IC153: Calculus 1
(Lecture 12)

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Recap

- Series of real numbers
- Convergence/divergence of a series
- Necessary condition for convergence of a series
- Necessary and sufficient condition for convergence of a series
- Absolute convergence $\Longrightarrow$ convergence
- Alternating series test (Leibniz test)

Comparison test

Theorem

Suppose $0 \le a_n \le b_n$ *for all* $n \ge k$ *for some* k. *Then*

1. $\sum_{n=1}^{\infty} b_n$ *converges* $\implies \sum_{n=1}^{\infty} a_n$ *converges.*
2. $\sum_{n=1}^{\infty} a_n$ *diverges* $\implies \sum_{n=1}^{\infty} b_n$ *diverges.*

Proof: (1) T_n = seq. of partial sums of $\sum_{n\geq k} b_n$

S_n = seq. of partial sums of $\sum_{n\geq k} a_n$

$\sum_{i=1}^{\infty} a_i$ converges $\iff$ $\sum_{i=k}^{\infty} a_i$ converges

$S_n \leq T_n \quad \forall \; n \geq k$

$\sum_{n=1}^{\infty} b_n$ is convergent $\Rightarrow \sum_{n\geq k} b_n$ is convergent $\Rightarrow T_n$ is bounded

$\Rightarrow S_n$ is bounded $\Rightarrow \sum_{n\geq 1} a_n$ converges

(2) Similar.

Note: We say that $a_n \leq b_n$ eventually if there exists $N \in \mathbb{N}$ such that $a_k \leq b_k$ for all $k \geq N$.

Example

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

   $\frac{1}{n^2} \leq \frac{1}{n(n-1)} \quad \forall \; n \geq 2$

   $\sum \frac{1}{n(n-1)}$ converges $\Rightarrow \sum \frac{1}{n^2}$ converges

2. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

   $\frac{1}{n!} \leq \frac{1}{n^2} \quad \forall \; n \geq 4$

   or $\frac{1}{n!} \leq \frac{1}{n(n-1)} \quad \forall \; n \geq 2$

Corollary

Let $a_n \geq 0$. Then both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converge or diverge together.

Proof:

$$0 \leq \frac{a_n}{1+a_n} \leq a_n$$

$\sum a_n$ converges $\Rightarrow$ $\sum \frac{a_n}{1+a_n}$ converges

($\Leftarrow$)

Let $\sum \frac{a_n}{1+a_n}$ converges $\Rightarrow$ $\frac{a_n}{1+a_n} \longrightarrow 0$

$\Rightarrow$ $a_n \longrightarrow 0$

$\Rightarrow$ $a_n \leq 1$ eventually

$\Rightarrow$ $1+a_n \leq 2$ eventually $\Rightarrow$ $\frac{a_n}{2} \leq \frac{a_n}{1+a_n}$ eventually. $\Rightarrow$ $\sum \frac{a_n}{2}$ converges

$\Rightarrow \frac{1}{2}\sum a_n$ converges.

Limit comparison test

Theorem

Suppose $a_n, b_n \geq 0$ eventually. Let $\frac{a_n}{b_n} \longrightarrow L$

1. *If $L \in (0, \infty)$, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.*
2. *If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*
3. *If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof: Since $L > 0$ $\exists$ $\epsilon > 0$ s.t. $L - \epsilon > 0$.

$\frac{a_n}{b_n} \longrightarrow L \Rightarrow \exists N \in \mathbb{N}$ s.t.

$$L - \epsilon \leq \frac{a_n}{b_n} \leq L + \epsilon \quad \text{—— (i)}$$

Proof (Cont.):

Eq (1) $\Rightarrow$ $(L-\epsilon)\, b_n \le a_n \le (L+\epsilon)\, b_n$ — (2)

(i) Let $\sum a_n$ converges. Eq (2) $\Rightarrow$ $(L-\epsilon)\, b_n \le a_n$

$\Rightarrow$ $\sum (L-\epsilon) b_n$ converges

$\Rightarrow$ $(L-\epsilon) \sum b_n$ converges

$\Rightarrow$ $\sum b_n$ converges

(ii) Let $\sum a_n$ diverges. $a_n \le (L+\epsilon)\, b_n$

$\Rightarrow$ $\sum (L+\epsilon)\, b_n$ diverges

$\Rightarrow$ $\sum b_n$ diverges

All other cases can be done similarly.

Example

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n}$ converges. Let $a_n = \frac{1}{3^n}$

$$\frac{a_n}{b_n} = \frac{3^n - n}{3^n} = 1 - \frac{n}{3^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \xrightarrow{\text{by LCT}} \sum \frac{1}{3^n - n} \text{ converge b/c } \sum \frac{1}{3^n} \text{ converges}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \log\left(1 + \frac{1}{n}\right)$ converges. Let $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{n}\log\left(1+\frac{1}{n}\right)}{1/n^2} = n\log\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \log\left(1+\frac{1}{n}\right)^n = \log e = 1$$

$$\Rightarrow \sum \frac{1}{n}\log\left(1+\frac{1}{n}\right) \text{ converges (by LCT)}$$

Questions?

Cauchy test

Theorem (Cauchy Condensation Test)

If $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof: Let $S_n = a_1 + a_2 + \dots + a_n$ & $T_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$

($\Leftarrow$) Suppose (T_k) converges. For a fixed n, choose k s.t. $2^k \geq n$,

Then
$$\begin{aligned} S_n &= a_1 + a_2 + \dots + a_n \\ &\leq a_1 + (a_2 + a_3) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \dots + 2^k a_{2^k} = T_k \end{aligned}$$

$\Rightarrow (S_n)$ is bounded above $\Rightarrow (S_n)$ is convergent.

Proof (Cont.): $(\Rightarrow)$ Let (S_n) converges. For a fixed k, choose n s.t. $n \geq 2^k$. Then

$$S_n = a_1 + \cdots + a_n$$

$$\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$

$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k}$$

$$= \frac{1}{2}T_k \Rightarrow S_n \geq \frac{1}{2}T_k \Rightarrow T_k \leq 2S_n$$

$\Rightarrow (T_k)$ is bounded above $\Rightarrow (T_k)$ is convergent.

Example

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

$$\sum 2^k a_{2^k} = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum 2^k \cdot 2^{-pk}$$

$$= \sum_{k=1}^{\infty} 2^{k(1-p)}$$ converges if & only if $p > 1$.

2. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

$$\sum 2^k \cdot \frac{1}{2^k \cdot (\log 2^k)^p} = \sum \frac{1}{(k \log 2)^p}$$

$$= \frac{1}{(\log 2)^p} \sum \frac{1}{k^p}$$ converges if & only if $p > 1$

Ratio test

Pf ②: Since $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$, $|a_n| \not\to 0 \Rightarrow a_n \not\to 0$

$\Rightarrow \sum a_n$ diverges.

Theorem

Let $a_n \neq 0$ for all n.

1. *If $\left|\frac{a_{n+1}}{a_n}\right| \leq q$ eventually for some $0 < q < 1$, then the series $\sum_{n=1}^{\infty} |a_n|$ converges.*
2. *If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ eventually, then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof: ①. $\left|\frac{a_{n+1}}{a_n}\right| \leq q \quad \forall\ n \geq N$

$\Rightarrow |a_{N+1}| \leq q\,|a_N| \Rightarrow |a_{N+p}| \leq q^p |a_N|$

$$\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{N-1} |a_n|}_{P} + \sum_{n \geq N} |a_n| \leq P + \sum_{k=1}^{\infty} q^k |a_N|$$

$$= P + |a_N| \sum_{k=1}^{\infty} q^k$$

$\Rightarrow \sum_{n \geq 1} |a_n|$ converges.

Ratio test

Corollary

Let $a_n \neq 0$ for all n and $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$ for some L.

1. *If $L < 1$ then the series $\sum_{n=1}^{\infty} |a_n|$ converges.*
2. *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $L = 1$ then we can not say anything.*

Proof: 3. $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

here $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right|$

$= 1$

$\lim \frac{1/(n+1)^2}{1/n^2}$

$= \lim \left(\frac{n}{n+1}\right)^2$

$= 1$

Proof (Cont.):

1. $L < 1$, & $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq L + \frac{1-L}{2} = q < 1$

$\Rightarrow \sum |a_n|$ converges

2. Do it yourself.

Example

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\Rightarrow$ Series converges

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = 0$$

$\Rightarrow$ Series converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$\Rightarrow$ Series diverges.

Root test

Theorem

Let $|a_n|^{1/n} \longrightarrow L$ for some L.

1. *If $L < 1$ then the series $\sum_{n=1}^{\infty} |a_n|$ converges.*
2. *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *If $L = 1$ then we can not say anything.*

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$\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}}$

$= 1$

$\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1$

Example

1. $$\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$$

$\lim_{n\to\infty} \left|\frac{5n-3n^3}{7n^3+2}\right| = \frac{3}{7} < 1$

$\Rightarrow$ Series converges.

Example

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

$$\lim_{n\to\infty}\left(\frac{n^n}{3^{1+2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \infty$$

$\Rightarrow$ Series diverges

3. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{1+n}\right)^{n^2}$

$$\lim_{n\to\infty}\left(\frac{n}{1+n}\right)^{n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

$\Rightarrow$ Series converges.

Questions?