Kuhn Tucker condition (KKT)

In the previous lecture the optimization of functions of several variables subject to equality constraints using the method of Lagrange multipliers. In this lecture the Kuhn Tucker condition will be discussed to solve non-linear optimization problem with inequality constraints.

Consider the optimization problem

$$\min \quad f(x_1, x_2, \dots x_n)$$

$$\text{subject to} \quad g_1(x_1, x_2 \dots x_n) \le 0$$

$$g_2(x_1, x_2, \dots x_n) \le 0$$

Then the Kuhn Tucker conditions for $x^* = (x_1^*, \dots x_n^*)$ to a minimum points are

$$\boxed{\begin{aligned} &\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ &\lambda g_1 = 0 \\ &\mu g_2 = 0 \\ &g_1(x_1, x_2, \dots x_n) \le 0 \\ &g_2(x_1, \dots x_n) \le 0 \\ &\lambda \le 0 \text{ and } \mu \le 0 \end{aligned}}$$

Prob 1 Solve the following optimization problem

$$\text{Min } f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2$$

subject to

$$x_1 - x_2 - 2x_3 \le 12$$

$$x_1 + 2x_2 - 3x_3 \le 8$$

Soln The Kuhn Tucker conditions for this problem are

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \quad \text{where}$$

$$\lambda g_1 = 0$$

$$\mu g_2 = 0$$

$$g_1 \le 0$$

$$g_2 \le 0$$

$$\lambda, \mu \le 0$$

$g_1(x_1, x_2, x_3)$
$= x_1 - x_2 - 2x_3 - 12$
and $g_2(x_1, x_2, x_3)$
$= x_1 + 2x_2 - 3x_3 - 8$

$$\Rightarrow (2x_1, 4x_2, 6x_3) = \lambda(1, -1, -2) + \mu(1, 2, -3)$$

$$\lambda(x_1 - x_2 - 2x_3 - 12) = 0$$

$$\mu(x_1 + 2x_2 - 3x_3 - 8) = 0$$

$$x_1 - x_2 - 2x_3 - 12 \le 0$$

$$x_1 + 2x_2 - 3x_3 - 8 \le 0$$

$$\lambda \le 0 \ \& \ \mu \le 0$$

$$\Rightarrow \quad 2x_1 - \lambda - \mu = 0 \quad \text{--- (1)}$$

$$4x_2 + \lambda - 2\mu = 0 \quad \text{--- (2)}$$

$$6x_3 + 2\lambda + 3\mu = 0 \quad \text{--- (3)}$$

$$\lambda(x_1 - x_2 - 2x_3 - 12) = 0 \quad \text{--- (4)}$$

$$\mu(x_1 + 2x_2 - 3x_3 - 8) = 0 \quad \text{--- (5)}$$

$$x_1 - x_2 - 2x_3 - 12 \le 0 \quad \text{--- (7)}$$

$$x_1 + 2x_2 - 3x_3 - 8 \le 0 \quad \text{--- (8)}$$

$$\lambda, \mu \le 0 \quad \text{--- (9)}$$

By (4), $\lambda = 0$ or $x_1 - x_2 - 2x_3 - 12 = 0$

Case – I $\quad \lambda = 0$, From (1), (2) & (3) we get

$$x_1 = \frac{\mu}{2}, \quad x_2 = \frac{\mu}{2} \quad \text{and} \quad x_3 = -\frac{\mu}{2}$$

Substituting the value of x_1, x_2 & x_3 in (5), we see

$$\mu\left(\frac{\mu}{2} + \mu + \frac{3\mu}{2} - 8\right) = 0$$

$$\Rightarrow \mu(3\mu - 8) = 0 \Rightarrow \mu = 0 \text{ or } \mu = \frac{8}{3}$$

which is not possible as $\mu \le 0$

$\therefore$ $x^* = (0, 0, 0)$ is an optimum solⁿ.

Case – II $\quad x_1 - x_2 - 2x_3 - 12 = 0$

From (1), (2) & 3, we have $x_1 = \frac{\lambda + \mu}{2}$, $x_2 = \frac{2\mu - \lambda}{4}$

and $x_3 = \frac{-(2\lambda + 3\mu)}{6}$

substituting these values in $x_1 - x_2 - 2x_3 - 12 = 0$

$$\Rightarrow \quad \frac{\lambda+\mu}{2} - \frac{2\mu-\lambda}{4} + \frac{2\lambda+3\mu}{3} - 12 = 0$$

$$\Rightarrow \quad \frac{6\lambda+6\mu-6\mu+3\lambda+8\lambda+12\mu}{12} = 12$$

$$\Rightarrow \quad 17\lambda + 12\mu = 144$$

this is not possible as μ and λ have to take non-positive value.

$\therefore$ $(0,0,0)$ is the optimum sol^n.

consider the optimization problem of the form

$$\min f(x_1, x_2, \ldots x_n)$$

$$\text{subject to} \quad g_1(x_1, x_2 \ldots x_n) \geq 0$$

$$g_2(x_1, \ldots x_n) \geq 0$$

Then the Kuhn Tucker condition for $x^* = (x_1^*, x_2^* \ldots x_n^*)$ to a minimum point are

$$\boxed{\begin{array}{c} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ \lambda g_1 = 0 \quad , \quad \mu g_2 = 0 \\ g_1 \geq 0 \quad , \quad g_2 \geq 0 \\ \mu, \lambda \geq 0 \end{array}}$$

Prob 2 Solve the optimization problem

$$\min f(x_1, x_2) = x_1^2 + x_2^2 + 60x_1 \text{ s.t}$$

$$x_1 \geq 80$$

$$x_1 + x_2 - 120 \geq 0$$

Sol^n

$$g_1(x_1, x_2) = x_1 - 80 \geq 0$$

$$g_2(x_1, x_2) = x_1 + x_2 - 120 \geq 0$$

$$\Rightarrow \quad (2x_1 + 60, 2x_2) = \lambda(1, 0) + \mu(1, 1)$$

$$\lambda(x_1 - 80) = 0$$

$$\mu(x_1 + x_2 - 120) = 0$$

$$x_1 - 80 \geq 0$$

$$x_1 + x_2 - 120 \geq 0$$

$$\lambda, \mu \geq 0$$

$$\Rightarrow \quad 2x_1 + 60 = \lambda + \mu \quad \text{— (1)}$$

$$2x_2 = \mu \quad \text{— (2)}$$

$$\lambda(x_1 - 80) = 0 \quad \text{— (3)}$$

$$\mu(x_1 + x_2 - 120) = 0 \quad \text{— (4)}$$

$$x_1 \geq 80 \quad \text{— (5)}$$

$$x_1 + x_2 - 120 \geq 0 \quad \text{— (6)}$$

$$\lambda, \mu \geq 0 \quad \text{— (7)}$$

From (3), $\lambda = 0$ or $x_1 = 80$

$\lambda = 0$

$$x_1 = \frac{\mu}{2} - 30, \quad x_2 = \frac{\mu}{2}$$

Substituting this in (4),

$$\mu(\mu - 150) = 0 \Rightarrow \mu = 0, \ \mu = 150$$

$\mu = 0$ lead us to $x_1 = -30$, $x_2 = 0$, this is not possible as $x_1 \geq 80$

$\mu = 150$

$$x_1 = 45, \quad x_2 = 75$$

this contradicts $x_1 \geq 80$

Case - II $\quad x_1 = 80$

By (2), $x_2 = \frac{\mu}{2} \Rightarrow \mu = 2x_2$

Substitute $x_2 = \frac{\mu}{2}$, in (4)

$$2x_2(80 + x_2 - 120) = 0$$

$$\Rightarrow x_2 = 0 \text{ or } x_2 = 40$$

$x_2 = 0 \Rightarrow x_1 \geq 120$ but $x_1 = 80$

$x_2 = 40$

consider the optimization problem of the form

$$\min f(x_1, x_2 \ldots x_n)$$

$$\text{s.t} \quad g_1(x_1 \ldots x_n) \leq 0$$

$$g_2(x_2 \ldots x_n) \leq 0$$

$$h(x_1, \ldots x_n) = 0$$

Then the KKT conditions for optimality are

①
$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \mu h$$

$$\lambda_1 g_1 = 0$$

$$\lambda_2 g_2 = 0$$

$$g_1 \leq 0$$

$$g_2 \leq 0$$

$$h = 0$$

$$\lambda_1 \text{ and } \lambda_2 \leq 0$$

Prob 3 Find the minimizer of the optimization problem

$$f(x_1, x_2) = (x_1 - 1)^2 + x_2 - 2$$

$$\text{s.t} \quad x_1 + x_2 \leq 2$$

$$x_2 - x_1 - 1 = 0 \quad 1/2, \; 3/2$$

$-5/4$

$-7/4$

Soln

Prob 4 consider the following optimization problem

$$\min f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

s.t $\quad 2x_1 + x_2 - 5 \leq 0$

$$x_1 + x_3 \leq 2$$

$$1 - x_1 \leq 0$$

$$x_2 \geq 2$$