Lecture 19: Transformation of Variable Technique

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19.0.1. Distribution Function Technique

Let $\underline{X} = (X_1, X_2, \ldots, X_p)$ be a p-dimensional random vector with d.f. F and p.m.f. / p.d.f. $f(\cdot)$. Also, let $\underline{g} : \mathbb{R}^p \to \mathbb{R}^q : \underline{g} = (g_1, g_2, \ldots, g_q)$, $\underline{Y} = (Y_1, Y_2, \ldots, Y_q) = (g_1(\underline{X}), g_2(\underline{X}), \ldots, g_q(\underline{X}))$. We are interested in the distribution of random vector $\underline{Y}$.

One can first find the d.f. of $\underline{Y} = (Y_1, Y_2, \ldots, Y_q)$

$$F_{\underline{Y}}(y_1, y_2, \ldots, y_q) = \Pr(g_1(\underline{X}) \le y_1, g_2(\underline{X}) \le y_2, \ldots, g_q(\underline{X}) \le y_q), \;\; \underline{y} = (y_1, y_2, \ldots, y_q) \in \mathbb{R}^q,$$

and then find the p.m.f. / p.d.f. of $\underline{Y} = (Y_1, Y_2, \ldots, Y_q)$.

Example 19.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution having d.f. F, p.m.f. / p.d.f. f and support S. Let $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_2 = \max\{X_1, X_2, \ldots, X_n\}$.*

(a) Find the joint d.f. of $\underline{Y} = (Y_1, Y_2)$.

(b) Find the marginal d.f.s of Y_1 and Y_2 using findings of (a).

(c) Find the marginal d.f.s of Y_1 and Y_2 directly (that is, without using (a)).

(d) Find the marginal p.m.f. / p.d.f. $\underline{Y} = (Y_1, Y_2)$ using findings in (b).

Solution: (a) For $(y_1, y_2) \in \mathbb{R}^2$,

$$\begin{aligned}
F_{\underline{Y}}(y_1, y_2) &= \Pr(Y_1 \le y_1, Y_2 \le y_2) \\
&= \Pr(\min\{X_1, X_2, \ldots, X_n\} \le y_1, \max\{X_1, X_2, \ldots, X_n\} \le y_2) \\
&= \Pr(\max\{X_1, X_2, \ldots, X_n\} \le y_2) - \Pr(\min\{X_1, X_2, \ldots, X_n\} > y_1, \max\{X_1, X_2, \ldots, X_n\} \le y_2) \\
&= \Pr(X_i \le y_2,\ i = 1, 2, \ldots, n) - \Pr(X_i > y_1,\ i = 1, 2, \ldots, n, X_i \le y_2, i = 1, 2, \ldots, n) \\
&= \prod_{i=1}^{n} \Pr(X_i \le y_2) - \Pr(y_1 < X_i \le y_2, i = 1, 2, \ldots, n) \\
&= \prod_{i=1}^{n} \Pr(X_i \le y_2) - \prod_{i=1}^{n} \Pr(y_1 < X_i \le y_2) = \begin{cases} [F(y_2)]^n - [F(y_2) - F(y_1)]^n, & -\infty < y_1 < y_2 < \infty, \\ [F(y_2)]^n, & -\infty < y_2 < y_1 < \infty. \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
F_{Y_1}(y_1) &= \lim_{y_2 \to \infty} F_{\underline{Y}}(y_1, y_2) = 1 - [1 - F(y_1)]^n, -\infty < y_1 < \infty, \\
F_{Y_2}(y_2) &= \lim_{y_1 \to \infty} F_{\underline{Y}}(y_1, y_2) = [F(y_2)]^n, -\infty < y_2 < \infty.
\end{aligned}$$

(c)

$$
\begin{aligned}
F_{Y_1}(y_1) &= \Pr(Y_1 \le y_1) \\
&= \Pr(\min\{X_1, X_2, \ldots, X_n\} \le y_1) \\
&= 1 - \Pr(\min\{X_1, X_2, \ldots, X_n\} > y_1) \\
&= 1 - \Pr(X_i > y_1, i = 1, 2, \ldots, n) = 1 - \prod_{i=1}^{n} \Pr(X_i > y_1) = 1 - [1 - F(y_1)]^n, -\infty < y_1 < \infty.
\end{aligned}
$$

$$
\begin{aligned}
F_{Y_2}(y_2) &= \Pr(Y_2 \le y_2) \\
&= \Pr(\max\{X_1, X_2, \ldots, X_n\} \le y_2) \\
&= \Pr(X_i \le y_2, i = 1, 2, \ldots, n) = \prod_{i=1}^{n} \Pr(X_i \le y_2) = [F(y_2)]^n, -\infty < y_2 < \infty.
\end{aligned}
$$

(d) **Case I:** X_1 is a discrete r.v. Then $S_{X_1} = S_{Y_1} = S_{Y_2}$. For $y_1 \in S_{X_1}$

$$f_{Y_1}(y_1) = \Pr(Y_1 = y_1) = F_{Y_1}(y_1) - F_{Y_1}(y_1-) = [1 - F(y_1-)]^n - [1 - F(y_1)]^n.$$

Thus,

$$f_{Y_1}(y_1) = \begin{cases} [1 - F(y_1-)]^n - [1 - F(y_1)]^n, \text{ if } y_1 \in S_{X_1}, \\ 0, \text{ otherwise.} \end{cases}$$

Similarly,

$$f_{Y_2}(y_2) = F_{Y_2}(y_2) - F_{Y_2}(y_2-) = \begin{cases} [F(y_2)]^n - [F(y_2-)]^n, & \text{if } y_2 \in S_{X_1} \\ 0, & \text{otherwise} \end{cases}$$

Case II: X_1 is a continuous r.v.

Let $F(\cdot)$ be differentiable everywhere (except possibly on a set having length zero (that is, it does not contain any open interval)

$$
\begin{aligned}
f_{Y_1}(y) &= \frac{\mathrm{d}}{\mathrm{d}y} \left(1 - [1 - F(y)]^n\right) = n\,[1 - F(y)]^{n-1} f(y), -\infty < y < \infty, \\
f_{Y_2}(y) &= \frac{\mathrm{d}}{\mathrm{d}y} [F(y)]^n = n\,[F(y)]^{n-1} f(y), -\infty < y < \infty.
\end{aligned}
$$

Example 19.2. *Let X_1 and X_2 be iid r.v.s with common p.d.f.*

$$f(x) = \begin{cases} 2x, \ 0 < x < 1, \\ 0, \ \textit{otherwise}. \end{cases}$$

Find the d.f. of $Y = X_1 + X_2$. Hence find the p.d.f. of Y.

Solution: The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is

$$f_{\underline{X}}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = f(x_1) f(x_2) = \begin{cases} 4x_1x_2, \ 0 < x_1 < 1, \ 0 < x_2 < 1, \\ 0, \text{ otherwise.} \end{cases}$$

For $y \in \mathbb{R}$,

$$F_Y(y) = \Pr(Y \le y) = \Pr(X_1 + X_2 \le y) = \underset{x_1 + x_2 \le y}{\int_0^1 \int_0^1} 4x_1x_2 \mathrm{d}x_1 \mathrm{d}x_2.$$

Clearly for $y<0$, $F_Y(y)=0$ and for $y\geq 2$, $F_Y(y)=1$. Now consider $y\in[0,1)$,

$$F_Y(y)=\int_0^y\int_0^{y-x_1}4x_1x_2\mathrm{d}x_2\mathrm{d}y_1=\frac{y^4}{6}.$$

For $y\in[1,2)$,

$$F_Y(y)=\int_0^{y-1}\int_0^1 4x_1x_2\mathrm{d}x_2\mathrm{d}y_1+\int_{y-1}^1\int_0^{y-x_1}4x_1x_2\mathrm{d}x_2\mathrm{d}x_1=(y-1)^2+\frac{(4y-3)-(y+3)(y-1)^3}{6}.$$

Thus,

$$F_Y(y)=\begin{cases}0, \ y<0,\\ \frac{y^4}{6}, \ 0\leq y<1,\\ (y-1)^2+\frac{(4y-3)-(y+3)(y-1)^3}{6}, \ 1\leq y<2,\\ 1, \ y\geq 2.\end{cases}$$

Clearly, Y is continuous r.v. with p.d.f.

$$f_Y(y)=\begin{cases}\frac{2}{3}y^3, \ 0<y<1,\\ 2(y-1)+\frac{2}{3[1-(y+2)(y-1)^2]}, \ 1<y<2,\\ 0, \ \text{otherwise}.\end{cases}$$

19.0.2. Transformation of Variable Technique

Let $\underline{X}=(X_1,X_2,\dots,X_p)$ be a p-dimensional discrete random vector with support S and p.m.f. $f(\cdot)$. Let $g_i:\mathbb{R}^p\to\mathbb{R}$, $i=1,2,\dots,k$ and $Y_i=g_i(\underline{X})$, $i=1,2,\dots,k$ where $1\leq k\leq p$ is an integer. Then $\underline{Y}=(Y_1,Y_2,\dots,Y_k)$ is discrete random vector with support

$$T=\{(y_1,y_2,\dots,y_k): y_i=g_i(x_1,x_2,\dots,x_p), i=1,2,\dots,k \ \text{ for some } \underline{x}=(x_1,x_2,\dots,x_p)\in S\},$$

d.f. $G(\underline{y})=G(y_1,y_2,\dots,y_k)=\sum_{\underline{x}\in A_{\underline{y}}}f(\underline{x})$, $\underline{y}\in\mathbb{R}^k$ and p.m.f.

$$g(\underline{y})=\begin{cases}\sum_{\underline{x}\in B_{\underline{y}}}f(\underline{x}), \ \text{if } \underline{y}\in T,\\ 0, \ \text{otherwise},\end{cases}$$

where $A_{\underline{y}}=\{\underline{x}:(x_1,x_2,\dots,x_p)\in S: g_i(\underline{x})\leq y_i, \ i=1,2,\dots,k\}$ and $B_{\underline{y}}=\{\underline{x}:(x_1,x_2,\dots,x_p)\in S: g_i(\underline{x})=y_i, \ i=1,2,\dots,k\}$.

Example 19.3. *Let $X_1,X_2,\dots,X_p$ be independent r.v.s with X_i having the p.m.f. (Binomial distribution), that is,*

$$f_i(x)=\begin{cases}\binom{n_i}{x}\theta^x(1-\theta)^{n_i-x}, \ x\in\{0,1,\dots,n_i\},\\ 0, \ \textit{otherwise},\end{cases}$$

$i=1,2,\dots,k$, where $\theta\in(0,1)$ and $n_i\in\{1,2,\dots\}$, $i=1,2,\dots,k$ are fixed real constants. Let $Y=X_1+X_2+\cdots+X_p$. Find the p.m.f. of Y.

Solutions: The joint p.m.f. of $\underline{X}=(X_1,X_2,\dots,X_p)$ is

$$f_{\underline{X}}(\underline{x})=\prod_{i=1}^p f_i(x_i)=\begin{cases}\left(\prod_{i=1}^p\binom{n_i}{x_i}\right)\theta^{\sum_{i=1}^p x_i}(1-\theta)^{n-\sum_{i=1}^p x_i}, \ \underline{x}\in\prod_{i=1}^p\{0,1,\dots,n_i\},\\ 0, \ \text{otherwise},\end{cases}$$

where $n = \sum_{i=1}^{p} n_i$. Clearly, $f_Y(y) = \Pr(X_1 + \cdots + X_p = y) = 0$, if $y \neq \{0, 1, \ldots, n\}$. For $y \in \{0, 1, \ldots, n\}$

$$\begin{aligned} f_Y(y) &= \Pr(Y = y) \\ &= \Pr(X_1 + \cdots + X_p = y) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \cdots \sum_{\substack{x_p=0 \\ x_1+\cdots+x_p=y}}^{n_p} \left(\prod_{i=1}^{p} \binom{n_i}{x_i} \right) \theta^{\sum_{i=1}^{p} x_i} (1-\theta)^{n - \sum_{i=1}^{p} x_i} \\ &= \theta^y (1-\theta)^{n-y} \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \cdots \sum_{\substack{x_p=0 \\ x_1+\cdots+x_p=y}}^{n_p} \left(\prod_{i=1}^{p} \binom{n_i}{x_i} \right) = \binom{n}{y} \theta^y (1-\theta)^{n-y}. \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, \ y \in \{0, 1, \ldots, n\}, \\ 0, \ \text{otherwise}. \end{cases}$$

Exercise: Let $X_1, X_2, \ldots, X_p$ be independent r.v.s with X_i having the p.m.f. (Poisson distribution)

$$f_i(x) = \begin{cases} \frac{e^{-\lambda_i} \lambda_i^x}{x!}, \ x \in \{0, 1, \ldots\}, \\ 0, \ \text{otherwise}, \end{cases}$$

where $\lambda_i > 0$, $i = 1, 2, \ldots, p$ are fixed real constants. Show that the p.m.f. of $Y = X_1 + X_2 + \cdots + X_p$ is

$$f_Y(y) = \begin{cases} \frac{e^{-\lambda} \lambda^y}{y!}, \ y \in \{0, 1, \ldots\}, \\ 0, \ \text{otherwise}, \end{cases}$$

where $\lambda = \sum_{i=1}^{p} \lambda_i$.

Theorem 19.4. *Let $\underline{X} = (X_1, X_2, \ldots, X_p)$ be a continuous random vector with support S and joint p.d.f. $f(\cdot)$. Let $S_i \subseteq \mathbb{R}^p$, $i \in \Delta$ be a countable partition of S ($S_i \cap S_j \ \forall \ i \neq j$ and $\cup_{i \in \Delta} S_i = S$). Suppose that $h_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, 2, \ldots, p$ are functions such that in each S_i^0, $\underline{h} = (h_1, h_2, \ldots, h_p) : S_i^0 \to \mathbb{R}$ is one-to-one with inverse transformation $\underline{h}_i^{-1}(\underline{t}) = (h_{1,i}^{-1}(\underline{t}), h_{2,i}^{-1}(\underline{t}), \ldots, h_{p,i}^{-1})(\underline{t})$, $i \in \Delta$, here S_1^0 denotes the interior of S_i, $i \in \Delta$. Further suppose that $h_{j,i}^{-1}(\underline{t})$, $j = 1, 2, \ldots, p$, $i \in \Delta$ have continuous partial derivatives and the Jacobian determinants*

$$J_i = \begin{vmatrix} \frac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0, \ \ i \in \Delta.$$

Define $\underline{h}(S_j^0) = \{\underline{h}(\underline{x}) = (h_1(\underline{x}), h_2(\underline{x}), \ldots, h_p(\underline{x})) \in \mathbb{R}^p : \underline{x} \in S_j\}$, $j \in \Delta$ and $T_l = h_l(X_1, X_2, \ldots, X_p)$, $l = 1, 2, \ldots, p$. Then the random vector $\underline{T} = (T_1, T_2, \ldots, T_p)$ is a continuous random vector with p.d.f.

$$f_{\underline{T}}(\underline{t}) = \sum_{j \in \Delta} f(h_{1,j}^{-1}(\underline{t}), h_{2,j}^{-1}(\underline{t}), \ldots, h_{p,j}^{-1}(\underline{t})) |J_j| I_{\underline{h}(S_j^0)}(\underline{t}).$$

Corollary 19.5. *Under the notation and assumption of the above theorem suppose that $\underline{h} = (h_1, h_2, \ldots, h_p) : S^0 \to \mathbb{R}^p$ is one-to-one with inverse transformation $\underline{h}^{-1}(\underline{t}) = (h_1^{-1}(\underline{t}), h_2^{-1}(\underline{t}), \ldots, h_p^{-1})(\underline{t})$ (say), here S^0 denotes the interior of S. Furthermore suppose that $h_i^{-1}(\underline{t})$, $i = 1, 2, \ldots, p$ have continuous partial derivatives and the jacobian determinant*

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_1^{-1}(\underline{t})}{\partial t_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_p^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_p^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0.$$

Define $\underline{h}(S^0) = \{h(\underline{x}) : \underline{x} \in S\}$ *and* $T_j = h_j(X_1, X_2, \ldots, X_p)$, $j = 1, 2, \ldots, p$. *Then the random vector* $\underline{T} = (T_1, T_2, \ldots, T_p)$ *is a continuous random vector with joint p.d.f.*

$$f_{\underline{T}}(\underline{t}) = f(h_1^{-1}(\underline{t}), h_2^{-1}(\underline{t}), \ldots, h_p^{-1}(\underline{t}))|J|I_{\underline{h}(S^0)}(\underline{t}).$$

Example 19.6. *Let* X_1 *and* X_2 *be iid r.v.s with common p.d.f.* $f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \textit{otherwise}. \end{cases}$ *Find the p.d.f. of* $Y = \frac{X_1}{X_1+X_2}$.

Solution: The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is

$$f_{\underline{X}}(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1 > 0,\ x_2 > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Here $S = [0, \infty) \times [0, \infty)$, $S^0 = (0, \infty) \times (0, \infty)$. Define $Z = X_1 + X_2$, $h_1(x_1, x_2) = \frac{x_1}{x_1+x_2}$ and $h_2(x_1, x_2) = x_1 + x_2$. Then $\underline{h} : S^0 \to \mathbb{R}^2$ as 1-1; have $\underline{h} = (h_1, h_2)$. We have

$$h_1(x_1, x_2) = \frac{x_1}{x_1 + x_2} = y,\ h_2(x_1, x_2) = x_1 + x_2 = z \implies x_1 = h_1^{-1}(y, z) = yz \text{ and } x_2 = h_2^{-1}(y, z) = z(1 - y).$$

$\underline{x} \in S^0 \iff x_1 > 0, x_2 > 0 \iff yz > 0,\ z(1-y) > 0 \iff 0 < y < 1,\ z > 0$. Thus, $\underline{h}(S^0) = (0, 1) \times (0, \infty)$,

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(y,z)}{\partial y} & \frac{\partial h_1^{-1}(y,z)}{\partial z} \\ \frac{\partial h_p^{-1}(y,z)}{\partial y} & \frac{\partial h_p^{-1}(y,z)}{\partial z} \end{vmatrix} = \begin{vmatrix} z & y \\ -z & 1-y \end{vmatrix} = z.$$

Thus the joint p.d.f. of (Y, Z) is

$$f_{Y,Z}(y, z) = f_{\underline{X}}(yz, z(1-y))|z|I_{(0,1)\times(0,\infty)}(y, z) = \begin{cases} ze^{-z}, & 0 < y < 1,\ z > 0 \\ 0, & \text{otherwise} \end{cases} = f_Y(y)f_Z(z),$$

where

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad f_Z(z) = \begin{cases} ze^{-z}, & z > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, Y and Z are independent r.v.s with p.d.f.s given above. In particular the p.d.f. of Y is $f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{otherwise}. \end{cases}$

Exercise: Let X_1 and X_2 be iid r.v.s with common p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & -2 < x < -1, \\ \frac{1}{6}, & 0 < x < 3, \\ 0, & \text{otherwise}. \end{cases}$$

Find the p.d.f. of $Y_1 = |X_1| + |X_2|$.

Hint: Define auxiliary variable $Y_2 = |X_1|$. Here $S = ([-2, -1] \cup [0, 3]) \times ([-2, -1] \cup [0, 3])$ and

$$S^0 = (((-2, -1) \cup (0, 3)) \times ((-2, -1) \cup (0, 3))) = S_1^0 \cup S_2^0 \cup S_3^0 \cup S_4^0,$$

where $S_1^0 = (-2, -1) \times (-2, -1)$, $S_2^0 = (-2, -1) \times (0, 3)$, $S_3^0 = (0, 3) \times (-2, -1)$ and $S_4^0 = (0, 3) \times (0, 3)$. On each S_1^0, $\underline{h}(\underline{x}) = (h_1(x_1, x_2), h_2(x_1, x_2)) = (y_1, y_2) = (|x_1| + |x_2|, |x_1|)$ is 1-1. Now proceed.

19.0.3. Moment Generating Function Technique

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with p.m.f. / p.d.f. $f_{\underline{X}}(\cdot)$ and let $g : \mathbb{R}^p \to \mathbb{R}^q$ be a given function. Suppose that we seek probability distribution (p.m.f. / p.d.f.) of $\underline{Y} = g(\underline{X})$. Under the m.g.f. technique, we try to identify the m.g.f. $M_{\underline{Y}}(t)$ of random vector $\underline{Y}$ with the m.g.f. of some known distribution on a rectangle containing origin. Then the uniqueness of m.g.f. as stated in the following theorem, ascertains that $\underline{Y}$ has that known distribution.

Theorem 19.7. *Let $\underline{X}$ and $\underline{Y}$ be 1-dimensional random vectors. Suppose that there exists an $h > 0$ such that*

$$M_{\underline{X}}(\underline{t}) = M_{\underline{Y}}(\underline{t}) \; \forall \, \underline{t} \in (-h, h) \times (-h, h) \times \cdots \times (-h, h).$$

Then $\underline{X} \stackrel{d}{=} \underline{Y}$.

19.0.4. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample (of continuous r.v.s) from a distribution having d.f. F, p.d.f. f and support S. Let $Y_r = r$-th smallest of $X_1, X_2, \ldots, X_n$, $r = 1, 2, \ldots, n$. The Y_r is called the r-th order statistic based on random sample $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are called order statistics based on random sample $X_1, X_2, \ldots, X_n$.

Note that if $X_1, X_2, \ldots, X_n$ are continuous r.v.s then $\Pr(Y_1 < Y_2 < \cdots < Y_n) = 1$ and thus $Y_1, Y_2, \ldots, Y_n$ are uniquely defined with probability one.

Theorem 19.8. *Under the above notation,*

(a) the joint p.d.f. of $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ is

$$g(y_1, y_2, \ldots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f(y_i), & -\infty < y_1 < y_2 < \cdots < y_n < \infty, \\ 0, & \textit{otherwise}, \end{cases}$$

(b) the marginal p.d.f. of Y_r, $r = 1, 2, \ldots, n$ is

$$g_r(y) = \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y), \quad -\infty < y < \infty.$$

Proof. Since $\underline{X} = (X_1, X_2, \ldots, X_n)$ is a continuous random vector $\Pr(Y_1 < Y_2 < \cdots < Y_n) = 1$ (why?). Define $S_n = S \times S \times \cdots \times S$, so that support of $\underline{X} = (X_1, X_2, \ldots, X_n)$ is S_n. Define

$$\begin{aligned} S_1^0 &= \{\underline{x} \in S_n : x_1 < x_2 < \cdots < x_n\}, \\ S_2^0 &= \{\underline{x} \in S_n : x_1 < x_2 < \ldots x_n < x_{n-1}\}, \\ &\vdots \\ S_{n!}^0 &= \{\underline{x} \in S_n : x_n < x_{n-1} < \cdots < x_1\}. \end{aligned}$$

On each S_i^0, $\underline{Y} = (Y_1, Y_2, \ldots, Y_n) = (h_{1,i}(\underline{X}), h_{2,i}(\underline{X}), \ldots, h_{n,i}(\underline{X}))$ is 1-1 with inverse transformation $\underline{h}_i^{-1} = (h_{1,i}^{-1}, h_{2,i}^{-1}, \ldots, h_{n,i}^{-1})$, $i = 1, 2, \ldots, n!$. Note that as a set

$$\{h_{1,i}^{-1}, h_{2,i}^{-1}, \ldots, h_{n,i}^{-1}\} = \{y_1, y_2, \ldots, y_n\}, \quad i = 1, 2, \ldots, n!.$$

Therefore the Jacobian of inverse transformation in each S_i is ± 1.

$$\underline{h}(S_1^0) = \{\underline{y} \in S_n : y_1 < y_2 < \cdots < y_n\} = T_1, \text{ say}.$$

Then the joint p.d.f. of $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$ is

$$g(\underline{y}) = \sum_{j=1}^{n!} f_{\underline{X}}(h_{1,j}^{-1}(\underline{y}), \dots, h_{n,j}^{-1}(\underline{y}))|J_j| I_{\underline{h}(S_j^0)}(\underline{y})$$

$$= \sum_{j=1}^{n!} \left(\prod_{i=1}^{n} f(h_{ij}^{-1}(\underline{y})) \right) | \pm 1| I_{T_j}(\underline{y}) = n! \prod_{i=1}^{n} f(y_i), \ -\infty < y_1 < y_2 < \cdots < y_n < \infty.$$

(b) The marginal p.d.f. of Y_r is

$$g_r(y) = \int_{-\infty}^{y} \int_{-\infty}^{y_{r-1}} \cdots \int_{-\infty}^{y_2} \int_{y}^{\infty} \int_{y_{r+1}}^{\infty} \cdots \int_{y_n}^{\infty} n! f(y_1) \cdots f(y_{r-1}) f(y) f(y_{r+1}) \cdots f(y_n) \mathrm{d}y_n \cdots \mathrm{d}y_{r+1} \mathrm{d}y_1 \cdots \mathrm{d}y_r$$

$$= \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y), \ \ -\infty < y < \infty.$$

Similarly, for $1 \le r < s \le n$, the joint p.d.f. of (Y_r, Y_s) is

$$f_{Y_r} f_{Y_s}(y, z) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(y)]^{r-1} [F(z) - F(y)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y),$$

$-\infty < x < y < \infty$. This completes the proof. □