1. We have $f_{X,Y}(x,y) = \begin{cases} K(1-x-y), & x>0,\ y>0,\ x+y<1 \\ 0, & o/w \end{cases}$

$K = 6$. The marginal of X & Y are

$$f_X(x) = \begin{cases} 3(1-x)^2, & 0<x<1 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = \begin{cases} 3(1-y)^2, & 0<y<1 \\ 0, & o/w. \end{cases}$$

So for a given $y \in (0,1)$ the condition of X given $Y=y$

$$f_{X|Y}(x|Y=y) = \begin{cases} \dfrac{6(1-x-y)}{3(1-y)^2}, & 0<x<1-y \\ 6, & o/w. \end{cases}$$

$$E(X|Y=\tfrac{1}{2}) = \int_0^{\frac{1}{2}} \frac{x\,6(1-x-\frac{1}{2})}{3\cdot(\frac{1}{2})^2}\,dx$$

$$= 4\int_0^{1/2} x(1-2x)\,dx = -\frac{1}{6}.$$

For a given $x \in (0,1)$, the conditional pdf of

$\cancel{X}$ given $X = x$

$$f_{Y|X}(y|X=x) = \begin{cases} \dfrac{6(1-x-y)}{3(1-x)^2}, & 0 < y < 1-x \\ 0, & o/w \end{cases}$$

$$E(Y|X=x) = \int_0^{1-x} \frac{y\,6(1-x-y)}{3(1-x)^2}\,dy$$

$$E(Y|X=1/2) = \int_0^{1/2} \frac{6y(1-y-\frac{1}{2})}{3(\frac{1}{2})^2}\,dy = -1/6.$$

$$\text{Var}(X|Y=\tfrac{1}{2}) = \int_0^{1/2} \frac{(x+\frac{1}{6})^2\,6(1-x-1/2)}{3(\frac{1}{2})^2}\,dx$$

$$\text{var}(Y|X=1/2) = \int_0^{1/2} \frac{(y+\frac{1}{6})^2\,6(1-1/2-y)}{3(\frac{1}{2})^2}\,dy.$$

(2) We have

$$f_{X,Y}(x,y) = P(X=x, Y=y) = \begin{cases} c(3x+4y), & x=0,1,2,3;\ y=1,2,3,4 \\ 0, & o/w. \end{cases}$$

Here $c = \frac{1}{232}$

We have

$$f_X(x) = P(X=x) = \begin{cases} \frac{1}{58}(3x+10), & x=0,1,2,3 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = P(Y=y) = \begin{cases} \frac{1}{116}(9+8y), & y=1,2,3,4 \\ 0, & o/w. \end{cases}$$

For a given y,

$$E(X|Y=y) = \sum_{x=0}^{3} x\, P(X=x|Y=y)$$

$$= \sum_{x=0}^{3} x \frac{P(X=x, Y=y)}{P(Y=y)}$$

In particular

$$E(X|Y=1) = \sum_{x=0}^{3} \frac{x P(X=x, Y=1)}{P(Y=1)} = \sum_{x=0}^{3} \frac{\frac{x}{232}(3x+4)}{\frac{17}{116}}$$

$$= \sum_{x=0}^{3} \frac{17}{2} x(3x+4)$$

For a given x.

$$E(Y|X=x) = \sum_{y=1}^{4} y \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \sum_{y=1}^{4} y \frac{\frac{1}{232}(3x+4y)}{\frac{1}{58}(3x+10)}$$

In particular $x=0$

$$E(Y|X=0) = \sum_{y=1}^{4} \frac{y}{4} \cdot \frac{4y}{10} = \sum_{y=1}^{4} \frac{y^2}{10}.$$

③ Try yourself.

④ ⓐ X and Y are independent and $X \sim U(0,1)$, $Y \sim U(0,1)$

Then the joint of X & Y is

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & o/w. \end{cases}$$

Take $U = XY$, $V = X$. $\Rightarrow$ in real variables

$u = xy$, $v = x$

$\Rightarrow$ $x = v$, $y = u/v$.

$$J = \begin{vmatrix} 0 & \frac{1}{v} \\ 1 & -\frac{u^2}{v} \end{vmatrix} = -\frac{1}{v} < 0,$$

Since $x \in (0,1)$, $y \in (0,1)$, $u \in (0,1)$, $v \in (0,1)$

So the joint of U, V is

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{v}, & 0 < v < 1,\ 0 < \frac{u}{v} < 1, \quad 0<u<1,\ 0<v< \\ 0, & o/w \end{cases}$$

Then the marginal of U is given as

$$f_U(u) = \int_u^1 \frac{1}{v}\, dv \quad , \qquad 0 < u < 1$$

$$= -\log u, \qquad 0 < u < 1.$$

So $$f_U(u) = \begin{cases} -\log u, & 0 < u < 1 \\ 0, & o/w. \end{cases}$$

(b) Take $U = \frac{X}{Y}$, $V = X$.

Then in real variable $u = \frac{x}{y}$, $v = x$.

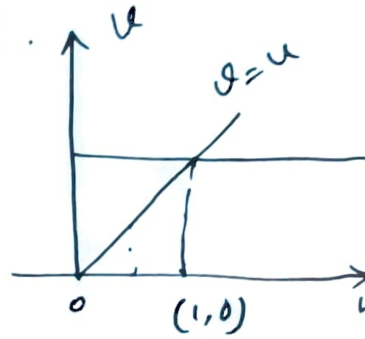
$\Rightarrow x = v$, $y = \frac{v}{u}$, we have

$J = \frac{v}{u^2}$.

$0 < v < 1$, $0 < \frac{v}{u} < 1$, also $v \in (0,1)$, $v > 0$

So the joint of U, V is

$$f_{U,V}(u,v) = \begin{cases} \frac{v}{u^2}, & 0 < v < 1,\ u > 0. \\ & 0 < \frac{v}{u} < 1. \\ 0, & o/w. \end{cases}$$



So the marginal of U is

$$f_U(u) = \int_0^u \frac{v}{u^2}\, dv. \quad , \quad 0 < u < 1$$

$$= \frac{1}{u^2}\left[\frac{v^2}{2}\right]_0^u \quad , \quad 0 < u < 1$$

$$= \frac{1}{2}, \quad 0 < u < 1.$$

$$f_U(u) = \int_0^1 \frac{v}{u^2}\, dv, \quad u > 1 \qquad = \frac{1}{2u^2}, \quad u > 1$$

$$\text{So } f_U(u) = \begin{cases} \frac{1}{2}, & 0 < u < 1 \\ \frac{1}{2u^2}, & u > 1. \\ 0, & o/w. \end{cases}$$

Which is the density of $U = \frac{X}{Y}$.

(5) $P(XY=0) = P(X=0 \text{ or } Y=0)$

$= P(X=0) + P(Y=0) - P(X=0, Y=0)$

$= P(X=0) + P(Y=0) - P(X=0)\,P(Y=0)$

$= P(X=0) + P(Y=0)\,(1 - P(X=0))$

$= e^{-2} + \left(\frac{1}{4}\right)^{10}(1-e^{-2})$.

(6) The p.m.f. of (X,Y) is given as

$P(X=x, Y=y)$

$Y \backslash X$	-1	0	1	$f_Y(y)$
-2	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$	$5/12$
1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$	$5/12$
2	$\frac{1}{12}$	0	$\frac{1}{12}$	$1/6$
$f_X(x)$	$5/12$	$1/6$	$5/12$	1

Then the distⁿ of (U,V), where $U = |X|$, $V = Y^2$

$P(U=u, V=v)$

$V \backslash U$	0	1	$f_V(v)$
1	$1/12$	$1/3$	$5/12$
4	$1/12$	$1/2$	$7/12$
$f_U(u)$	$1/6$	$5/6$.	

(7) The joint density of X & Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x < 1,\ 0 < y < 1 \\ 0, & o/w . \end{cases}$$

The required prob is

$$P(9x^2 + 9Sx + 1 = 0 \text{ has no real root})$$

$$= P(81S^2 - 36 < 0)$$

$$= P\left(-\frac{2}{3} < S < 2/3\right) = P(-2/3 < S < 0) +$$

$$P(0 < S < 2/3)$$

$$= P\left(-\frac{2}{3} < X+Y < 0\right) + P(0 < X+Y < 2/3)$$

$$= 0 + \int_0^{2/3} \int_0^{2/3 - x} 1\, dy\, dx$$

$$= 0 + \int_0^{2/3} \left(\frac{2}{3} - x\right) dx = 2/9 .$$

(8) The joint density of X & Y is given as

$$f(x,y) = \begin{cases} ce^{-(x+y)} & \text{if } y > x > 0 \\ 0 & o/w . \end{cases}$$

We know

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1 \Rightarrow c\int_{0}^{\infty}\int_{x}^{\infty} ce^{-(x+y)}\,dy\,dx = 1$$

$$\Rightarrow c\int_{0}^{\infty} e^{-2x}\,dx = 1 \Rightarrow c = 2.$$

The marginal pdf of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= 2\int_{x}^{\infty} e^{-(x+y)}\,dy, \qquad x > 0.$$

$$= 2e^{-2x}, \qquad x > 0$$

So

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & o/w. \end{cases}$$

Now for a fixed $x \in (0,\infty)$ the conditional pdf of Y given $X = x$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} e^{-(y-x)}, & y > x \\ 0, & o/w. \end{cases}$$

So $E(Y|X=2) = \int_2^\infty y\, e^{-(y-2)}\, dy$

$= \int_0^\infty (t+2)\, e^{-t}\, dt = \Gamma(2) + 2\Gamma(1)$

$= 3.$

(9) $X_1, X_2 \overset{iid}{\sim} \exp(\lambda)$. Then the joint density of X_1, X_2 is given by

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda^2 e^{-\lambda(x_1+x_2)}, & x_1>0,\ x_2>0 \\ 0 & , \text{o/w}. \end{cases}$$

consider the transformation $Y_1 = X_1,\ Y_2 = X_1 + X_2$

Then the inverse transformation in real variable

$x_1 = y_1, \quad x_2 = (y_2 - y_1)$

$y_1 > 0, \quad y_2 - y_1 > 0 \Rightarrow y_2 > y_1 > 0.$

Jacobian of the inverse transformation is

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

The joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \lambda^2 e^{-\lambda y_2}, & y_1>0,\ y_2>y_1 \\ 0, & \text{o/w}. \end{cases}$$

The marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \int_0^{y_2} f_{Y_1,Y_2}(y_1,y_2)\, dy_1, \qquad y_2 > 0.$$

$$= \int_0^{y_2} \lambda^2 e^{-\lambda y_2}\, dy_1, \quad y_2 > 0$$

$$= \lambda^2 y_2 e^{-\lambda y_2}, \quad y_2 > 0.$$

So $f_{Y_2}(y_2) = \begin{cases} \lambda^2 y_2 e^{-\lambda y_2}, & y_2 > 0 \\ 0, & o/w. \end{cases}$

clearly Y_2 is a Gamma random variable.

Now for given $y_2 \in (0, \infty)$, the conditional distn of Y_1 given Y_2 is

$$f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} \dfrac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)}, & 0 < y_1 < y_2 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} \dfrac{\cancel{\lambda^2} \cancel{e^{-\lambda y_2}}}{\cancel{\lambda^2} y_2 \cancel{e^{-\lambda y_2}}}, & 0 < y_1 < y_2 \\ 0, & o/w. \end{cases}$$

$$= \begin{cases} \frac{1}{y_2}, & 0 < y_1 < y_2 \\ 0, & o/w. \end{cases}$$

The conditional prob density fun of X given $Y=y\ (>0)$ is

$$f_{X|Y}(x|y) = \begin{cases} y e^{-yx}, & x>0 \\ 0, & o/w. \end{cases}$$

Marginal of Y is $$f_Y(y) = \begin{cases} \alpha e^{-\alpha y}, & y>0 \\ 0, & o/w. \end{cases}$$

Then the joint of X and Y is

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

$$= \begin{cases} \alpha y e^{-y(x+\alpha)}, & x>0,\ y>0 \\ 0, & o/w. \end{cases}$$

Marginal of X is

$$f_X(x) = \int_0^\infty \alpha y e^{-y(x+\alpha)} dy, \quad x>0$$

$$= \frac{\alpha}{(x+\alpha)^2} \Gamma(2), \quad x>0$$

$$f_X(x) = \begin{cases} \frac{\alpha}{(x+\alpha)^2}, & x>0 \\ 0, & o/w. \end{cases}$$

Now for a fixed $x>0$. the conditional pdf of Y given $X=x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{\alpha y e^{-y(x+\alpha)}}{\alpha/(x+\alpha)^2}, & y>0 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} y(x+\alpha)^2 e^{-y(x+\alpha)}, & y>0 \\ 0, & o/w. \end{cases}$$

(11) $X_1 \sim N(200, 8)$, $X_2 \sim N(104, 8)$, $X_3 \sim N(108, 15)$
$X_4 \sim N(120, 15)$, $X_5 \sim N(210, 15)$. They are also independent.

$$U = \frac{X_1+X_2}{2} \sim N(152, 4)$$

$$V = \frac{X_1+X_2+X_3}{3} \sim N\left(\frac{108+120+210}{3}, \frac{1}{9}(15+15+15)\right)$$

$$= N(146, 5)$$

clearly U and V are independent

$$W = U-V \sim N(6, 9)$$

$$P(U>V) = P(W>0) = P\left(\frac{W-6}{3} > -\frac{6}{3}\right)$$

$$= P(Z>-2) = 1-\Phi(-2)$$

$$= \Phi(2) = 0.9772$$

12 (a) Define the random variable

$$X_i = \begin{cases} 1, & \text{if the } i\text{th die show even number on its upper face, } i=1,2,\cdots 6 \\ 0, & \text{o/w} \end{cases}$$

$$P(\text{show even number}) = \frac{3}{6} = \frac{1}{2}$$

So $X_i \sim$ Bernoulli, distribution with success prob $= \frac{1}{2}$.

Also X_i's are independent, $i=1,2,\cdots,6$.

$$S = \sum_{i=1}^{6} X_i \sim Bin(6, 1/2)$$

$$E(S) = 6\cdot\frac{1}{2} = 3, \quad Var(S) = npq = 6\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{2}$$

(b) X_1 and X_2 are independent with m.g.f

$$M_1(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3 \text{ and } M_2(t) = e^{2(e^t-1)}$$

It can be follows from uniqueness of mgf

$$X_1 \sim Bin\left(3, \frac{1}{4}\right), \quad X_2 \sim \mathcal{P}(2)$$

$$P(X_1+X_2=1) = P(X_1=0, X_2=1) + P(X_1=1, X_2=0)$$

$$= P(X_1=0)P(X_2=1) + P(X_1=1)P(X_2=0)$$

(By independence)

$$= \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 \frac{e^{-2}2^1}{1!} + \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 e^{-2}$$

$$= \frac{81}{64}e^{-2}$$