Phonon gas :→

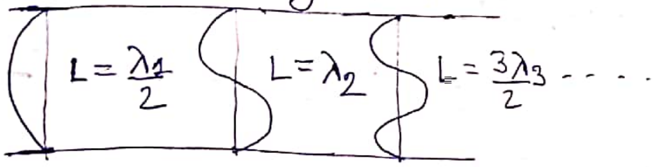
Photon → Quantize particles of Light wave

Similarly, Phonon → " " " Sound wave

In solid, vibration of lattice can be realized in terms of billions of phonons.

N atoms → 3N no. of phonons

Classical stationary waves



$L = \frac{n\lambda_n}{2}$ → wave lengths of stationary waves

$$y = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2\pi}{\lambda_n} x\right) \qquad \lambda_n = \frac{2L}{n}$$

phonons in solid with length L or volume V

$L \sim \lambda_1 = \frac{h}{P_1} = \frac{2\pi}{K_1}$

$L \approx 2\lambda_2 = \frac{2h}{P_2} = \frac{4\pi}{K_2}$

$L = n\lambda_n = n\frac{h}{P_n} = n\frac{2\pi}{K_n}$

$\Rightarrow P_n = \frac{nh}{L}$ or $K_n = n\left(\frac{2\pi}{L}\right)$

Lowest momentum = momentum pixel (ΔP or Δk)

$\Delta P = \frac{h}{L}$ or $\Delta K = \frac{2\pi}{L}$

Debye assume

an upper momentum P_D or K_D or upper energy E_D or ω_D. So

$$\cancel{3}\int_0^{K_D} \frac{d^3K}{(\Delta K)^3} = \cancel{3}N$$

$$\Rightarrow \int_0^{K_D} \frac{4\pi K^2 dK}{\left(\frac{2\pi}{L}\right)^3} = N \Rightarrow \frac{V}{2\pi^2}\int_0^{K_D} K^2 dK = N$$

$$\Rightarrow \boxed{K_D = \left[\frac{6\pi^2}{v}\right]^{1/3}}$$

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Assuming an upper energy ϵ_D, proposed by Debye,
we can get total no. of states (= 3N = no of phonons)

$$3\int_0^{\epsilon_D} \frac{d^3x\, d^3p}{h^3} = 3N$$

$$\Rightarrow \frac{V3}{h^3}\int_0^{\epsilon_D} 4\pi p^2 dp = 3N \qquad \epsilon = pc$$

$$\Rightarrow \frac{3V4\pi}{h^3c^3}\left[\frac{\epsilon_D^3}{3}\right] = 3N \Longrightarrow \boxed{N = \frac{4\pi V}{h^3c^3}\frac{\epsilon_D^3}{3}}$$

$$\Rightarrow \epsilon_D = \left[\frac{3N}{V}\frac{h^3c^3}{4\pi}\right]^{1/3}$$

$$\hbar\omega_D = hc\left[\frac{3}{v4\pi}\right]^{1/3} \Rightarrow \boxed{\omega_D = c\left[\frac{6\pi^2}{v}\right]^{1/3}} \qquad v = \frac{V}{N}$$

$$\boxed{K_D = \frac{\omega_D}{c} = \left[\frac{6\pi^2}{v}\right]^{1/3}} \qquad \boxed{p_D = \hbar K_D = \frac{h}{2\pi}\left(\frac{6\pi^2}{v}\right)^{1/3}}$$

Internal energy $U = 3\int \frac{d^3x\, d^3p}{h^3}\frac{\epsilon}{e^{\beta\epsilon}-1}$

$$= \frac{3V}{h^3}\int_0^{\epsilon_D} 4\pi\left(\frac{\epsilon}{c}\right)^2\frac{d\epsilon}{c}\,\frac{\epsilon}{e^{\beta\epsilon}-1}$$

$$= \frac{12\pi V}{h^3}\frac{(KT)^4}{c^3}\underbrace{\int_0^{\beta\epsilon_D}\frac{x^3\,dx}{e^x-1}}$$ $\qquad \beta\epsilon = x$

$$\frac{U}{N} = \frac{9(KT)^4}{\epsilon_D^3}\int_0^{\beta\epsilon_D}\frac{x^3\,dx}{e^x-1}$$

$$\frac{U}{N} = 3KT\left[\frac{3}{(\beta\epsilon_D)^3}\int_0^{\beta\epsilon_D}\frac{x^3}{e^x-1}\right]$$

$D(\beta\epsilon_D)$ is Debye function.

~~No~~ Now Debye function $D(t)$ is defined as

$$D(t) = \frac{3}{t^3}\int_0^t \frac{x^3\,dx}{e^x - 1} = \begin{cases} 1 - \frac{3}{8}t + \frac{1}{20}t^2 + \cdots & (t \ll 1) \\ \frac{\pi^4}{5t^3} + O(e^{-t}) & (t \gg 1) \end{cases}$$

Here $t = \beta \epsilon_D = \frac{\epsilon_D}{KT} = \frac{T_D}{T}$ where $T_D = \frac{\epsilon_D}{k} = \frac{\hbar \omega_D}{k}$ is Debye temperature

So in terms of T_D, we can say

$$D\left(\frac{T_D}{T}\right) = \frac{3}{\left(\frac{T_D}{T}\right)^3}\int_0^{T_D/T} \frac{x^3\,dx}{e^x - 1} = \begin{cases} 1 - \frac{3}{8}\frac{T_D}{T} + \cdots & (T_D \ll T) \\ \frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3 + O(e^{-T_D/T}) & (T_D \gg T) \end{cases}$$

So for large T, $\frac{U}{N} = 3KT\,D\left(\frac{T_D}{T}\right) \approx 3KT$

for small T, $\frac{U}{N} = 3KT\,D\left(\frac{T_D}{T}\right) \approx 3KT\left[\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3\right]$