CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Circuits and Protocols

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Quantum Circuits

- Time goes from left to right.
- Horizontal lines represent qubits.
- Operations and measurements are represented by various symbols.
- Hadamard transform applied to a single qubit

- Double lines indicate classical bits

Example

Consider two qubits X and Y in the superposition

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- Using the Dirac Notation:

$$\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle$$

Operation: $H \otimes I$

A Hadamard transform to the first qubit and nothing to the second qubit

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

- In Dirac notation: $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
- Can perform this computation directly with the Dirac notation?

Note

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

- Starting superposition:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

- Superposition after $H \otimes I$

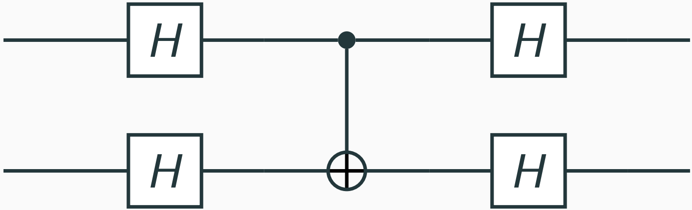
$$\frac{1}{\sqrt{2}}(H|0\rangle)|0\rangle + \frac{1}{\sqrt{2}}(H|1\rangle)|1\rangle$$

- Using values of $H|0\rangle$ and $H|1\rangle$

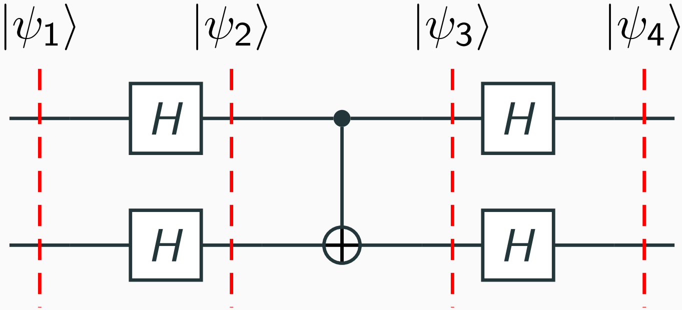
$$
\begin{aligned}
&\frac{1}{\sqrt{2}}(H|0\rangle)|0\rangle + \frac{1}{\sqrt{2}}(H|1\rangle)|1\rangle \\
=&\frac{1}{\sqrt{2}}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}|0\rangle + \frac{1}{\sqrt{2}}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}|1\rangle \\
=&\frac{1}{2}|0\rangle|0\rangle + \frac{1}{2}|0\rangle|1\rangle + \frac{1}{2}|1\rangle|0\rangle - \frac{1}{2}|1\rangle|1\rangle \\
=&\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle
\end{aligned}
$$

Verdict

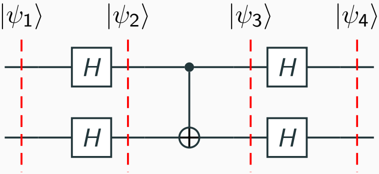
Dirac Notation can be directly employed to handle operations in quantum circuits



- What does this circuit do?

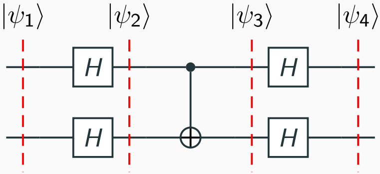

$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
$|\psi_4\rangle$
H
H
H
H

Case-1: $|\psi_1\rangle = |00\rangle$ Circuit Slicing

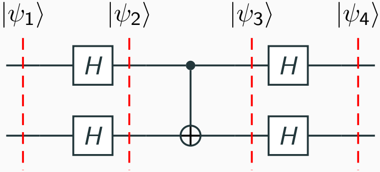


$$
\begin{aligned}
|\psi_2\rangle &= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)} \\
&= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}\boxed{|10\rangle} + \frac{1}{2}\boxed{|11\rangle} \\
|\psi_3\rangle &= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}\boxed{|11\rangle} + \frac{1}{2}\boxed{|10\rangle} \\
&= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)} \\
|\psi_4\rangle &= |00\rangle
\end{aligned}
$$

Case-2: $|\psi_1\rangle = |01\rangle$ — Circuit Slicing

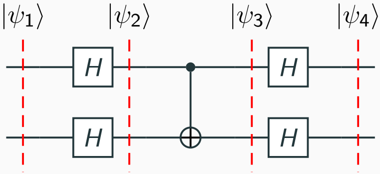


$$
\begin{aligned}
|\psi_2\rangle &= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)} \\
&= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}\boxed{|10\rangle} - \frac{1}{2}\boxed{|11\rangle} \\
|\psi_3\rangle &= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}\boxed{|11\rangle} - \frac{1}{2}\boxed{|10\rangle} \\
&= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)} \\
|\psi_4\rangle &= |11\rangle
\end{aligned}
$$



$$
\begin{aligned}
|\psi_2\rangle &= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)} \\
&= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}\boxed{|10\rangle} - \frac{1}{2}\boxed{|11\rangle} \\
|\psi_3\rangle &= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}\boxed{|11\rangle} - \frac{1}{2}\boxed{|10\rangle} \\
&= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)} \\
|\psi_4\rangle &= |10\rangle
\end{aligned}
$$

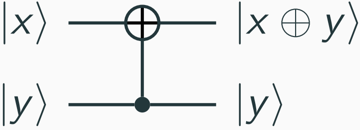
Case-4: $|\psi_1\rangle = |11\rangle$



$$
\begin{aligned}
|\psi_2\rangle &= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)} \\
&= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}\boxed{|10\rangle} + \frac{1}{2}\boxed{|11\rangle} \\
|\psi_3\rangle &= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}\boxed{|11\rangle} + \frac{1}{2}\boxed{|10\rangle} \\
&= \boxed{\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)}\boxed{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)} \\
|\psi_4\rangle &= |01\rangle
\end{aligned}
$$

Circuit Equivalent

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|11\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$\|01\rangle$



- The roles of **control** and **target** effectively **switched** in the CNOT gate because of the Hadamard transforms

References

1. Quantum Computing for Computer Scientists, by Noson S. Yanofsky, Mirco A. Mannucci
2. Quantum Computing Explained, David Mcmahon. John Wiley & Sons
3. Lecture Notes on Quantum Computation, John Watrous, University of Calgary
    - https://cs.uwaterloo.ca/~watrous/QC-notes/