

CS621/CSL611

Quantum Computing For Computer Scientists

Quantum Search

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Quantum Search

Grover's Algorithm

- Given an unordered array of m elements, find a particular element.
- Classically, in the worst case, this takes m queries.
- On average, we will find the desired element in $m/2$ queries.

Can we do better?

- Lov Grover's search algorithm does the job in \sqrt{m} queries.
- Grover's algorithm has many applications to database theory and other areas.

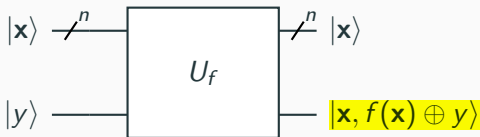
- Let us look at the search problem from the point of view of functions.
- **Given:** A function $f : \{0, 1\}^n \rightarrow \{0, 1\} : \exists \mathbf{x}_0 \in \{0, 1\}^n$

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} = \mathbf{x}_0, \\ 0, & \text{if } \mathbf{x} \neq \mathbf{x}_0. \end{cases}$$

- **Goal:** To find \mathbf{x}_0
- Worst case effort:
 - **Classical Search:** 2^n evaluations of f
 - **Grover's Search:** $\sqrt{2^n} = 2^{\frac{n}{2}}$ evaluations of f

$$|x, y\rangle \xrightarrow{U_f} |x, f(x) \oplus y\rangle$$

Quantum Circuit for f



Example (For $n = 2$, if f “picks out” 10, then U_f is)

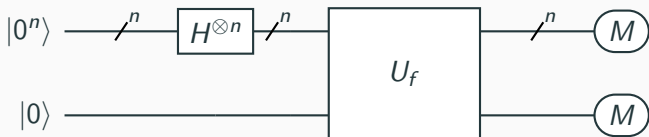
	00, 0	00, 1	01, 0	01, 1	10, 0	10, 1	11, 0	11, 1
00, 0	1							
00, 1		1						
01, 0			1					
01, 1				1				
10, 0					1			
10, 1						1		
11, 0							1	
11, 1								1

$$f : \{0, 1\}^2 \rightarrow \{0, 1\}$$

Exercise

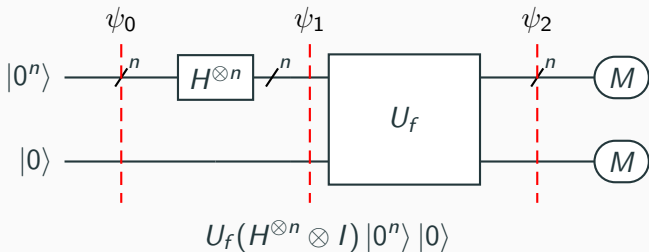
- Find U_f , if f “picks out” 11.

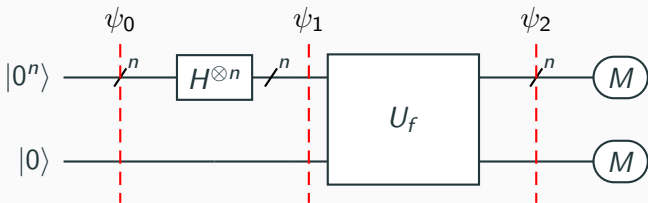
	00, 0	00, 1	01, 0	01, 1	10, 0	10, 1	11, 0	11, 1
00, 0								
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01, 1								
10, 0								
10, 1								
11, 0								
11, 1								



- Try placing $|\mathbf{x}\rangle$ into a superposition of all possible strings
- Then evaluate U_f
- In terms of matrices this becomes

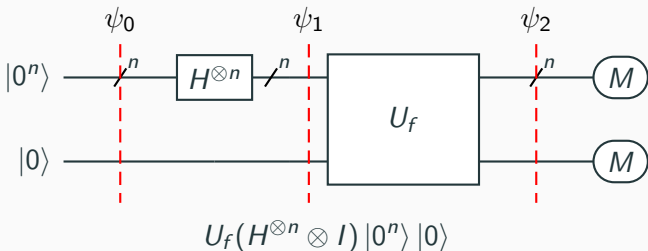
$$U_f(H^{\otimes n} \otimes I) |0^n\rangle |0\rangle$$





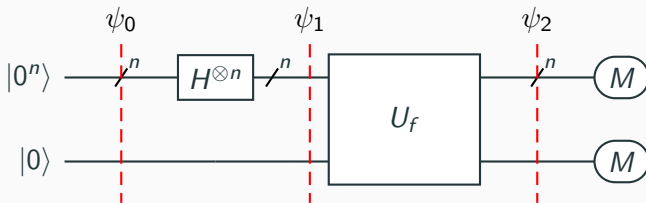
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$$|\psi_0\rangle = |0^n\rangle |0\rangle$$



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$$|\psi_1\rangle = \left(\frac{\sum_{x \in \{0,1\}^n} |\mathbf{x}\rangle}{\sqrt{2^n}} \right) |0\rangle$$



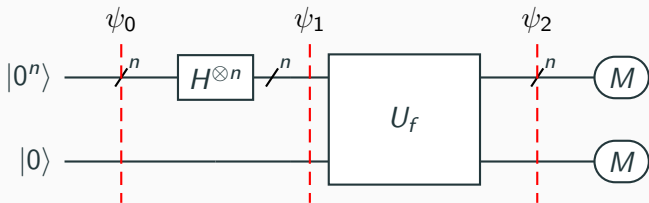
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$$|\psi_2\rangle = \left(\frac{\sum_{x \in \{0,1\}^n} |\mathbf{x}\rangle |f(\mathbf{x})\rangle}{\sqrt{2^n}} \right)$$

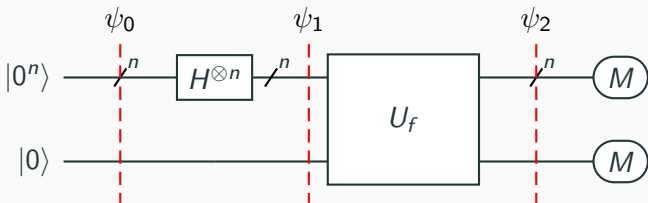
Recall, $|\mathbf{x}, y\rangle \xrightarrow{U_f} |\mathbf{x}, f(\mathbf{x}) \oplus y\rangle$



$$|\psi_2\rangle = \left(\frac{\sum_{\mathbf{x} \in \{0,1\}^n} |\mathbf{x}\rangle |f(\mathbf{x})\rangle}{\sqrt{2^n}} \right)$$

- Measuring the top qubits will, with equal probability, give one of the 2^n binary strings
- Measuring the bottom qubit will give
 - $|0\rangle$ with probability $\frac{2^n-1}{2^n}$
 - $|1\rangle$ with probability $\frac{1}{2^n} \implies$ the top qubit will reveal x_0 ? *How?*

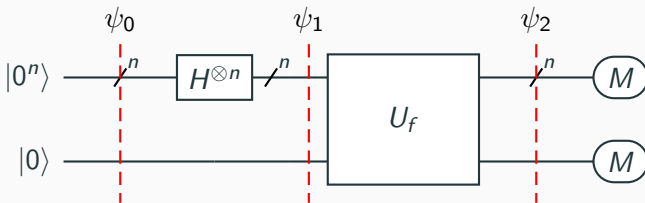
Measuring $|1\rangle$ **highly** improbable, Need better solution



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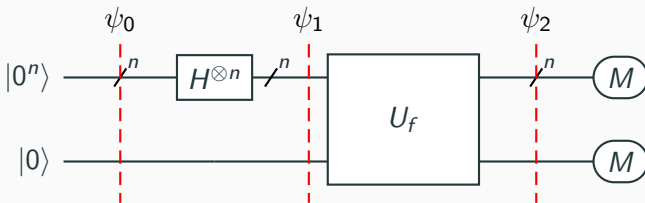
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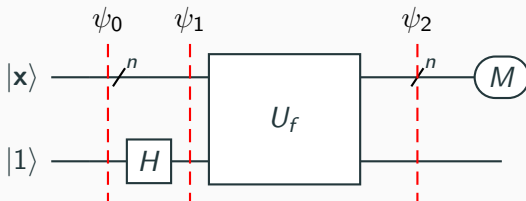
Measuring $|1\rangle$ **highly** improbable, Need better solution

- **Trick-1: Phase Inversion:**
 - Changes the phase of the desired state
- **Trick-2: Inversion About The Mean/Average:**
 - This is a way of boosting the separation of the phases.

Idea

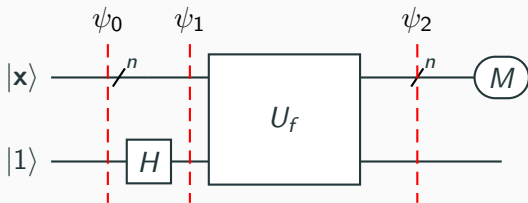
Take U_f and place the bottom qubit in the superposition $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

- Corresponding quantum circuit for an arbitrary x

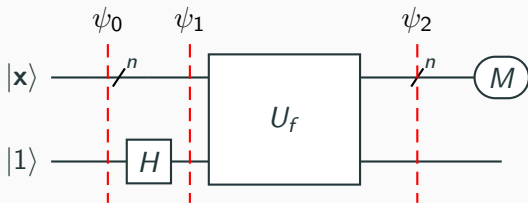


$$U_f(I_n \otimes H) |x\rangle |1\rangle$$

- Note:** Phase inversion, by construction, is a unitary operation.

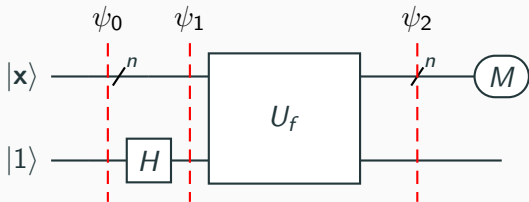


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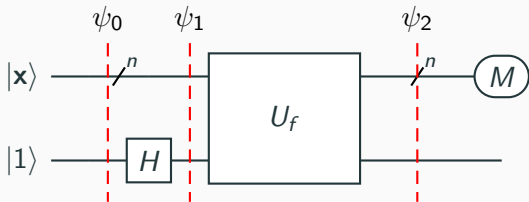
$$|\psi_1\rangle = |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \left(\frac{|x\rangle |0\rangle - |x\rangle |1\rangle}{\sqrt{2}} \right)$$



$$|\psi_0\rangle = |x\rangle |1\rangle$$

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$$|\psi_2\rangle = |x\rangle \left(\frac{|f(x) \oplus 0\rangle - |f(x) \oplus 1\rangle}{\sqrt{2}} \right) = |x\rangle \left(\frac{|f(x)\rangle - \overline{|f(x)\rangle}}{\sqrt{2}} \right)$$



$$|\psi_0\rangle = |x\rangle |1\rangle$$

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$$= (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \begin{cases} -1 |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right), & \text{if } x = x_0 \\ +1 |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right), & \text{if } x \neq x_0 \end{cases}$$

- Let $|\mathbf{x}\rangle$ start in a equal superposition of four different states:

$$|\mathbf{x}\rangle \rightarrow \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right]^T$$

- Let f be the function that uniformly chooses “10”
- After phase inversion state looks like:

$$\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]^T$$

- Measurement does not help since $\left| \frac{1}{2} \right|^2 = \left| -\frac{1}{2} \right|^2 = \frac{1}{4}$
- Implication:** Just separating phases is **not** enough

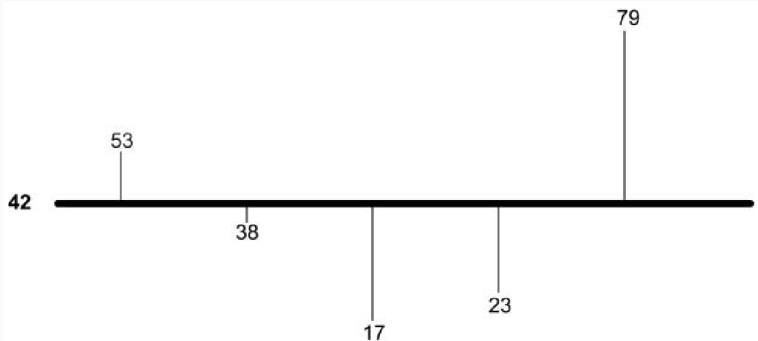
What next?

- **Boosting Phase Separation** between desired binary string from the other binary strings
- **Inversion About The Mean/Average**
- A way of boosting the **separation of the phases.**
- How? Let us see an example

Example

Inversion About The Mean/Average

- Consider a sequence of integers: 53, 38, 17, 23, and 79.



Interpreting the Average

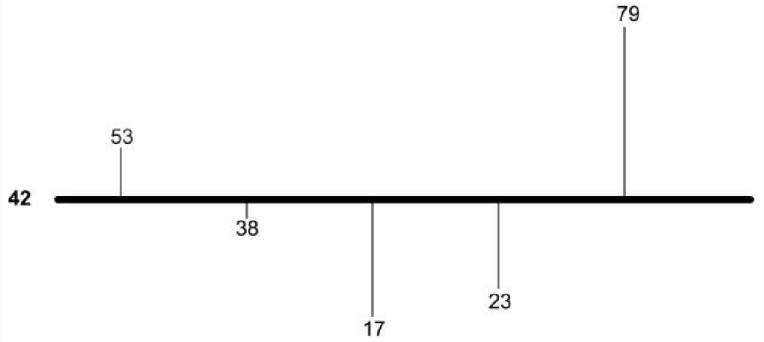
Average is $a = 42$

The average is the number such that the **sum** of the lengths of the **lines above** the average is the same as the **sum** of the lengths of the **lines below**.

Example

Inversion About The Mean/Average

- Consider a sequence of integers: 53, 38, 17, 23, and 79.



Target

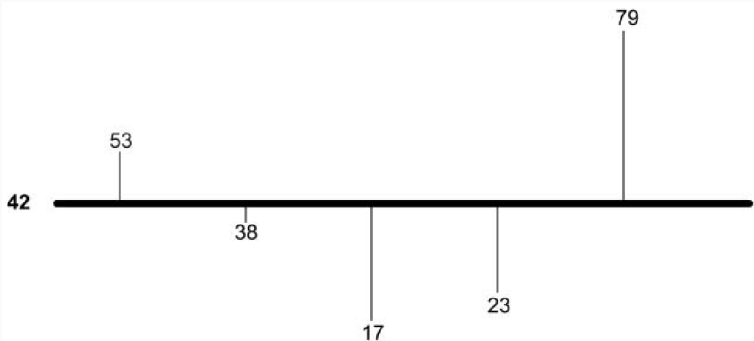
Change The Sequence

- Invert elements around the average
 - Above \leftrightarrow Below
 - Absolute distance** of from average is preserved

Example

Inversion About The Mean/Average

- Consider a sequence of integers: 53, 38, 17, 23, and 79.



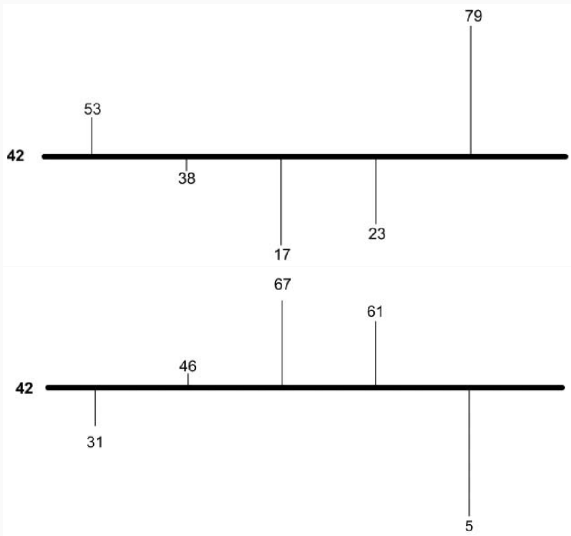
Example ($v' = a + (a - v) \implies v' = -v + 2a$)

53 (Above) \rightarrow 31 (Below) [$a + (a - 53) = 31$]

38 (Below) \rightarrow 46 (Above) [$a + (a - 38) = 46$]

Example

Inversion About The Mean/Average



$$\{53, 38, 17, 23, 79\} \xrightarrow{\text{Average, } a = 42} \{31, 46, 67, 61, 5\}$$

- Consider the following numbers: 5, 38, 62, 58, 21, and 35.
Invert these numbers around their mean.

Unitary Matrix For Inversion About The Mean

- Continuing with the example: $V = [53, 38, 17, 23, 79]^T$
- Consider the following operation: AV

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 53 \\ 38 \\ 17 \\ 23 \\ 79 \end{bmatrix} = \begin{bmatrix} 42 \\ 42 \\ 42 \\ 42 \\ 42 \end{bmatrix}$$

- So matrix A finds the **average** of a sequence.
- In terms of matrices, the formula $v' = -v + 2a$ becomes

$$V' = -V + 2AV = (-I + 2A)V$$

Unitary Matrix For Inversion About The Mean

$$(-I+2A) = \begin{bmatrix} (-1 + \frac{2}{5}) & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & (-1 + \frac{2}{5}) & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & (-1 + \frac{2}{5}) & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & (-1 + \frac{2}{5}) & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5} & (-1 + \frac{2}{5}) \end{bmatrix}$$

- In our case:

$$(-I + 2A)[53, 38, 17, 23, 79]^T = [31, 46, 67, 61, 5]^T$$

Generalization for 2^n Numbers

$$A = \begin{bmatrix} \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} \\ \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} \\ \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} \\ \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} \\ \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} & \frac{1}{2^n} \end{bmatrix}$$

$$(-I + 2A)$$

$$\begin{bmatrix} \left(-1 + \frac{2}{2^n}\right) & \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} \\ \frac{2}{2^n} & \left(-1 + \frac{2}{2^n}\right) & \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} \\ \frac{2}{2^n} & \frac{2}{2^n} & \left(-1 + \frac{2}{2^n}\right) & \frac{2}{2^n} & \frac{2}{2^n} \\ \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} & \left(-1 + \frac{2}{2^n}\right) & \frac{2}{2^n} \\ \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} & \frac{2}{2^n} & \left(-1 + \frac{2}{2^n}\right) \end{bmatrix}$$

- Prove that $-I + 2A$ is a unitary matrix.

Combining Trick-1 and Trick-2

- When considered separately, **phase inversion** and **inversion about the mean** are each **innocuous** operations.
- However, when **combined**, they are a **very powerful** operation that separates the **amplitude** of the desired state from those of all the **other states**.
- Let us do an example to understand this interplay of Trick-1 and Trick-2

- Consider the vector $[10, 10, 10, 10, 10]^T$.
- Target the fourth element:
- State after phase inversion:

$$[\quad , \quad , \quad , \quad , \quad]$$

- Average is:
- State after inversion about mean:

$$[\quad , \quad , \quad , \quad , \quad]$$

- The difference between the fourth element and all the others is:

- State after phase inversion:

[, , , ,]

- Average is:
- State after inversion about mean:

[, , , ,]

- The difference between the fourth element and all the others is:

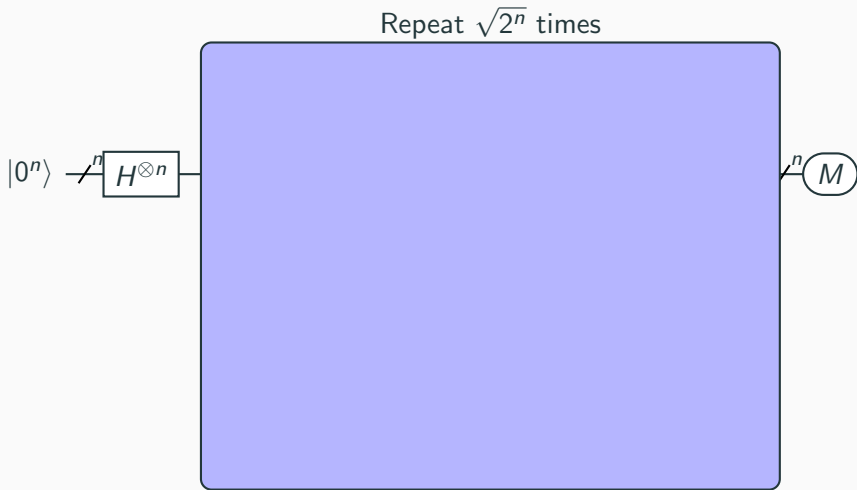
- **Step 1.** Start with a state $|0^n\rangle$
- **Step 2.** Apply $H^{\otimes n} |0^n\rangle$
- **Step 3.** Repeat $\sqrt{2^n}$ times:
 - **Step 3a.** Apply the phase inversion operation:

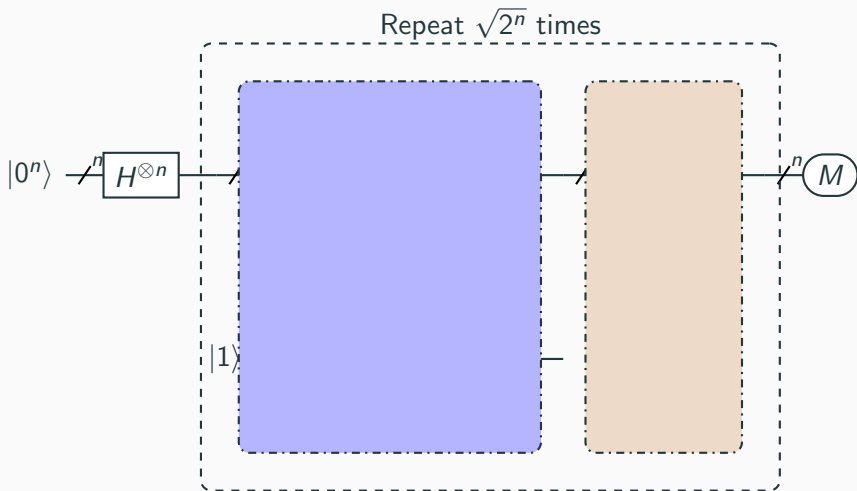
$$U_f(I \otimes H)$$

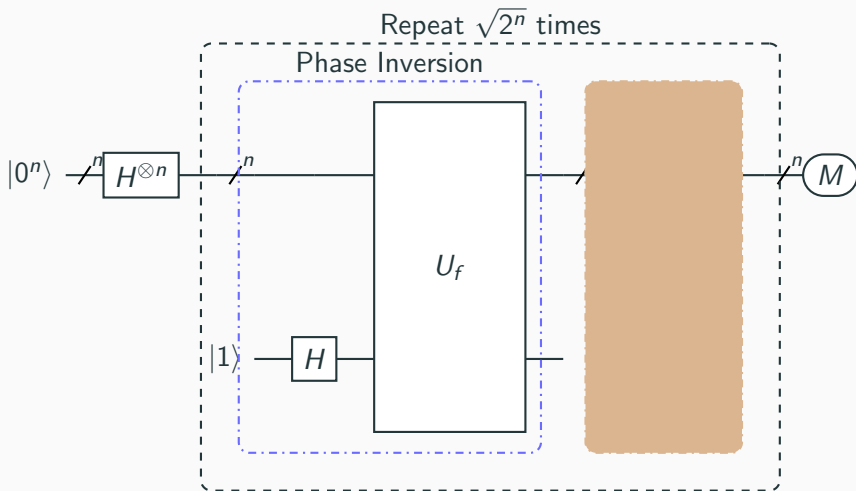
- **Step 3b.** Apply the inversion about the mean operation:

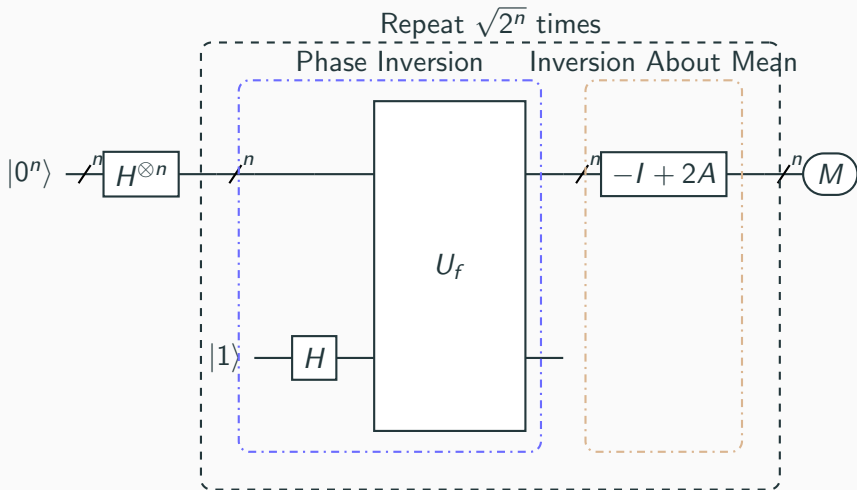
$$-I + 2A$$

- **Step 4.** Measure the qubits









- Let f be a function that picks out the string “101”.
- Initial state:

$$|\phi_1\rangle = \left[\overbrace{1}^{000}, \overbrace{0}^{001}, \overbrace{0}^{010}, \overbrace{0}^{011}, \overbrace{0}^{100}, \overbrace{0}^{101}, \overbrace{0}^{110}, \overbrace{0}^{111} \right]^T$$

- After $H^{\otimes n}$

$$|\phi_2\rangle = \left[\frac{\overbrace{1}^{000}}{\sqrt{8}}, \frac{\overbrace{1}^{001}}{\sqrt{8}}, \frac{\overbrace{1}^{010}}{\sqrt{8}}, \frac{\overbrace{1}^{011}}{\sqrt{8}}, \frac{\overbrace{1}^{100}}{\sqrt{8}}, \frac{\overbrace{1}^{101}}{\sqrt{8}}, \frac{\overbrace{1}^{110}}{\sqrt{8}}, \frac{\overbrace{1}^{111}}{\sqrt{8}} \right]^T$$

- After Phase Inversion

$$|\phi_{3a}\rangle = \left[\begin{array}{ccccccc} \overbrace{1}^{000} & \overbrace{1}^{001} & \overbrace{1}^{010} & \overbrace{1}^{011} & \overbrace{1}^{100} & \overbrace{-1}^{101} & \overbrace{1}^{110} & \overbrace{1}^{111} \\ \hline \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \end{array} \right]^T$$

- Average of the numbers:

$$a = \text{-----} =$$

- Calculating the inversion about the mean:

$$\text{For, } \frac{1}{\sqrt{8}}, -v + 2a =$$

$$\text{For, } -\frac{1}{\sqrt{8}}, -v + 2a =$$

- After this step,

$$|\phi_{3b}\rangle = \left[\underbrace{000}, \underbrace{001}, \underbrace{010}, \underbrace{011}, \underbrace{100}, \underbrace{101}, \underbrace{110}, \underbrace{111} \right]^T$$

- Repeating... After phase inversion:

$$|\phi_{3a}\rangle = \left[\underbrace{000}, \underbrace{001}, \underbrace{010}, \underbrace{011}, \underbrace{100}, \underbrace{101}, \underbrace{110}, \underbrace{111} \right]^T$$

- Average of the numbers:

$$a = \text{-----} =$$

- Calculating the inversion about the mean:

For, Case - 1 $\therefore -v + 2a =$

For, Case - 2 $\therefore -v + 2a =$

- After this step,

$$|\phi_{3b}\rangle = \left[\underbrace{000}, \underbrace{001}, \underbrace{010}, \underbrace{011}, \underbrace{100}, \underbrace{101}, \underbrace{110}, \underbrace{111} \right]^T$$

- What are the number values of the numbers in $|\phi_{3b}\rangle$?

Case – 1 : , Case – 2 :

- Let us square the numbers in $|\phi_{3b}\rangle$ to get the probabilities.

Case – 1 : , Case – 2 :

- Most probable state after measurement:

$$|\phi_4\rangle = \left[\underbrace{000}_0, \underbrace{001}_0, \underbrace{010}_0, \underbrace{011}_0, \underbrace{100}_0, \underbrace{101}_1, \underbrace{110}_0, \underbrace{111}_0 \right]^T$$

- Write a program in your favorite programming language to do a similar analysis for the case where $n = 4$ and f chooses the “1101” string?

- A classical algorithm will search an unordered array of size n in n steps.
- Grover's algorithm will take time \sqrt{n} .
- This is what is referred to as a **quadratic speedup**.

What happens if we repeat beyond \sqrt{n}

- The phase difference will degrade.
- The **overcooking** effect¹!
- Let us look at Bernstein's Indocrypt 2021 Example to see a demonstration of this effect.

¹Proof makes use of geometry. Needs another view of the systems

Quantum Search

Grover's Search Demo

Adapted from Bernstein's Invited Talk at Indocrypt 2021

Grover's algorithm

Assume: unique $s \in \{0, 1\}^n$
has $f(s) = 0$.

Goal: Figure out s .

Non-quantum algorithm to find s :
compute f for many inputs,
hope to find output 0.
Success probability is very low
until #tries approaches 2^n .

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Grover's algorithm takes only $2^{n/2}$
quantum evaluations of f .
e.g. 2^{64} instead of 2^{128} .

Start from uniform superposition
over n -bit strings u : each $a_u = 1$.

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over n -bit strings u : each $a_u = 1$.

Step 1: Set $a \leftarrow b$ where
 $b_u = -a_u$ if $f(u) = 0$,
 $b_u = a_u$ otherwise.

This is fast if f is fast.

Start from uniform superposition over n -bit strings u : each $a_u = 1$.

Step 1: Set $a \leftarrow b$ where
 $b_u = -a_u$ if $f(u) = 0$,
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Step 2: “Grover diffusion”.
Negate a around its average.
This is also fast.

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Step 2: “Grover diffusion”.
Negate a around its average.
This is also fast.

Repeat Step 1 + Step 2
about $0.58 \cdot 2^{0.5n}$ times.

Start from uniform superposition
over n -bit strings u : each $a_u = 1$.

Step 1: Set $a \leftarrow b$ where
 $b_u = -a_u$ if $f(u) = 0$,
 $b_u = a_u$ otherwise.

This is fast if f is fast.

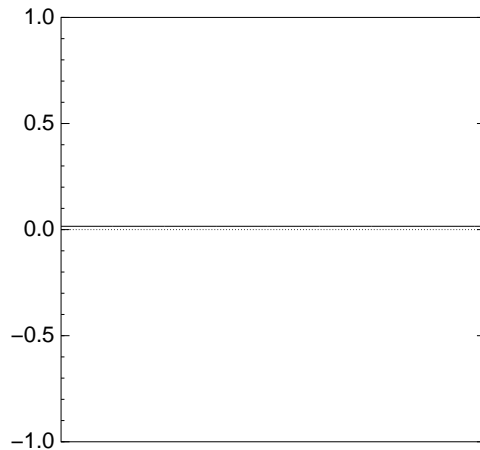
Step 2: “Grover diffusion”.
Negate a around its average.
This is also fast.

Repeat Step 1 + Step 2
about $0.58 \cdot 2^{0.5n}$ times.

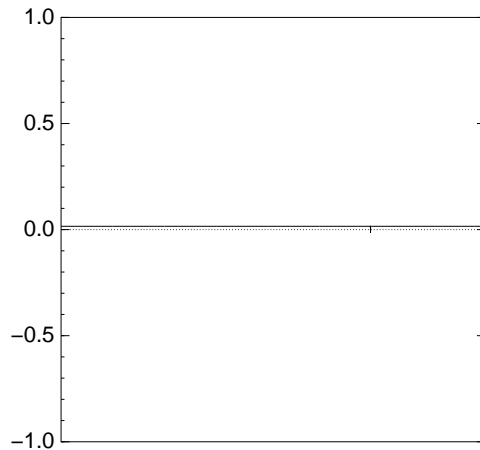
Measure the n qubits.

With high probability this finds s .

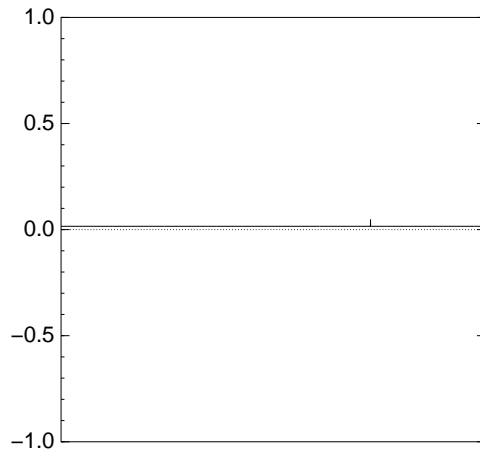
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after 0 steps:



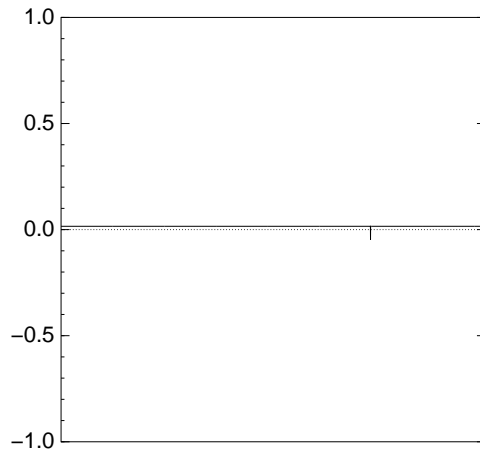
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after Step 1:



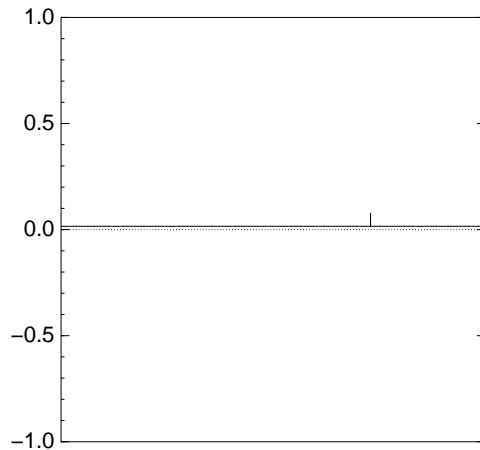
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after Step 1 + Step 2:



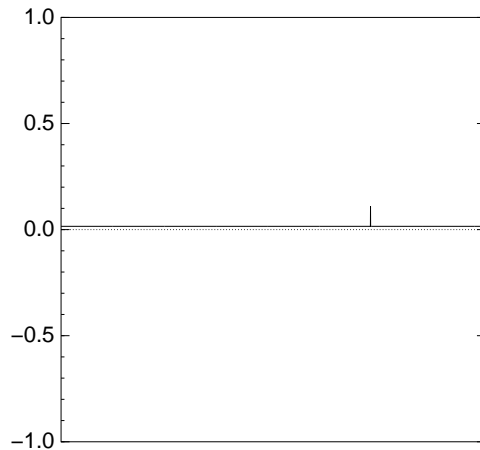
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after Step 1 + Step 2 + Step 1:



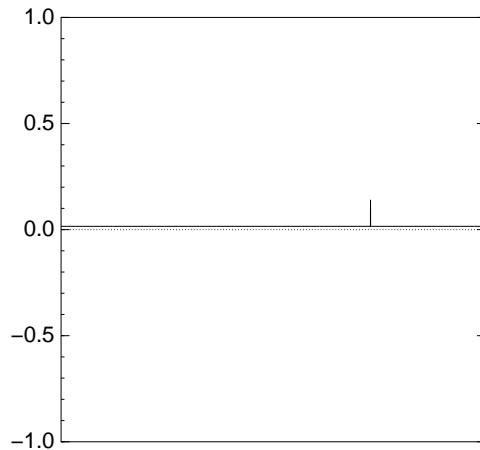
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $2 \times (\text{Step 1} + \text{Step 2})$:



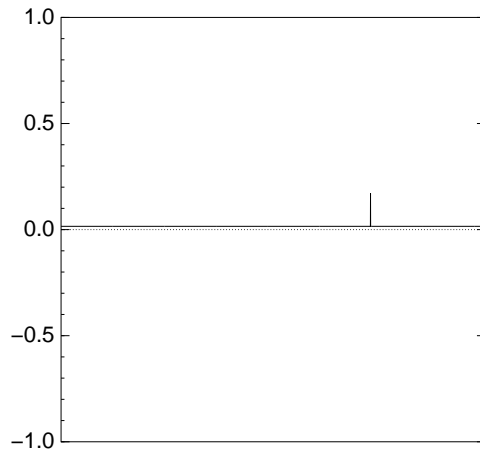
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $3 \times (\text{Step 1} + \text{Step 2})$:



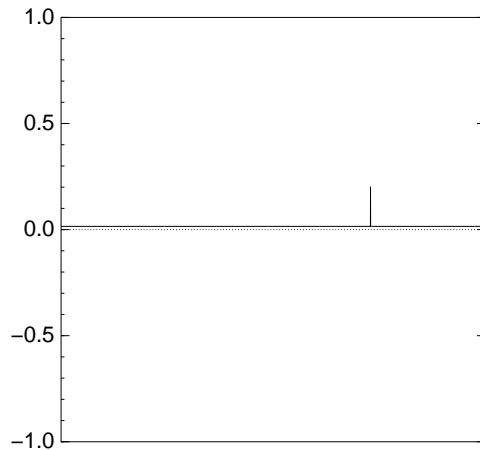
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $4 \times (\text{Step 1} + \text{Step 2})$:



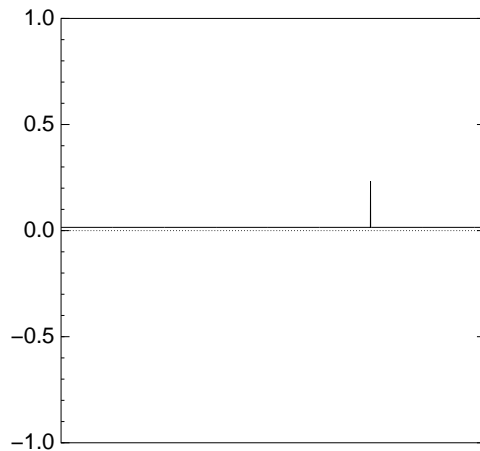
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $5 \times (\text{Step 1} + \text{Step 2})$:



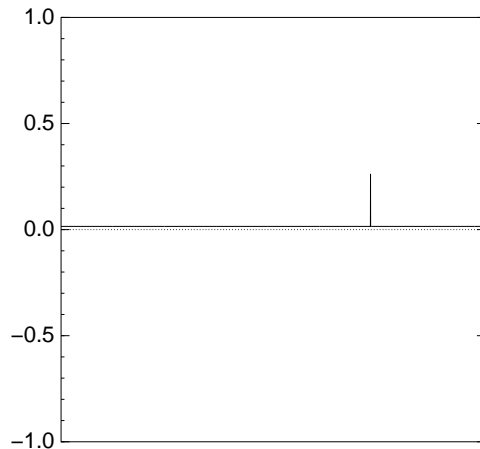
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $6 \times (\text{Step 1} + \text{Step 2})$:



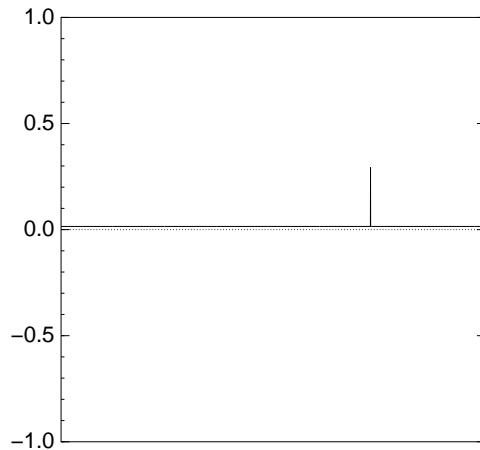
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $7 \times (\text{Step 1} + \text{Step 2})$:



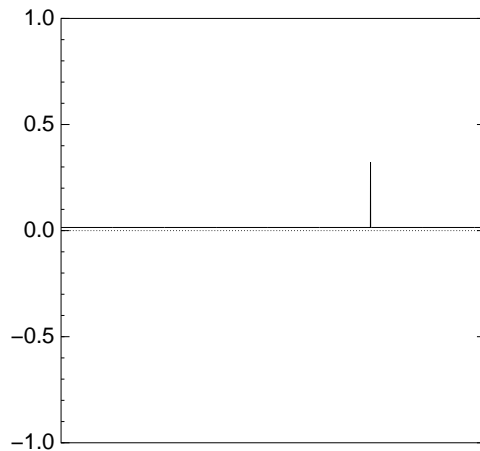
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $8 \times (\text{Step 1} + \text{Step 2})$:



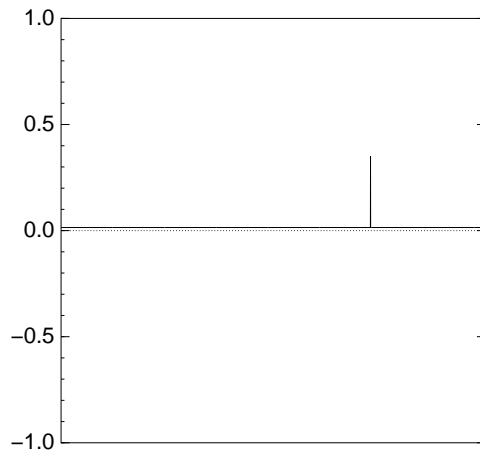
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $9 \times (\text{Step 1} + \text{Step 2})$:



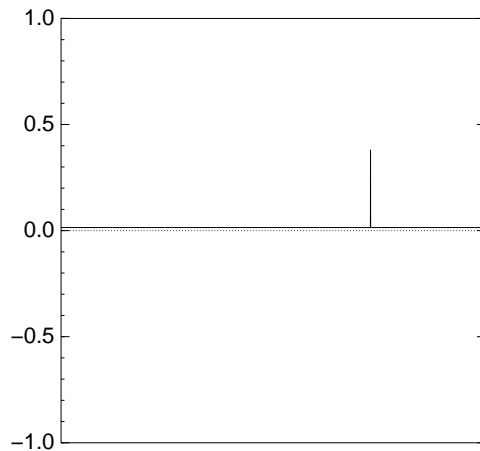
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $10 \times (\text{Step 1} + \text{Step 2})$:



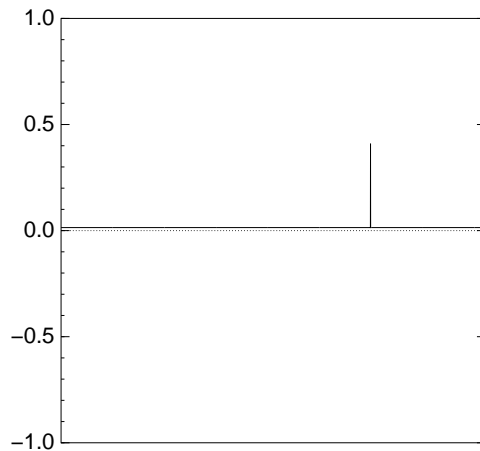
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $11 \times (\text{Step 1} + \text{Step 2})$:



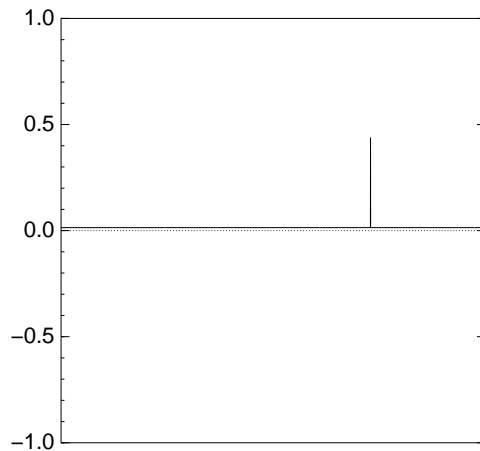
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $12 \times (\text{Step 1} + \text{Step 2})$:



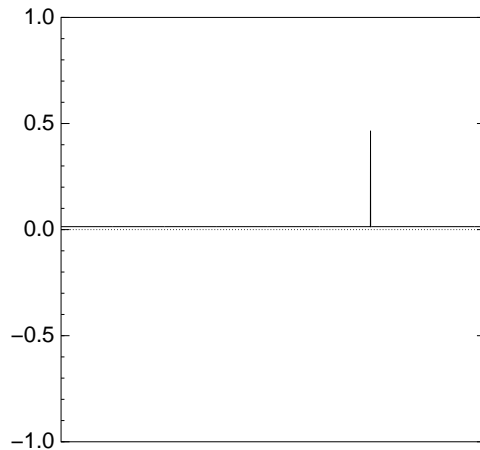
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $13 \times (\text{Step 1} + \text{Step 2})$:



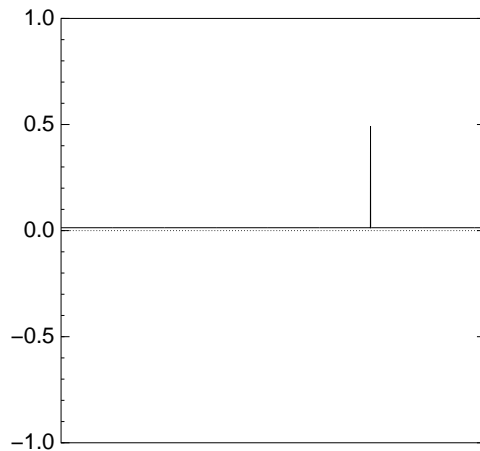
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $14 \times (\text{Step 1} + \text{Step 2})$:



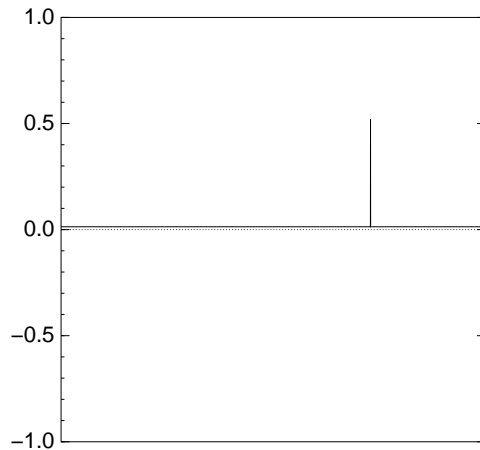
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $15 \times (\text{Step 1} + \text{Step 2})$:



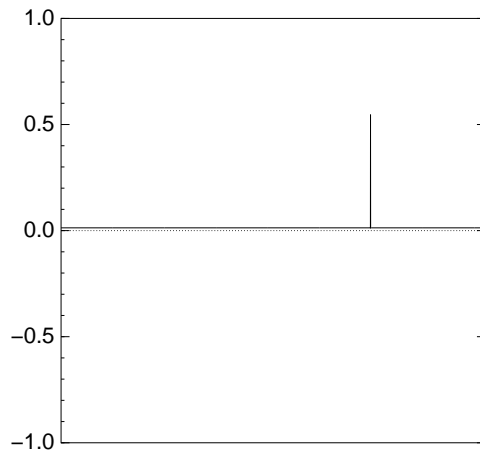
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $16 \times (\text{Step 1} + \text{Step 2})$:



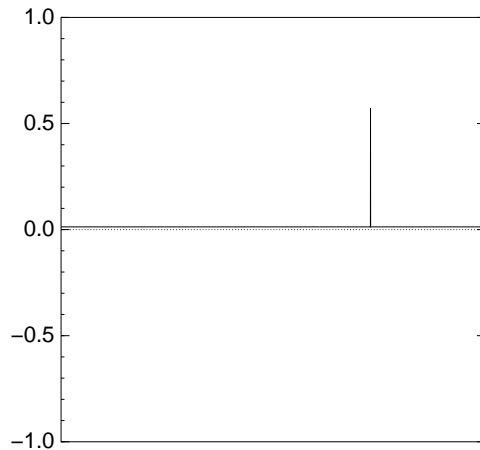
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $17 \times (\text{Step 1} + \text{Step 2})$:



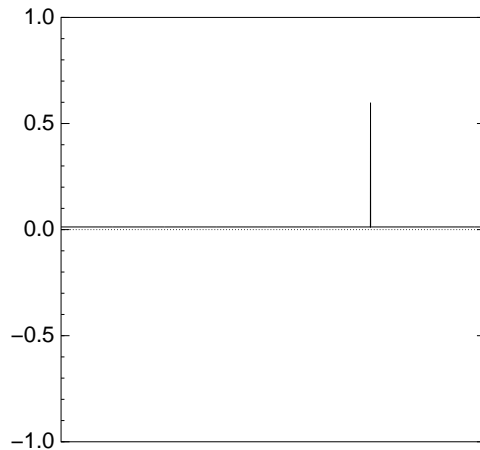
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $18 \times (\text{Step 1} + \text{Step 2})$:



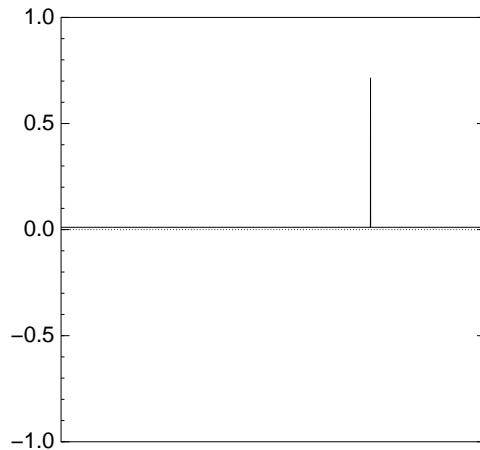
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $19 \times (\text{Step 1} + \text{Step 2})$:



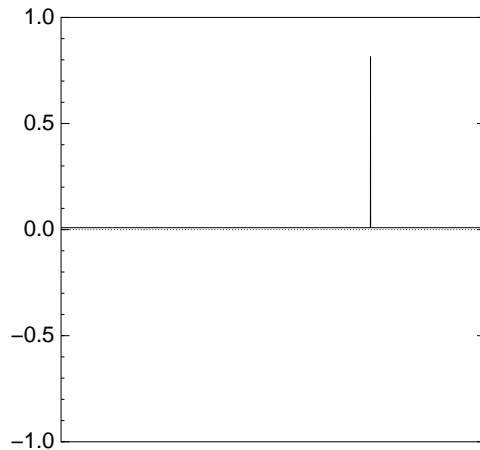
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $20 \times (\text{Step 1} + \text{Step 2})$:



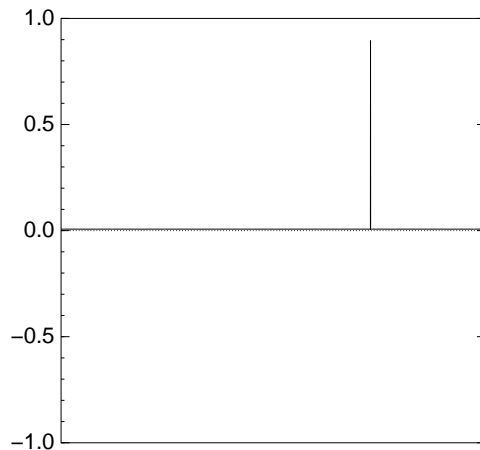
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $25 \times (\text{Step 1} + \text{Step 2})$:



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $30 \times (\text{Step 1} + \text{Step 2})$:

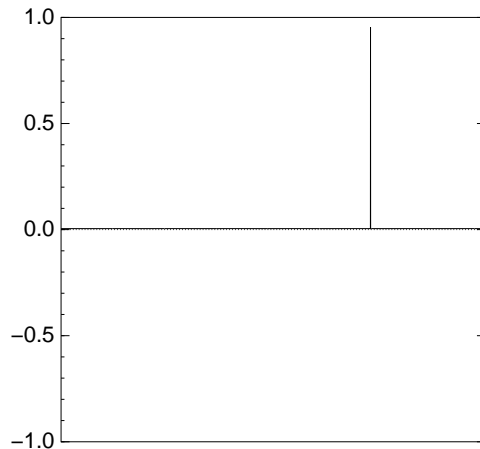


Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $35 \times (\text{Step 1} + \text{Step 2})$:

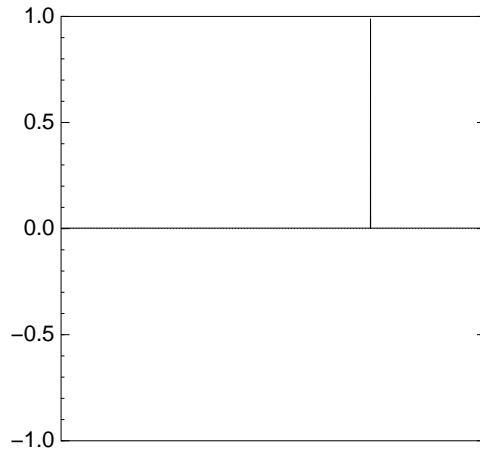


Good moment to stop, measure.

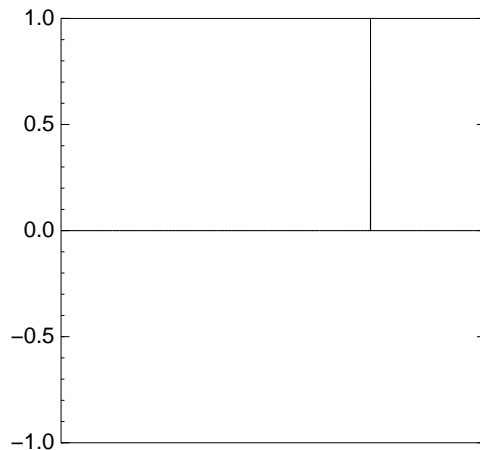
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $40 \times (\text{Step 1} + \text{Step 2})$:



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $45 \times (\text{Step 1} + \text{Step 2})$:

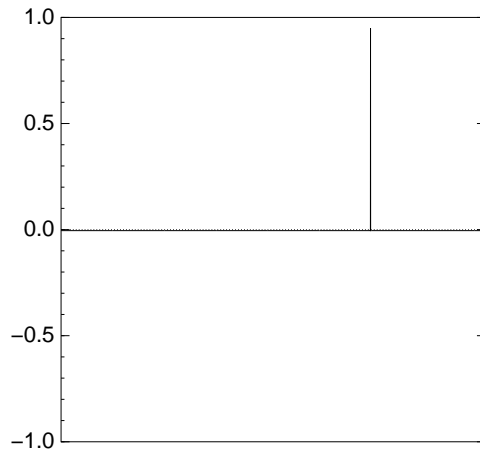


Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $50 \times (\text{Step 1} + \text{Step 2})$:

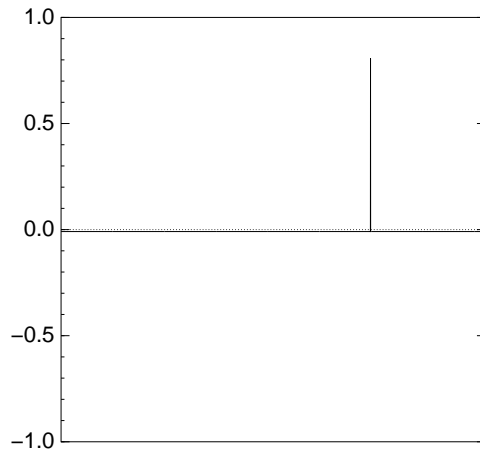


Traditional stopping point.

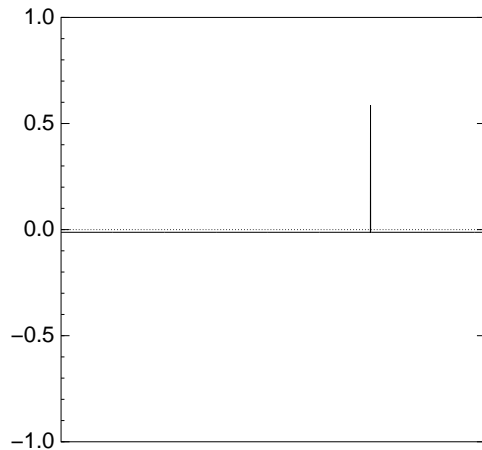
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $60 \times (\text{Step 1} + \text{Step 2})$:



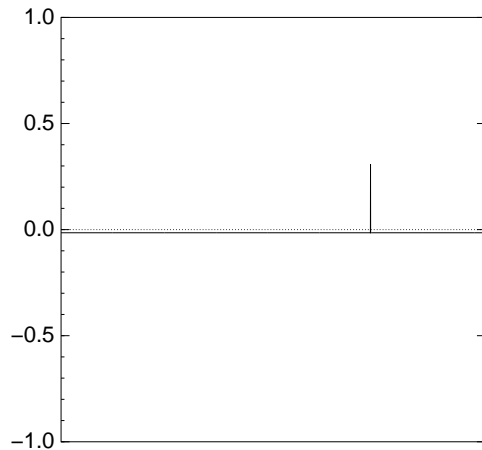
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $70 \times (\text{Step 1} + \text{Step 2})$:



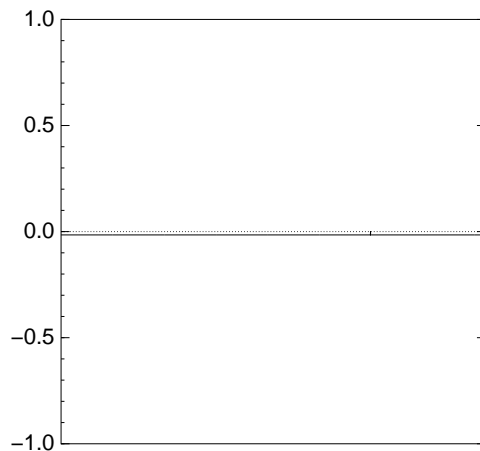
Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $80 \times (\text{Step 1} + \text{Step 2})$:



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $90 \times (\text{Step 1} + \text{Step 2})$:



Normalized graph of $u \mapsto a_u$
for an example with $n = 12$
after $100 \times$ (Step 1 + Step 2):



Very bad stopping point.