Now if we take average mass density

$$\rho(r) \approx \langle\rho\rangle = \frac{M}{\frac{4}{3}\pi R^3} \qquad \langle\rho\rangle = \frac{\int_0^R \rho(r)\, 4\pi r^2 dr}{\int_0^R 4\pi r^2 dr} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow \Omega = -\int_0^R \frac{G M(r)}{r} 4\pi r^2 \rho(r)\, dr \approx -\int_0^R G \frac{4}{3}\pi r^3 \rho(r)\, 4\pi r \rho(r)\, dr$$

$$= -G \frac{4\pi}{3} \times 4\pi \langle\rho\rangle^2 \left[\frac{R^5}{5}\right] = -\frac{3}{5R}\left[\frac{4\pi R^3 \langle\rho\rangle}{3}\right]^2$$

$$\boxed{\Omega = -\frac{3}{5}\frac{GM^2}{R}} \quad \text{--- (4)}$$

using $M(r) = \frac{4}{3}\pi r^3 \rho(r)$

and $\rho(r) \approx \langle\rho\rangle$

Now internal energy /Kinetic energy

$$U = \frac{3}{2} PV = \frac{3}{2}\int_0^R P(r) 4\pi r^2 dr \qquad P = \frac{\langle\rho\rangle K_B T}{m}$$

$m \rightarrow$ mass of gas particles

$$= \frac{3}{2}\frac{K_B}{m}\int_0^R T(r) \langle\rho\rangle 4\pi r^2 dr$$

$$\langle T\rangle = \frac{1}{M}\int_0^R 4\pi r^2 dr \langle\rho\rangle T(r).$$

$$= \frac{3}{2}\frac{K_B M \langle T\rangle}{m} \quad \text{--- (5)}$$

Virial theorem (3) become $-2U = \Omega$

$$\Rightarrow -2\left(\frac{3}{2} K_B \langle T\rangle \frac{M}{m}\right) = -\frac{3}{5}\frac{GM^2}{R}$$

$$\langle T\rangle = \frac{1}{5}\frac{GMm}{K_B R}$$

$$\langle\rho\rangle \frac{4}{3}\pi R^3 = M$$

$$\Rightarrow R^3 = \frac{3M}{4\pi\langle\rho\rangle}$$

$$= \frac{1}{5}\frac{Gm}{K_B} M \left(\frac{4\pi\langle\rho\rangle}{3M}\right)^{1/3}$$

$$\propto M^{2/3}\langle\rho\rangle^{1/3}$$

$m_H = 1.6 \times 10^{-27}$

$G = 6.6 \times 10^{-11}\ m^3/Kg/s^2$, $K_B = 1.38 \times 10^{-23}\ m^2 Kg\, s^{-2} K^{-1}$

Sun:- $M = 1.9 \times 10^{30}$ Kg, $R = 6.9 \times 10^{8}$ m

$$\langle T \rangle = \frac{1}{5} \frac{6.6 \times 10^{-11} \cancel{m^3/Kg/s^2} \times 1.9 \times 10^{30}}{1.38 \times 10^{-23} \times 6.9 \times 10^{8}} \times 1.6 \times 10^{-27}$$

$$= \left(\frac{6.6 \times 1.9 \times 1.6}{5 \times 1.38 \times 6.9}\right) \frac{\times 10^{-8}}{10^{-15}}$$

$$= 4 \times 10^{6}\ ^\circ K$$

$$n = g\int_0^{p_F} \frac{d^3p}{h^3}\cdot 1 = g\,\frac{4\pi}{h^3}\left(\frac{p_F^3}{3}\right) \qquad E_F = \frac{p_F^2}{2m}$$

$$= \sqrt{p_F^2c^2 + m^2c^4}$$

$$\Rightarrow p_F = \left[\frac{3h^3}{4\pi g}\,n\right]^{1/3} \qquad = g\,\frac{4\pi}{3h^3}\,(2mE_F)^{3/2} \quad \text{N.R.}$$

$$= g\,\frac{4\pi}{3h^3}\left(\frac{E_F^2}{c^2} - m^2c^2\right)^{3/2} \quad \text{R}$$

$$= g\,\frac{4\pi}{3h^3}\,\frac{E_F^3}{c^3} \quad \text{R for } m=0$$

$$\epsilon = g\int_0^{p_F}\frac{d^3p}{h^3}\,E = \frac{g}{h^3}\int_0^{p_F} 4\pi p^2\,dp\left(\frac{p^2}{2m}\right)$$

$$= \frac{g}{h^3}\,\frac{4\pi}{2m}\,\frac{p_F^5}{5} \quad \text{NR}$$

$$= \frac{g}{5h^3}\,\frac{4\pi}{2m}\,(2mE_F)^{5/2}$$

$$\boxed{\langle E\rangle = \frac{\epsilon}{n} = \frac{3}{5}E_F}$$

$$P = g\int_0^{p_F}\frac{d^3p}{h^3}\left(\frac{p\,v}{3}\right) \qquad v = \frac{p}{m}$$

$$= \frac{g}{3h^3}\int_0^{p_F} 4\pi p^2\left(p\,\frac{p}{m}\right)$$

$$= \frac{g}{3h^3}\,\frac{4\pi}{m}\,\frac{p_F^5}{5} \;\Rightarrow\; p_F = \left[\frac{15\,P\,h^3 m}{g\,4\pi}\right]^{1/5}$$

$$= \frac{g\,4\pi}{15h^3 m}\left[\frac{3h^3}{4\pi g}\,n\right]^{5/3}$$

$$= \underbrace{\frac{1}{5m}\left[\frac{3h^3}{4\pi g}\right]^{2/3}}_{K} n^{5/3}$$

Hydrodynamical Equilibrium

$$\frac{1}{\rho_e}\frac{dP_e}{dr} = \frac{GM}{R^2} \qquad M = \frac{4}{3}\pi R^3 \rho_e$$

$$\rho_e = \frac{M}{\frac{4}{3}\pi R^3} \propto \frac{M}{R^3}$$

$$\int_0^R \frac{dP_e}{dr}dr = \int_0^R \frac{GM}{R^2} \frac{M}{\frac{4}{3}\pi R^3} dr$$

$$P = \frac{GM^2}{\frac{4}{3}\pi} \frac{R^{-5+1}}{-5+1}$$

$$P_e \propto \frac{M^2}{R^4} \quad \text{—— (1)}$$

But degenerate pressure $P_e \propto \rho_e^{5/3}$

$$\propto \frac{M^{5/3}}{R^5} \quad \text{—— (2)}$$

$$\Rightarrow \frac{M^{1/3}}{R} \propto \frac{P_e}{P_e} \Rightarrow R \propto \frac{1}{M^{1/3}} \Longrightarrow \rho_e \propto \frac{M}{R^3}$$

$$\rho_e \propto M^2$$

For Relativistic

$$P = \frac{m_e c^2}{8\pi^2 \lambda_e^3} \phi(x_e) \qquad x_e = \frac{p_F}{m_e c}$$

$$\rho = \hat{\mu} m_p c^2 n_e =$$

$$\propto \begin{cases} P \propto \rho^{5/3} & \text{for } x_e \ll 1 \\ \propto \rho^{4/3} & \text{for } x_e \gg 1 \end{cases}$$

$$\underbrace{\left(\frac{9\pi M}{4\hat{\mu} m_p}\right)^{4/3} - \frac{3\pi\alpha}{\hbar c} GM^2}_{>0} = \left(\frac{R}{\lambda_e}\right)^2 \left(\frac{9\pi M}{4\hat{\mu} m_p}\right)^{2/3}$$

$$>0$$

$$M \le \left(\frac{9\pi}{4\hat{\mu} m_p}\right)^2 \left(\frac{\hbar c}{3\pi\alpha G}\right) \qquad m_p = \sqrt{\frac{\hbar c}{G}}$$

$$= 10^{19}\ \mathrm{GeV}/c^2$$

$$= \ldots\ 10^{15}\ \mathrm{GeV}$$