SYDE 312 Jan 28

basis, dimension, coordinates

Shopping list:
dimension
basis
independance
coordinates

I-3 Independance & Basis (cont.)

V v.s. over F

$\{u_1, \dots, u_r\}$ set of vectors $\quad V = S_p\{u_1, \dots, u_r\}$

1) If $\{u_1, \dots, u_r\}$ dep find a vec = l.c of the others and discard it ("cast out"). Span doesn't change

2) check independance. If no, go back to 1)
If yes, called a 'basis' of V. A basis of V is a minimal spanning set or linearly independant spanning set or maximal independant set.

Important result (not proved)

The number of vectors in any basis is always the same, called the dimension of V, $\dim(V)$. plural of basis → bases

$B = \{u_1, \dots, u_n\}$ basis of V. Any vec in $v \in V$ can be written as a l.c.

$v = K_1 u_1 + \dots + K_n u_n$ // since B is a sp. set

Suppose $v = K'_1 u_1 + \dots + K'_n u_n$ is some other lc for v

$\Rightarrow K_1 u_1 + \dots + K_n u_n = K' u_1 + \dots + K'_n u_n$

$\Rightarrow (K_1 - K'_1) u_1 + \dots + (K_n - K'_n) u_n = 0$

$\Rightarrow K_1 - K'_1 = 0$ (B is lin. ind.) ie $K_i = K_i'$

The expression for a vec. $v \in V$ is a l.c. of basis vectors is <u>unique</u>

$v = a_1 u_1 + \dots + a_n u_n$ (say)

where $B = \{u_1, \dots, u_n\}$ is the basis chosen $v \in V$

The $a_1, \dots, a_n$ are called the coordinates of v w.r.t. B notation.

$[v]_B = [a_1, \dots, a_n]_B$

Choose a different B' basis $B' = \{u_1', \dots, u_n'\}$

$v = b_1 u_1' + \dots + b_n u_n'$ So $[v]_B \neq [v]_{B'}$ in general

Note: B vectors are assumed to have an order i.e. B is an ordered basis

<u>Standard Bases</u>

$\mathbb{R}^n : \{e_1, \dots, e_n\} = S$

$e_i = (0, \dots, 0, \underset{\uparrow\, i\text{th}}{1}, 0, \dots, 0)$

$\mathbb{P}_n : \{1, t, t^2, \dots, t^n\} = S$

mono nomial basis

$\dim(\mathbb{P}_n) = n+1$

$M_{m,n} : E_{ij} = \begin{array}{cc} & j \\ i & \left[\begin{array}{c} \\ 1 \\ \\ \end{array}\right] \end{array}$ $\{E_{11}, E_{12}, \dots, E_{mn}\} = S$

$\dim(M_{m,n}) = mn$

bin v.s. on n things

$\{1, 2, \ldots, n\} = S$ Recall this means $\{\{1\}, \{2\}, \ldots, \{n\}\}$

In say bin v.s. on $\{1,2,3,4,5\}$
we have

$$[1,3,5]_S = [1,0,1,0,1]_S$$

means $[\{1,3,5\}]_S = [1,0,1,0,1]_S$

Ex. Is $B = \{(1,1,1), (1,2,3), (2,-1,1)\}$ a basis for $\mathbb{R}^3$?

Solution: Any 3 ind. vectors in $\mathbb{R}^3$ would be a basis, so check independance:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$ ind. $\therefore$ a basis //

Ex. Find a basis for $W = Sp\{(1,-2,5,3), (2,3,1,-4), (3,8,-3,-5)\} < \mathbb{R}^4$ and extend it to a basis for $\mathbb{R}^4$

Solution: Row space method

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W = Sp\underbrace{\{(1,-2,5,-3), (0,7,9,-2)\}}_{\text{basis for } W}$$ $\dim(W) = 2$

To extend to all of $\mathbb{R}^4$, include vecs $\{(0,0,1,0), (0,0,0,1)\}$

new pivot

Coordinates

In $\mathbb{R}^n$: $v = (v_1, \dots, v_n)$ and $[v]_S = [v_1, \dots, v_n]_S$

means: $v = v_1 e_1 + \dots + v_n e_n$

In $\mathbb{R}^3$: $[(x,y,z)]_S = [x,y,z]_S$

because $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

Coordinates with respect to the standard basis in $\mathbb{R}^n$ are assumed by the notation $(v_1, \dots, v_n)$ for a vec. v.

Note again in general: $[v]_B \neq [v]_{B'}$ for a given $v \in V$

Q: How can we find $[v]_{B'}$ from $[v]_B$ easily?

Change of basis problem

Ex Find the coords. of $v = (2,3,4)$ w.r.t.
$B = \{(1,1,1), (1,1,0), (1,0,0)\}$ in $\mathbb{R}^3$

Solution: want $(2,3,4) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

Solve: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 4 \end{array}\right] \sim$ Ans: $a = 4$, $b = -1$, $c = -1$

$[(2,3,4)]_B = [4,-1,-1]_B$

Change of basis

how to get $[v]_B$ given $[v]_S$?

$[(1,1,1)]_B = [1,0,0]_B$ in the previous example because

$(1,1,1) = 1(1,1,1) + \cancel{0(1,1,0)} + \cancel{0(1,0,0)}$

Sim. for other vecs. in B.

Cont. with the previous ex.

Put $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $[v]_S = P[v]_B$

$$[v]_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} [v]_B \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_B \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_B$$

In general, $B = \{u_1, \dots, u_n\}$ basis of V

The change of basis matrix

from between S & B is

$P = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$ where $u_1, \dots, u_n$ are expressed in S coords

Then $\boxed{[v]_S = P[v]_B}$ for any $v \in V$

Usually we want $[v]_B = P^{-1}[v]_S$

given $[v]_S$ }

P^{-1} exists because the cols are basis vectors $\Rightarrow$ full rank

$$\begin{tikzcd} S \arrow[r, bend right, "P^{-1}"'] & B \arrow[l, bend right, "P"'] \end{tikzcd}$$

Ex: previous ex. done with P

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_S}_{[v]_S} = \underbrace{\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}_B}_{[v]_B} \quad //$$

<u>Change of basis where neither are standard</u>

$$\begin{tikzcd} B \arrow[rr, bend left, "Q^{-1}P"] \arrow[dr, bend right, "P"'] & & B' \arrow[dl, bend right, "Q"'] \\ & S \arrow[ul, bend right, "P^{-1}"'] \arrow[ur, bend right, "Q^{-1}"'] & \end{tikzcd}$$

Ex $B = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B' = \{(1,0,1), (1,1,0), (0,1,1)\}$

Find $[v]_{B'}$ given $[v]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B$ and $[v]_S$

Solution:

$$\begin{tikzcd} B \arrow[rr, "Q^{-1}P"] \arrow[dr, "P"'] & & B' \\ & S \arrow[ur, "Q^{-1}"'] & \end{tikzcd}$$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

$[v]_S = P[v]_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

$Q^{-1}P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

$[v]_{B'} = Q^{-1}P[v]_B = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}_{B'}$