Interactive Computer Graphics

Unit 2: Math for Computer Graphics

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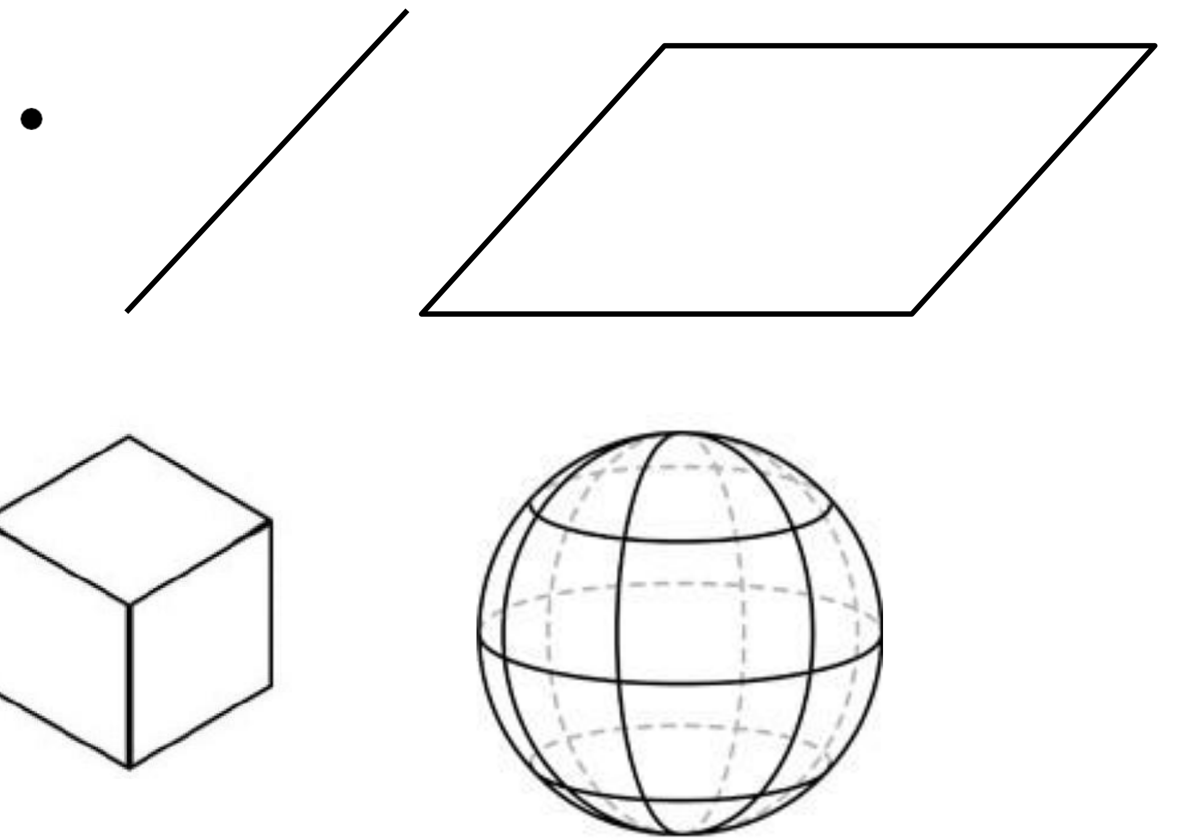
Unit 2: Math for Computer Graphics

- Analytic Geometry
- Vectors
- Matrices
- Math problem set will be assigned today

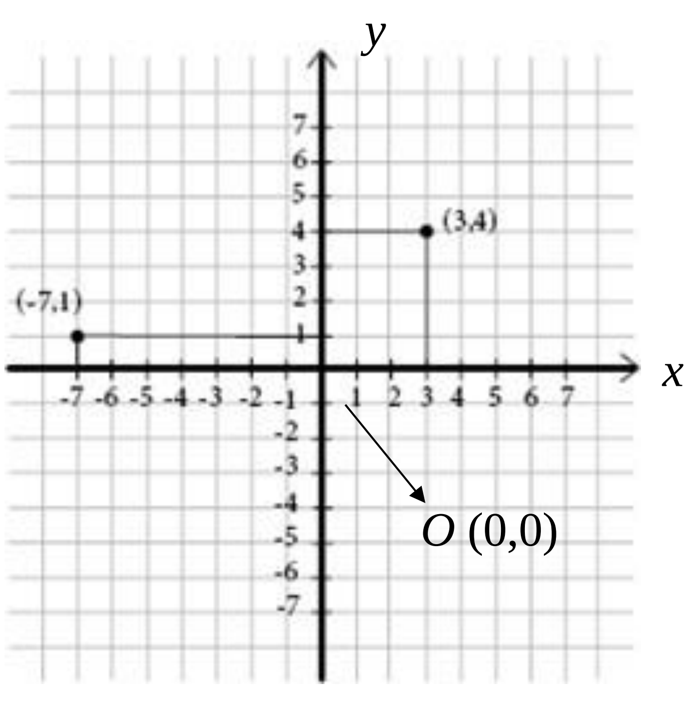
Build A 3D World

- What geometric primitives will you need?
  - Point
  - Line
  - Plane
  - Cube
  - Sphere
  - …



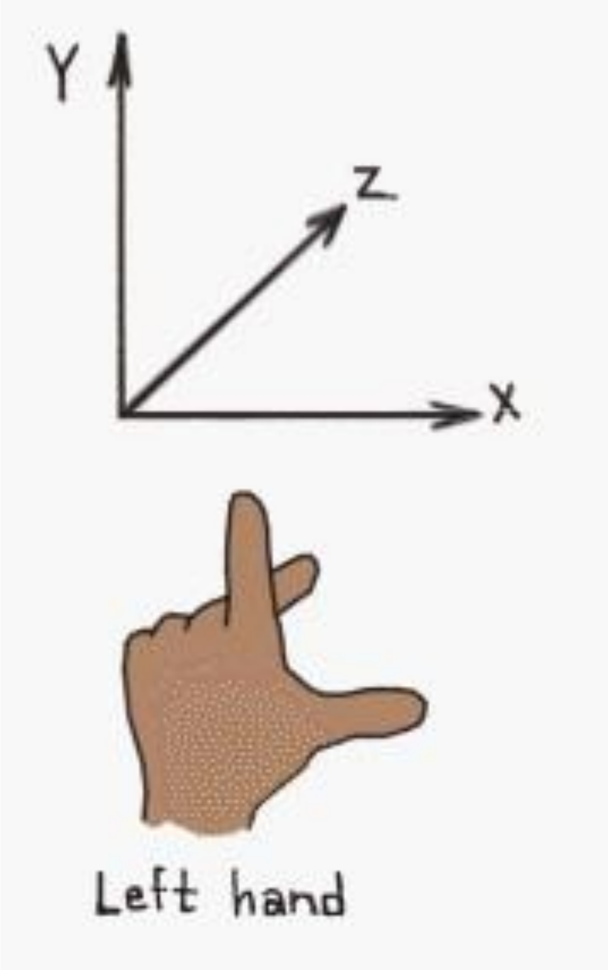
How to represent a point?

- Coordinate system (Cartesian)
  - Origin
  - Orthogonal Axes
  - Unit

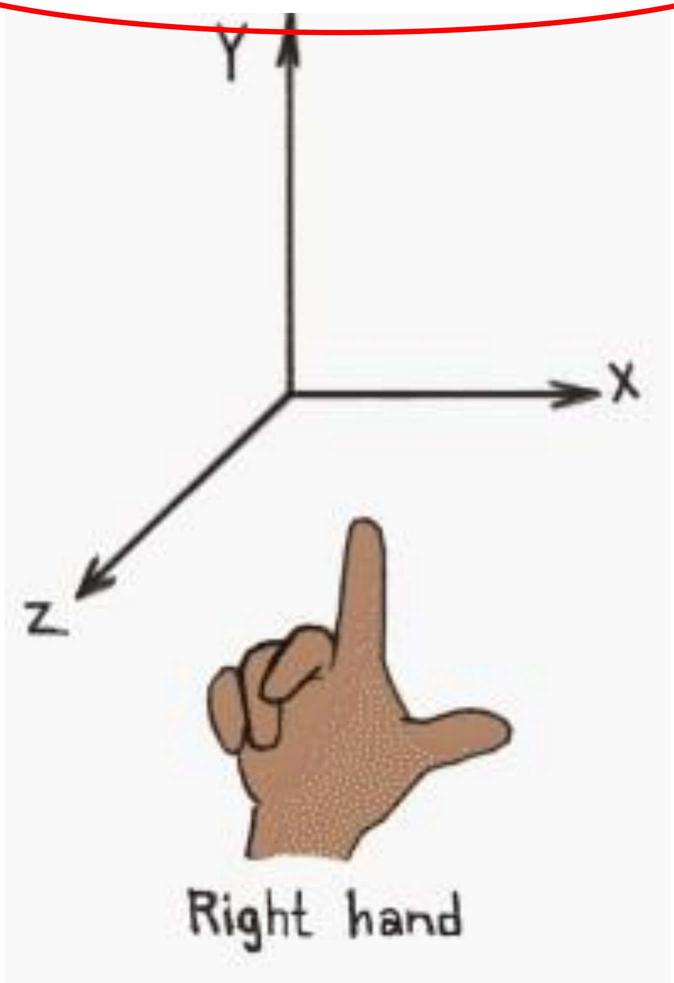


3D Coordinate

- Left-handed system

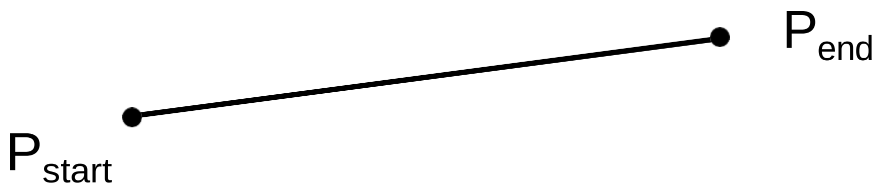


- Right-handed system

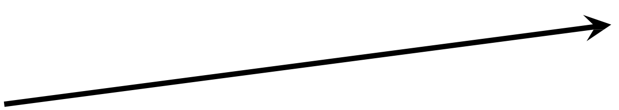


Lines?

- Line Segment
  - Defined by two end points

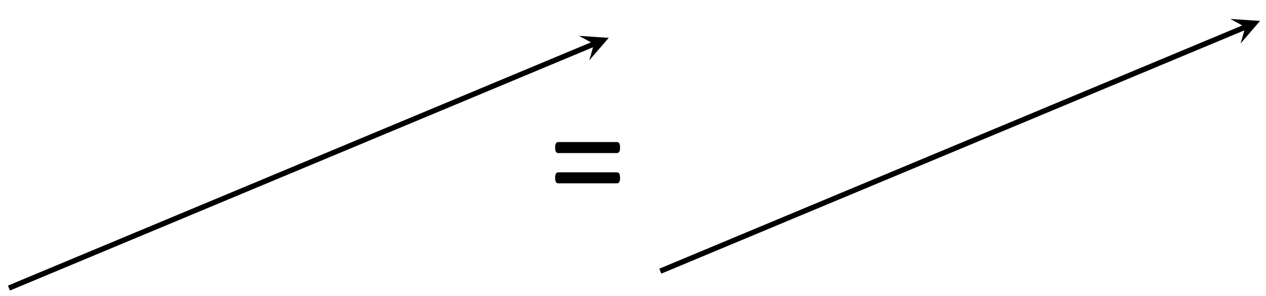


- Vector
  - Line with direction!



Vector

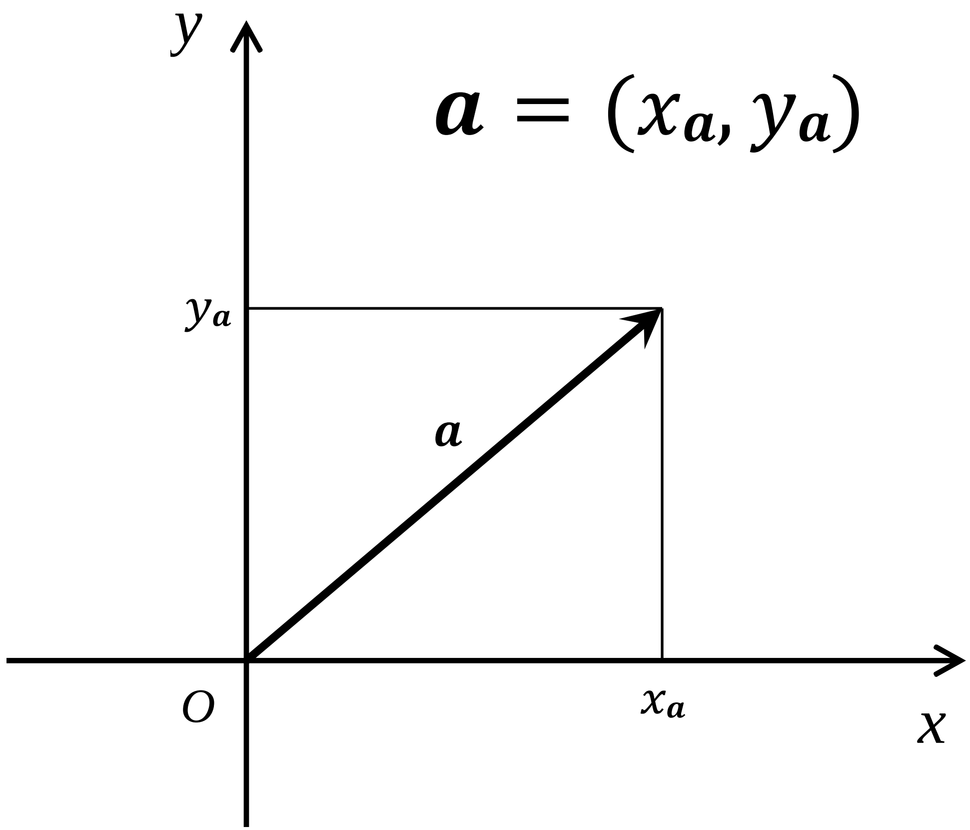
- Vector looks like an arrow
- Defined by *direction* and *length*
  - Absolute location is not important
- Usually written with bold letters or letters with arrow top
  - For example: $\boldsymbol{a}$ or $\vec{a}$



Vector $\boldsymbol{a}$ Vector $\boldsymbol{b}$

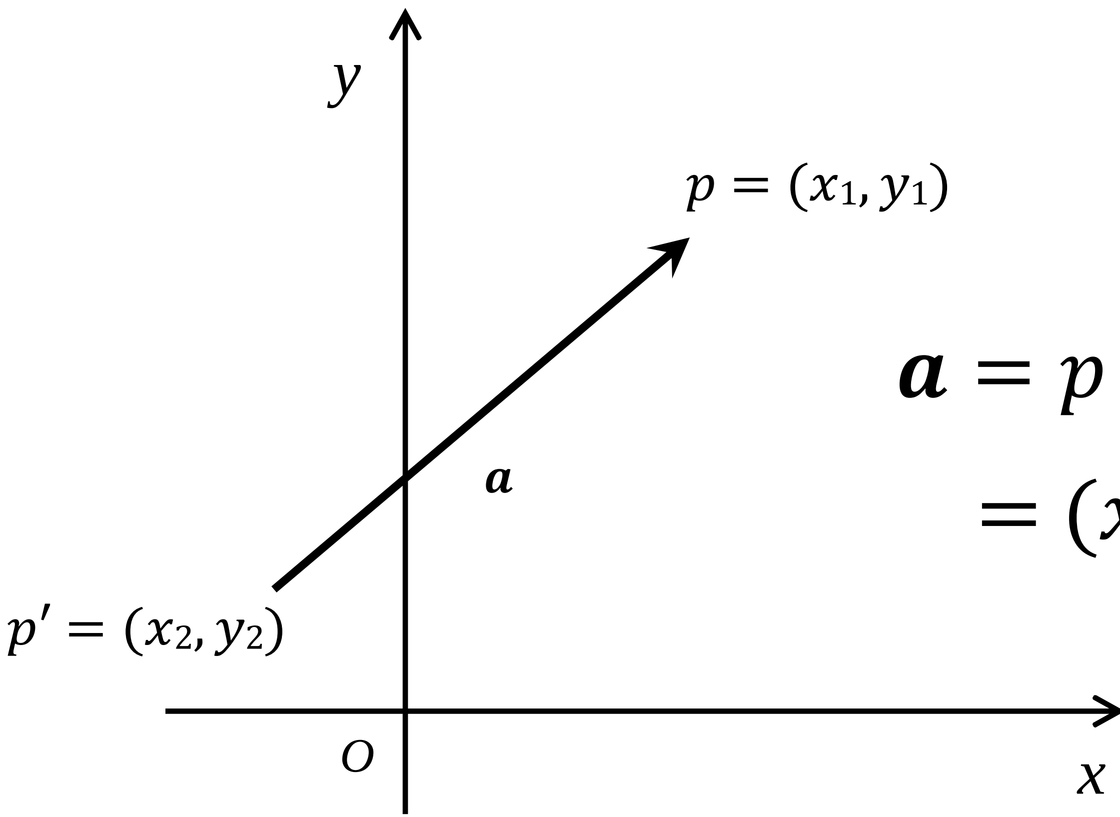
Vector Representation

- Vectors are represented by their coordinates on x and y axes



Vector Representation

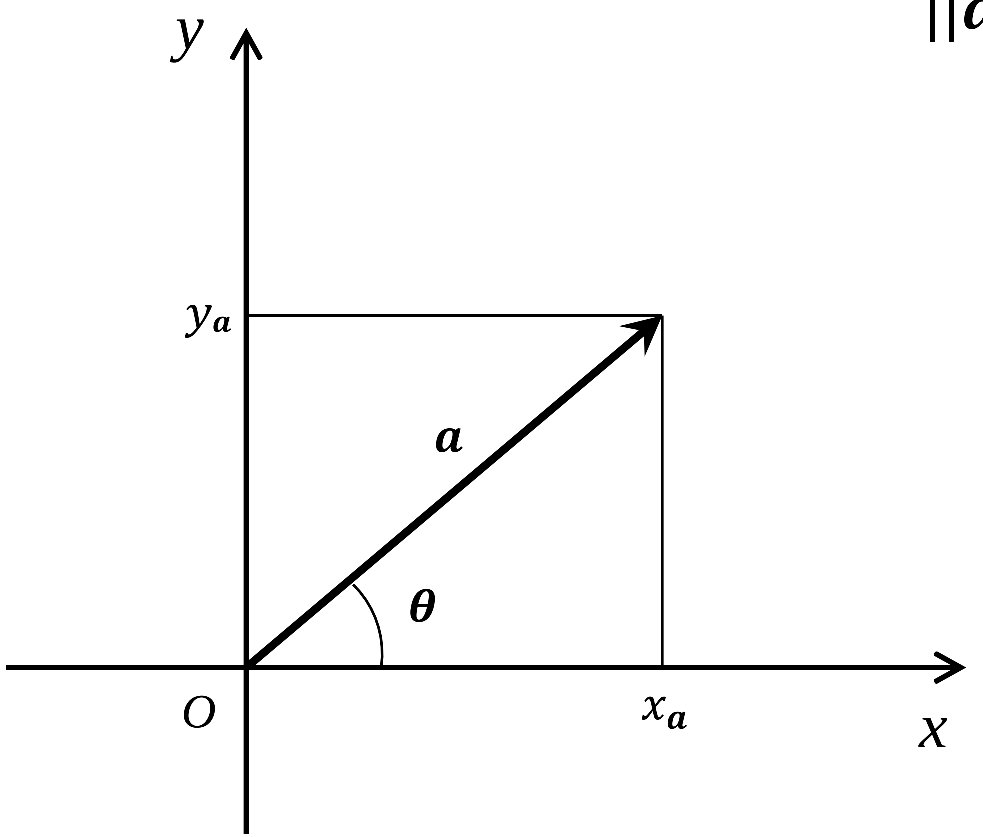
- What if the starting point is not at origin?
  - Take difference between the two end points



$$\boldsymbol{a} = p - p'$$
$$= (x_1 - x_2, y_1 - y_2)$$

Vector Properties

- Length: $||\boldsymbol{a}|| = \sqrt{x_{\boldsymbol{a}}{}^2 + y_{\boldsymbol{a}}{}^2}$
- Direction in angle: $\theta = \cos^{-1}(\frac{x\boldsymbol{a}}{||\boldsymbol{a}||})$

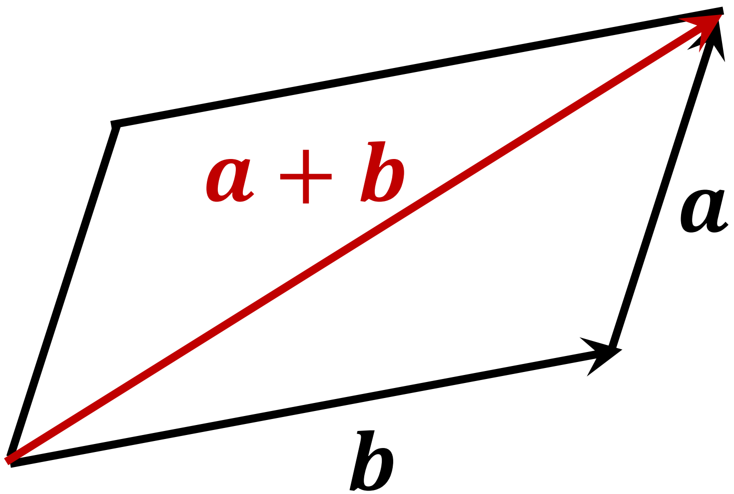


Vector Operations

- Addition
- Subtraction
- Scaling
- Multiplication
  - Dot product
  - Cross product

Vector Addition

- Geometrically: Parallelogram rule
- Arithmetically: Simply add coordinates
- Vector addition is commutative



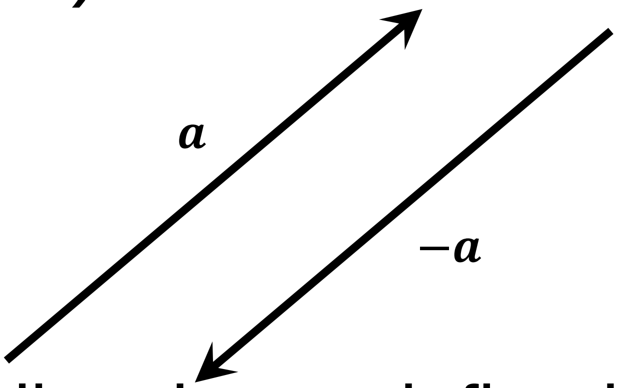
$$\boldsymbol{a} = (x_{\boldsymbol{a}}, y_{\boldsymbol{a}})$$
$$\boldsymbol{b} = (x_{\boldsymbol{b}}, y_{\boldsymbol{b}})$$

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} = (x_{\boldsymbol{a}} + x_{\boldsymbol{b}}, y_{\boldsymbol{a}} + y_{\boldsymbol{b}})$$

Vector Subtraction

- We define minus (or negative) such that

  $$-\boldsymbol{a} + \boldsymbol{a} = \boldsymbol{0}$$



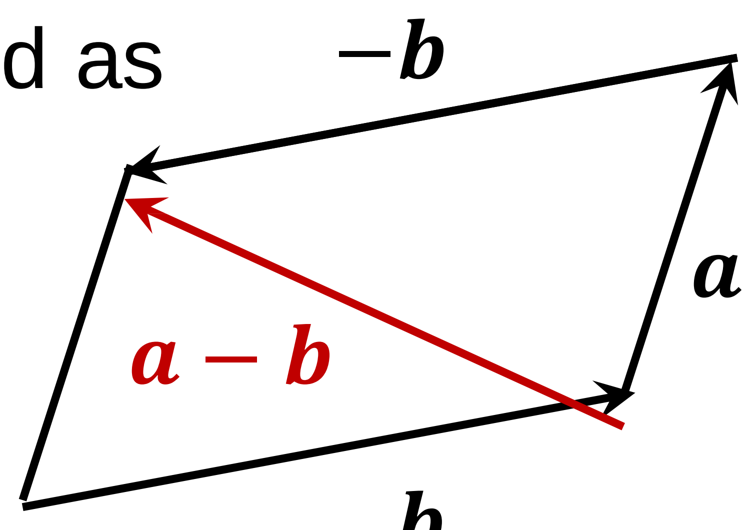
  – Simply reverse the direction

  – Length keeps the same

  – Zero vector $\boldsymbol{0}$: length = 0 & direction undefined

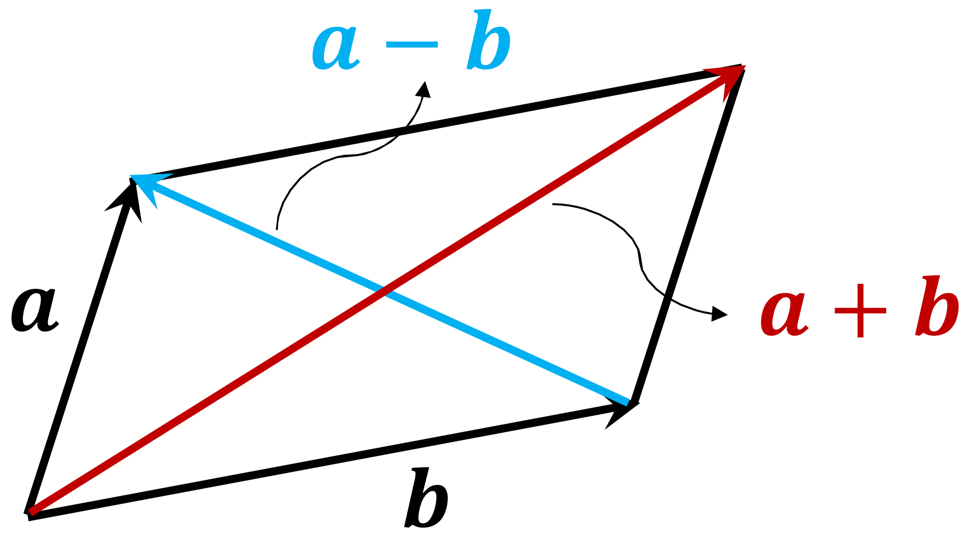
- Subtraction is defined as

  $$\boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{b} + \boldsymbol{a}$$



Parallelogram Rule

- If two vectors share the same origin, the two diagonals in the formed parallelogram give $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$

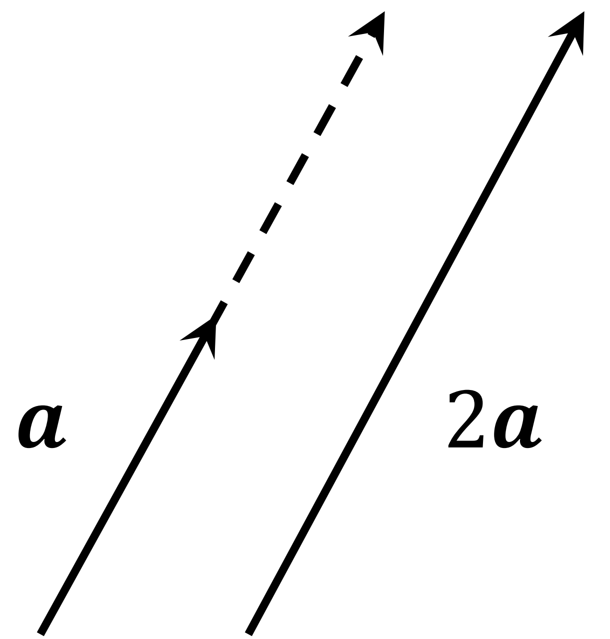


Vector Scaling

- Scaling operation scales the vector's length while the direction remains the same

$$||k\boldsymbol{a}|| = k||\boldsymbol{a}||$$

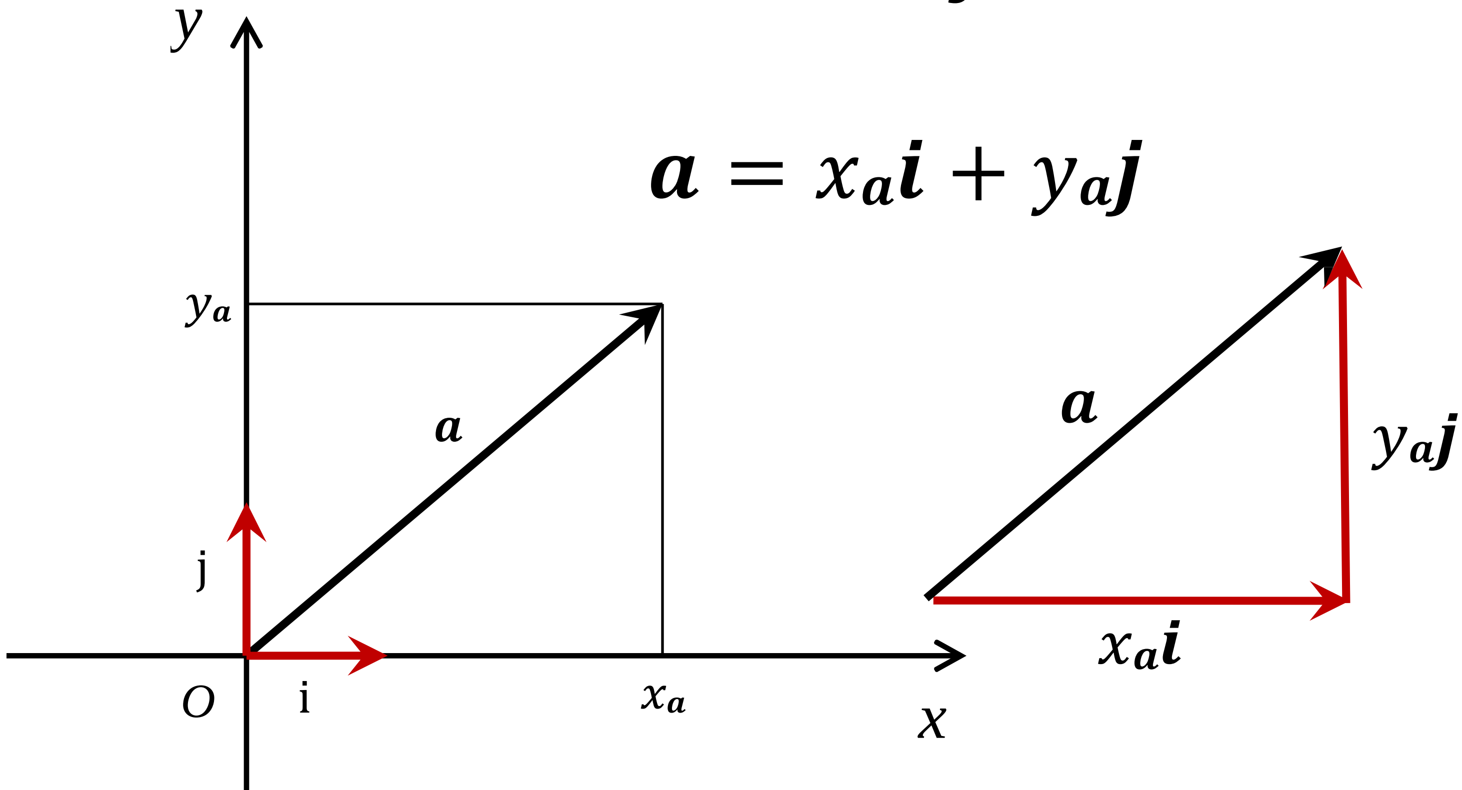
- Scaling can be combined with addition and subtraction
  - Linear combination



Vector Coordinate

- Vector coordinate can be expressed by linear combination (scaling and addition) of two unit basis vectors ***i*** and ***j***

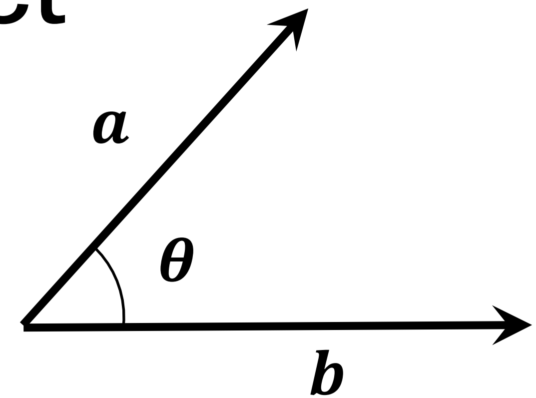
$$\boldsymbol{a} = x_a \boldsymbol{i} + y_a \boldsymbol{j}$$



Vector Multiplication

- Dot product
  - Produce a scalar
- Cross product
  - Produce a new vector

Dot Product



- Dot product is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos \boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is the angle between the two vectors

- Dot product is also called scalar product
  - because its result is a scalar
- Dot product is commutative

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

- Dot product is distributive

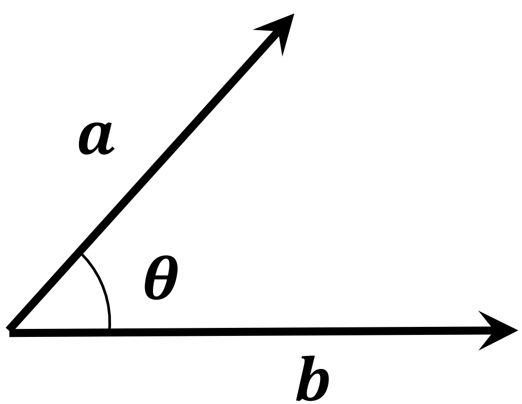
$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

Dot Products in Coordinate

- Dot product is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos \boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is the angle between the two vectors



$$\boldsymbol{a} = (x_{\boldsymbol{a}}, y_{\boldsymbol{a}})$$
$$\boldsymbol{b} = (x_{\boldsymbol{b}}, y_{\boldsymbol{b}})$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = x_{\boldsymbol{a}} x_{\boldsymbol{b}} + y_{\boldsymbol{a}} y_{\boldsymbol{b}}$$

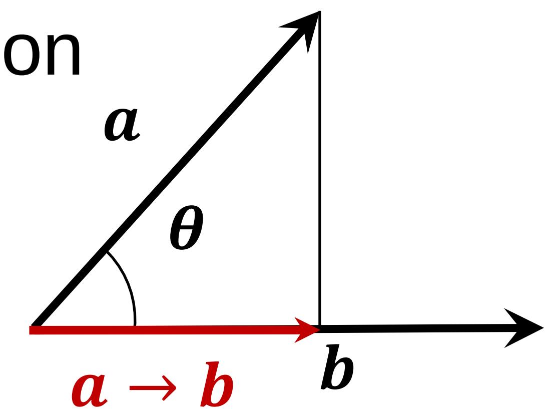
Dot Product: Applications

- Find angle between two vectors

$$\boldsymbol{\theta} = \cos^{-\mathbf{1}}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|}\right)$$

- Finding *projection* of one vector on another
  - Useful in coordinate transformation
  - Projection of **a** on **b**

$$\|\boldsymbol{a} \to \boldsymbol{b}\| = \|\boldsymbol{a}\| \cos\boldsymbol{\theta} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|}$$

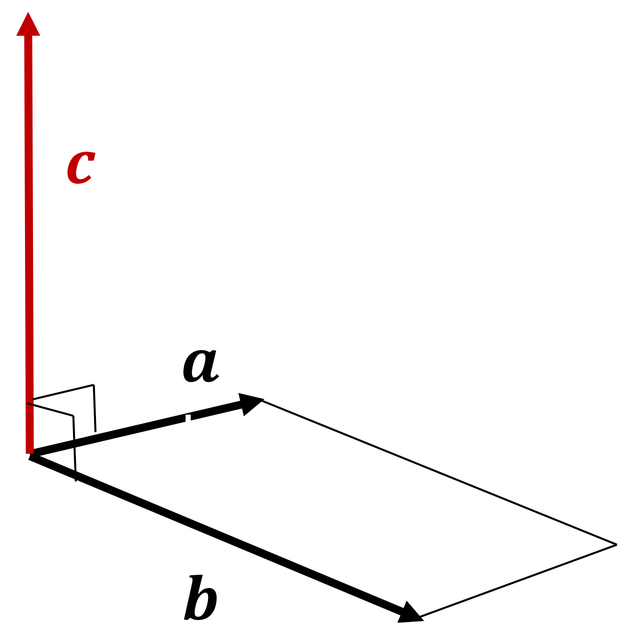


Cross Product

- Cross product is only used for 3D vectors
- Cross product results in a new vector

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$

- Direction of $\boldsymbol{c}$ is orthogonal to the two initial vectors
- Direction is determined by right-hand rule

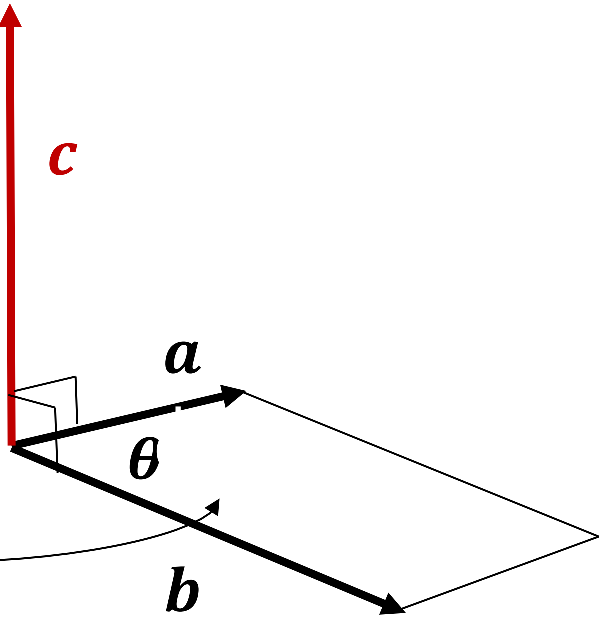


Cross Product

- Length of cross product?

$$\|\boldsymbol{c}\| = \|\boldsymbol{a}\times\boldsymbol{b}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \sin \boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is the angle formed the initial two vectors



Area of the parallelogram formed by the two vectors!

Cross Product: Properties

- Cross product is distributive

$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$$

$$(k\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{a}\times(k\boldsymbol{b})=k(\boldsymbol{a}\times\boldsymbol{b})$$

- Cross product is NOT commutative

$$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$$

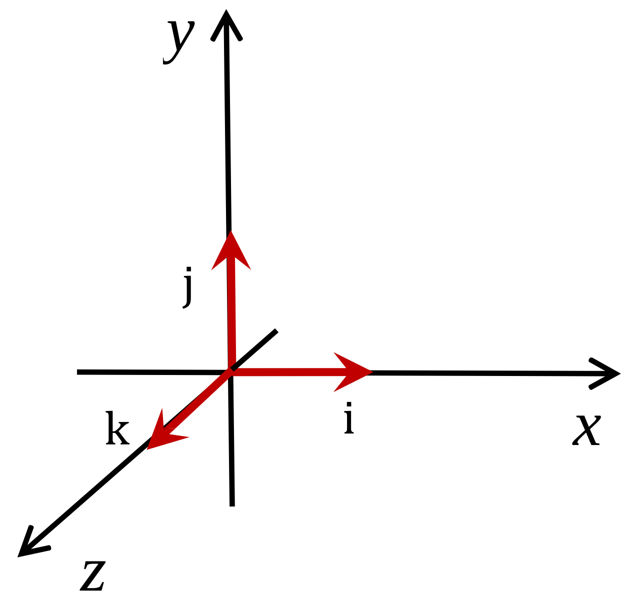
  – *Order matter!*
  – Think about the right-hand rule

Cross Product in Coordinate

- Given $\boldsymbol{a} = (x_{\boldsymbol{a}}, y_{\boldsymbol{a}}, z_{\boldsymbol{a}})$ & $\boldsymbol{b} = (x_{\boldsymbol{b}}, y_{\boldsymbol{b}}, z_{\boldsymbol{b}})$
- Calculated by determinant

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_{\boldsymbol{a}} & y_{\boldsymbol{a}} & z_{\boldsymbol{a}} \\ x_{\boldsymbol{b}} & y_{\boldsymbol{b}} & z_{\boldsymbol{b}} \end{vmatrix}$$

$$= (y_{\boldsymbol{a}} z_{\boldsymbol{b}} - z_{\boldsymbol{a}} y_{\boldsymbol{b}})\boldsymbol{i}$$
$$+ (z_{\boldsymbol{a}} x_{\boldsymbol{b}} - x_{\boldsymbol{a}} z_{\boldsymbol{b}})\boldsymbol{j}$$
$$+ (x_{\boldsymbol{a}} y_{\boldsymbol{b}} - y_{\boldsymbol{a}} x_{\boldsymbol{b}})\boldsymbol{k}$$

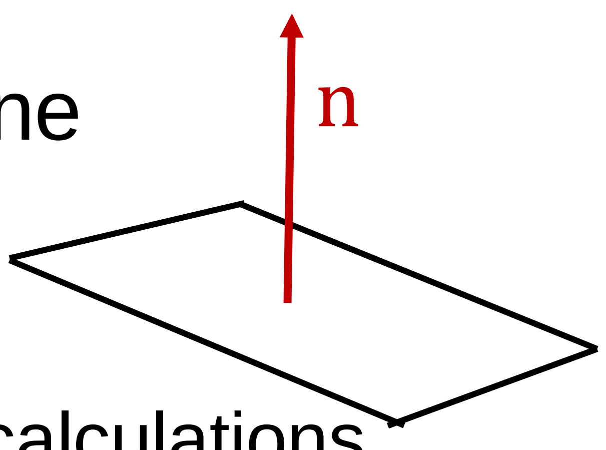


Applications in Computer Graphics

- Constructing a coordinate system
  - Axes are mutually orthogonal
  - Follow right-hand rule
  - For any 3D Cartesian system (x-y-z):

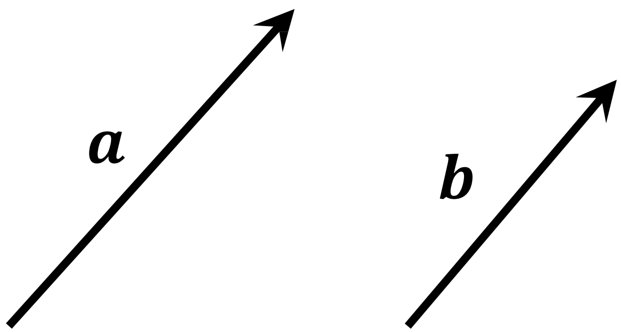
$$\boldsymbol{z} = \boldsymbol{x} \times \boldsymbol{y};\ \boldsymbol{x} = \boldsymbol{y} \times \boldsymbol{z};\ \boldsymbol{y} = \boldsymbol{z} \times \boldsymbol{x}$$

- Find *Normal Vector* of a plane
  - Normal Vector **n** is a vector *perpendicular* to a plane
  - Important to many graphics calculations



Quiz

- Cross product of two parallel vectors?



$$\boldsymbol{a} \parallel \boldsymbol{b}$$

$$\boldsymbol{a} \times \boldsymbol{b} = ?$$

$$\boldsymbol{a} \parallel \boldsymbol{b} \Rightarrow \sin\theta = 0$$
$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\boldsymbol{\theta} = 0$$
$$\boldsymbol{a} \times \boldsymbol{b} = \mathbf{0}$$

Matrices

- What are matrices?
  – Array of numbers (m x n = m rows, n columns)

$$\begin{bmatrix} 2 & 10 \\ 5 & 6 \\ 8 & 7 \\ 1 & 3 \end{bmatrix} \quad (4 \times 2)$$

- Important to geometric transformations
  – Translation, rotation, shear, scale

  (will talk about next time)

Matrix Operations

- Addition/Subtraction
  - per-element operation
- Multiply by scalar
  - per-element operation
- Matrix-matrix multiplication
  - A little tricky… we'll see how

Matrix-Matrix Multiplication

- To compute the (i,j)th element in the result:
  - Multiply the *ith row* in the first matrix with the *jth column* in the second matrix
  - Then sum them up

$$\begin{bmatrix} 2 & 10 \\ 5 & 6 \\ 8 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 9 & 4 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 58 & 28 \\ 21 & 69 & 32 \\ 31 & 100 & 46 \\ 6 & 21 & 10 \end{bmatrix}$$

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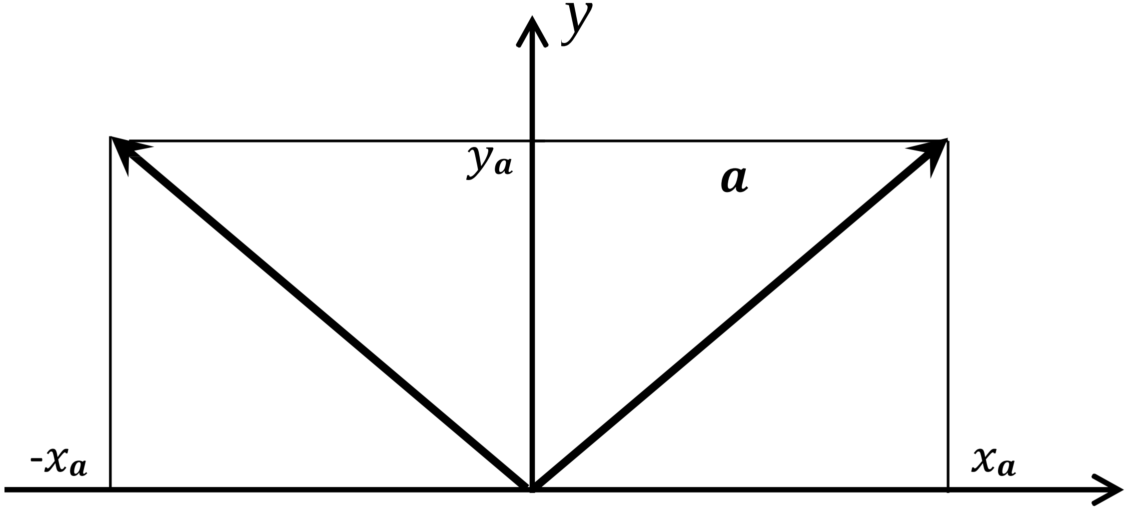
Multiplication Properties

- Number of columns in the first matrix must equal to the number of rows in the second matrix
- Non-commutative!
  - $AB \neq BA$
  - Sometimes changing order even makes the operation illegal!
- Associative
  - $(AB)C=A(BC)$
- Distributive
  - $A(B+C)=AB+AC$
  - $(A+B)C=AC+BC$

Matrix-Vector Multiplication

- Later we'll use a lot!
- Treat Vector as a *column* matrix
- Example: given vector $\boldsymbol{a} = (x_a, y_a)$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -x_a \\ y_a \end{bmatrix}$$



Matrix Transpose

- Switch rows and column indices
- If A is a m x n matrix, its transpose A^T is a n x m matrix
- Example:

$$\begin{bmatrix} 2 & 10 \\ 5 & 6 \\ 8 & 7 \\ 1 & 3 \end{bmatrix}^{\backslash} = \begin{bmatrix} 2 & 5 & 8 & 1 \\ 10 & 6 & 7 & 3 \end{bmatrix}$$

Identity Matrix & Inversion

- Identity matrix is a square matrix (row = column) with all its diagonal elements equal to 1 and the rest 0

$$I_{(3x3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverse Matrix A^{-1}

$$AA^{-1} = A^{-1}A = I$$

Compute Inverse Matrix

- Only square matrix can be inversed
- Matrix inversion formula:

$$\mathrm{A}^{-1} = \frac{1}{det(\mathrm{A})} adj(\mathrm{A})$$

where *det*(A) is the determinant of A

adj(A) is the adjugate matrix of A

Example

- Given a 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- Let's compute its inverse matrix A^{-1}

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$det(A) = ad - cb$$

$$A^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-b}{ad-cb} \\ \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$$

Math Problem Set

- Due on next Thursday (8/31)
- Turn in by *yourself* in class
- Submission will not be accepted after Thursday class
- Your solution will be graded based on each step! (only writing the final result will earn a small portion of the score)
- Write clearly and staple all paper together