MA202: Calculus II

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Lecture Notes

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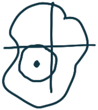
Lecture 2

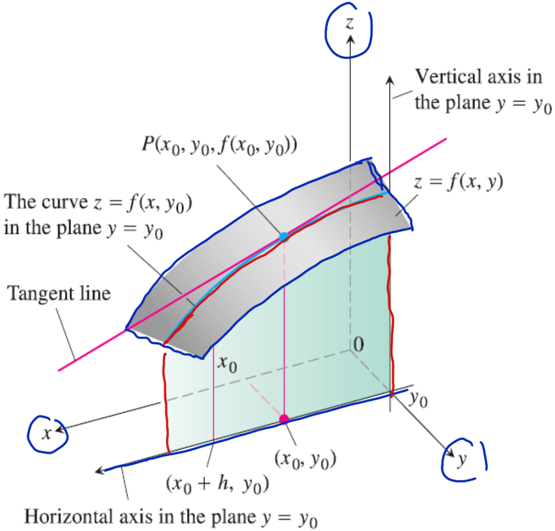
Partial Differentiation

Recall: Let $D \subset \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to have a **derivative** at c if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, and then it is denoted by $f'(c)$.



1. If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$.
2. We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.
3. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used.

Partial Differentiation



Partial Differentiation

Definition

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D.
A function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** x at (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, and then it is denoted by $f_x(x_0, y_0)$ or by $\boxed{\dfrac{\partial f}{\partial x}(x_0, y_0).}$

Similarly, a function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** y at (x_0, y_0) if

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

exists, and then it is denoted by $f_y(x_0, y_0)$ or by $\boxed{\dfrac{\partial f}{\partial y}(x_0, y_0).}$

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$. Partial derivatives of f exist at all points of $\mathbb{R}^2$. In fact, for $(x_0, y_0) \in \mathbb{R}^2$,

$$f_x(x_0, y_0) = 2x_0 \quad \text{and} \quad f_y(x_0, y_0) = 2y_0.$$

(ii) Let $f(x,y) := \sqrt{x^2 + y^2}$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0, y_0) \neq (0,0)$. Then

$$f_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \quad \text{and} \quad f_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}.$$

But f does not have either partial derivative at $(0,0)$ since $\lim_{h \to 0} |h|/h$ does not exist. Note: f is continuous at $(0,0)$.

(iii) Let $f(x,y) := xy/(x^2 + y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) = 0$. It is easy to see that $f_x(0,0) = 0 = f_y(0,0)$. We have already seen that f is not continuous at $(0,0)$.

Let $D \subset \mathbb{R}^2$, and $f : D \to \mathbb{R}$. Suppose $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$.

Also, if $f_x : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{xy}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

Similarly, we define $f_{yy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$, or $\dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$.

In general, the **mixed partial derivatives** $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ may not be equal.

$$f(x,y) = \begin{cases} xy\dfrac{x^2+y^2}{x^2-y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Calculate $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

$$f_{xy}(0,0) = \lim_{K\to 0} \frac{f_x(0,K) - f_x(0,0)}{K} = \lim \frac{-K-0}{K} = -1$$

$$f_x(0,K) = \lim_{h\to 0} \frac{f(h,K) - f(0,K)}{h}$$

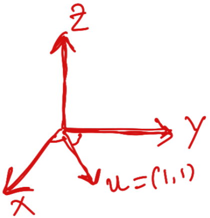
$x_0 = 0$

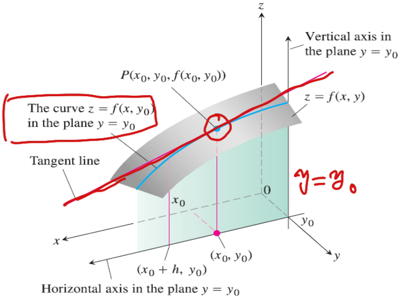
$y_0 = K$ $= \lim_{h\to 0} K\dfrac{h^2+K^2}{h^2-K^2} = -K$

Partial Differentiation

Let $f : D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function where both the partial derivatives exist at (x_0, y_0).

1. Geometrically, $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the tangent to the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$ at $(x_0, y_0, f(x_0, y_0))$.





2. Physically, $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$ is the rate of change in f at (x_0, y_0) along x-axis.

$\frac{\partial f}{\partial y}$

Gradien

If both the partial derivative exist of $f(x, y)$ at (x_0, y_0) then the column vector

$$\begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{pmatrix}$$

is called the gradient of f and denoted by $\bigtriangledown f(x_0, y_0)$.

Directional Derivative

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. $\sqrt{u_1^2+u_2^2}=1$

Let $\boldsymbol{u} := (u_1, u_2) \in \mathbb{R}^2$ be a **unit vector**, that is, $\boxed{\|\boldsymbol{u}\|} = 1$.

Definition

A function $f : D \to \mathbb{R}$ is said to have a **directional derivative along $\boldsymbol{u}$** at (x_0, y_0) if

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

exists, and then it is denoted by $(\boldsymbol{D_u} f)(x_0, y_0)$.

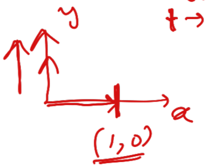
Directional Derivative

Important points

1. Directional derivative of f along u is the measure of rate of change of f as (x, y) changes in the direction u.
2. If $u = (1, 0)$ then directional derivative along u (i.e., x-axis) is same as $f_x(x_0, y_0)$ and if $u = (0, 1)$ then directional derivative along u (i.e., x-axis) is same as $\boxed{f_y(x_0, y_0.)}$

$u = (1,0)$

$$\underset{t \to 0}{Lt} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}$$



Examples:

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0,y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1,u_2)$ be a unit vector. For $t \neq 0$,

$u_1^2+u_2^2=1$

$$\frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t}$$

$$= \frac{(x_0+tu_1)^2 + (y_0+tu_2)^2 - x_0^2 - y_0^2}{t}$$

$$= \boxed{2x_0u_1 + 2y_0u_2 + t.}$$

$f_x(x_0,y_0)$ $\quad u_1=1,\ u_2=0 \quad (1,0)$

Letting $t \to 0$, we obtain

$$(\boldsymbol{D}_{\boldsymbol{u}}f)(x_0,y_0) = \boxed{2x_0u_1 + 2y_0u_2.} = 2x_0$$

Directional Derivative

(ii) Let $f(x,y) := \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$. Let $(x_0,y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1,u_2)$ be a unit vector. For $t \neq 0$,

$$\boxed{\frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t}}$$
$$= \frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} - \sqrt{x_0^2+y_0^2}}{t}$$
$$= \boxed{\frac{2x_0u_1 + 2y_0u_2 + t}{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} + \sqrt{x_0^2+y_0^2}}}.$$

Hence if $(x_0,y_0) \neq (0,0)$, then

$$(\boldsymbol{D}_{\boldsymbol{u}}f)(x_0,y_0) = \boxed{\frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}}} = (\nabla f)(x_0,y_0)\cdot \boldsymbol{u}.$$

$$\frac{t}{\sqrt{t^2u_1^2+t^2u_2^2}} = \frac{t}{|t|}$$

But $\boxed{(D_u f)(0,0)}$ does not exists.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that f has directional derivative at $(0,0)$ along any direction $u = (u_1, u_2)$ where $u_1^2 + u_2^2 = 1$ but f is not continuous at $(0,0)$.

$$(D_u f)(0,0) = \lim_{t \to 0} \boxed{\frac{f(tu_1, tu_2) - f(0,0)}{t}} = \begin{cases} \dfrac{u_1^2}{u_2}, & \boxed{u_2 \neq 0} \\ 0, & u_2 = 0 \end{cases}$$

$$= \lim_{t \to 0} \frac{t^3 u_1^2 u_2}{t\,(t^4 u_1^4 + t^2 u_2^2)}$$

Important note

Even when f has directional derivatives at a point (x_0, y_0) along all arbitrary direction still f may not be continuous!

Differentiability of a function of two variables

1. Directional derivative of f at a point (x_0, y_0) gives us the rate of change of f at (x_0, y_0) in a particular direction,
2. Now we are interested in the complete information about the rate of change in f at (x_0, y_0)

Let us recall the one variable situation. If $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and c is an interior point of D, then the derivative of f at c is

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

We note that if $f : D \to \mathbb{R}$, c is an interior point of D and $\alpha \in \mathbb{R}$, then $\boxed{f'(c) = \alpha} \in \mathbb{R}$ if and only if

$$\boxed{\lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{h} = 0,}$$

that is,

$$\boxed{\lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{|h|} = 0.}$$

$|h| =: h$

Differentiability of a function of two variables

If $D \subseteq \mathbb{R}^2$, and (x_0, y_0) is an interior point of D then in a similar fashion as above the following equation is well-defined (where $\alpha, \beta \in \mathbb{R}$)

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \boxed{\alpha h - \beta k}}{||(h,k)||} = 0.$$

Definition

We say that f is differentiable at (x_0, y_0) if there is $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0.$$

In this case the pair $(\alpha, \beta) \in \mathbb{R}^2$ is called the total derivative of f at (x_0, y_0).

Differentiability of a function of two variables

1. In the case of single variable we assume that $\alpha = f'(c)$.
2. Natural question arises about the quantities α and β
3. Letting $(h, k) \to (0, 0)$ along x-axis (i,e, $h \to 0$ and $k = 0$) we get

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0) - \alpha h}{|h|} = 0$$

This is same as

$$\boxed{\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{|h|} = \alpha} \quad f_x(x_0, y_0)$$

Hence $f_x(x_0, y_0) = \alpha$

4. Similarly letting $(h, k) \to (0, 0)$ along y-axis we can get $\boxed{\beta = f_y(x_0, y_0)}$.
5. If f is differentiable at (x_0, y_0) then $(\alpha, \beta)^t = (f_x(x_0, y_0), f_y(x_0, y_0))^t$ is called the total derivative of f. (This is the gradient of f if we write as a column vector.)

Differentiability of a function of two variables

Proposition (Increment Lemma)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Then $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) if and only if there exist functions $f_1, f_2 : D \to \mathbb{R}$ such that f_1 and f_2 are continuous at (x_0, y_0), and for all $(x, y) \in D$,

$$f(x,y) - f(x_0, y_0) = (x - x_0)f_1(x,y) + (y - y_0)f_2(x,y).$$

In this case, $\nabla f(x_0, y_0) = \left(f_1(x_0, y_0), f_2(x_0, y_0)\right)$.

A pair (f_1, f_2) of functions stated in the Increment Lemma is called a pair of increment functions associated with the function f and the point (x_0, y_0).

Differentiability of a function of two variables

Proposition (Differentiability $\Longrightarrow$ Continuity)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). Then f is continuous at (x_0, y_0).

Proof:

If (f_1, f_2) is a pair of increment functions associated with f and (x_0, y_0), then

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y)$$

for all $(x, y) \in D$. Consequently, the continuity of f at (x_0, y_0) follows from the continuity of f_1 and f_2 at (x_0, y_0). $\square$

Differentiability: Necessary & sufficient conditions

If $D \subseteq \mathbb{R}^2$, where $f : D \to \mathbb{R}$ and (x_0, y_0) is an interior point of D. Then if f is differentiable at (x_0, y_0), the following conditions are necessary

Necessary conditions

1. Both the partial derivatives f_x and f_y exist at (x_0, y_0)
2. f is continuous at (x_0, y_0)

Sufficient condition

Let $D \subseteq \mathbb{R}^2$, where $f : D \to \mathbb{R}$ and (x_0, y_0) is an interior point of D. Suppose one of the partial derivatives exists at (x_0, y_0) and other is continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).

How to check differentiability

1. Whether f violates any necessary conditions (to check f is not differentiable)
2. whether f satisfies all the sufficient conditions. Also the definition of differentiability can be checked.

1. $f(x,y) = \sqrt{(x^2+y^2)}$, $(x,y) \in \mathbb{R}^2$. f is continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ do not exist. So f is not differentiable at $(0,0)$. However f is differentiable all other point in $\mathbb{R}^2$.
2. $f(x,y) = 1$ if $0 < y < x^2$ and $f(x,y) = 0$, otherwise. f is not continuous at $(0,0)$. (check it!) So f is not differentiable at $(0,0)$.

Check whether $\boxed{f(x,y) = \sqrt{|xy|}}$ is differentiable at $(0,0)$ or not. We use the definition of differentiability here.

$$f(x,y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y}, & (x,y) \neq (0,0) \\ x^2 \sin\dfrac{1}{x}, & y = 0, \quad x \neq 0 \\ y^2 \sin\dfrac{1}{y}, & x = 0, \quad y \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$$

Calculate $f_x(x,y)$ and $f_y(x,y)$. Also check the differentiability of f.