EXAMPLE 1.11 Find the Newton's Method formula for the equation $x^3 + x - 1 = 0$.

Since $f'(x) = 3x^2 + 1$, the formula is given by

$$
\begin{aligned}
x_{i+1} &= x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1} \\
&= \frac{2x_i^3 + 1}{3x_i^2 + 1}.
\end{aligned}
$$

Iterating this formula from initial guess $x_0 = -0.7$ yields

$$
\begin{aligned}
x_1 &= \frac{2x_0^3 + 1}{3x_0^2 + 1} = \frac{2(-0.7)^3 + 1}{3(-0.7)^2 + 1} \approx 0.1271 \\
x_2 &= \frac{2x_1^3 + 1}{3x_1^2 + 1} \approx 0.9577.
\end{aligned}
$$

These steps are shown geometrically in Figure 1.9. Further steps are given in the following table:

i	x_i	$e_i = \|x_i - r\|$	e_i/e_{i-1}^2
0	−0.70000000	1.38232780	
1	0.12712551	0.55520300	0.2906
2	0.95767812	0.27535032	0.8933
3	0.73482779	0.05249999	0.6924
4	0.68459177	0.00226397	0.8214
5	0.68233217	0.00000437	0.8527
6	0.68232780	0.00000000	0.8541
7	0.68232780	0.00000000	

After only six steps, the root is known to eight correct digits. There is a bit more we can say about the error and how fast it becomes small. Note in the table that, once convergence starts to take hold, the number of correct places in x_i approximately doubles on each iteration. This is characteristic of "quadratically convergent" methods, as we shall see next.