cancel{1}$

Characteristic polynomial of $2A$

$$f(x) = \begin{vmatrix} x-1 & i & 1-i \\ -i & x-1 & -1-i \\ -1-i & 1-i & x \end{vmatrix} = \underline{x^3 - 2x^2 + 4x - 8}$$

Char$_A$: $x^3 - x^2 + x - 1$

$(x-1)(x^2+1)$

$x = 1, i, -i$ ✓

$\lambda \to A$

$c\lambda \to cA$

$\frac{1}{2}A \to \lambda/2$

$$= (x-2)(x^2+4)$$

Eigen values of $2A$: $x = 2$, $x = \pm 2i$

Eigenvalues of $A \Rightarrow 1, i, -i$ ✓

Observe that $|\lambda| = 1 \;\forall\; \lambda = 1, i, -i$

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 1+i & 1 \\ -1+i & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & -1-i & -1 \\ 1-i & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Check it!! $AA^* = A^*A \Rightarrow A$ is normal matrix

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**Diagonalizability Theorem**:- Let $T: V \to V$ linear operator on finite dimensional vector space. Let $c_1, c_2, \dots c_k$ be the distinct eigenvalues of $T$ and $E_1, E_2, \dots E_k$ are eigen space associated with eigenvalues $c_1, c_2, c_3, \dots c_k$. Then $T$ is diagonalizable iff

$$\underline{V} = E_1 + E_2 + \dots E_k$$

Recall

$E_1 \& E_2 \subset V$

$E_1 + E_2 = \{\alpha + \beta : \alpha \in E_1, \beta \in E_2\}$ ✓

$\mathcal{B}_1$ basis of $E_1$, $\mathcal{B}_2$ basis of $E_2$

then $\mathcal{B}_1 \cup \mathcal{B}_2$ spans $E_1 + E_2$

$\dim E_1 + E_2 \leq \dim E_1 + \dim E_2$

Proof: Aim

$\beta_1 + \beta_2 + \dots \beta_k = 0$ ✓

$\mathcal{B}_1 = \{\alpha_1^1, \alpha_2^1, \dots \alpha_{d_1}^1\}$

$\mathcal{B}_2 = \{\alpha_1^2, \alpha_2^2, \dots \alpha_{d_2}^2\}$

$\underbrace{\boxed{c_1^1 \alpha_1^1 + c_2^1 \alpha_2^1 + \dots c_{d_1}^1 \alpha_{d_1}^1}}_{\beta_1} + (c_1^2 \alpha_1^2 + c_2^2 \alpha_2^2 + \dots c_{d_2}^2 \alpha_{d_2}^2) + \dots \dots c_1^k \alpha_1^k + c_2^k \alpha_2^k + \dots c_{d_k}^k \alpha_{d_k}^k = 0$

$\beta_1 = 0$ ✓

✓ $\beta_1 + \beta_2 + \cdots \quad \beta_k$

$$\overset{?}{\Longrightarrow} \beta_i = 0 \quad \forall\, i = 1, 2, \ldots k$$

$$\Longrightarrow \dim \sum_{i=1}^{k} E_i = \sum_{i=1}^{k} \underline{\dim E_i} \quad ✓$$

We will see a proof in next class.