Photon gas : Black body radiation

photon energy-momentum relation $\epsilon = pc$

no of photons $$N = 2\int \frac{d^3x\, d^3p}{h^3} \frac{1}{e^{\beta\epsilon}-1}$$

$$= \frac{2V}{h^3}\int 4\pi p^2 dp \frac{1}{e^{\beta pc}-1}$$

$\beta pc = x$

$dp = \frac{KT}{c}dx$

$$= \frac{8\pi V (KT)^3}{h^3 c^3}\int_0^\infty \frac{x^2 dx}{e^x - 1}$$

Riemann zeta function

$$\zeta_n = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{e^x-1}dx$$

$$\frac{N}{V} = 8\pi\left(\frac{KT}{hc}\right)^3 \zeta_3 \Gamma(3)$$

$$= 16\pi\left(\frac{KT}{hc}\right)^3 \zeta_3 \propto T^3$$

Internal energy $$U = 2\int\frac{d^3x\, d^3p}{h^3}\frac{\epsilon}{e^{\beta\epsilon}-1}$$

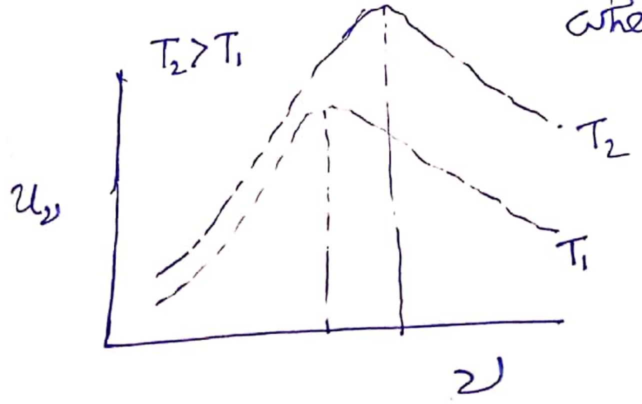
$$U = \frac{2V}{h^3}\int 4\pi p^2 dp \frac{pc}{e^{\beta pc}-1}$$

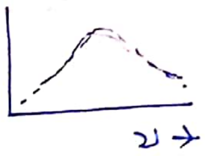
$$\frac{U}{V} = \frac{8\pi c}{h^3}\int\frac{p^3 dp}{e^{\beta pc}-1} = \int u_\nu d\nu$$

where $$u_\nu d\nu = \frac{8\pi c}{h^3}\frac{p^3 dp}{e^{\beta pc}-1}$$

$$= \frac{8\pi}{h^3} c. \left(\frac{h\nu}{c}\right)^3 \frac{h d\nu}{c} \frac{1}{e^{\beta h\nu}-1}$$

$$= \left(\frac{8\pi h}{c^3}\right)\frac{d\nu}{e^{\beta h\nu}-1}$$



$$\text{So } \frac{U}{V} = \int$$  $$d\nu$$

$$= \int u_\nu \, d\nu$$

$$= \frac{8\pi c}{h^3} \int \frac{p^3 dp}{e^{\beta pc} - 1} \qquad \beta pc = x, \quad dp = \frac{KT}{c} dx$$

$$= \frac{8\pi c}{h^3} \left(\frac{KT}{c}\right)^4 \int_0^\infty \frac{x^{4-1}}{e^x - 1} dx$$

$$= \frac{8\pi}{h^3 c^3} (KT)^4 \zeta_4 \, \Gamma(4) \quad \propto T^4$$

$$= \frac{48\pi}{h^3 c^3} (KT)^4 \zeta_4$$

photon ~~P~~ ~~pressure~~ ~~P~~

$$= \frac{8\pi^5}{15 h^3 c^3} (KT^4) \qquad \because \zeta_4 = \frac{\pi^4}{90}$$

$$= \frac{\pi^2}{15 (\hbar c)^3} (KT)^4 \quad \propto T^4$$

Stefan constant σ?

~~To~~

Intensity from blackbody $I = \frac{c}{4} \frac{U}{V}$

$$I(T) = \sigma T^4 \quad \text{where } \sigma = \frac{\pi^2 K^4}{60 \hbar^3 c^2}$$

$$= \frac{8\pi^5 K^4}{60 h^3 c^2}$$

$$= \frac{12\pi K^4 \zeta_4}{h^3 c^2}$$

Pressure $P = -\frac{\Phi}{V} = -\frac{1}{V}\left[-\frac{KT}{-1}\int \frac{2 d^3p\, d^3x}{h^3} \ln\{1 - ze^{-\beta\epsilon}\}\right]$

$$= -\frac{2KT}{h^3}\int d^3p \ln\{1 - e^{-\beta\epsilon}\} \qquad \text{for } z = 1$$

$$\frac{P}{KT} = -\frac{\cancel{4}8\pi}{h^3}\int_0^\infty p^2 dp \ln\{1 - e^{-\beta pc}\}$$

$$\epsilon = pc$$

$$= -\frac{8\pi}{h^3}\left[\underbrace{\left|\frac{p^3}{3}\ln\{1 - e^{-\beta pc}\}\right|_0^\infty}_{\to 0} - \int_0^\infty \frac{p^3}{3}\frac{\beta c\, e^{-\beta pc}}{1 - e^{-\beta pc}}dp\right]$$

$$= \frac{8\pi}{h^3}\int_0^\infty \frac{\beta p^3}{3}\frac{c}{e^{\beta pc} - 1}$$

$$\beta pc = x$$

$$dp = \frac{KT}{c}dx$$

$$= \frac{8\pi}{h^3}\left(\frac{\beta c}{3}\right)\left(\frac{KT}{c}\right)^4\int_0^\infty \frac{x^{4-1}dx}{e^x - 1}$$

$$P = \frac{8\pi}{3}\frac{1}{h^3c^3}(KT)^4\,\Gamma(4)\,\zeta(4) = \frac{1}{3}\frac{U}{V} \qquad \boxed{PV = \frac{1}{3}U}$$

$$= \frac{16\pi}{h^3c^3}K^4T^4\left(\frac{\pi^4}{90}\right)$$

$$= \left(\frac{8\pi^5 K^4}{45\, h^3c^3}\right)T^4$$