29/09/16

Lecture 7

Convergence of sequences $\{x_k\}$ $k=0,1,\ldots$ to a limit L

$\{x_k\} \to L$ if for all $\delta > 0$ $\exists$ $K(\delta)$, an index, satisfying

$$|x_k - L| < \delta \quad \forall\, k \geq K(\delta)$$

Absolute error (root finding)

$e_k = r - x_k \quad k = 0, 1, \ldots$

~~$M \in [0,1]$ $n < 1$ or~~

Order of convergence

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^n} = M \Rightarrow |e_{k+1}| = M|e_k|^n$$

$M \in [0,1]$, $n = 1$ $\Rightarrow$ order of convergence is linear.

if $M = 0$, its called super linear

$n = 2$ $\Rightarrow$ quadratic order

Consider the FPI

$$x_{k+1} = g(x_k) \quad k = 0, 1, 2, \ldots$$

where

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \quad a > 0$$

Suppose $x = p$ is a fixed pt of $g(x)$

$$p = \frac{1}{2}\left(p + \frac{a}{p}\right), \quad 2p = p + \frac{a}{p} \quad \rightarrow$$

$$2p^2 = p^2 + a$$

$$p^2 = a$$

$$p = \pm\sqrt{a}$$

$f(x) = 0$

$f(x) = x - \sqrt{a}$

or $f(x) = x^2 - a$

What is the order of convergence? (of this FPI)

$$e_{k+1} = \sqrt{a} - x_{k+1}$$

$$= \sqrt{a} - \left(\frac{1}{2}\left(x_k + \frac{a}{x_k}\right)\right)$$

$$= \frac{2x_k\sqrt{a} - x_k^2 - a}{2x_k}$$

$$= -\frac{(\sqrt{a} - x_k)^2}{2x_k} = -\frac{e_k^2}{2x_k}$$

$$\frac{|e_{k+1}|}{|e_k|^2} = \frac{1}{2x_k}$$

$$\therefore \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = \frac{1}{2\sqrt{a}} = M$$

$$\Rightarrow p = 2$$

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411 lecture 8

Error in rootfinding

Consider the following root finding problems

$$f(x) = x^3 - 2x^2 + \frac{4}{3}x - \frac{8}{27} = 0 \quad \text{on the interval } [0, 1]$$

and the root $r = \frac{2}{3}$

Terminate with

$x = m_k$ when $f(m_k) < \varepsilon_m$ where ε_m is machine precision in double - precision arithmetic

$$\varepsilon_m = 2^{-52} \approx 2.2 \times 10^{-16}$$

After running this we get

$$x_c = m_{16} = \underbrace{0.666664}_{5 \text{ correct digits}}1$$

Find r based on $f(x) = 0$

Estimate m

Find $g(x)$ from $f(x) = (x-r)^m g(x)$

Find r_2, a root of $g(x)$

use r_2 as

Jenkins - Traub → garuntees to find all roots of a polynomial

Solving systems of non linear equations

↳ Computationaly much more challenging

Example $(N = 2)$

Solve $f_1(x) = X_1 + X_2 - 3 = 0$
$f_2(x) = X_1^2 + X_2^2 - 9 = 0$

For these systems there is no generalization of IVT that could possible answer the question of existence.

Uniqueness is also a much more difficult question to answer.

Solve

$$f_1(X_1, \ldots, X_N) = 0$$
$$f_2(X_1, \ldots, X_N) = 0$$
$$\vdots$$
$$f_N(X_1, \ldots, X_N) = 0$$

N equations in N unknowns $N > 1$

$F(X) = 0$ $\quad F: \mathbb{R}^N \to \mathbb{R}^N$

$$F = \begin{bmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_N(X) \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

Notes

1) Newton's method is *locally* convergent

2) If it does converge then it does so ~~at~~ at a quadratic rate

3) Solving $J(x^{(n)})d = -F(x^{(k)})$ for $d = d^{(k)}$ in order to obtain $x^{(k+1)} = x^{(k)} + d^{(k)}$

can be prohibitively expensive

Secant

line through (x_{k-1}, f_{k-1}) & (x_k, f_k)

$\{x_k\}_{k=0}^{\infty} \to r$

$\{|x_k - x_{k-1}|\}_{k=0}^{\infty} \to 0$

$\left\{\dfrac{f_{k-1} - f_k}{x_{k-1} - x_k}\right\} \to$

Quasi Newton Methods

Replace

$$J(x^{(k)})d = -F(x^{(k)})$$

with $A_k d = -F(x^{(k)})$ with aim to capture the same preformance (in the tail of the sequence) where

1) A_k is easier to compute than $J(x^{(a)})$
2) $A_k d = -F(x^{(k)})$
is easier to solve.

8/10/16

411 Lecture 9

Recap (Newton's Method)

Solve $F(x)=0$ $\quad F: \mathbb{R}^N \to \mathbb{R}^N$
Model $F(x)$ at $x = x^{(k)}$, with
$L(x) = F(x^{(k)}) + J(x^{(k)})(x - x^{(k)})$
Solve:
$L(x)=0$ to obtain $x^{(k+1)}$

$\Rightarrow$ Solve $J(x^{(k)})d = -F(x^{(k)})$ for $d^{(k)}$
set $x^{(k+1)} = x^{(k)} + d^{(k)}$

$J(x^{(k)})$

$J(x^{(k)})d = -F(x^{(k)})$ for $d = d^k$

epsilon chang in x vector in jth component

$$[J(x^{(k)})]_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x=x^{(k)}} \approx \frac{f_i(x^{(k)} - e_j) - f(x^{(k)})}{|e_j|}$$

$i = 1, ----, N$
$j = 1, ----, N$

$A_k d = -F(x^{(k)})$

what should A_k be?

Univariate Case $f: R \to R$

In N-R we modelled $f(x)$ at $x = x_k$

$l(x) = f(x^{(k)}) + (x - x_k) f'(x_k)$ tangent line to f at $x^{(k)}$

$l(x) = 0 \Rightarrow x^{(k+1)} = x^{(k)} - \frac{1}{f'(x^{(k)})} f(x^{(k)})$ N-R

Secant Method (univariate)

Model $f(x)$ at $x = x_k$ with

$$l(x) = f_k \frac{(x - x_{k-1})}{(x_k - x_{k-1})} + f_{k-1} \frac{(x - x_k)}{(x_{k-1} - x_k)}$$

Lagrange form of secant line through (x_{k-1}, f_{k-1}) & (x_k, f_k)

Set x_{k+1} to solution of $l(x) = 0$

$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k$$

$$= x_k - \frac{1}{\frac{(f_k - f_{k-1})}{(x_k - x_{k-1})}} f(x_k)$$

$$f'(x_k) \approx \frac{f_k - f_{k-1}}{x_k - x_{k-1}} = \frac{\Delta_k}{\delta_k}$$

$$f'(x_k)\, \delta_k = \Delta_k$$

$\Downarrow$ suggests where $\delta_k = x^{(k)} - x^{(k-1)}$

$$A_k \delta_k = \Delta_k \qquad \Delta_k = F(x^{(k)}) - F(x^{(k-1)})$$

$= J(x^{(k)})\, \delta_k$ (B1 - Secant Conditions)

- B1 gives us N linear equations in N^2 unknowns

B2 $A_k \omega = A_{k-1} \omega$ where $\omega^T \delta_k = 0$

(B1)+(B2) $\Rightarrow$ Quasi - Newton Method

Broyden's First Method

$$A_k = A_{k-1} + \frac{(\Delta_k - A_{k-1}\delta_k)\delta_k^T}{\delta_k^T \delta_k}$$

$\hookrightarrow$ scalar