IC105: Probability and Statistics

Tutorial 1 (Solutions)

① Let A be an event that the 7 cards include exactly 3 aces.

$$P(A) = \frac{\#\text{ ways favourable to } A}{\#\text{ ways to choose 7 cards}}$$

$$= \frac{(\#\text{ ways to choose 3 aces}) \cdot (\#\text{ ways to choose other 4 cards})}{\#\text{ ways to choose 7 cards}}$$

$$= \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}.$$

$\square$

② Let the three classes be denoted by C_1, C_2, C_3.

Let A be the event that Joe and Jane end up in the same classroom.

Also, let A_i denote the event that Joe and Jane end up in class C_i, $i = 1,2,3$.

$$\therefore \quad P(A) = P(A_1) + P(A_2) + P(A_3)$$

$$= 3P(A_1) \quad (\text{as } P(A_1) = P(A_2) = P(A_3)).$$

$$= 3\frac{\binom{88}{28}}{\binom{90}{30}} = 3 \times \frac{88!}{28!\,60!} \times \frac{30!\,60!}{90!} = \frac{3 \times 30 \times 29}{90 \times 89} = \frac{29}{89},$$

$\square$

(3) Given $A \subset B$. Then, $A \cap B = A \longrightarrow (i)$

If $P(A \cap B) = P(A)P(B)$

$\Leftrightarrow \quad P(A) = P(A)P(B)$ by (i)

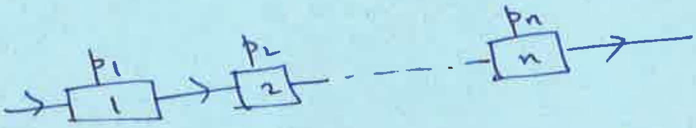
$\Leftrightarrow \quad P(A)(1 - P(B)) = 0$

$\Leftrightarrow \quad P(A) = 0$ or $P(B) = 1$

So, A and B can be independent iff either $P(A) = 0$ or $P(B) = 1$. $\square$

Example: For $A = \phi$ or $B = \Omega$ it will hold.

(4) (a) Series system:

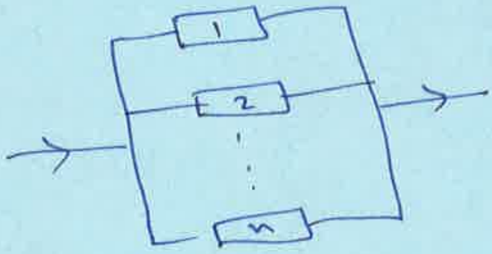


(n independent components)

Here, i^{th} component works with probability p_i for all $1 \le i \le n$.

$P(\text{series system works}) = P(\text{each component works}) = p_1 p_2 \cdots p_n = \prod_{i=1}^{n} p_i$.

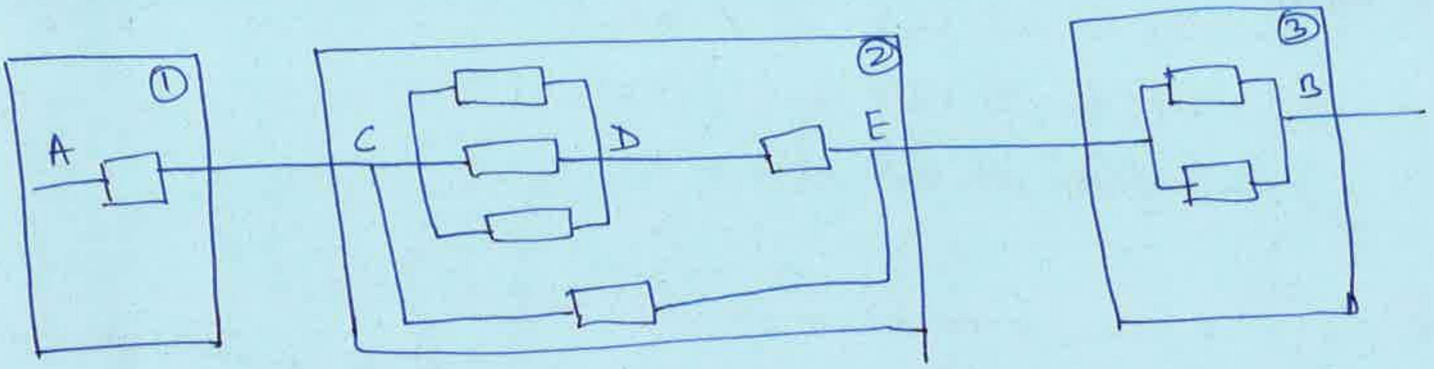
(b) Parallel system:



(n independent components).

Here, i^{th} component works with probability p_i for all $1 \le i \le n$.

$$\begin{aligned} P(\text{parallel system works}) &= P(\text{at least one component works}) \\ &= 1 - P(\text{no. component works}) \\ &= 1 - (1-p_1)(1-p_2) \cdots (1-p_n) \\ &= 1 - \prod_{i=1}^{n} (1-p_i). \end{aligned}$$



Let $P(X \to Y)$ denote the probability of a successful connection between node X and Y. Then,

$$P(A \to B) = P(A \to C)\, P(C \to E)\, P(E \to B) \quad \text{(Series system)}$$

- $P(A \to C) = p$
- $P(C \to E) = 1 - (1-p)\left(1 - P(C \to D)\, P(D \to E)\right)$

  But $P(C \to D) = 1 - (1-p)^3$

  $P(D \to E) = p.$

  $\therefore P(C \to E) = 1 - (1-p)\left(1 - p(1-(1-p)^3)\right).$

- $P(E \to B) = 1 - (1-p)^2.$

Finally,

$$\therefore P(A \to B) = p\left(1 - (1-p)\left(1 - p\left(1-(1-p)^3\right)\right)\right)\left(1-(1-p)^2\right) \qquad \square$$

(5) Given

$$P(\text{Bo will beat Ci}) = 0.6$$

$$P(\text{Al will beat Bo}) = 0.5$$

$$P(\text{Al will beat Ci}) = 0.7$$

(a) i) $P(\text{second round will be required})$

$= P(\text{Bo wins both games with Ci}) + P(\text{Ci wins both games with Bo})$

$= (0.6)^2 + (0.4)^2 = 0.52$

ii) $P(\text{Bo wins 1st rounds}) = (0.6)^2 = 0.36$.

iii) $P(\text{Al retains chamionship}) = 1 - P(\text{Al does not retains championship})$

$= 1 - P(\text{Bo champ}) - P(\text{Ci champ})$.

$= 1 - (0.6)^2 (0.5)^2 - (0.4)^2 (0.3)^2$

$= 0.8956$.

(b) i) $P(\text{Bo surviving challenger} \mid \text{second round req.}) = \dfrac{(0.6)^2}{0.52} = 0.6923$.

ii) $P(\text{Al retains championship} \mid \text{second round req.})$

$$= \frac{P(\text{Al retains championship and second round req.})}{P(\text{second round req.})}$$

$$= \frac{\begin{array}{l} P(\text{Al retains championship and second round req. and Bo surviving challenger}) \\ + P(\text{Al retains " " " and Ci " "}) \end{array}}{P(\text{second round req.})}$$

$$= \frac{(1-(0.5)^2)(0.6)^2 + (1-(0.3)^2)(0.4)^2}{0.52} = 0.7992.$$

(c) $P(\text{Bo surviving challenger} \mid \text{second round req. and one game in second round})$

$$= \frac{P(\text{second round req. and One game in 2nd round} \mid \text{Bo surviving challenger}) \cdot P(\text{Bo surv chall})}{P(\text{second round req. and One game} \mid \text{Bo chall.}) P(\text{Bo challenger}) + P(\text{2nd rnd req. and one game} \mid \text{Ci chall}) \times P(\text{Ci challenger})}$$

$$= \frac{0.5 \times (0.6)^2}{0.5 \times (0.6)^2 + 0.7 \times (0.4)^2} = 0.6164$$

□