Machine Learning

Homework 4

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CS550 Machine Learning

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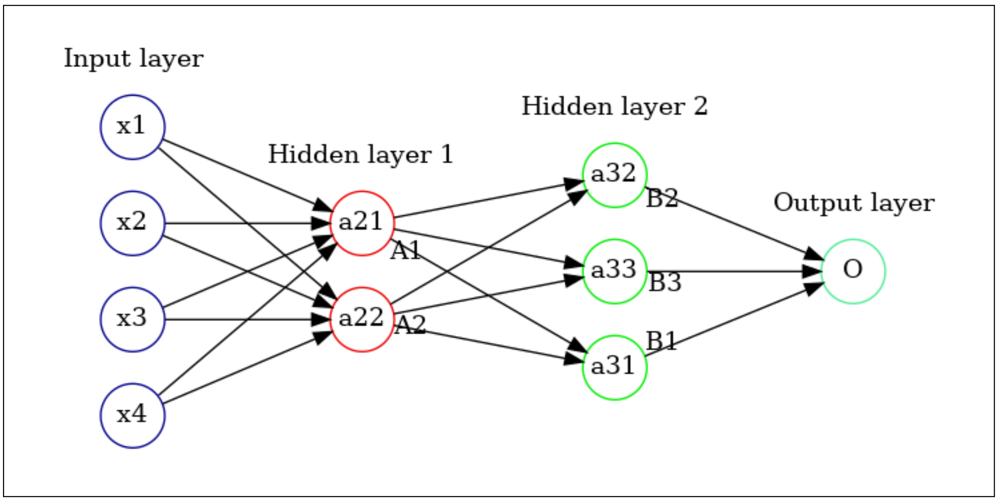
Problem 1

Ch10_Q1

Solution.

a) Neural network described with p = 4 input units, 2 units in the first hidden layer, 3 units in the second hidden layer, and a single output



b) Assinging weights and biases to each nueron whcih is shown in figure 1

Let's break down the calculations step by step. We'll assume ReLU activation functions and denote weights as w_i and biases as b_i for simplicity.

Step 1: Calculate the inputs for the first hidden layer:

$$\text{Input to } a_{21} = (x_1 \cdot w_{11}) + (x_2 \cdot w_{12}) + (x_3 \cdot w_{13}) + (x_4 \cdot w_{14}) + b_{21}$$
$$\text{Input to } a_{22} = (x_1 \cdot w_{21}) + (x_2 \cdot w_{22}) + (x_3 \cdot w_{23}) + (x_4 \cdot w_{24}) + b_{22}$$

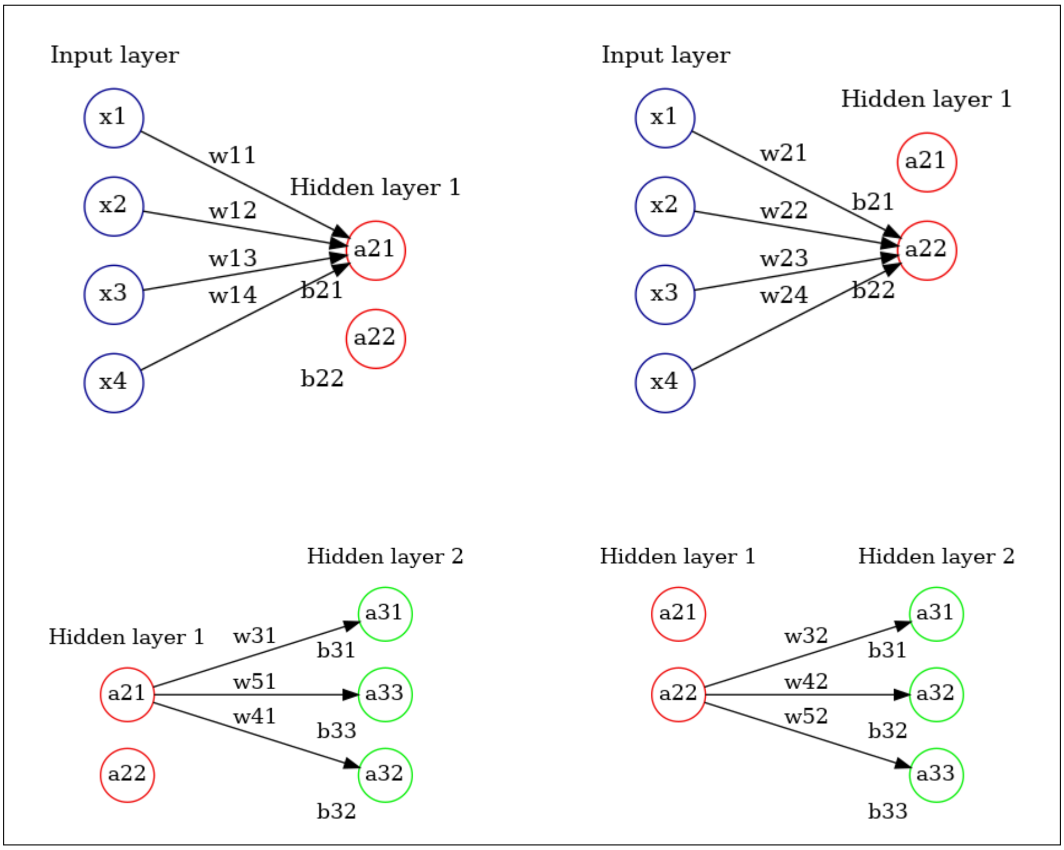
Step 2: Apply the ReLU activation function to the inputs:

$$A_1 = \text{ReLU}(\text{Input to } a_{21})$$
$$A_2 = \text{ReLU}(\text{Input to } a_{22})$$

Step 3: Calculate the inputs for the second hidden layer using the activations from the first hidden layer:

$$\text{Input to } a_{31} = (A_1 \cdot w_{31}) + (A_2 \cdot w_{32}) + b_{31}$$
$$\text{Input to } a_{32} = (A_1 \cdot w_{41}) + (A_2 \cdot w_{42}) + b_{32}$$
$$\text{Input to } a_{33} = (A_1 \cdot w_{51}) + (A_2 \cdot w_{52}) + b_{33}$$

Figure 1:



Step 4: Apply the ReLU activation function to the inputs:

$$B_1 = \text{ReLU}(\text{Input to } a_{31})$$
$$B_2 = \text{ReLU}(\text{Input to } a_{32})$$
$$B_3 = \text{ReLU}(\text{Input to } a_{33})$$

Now, we have the outputs of the second hidden layer as B_1, B_2, and B_3 in terms of x_1, x_2, x_3, x_4, weights (w), and biases (b).

Step 5: Calculate the input to the output layer using the activations from the second hidden layer:

$$\text{Input to } O = (B_1 \cdot w_{61}) + (B_2 \cdot w_{62}) + (B_3 \cdot w_{63}) + b_4$$

Step 6: Apply the ReLU activation function to obtain the final output O:

$$f(x) = \text{ReLU}(\text{Input to } O)$$

So, the final output $f(x)$ is a function of the weights (w_i),(x_i) and biases (b_i) for all layers, as well as the ReLU activation functions applied to the intermediate inputs.

c) Let's plug in some values for the coefficients and calculate the value of $f(X)$. We'll assume the following coefficients for simplicity:

For the first hidden layer:

$$\begin{aligned}
&w_{11} = 0.5, \quad w_{12} = -0.3, \quad w_{13} = 0.2, \quad w_{14} = 0.1,\\
&w_{21} = -0.1, \quad w_{22} = 0.2, \quad w_{23} = 0.4, \quad w_{24} = -0.2,\\
&b_{21} = 0.2, \quad b_{22} = 0.1.
\end{aligned}$$

For the second hidden layer:

$$\begin{aligned}
&w_{31} = 0.3, \quad w_{32} = -0.1, \quad b_{31} = 0.1,\\
&w_{41} = -0.2, \quad w_{42} = 0.3, \quad b_{32} = -0.1,\\
&w_{51} = 0.2, \quad w_{52} = 0.1, \quad b_{33} = 0.2.
\end{aligned}$$

For the output layer:

$$w_{61} = -0.3, \quad w_{62} = 0.2, \quad w_{63} = -0.1, \quad b_4 = 0.3.$$

Now, let's calculate the value of $f(X)$ by following the steps mentioned in part (b) with these coefficient values. We'll substitute the given coefficients into the equations for each layer to obtain the final output $f(X)$.

$$\begin{aligned}
\text{Input to } a_{21} &= (x_1 \cdot 0.5) + (x_2 \cdot (-0.3)) + (x_3 \cdot 0.2) + (x_4 \cdot 0.1) + 0.2\\
&= 0.5x_1 - 0.3x_2 + 0.2x_3 + 0.1x_4 + 0.2\\
&= 0.5 \cdot (1) - 0.3 \cdot (1) + 0.2(1) + 0.1 \cdot (1) + 0.2\\
&= 0.7\\
\text{Input to } a_{22} &= (x_1 \cdot (-0.1)) + (x_2 \cdot 0.2) + (x_3 \cdot 0.4) + (x_4 \cdot (-0.2)) + 0.1\\
&= -0.1x_1 + 0.2x_2 + 0.4x_3 - 0.2x_4 + 0.1\\
&= -0.1 \cdot (1) + 0.2 \cdot (1) + 0.4 \cdot (1) - 0.2 \cdot (1) + 0.1\\
&= 0.4
\end{aligned}$$

Now, apply the ReLU activation function to the inputs:

$$\begin{aligned} A_1 &= \max(0, 0.7) \\ &= 0.7 \\ A_2 &= \max(0, 0.4) \\ &= 0.4 \end{aligned}$$

For the second hidden layer, we'll use the activations from the first hidden layer:

$$\begin{aligned} \text{Input to } a_{31} &= (A_1 \cdot 0.3) + (A_2 \cdot (-0.1)) + 0.1 \\ &= 0.3A_1 - 0.1A_2 + 0.1 \\ &= 0.3 \cdot (0.7) - 0.1 \cdot (0.4) + 0.1 \\ &= 0.21 - 0.04 + 0.1 \\ &= 0.27 \\ \text{Input to } a_{32} &= (A_1 \cdot (-0.2)) + (A_2 \cdot 0.3) - 0.1 \\ &= -0.2A_1 + 0.3A_2 - 0.1 \\ &= -0.2 \cdot (0.7) + 0.3 \cdot (0.4) - 0.1 \\ &= -0.14 + 0.12 - 0.1 \\ &= -0.12 \\ \text{Input to } a_{33} &= (A_1 \cdot 0.2) + (A_2 \cdot 0.1) + 0.2 \\ &= 0.2A_1 + 0.1A_2 + 0.2 \\ &= 0.2 \cdot (0.7) + 0.1 \cdot (0.4) + 0.2 \\ &= 0.14 + 0.04 + 0.2 \\ &= 0.38 \end{aligned}$$

Apply the ReLU activation function to the inputs:

$$\begin{aligned} B_1 &= \max(0, 0.27) &= 0.27 \\ B_2 &= \max(0, -0.12) &= 0 \\ B_3 &= \max(0, 0.38) &= 0.38 \end{aligned}$$

Now, for the output layer, use the activations from the second hidden layer:

$$\begin{aligned} \text{Input to } O &= (B_1 \cdot (-0.3)) + (B_2 \cdot 0.2) + (B_3 \cdot (-0.1)) + 0.3 \\ &= -0.3B_1 + 0.2B_2 - 0.1B_3 + 0.3 \\ &= -0.3 \cdot (0.27) + 0.2 \cdot (0) - 0.1 \cdot (0.38) + 0.3 \\ &= 0.181 \end{aligned}$$

Finally, apply the ReLU activation function to obtain the final output O:

$$f(x) = \max(0, 0.181)$$
$$f(x) = 0.181$$

d) Number of Parameters:

To calculate the number of parameters in the neural network, we need to count the weights and biases. Here's the breakdown:

- **First Hidden Layer:**
  - 2 neurons in the first hidden layer.
  - Each neuron has 4 weights ($w_{11}, w_{12}, w_{13}, w_{14}$) and 1 bias ($b_{21}$).
  - Total parameters for the first hidden layer: $2 \times (4 \text{ weights} + 1 \text{ bias}) = 10$ parameters.
- **Second Hidden Layer:**
  - 3 neurons in the second hidden layer.
  - Each neuron has 2 weights (w_{31}, w_{32}) and 1 bias (b_{31}).
  - Total parameters for the second hidden layer: $3 \times (2 \text{ weights} + 1 \text{ bias}) = 9$ parameters.
- **Output Layer:**
  - 1 neuron in the output layer.
  - This neuron has 3 weights (w_{61}, w_{62}, w_{63}) and 1 bias (b_4).
  - Total parameters for the output layer: $1 \times (3 \text{ weights} + 1 \text{ bias}) = 4$ parameters.

Now, sum up the parameters from all layers:

$$\begin{aligned}
\text{Total Parameters} &= \text{Parameters in the first hidden layer} \\
&+ \text{Parameters in the second hidden layer} \\
&+ \text{Parameters in the output layer} \\
&= 10 + 9 + 4 \\
&= 23 \text{ parameters}
\end{aligned}$$

So, there are a total of 23 parameters in this neural network.

Problem 2

Ch10_Q2

Solution.
Equation 4.13

$$\log\left(Pr(Y = k|X = x)\right) = \frac{e^{\beta_{k0}+\beta_{k1}x_1+\ldots+\beta_{kp}x_p}}{\sum_{l=1}^{K} e^{\beta_{l0}+\beta_{l1}x_1+\ldots+\beta_{lp}x_p}}$$

Equation 10.13

$$f_m(X) = Pr(Y = m|X) = \frac{e^{Z_m}}{\sum_{l=0}^{9} e^{Z_l}}$$

(a) In equation (10.13):

$$f_m(X) = Pr(Y = m|X) = \frac{e^{Z_m}}{\sum_{l=0}^{9} e^{Z_l}}$$

If we add a constant c to each of the z_l values, the probability remains unchanged. Proof :

Let $Z'_l = Z_l + c$ for $l = 0, 1, \ldots, 9$, where c is a constant.

Now, we'll compute the new probability $f'_m(X)$ with Z'_l:

$$f'_m(X) = Pr(Y = m|X) = \frac{e^{Z'_m}}{\sum_{l=0}^{9} e^{Z'_l}}$$

Substitute $Z'_l = Z_l + c$ into the equation:

$$f'_m(X) = \frac{e^{Z_m+c}}{\sum_{l=0}^{9} e^{Z_l+c}}$$

Now, we can factor out e^c from the numerator and denominator:

$$f'_m(X) = \frac{e^c \cdot e^{Z_m}}{e^c \cdot \sum_{l=0}^{9} e^{Z_l}}$$

we can remove e^c from the numerator and denominator:

$$f'_m(X) = \frac{e^{Z_m}}{\sum_{l=0}^{9} e^{Z_l}}$$

This is exactly the same as the original probability $f_m(X)$. Therefore, adding a constant c to each of the z_l values in equation (10.13) does not change the probability.

(b) Starting with Equation (4.13) for class k:

$$\log\left(Pr(Y=k|X=x)\right) = \frac{e^{\beta_{k0}+\beta_{k1}x_1+\ldots+\beta_{kp}x_p}}{\sum_{l=1}^{K} e^{\beta_{l0}+\beta_{l1}x_1+\ldots+\beta_{lp}x_p}}$$

Add constants c_j to coefficients for each class and feature:

$$\log\left(Pr(Y=k|X=x)\right) = \frac{e^{(\beta_{k0}+c_0)+(\beta_{k1}+c_1)x_1+\ldots+(\beta_{kp}+c_p)x_p}}{\sum_{l=1}^{K} e^{(\beta_{l0}+c_0)+(\beta_{l1}+c_1)x_1+\ldots+(\beta_{lp}+c_p)x_p}}$$

Now, simplify the terms in the numerator and denominator:

Numerator:

$$e^{(\beta_{k0}+c_0)} \cdot e^{(\beta_{k1}+c_1)x_1} \cdot \ldots = e^{(c_0)} \cdot e^{(c_1)x_1} \cdot \ldots \times e^{\beta_{k0}} \cdot e^{\beta_{k1}\times x_1} \cdot \ldots$$

Denominator:

$$\sum_{l=1}^{K} e^{(\beta_{l0}+c_0)} \cdot e^{(\beta_{l1}+c_1)x_1} \cdot \ldots = \sum_{l=1}^{K} e^{c_0} \cdot e^{c_1 \cdot x_1} \cdot \ldots \times e^{(\beta_{l0})} \cdot e^{(\beta_{l1})x_1} \cdot \ldots$$

The added constants c_j cancel out in both the numerator and denominator:

$$\begin{aligned}
\log\left(Pr(Y=k|X=x)\right) &= \frac{e^{(\beta_{k0}+c_0)} \cdot e^{(\beta_{k1}+c_1)x_1} \cdot \ldots \cdot e^{(\beta_{kp}+c_p)x_p}}{\sum_{l=1}^{K} e^{(\beta_{l0}+c_0)} \cdot e^{(\beta_{l1}+c_1)x_1} \cdot \ldots \cdot e^{(\beta_{lp}+c_p)x_p}} \\
&= \frac{e^{(c_0)} \cdot e^{(c_1)x_1} \cdot \ldots \times e^{\beta_{k0}} \cdot e^{\beta_{k1}\times x_1} \cdot \ldots}{\sum_{l=1}^{K} e^{c_0} \cdot e^{c_1 \cdot x_1} \cdot \ldots \times e^{(\beta_{l0})} \cdot e^{(\beta_{l1})x_1} \cdot \ldots} \\
&= \frac{(e^{(c_0)} \cdot e^{(c_1)x_1} \cdot \ldots) \times (e^{\beta_{k0}} \cdot e^{\beta_{k1}\times x_1} \cdot \ldots)}{(e^{c_0} \cdot e^{c_1 \cdot x_1} \cdot \ldots) \times (\sum_{l=1}^{K} \times e^{(\beta_{l0})} \cdot e^{(\beta_{l1})x_1} \cdot \ldots)} \\
&= \frac{e^{\beta_{k0}+\beta_{k1}x_1+\ldots+\beta_{kp}x_p}}{\sum_{l=1}^{K} e^{\beta_{l0}+\beta_{l1}x_1+\ldots+\beta_{lp}x_p}} \\
&= \log\left(Pr(Y=k|X=x)\right)
\end{aligned}$$

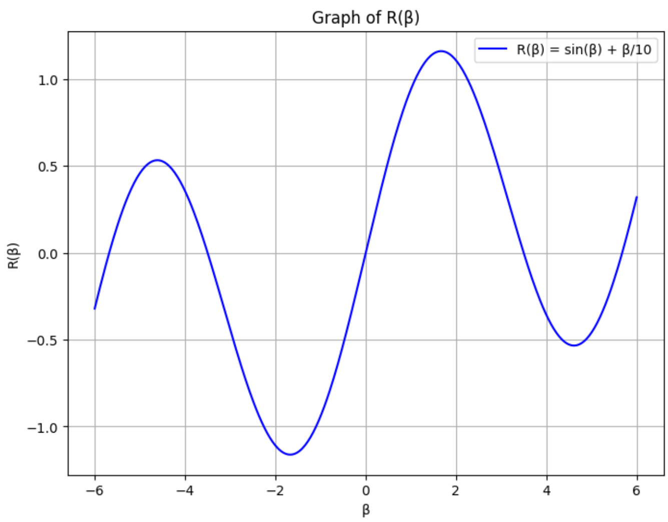
This shows that adding constants c_j to coefficients does not change the predictions at any new point x, and the probabilities remain the same.

Problem 3

Ch10_Q6

Solution.

(a) Graph of the function $R(\beta) = sin(\beta) + \frac{\beta}{10}$ over the range $\beta\epsilon[6, 6]$.



(b) The derivative of the function $R(\beta) = \sin(\beta) + \frac{\beta}{10}$ with respect to β

- The derivative of $\sin(\beta)$ with respect to β is $\cos(\beta)$.
- The derivative of $\frac{\beta}{10}$ with respect to β is $\frac{1}{10}$.

Now, combining these derivatives, we get the derivative of $R(\beta)$:

$$R'(\beta) = \cos(\beta) + \frac{1}{10}$$

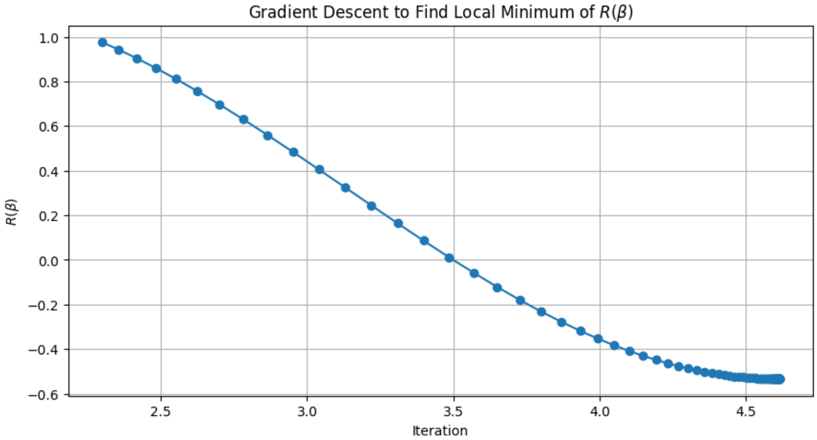
So, the derivative of the function $R(\beta) = \sin(\beta) + \frac{\beta}{10}$ is $R'(\beta) = \cos(\beta) + \frac{1}{10}$.

(c) Running a gradient descent to find a local minimum of $R(\beta) = \sin(\beta)+\frac{\beta}{10}$ with $\beta_0 = 2.3$ and a learning rate $\rho = 0.1$ involves iteratively updating β using the following formula:

$$\beta_{i+1} = \beta_i - \rho \cdot \frac{dR}{d\beta}$$

where $\frac{dR}{d\beta}$ is the derivative of $R(\beta)$.

We'll start with $\beta_0 = 2.3$ and update it iteratively until convergence.



Final value of β : 4.612220565617592
Final value of $R(\beta)$: −0.5337652811838157

So the local minima $R(\beta) = -0.53$ occurs at $\beta = 4.61$ approximately

(d) for $B^o = 1.4$

Final value of β : −1.6709610375631647
Final value of $R(\beta)$: −1.162083811898611
So the local minima $R(\beta) = -1.162$ occurs at $\beta = -1.67$ approximately