Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Tutorial Sheet 3

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1. Show that the following matrix A is Hermitian

$$A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}.$$

Prove that there exists a unitary matrix U such that A can be written as $A = UDU^{-1}$ for a diagonal matrix D.

2. Let A be an $n \times n$ complex matrix. Prove that A is Hermitian if and only if X^*AX is real for all vectors X in $\mathbb{C}^n$.

3. Find out a real symmetric matrix B and a real skew-symmetric matrix C such that the following matrix A can be written as $A = B + iC$

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

Can every Hermitian matrix A can be written in a similar fashion?

4. Find out Hermitian matrices B and C such that the following matrix A can be written as $A = B + iC$

$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}.$$

Generalize it for any complex $n \times n$ matrix.

5. Find the minimal polynomial for the following linear operators

   (i) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined as $Tf = f'$. $x^3 = 0$

   (ii) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$, defined as $T(A) = A^t$

6. Let V be an n-dimensional vector space and let T be a linear operator on V. Suppose that there exists some positive integer k so that $T^k = 0$. Prove that $T^n = 0$.

7. Find a minimal polynomial of the following matrix without finding characteristic polynomial

$x^2 = 0$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

8. Let $a, b, c \in \mathbb{R}$, then show that for the following matrix characteristic and minimal polynomials are same

$$A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}.$$

9. Prove that if $T \in L(V, V)$ is annihilated by a polynomial over $\mathbb{C}$ having distinct roots, then T is diagonalizable. As a direct application of this result, show the following

   (a) Let T be a linear operator on a complex vector space such that $T^k = I$ for some positive integer k. Then T is diagonalizable.

   (b) Prove that every matrix A satisfying $A^2 = A$ is diagonalizable.

10. Compute the minimal polynomial for the following matrices

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

11. Verify Cayley-Hamilton theorem for the following

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (2x + 5y, 6x + y)$

   (b) $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

12. Let characteristic polynomial of a matrix A be $x^2 - x + 1$. Compute A^3 and A^5.