

CS621/CSL611

Quantum Computing For Computer Scientists

Quantum Search

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Quantum Search

A Simple Searching Problem

$$f : \{0, 1\}^2 \rightarrow \{0, 1\}$$

A Special Family of Functions

f_{00}	
input	output
00	1
01	0
10	0
11	0

f_{01}	
input	output
00	0
01	1
10	0
11	0

f_{10}	
input	output
00	0
01	0
10	1
11	0

f_{11}	
input	output
00	0
01	0
10	0
11	1

- $f \in \{f_{00}, f_{01}, f_{10}, f_{11}\}$
- f takes value 1 on one input and value 0 on all others

Goal

A Simple Search Problem

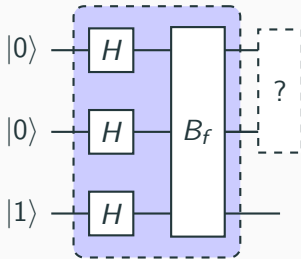
Find the input on which the function takes value 1.

- Constraint: Access to the function is restricted to evaluations of the equivalent quantum transformation B_f

Definition

An evaluation of B_f on some input (quantum or classical) is a query.

- How many queries are necessary and sufficient to solve the problem?
 - Classical: **Three.**
 - Quantum: **One!**



Approach similar to Deutsch's Circuit

Circuit Diagram

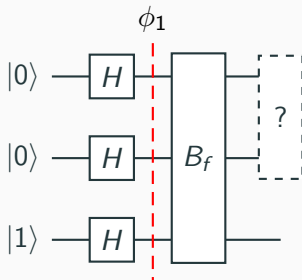
Part of the circuit to be revealed later.

A More General Formulation

- Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ be any function for +ve integers n and m
- The associated quantum transformation U_f is given as:

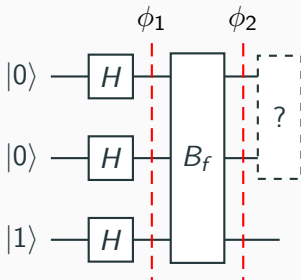
$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$$

- The associated matrix will always be a **permutation** matrix, and is therefore **unitary**.



- Initial Step: $|001\rangle$
- After H-transforms:

$$\begin{aligned}
 H \otimes H \otimes H |001\rangle &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
 &= \frac{1}{2} \left(|00\rangle + |01\rangle + |10\rangle + |11\rangle \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)
 \end{aligned}$$



- After applying B_f

$$\left(\frac{1}{2}(-1)^{f(00)} |00\rangle + \frac{1}{2}(-1)^{f(01)} |01\rangle + \frac{1}{2}(-1)^{f(10)} |10\rangle + \frac{1}{2}(-1)^{f(11)} |11\rangle \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Unaltered by B_f

Following arguments similar to the analysis of Deutsch's Algorithm. Invoking the phase kickback effect

If $f = f_{00}$

Understanding ϕ_2

$$\begin{array}{ccccccc} & \overset{1}{=} & & \overset{0}{=} & & \overset{0}{=} & & \overset{0}{=} \\ & | & & | & & | & & | \\ (\frac{1}{2}(-1)^{f(00)} |00\rangle + \frac{1}{2}(-1)^{f(01)} |01\rangle + \frac{1}{2}(-1)^{f(10)} |10\rangle + \frac{1}{2}(-1)^{f(11)} |11\rangle) \end{array}$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = -\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

If $f = f_{01}$

Understanding ϕ_2

$$\left(\frac{1}{2}(-1)^{f(00)} |00\rangle + \frac{1}{2}(-1)^{f(01)} |01\rangle + \frac{1}{2}(-1)^{f(10)} |10\rangle + \frac{1}{2}(-1)^{f(11)} |11\rangle \right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = -\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

If $f = f_{10}$

Understanding ϕ_2

$$\left(\frac{1}{2}(-1)^{f(00)} |00\rangle + \frac{1}{2}(-1)^{f(01)} |01\rangle + \frac{1}{2}(-1)^{f(10)} |10\rangle + \frac{1}{2}(-1)^{f(11)} |11\rangle \right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = -\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{10} \implies |\phi_{10}\rangle = +\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle - \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

If $f = f_{11}$

Understanding ϕ_2

$$\left(\frac{1}{2}(-1)^{f(00)} |00\rangle + \frac{1}{2}(-1)^{f(01)} |01\rangle + \frac{1}{2}(-1)^{f(10)} |10\rangle + \frac{1}{2}(-1)^{f(11)} |11\rangle \right)$$

State of first two qubits

$$f = f_{00} \implies |\phi_{00}\rangle = -\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{01} \implies |\phi_{01}\rangle = +\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{10} \implies |\phi_{10}\rangle = +\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle - \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$f = f_{11} \implies |\phi_{11}\rangle = +\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle - \frac{1}{2} |11\rangle$$

- $\{|\phi_{00}\rangle, |\phi_{01}\rangle, |\phi_{10}\rangle, |\phi_{11}\rangle\}$ forms an *orthonormal set*

$$|\phi_{00}\rangle = -\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$|\phi_{01}\rangle = +\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$|\phi_{10}\rangle = +\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle - \frac{1}{2} |10\rangle + \frac{1}{2} |11\rangle$$

$$|\phi_{11}\rangle = +\frac{1}{2} |00\rangle + \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle - \frac{1}{2} |11\rangle$$

Definition (Orthonormal set)

Unit vectors¹ that are *pairwise orthogonal*:

$$\langle \phi_{ab} | \phi_{cd} \rangle = \begin{cases} 1 & \text{if } a = c \text{ and } b = d \\ 0 & \text{otherwise} \end{cases}$$

¹Norm = 1, Verify $\| |\phi_{ab}\rangle \| = 1$

- Inner product helps compute length of a vector
- Is a generalization of the dot product
- Takes two vectors from \mathbb{C}^n and map them to a complex number
- Inner product between two vectors $|u\rangle$ and $|v\rangle$ is denoted by

$\langle u|v\rangle \leftarrow$ Recall Bra-Ket Notation

Definition (Orthogonal Vector)

If the inner product between two vectors is zero, then the vectors are called **orthogonal** to each other.

$$\langle u|v\rangle = 0 \implies |u\rangle, |v\rangle \rightarrow \text{orthogonal}$$

- Interesting property: $\langle u|v\rangle^* = \langle v|u\rangle$

Definition (Norm)

The inner product of a vector with itself defines the norm or length of the vector. The norm is a real number

$$\|u\| = \sqrt{\langle u|u\rangle}$$

Note: $\sqrt{\langle u|u\rangle} \geq 0$ with equality iff $|u\rangle = 0$

- Following relations are hold w.r.t linear combinations (superpositions) of vectors

$$\langle u | \alpha v + \beta w \rangle = \alpha \langle u | v \rangle + \beta \langle u | w \rangle$$

$$\langle \alpha u + \beta v | w \rangle = \alpha^* \langle u | w \rangle + \beta^* \langle v | w \rangle$$

Definition (Normalized Vector)

When the **norm of a vector is unity**, we say that vector is **normalized**.

If $\langle a|a \rangle = 1$, then $|a\rangle$ is normalized.

Recall, even if a vector is not normalized, we can make it normalized by dividing the vector by its **norm**.

Example (Normalization)

Let $|u\rangle = \begin{pmatrix} 2 \\ 4i \end{pmatrix}$ Then, $\langle u| = (|u\rangle)^\dagger = (|u\rangle^T)^* = (2 \ -4i)$

$\langle u|u \rangle = (2 \ -4i) \begin{pmatrix} 2 \\ 4i \end{pmatrix} = 20$, Norm $\rightarrow \|u\| = \sqrt{\langle u|u \rangle} = \sqrt{20}$

Normalization $\xrightarrow{\text{Divide by Norm}} |\tilde{u}\rangle = \frac{|u\rangle}{\|u\|} = \frac{1}{\sqrt{20}} |u\rangle$

Normalization Now, $\langle \tilde{u}|\tilde{u} \rangle = \left(\frac{1}{\sqrt{20}} \langle u| \right) \left(\frac{1}{\sqrt{20}} |u\rangle \right) = \frac{1}{20} \langle u|u \rangle = \frac{20}{20} = 1$

Definition (Orthonormal Set)

If each element of a set of vectors is **normalized** and the elements are *orthogonal* with respect to each other², we say the set is **orthonormal**

- Example: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} \langle 0|0\rangle &= (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \\ \langle 0|1\rangle &= (1 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \end{aligned}$$

$\implies \text{Normalized}$

$$\begin{aligned} \langle 0|1\rangle &= (1 \ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ \langle 1|0\rangle &= (0 \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

$\implies \text{Orthogonal}$

²Pairwise Orthogonal

If an orthonormal set is also a basis, then it is called an **orthonormal basis set**.

- **Exercise:** Is this following set orthonormal?

$$|u_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |u_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

How to Exploit an Orthonormal Set?

Interesting Property

For an orthonormal set like $\{|\phi_{00}\rangle, |\phi_{01}\rangle, |\phi_{10}\rangle, |\phi_{11}\rangle\}$, it is **always** possible to build a quantum circuit that **exactly distinguishes** the states

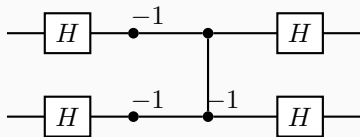
- The corresponding unitary transformation:

$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

- How to get U ?
 - Let the vectors form the columns of a matrix
 - Then take the conjugate transform

Verify: $U|\phi_{ab}\rangle = |ab\rangle, \forall a, b \in \{0, 1\}$

$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$



- Here -1 represents the *phase-flip* or Pauli- σ_z gate.

Complete Quantum Circuit

