Lecture #14 (IC152)

(Gram-Schmidt Process)

Given a linearly independent subset S of an ips V

$$\Downarrow$$

an orthogonal subset S' of V s.t. $\underline{\text{span}(S) = \text{span}\, S'}$

Theorem Let V be a nonzero finite dimensional ips. Then there exists an orthonormal basis for V.

Proof:- As V is vector space of finite dimension ($\dim V = n$)

$\Rightarrow \exists\ \mathcal{B} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ a basis of V

$\Rightarrow \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is a linearly independent set of V

$\Rightarrow \exists$ an orthogonal set $\{\beta_1, \beta_2, \ldots \beta_n\} = \mathcal{B}'$

s.t. $\text{span}(\mathcal{B}') = \text{span}(\mathcal{B}) = \underline{V}$

$\Rightarrow$ $\mathcal{B}'$ is an orthogonal basis for V.

Now by dividing each vector β_i of $\mathcal{B}'$ by it's length $\|\beta_i\|$, we get required orthonormal basis.

$\mathbb{R}^3$

$\mathcal{B} = \{e_1, e_2, e_3\}$

$\|e_i\| = 1 \quad i = 1, 2, 3$

$\langle e_i, e_j \rangle = 0$

Ex:- $V = P_2(\mathbb{R})$, $\mathcal{B} = \{1, x, x^2\}$ find out an orthonormal basis for V. $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

Solution: $\alpha_1 = 1, \ \alpha_2 = x, \ \alpha_3 = x^2$

$$\int_{-1}^{1} x \, dx = 0$$

$$\beta_1 = \alpha_1 = 1$$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 = x$$

$$\beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\|\beta_2\|^2} \beta_2$$

$$= x^2 - \frac{1}{3}$$

$$\int^{1} x^2 = 2/3$$

$$\mathcal{B}' = \left\{ 1,\ x,\ x^2 - \frac{1}{3} \right\}$$

$$\int_{-1}^{1} x^2 \cdot x = 0$$

$$\|\beta_1\|^2 = 2$$

Orthonormal basis

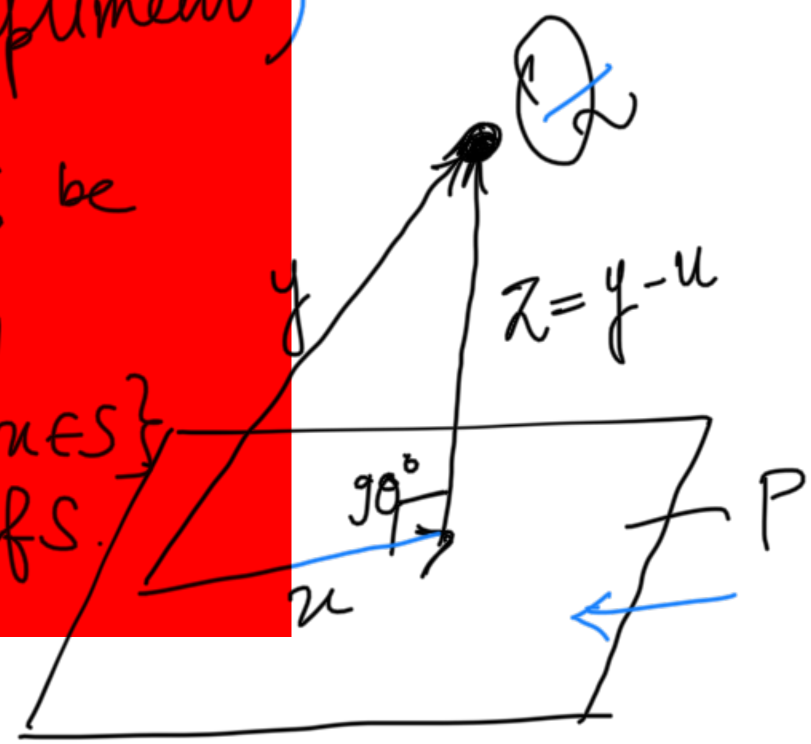
$$\mathcal{B}'' = \left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\,x,\ \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\} \ !!$$

Definition :- (Orthogonal compliment)

Let V be a non-empty ips & S be a subset (non-empty) of V then

$$S^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \ \forall u \in S \}$$

$S^{\perp}$ is called orthogonal compliment of S.



Ex: $V = \mathbb{R}^3$

$$S = \langle \{ e_1, e_2 \} \rangle$$

$$S^{\perp} = \langle \{ e_3 \} \rangle$$

$S \subset V$

Trivial examples $S = \{0\}$,

$S^{\perp} = V$

If $S = V \Rightarrow S^{\perp} = \{0\}$

Theorem :- $S^{\perp}$ is a vector subspace of V !!

Let $\alpha, \beta \in S^{\perp}$ & $c \in \mathbb{F}$

To show $\langle c\alpha + \beta, \gamma \rangle = 0 \ \forall \ \gamma \in S$

L.H.S. $c\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle$

$= 0 + 0 = 0$

$0 \in S^{\perp}$.

Theorem :- Let W be a finite dimensional subspace of an ips V and let $y \in V$ then $\exists$ a unique vector u in W & $z \in W^{\perp}$ s.t.

$y = u + z$. ✓

Furthermore if $\{v_1, v_2, \dots v_k\}$ be an orthonormal basis for W, then

$$\rightarrow \quad u = \sum_{i=1}^{k} \langle y, v_i \rangle \underline{\underline{v_i}} \leftarrow$$

<u>Proof</u> :- Uniqueness: Let $y = u_1 + z_1 = u_2 + z_2$
where $u_1, u_2 \in W$ & $z_1, z_2 \in W^\perp$

& $z_2 - z_1 \in W^\perp$

$$\Rightarrow \quad u_1 - u_2 \in W$$

$$u_1 - u_2 = z_2 - z_1$$

$$u_1 - u_2 \in W \cap W^\perp = \{0\}$$

$$\Rightarrow u_1 = u_2 \quad |||^{ly} \quad z_1 = z_2$$

Let u be as

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$ & y be a given vector in V then can we show that

$$\rightarrow z = y - u \in W^\perp \,?$$

... that $u \in W$.

It is clear ...

If $\langle \alpha, v_j \rangle = 0 \ \forall\ j = 1, 2, \dots k$

L.H.S. $\left\langle y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i \,,\, v_j \right\rangle$

$= \langle y, v_j \rangle - \left\langle \sum_{i=1}^{k} \langle y, v_i \rangle v_i \,,\, v_j \right\rangle$

$= \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \underline{\langle v_i, v_j \rangle}$

$= \langle y, v_j \rangle - \langle y, v_j \rangle \underbrace{\langle v_j, v_j \rangle}_{=1}$

$= 0$

<u>Example</u> :- $V = P_3(\mathbb{R})$, $W = P_2(\mathbb{R})$ $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

$y = x^3$

$y = u + z$, $u \in W$, $z \in W^{\perp}$?

$$\text{OB of } W = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

$$u = \left\langle x^3, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \left\langle x^3, \sqrt{\frac{3}{2}}\,x \right\rangle \sqrt{\frac{3}{2}}\,x$$

$$+ \left\langle x^3, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\rangle \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$= \frac{3}{5}x$$

$$z = x^3 - u = x^3 - \frac{3}{5}x$$

$$\boxed{x^3 = \underbrace{\frac{3}{5}x}_{u} + \underbrace{\left(x^3 - \frac{3}{5}x\right)}_{z}}$$

Corollary :- $\|y - u\| \leq \|y - w\| \quad \forall\, w \in W$

$$y - u = z \in W^{\perp}$$

$w \in W$

$u - w \in W \Rightarrow \langle u-w, z \rangle = 0$

$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$ if $\langle \alpha, \beta \rangle = 0$

$$\|y-w\|^2 = \|u+z-w\|^2 = \|\underline{\underline{u-w}} + \underline{\underline{z}}\|^2$$

$$= \|u-w\|^2 + \|z\|^2$$

$$\geq \|z\|^2 = \|y-u\|^2$$

$$\Rightarrow \|y-u\| \leq \|y-w\| \ \forall\, w \in W.$$

Theorem :- If V be a finite-dimensional vector space then

$$\dim V = \dim W + \dim W^{\perp}$$

for any subspace W of V.
