CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Search

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Quantum Search

A Simple Searching Problem

A Special Family of Functions

f_{00}

input	output
00	1
01	0
10	0
11	0

f_{01}

input	output
00	0
01	1
10	0
11	0

f_{10}

input	output
00	0
01	0
10	1
11	0

f_{11}

input	output
00	0
01	0
10	0
11	1

- $f \in \{f_{00}, f_{01}, f_{10}, f_{11}\}$
- f takes value 1 on one input and value 0 on all others

Goal — **A Simple Search Problem**

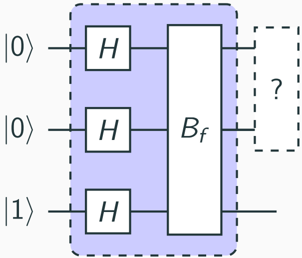
Find the input on which the function takes value 1.

- Constraint: Access to the function is restricted to evaluations of the equivalent quantum transformation B_f

Definition

An evaluation of B_f on some input (quantum or classical) is a query.

- How many queries are necessary and sufficient to solve the problem?
  - Classical: **Three.**
  - Quantum: **One!**



Approach similar to Deutsch's Circuit

Circuit Diagram

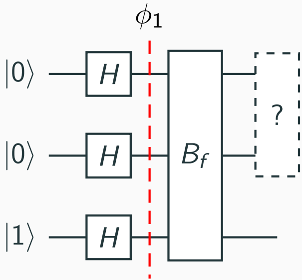
Part of the circuit to be revealed later.

A More General Formulation

- Let $f : \{0,1\}^n \rightarrow \{0,1\}^m$ be any function for +ve integers n and m
- The associated quantum transformation U_f is given as:

$$U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus f(x)\right\rangle$$

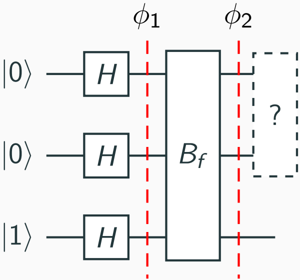
- The associated matrix will always be a **permutation** matrix, and is therefore **unitary**.



- Initial Step: $|001\rangle$
- After H-transforms:

$$
\begin{aligned}
H \otimes H \otimes H\,|001\rangle &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2}\Big(\,|00\rangle + |01\rangle + |10\rangle + |11\rangle\,\Big)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)
\end{aligned}
$$

Step-2: ϕ_2 Circuit Slicing



- After applying B_f

$$\left(\frac{1}{2}(-1)^{f(00)}|00\rangle + \frac{1}{2}(-1)^{f(01)}|01\rangle + \frac{1}{2}(-1)^{f(10)}|10\rangle + \frac{1}{2}(-1)^{f(11)}|11\rangle\right)\overbrace{\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)}^{\text{Unaltered by } B_f}$$

Following arguments similar to the analysis of Deutsch's Algorithm. Invoking the phase kickback effect

$$\left(\frac{1}{2}(-1)^{\overset{=1}{f(00)}} |00\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(01)}} |01\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(10)}} |10\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(11)}} |11\rangle\right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = \boxed{-\frac{1}{2}} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$\left(\frac{1}{2}(-1)^{\overset{=0}{f(00)}}|00\rangle + \frac{1}{2}(-1)^{\overset{=1}{f(01)}}|01\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(10)}}|10\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(11)}}|11\rangle\right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = \boxed{-\frac{1}{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2}|00\rangle \boxed{-\frac{1}{2}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

If $f = f_{10}$ — Understanding ϕ_2

$$\left(\frac{1}{2}(-1)^{\overset{=0}{f(00)}}|00\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(01)}}|01\rangle + \frac{1}{2}(-1)^{\overset{=1}{f(10)}}|10\rangle + \frac{1}{2}(-1)^{\overset{=0}{f(11)}}|11\rangle\right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = \boxed{-\frac{1}{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2}|00\rangle \boxed{-\frac{1}{2}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{10} \implies |\phi_{10}\rangle = +\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle \boxed{-\frac{1}{2}}|10\rangle + \frac{1}{2}|11\rangle$$

$$\left(\tfrac{1}{2}(-1)^{\overbrace{f(00)}^{=0}}|00\rangle + \tfrac{1}{2}(-1)^{\overbrace{f(01)}^{=0}}|01\rangle + \tfrac{1}{2}(-1)^{\overbrace{f(10)}^{=0}}|10\rangle + \tfrac{1}{2}(-1)^{\overbrace{f(11)}^{=1}}|11\rangle\right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = \boxed{-\frac{1}{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2}|00\rangle \boxed{-\frac{1}{2}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{10} \implies |\phi_{10}\rangle = +\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle \boxed{-\frac{1}{2}}|10\rangle + \frac{1}{2}|11\rangle$$

$$f = f_{11} \implies |\phi_{11}\rangle = +\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle \boxed{-\frac{1}{2}}|11\rangle$$

- $\{|\phi_{00}\rangle, |\phi_{01}\rangle, |\phi_{10}\rangle, |\phi_{11}\rangle\}$ forms an *orthonormal set*

$$|\phi_{00}\rangle = -\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$|\phi_{01}\rangle = +\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$|\phi_{10}\rangle = +\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$|\phi_{11}\rangle = +\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Definition (Orthonormal set)

Unit vectors[1] that are *pairwise orthogonal*:

$$\langle\phi_{ab}|\phi_{cd}\rangle = \begin{cases} 1 \text{ if } a = c \text{ and } b = d \\ 0 \text{ otherwise} \end{cases}$$

[1] Norm = 1, Verify $\| |\phi_{ab}\rangle \| = 1$

- Inner product helps compute length of a vector
- Is a generalization of the **dot product**
- Takes two vectors from $\mathbb{C}^n$ and map them to a complex number
- Inner product between two vectors $|u\rangle$ and $|v\rangle$ is denoted by

$$\langle u|v\rangle \leftarrow \text{ Recall Bra-Ket Notation}$$

Definition (Orthogonal Vector)

If the inner product between two vectors is zero, then the vectors are called **orthogonal** to each other.

$$\langle u|v\rangle = 0 \implies |u\rangle, |v\rangle \rightarrow \text{orthogonal}$$

- Interesting property: $\langle u|v\rangle^* = \langle v|u\rangle$

$* \rightarrow$ complex conjugate

Definition (Norm)

The inner product of a vector with **itself** defines the norm or length of the vector. The norm is a real number

$$\|u\| = \sqrt{\langle u|u\rangle}$$

Note: $\sqrt{\langle u|u\rangle} \geq 0$ with equality iff $|u\rangle = 0$

- Following relations are hold w.r.t linear combinations (superpositions) of vectors

$$\left\langle u \middle| \boxed{\alpha v + \beta w} \right\rangle = \alpha \langle u|v\rangle + \beta \langle u|w\rangle$$

$$\left\langle \boxed{\alpha u + \beta v} \middle| w \right\rangle = \alpha^* \langle u|w\rangle + \beta^* \langle v|w\rangle$$

Definition (Normalized Vector)

When the norm of a vector is **unity**, we say that vector is *normalized*.

If $\langle a|a\rangle = 1$, then $|a\rangle$ is normalized.

Recall, even if a vector is not normalized, we can make it normalized by dividing the vector by its **norm**.

Example (Normalization)

Let $|u\rangle = \begin{pmatrix} 2 \\ 4i \end{pmatrix}$ Then, $\langle u| = (|u\rangle)^\dagger = (|u\rangle^T)^* = (2 \ \ -4i)$

$\langle u|u\rangle = (2 \ \ -4i)\begin{pmatrix} 2 \\ 4i \end{pmatrix} = 20, \quad$ Norm $\rightarrow \|u\| = \sqrt{\langle u|u\rangle} = \sqrt{20}$

Normalization $\xrightarrow{\text{Divide by Norm}} |\tilde{u}\rangle = \frac{|u\rangle}{\|u\|} = \frac{1}{\sqrt{20}}|u\rangle$

Normalization Now, $\langle\tilde{u}|\tilde{u}\rangle = \left(\frac{1}{\sqrt{20}}|u\rangle\right)\left(\frac{1}{\sqrt{20}}\langle u|\right) = \frac{1}{20}\langle u|u\rangle = \frac{20}{20} = 1$

Definition (Orthonormal Set)

If each element of a set of vectors is **normalized** and the elements are *orthogonal* with respect to each other[2], we say the set is **orthonormal**

- Example: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\underbrace{\begin{aligned} \langle 0|0\rangle &= (1 \ \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \\ \langle 0|1\rangle &= (1 \ \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \end{aligned}}_{\implies \textit{Normalized}}$$

$$\underbrace{\begin{aligned} \langle 0|1\rangle &= (1 \ \ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ \langle 1|0\rangle &= (0 \ \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \end{aligned}}_{\implies \textit{Orthogonal}}$$

[2] Pairwise Orthogonal

If an othonormal set is also a basis, then it is called an *orthonormal basis set*.

- **Exercise**: Is this following set orthonormal?

$$|u_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |u_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

How to Exploit an Orthonormal Set?

Interesting Property

For an orthonormal set like $\{|\phi_{00}\rangle, |\phi_{01}\rangle, |\phi_{10}\rangle, |\phi_{11}\rangle\}$, it is **always** possible to build a quantum circuit that **exactly** *distinguishes* the states

- The corresponding unitary transformation:

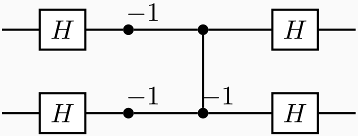
$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

- How to get U?
    - Let the vectors form the columns of a matrix
    - Then take the conjugate transform

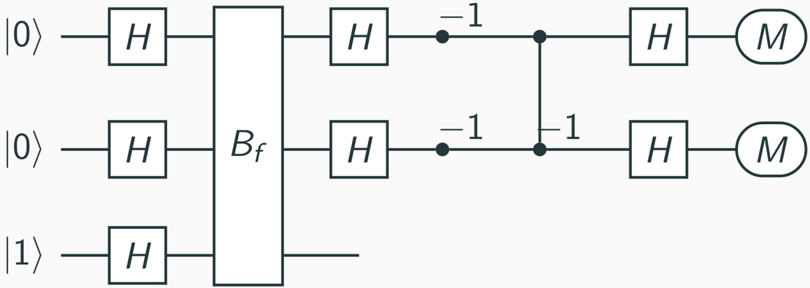
Verify: $U|\phi_{ab}\rangle = |ab\rangle, \forall a, b \in \{0, 1\}$

Quantum Circuit for U

$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$



- Here -1 represents the *phase-flip* or Paul$-\sigma_z$ gate.


$|0\rangle$
$|0\rangle$
$|1\rangle$
H
H
H
B_f
H
H
-1
-1
-1
H
H
M
M