CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Search

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Winter 2024

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Quantum Search

The Deutsch-Jozsa algorithm

- Generalization of Deutsch's Algorithm

Problem Definition

- Assume $n-$bit Boolean functions $f(x) : \{0,1\}^n \rightarrow \{0,1\}$
- Restriction (Any **one** holds):
  - f is **constant** $\implies$

$$f(x) = 0, \forall x \in \{0,1\}^n \text{ or, } f(x) = 1, \forall x \in \{0,1\}^n$$

  - f is **balanced** $\implies$

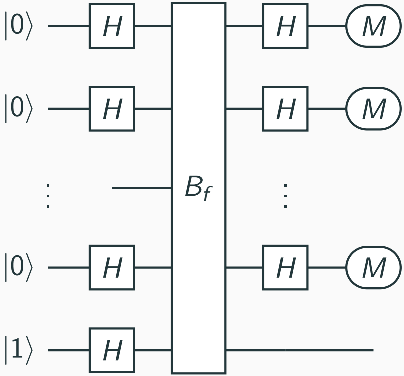
$$\Big|\{x \in \{0,1\}^n : f(x) = 0\}\Big| = \Big|\{x \in \{0,1\}^n : f(x) = 1\}\Big| = 2^{n-1}$$

  - Goal: To determine **which** of the **two** possibilities holds

- Classical
  - Probabilistic: Evaluate at random points, Check constancy. Reasonable success probability with **few** trials.
  - Deterministic: At least $(2^{n-1} + 1)$ queries needed
- Quantum - **One query will be sufficient**
  - Courtesy: The Deutsch-Jozsa Algorithm

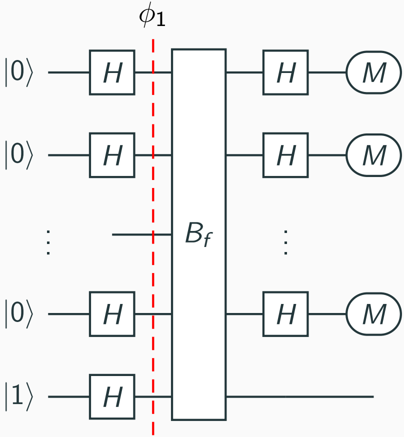
Quantum Transformation

Access to the function f is restricted to queries to a device corresponding to the transformation B_f defined as
$B_f \left|x\right\rangle \left|b\right\rangle = \left|x\right\rangle \left|b \oplus f(x)\right\rangle$



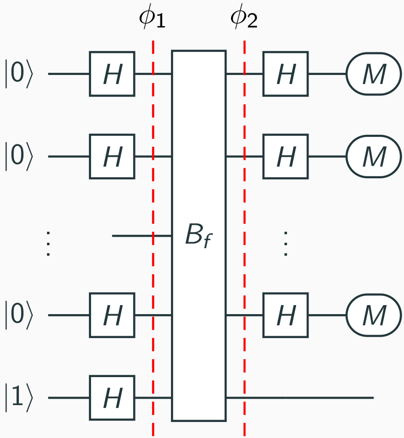
$$B_f\,|x\rangle\,|b\rangle = |x\rangle\,|b \oplus f(x)\rangle$$

If all n measurement results are 0, we conclude that the function was constant. Otherwise, if at least one of the measurement outcomes is 1, we conclude that the function was balanced.



- After $H-$transforms

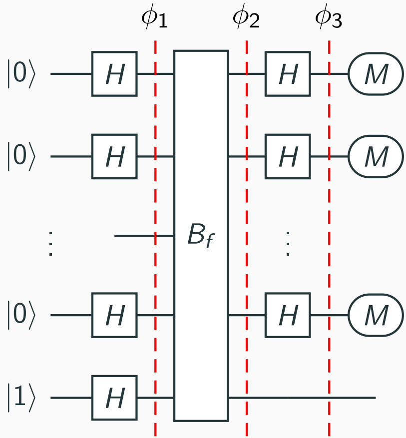
$$\phi_1 : H^{\otimes n} |0\rangle \otimes H |1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$



- After B_f-transform

$$\phi_2 : \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Step-3: ϕ_3 — The Deutsch-Jozsa Algorithm



- After second level $H-$transforms

$$\phi_3 : \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right)$$

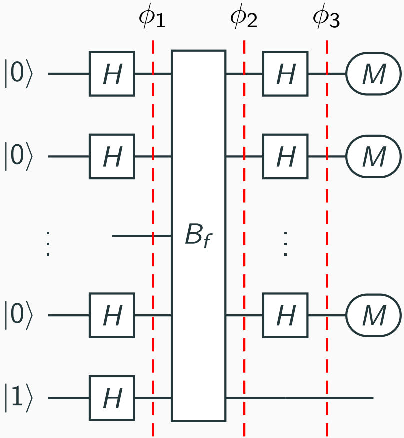
Recall, $H^{\oplus n}$

- After second level $H-$transforms

$$\phi_3 : \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right)$$
$$= \sum_{y \in \{0,1\}^n} \left(\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot y} \right) |y\rangle$$

- What is the probability that the measurements all give outcome 0?
- The amplitude associated with the classical state $|0^n\rangle$ is

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$



- The probability that the measurements all give outcome 0 is

$$\left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right|^2 = \begin{cases} 1 \text{ if } f \text{ is constanst} \\ 0 \text{ if } f \text{ is balanced} \end{cases}$$

References

1. Quantum Computing for Computer Scientists, by Noson S. Yanofsky, Mirco A. Mannucci
2. Quantum Computing Explained, David Mcmahon. John Wiley & Sons
3. Lecture Notes on Quantum Computation, John Watrous, University of Calgary
    - `https://cs.uwaterloo.ca/~watrous/QC-notes/`