CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Architecture

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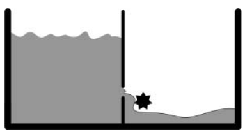
Reversible Gates

Reversibility of Writing

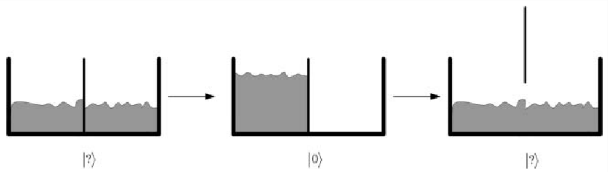
- Reversible gates predates the idea of quantum computing
  'In the 1960s, Rolf Landauer analyzed computational processes and showed that erasing information, as opposed to writing information, is what causes energy loss and heat.'

  Landauers principle

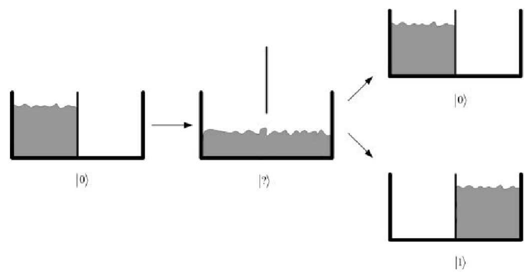


- Losing information means energy is being dissipated

- Writing is reversible.



- **Erasing** information is an irreversible, energy-dissipating operation. How?



Point-to-Ponder

If erasing information is the only operation that uses energy, then a computer that is reversible and does not erase would not use any energy.

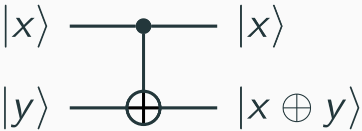
Charles H. Bennett was first known to have started working on reversible circuits and programs in the 1970s.

Are Classical Gates Reversible?

- The NOT gate and the identity gates are reversible.
- They are their own inverses

$$NOT \times NOT = I_2 \qquad I_n \times I_n = I_n$$

cNOT — Controlled-Not Gate

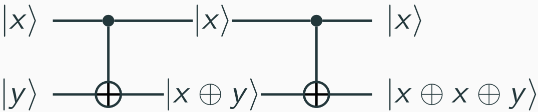


$$\begin{array}{c c} & \begin{array}{cccc} \mathbf{00} & \mathbf{01} & \mathbf{10} & \mathbf{11} \end{array} \\ \begin{array}{c} \mathbf{00} \\ \mathbf{01} \\ \mathbf{10} \\ \mathbf{11} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{array}$$

- Gate has two inputs and two output
- The top input is the **control** bit
- The bottom input is the **target** bit
- cNOT gate: $|x, y\rangle \to |x, x \oplus y\rangle$
- Now compute cNot $|10\rangle$

cNOT is Reversible

- cNOT $\times$ cNOT

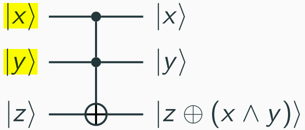


- Verify the same using the cNOT matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

ccNOT

Toffoli Gate



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\begin{array}{c c}
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\begin{bmatrix}
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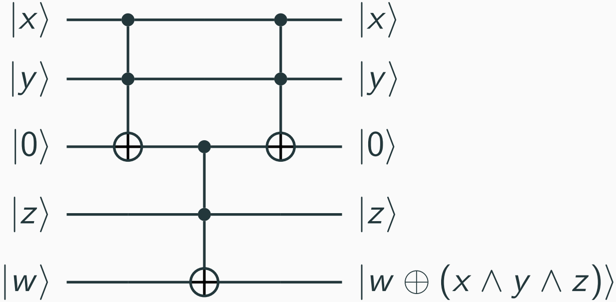
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

- Similar to the controlled-NOT gate
- But with **two** controlling bits
- Target bit flips only when both controlling bits are in state $|1\rangle$
- Toffoli gate: $|x, y, z\rangle \rightarrow |x, y, z \oplus (x \wedge y)\rangle$
- The Toffoli gate is also reversible

A Derived Gate

- A gate with three controlling bits can be constructed from three Toffoli gates

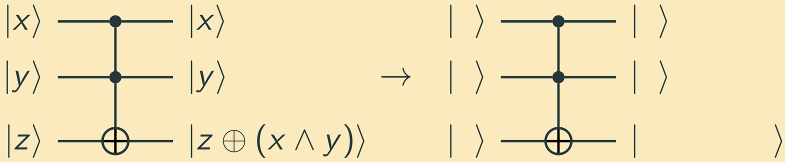


- Any logical gate can be derived using copies of the Toffoli gates
- This makes the Toffoli gate an universal gate
- One can make a reversible computer using only Toffoli gates
- Such a computer would, in theory, neither use any energy nor give off any heat.

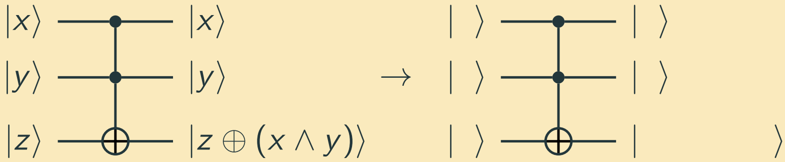
How to show that the Toffoli gate is universal?

One needs to show that the Toffoli gate can be used to make both the AND and NOT gates.

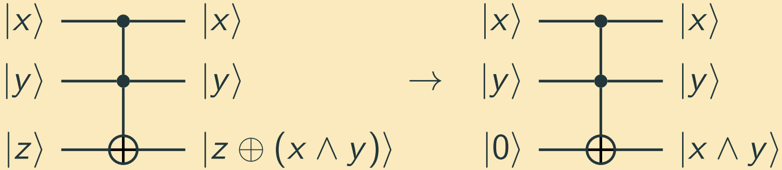
Toffoli $\rightarrow$ AND



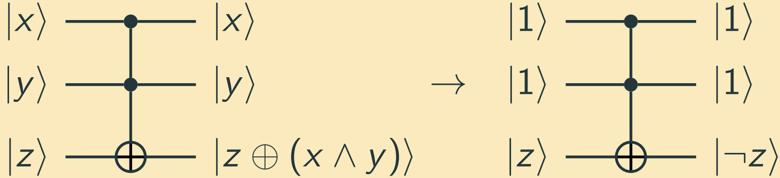
Toffoli $\rightarrow$ NOT



Toffoli $\rightarrow$ AND



Toffoli $\rightarrow$ NOT



Trick to Construct All Gates

- A way of producing a fanout of values
- A gate is needed that inputs a value and outputs two of the same values

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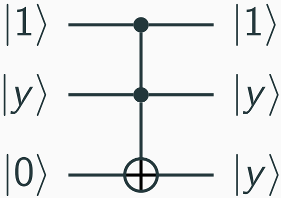
- A gate is needed that inputs a value and outputs two of the same values

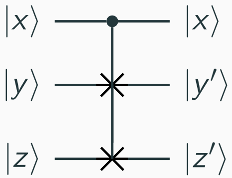


Exercise

1. Construct the NAND with one Toffoli gate.
2. Construct the XOR with one Toffoli gate.
3. Construct the OR gate with two Toffoli gates.

Conditional Swap

The Fredkin Gate



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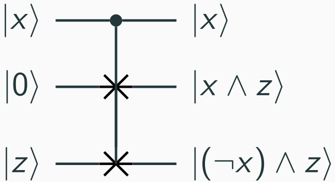
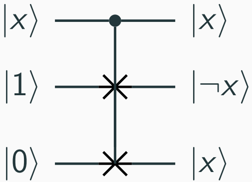
\begin{array}{c} \mathbf{000} \\ \mathbf{001} \\ \mathbf{010} \\ \mathbf{011} \\ \mathbf{100} \\ \mathbf{101} \\ \mathbf{110} \\ \mathbf{111} \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

- Top $|x\rangle$ input is control input
- If $|x\rangle$ is set to $|0\rangle$, then $|y'\rangle = |y\rangle$ and $|z'\rangle = |z\rangle$
- If $|x\rangle$ is set to $|1\rangle$, then $|y'\rangle = |z\rangle$ and $|z'\rangle = |y\rangle$
- Fredkin gate: $|0, y, z\rangle \rightarrow |0, y, z\rangle$ and $|1, y, z\rangle \rightarrow |1, z, y\rangle$
- The Fredkin gate is also reversible



Fredkin $\rightarrow$ AND



Fredkin $\rightarrow$ NOT

$$
\begin{array}{c|cccccccc|}
 & \mathbf{000} & \mathbf{001} & \mathbf{010} & \mathbf{011} & \mathbf{100} & \mathbf{101} & \mathbf{110} & \mathbf{111} \\
\mathbf{000} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{001} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{010} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{011} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\mathbf{100} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\mathbf{101} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\mathbf{110} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\mathbf{111} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

$$
\begin{array}{c|cccccccc|}
 & \mathbf{000} & \mathbf{001} & \mathbf{010} & \mathbf{011} & \mathbf{100} & \mathbf{101} & \mathbf{110} & \mathbf{111} \\
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\mathbf{010} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{011} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\mathbf{100} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\mathbf{101} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\mathbf{110} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\mathbf{111} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

- Toffoli and Fredkin gates are not only reversible gates; a glance at their matrices indicates that they are also **unitary**
- Recall the significance of unitary matrices with regards to operations in quantum theory
- Points us in the direction of their usage as quantum gates

Quantum Gates

Definition (Quantum Gate)

A *quantum gate* is simply an operator that acts on qubits. Such operators will be represented by unitary matrices.

- Identity operator I
- Hadamard gate H
- NOT gate
- Controlled-NOT gate
- Toffoli gate
- Fredkin gate

- Some other[1] quantum gates: **Pauli** matrices

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad \boxed{\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \quad \boxed{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

- Verify if each of the quantum gates have unitary matrices

[1] There are other quantum gates too which we will encounter later.

- This is not unitary or, in general, even reversible.
- This operation is usually performed at the end of a computation when we want to measure qubits (and find bits).
- Following symbols are generally used to denote a measurement.

- We will learn about measurements and partial measurement in the upcoming parts of the course.

References

1. Quantum Computing for Computer Scientists, by Noson S. Yanofsky, Mirco A. Mannucci
2. Quantum Computing Explained, David Mcmahon. John Wiley & Sons
3. Lecture Notes on Quantum Computation, John Watrous, University of Calgary
    - https://cs.uwaterloo.ca/~watrous/QC-notes/