TeX Gyre Termes Math

Machine Learning

Homework 1

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Problem 1

Solution.

Question 12

Given

b = 0, 8, 8, 20

t = 0, 1, 3, 4

line equation

$$C + Dt = b$$

So Matrix A and b are

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Now solving

$$A^T A x = A^T b \quad (1)$$

$$x = (A^T A)^{-1} A^T b \quad (2)$$

$$x = \begin{bmatrix} C \\ D \end{bmatrix}$$

let's calculate separately

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 1+3+4 \\ 1+3+4 & 1+9+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$$

$$
\begin{aligned}
(A^TA)^{-1} &= \frac{1}{\text{mod }(A^TA)} cofac \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}^T \\
&= \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}^T \\
&= \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^Tb &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \\
&= \begin{bmatrix} 36 \\ 112 \end{bmatrix}
\end{aligned}
$$

from equation (2)

$$
\begin{aligned}
x &= (A^TA)^{-1}A^Tb \\
&= \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 36 \\ 112 \end{bmatrix} \\
&= \frac{1}{40} \begin{bmatrix} 26 * 36 & (-8) * 112 \\ 36 * (-8) & 4 * 112 \end{bmatrix} \\
&= \frac{1}{40} \begin{bmatrix} 40 \\ 160 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\end{aligned} \tag{3}
$$

Now calculate P

we know that

$$
\begin{aligned}
p &= Ax \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 1+0 \\ 1+4 \\ 1+12 \\ 1+16 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \qquad (4)
\end{aligned}
$$

So from (4) we can calculate errors as follows

$$
\begin{aligned}
e &= b - p \\
&= \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix} \qquad (5)
\end{aligned}
$$

from equation (5) we can calculate

$$
\begin{aligned}
E &= e_1^2 + e_2^2 + e_3^2 + e_4^2 \\
&= (-1)^2 + (3)^2 + (-5)^2 + (3)^2 \\
&= 1 + 9 + 25 + 9 \\
&= 44
\end{aligned}
$$

So value of minimum squared error is 44

Problem 2

Solution.

Question 13

Given

b = 0, 8, 8, 20

t = 0, 1, 3, 4

four equations are(unsolvable)

$$Ax = b$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$C = 0 \tag{1}$$

$$C + D = 8 \tag{2}$$

$$C + 3D = 8 \tag{3}$$

$$C + 4D = 20 \tag{4}$$

Now we have p = 1, 5, 13, 17

let's find value of x

$$\begin{aligned} Ax &= p \\ A^T Ax &= A^T p \\ x &= (A^T A)^{-1} A^T p \end{aligned}$$

from Question 12 we know value of $(A^TA)^{-1}$

$$x = \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 26 & 26-8 & 26-24 & 26-32 \\ -8 & -8+4 & -8+12 & -8+16 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 26 & 18 & 2 & -6 \\ -8 & -4 & 4 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 26+90+26-102 \\ -8-20+52+136 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 160 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So the value of x is (1,4) which is same as previous question as we are putting same p from previous question

t = (0, 1, 3, 4)
b = (0, 8, 8, 20)

Now matrix A is (as co-effecient of C are zeros)

$$A = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

To find

$$A^TA\hat{x} = A^Tb$$
$$\hat{x} = (A^TA)^{-1}A^Tb$$

let's calculate

$$A^T A = \begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} = [26] \tag{5}$$

$$A^T b = \begin{bmatrix} 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = [112] \tag{6}$$

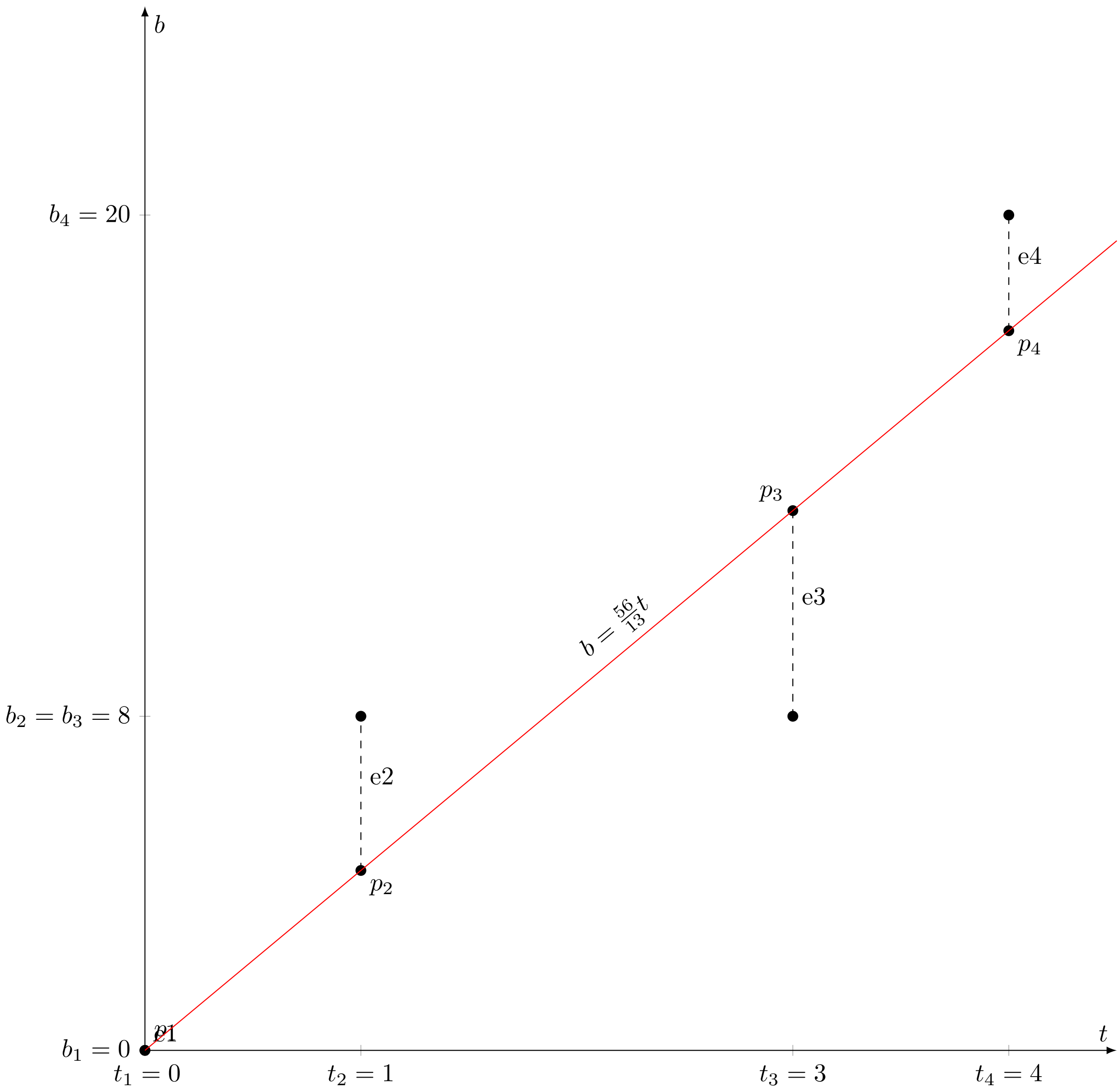
from equation(5) and (6)

$$\begin{aligned} \hat{x} &= (A^T A)^{-1} A^T b \\ &= \frac{1}{26} \times [112] \quad = \frac{56}{13} \end{aligned} \tag{7}$$

So the required line equation is

$$\begin{aligned} Dt &= b \\ \left(\frac{56}{13}\right) t &= b \end{aligned}$$

Graphical Representation



Problem 3

Solution.

Question 19

Given vectors

b = (0, 8, 8, 20)

a = (0, 1, 3, 4)

To find projection of b along a :-

$$a.b = |a||b|cos\theta \qquad \text{(dot product)}$$
$$cos\theta = \frac{a.b}{|a||b|} \tag{1}$$

component of b along a

$$= (|b|cos\theta)\hat{a}$$
$$= |b|\left(\frac{a.b}{|a||b|}\right)\left(\frac{a}{|a|}\right)$$
$$= \frac{a.b}{|a||a|}a \tag{2}$$

we can also write equation(2) as

$$= \left(\frac{a^Tb}{a^Ta}\right)a \tag{3}$$

from above equations

$$p = \hat{x}a$$
$$\therefore \hat{x} = \frac{a^Tb}{a^Ta}$$

let's calculate $\hat{x}$ (taking values of a^Tb and a^Ta from question 18 [equation(5) and (6)])

$$\hat{x} = \frac{a^Tb}{a^Ta} \quad = \frac{112}{26} \quad = \frac{56}{13}$$

Calculating p

$$p = \hat{x}a$$
$$= \frac{56}{13}\begin{bmatrix}0\\1\\3\\4\end{bmatrix}$$

from Problem 16 and Problem 18

(C,D) = (9,56/13)

from Problem 11-14

(C,D) = (1,4)

These are not same because vector $(1,1,1,1)$ and $(0,1,3,4)$ are not perpendicular

Problem 4

Solution.

Problem 20

Given

equation of Parabola

$$b = C + Dt + Et^2$$

vectors are

b = (0, 8, 8, 20)

t = (0, 1, 3, 4)

So out unsolvable equation will be

$$Ax = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Now solve for equation

$$A^T Ax = A^T b$$

Calculating both separately

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 0+1+3+4 & 0+1+9+16 \\ 0+1+3+4 & 0+1+9+16 & 0+1+27+64 \\ 0+1+9+16 & 0+1+27+64 & 0+1+81+256 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0+8+8+20 \\ 0+8+32+80 \\ 0+8+72+320 \end{bmatrix}$$

$$= \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

So our Normal equation will be

$$A^TAx = A^Tb$$

$$\begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

Figure II.3b will not change as it's just projection in 4 dimension vector $[R^4]$ **same as given figure**

Problem 5

Solution.

Question 21

Given

Cubic equation

$$b = C + Dt + Et^2 + Ft^3$$

Writing in form of $Ax = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

We have system of linear equation as follows

$$C = 0 \tag{1}$$
$$C + D + E + F = 8 \tag{2}$$
$$C + 3D + 9E + 27E = 8 \tag{3}$$
$$C + 4D + 16E + 64F = 20 \tag{4}$$

solving system of linear equations by elimination method

Step 1: Subtract the first row from the second, third, and fourth rows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 9 & 27 \\ 0 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Step 2: Subtract 3 times the second row from the third row and 4 times the second row from the fourth row:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & 12 & 60 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -16 \\ -12 \end{bmatrix}$$

Step 3: Divide the third row by 6:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 12 & 60 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ \frac{-16}{6} \\ -12 \end{bmatrix}$$

Step 4: Devide by 12 to the fourth row:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ \frac{-8}{3} \\ -1 \end{bmatrix}$$

Step 5: Subtract third row from fourth row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ \frac{-8}{3} \\ \frac{5}{3} \end{bmatrix}$$

Step 6: Subtract the four times of fourth row from the third row :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ \frac{-28}{3} \\ \frac{5}{3} \end{bmatrix}$$

Step 7: Subtract the fourth row from the second row and Subtract the third row from second row :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{47}{3} \\ \frac{-8}{3} \\ \frac{5}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$$

Solution is

$$\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$$

As we are getting solution to the given equations that means all points lies on the Cubic part, so error(e) is zero and p in nothing but b

e =(0,0,0,0), p=(0,8,8,20)

Problem 6

Solution.

Question 22

Given

Part a]

$$\bar{t} = 2$$
$$\bar{b} = 9$$

To verify

$$C + D\bar{t} = \bar{b} \tag{1}$$

from Question 12 we can take

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} 1 + 4 \times \bar{t} &= \bar{b} \\ 1 + 4 \times (2) &= (9) \\ 1 + 8 &= 9 \\ 9 &= 9 \qquad \text{(verified)} \end{aligned}$$

Explanation

we know that for best line

$$\sum e_i = 0$$

also

$$\begin{aligned} e_i &= b_i - p_i \\ p_i &= b_i - e_i \sum p_i & &= \sum b_i - \sum e_i `\sum p_i = \sum b_i \end{aligned}$$

best line equation is

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

by adding all equations we get

$$4C + D\sum t)i = \sum p_i$$
$$4C + D\sum t)i = \sum b_i$$
$$C + D\frac{\sum t_i}{4} = \frac{\sum b_i}{4}$$
$$C + D\bar{t} = \bar{b}$$

So we can conclude that point $(\bar{t}, \bar{b})$ lie on best fit line

Part b]

Given equation

$$A^T A\hat{x} = A^T b$$

where

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix} \quad \begin{bmatrix} 1+1+1+1 & t_1+t_2+t_3+t_4 \\ t_1+t_2+t_3+t_4 & t_1^2+t_2^2+t_3^2+t_4^2 \end{bmatrix}$$

$$= \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad = \begin{bmatrix} b_1+b_2+b_3+b_4 \\ b_1t_1+b_2t_2+b_3t_3+b_4 \end{bmatrix}$$

$$= \begin{bmatrix} \sum b_i \\ \sum b_i t_i \end{bmatrix}$$

Combining all we get $==>$

$$A^T A\hat{x} = A^T b$$
$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum b_i t_i \end{bmatrix}$$

First equation is

$$
\begin{aligned}
mC + D\sum t_i &= \sum b_i \\
C + D\frac{\sum t_i}{m} &= \frac{\sum b_i}{m} \\
C + D\bar{t} &= \bar{b}
\end{aligned}
$$