CS621/CSL611
Quantum Computing For Computer Scientists

The Leap from Classical to Quantum

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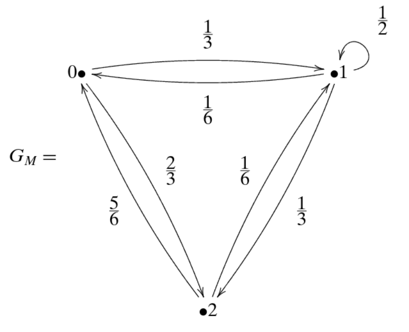
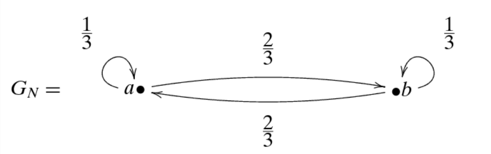
Assembling Systems

Dealing with Composite Systems





$$M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Two-marble System: G_M for red-marble, G_N for blue marble

How does a state for a two-marble system look?

- There are $3 \times 2 = 6$ possible states of the combined system. Why?
  - The red marble can be on one of three vertices and
  - The blue marble can be on one of two vertices,
- This is the **tensor product**[1] of a 3-by-1 vector with a 2-by-1 vector

Example (A typical combined state)

$$X = \begin{array}{c} 0a \\ 0b \\ 1a \\ 1b \\ 2a \\ 2b \end{array} \begin{bmatrix} \frac{1}{18} \\ 0 \\ \frac{2}{18} \\ \frac{1}{3} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$ $\leftarrow$ How to interpret this?

[1] More on tensor product in up-coming lectures.

- For a system to go from state ij to a state $i'j'$ we must multiply the probability of going from state i to state i' with the probability of going from state j to state j'.

$$ij \xrightarrow{M[i',i] \times N[j',j]} i'j' \text{ [Provided systems are independent]}$$

- What is the probability of going from state $1a$ to state $2b$?

$$X = \begin{array}{c} 0a \\ 0b \\ 1a \\ 1b \\ 2a \\ 2b \end{array} \begin{bmatrix} \frac{1}{18} \\ 0 \\ \frac{2}{18} \\ \frac{1}{3} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}. \qquad N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

- What does it mean for the entire system? Tensor Product

$$M \otimes N = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{ccc} \quad\quad\mathbf{0} & \quad\quad\quad\quad\mathbf{1} & \quad\quad\quad\quad\mathbf{2} \end{array} \\ \begin{bmatrix} 0\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{5}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ \frac{1}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{2}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{6}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ \frac{2}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & \frac{1}{3}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} & 0\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{bmatrix} \end{array}$$

$$= \begin{array}{c|cccccc} & \mathbf{0}a & \mathbf{0}b & \mathbf{1}a & \mathbf{1}b & \mathbf{2}a & \mathbf{2}b \\ \hline \mathbf{0}a & 0 & 0 & \frac{1}{18} & \frac{2}{18} & \frac{5}{18} & \frac{10}{18} \\ \mathbf{0}b & 0 & 0 & \frac{2}{18} & \frac{1}{18} & \frac{10}{18} & \frac{5}{18} \\ \mathbf{1}a & \frac{1}{9} & \frac{2}{9} & \frac{1}{6} & \frac{2}{6} & \frac{1}{18} & \frac{2}{18} \\ \mathbf{1}b & \frac{2}{9} & \frac{1}{9} & \frac{2}{6} & \frac{1}{6} & \frac{2}{18} & \frac{1}{18} \\ \mathbf{2}a & \frac{2}{9} & \frac{4}{9} & \frac{1}{9} & \frac{2}{9} & 0 & 0 \\ \mathbf{2}b & \frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & 0 \end{array}.$$

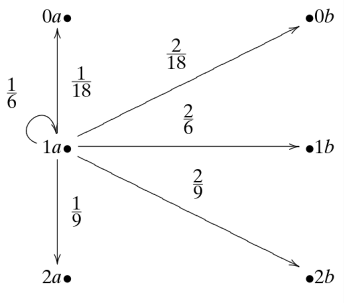
- The graph that corresponds to this matrix, $G_M \times G_N$ - called the **Cartesian product** of two weighted graphs has 28 weighted arrows.

- Find the matrix and the graph that correspond to $N \otimes N$.

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Visualizing $M \otimes N$ (Third Column)

$$
\begin{array}{c@{\quad}c}
 & \begin{array}{cccccc} \mathbf{0a} & \mathbf{0b} & \mathbf{1a} & \mathbf{1b} & \mathbf{2a} & \mathbf{2b} \end{array} \\
\begin{array}{c} \mathbf{0a} \\ \mathbf{0b} \\ \mathbf{1a} \\ \mathbf{1b} \\ \mathbf{2a} \\ \mathbf{2b} \end{array} &
\begin{bmatrix}
0 & 0 & \frac{1}{18} & \frac{2}{18} & \frac{5}{18} & \frac{10}{18} \\
0 & 0 & \frac{2}{18} & \frac{1}{18} & \frac{10}{18} & \frac{5}{18} \\
\frac{1}{9} & \frac{2}{9} & \frac{1}{6} & \frac{2}{6} & \frac{1}{18} & \frac{2}{18} \\
\frac{2}{9} & \frac{1}{9} & \frac{2}{6} & \frac{1}{6} & \frac{2}{18} & \frac{1}{18} \\
\frac{2}{9} & \frac{4}{9} & \frac{1}{9} & \frac{2}{9} & 0 & 0 \\
\frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & 0
\end{bmatrix}
\end{array}
$$



Attempt this again

- What is the probability of going from state $1a$ to state $2b$?

- Tensor product of the matrices will then act on the tensor product of the vectors.

$$
\begin{array}{c c}
 & \begin{array}{cccccc} \mathbf{0}a & \mathbf{0}b & \mathbf{1}a & \mathbf{1}b & \mathbf{2}a & \mathbf{2}b \end{array} \\
\begin{array}{c} \mathbf{0}a \\ \mathbf{0}b \\ \mathbf{1}a \\ \mathbf{1}b \\ \mathbf{2}a \\ \mathbf{2}b \end{array} &
\begin{bmatrix}
0 & 0 & \frac{1}{18} & \frac{2}{18} & \frac{5}{18} & \frac{10}{18} \\
0 & 0 & \frac{2}{18} & \frac{1}{18} & \frac{10}{18} & \frac{5}{18} \\
\frac{1}{9} & \frac{2}{9} & \frac{1}{6} & \frac{2}{6} & \frac{1}{18} & \frac{2}{18} \\
\frac{2}{9} & \frac{1}{9} & \frac{2}{6} & \frac{1}{6} & \frac{2}{18} & \frac{1}{18} \\
\frac{2}{9} & \frac{4}{9} & \frac{1}{9} & \frac{2}{9} & 0 & 0 \\
\frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & 0
\end{bmatrix}
\end{array}
\qquad
\begin{array}{c c}
\begin{array}{c} 0a \\ 0b \\ 1a \\ 1b \\ 2a \\ 2b \end{array} &
\begin{bmatrix}
\frac{1}{18} \\ 0 \\ \frac{2}{18} \\ \frac{1}{3} \\ 0 \\ \frac{1}{2}
\end{bmatrix}
\end{array}
$$

What happens to Composite Systems in Quantum Theory?

- Similar to probabilistic systems but has more!
  - The states of two separate systems can be combined using the tensor product of two vectors
  - The changes of two systems are combined by using the tensor product of two matrices
  - The tensor product of the matrices will then act on the tensor product of the vectors

Whats More? Entangled States

- In the quantum world there are many more possible states than just states that can be **combined from smaller ones**.
- There can be states that cannot be expressed as the tensor product of the smaller states
- These are the more interesting ones $\rightarrow$ **Entangled States**

- Similarly, arguments are there for operations on a combined quantum system

- How many vertices for one-bit with a marble?
- How many vertices for m-bits?
- Size of transition matrix?
- Can you see the **exponential growth**[2] in resources required?
- Can a classical computer simulate such a system?

The Quantum Computer

A prospective quantum computer, with its inherent ability to perform massive parallel processing, might be able to accomplish the task.

[2] This exponential growth is actually one of the main reasons Richard Feynman started talking (Feynman, 1982) about quantum computing in the first place.

Take-Away So Far

- A composite system is represented by the Cartesian product of the transition graphs of its subsystems.
- If two matrices act on the subsystems independently, then their tensor product acts on the states of their combined system.
- There is an exponential growth in the amount of resources needed to describe larger and larger composite systems