

CS 553

Lecture 9
Linear Cryptanalysis

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Linear Cryptanalysis

- Less effective than Differential Cryptanalysis
- But is a Known Plaintext Attack (Recall DC is CPA)



- Credited to Matsui for applications on DES
- Earlier references on FEAL-4
 - By Tardy-Corfdir and Gilbert

Uses a linear relation between inputs and outputs of an encryption algorithm that holds with a certain probability

This approximation can be used to assign probabilities to the possible keys and locate the most probable one.

Linear Cryptanalysis

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Linear Approximation

Basic Idea

Uses a linear relation between inputs and outputs of an encryption algorithm that holds with a certain probability

This approximation can be used to assign probabilities to the possible keys and locate the most probable one.

► Consider the encryption scheme

$$c = m \oplus k$$
 $c, k, m \in \{0, 1\}^b$

► Bit expansion

$$\begin{bmatrix} c_0 & = & m_0 \oplus k_0 \\ c_1 & = & m_1 \oplus k_1 \\ & \vdots & & \\ c_{b-1} & = & m_{b-1} \oplus k_{b-1} \end{bmatrix} \rightarrow \begin{bmatrix} k_0 & = & m_0 \oplus c_0 \\ k_1 & = & m_1 \oplus c_1 \\ & \vdots & & \\ k_{b-1} & = & m_{b-1} \oplus c_{b-1} \end{bmatrix}$$

▶ Vulnerability if k reused \triangle What about KPA?

What did we do here?

Key expressed as a (linear) relation between plaintext and ciphertext

Consider the following 4-bit cryptosystem

$$c_{3} = m_{3} \oplus m_{1} \oplus m_{0} \oplus k_{3} \oplus k_{1} \oplus k_{0}$$

$$c_{2} = m_{2} \oplus m_{0} \oplus k_{2} \oplus k_{0}$$

$$c_{1} = m_{3} \oplus m_{2} \oplus k_{3} \oplus k_{2}$$

$$k_{3} = m_{3} \oplus c_{0} \oplus c_{3}$$

$$k_{2} = m_{2} \oplus c_{3} \oplus c_{1} \oplus c_{0}$$

$$k_{1} = m_{1} \oplus c_{3} \oplus c_{2} \oplus c_{1}$$

$$k_{0} = m_{0} \oplus c_{3} \oplus c_{2} \oplus c_{1} \oplus c_{0}$$

Reiterates the basic aim of LC

Constructing equations that express bits of the key in terms of bits of the message and ciphertext.

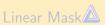
The idea of masking

Extracting specific bits using the mask vector

$$(1,0,0,0) imes egin{pmatrix} m_3 \ m_2 \ m_1 \ m_0 \end{pmatrix} = m_3 \qquad (1,0,1,0) imes egin{pmatrix} m_3 \ m_2 \ m_1 \ m_0 \end{pmatrix} = m_3 \oplus m_1$$

Linear combination using the mask vector

$$(1,0,1,1) \times \begin{pmatrix} m_3 \\ m_2 \\ m_1 \\ m_0 \end{pmatrix} \oplus (1,0,1,1) \times \begin{pmatrix} k_3 \\ k_2 \\ k_1 \\ k_0 \end{pmatrix} = m_3 \oplus m_1 \oplus m_0 \oplus k_3 \oplus k_1 \oplus k_0$$



Extracting specific bits using the mask vector

$$(1,0,0,0) imesegin{pmatrix} m_3 \ m_2 \ m_1 \ m_0 \end{pmatrix} = m_3 \qquad (1,0,1,0) imesegin{pmatrix} m_3 \ m_2 \ m_1 \ m_0 \end{pmatrix} = m_3\oplus m_1$$

► Linear combination using the mask vector

$$(1,0,1,1) imesegin{pmatrix} m_3 \ m_2 \ m_1 \ m_0 \end{pmatrix} \oplus (1,0,1,1) imesegin{pmatrix} k_3 \ k_2 \ k_1 \ k_0 \end{pmatrix} = m_3 \oplus m_1 \oplus m_0 \oplus k_3 \oplus k_1 \oplus k_0$$

Linear Mask 🕮

Linear Mask in Action

Recall our 4-bit cryptosystem

$$c_3 = m_3 \oplus m_1 \oplus m_0 \oplus k_3 \oplus k_1 \oplus k_0,$$

$$c_2 = m_2 \oplus m_0 \oplus k_2 \oplus k_0,$$

$$c_1 = m_3 \oplus m_2 \oplus k_3 \oplus k_2, \text{ and }$$

$$c_0 = m_1 \oplus m_0 \oplus k_1 \oplus k_0.$$

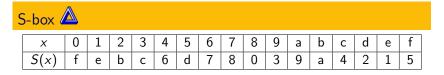
Consider the first equation

$$c_3 = m_3 \oplus m_1 \oplus m_0 \oplus k_3 \oplus k_1 \oplus k_0$$

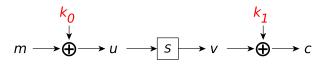
 \blacktriangleright With masks α, β this can be written as:

$$\alpha \cdot c = \beta \cdot m \oplus \beta \cdot k$$
, where $\alpha = \{1, 0, 0, 0\}, \ \beta = \{1, 0, 1, 1\}$

► Sypher00A encrypts 4 bits with two 4 bit keys



Encryption



► Same as Sypher001 in DC but different Sbox

▶ Linear approximation of $S[\cdot]$

To find some (α, β) such that

$$\Pr\left[\alpha \cdot x = \beta \cdot S[x]\right] \neq \frac{1}{2}$$

- ► Implication:
 - ► XOR of certain bits of the input to *S* equals
 - ightharpoonup XOR of certain bits in the output of S
 - ► With a probability "different from the random case"

$$m \longrightarrow \bigoplus^{k_0} \downarrow \qquad \qquad \downarrow^{k_1} \downarrow \qquad \qquad \downarrow^{k_2}$$

$$\alpha \cdot m = \alpha \cdot k_0 \oplus \alpha \cdot u \longrightarrow \text{Holds with prob. 1}$$
 (1)

$$\alpha \cdot u = \beta \cdot v$$
 \rightarrow Holds with prob. p (2)

$$\beta \cdot v = \beta \cdot k_1 \oplus \beta \cdot c \qquad \rightarrow \text{Holds with prob. 1}$$
 (3)

Adding Eqn. (1-3):

$$(\alpha \cdot m) \oplus (\alpha \cdot u) \oplus (\beta \cdot v) = (\alpha \cdot k_0) \oplus (\alpha \cdot u) \oplus (\beta \cdot v) \oplus (\beta \cdot k_1) \oplus (\beta \cdot c)$$

Simplifying

$$(\alpha \cdot m) \oplus (\beta \cdot c) = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$$

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Simplifying

$$(\alpha \cdot m) \oplus (\beta \cdot c) = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$$

$$\alpha \cdot m = \alpha \cdot k_0 \oplus \alpha \cdot u$$
 \rightarrow Holds with prob. 1 (1)
 $\alpha \cdot u = \beta \cdot v$ \rightarrow Holds with prob. p (2)
 $\beta \cdot v = \beta \cdot k_1 \oplus \beta \cdot c$ \rightarrow Holds with prob. 1 (3)

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 (1)

$$\alpha \cdot u = \beta \cdot v \qquad \rightarrow \text{Holds with prob. } p \qquad (2)$$

$$\beta \cdot v = \beta \cdot k_1 \oplus \beta \cdot c \qquad \rightarrow \text{Holds with prob. 1}$$
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Simplifying

$$(\alpha \cdot m) \oplus (\beta \cdot c) = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$$

$$m \xrightarrow{k_0} \downarrow \qquad \qquad \downarrow \downarrow \downarrow \\ m \xrightarrow{\downarrow} u \xrightarrow{\downarrow} s \xrightarrow{\downarrow} v \xrightarrow{\downarrow} c$$

► Adding Eqn. (1 – 3):

$$(\alpha \cdot m) \oplus (\alpha \cdot u) \oplus (\beta \cdot v) = (\alpha \cdot k_0) \oplus (\alpha \cdot u) \oplus (\beta \cdot v) \oplus (\beta \cdot k_1) \oplus (\beta \cdot c)$$

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- ightharpoonup p = 0 or p = 1, equally useful for attacker
- ▶ if p = 1, attacker recovers a key-bit using multiple (m, c)▶ Recall this a KPA
- ightharpoonup if p=0, attacker uses the same strategy with

$$(\alpha \cdot m) \oplus (\beta \cdot c) \oplus \mathbf{1} = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$$

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$$(\alpha \cdot m) \oplus (\beta \cdot c) \oplus \mathbf{1} = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$$



- ► Worst-case scenario for attacker: $p = \frac{1}{2}$
- Attacker gets no extra info about key-bit
- \triangleright 0/1 is equally probable
- Ideal from designer's perspective

Choose masks α and β so that equations in linear approximation

$$p = \frac{1}{2} + \epsilon$$

where ϵ , which is known as the bias is non-zero ("non-negligible")

- ▶ Target: $0 < |\epsilon| \le \frac{1}{2}$
- ightharpoonup Larger $|\epsilon| \Longrightarrow$ better attack

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Non-zero Bias

Aim of I C

Choose masks α and β so that equations in linear approximation hold with probability

$$p = \frac{1}{2} + \epsilon$$

where ϵ , which is known as the bias is non-zero ("non-negligible")

- ▶ Target: $0 < |\epsilon| \le \frac{1}{2}$
- ightharpoonup Larger $|\epsilon| \Longrightarrow$ better attack

$$\alpha = (1, 0, 0, 1), \beta = (0, 0, 1, 0)$$

Example (Sypher00A)

$$m \xrightarrow{k_0} u \xrightarrow{s} v \xrightarrow{k_1} c$$

X	0	1	2	3	4	5	6	7	8	9	а	b	С	d	е	f
S[x]	f	е	b	С	6	d	7	8	0	3	9	а	4	2	1	5
$\alpha \cdot x$	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0

$$\Pr\left[\alpha \cdot x = \beta \cdot S[x]\right] = \frac{2}{16}$$

or
$$\Pr \Big[lpha \cdot x \oplus 1 = eta \cdot S[x] \Big] = rac{14}{16}$$

$$\begin{array}{c}
k_0 \\
\downarrow \\
m \longrightarrow \bigoplus \longrightarrow u \longrightarrow \boxed{s} \longrightarrow v \longrightarrow \bigoplus \longrightarrow c
\end{array}$$

- ▶ In terms of Sypher00A $\longrightarrow \Pr\Big[\alpha \cdot u \oplus 1 = \beta \cdot v\Big] = \frac{14}{16}$
- ▶ i.e., $\Pr\Big[(\alpha \cdot m) \oplus (\beta \cdot c) \oplus 1 = (\alpha \cdot k_0) \oplus (\beta \cdot k_1)\Big] = \frac{14}{16}$

- ▶ Initialize counters T_0 and T_1 to 0
- ▶ Request the encryptions of *N* known plaintexts.
- For each plaintext-ciphertext pair, we compute the **left-hand** side of the equation: $(\alpha \cdot m) \oplus (\beta \cdot c) \oplus 1$,
 - ► Which is either 0 or 1.
- ▶ Gives an estimate for the value of $(\alpha \cdot k_0) \oplus (\beta \cdot k_1)$
- $ightharpoonup T_0++$ if LHS evaluates to 0; T_1++ if LHS evaluates to 1

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m \longrightarrow \bigoplus u \longrightarrow \boxed{s} \longrightarrow v \longrightarrow \bigoplus \cdots
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- $(\alpha \cdot k_0) \oplus (\beta \cdot k_1) \stackrel{?}{=} 0/1$
- ► Key-bit estimation correct with prob. $\frac{14}{16}$
- \blacktriangleright What to expect at T_0/T_1 after N KP encryptions

If
$$(\alpha \cdot k_0) \oplus (\beta \cdot k_1) = 1$$

$$T_0 \leftarrow \frac{2N}{16}$$

$$T_1 \leftarrow \frac{14N}{16}$$

If
$$(\alpha \cdot k_0) \oplus (\beta \cdot k_1) = 0$$

$$T_0 \leftarrow \frac{14N}{16}$$

$$T_1 \leftarrow \frac{2N}{16}$$

- ▶ Verifying any one counter say, T_0 for $\frac{2N}{16}$ or $\frac{14N}{16}$
 - ► Reveals one bit \rightarrow $(\alpha \cdot k_0) \oplus (\beta \cdot k_1)$
 - ▶ Increase $N \rightarrow$ better success prob.

- $(\alpha \cdot k_0) \oplus (\beta \cdot k_1) \stackrel{?}{=} 0/1$
- ightharpoonup Key-bit estimation correct with prob. $\frac{14}{16}$
- ▶ What to expect at T_0/T_1 after N KP encryptions

If
$$(\alpha \cdot k_0) \oplus (\beta \cdot k_1) = 1$$

$$T_0 \leftarrow \frac{2N}{16}$$
 $T_1 \leftarrow \frac{14N}{16}$

If
$$(\alpha \cdot k_0) \oplus (\beta \cdot k_1) = 0$$

$$T_0 \leftarrow \frac{14N}{16}$$
 $T_1 \leftarrow \frac{2N}{16}$

- ▶ Verifying any one counter say, T_0 for $\frac{2N}{16}$ or $\frac{14N}{16}$
 - Reveals one bit $\rightarrow (\alpha \cdot k_0) \oplus (\beta \cdot k_1)$
 - ▶ Increase $N \rightarrow$ better success prob.

In General

- ightharpoonup s
 ightharpoonup s
 ightharpoonup s value of RHS of target equation involving secret key
- ▶ Counters $\rightarrow T_s, T_{s \oplus 1}$
- ► Expected values after using *N* texts

$$T_s \leftarrow pN$$
 $T_{s\oplus 1} \leftarrow (1-p)N$

- For $p \neq \frac{1}{2}$ and enough N
 - Possible to determine s
 - Correspondingly recover 1 bit of key info.

The Linear Approximation Table 🛆



	1	2	3	4	5	6	7	8	9	а	b	С	d	е	f
1	-2		2		-2	4	-2	2	4	2		-2		2	
2	2	-2		-2			2	2	4		2	4	-2	-2	.
3	4	2	2	-2	2					2	-2	-2	-2		4
4		-2	2	2	-2			-4		2	2	2	2		4
5	-2	2		2	4		2	-2	4		-2		2	-2	.
6	-2		2		2	4	2	2	-4	2		2		-2	
7				4		-4					4		4		
8		-2	2	-4		2	2	-4		-2	-2			2	-2
9	-2	-6			2	-2		2			-2	-2			2
а	-2		-6	-2		2		-2		2			-2		2
b				2	-2	2	-2			-4	-4	2	-2	-2	2
С				-2	-2	-2	-2			4	-4	2	2	-2	-2
d	-2		2	2		-2		-2		2			-6		-2
е	2	-2			2	2	-4	-2			2	-2		-4	-2
f	-4	2	2	-4		-2	-2			-2	2			-2	2

How to interpret it¹?

¹Will be discussed in details in next class