3 3rd Assignments

3.1 Group E - Phonon Gas

Photon $\rightarrow$ Quantize particles of light wave
Similarly, Phonon $\rightarrow$ Quantize particles of sound wave

In solid, vibration of lattice can be realized in terms of billions of phonons.
N atoms $\rightarrow$ 3N no of atoms

Classical Stationary waves
$L = \frac{n\lambda_n}{2}$
where λ_n is wavelength of stationary waves
$y = \sum_{n=1}^{\alpha} A_n \sin \frac{2\pi x}{\lambda_n}$ $\qquad\qquad \lambda_n = \frac{2L}{n}$

Phonons in solid with length L or volume V
$L \sim \lambda_1 = \frac{h}{p_1} = \frac{2\pi}{k_1}$
$L \sim 2\lambda_2 = \frac{2h}{p_2} = \frac{4\pi}{k_2}$
$L \sim n\lambda_n = \frac{nh}{p_n} = \frac{n2\pi}{k_n}$
$\implies p_n = \frac{nh}{L}$ or $K_n = n\frac{2\pi}{L}$
Lowest momentum = momentum pixel(Δp)
$\Delta p = \frac{h}{L}$ or $\Delta K = \frac{2\pi}{L}$

Debye assumes an upper momentum P_D or K_D or upper energy E_D or W_D. So
$3\int_0^{K_D} \frac{d^3K}{(\Delta K)^3} = 3N$
$\implies \int_0^{K_D} \frac{4\pi K^2 dK}{(\frac{2\pi}{L})^3} = N \implies \frac{V}{2\pi^2}\int_0^{K_D} K^2 dK = N$
$\implies K_D = \left[\frac{6\pi^2}{\nu}\right]^{\frac{1}{3}}$

Assuming an upper energy ϵ_D, proposed by Debye, we can get total number of states(=3N = no of phonons)

$$3\int_0^{\epsilon_D} \frac{d^3x d^3P}{h^3} = 3N$$

$$\frac{3V4\pi}{h^2C^3}\left[\frac{\epsilon_D^3}{3}\right] = 3N$$

$$\implies N = \frac{4\pi V}{h^3C^3}\frac{\epsilon_D^3}{3}$$

$$\implies \epsilon_D = \left[\frac{3N}{V}\frac{h^3C^3}{4\pi}\right]^{\frac{1}{3}}$$

$\hbar\omega_D = hc\left[\frac{3}{v4\pi}\right]^{\frac{1}{3}} \implies \omega_D = c\left[\frac{6\pi^2}{v}\right]^{\frac{1}{3}}$

$$K_D = \frac{\omega_D}{c} = \left[\frac{6\pi^2}{v}\right]^{\frac{1}{3}}$$

$$P_D = \hbar K_D = \frac{h}{2\pi}\left(\frac{6\pi^2}{v}\right)^{\frac{1}{3}}$$

$$\begin{aligned}
Internal Energy, U &= 3\int \frac{d^3x d^3P}{h^3}\frac{\epsilon}{e^{\beta\epsilon}-1} \\
&= \frac{3V}{h^3}\int_0^{\epsilon_D} 4\pi\frac{\epsilon^2}{c^2}\frac{d\epsilon}{c}\frac{\epsilon}{e^{\beta\epsilon}-1} \\
&= \frac{12\pi V}{h^3}\frac{(KT)^4}{c^3}\int_0^{\beta\epsilon_D}\frac{x^3dx}{e^x-1} \\
\frac{U}{N} &= \frac{g(KT)^4}{\epsilon_D^3}\int_0^{\beta\epsilon_D}\frac{x^3dx}{e^x-1}
\end{aligned}$$

$$\frac{U}{N} = 3KT\left[\frac{3}{(\beta\epsilon_D)^3}\int_0^{\beta\epsilon_D}\frac{x^3}{e^x-1}\right], \text{where } D(\beta\epsilon_D) \text{ is Debye function}$$

Now Debye function $D(t)$ is defined as

$$D(t) = \frac{3}{(t)^3}\int_0^t \frac{x^3}{e^x-1} = \begin{cases} 1-\frac{3t}{8}+\frac{1t^2}{20}+\cdots (t \ll 1) \\ \frac{\pi^4}{5t^3}+O\left(e^{-t}\right) \\ \quad (t \gg 1) \end{cases}$$

Here $t = \beta\epsilon_D = \frac{\epsilon_D}{KT} = \frac{T_D}{T}$ where $T_D = \frac{\epsilon_D}{K} = \frac{\hbar\omega_D}{k}$ is Debye temperature. So in terns of T_D, we can say

$$D\left(\frac{T_D}{T}\right) = \frac{3}{\left(\frac{T_D}{T}\right)_0^3}\int_0^{T_D/T}\frac{x^3dx}{e^x-1} = \begin{cases} 1-\frac{3}{8}\frac{T_D}{T}+\cdots (T_D \ll T) \\ \frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3+O\left(e^{-T_D/T}\right) \\ \quad (T_D \gg T) \end{cases}$$

so for large T, $\frac{U}{N} = 3KTD\left(\frac{T_D}{T}\right) \approx 3KT$
for small T, $\frac{U}{N} = 3KTD\left(\frac{T_D}{T}\right) \approx 3KT\left[\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3\right]$

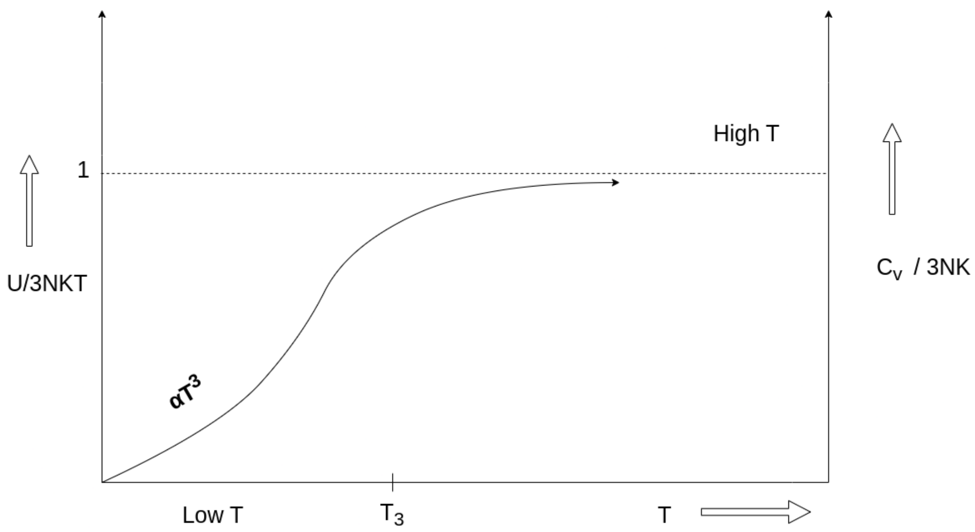


Figure 1: Phonon Gas

3.2 Group C

Now if we take average mass density,

$$\rho(r) \approx \langle\rho\rangle = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\langle\rho\rangle = \frac{\int_0^R \rho(r) 4\pi r^2\, dr}{\int_0^R 4\pi r^2\, dr} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\Omega = -\int_0^R \frac{GM(r)}{r} 4\pi r^2 \rho(r)\, dr \approx -\int_0^R G\frac{4}{3}\pi r^3 \rho(r) 4\pi r \rho(r)\, dr$$

$$\Omega = -G\frac{4\pi}{3} \times 4\pi\langle\rho\rangle^2 \left[\frac{R^5}{5}\right]$$

$$\Omega = -\frac{3}{5R}\left[\frac{4\pi}{3}R^3\langle\rho\rangle\right]^2$$

Using $M(r) = \frac{4}{3}\pi r^3 \rho(r)$ and $\rho(r) \approx \langle\rho\rangle$

$$\Omega = -\frac{3}{5}\frac{GM^2}{R} \tag{4}$$

Now internal energy/kinetic energy

$$U = \frac{3}{2}PV = \int_0^R \frac{3}{2}\rho(r) 4\pi r^2\, dr$$

Using $P = \frac{\langle\rho\rangle}{m}K_B T$

Where m is the mass of gas particles

$$U = \frac{3}{2m}K_B \int_0^R T(r)\langle\rho\rangle 4\pi r^2\, dr$$

Using $\langle T\rangle = \frac{1}{M}\int_0^R T(r)\langle\rho\rangle 4\pi r^2\, dr$

$$U = \frac{3}{2}\frac{K_B}{m}M\langle T\rangle \tag{5}$$

Virial theorem 3 becomes $-2U = \Omega$

$$\Rightarrow -2\left(\frac{3}{2}K_B\langle T\rangle\frac{M}{m}\right) = -\frac{3}{5}\frac{GM^2}{R}$$
$$\langle T\rangle = \frac{1}{5}\frac{GMm}{K_B R}$$

As we know,

$$\langle\rho\rangle\frac{4}{3}\pi R^3 = M$$
$$\Rightarrow R^3 = \frac{3M}{4\pi\langle\rho\rangle}$$

Thus,

$$\langle T\rangle = \frac{1}{5}\frac{Gm}{K_B}M\left(\frac{4\pi\langle\rho\rangle}{3M}\right)^{\frac{1}{3}}$$
$$\langle T\rangle \propto M^{\frac{2}{3}}\langle\rho\rangle^{\frac{1}{3}}$$

Now,

$$m_H = 1.6\times 10^{-27}$$
$$G = 6.6\times 10^{-11}m^3/kg/s^2$$
$$K_B = 1.38\times 10^{-23}m^2kgs^{-2}K^{-1}$$

For Sun,

$$M = 1.9\times 10^{30}\text{kg}$$
$$R = 6.9\times 10^{8}\text{m}$$

Calculating value of $\langle T\rangle$

$$\langle T\rangle = \frac{1}{5}\times\frac{6.6\times 10^{-11}\times 1.9\times 10^{30}}{1.38\times 10^{-23}\times 6.9\times 10^{8}}\times 1.6\times 10^{-27}$$
$$= \left(\frac{6.6\times 1.9\times 1.6}{5\times 1.38\times 6.9}\right)\times\frac{10^{-8}}{10^{-15}}$$
$$= 4\times 10^{6}$$

3.3 Group D : ******

Number Density is represented by n

$$n = g\int_0^{P_f} \frac{d^3P}{h^3}1 = g\frac{4\pi}{h^3}(\frac{P_f^3}{3}) \tag{4}$$

$$E_f = \frac{P_f^2}{2m} = \sqrt{P^2c^2 + m^2c^2}$$

$$\Rightarrow P_F\left[\frac{3h^3}{4\pi g}n\right]^{1/3}$$

Non Relative,

$$n = g\frac{4\pi}{3h^3}\left(2mE_F\right)^{3/2} \qquad N \cdot R \tag{5}$$

Relative,

$$= g\frac{4\pi}{3h^3}\left(\frac{E_F^2}{c^2} - m^2c^2\right)^{3/2} \qquad R \tag{6}$$

$$= 8\frac{4\pi}{3h^3}\frac{E_F^3}{c^3}$$

R for m=0

$$\epsilon = g\int_0^{p_F}\frac{d^3p}{h^3}E = \frac{g}{h^3}\int_0^{P_F} 4\pi P^2 dP\left(\frac{P^2}{2m}\right)$$

Non Relative,

$$= \frac{g}{h^3}\frac{4\pi}{2m}\frac{P_F^5}{5} \qquad NR$$

$$= \frac{g}{5h^3}\frac{4\pi}{2m}\left(2mE_p\right)^{5/2}$$

$$\langle E\rangle = \frac{\epsilon}{n} = \frac{3}{5}E_F$$

$$p = g\int_0^{p_F}\frac{d^3P}{h^3}\left(\frac{pv}{3}\right) \qquad v = \frac{p}{m}$$

$$= \frac{g}{3h^3}\int_0^{P_f} 4\pi p^2\left(\frac{pp}{m}\right)$$

$$= \frac{g}{3h^3}\frac{4\pi}{m}\frac{P_F^5}{5} \qquad \Rightarrow P_F = \left[\frac{15Ph^3m}{g4r}\right]^{\frac{1}{5}} \tag{7}$$

$$= \frac{g4n}{15h^3m}\left[\frac{3h^3}{4\pi g}n\right]^{5/3}$$

$$= \underbrace{\frac{1}{5m}\left[\frac{3h^3}{4\pi g}\right]^{2/3}}_{k} n^{5/3}$$

Hydrodynamical Equlibriun

$$\frac{1}{\rho_e}\frac{dP_e}{dr} = \frac{GM}{R^2}$$

$$M = \frac{4}{3}nR^3\rho_e$$

$$\rho_e = \frac{M}{\frac{4}{3}\pi^3}$$

$$\propto \frac{M}{R^3}$$

$$\int_0^R \frac{dP_e}{dr}dr = \int_0^R \frac{GM}{R^2}\frac{M}{\frac{4\pi R^3}{3}}dr$$

$$P = \frac{GM^2}{\frac{4\pi}{3}}\frac{R^{-5+1}}{-5+1} \tag{8}$$

$$P_e \propto \frac{M^2}{R^4} \tag{9}$$

But degenerate pressure,

$$P_e \propto \rho_e^{5/3}$$

$$\propto \frac{M^{5/3}}{R^5} \tag{10}$$

$$\Rightarrow \frac{M^{1/3}}{R} \propto \frac{\rho_e}{P_e} \Rightarrow R \propto \frac{1}{M^{1/3}}$$

$$\Rightarrow \rho \propto \frac{M}{R^3}$$

$$\rho_e \propto M^2$$

For Relativistic

$$P = \frac{m_e c^2}{8\pi^2 x_e^3}\phi\left(x_e\right) \qquad x_e = \frac{P_F}{m_e c}$$

$$\rho = \hat{\mu} m_p c^2 n_e$$

$$\rho = \begin{cases} P \propto \rho^{\frac{5}{3}}, & \text{for } x_e \ll 1 \\ \propto \rho^{\frac{4}{3}}, & \text{for } x_e \gg 1 \end{cases}$$

$$\underbrace{\left(\frac{9\pi M}{4\hat{\kappa} m_p}\right)^{4/3} - \frac{3\pi\alpha}{\hbar C} GM^2}_{>0} = \left(\frac{R}{x_e}\right)^2 \left(\frac{9\pi M}{4\hat{\mu} m_p}\right)^{2/3}$$

$$M \leq (\frac{9\pi}{4\hat{\mu} m_p})^2 (\frac{\hbar c}{3\pi\alpha G}$$

$$m_p = \sqrt{\frac{\hbar c}{G}} = 10^{19} GeVe^{-2} \tag{11}$$

3.4 Group F : ******

$$number\ N = \Sigma \left(\frac{1}{e^{\beta(\epsilon-\mu)} + \eta}\right)$$

where if
$\eta = 0$ MB
= 1 FD
= -1 BE

$$Internal\ Energy\ U = \Sigma \left(\frac{\epsilon}{e^{\beta(\epsilon-\mu)} + \eta}\right)$$

$$Grand\ potential\ \phi = \frac{-KT}{\eta} \Sigma ln(1 + \eta e^{-\beta(\epsilon-\mu)})$$

fugacity z = $e^{\beta\mu} \rightarrow \int \frac{d^3 P d^3 x}{h^3}$

In integration form, $\Sigma \rightarrow V \int D(\epsilon) d\epsilon$, where D($\epsilon$) is density of states.

Therefore,

$$N = V \int \frac{d^3 P}{h^3} \frac{1}{e^{\beta(\epsilon-\mu)} + \eta}$$

$$D(\epsilon) d\epsilon = D(P) dP = \frac{d^3 P}{h^3} = \frac{4\pi P^2 dP}{h^3}$$

$$U = V \int \frac{d^3 P}{h^3} \frac{\epsilon}{e^{\beta(\epsilon-\mu)} + \eta}$$

$$\phi = -PV = \frac{-KT}{\eta} V \int \frac{d^3P}{h^3} ln(1 + \eta e^{-\beta(\epsilon-\mu)})$$

Therefore,

$$P = \frac{KT}{\eta} \int \frac{d^3P}{h^3} ln(1 + \eta e^{-\beta(\epsilon-\mu)}) \tag{12}$$

Bose Gas

$$N = \sum_{\epsilon} \frac{1}{e^{\beta(\epsilon-\mu)} - 1} = \frac{1}{z^{-1}e^{\beta\epsilon} - 1}$$

$$N = \sum_{\epsilon>0} \frac{1}{e^{\beta(\epsilon-\mu)} - 1} + \frac{1}{e^{-\beta\epsilon} - 1} = \sum_{\epsilon>0} \frac{1}{z^{-1}e^{\beta\epsilon} - 1} + \frac{1}{z^{-1} + 1}$$

where the second parts are contribution for $\epsilon = 0$

$$N = \int \frac{d^3Pd^3x}{h^3} \frac{1}{e^{\beta(\epsilon-\mu)} - 1} + \frac{z}{1-z}$$

$$N = \frac{V}{h^3} \int_0^{\alpha} 2\pi(2m)^{3/2} \epsilon^{1/2} d\epsilon \frac{1}{e^{\beta(\epsilon-\mu)} - 1} + \frac{z}{1-z}$$

Now, for non relativistic scenario, $\epsilon = \frac{p^2}{2m}$
$d^3P = 4\pi P^2 dP = 4\pi\sqrt{2m\epsilon}$
PdP = mdϵ

Look: First term at $\epsilon = 0$, the integrand gets vanished, therefore the 2nd term is added separately.

$$PV = -\Phi = -KT \sum_{\epsilon} ln\{1 - e^{-\beta(\epsilon-\mu)}\}$$

$$PV = -KT \sum_{\epsilon>0} ln\{1 - ze^{-\beta\epsilon}\} + ln(1-z)]$$

$$PV = -KT[\frac{V}{h^3} 2\pi(2m)^{3/2} \int_0^{\alpha} \epsilon^{1/2} d\epsilon ln(1 - ze^{-\beta\epsilon}) + ln(1-z)]$$

$$N = V \int_0^{\infty} \frac{d^3P}{h^3} \frac{1}{(z^{-1}e^{\beta\epsilon} - 1)} + \frac{z}{1-z} \tag{13}$$

$$P = -KT \int \frac{d^3P}{h^3} ln(1 - ze^{-\beta\epsilon}) - \frac{KT}{V} ln(1-z) \tag{14}$$

3.5 Group G : Bose

Let us call $N_0 = \frac{\cancel{z}}{1-\cancel{z}}$ is the number of particles at $\epsilon = 0$

Pressure:-

So the pressure $P_0 = -\frac{KT}{V} ln(1-\cancel{z})$ at ground state

$= +\frac{KT}{V} ln(1+N_0)$ $\qquad [\because \cancel{z} = \frac{N_0}{1+N_0}]$

$\approx KT \frac{1}{N} lnN \longrightarrow$ Negligible for large N

So $\frac{P}{KT} = -\frac{2\pi}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \epsilon^{\frac{1}{2}} ln(1-\cancel{z}\ e^{-\beta\epsilon})\, d\epsilon$

$= \frac{g_{5/2}(\cancel{z})}{\lambda^3}$

where $\lambda = \frac{h}{(2\pi mKT)^{1/2}} \Rightarrow$ Thermal deBroglie Wavelength

$$g_n(\cancel{z}) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{\cancel{z}^{-1} e^x - 1} = \cancel{z} + \frac{\cancel{z}^2}{2^n} + \frac{\cancel{z}^3}{3^n} + \ldots\ldots \tag{15}$$

where $g_n(\cancel{z})$ is the Bose-Einstein function

No. of particles:-

$\frac{N}{V} - \frac{N_0}{V} = \frac{2\pi(2m)^{\frac{3}{2}}}{h^3} \int_0^\infty \epsilon^{\frac{1}{2}} \frac{1}{\cancel{z}^{-1} e^{\beta\epsilon} - 1}\, d\epsilon$

$= \frac{g_{1/2}(\cancel{z})}{\lambda^3} = \frac{N_e}{V}$

where $N_0 \longrightarrow$ No. of particles at ϵ and

$N_e \longrightarrow$ No. of particles at ϵ or in excited state

Since $\cancel{z} = e^{\beta\mu} \leq 1$ for Bose gas,

$\Rightarrow \frac{N-N_0}{V} = \frac{N_e}{V} \leq \frac{g_{\frac{3}{2}}(\cancel{z}=1)}{\lambda^3} = \frac{\zeta_{\frac{3}{2}}}{\lambda^3}$

3.6 Group I : ******

$$\frac{P}{KT} = -\frac{2\pi}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \epsilon^{\frac{1}{2}} ln(1 - Ze^{-\beta\epsilon})\, d\epsilon$$

$$\Rightarrow \beta\varepsilon = x$$

$$\Rightarrow d\epsilon = \frac{dx}{\beta} = KT\, dx$$

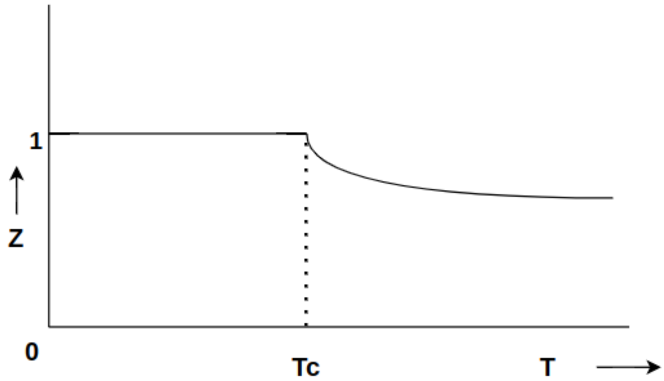
$$\begin{aligned}
\frac{P}{KT} &= -2\pi\left(\frac{2mKT}{h^2}\right)^{\frac{3}{2}} \int_0^\infty x^{\frac{1}{2}} ln(1 - Ze^{-x})\, dx \\
&= -2\pi\left(\frac{2mKT}{h^2}\right)^{\frac{3}{2}} \left[\left| \frac{x^{\frac{3}{2}}}{\frac{3}{2}} ln(1 - Ze^{-x}) \right|_0^\infty - \int_0^\infty \frac{Ze^{-x}}{1 - Ze^{-x}} \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\, dx \right] \\
&= 2\pi\left(\frac{2mKT}{h^2}\right)^{\frac{3}{2}} \frac{1}{\frac{3}{2}} \int_0^\infty \frac{x^{\frac{3}{2}}}{Z^{-1}e^x - 1}\, dx \\
&= \underbrace{\left(\frac{2\pi mKT}{h^2}\right)^{\frac{3}{2}}}_{\lambda^3} \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} \underbrace{\int_0^\infty \frac{x^{\frac{5}{2}-1}}{Z^{-1}e^x - 1}\, dx}_{g_{\frac{5}{2}(z)}} \qquad \text{as } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}
\end{aligned}$$

Thermal debroglie wavelength BE function

BE function	$\longleftrightarrow$	Rieman Zeta function
$g_n(Z)$		$\zeta_n = \sum_{l=1}^\infty \frac{1}{l^n}$

$$\Rightarrow \boxed{g_n(Z = 1) = \zeta_n} \tag{16}$$

If we assume a picture, as $T \downarrow \Rightarrow Z \downarrow$



$$\begin{array}{ccc} \text{or High T} & & Z<1 \\ \downarrow & \Rightarrow & \downarrow \\ \text{low T} & & \text{Z} = 1 \end{array}$$

and let us assume a characteristic temperature T_c, where Z becomes unity.

Before coming to $T = T_c$, i.e. $T > T_c$, we can assume

$$\begin{aligned} N_o &\to low \\ &\qquad\qquad \Rightarrow N_e \approx N\,(\text{where} N = N_o + N_e) \\ N_e &\to high \end{aligned}$$

Now at $T = T_c$, Z $= 1$

$$\Rightarrow g_{\frac{3}{2}}(Z=1) = \zeta_{\frac{3}{2}}$$
$$\Rightarrow \frac{N_e}{V} = \frac{g_{\frac{3}{2}}(Z=1)}{\lambda^3} = \frac{\zeta_{\frac{3}{2}}}{\lambda^3}$$
$$\Rightarrow \frac{N_e}{V} = \zeta_{\frac{3}{2}} \frac{(2\pi mKT_c)^{\frac{3}{2}}}{h^3}$$

$$\Rightarrow \frac{N_e}{V} = \zeta_{\frac{3}{2}} \frac{(2\pi mK)^{\frac{3}{2}}}{h^3} T_c^{\frac{3}{2}}$$

$$\Rightarrow T_c^{\frac{3}{2}} = \frac{h^3}{(2\pi mK)^{\frac{3}{2}} \zeta_{\frac{3}{2}}} \left(\frac{N_e}{V} \right)$$

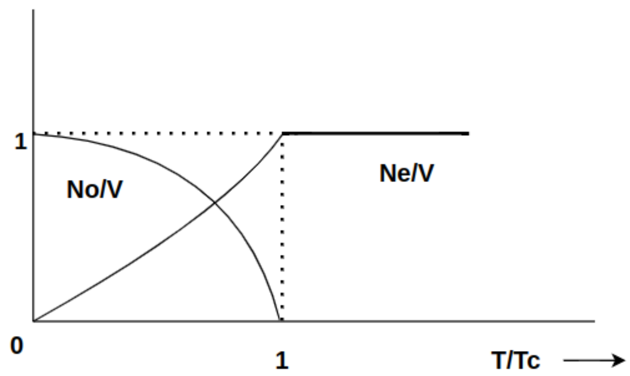
$$\Rightarrow T_c = \frac{h^2}{2\pi mK} \left\{ \frac{N}{V\mathcal{G}_{\frac{3}{2}}} \right\}^{\frac{2}{3}} \qquad \text{assuming } N_e \approx N$$

$$\frac{T_c}{T} = (\frac{h^2}{2\pi mKT}) \left\{ \frac{N}{V\zeta_{\frac{3}{2}}} \right\}^{\frac{2}{3}}$$

$$= \lambda^2 \left\{ \frac{N}{V\zeta_{\frac{3}{2}}} \right\}^{\frac{2}{3}}$$

$$= \left\{ \frac{\lambda^3 N}{V\zeta_{\frac{3}{2}}} \right\}^{\frac{2}{3}}$$

$$\frac{\lambda^3 N}{V\zeta_{\frac{3}{2}}} = \left(\frac{T_c}{T} \right)^{\frac{3}{2}} \tag{17}$$



So for $0 \leq$ T $\leq T_C$, Z remains unity.

$$
\begin{aligned}
\frac{N}{V} - \frac{N_o}{V} &= \frac{N_e(Z=1)}{V} \\
&= \frac{g_{\frac{3}{2}}(Z=1)}{\lambda^3} \\
\frac{N}{N} - \frac{N_o}{N} &= \frac{V}{N}\frac{\zeta_{\frac{3}{2}}}{\lambda^3}
\end{aligned}
$$

$$
\Rightarrow \frac{N_o}{N} = 1 - (\frac{T}{T_c})^{\frac{3}{2}} \qquad T \leq T_c
$$

$$
\Rightarrow \frac{N_e}{N} = 1 - \frac{N_o}{N} = (\frac{T}{T_c})^{\frac{3}{2}} \quad T \leq T_c \tag{18}
$$

3.7 Group J : ******

3.8 Group K : ******

$$
\begin{aligned}
\text{Internal Energy } U &= \int \frac{d^3xd^3\rho}{h^3}\frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1} \\
\implies U &= \frac{V}{h^3}2\pi(2m)^{\frac{3}{2}}\int_0^{\inf}\frac{\epsilon^{\frac{3}{2}}}{z^{-1}e^{\beta\epsilon}-1} \\
&= \frac{2\pi V}{h^3}(2mKT)^{\frac{3}{2}}\int_0^{\inf}\frac{x^{\frac{5}{2}-1}z^{-1}e^x-1}{d}x \\
\beta\epsilon &= x \\
d\epsilon &= KTdx \\
\frac{U}{KT} &= V(\frac{2mnKT}{h^2})^{\frac{3}{2}}\frac{1}{\frac{1}{2\sqrt{\pi}}}\int_0^{\inf}\frac{x^{\frac{5}{2}-1}}{z^{-1}e^x-1}dx \\
&= \frac{V}{\lambda^3}\frac{3}{2}\frac{1}{\Gamma(\frac{5}{2})} \\
&= \frac{3}{2}V\frac{g_{\frac{5}{2}}(z)}{\lambda^3}
\end{aligned}
$$

$$\text{Since } \frac{P}{KT} = \frac{g_{\frac{5}{2}}(z)}{\lambda^3}$$

So, $$\boxed{U = \tfrac{3}{2}PV} \quad ......(1)$$

$$\text{Relation in also valid for ideal gas - } \mho = \frac{3}{2}NKT$$

$$PV = NKT$$

$$\implies U = \frac{3}{2}PV$$

$$\frac{U}{NK} = \frac{3}{2}\nu\frac{g_{\frac{5}{2}}(z)}{\lambda^3}T,\ T > T_e \text{ or } z < 1$$

$$= \frac{3}{2}\nu\frac{g_{\frac{5}{2}}}{\lambda^3}T,\ T \leq T_e \text{ or } z = 1$$

$$\frac{T}{\lambda^3} = aT^{\frac{5}{2}}, \text{ where } a = \left(\frac{2m\pi K}{h^2}\right)^{\frac{3}{2}}$$

$$\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{5}{2}aT^{3/2} = \frac{5}{2}\lambda^3 \qquad ......(2)$$

Now for $T \leq T_c$ (when $Z = 1$),

$$\frac{U}{NK} = \frac{3}{2}\nu\zeta_{5/2}\left(\frac{T}{\lambda^3}\right)$$

$$\Rightarrow \frac{C_v}{NK} = \frac{1}{NK}\frac{dU}{dT} = \frac{3}{2}\nu\zeta_{5/2}\frac{d}{dT}\left(\frac{T}{\lambda^3}\right)$$

$$= \frac{15}{4}\nu\left(\frac{\zeta_{5/2}}{\lambda^3}\right) \propto T^{3/2} \qquad ......(3)$$

Now at high T, when $z \ll 1$,

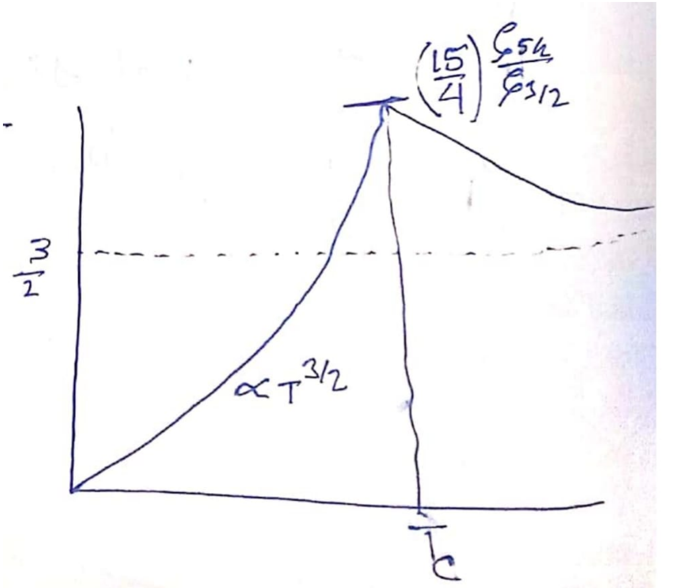
i.e. $g_{\frac{5}{2}}(x) \approx z \approx \frac{\lambda^3}{v}$,

So, $\frac{U}{NK} = \frac{3}{2}v\frac{g_{\frac{5}{2}}(x)}{\lambda^3}T \approx \frac{3}{2}T$,

$\Rightarrow$ $\frac{C_v}{NK} = \frac{1}{NK}\left(\frac{\partial U}{\partial T}\right) \approx \frac{3}{2}$,

At, $T = T_c$, $\left(\frac{v_c,\xi_{3/2}}{\lambda^3}\right) = 1$,

$\Rightarrow$ $\frac{C_v}{NK}(T = T_c) = \frac{15}{4}\frac{\xi_{5/2}}{\xi_{3/2}}$ ......(4).


$\left(\frac{15}{4}\right)\frac{\zeta_{5/2}}{\zeta_{3/2}}$
$\frac{3}{2}$
$\propto T^{3/2}$
T_c