CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Circuits and Protocols

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Quantum Teleportation

Opposite Analogue of Superdense Coding

Problem Definition

- Alice and Bob are in different parts of the world.
- **Need**: Alice wants to send a qubit $(\alpha |0\rangle + \beta |1\rangle)$ to Bob.
- **Constraint:** Alice can only send **classical bits** to accomplish this task. How many bits are required?
- **Fact:** *Alice could send approximations of α and β to Bob. However, Alice may not know α and β, and she may not be able to perform measurements on her qubit that would reveal these numbers. (And then there is entanglement.)*
- **Verdict:** There is no way Alice can do this with **only** classical bits without *additional resources*

The first reference to the investigation of this protocol was due to Bennett et al. in 1993. It was experimentally realized in 1997 by two research groups, led by Sandu Popescu and Anton Zeilinger, respectively.

Pre-shared Entangled Resource

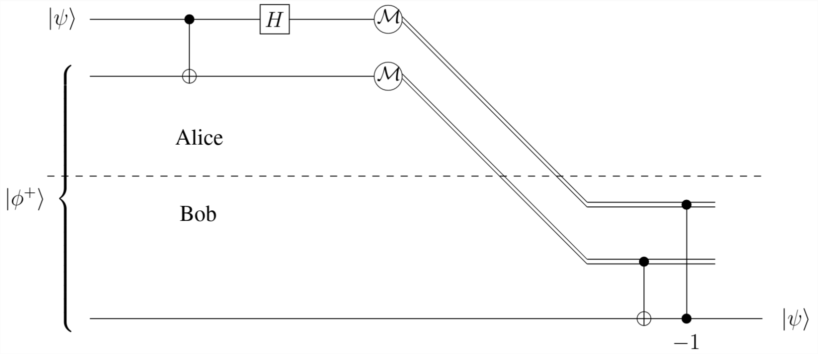
Share an $e-$bit

- **Additional Resource:** One pre-shared unit of entanglement
- Alice and Bob prepare two qubits A and B in the superposition:

$$\boxed{\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle}$$

  Alice takes qubit A and Bob takes the qubit B.
- **Implication:** *Given the additional resource of a shared e−bit of entanglement, Alice will be able to transmit a qubit to Bob using two bits of classical information.*

Quantum Circuit Diagram



Initial State

- State of qubit to be transmitted to Bob:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Initial state:

$$\overbrace{(\alpha\,|0\rangle + \beta\,|1\rangle)}^{\text{To Transmit}}\overbrace{\left(\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle\right)}^{\text{Shared e-bit}}$$

- State of qubit to be transmitted to Bob:

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$$\boxed{=\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)}$$

Step-1: Apply CNOT

- Initial state:

$$\frac{1}{\sqrt{2}}\left(\alpha\,|\overset{\text{Control}}{0}\overset{\text{Target}}{0}0\rangle+\alpha\,|\overset{\text{Control}}{0}\overset{\text{Target}}{1}1\rangle+\beta\,|\overset{\text{Control}}{1}\overset{\text{Target}}{0}0\rangle+\beta\,|\overset{\text{Control}}{1}\overset{\text{Target}}{1}1\rangle\right)$$

- After CNOT

$$\frac{1}{\sqrt{2}}(\alpha\,|000\rangle+\alpha\,|011\rangle+\beta\,|110\rangle+\beta\,|101\rangle)$$

- H on Qubit-1
- After CNOT

$$\frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle)$$

- After Hadamard transform:

- H on Qubit-1
- After CNOT

$$\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

- After Hadamard transform:

$$\frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle$$
$$+ \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$$

$$\boxed{\begin{aligned} =&\frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \\ &\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle) \end{aligned}}$$

Bob's qubit seems to depend on α and β

- State after $H-$transform

$$\frac{1}{2}|00\rangle \overbrace{(\alpha|0\rangle + \beta|1\rangle)}^{\text{Bob's Qubit}} + \frac{1}{2}|01\rangle \overbrace{(\alpha|1\rangle + \beta|0\rangle)}^{\text{Bob's Qubit}} +$$
$$\frac{1}{2}|10\rangle \overbrace{(\alpha|0\rangle - \beta|1\rangle)}^{\text{Bob's Qubit}} + \frac{1}{2}|11\rangle \overbrace{(\alpha|1\rangle - \beta|0\rangle)}^{\text{Bob's Qubit}}$$

- Can Bob exploit this? **Answer**: Not Actually
- Bob *cannot*[1] transform and measure his qubit **alone** at this point to learn anything about α and β.

[1] No measurements or communication have been made at this point.

- State after $H-$transform

$$\frac{1}{2}\overset{\overset{M}{\downarrow}}{|00\rangle}(\alpha|0\rangle+\beta|1\rangle)+\frac{1}{2}\overset{\overset{M}{\downarrow}}{|01\rangle}(\alpha|1\rangle+\beta|0\rangle)+$$
$$\frac{1}{2}\overset{\overset{M}{\downarrow}}{|10\rangle}(\alpha|0\rangle-\beta|1\rangle)+\frac{1}{2}\overset{\overset{M}{\downarrow}}{|11\rangle}(\alpha|1\rangle-\beta|0\rangle)$$

- Alice needs to measure[2] her qubits
- She communicates the resultant **classical bits** to Bob

[2]Note: This is a partial measurement considering the 3-qubit system
The **distribution** of measurement outcomes and the **resulting state** of Bob's qubit after the measurements hold the key to the success of this protocol

Step-4: Case 1 Alice measures 00

- Happens with probability

$$\left\| \frac{1}{2}(\alpha |0\rangle + \beta |1\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|00\rangle (\alpha |0\rangle + \beta |1\rangle)$$

- Alice transmits the classical bits 00 to Bob

Step-4: Case 1 — Alice measures 00

- Happens with probability

$$\left\| \frac{1}{2}(\alpha\,|0\rangle + \beta\,|1\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|00\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle)$$

- Alice transmits the classical bits 00 to Bob
- Bob performs no other operations since both received bits are zero.
- **Final State**: Bob's qubit remains in the state $(\alpha\,|0\rangle + \beta\,|1\rangle)$ at the end of the protocol.

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \xrightarrow{I} (\alpha\,|0\rangle + \beta\,|1\rangle)$$

- Happens with probability

$$\left\| \frac{1}{2}(\alpha |1\rangle + \beta |0\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|01\rangle (\alpha |1\rangle + \beta |0\rangle)$$

- Alice transmits the classical bits 01 to Bob

Step-4: Case 2 — Alice measures 01

- Happens with probability

$$\left\|\frac{1}{2}(\alpha\,|1\rangle + \beta\,|0\rangle)\right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|01\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle)$$

- Alice transmits the classical bits 01 to Bob
- Bob performs a NOT operation on his qubit as first bit received $\leftarrow 0$ and second bit $\leftarrow 1$
- **Final State**: Bob's qubit remains in the state $(\alpha\,|0\rangle + \beta\,|1\rangle)$ at the end of the protocol.

$$(\alpha\,|1\rangle + \beta\,|0\rangle) \xrightarrow{NOT} (\alpha\,|0\rangle + \beta\,|1\rangle)$$

Step-4: Case 3 Alice measures 10

- Happens with probability

$$\left\| \frac{1}{2}(\alpha |0\rangle - \beta |1\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|10\rangle (\alpha |0\rangle - \beta |1\rangle)$$

- Alice transmits the classical bits 10 to Bob

Step-4: Case 3 — Alice measures 10

- Happens with probability

$$\left\| \frac{1}{2}(\alpha \left|0\right\rangle - \beta \left|1\right\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$\left|10\right\rangle (\alpha \left|0\right\rangle - \beta \left|1\right\rangle)$$

- Alice transmits the classical bits 10 to Bob
- Bob performs a σ_z operation on his qubit as first bit received $\leftarrow 1$ and second bit $\leftarrow 0$
- **Final State**: State of Bob's qubit becomes $(\alpha \left|0\right\rangle + \beta \left|1\right\rangle)$ at the end of the protocol.

$$(\alpha \left|0\right\rangle - \beta \left|1\right\rangle) \xrightarrow{\sigma_z} (\alpha \left|0\right\rangle + \beta \left|1\right\rangle)$$

- Happens with probability

$$\left\| \frac{1}{2}(\alpha |1\rangle - \beta |0\rangle) \right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$|11\rangle (\alpha |1\rangle - \beta |0\rangle)$$

- Alice transmits the classical bits 11 to Bob

Step-4: Case 4 — Alice measures 11

- Happens with probability

$$\left\|\frac{1}{2}(\alpha\left|1\right\rangle - \beta\left|0\right\rangle)\right\|^2 = \frac{1}{4}$$

- Implication: State of the three qubits now becomes

$$\left|11\right\rangle(\alpha\left|1\right\rangle - \beta\left|0\right\rangle)$$

- Alice transmits the classical bits 11 to Bob
- Bob performs a NOT operation on his qubit and then performs a σ_z as both bits received are 1.
- **Final State**: Bob's qubit changes state

$$(\alpha\left|1\right\rangle - \beta\left|0\right\rangle) \xrightarrow{NOT} (\alpha\left|0\right\rangle - \beta\left|1\right\rangle) \xrightarrow{\sigma_z} (\alpha\left|0\right\rangle + \beta\left|1\right\rangle)$$

Verdict

Final State: $\alpha |0\rangle + \beta |1\rangle$

- In all four cases, Bob's qubit is in the state $(\alpha |0\rangle + \beta |1\rangle)$ at the end of the protocol

Bits	State After M	State After CNOT[3]	State After C-σ_z[4]
00	$\|00\rangle (\alpha \|0\rangle + \beta \|1\rangle)$	$\|00\rangle (\alpha \|0\rangle + \beta \|1\rangle)$	$\|00\rangle (\alpha \|0\rangle + \beta \|1\rangle)$
01	$\|01\rangle (\alpha \|1\rangle + \beta \|0\rangle)$	$\|01\rangle (\alpha \|0\rangle + \beta \|1\rangle)$	$\|01\rangle (\alpha \|0\rangle + \beta \|1\rangle)$
10	$\|10\rangle (\alpha \|0\rangle - \beta \|1\rangle)$	$\|10\rangle (\alpha \|0\rangle - \beta \|1\rangle)$	$\|10\rangle (\alpha \|0\rangle + \beta \|1\rangle)$
11	$\|11\rangle (\alpha \|1\rangle - \beta \|0\rangle)$	$\|11\rangle (\alpha \|0\rangle - \beta \|1\rangle)$	$\|11\rangle (\alpha \|0\rangle + \beta \|1\rangle)$

Stronger Implication

- Prior entanglement of Alice's initial qubit will be *preserved*.
  - Teleportation $\implies$ a **perfect** quantum channel[5]

[3] Control - Second bit, Traget - Third Qubit

[4] Control - First Qubit, Target - Third Qubit

[5] Exactly like the physical transport of Alice's qubit to Bob.

- Repeat the QT Protocol by considering the scenario where Alice's qubit has a prior entanglement.

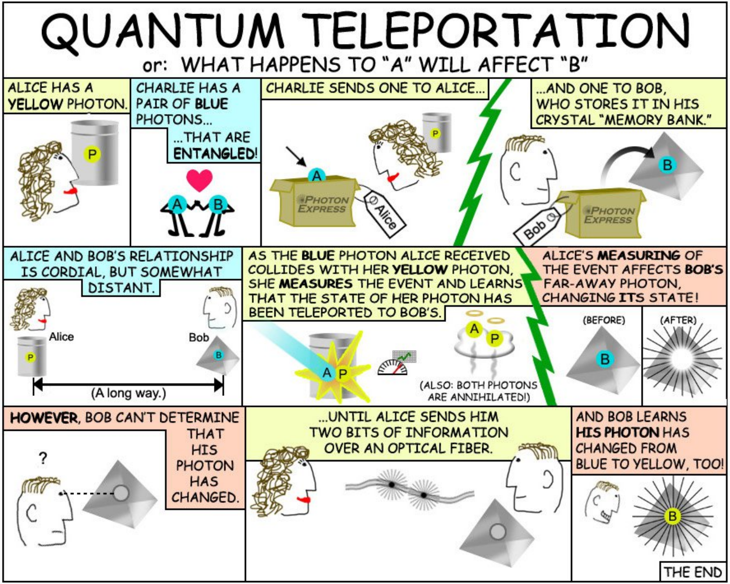

Image Source: https://www.jpl.nasa.gov/news/

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