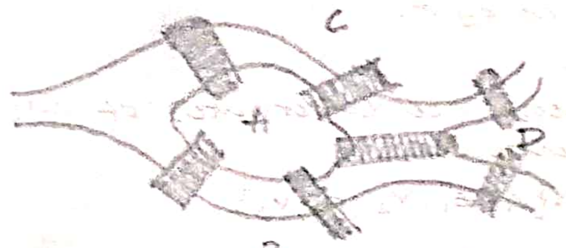
02/08/2022

Discrete Structures II

Quiz - 20
Assignment - 20
Tierce - 60

Book : Rosen

Konigsberg Bridge Problem:



[not a graph as per def. as there are 2 A→B]

A graph is a tuple (V, E) where
V is called vertex set
$E \subseteq V \times V$

Eg: $V = \{1, 2, 3, 4\}$ $E = \{(1,2), (2,3), (3,4), (4,1)\}$

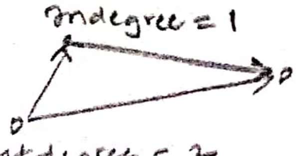
UnDirected graph (facebook) Directed graph (instagram)

Path: A sequence of vertices $s = \{v_1, v_2 ..\}$ such that $(v_i, v_j) \in E \ \forall \ v \in s$
A path is a sequence of eigen $\{e_1, \dots e_n\}$ s.t $e_i = (a, b)$ $e_{i+1} = (b, c) \ \forall \ c \in V$

(.) Two vertices A & B are called adjacent if $(A, B) \in E$

(.) an edge is said to be incident to a vertex V if $e = (v, u)$ for some u.

(.) Degree
Directed:



Total degree = 2 (no of edges)

(Q) can there be a graph where odd number of nodes are there like 2, 3, 2
⟹ NO
(sum of nodal degrees is to be even)

cycle: closed path

connectedness: existance of path between every two points

How to store a graph? (2D array)

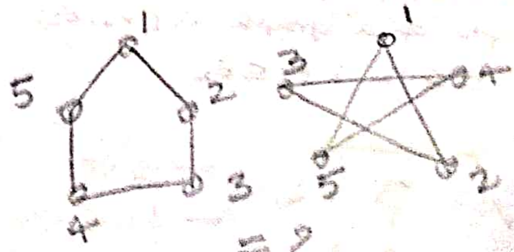
$$\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = s \end{array}$$

No. of max paths from A to C using 2 edges $= s^2$

(Q) Can there be a graph with every ~~matrix~~ node of unique degree? (vertices > 1)

⟹ NO
[Pigeon hole principle, ~~every~~ atleast one pair of points has same degree as there cannot be one '0' and one 'n-1' noded vertex at a time]

$V = \{1, 2, 3, 4, 5\}$
$E = \{(1,2), (2,3), (3,4), (4,5), (5,1)\}$

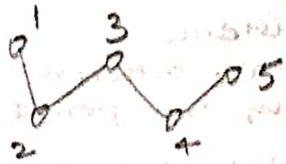


≡ ?

Graph Isomorphism:

Two graphs are isomers if ~~their vertices~~ are same there exists a bijection F that goes from $f: V(G) \to V(H)$ s.t. $(u,v) \in E(G)$ iff $(f(u), f(v)) \in E(H)$

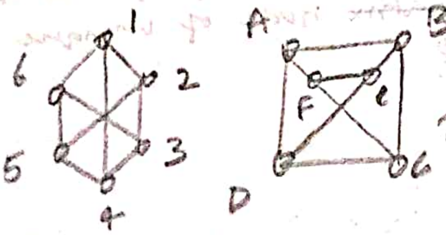
Eg:



$A \to 1$
$B \to 4$
$C \to 2$
$D \to 5$
$E \to 3$

isomorphous

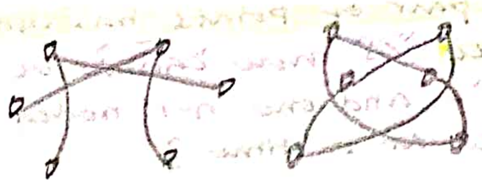
not isomorphous



$A \to 1$

isomorphous

incase of a dense graph, draw a complementary graph



05/08/22

Subgraph:

A graph $H = (V_1, E_1)$ is said to be a subgraph of a graph $G(V_2, E_2)$ if $V_1 \subseteq V_2$; $E_1 \subseteq E_2$

-- graph
— subgraph

Induced subgraph:

$G(V_1; E_1)$

induced subgraph induced by a subset $V_2 \subseteq V_1$ is subgraph (V_2, E_2) such that $(a,b) \in E$ if $a, b \in V_2$, $(a,b) \in E_2$

Note:

Induced subgraph must have all edges of picked vertices but a simple subgraph ~~doesn't~~ is not confined by it.

Walk:

A sequence (alternate) of edges and vertices.

Eg: $\{V_1, E_1, V_2, E_3, V_3\}$

~~Triat~~; Trail:

A walk where no edge is repeated.

Eg: $\{V_1, E_1, V_2, E_3, V_3, E_2, V_2\}$

Eulerian Graphs:

A graph ~~if it~~ that has closed trail that visits every edge of graph.

Properties:

→ If Eulerian, every ~~graph~~ vertex is of even degree

Proof: since it is a closed trial, any inedge must have a corresponding outedge.

$\Rightarrow 2n$

→ If all vertices are of even degree, graph must be Eulerian.

→ If there is a graph where every vertex have degree ≥ 2 then this graph must have a cycle.

Proof: ~~suppose~~ every ~~edge~~ graph edge with $< m$, edges

w $S = (V, E)$ s.t. $|E| = m$

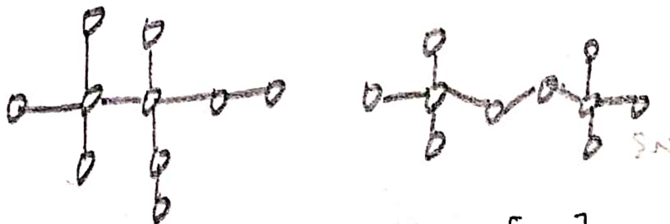
→ Every Eulerian graph can be written as union disjoint cycles of edges

(Q)

Are they isomorphic?

$\Rightarrow$ NO

(Q)



Are they isomorphic? [NO]

(Q) $G = (V, E)$

$|V| = 20$

$|E| = 62$

How many nodes are of degree 7? and of degree 3

$$7x + 3y = (62)2$$
$$x + y = 20$$
$$7x + 3(20 - x) = 124$$
$$4x = 64$$
$$x = 16, y = 4$$

(Q) Will there exist a graph of 07 vertices where every node has 3 degree? [NO]

Diameter:

max dist(u, v) $\forall u, v \in V$

Prove:

(Q) Every 5 vertices, 7 edge simple graph has diameter ≤ 2.

Proof:

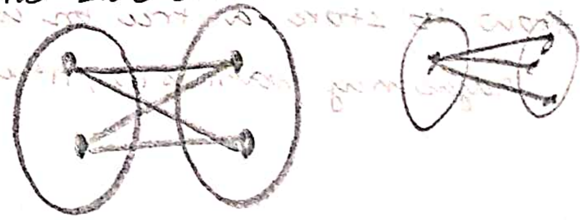
Special Type of Graphs:

Complete Graph:

No. of edges $= \frac{n(n-1)}{2}$ i.e. ${}^nC_2$ of n nodes

Bipartite Graph:

Partition such that one node on one side is connected to node on other side and never with those on same side exists



Bipartite graph

Theorem: All cycles are of even length

Theorem: If for a graph all simple cycles are of even length, then it is bipartite

12/08/22

Trees & Graphs:

Any connected graph without any cycle: Tree

A collection of trees is called a forest.

Note:

(i) B/w any two nodes in a tree, there is ~~to~~ unique

(ii) There is no ~~edge~~ edge in an n-vertex

(iii) A connected graph with n nodes, n-1 edges, must be a tree

(iv) For any tree with $n > 2$, then there is atleast 1 two ~~degree~~ ~~degree~~ degree vertex.

Case-1: ~~......~~ no node of degree 1

→ False

→ degree $\geq 2n$

Case-2: ~~...~~ one node of degree 1

$\sum deg(x) \geq 1 + 2(n-1)$
$= 2n-1 > 2$

How to store a tree on computer?

Adjacency matrices (+), n^2 entries

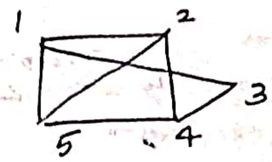
Spanning Tree:

A subgraph ~~...~~ graph if it is a tree & covers all vertices of the graph

Cut of a graph:

~~Partition~~ vertex

Partition in a graph



$\{(1,5), (2,3,4), (5,2)\}$

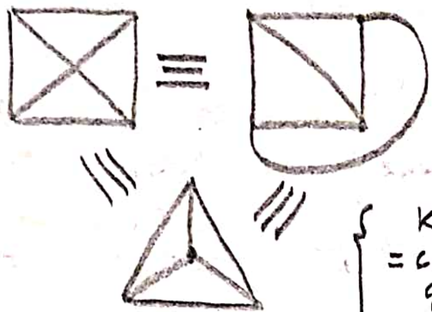
Note:

For any cut $(S, V-S)$ the minimum weighted edge that

23/8/22

Planar Graphs:

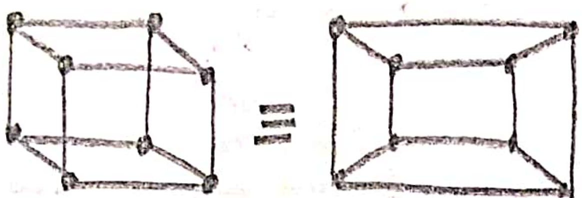
A graph is called planar if it can be drawn in such a way that no edges are crossed.



planar

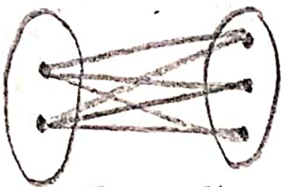
planar

{ K_n = complete graph with n vertices }



planar

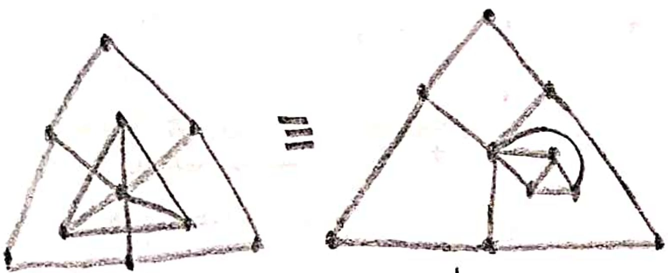
$K_{2,3}$



Bipartite

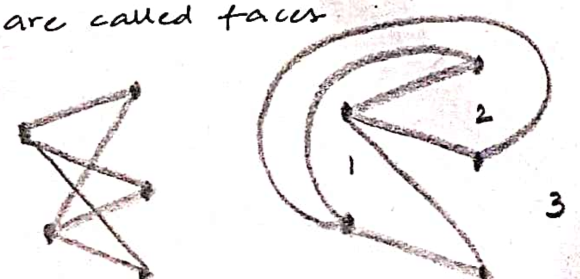
K_5

not planar



planar

(•) A planar graph partitions the plane into several regions, one of which is infinite. These regions are called faces.



3 faces

Euler's formula:

$$|V| + |F| = |E| + 2$$

Base case:

$$1 + 01 = 0 + 2$$

Holds true

Indn hypothesis:

let it holds for n vertices

for (n+1) vertices:

If a tree of 1 face, n viable vertices and corresponding edges, when (n+1)th vertex is removed, then an edge is lost, still preserving

$$|V| + |F| = |E| + 2$$

∴ Hence, proved.

(•) If $|V| \geq 3$, then: $|E| \leq 3|V| - 6$

$$\left[\begin{array}{l} \Rightarrow |V| + |F| = |E| + 2 \\ 3|V| - 6 \geq 3 \\ |V| \bullet |F| - 2 = |E| \geq 1 - |F| \\ \Rightarrow 3|V| - 6 \geq |E| \end{array}\right]$$

$$\boxed{\Sigma\, l(F) = 2E} \geq 3|F|$$

$$2E \geq 3|F|$$

$$2E \geq 3(-V + E + 2)$$

$$|E| \leq 3|V| - 6$$

(•) $|E| \leq 2|V| - 4$

(Q) If G is planar:

(i) Atleast one vertex has degree 8.

(ii) every vertex has degree atleast 5.

Then show that G has atleast 15 vertices.

Sol:

$$8 + 5(n-1) \leq \Sigma d_i$$

$$5n - 3 \leq 2|E|$$

$$|E| \geq \frac{5n-3}{2}$$

$$|E| \leq 3n - 6$$

$$5n + 3 \leq 6n - 12$$

$$\Rightarrow \boxed{n \geq 15}$$

(Q) Planar graph $|V| \geq 3$, then prove that $2n - 4 \geq |F|$

vertices

Sol: $3|V| - 6 \geq |E|$

$|V| + |F| = |E| + 2$

~~$|F| = |E| - |V| + 2$~~

$3|V| - 6$ ~~$|F| = |E| - 2 \geq 3$~~
$\geq |V| + |F| - 2$

~~$|F| = 2 \geq 3 \geq |E| \geq 3|V| - 6$~~

$2|V| - 4 \geq |F|$

~~$|F| \geq 2n - 4$~~

∴ Hence, proved.

~~$|E| + 2 - |F| \geq 3$~~

~~$|E| \geq 3 - 2 + |F|$~~

~~$3|V| - 6 \geq 3 - 2 + |F|$~~

~~$3|V| - 7 \geq |F|$~~

Graph Colouring:

$f : V \rightarrow C$

set $(u, v) \in E$; $f(u) \neq f(v)$

(·) Minimum colouring problem is unsolved. ~~for a planar graph.~~

(·) With $(k+1)$ colours, you can colour any ~~k_n~~ graph with a $(deg_{max} = k)$

(·) Any planar graph can be coloured with 4 colours.

Theorem:

Statement: Any planar graph is 5-colourable.

Proof: ~~For graphs with $n \leq 5$,~~

Base case — atleast one graph exists with 6 vertices

~~one node of degree ≤ 5 exists~~

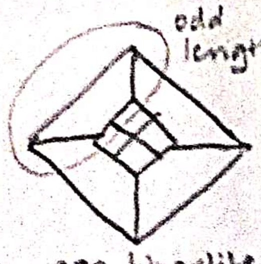
if 2 of neighbours are of same colour, then it is viable.

26/08/22

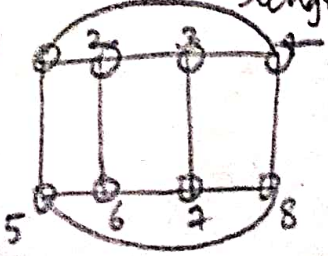
(Q) What is the largest possible number of vertices in a graph with 19 edges and all vertices having degree atleast 3.

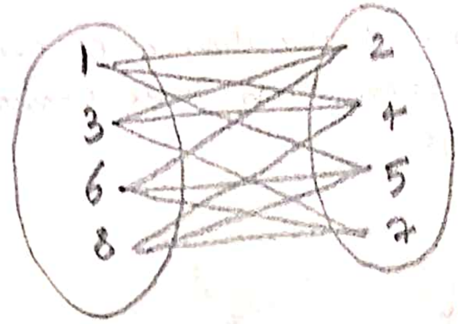
$2(E) = T \cdot d$

(Q) Verify whether Bipartite.

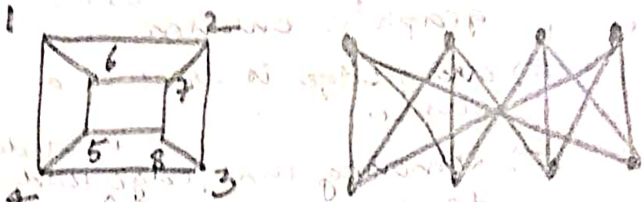


non bipartite

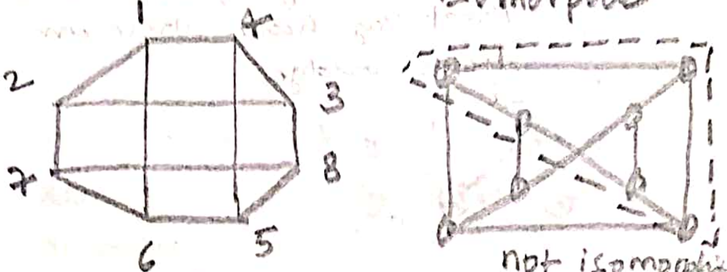




(Q) Check whether Isomorphic.



isomorphic



isomorphic

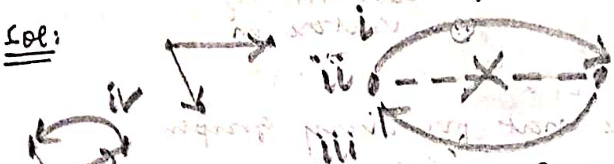
not isomorphic

{ there exists a cycle of odd length }

while others are Bipartite

(Q) How many simple graphs (directed) exist with n vertices?

Sol:



$^nC_2$ choices $\Rightarrow (4^{^nC_2})$

(Q) Let G be a graph where every vertex has degree atleast 'd'. Prove that G has a path of least length d.

Sol:

(Q) Prove that if the degree of every vertex is even, then there exists no bridge

Sol:

since $n \in$ even
graph is Eulerian
$\Rightarrow$ every edge is part of a cycle
$\Rightarrow$ Removing that edge (i.e. bridge) leads to restorating of even-order property, hence there can be no bridge.

(Q)

$\boxed{\text{Eccentricity: } \max (u,v) \text{ dist}}$

$\boxed{\text{Radius: min eccentricity of a vertex in } G}$

i) Prove that for every graph diameter of G:

$$\text{radius}(G) \le \text{diameter}(G) \le 2R(G)$$

ii) Give example for $R = D$ Ans: complete graphs

Sol: i) From definition;
Radius(G) $\le$ diameter(G)

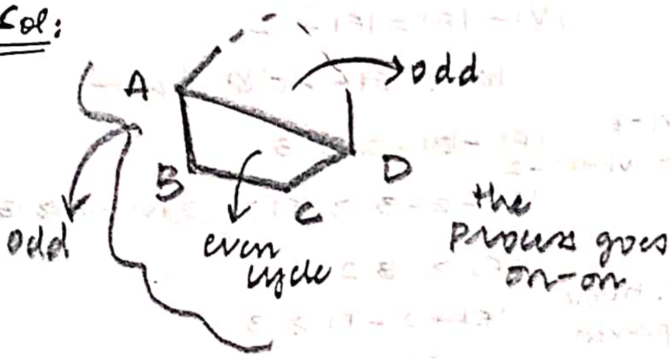
$d \le 2r$

(Q) G be a graph with no induced subgraph P_4 or C_3. Prove that G is bipartite

P_4:

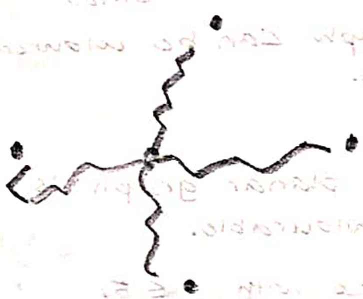
C_3:

Sol:



$\therefore$ At some point the odd cycle will be of length 3 which is a contradiction.
$\Rightarrow$ The graph should have been of even length
$\Rightarrow$ Bipartite

(Q) Tree, one vertex of degree k, P.T. there must be atleast k nodes of degree 1.



02/09/2022

Probabilistic methods in combinatrics:

Probabiliny space: discrete space

Cut of a graph:

Objective: Find the cut with max. number of cut edges

Theorem:
There exists a cut with atleast ($|E|/2$) edges

Theorem:
Every graph has a bipartite sub graph

Let G be a graph with every vertex placed in either of a set of 2 with a probability '1/2'.

$x_i = 1$; if a cut edge

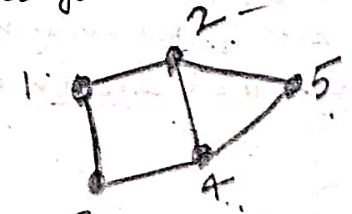
$x_i = 0$; otherwise

$$X = \sum_{e \in E} x_e$$

$$E(X) = \sum_{e \in E} E(x_e) = \sum_{e \in E} P(x_e = 1) + \sum_{e \in E} 0\, P(x_e = 0) = |E|/2$$

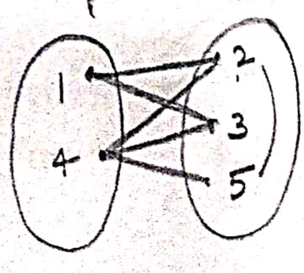
⇒ There exists atleast one cut with atleast $|E|/2$ edges.

(Q)



2 - 5 - 1
2 - 1 - 1
2 - 3 - 2
2 - 4 - 1

Sol:



⇒ $\deg(v) = d_{in}(v) + d_{out}(v)$

$\forall v \in S$

$d_{out}(v) \geq d_{in}(v)$

$d_{out}(v) \geq \deg(v)/2$

$$\sum_{v \in V} d_{out}(v) = 2 \times (\text{cut edge}(v))$$

$\sum d_{out}(v) \geq E$

$2 \times \text{cut edge} \geq E$

$\text{cut edges} \geq E/2$

Ramsey Number:

Plot a graph of 6 vertices. Upon colouring the graph with only red and blue, there shall always exist a triangle of red or blue.

The minimum number 'n' s.t. any colouring (two) of the edges of K_n have either a K_k ~~or~~ an induced subgraph or a K_l, as an induced subgraph, is called Ramsey Number.

Erdos Number:

Theorem: $R(x, x) \geq 2^{|x/2|}$

Proof: Fix n

Do a random two colouring of the edges of K_n

Let R be any subset of V

$|R| = k$

A_R = induced subgraph on A_R has all the edges of same colour.

$$P(A_R) = \frac{2}{2^{\binom{k}{2}}}$$

Prob that atleast one such R exists is:

$$P(\cup A_R) \leq \sum_{R \subseteq V} P(A_R) \leq \binom{n}{k} \cdot \frac{2}{2^{\binom{k}{2}}} < 1$$

$$\Rightarrow n = R(k, k) \geq 2^{|k/2|}$$

(Q) 1600 students, 16000 teams are formed. Each team has 80 stud. P.T. there always exist 2 teams where 4 members are common.

Sol:

$\frac{\cancel{1600}}{\cancel{80}} = \cancel{20 \text{ teams}}$

Pick 2 teams at random

$x_i = 1$ If ith student is in other team

$= 0$ otherwise.

$X = x_1 + x_2 + \dots + x_{1600}$

R.T.P: $E(X) \geq 4$

$E(X) = \sum_{1}^{1600} E(x_i)$

let n_i be the no of teams where ith student belongs.

$E(x_i) = ({}^{n_i}c_2) / ({}^{16000}c_2)$

$\sum_{i=1}^{1600} n_i = 16000 \times 80$

$\bar{n} = \frac{16000 \times 80}{1600} = 800$

$E(X) \geq \left(\frac{1600 \times {}^{\bar{n}}c_2}{{}^{16000}c_2} \right)$

$E(X) > 3$

$\implies E(X) = 4$

$\therefore E(X) \geq 4$

6/9/22 Independent set:

A subset $V' \subset V$ is said to be an independent set if $\forall\ u,v \in V'$, $(u,v) \notin E$

Maximum Independent Set Problem: (NP-hard)

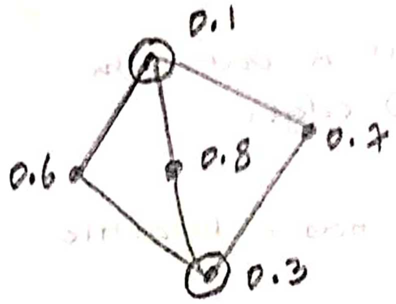
Theorem: Every graph has independent set of size atleast k.

$k = \sum_{v \in V} \frac{1}{\deg(v)+1}$

Proof: For every node $v \in V$ assign some weight to v uniformly randomly in $[0,1]$

Local min: $c(u) \geq c(v)$ $\forall (u,v) \in E$

claim: calculation of local minm forms an independent set.



Independent set = {0.1, 0.3}

Theorem:

let $X_i = 1$, if corresponding node is local minm

$= 0$, otherwise.

$X = \sum_{v_i \in V} x_i$

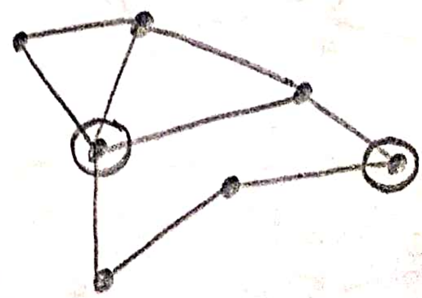
$E(X) = \sum_{v_i \in V} E(x_i) = \sum_{v_i \in V} P(x_i = 1)$

$= \sum_{v_i \in V} \frac{1}{\deg(v)+1}$

$\implies \exists$ an independent set of size atleast $\sum_{v \in V} \frac{1}{d(v)+1}$

Dominating Set:

A subset $V' \in V$ is said to be a dominating set if for every vertex $v \in V, V'$ $\exists$ a vertex $u \in V'$ s.t. $(u,v) \in E$



{⊙}, {•}

Minimum Dominating Set Problem:

Every graph with a minimum degree 'δ', has a dominating set of size $\leq \dfrac{n(1+\log(\delta+1))}{(1+\delta)}$

Proof: $D = \phi$ (let)

For every vertex s, put s in D with probability P.

let X be a set of nodes who don't have any neighbour in D.

Include X in D.

let $X_i = 1$, if V_i is in $D \cup X$

$= 0$ otherwise.

$$X = \sum x_i = \sum_D x_i + \sum_X x_i$$

$v \in D$, $E(x_i) = P = np$

$E(Y_i)$ = Prob that ~~vertex~~ node v_i and not of its neighbours ~~are~~ are also not picked.

$$= (1-P)^{(deg(v)+1)} \leq (1-p)^{\delta+1}$$

$$E(x) \leq np + n(1-p)^{\delta+1}$$

$$(1-p)^{\delta+1} \leq e^{-p(\delta+1)} \Rightarrow (1-p) \leq e^{-p}$$

$$\Rightarrow E(x) \leq np + n(e^{-p})^{\delta+1}$$

$$\leq n(p + e^{-p-p\delta})$$

$$f(p) = p + e^{(1+\delta)(-p)}$$

$$\Rightarrow 1 + -(1+\delta)e^{(1+\delta)(-p)} = 0$$

$$1 = (1+\delta)e^{(1+\delta)(-p)}$$

$$\frac{1}{(1+\delta)} = e^{-p(1+\delta)}$$

$$-\log(1+\delta) = -p(1+\delta)$$

$$\frac{\log(1+\delta)}{(1+\delta)} = p$$

$$E(x) \leq n\left(\frac{\log(1+\delta)+1}{1+\delta}\right)$$

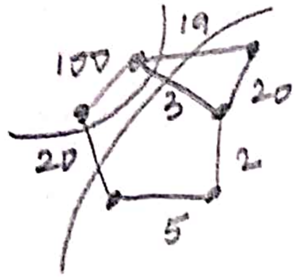
Min cut:

Sum of all the vertices on the cut vertices is minimum.

Partition of V into V_1, V_2 s.t.

$$\sum_{e \in (V_1, V_2)} w(e) \text{ is minimum}$$

s-t cut problem:



$120 > 66$

If 6 edges, 6 variables are req:

$x_e = 1$, if e is a cut edge

0, otherwise

$$\min \sum_{e \in E} x_e \cdot w(e)$$

~~Constants~~ constraints:

$$\forall P_{s,t} \sum_{e \in P_{s,t}} x_e \geq 1$$

$$\forall x_e \in \{0,1\}$$

ILP

LP

$\Rightarrow$ ILP $>$ LP

$$E(\text{size of cut}) = \sum_e E(x_e \text{ is a cut edge}) \times w_e$$

$$= \sum_e x_i w_e \leq ILP$$