Investigating the Brachistochrone problem in Vert Ramps

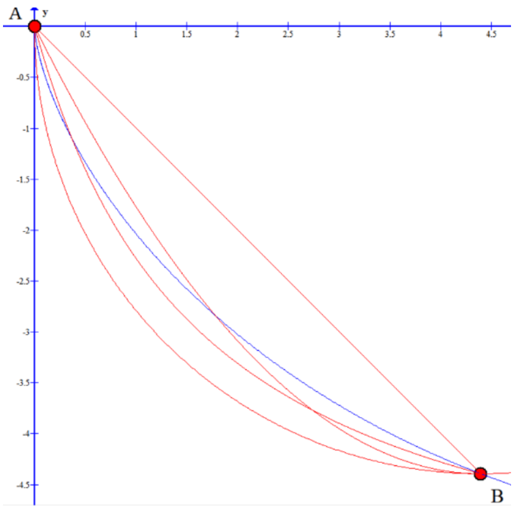
BRACHISTOCHRONE

Brachistochrone, derived by the words “brakhistos”, Greek for “shortest” and “khronos”, Greek for “time”, putting them together forming “shortest time”, hence “Brachistochrone”, is a curve of minimum time to reach from a starting point A to end point B, where B is a point which is not directly below A. In Variational Calculus, it is a path between two points which minimizes the time taken by a particle falling from point A to point B under the influence of gravity. This problem was posed and solved by Johann Bernoulli in the year 1696 and is probably one of the most interesting solved problems in the history of mathematics.

The main reason why I chose to investigate the problem of Brachistochrone was because this particular problem fascinated me, as I have always been interested in and intrigued by solving challenging problems, and I also have an interest in knowing how solutions to problems are derived, rather than just putting the ready solution to use for other purposes. Many prominent mathematicians thought the path for fastest descent was a straight line path, but it was proven wrong by Johann Bernoulli.

A solution to this problem was provided by 5 mathematicians: Ehrenfried Walther von Tschirnhaus, Isaac Newton, Jakob Bernoulli, Gottfried Leibniz, and Guillaume de l’Hôpital. Bernoulli had challenged various mathematicians throughout the world to solve this problem.

Now, I will be exploring the solution to this captivating problem and make sure it is explained thoroughly at each and every step.



The blue curve is the brachistochrone curve. It should take the least time to reach point B if the start point is A.

DERIVING THE SOLUTION TO THE BRACHISTOCHRONE PROBLEM

The main objective is to minimize the time taken for an object to reach from point A to B. Usually, time can be minimized either by minimizing the distance, or maximising the speed. This can be observed by using the formula for Time in terms of Distance and Speed, which is:

$$Time = \frac{Distance}{Speed}$$

As it can be observed, if the time is to be minimized, either the distance should be decreased, or speed should be increased. Although, it is important to understand that neither a straight line or a path made by a rational function will produce the minimum time path.

In a path created by a rational function, even though the velocity initially is almost maximum, time taken will not be minimum due to the object traveling a longer distance. The initial velocity will be almost maximum due to the assumption that gravity is effectively acting downwards on the path because of its steepness.

In a straight-line path, the distance is minimum, but the initial speed of the object will be lesser as gravity isn't as effective as it is in a path made by a rational function. This because a straight-line path isn't as steep as the path made by the rational function. Hence, the time taken will not be minimum.

It can be concluded that the minimum time path will be somewhere between these two paths. So basically, Brachistochrone path is a combination of maximum speed and minimum distance instead of just a path which computes either minimum distance or maximum speed. Also, the arc of a circle and the parabolic path have a very similar shape to that of the brachistochrone, which is why these paths will also be compared to the brachistochrone path.

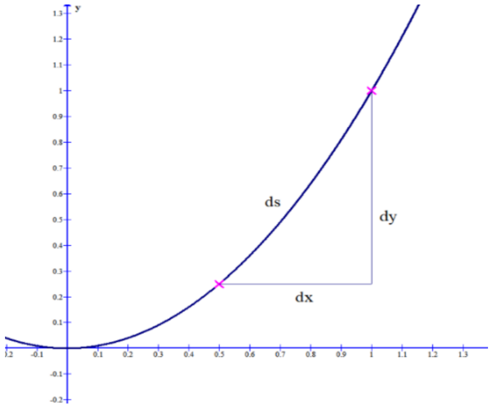
Now taking a point A and another point B, where A is the start point and B is the end point of the general path. Thus, the function for total time is given by:

$$T = \int_A^B dt$$

We know that velocity is the differentiation of displacement 's'. Using $v = \frac{ds}{dt}$, we can make 'dt' the subject, which gives us the equation $dt = \frac{ds}{v}$. We substitute this equation in the integrand to get:

$$T = \int_A^B \frac{ds}{v}$$

We know that 'ds' is a part of the path, so it can be observed that it is a path made by moving 'dx' units to the right and 'dy' units upwards from the starting point of the ds path. An example for this has been shown below, with the graph of $y = x^2$ which is the parabolic path. Therefore, using Pythagoras Theorem, we can find 'ds', which is:



$$ds \approx \sqrt{(dx)^2 + (dy)^2}$$

If an object is at height y, the potential energy is mgy, but if the object rolls down, the object's potential energy decreases, and as the law of conservation of energy states that "Energy cannot be created or destroyed; it can only be transformed or transferred from one form to another.", when the potential energy is maximum, the kinetic energy will be minimum and vice versa. So, at the top of the path, the potential energy will be maximum but the kinetic energy will be minimum, and when the object rolls down, the kinetic energy will be maximum, and the potential energy will be minimum.

$mgy = \frac{1}{2}mv^2$ cancelling out *'m'*

$v^2 = 2gy$

$v = \sqrt{2gy}$ taking square root of both sides of the equation

Now that we have equations for both 'ds' and 'v', we can substitute them into the total time function.

$$T = \int_A^B \frac{ds}{v} \text{ can now be written as } T = \int_A^B \frac{\sqrt{(dx)^2+(dy)^2}}{\sqrt{2gy}}$$

Now, from the expression $\sqrt{(dx)^2 + (dy)^2}$, taking $(dx)^2$ common, we get:

$$\sqrt{(dx)^2(1 + \left(\frac{dy}{dx}\right)^2})$$

This can now be written as $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

The function T can finally be written as

$$T = \int_A^B \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}} dx \ or\ T = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{y}} dx \ where\ g\ is\ constant$$

Finding the Stationary Time Function using the Euler-Lagrange Equation

Now, a function $y = F(x)$ has to be found such that it makes our time function stationary. In addition to making this function stationary, the boundary conditions also need to be considered. The proof of this equation will not be provided because it is out of scope of this exploration and will directly be utilised to solve the problem. This is the Euler Differential Equation, a partial differential equation which is used to optimize a function. In this case, the equation will be used to find the function which gives the minimum value of time.

It will be assumed that:

- The function $y = F(x)$ makes the function T stationary
- $y = f(x)$ satisfies the following boundary conditions.
  - If the points $A\ and\ B\ are\ (x_1, y_1)\ and\ (x_2, y_2)$:
    - $y(x_1) = y_1\ and\ y(x_2) = y_2$

The function $y = F(x)$ is found by solving the following equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0\ where\ F = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{y}}\ or\ \frac{\sqrt{1 + (y')^2}}{\sqrt{y}}$$

Steps in solving this equation:

$\frac{\partial F}{\partial y}$ is a partial differential equation. This is used when there are 2 or more variables used in the equation, and to differentiate this equation, all variables other than one have to be kept constant. In this case, all variables other than y' will be kept constant. Hence, $\frac{\partial F}{\partial y}$ is the differentiation of F in terms of y.

- Solving the first part of the equation, the partial differential of F in terms of y:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right)$$

Keeping $\sqrt{1+(y')^2}$ constant as the function F has to be differentiated in terms of y first:

$$\frac{\partial}{\partial y}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right) = \sqrt{1+(y')^2}\left(\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y}}\right)\right)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y}}\right) = \frac{-1}{2y^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial y}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right) = \frac{-\sqrt{1+(y')^2}}{2y^{\frac{3}{2}}} \quad \boxed{\text{Equation 1}}$$

- Solving for the next part of the equation, which is the partial differential of F in terms of y':

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right)$$

Keeping $\sqrt{y}$ constant as the function F has to be differentiated in terms of y'.

$$\frac{\partial}{\partial y'}\left(\frac{\sqrt{1+(y')^2}}{\sqrt{y}}\right) = \frac{1}{\sqrt{y}}\left(\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2}\right)\right)$$

$$\frac{\partial}{\partial y}\left(\sqrt{1+(y')^2}\right) = \frac{2y'}{2\sqrt{1+(y')^2}} = \frac{y'}{\sqrt{1+(y')^2}}$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y(1+(y')^2)}}$$

- Solving the last part of the equation, which is the derivative of the partial differential equation we just found above, F in terms of y':

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right) = \left(\frac{y''\left(\sqrt{y(1+(y')^2)}\right) - \left(\frac{y'\frac{d}{dx}(y(1+(y')^2))}{2\sqrt{y(1+(y')^2)}}\right)}{y(1+(y')^2)}\right)$$

$$\frac{d}{dx}(y + y(y')^2) = y' + 2yy'y'' + (y')^3$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)=\left(\frac{y''\left(\sqrt{y(1+(y')^2)}\right)-\left(\frac{y'(2yy'y''+y'(1+(y')^2))}{2\sqrt{y(1+(y')^2)}}\right)}{y(1+(y')^2)}\right)$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)=\left(\frac{2yy''(1+(y')^2)-2yy''(y')^2-(y')^2(1+(y')^2)}{2(y(1+(y')^2))^{\frac{3}{2}}}\right)$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)=\left(\frac{2yy''(1+(y')^2)-2yy''(y')^2-(y')^2(1+(y')^2)}{2(y(1+(y')^2))^{\frac{3}{2}}}\right)$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y(1+(y')^2)}}\right)=\left(\frac{2yy''-(y')^2-(y')^4}{2(y(1+(y')^2))^{\frac{3}{2}}}\right)$$ Equation 2

- Solving the Euler-Lagrange equation by substituting Equation 1 and Equation 2.

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

$$-\frac{\sqrt{1+(y')^2}}{2y^{\frac{3}{2}}}-\left(\frac{2yy''-(y')^2-(y')^4}{2(y(1+(y')^2))^{\frac{3}{2}}}\right)=0 \qquad LCM=2(y(1+(y')^2))^{\frac{3}{2}}$$

$$-(1+(y')^2)^2-2yy''+(y')^2+(y')^4=0$$

$$-1-2(y')^2-(y')^4-2yy''+(y')^2+(y')^4=0$$

$$1+(y')^2+2yy''=0$$

If this equation is multiplied by y':

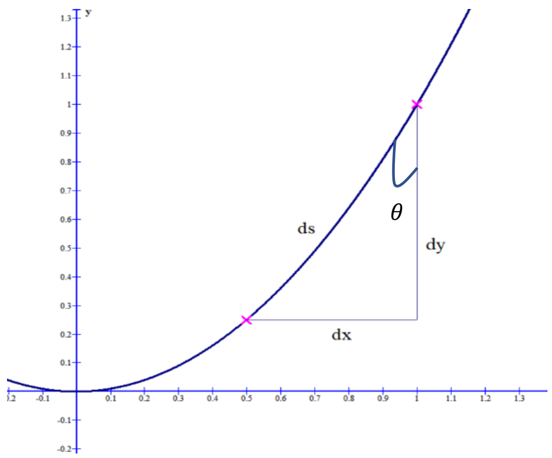
$$y'+(y')^3+2yy'y''=0$$

$$\int(y'+(y')^3+2yy'y'')dx=\int 0dx$$

$$\int\frac{d}{dx}(y+y(y')^2)\,dx=\int 0dx$$

$$y+y(y')^2=C$$

$$y'=\sqrt{\frac{C-y}{y}}\,and\ it\ is\ known\ that\ y'=\frac{dy}{dx};\ \frac{dy}{dx}=\sqrt{\frac{C-y}{y}}$$



$$\frac{dx}{dy}=\sqrt{\frac{y}{C-y}}$$

$$tan\theta=\frac{dx}{dy}$$

Now, a new variable θ is introduced and parametric equation for this curve using θ will be solved.

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{y}{C-y}$$

$$C\sin^2\theta - y\sin^2\theta = y\cos^2\theta$$

$$y = C\sin^2\theta \qquad\qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$\boxed{y = \frac{C}{2}(1-\cos 2\theta)}$$

Solving for parametric equation in the x-direction:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

$$\frac{dx}{d\theta} = \frac{dx}{dy}\times\frac{dy}{d\theta}$$

$$\frac{dy}{d\theta} = \frac{C}{2}\times 2\sin 2\theta = 2C\sin\theta\cos\theta \qquad\qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{C-y}}$$

$$\frac{dx}{d\theta} = \sqrt{\frac{y}{C-y}}\times 2C\sin\theta\cos\theta \qquad\qquad y = C\sin^2\theta$$

$$\frac{dx}{d\theta} = \sqrt{\frac{C\sin^2\theta}{C-C\sin^2\theta}}\times 2C\sin\theta\cos\theta = \sqrt{\frac{C\sin^2\theta}{C\cos^2\theta}}\times 2C\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}\times 2C\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = 2C\sin^2\theta = C(1-\cos 2\theta)$$

$$dx = C(1-\cos 2\theta)d\theta$$

$$\int dx = C\int 1d\theta - C\int \cos 2\theta\, d\theta$$

$$x = C(\theta - \frac{1}{2}\sin 2\theta)$$

$$\boxed{x = \frac{C}{2}(2\theta - \sin 2\theta)}$$

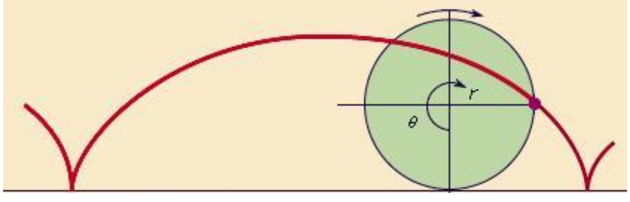
The 2 parametric equations are:

$$\boxed{x = m(2\theta - \sin 2\theta)\ and\ y = m(1-\cos 2\theta)}$$

Note: For the 2 parametric equations, the constant $\frac{C}{2}$ was replaced by m.

These parametric equations coincidentally represent the equations of a Cycloid. So, it can be observed that the function which makes the function T stationary is a segment on a cycloid.

A cycloid is a path generated by a point on the circle being rolled along a straight line with no friction. The path (in red) looks like something like this:



In this image, it can be seen that the path generated by a point on a circle resembles the shape of a cycloid.

The Brachistochrone is an inverted cycloid.

Deriving a function for Time Taken specifically for the Brachistochrone path

These parametric equations will be used to find a function that can calculate the time taken for the circular path and the brachistochrone path. When the parametric equations of ‘x’ and ‘y’ are differentiated, we get:

$$x = m(2\theta - \sin 2\theta) \longrightarrow \frac{dx}{d\theta} = m(2 - 2cos2\theta) = 2m(1 - cos2\theta)$$

$$\frac{d\theta}{dx} = \frac{1}{2m(1 - cos2\theta)}$$

$$y = m(1 - \cos 2\theta) \longrightarrow \frac{dy}{d\theta} = m(2sin2\theta) = 2m(sin2\theta)$$

Now, to find the derivative of the brachistochrone:

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2m(sin2\theta)}{2m(1 - cos2\theta)} = \frac{(sin2\theta)}{(1 - cos2\theta)}$$

Squaring both sides:

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{sin2\theta}{1 - cos2\theta}\right)^2 = \frac{\sin^2 2\theta}{(1 - cos2\theta)^2} = \frac{1 - \cos^2 2\theta}{(1 - cos2\theta)^2} = \frac{(1 + cos2\theta)(1 - cos2\theta)}{(1 - cos2\theta)^2}$$

After further simplification, we get:

$$\left(\frac{dy}{dx}\right)^2 = (y')^2 = \frac{1 + cos2\theta}{1 - cos2\theta}$$

We can make $cos2\theta$ the subject of the parametric equation ‘y’. We get it as:

$$\cos 2\theta = \frac{m - y}{m}$$

Now substituting $cos2\theta$ into $(k')^2$:

$$(y')^2 = \frac{1 + cos2\theta}{1 - cos2\theta} = \frac{1 + \left(\frac{m - y}{m}\right)}{1 - \left(\frac{m - y}{m}\right)} = \frac{2m - y}{y}$$

This can be written as: $(y')^2(y) - y = 2m$

Solving for the time taken to travel the path

$$\frac{dy}{dx} = \sqrt{\frac{2m - y}{y}}$$

Making dx the subject:

$$dx = \sqrt{\frac{y}{2m - y}}dy \quad \boxed{\textbf{(1)}}$$

Recalling the function for T:

$$T = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + (y')^2}}{\sqrt{y}} dx$$

Substituting (1) and $(y')^2$ into the integrand of the function T:

$$T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{1 + \frac{2m-y}{y}}{y} \times \frac{y}{2m-y}}\, dy$$

$$\therefore\ T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{2m}{y^2} \times \frac{y}{2m-y}}\, dy$$

$$T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{2m}{y(2m-y)}}\, dy$$

Now, it is important to notice that the denominator can be rewritten as:

$$(y)(2m-y) = -(m-y)^2 + (m)^2 \qquad \boxed{\textbf{(2)}}$$

Substituting $y = m(1 - \cos 2\theta)$:

$$-(m - m(1 - \cos 2\theta))^2 + (m)^2$$

$$-(m(\cos 2\theta))^2 + (m)^2$$

$$-m^2 \cos^2 2\theta + (m)^2$$

$$m^2(1 - \cos^2 2\theta) = m^2 \sin^2 2\theta$$

Substituting (2) into the integrand of the function T:

$$T = \frac{1}{\sqrt{2g}} \int_A^B \sqrt{\frac{2m}{(m)^2(\sin^2 2\theta)}}\, dy$$

Now substituting $\frac{dy}{d\theta} = 2m(sin2\theta)$ and rewriting the function as:

$$T = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{2m}}{m \sin 2\theta} 2m(sin2\theta)\, d\theta$$

$$T = \frac{2}{\sqrt{2g}} \int_A^B \sqrt{2m}\, d\theta$$

$$\int_A^B \sqrt{2m}\, d\theta = \sqrt{2m}\, \theta$$

$$T = \frac{2\theta}{\sqrt{2g}} \sqrt{2m} = 2\theta \sqrt{\frac{m}{g}}$$

∴ To calculate the time taken to travel the path, the formula is given by:

$$\boldsymbol{T = 2\theta \sqrt{\frac{m}{g}}}$$

Where:

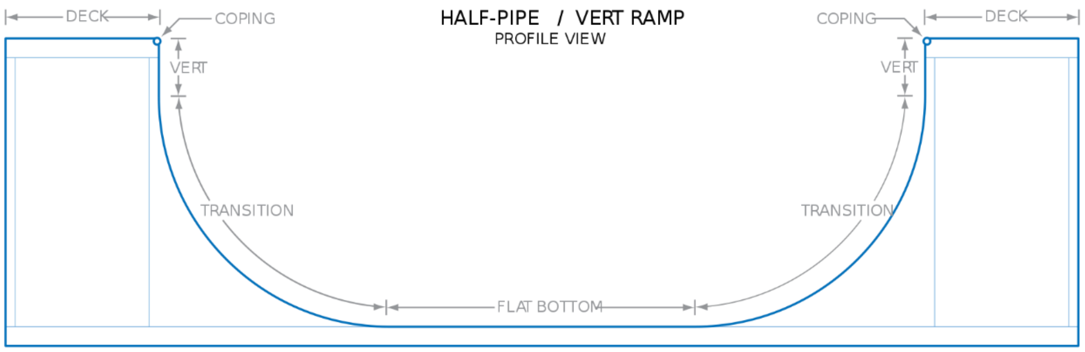
$'\theta'$ can be found by solving the 'x' and 'y' parametric equations

'm' is the constant in the parametric equations 'x' and 'y' that can be found by substituting the value of $'\theta'$ into any one of the parametric equations

'g' is the acceleration of gravity or the gravitational force. ($g \approx 9.81 m/s^2$)

BEST DESIGN FOR A 2013 HALF-PIPE RAMP

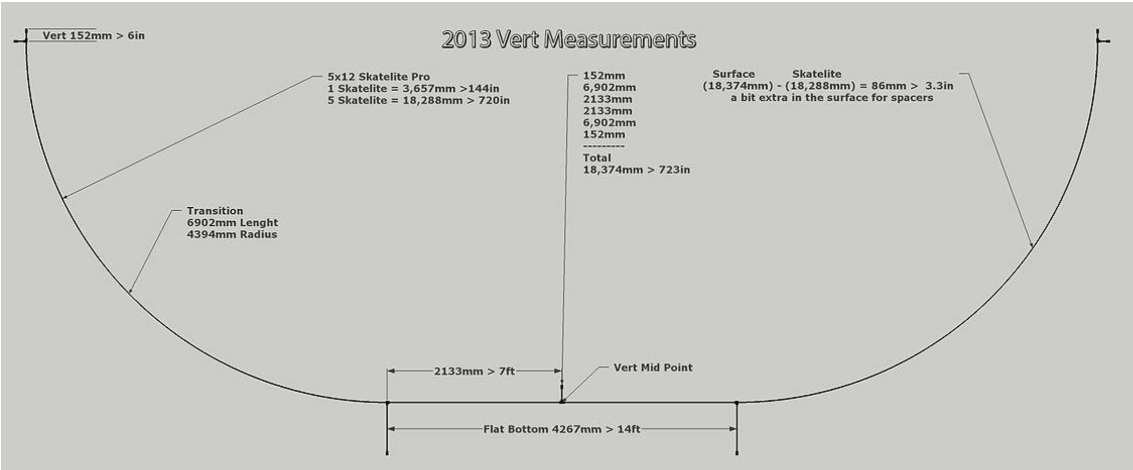
A half-pipe is a structure used in gravity extreme sports such as snowboarding, skateboarding, skiing, freestyle BMX, skating and scooter riding. Originally, half-pipes were considered to be half sections of a large diameter pipe. Since the past 4 decades, the structure of half-pipes is such that they contain an extended flat bottom between the quarter-pipes; the original style half-pipes are no longer built. The design of a Half-Pipe ramp looks similar to this:



Vert is the short path of the start of the ramp which is a vertical path, Transition refers to how the ramp is changing and Flat Bottom is the horizontal part of the ramp which helps the skateboarder to regain balance after landing and gives the skateboarder more time to prepare for the next trick.

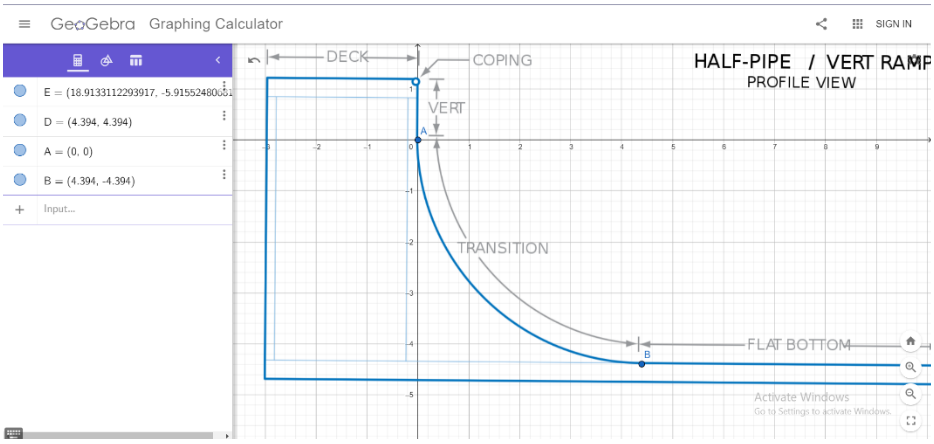
Different paths I will be testing:

- Straight Line Path
- Brachistochrone Path
- Path made by a Rational Function
- Parabolic Path
- Actual Path – Arc of a circle



Before starting the main calculations for these paths, the start point A is (0, 0) and the end point B is (4.394, -4.394) as I will be taking the dimensions of the 2013 Vert Ramps, which had a radius if 4394 mm or 4.394m. I noticed that the vertical part of the path had to be omitted to obtain an accurate start point, so I took the start point from the end of the vertical path.

This was decided by placing the image on the graph of GeoGebra and then points were traced along the image. The background of this image was removed using Adobe Photoshop CS6.



Time taken to travel each path will be calculated using the following integral:

$$T = \frac{1}{\sqrt{2g}} \int_{A}^{B} \sqrt{\frac{1 + (y'(x))^2}{y(x)}}\, dx$$

STRAIGHT LINE PATH (PATH A)

The equation of a straight line which passes through the points (0, 0) and (4.394, -4.394) is:

$$y(x) = -x$$

$$y'(x) = -1$$

The integral to calculate the time taken to travel this path:

$$T_A = \sqrt{\frac{1}{2 \times 9.80665}} \int_{0}^{4.394} \left| \sqrt{\frac{2}{-x}} \right| dx$$

$$T_A = 1.338749298\ s$$

PATH MADE BY A RATIONAL FUNCTION (PATH B)

For this path, a rational function was found which passed through the points (0, 0) and (4.394, -4.394). The steps I followed to obtain the equation have been shown below:

First, the general equation for a rational function was considered:

$y(x) = \frac{1}{a(x-b)} + c$, where the vertical asymptote is $x = b$ and the horizontal asymptote is $y = c$

I chose the value of 0.1 for ‘a’. For this value of ‘a’, we can form 2 simultaneous equations by substituting the value of ‘a’ as 1, and the 2 coordinates:

$$y(x) = \frac{1}{0.1(x-b)} + c;\ (0, 0)\ and\ (4.394, -4.394)$$

$$\text{Equation 1: } 0 = \frac{1}{-0.1b} + c;\ bc = 10$$

$$\text{Equation 2: } -4.394 = \frac{1}{0.1(4.394-b)} + c$$

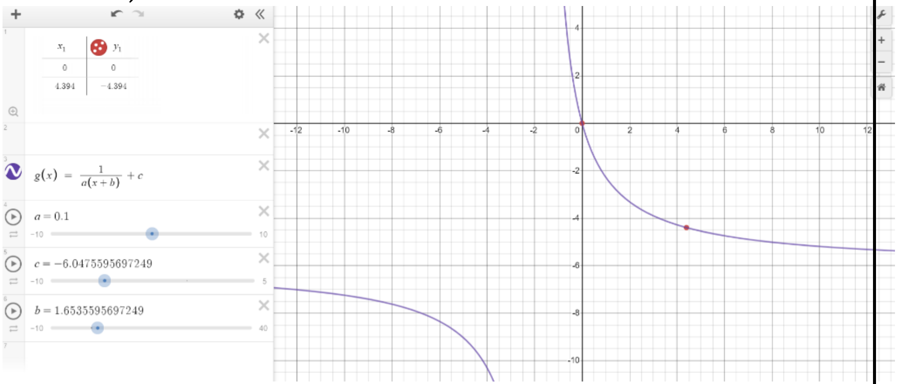
Using the solver on emathhelp.net, which has solvers for various equations, including rational simultaneous equations, I managed to solve these rational simultaneous equations. The values of 'b' and 'c' are:

$$b \approx 1.6535595697249, c \approx -6.0475595697249 \; OR \; b \approx -6.0475595697249, c \approx 1.6535595697249$$

However, the points I decided to choose were:

$$b \approx 1.6535595697249, c \approx -6.0475595697249$$

This is because the rational function graph with the other set of points doesn't go through the required points. Hence, the final rational function graph obtained is shown on the right:



$$y(x) = \frac{1}{0.1(x + 1.6535595697249)} - 6.0475595697249$$

$$y'(x) = -\frac{1}{0.1(x + 1.6535595697249)^2}$$

Using the Time function to calculate the time taken:

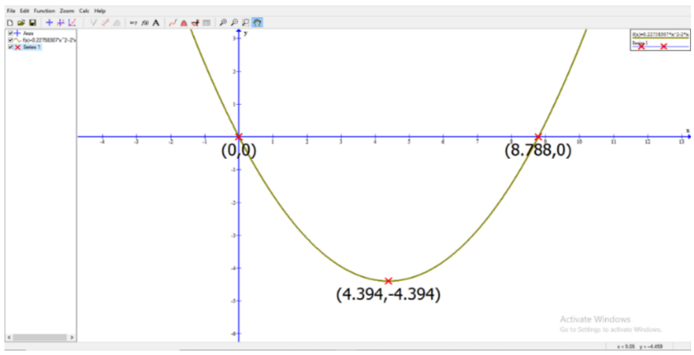
$$T_B = \sqrt{\frac{1}{2 \times 9.80665}} \int_0^{4.394} \left| \frac{\sqrt{1 + \left(-\frac{1}{0.1(x + 1.6535595697249)^2}\right)^2}}{\sqrt{\frac{1}{0.1(x + 1.6535595697249)} - 6.0475595697249}} \right| dx$$

This integral was solved using the online integral calculator as GDC and several other online websites failed to provide an answer. Using numerical integration, the value of T is:

$$T_B = 9.5102272477816 \text{ s}$$

<u>PARABOLIC PATH (PATH C)</u>

For a parabolic path from A to B, it is important to understand that there is no way to form a specific parabola with only 2 points, and in this case, I had only 2 points, A and B. I decided to choose another point, (8.788, 0), to form a parabola, graph is shown on the right:



The equation of this parabola is given by:

$$y(x) = 0.22758307x^2 - 2x$$

Derivative of this equation is:

$$y'(x) = 0.45516614x - 2$$

Using the T function to calculate time:

$$T_C = \frac{1}{\sqrt{2 \times 9.80665}} \int_0^{4.394} \left| \frac{\sqrt{1 + (0.45516614x - 2)^2}}{\sqrt{0.22758307x^2 - 2x}} \right| dx$$

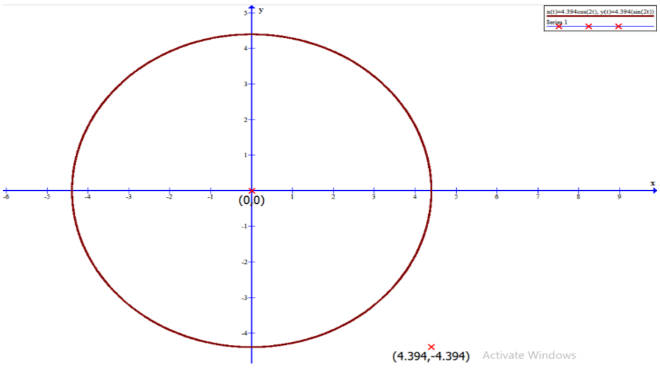
Solving this integral in my GDC gives the value of T as:

$$T_C = 1.247284617\ s$$

ARC OF A CIRCLE – ACTUAL PATH (PATH D)

This path was first thought of as the minimum time path by Galileo, however, it was proven to be wrong by Johann Bernoulli. For a circle's arc, first, the circle's equation has to be found. It can be noticed that the circle going through the points A and B forms exactly a quarter of a circle, which is also known as a quadrant. The general parametric equation of a circle is:

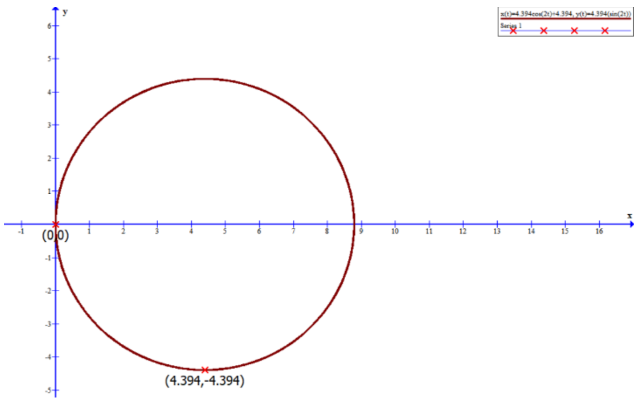
$$x = rcos\theta \ and \ y = rsin\theta$$



The circle's radius is 4.394, and the circle's centre will be such that the circle passes through the points A and B. When circle's centre is the origin, equation will be:

$x = 4.394cos\theta$

$y = 4.394sin\theta$



This circle has to be shifted to the right by 4.394 units for it to pass through the points A and B. So, the final equation of this circle is:

$x = 4.394cos\theta + 4.394$

$y = 4.394sin\theta$

General equation of a circle in terms of x and y:

$$x^2 + y^2 = r^2$$

$$y(x) = \pm\sqrt{r^2 - x^2}, where\ r = 4.394$$

$$y(x) = \sqrt{19.307236 - x^2}$$

Note: ± is insignificant as y(x) in the equation for time taken cannot be negative.

Derivative of the equation of this circle becomes:

$$\frac{dy}{dx} = -\frac{x}{\sqrt{19.307236 - x^2}}$$

$$T_D = \sqrt{\frac{1}{2 \times 9.80665}} \int_0^{4.394} \frac{\sqrt{1 + \left(-\frac{x}{\sqrt{19.307236 - x^2}}\right)^2}}{\sqrt{\sqrt{19.307236 - x^2}}} dx$$

$$T_D = 1.241071809589567\ s$$

BRACHISTOCHRONE PATH (PATH E)

The brachistochrone path will give us the shortest time for the path from A to B. For this, first the parametric equations have to be found. The general parametric equations for a cycloid are:

$$x = m(2\theta - \sin 2\theta); m = \frac{x}{2\theta - sin2\theta}\ and\ y = m(1 - \cos 2\theta);\ m = \frac{y}{1 - cos2\theta}$$

First, we will equate both equations for 'm' and obtain the value of 2θ. The point I chose is (4.394, -4.394).

$$\frac{4.394}{2\theta - sin2\theta} = \frac{-4.394}{1 - cos2\theta}$$

$$1 - cos2\theta = sin2\theta - 2\theta$$

$$2\theta = -2.412011rad$$

Substituting this value of 2θ to find the value of m:

$$m = \frac{4.394}{-2.412011 - \sin(-2.412011)} = -2.517397825$$

Therefore, the equations are:

$$x = -2.517397825(2\theta - \sin 2\theta)\ and\ y = -2.517397825(1 - \cos 2\theta)$$

The value of 2θ is negative because the cycloid is inverted. Using this negative value of 2θ, we also obtain the value of 'm' as negative. So, we will use the positive value of 2θ and 'm' to calculate for the time taken.

Solving for time T:

$$T_E = 2\theta \sqrt{\frac{m}{g}}\ where\ 2\theta = 2.412011\ rad, m = 2.517397825\ and\ g = 9.80665 m/s^2$$

$$T_E = 2.412011 \sqrt{\frac{2.517}{9.80665}} = 1.221970029\ s$$

CONCLUSION AND EVALUATION

As expected, the time taken to cover the distance between point A to B was least when the shape of the path was Brachistochrone. The times taken to travel the different paths are:

- $T_A = 1.338749298\ s$
- $T_B = 9.510227247$ s
- $T_C = 1.247284617\ s$
- $T_D = 1.241071810\ s$
- $T_E = 1.221970029\ s$

It can be noticed that path E, brachistochrone, took lowest time to reach from point A to B as expected. An interesting detail which shouldn't be missed is the time taken for the straight-line path. Straight line path, as assumed by some several great mathematicians before, was considered to be the fastest path between 2 points. However, as it is evident with the results obtained, it is the most time-consuming path, after the time taken for the rational function path. It can also be noticed that due to the shapes of brachistochrone and arc of a circle being quite similar, the difference in time taken to travel from A to B is quite less. Also, as it can be noticed, the circular and parabolic path take almost the same time to travel from point A to B as they have a mere difference of 0.006s, which is mostly unnoticeable in real life. Results obtained for time taken to travel from A to B for all paths except the rational function path were quite convincing, as circular path, which is the actual path, as expected, has a faster time than the straight-line path, but a slower time than the brachistochrone path.

Even though the path made by a rational function taking the most amount of time isn't a shocking result, one important detail that can be noticed is that the distance travelled in a rational path is not considerably larger when compared to other paths. And even with this, the time taken is significantly larger than the other paths. The effect of gravity may have a hand in this great difference, however, even with the effect of gravity, the time taken to travel the rational path should have, in my opinion, been closer to other values than the result obtained.

However, when a person skateboards on a vert ramp, there are many more factors that have to be taken into account, for example: friction forces, drag forces, etc. These forces have been neglected and it has been assumed that the gain in kinetic energy is equal to the loss in potential energy and vice versa. Also, actual ratios and dimensions of the vert ramp have to be taken into consideration if calculations for the real world are to be made.

WORKS CITED

"Cycloid | Mathematics". *Encyclopedia Britannica*, 2020, https://www.britannica.com/science/cycloid. Accessed 18 Aug 2019.

Mishra, P. (2018). Deriving and Deducing the Equation of the Curve of Quickest Descent. [online] Ijmaa.in. Available at: http://ijmaa.in/v4n3-a/99-121.pdf. Accessed 17 Aug 2019.

"Definite Integral Calculator - Symbolab". *Symbolab.Com*, 2020, https://www.symbolab.com/solver/definite-integral-calculator. Accessed 18 Mar 2020.

"Half-Pipe". *en.wikipedia.org*, 2020, https://en.wikipedia.org/wiki/Half-pipe. Accessed 15 Aug 2019.

Image links:

https://en.wikipedia.org/wiki/Half-pipe#/media/File:Half-Pipe_Vert_Ramp.svg

https://commons.wikimedia.org/wiki/File:Vert_ramp_measurements.jpg

"Rational Functions". *Desmos*, 2020, https://www.desmos.com/calculator/zyfcub6vhn. Accessed 18 Mar 2020.

"Simultaneous Equations Solver - Emathhelp". *Emathhelp.Net*, 2020, https://www.emathhelp.net/calculators/algebra-2/simultaneous-equations-solver. Accessed 17 Mar 2020.

Graph. 4.4.2 version, 2012.