$U+W$, $U\cap W$, $U \oplus W$

$U, W < V$

The sum $U+W$, is $\{u+w \mid u\in U, w \in W\} = Sp\{u,w\}$ when $u \in U$, $w \in W$

$\therefore U+W < V$

The ontersection $U\cap W$ is $\{v \mid v \in U \; v \in W\}$

important result ~~$\dim U + \dim W = \dim(U+W) + \dim(U\cap W)$~~

$$\boxed{\dim(U) + \dim(W) - \dim(U\cap W) = \dim(U+W)}$$

Any vector $v \in V$ can be written $\underbrace{\text{uniquely}}$ as ~~a~~ a liner combination.

$$v = u+w, \; u \in U, \; w \in W \underset{\text{if and only if.}}{\iff} U\cap W = \{0\}$$

The direct sum $U \oplus W$ of U, W...

Ex. Find bases for $U \overset{+}{\cancel{\oplus}} W \overset{\text{and}}{y} U\cap W$, ~~and~~ where

$U = Sp\{(1,4,0,-1), (2,-3,1,1)\}$

$W = Sp\{(0,1,1,1), (4,5,1,-1)\}$

$U+W$

method 1: put all in matrix, $\therefore U+W$, reduce to find basis

method 2: column space method

→ all 4 vectors @ colums of matix & row reduce.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 4 & -3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \dim(U+W) = 3$

$\therefore \dim(U\cap W) = 1.$

So, basis for $U+W$ is $\{(1,4,0,-1), (2,-3,1,1), (0,1,1,1)\}$

$U \cap W$

method 1: ? apparently not worth doing

~~method 2:~~ method 2

$(x, y, z, w) \in U \cap W$

$(\cancel{x}, y, z, w) = a(1,4,0,-1) + b(2,-3,1,1)$ sp of U

$= -c(0,1,1,1) + (-d)(4,5,1,-1)$ sp of W

∴ they're equal...

$a(1,4,0,-1) + b(2,-3,1,1) = c(0,\cancel{d},1,1) + d(4,5,1,-1)$

if we find the constraints of a, b, c, d
↳ solve.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 4 & -3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

→ row redue to get as done for U+W method 2..

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d = free

$c = 0$

$b = -c - d = -d$

$a = -2d$.

$$\therefore \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = d \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

} from this $(x, y, z, w) = \overset{(-2)}{a}(1,4,0,-1) + \overset{(-1)}{b}(2,-3,1,1)$
$= (-4, -5, -1, 1)$

} from this $(x, y, z, w) = (4, 5, 1, -1)$

conclusion: A basis for $U \cap W$ is $\{(4,5,1,-1)\}$

$\underset{2}{\dim(U)} + \underset{2}{\dim(W)} - \underset{1}{\dim(U \cap W)} = \underset{3}{\dim(U+W)}$

I-4: More Linear Maps

V, U as v.s. (same scala)

$\{v_1, v_2, \ldots v_n\}$ basis of V

$\{u_1, \ldots u_n\}$ ANY vector in U

there is a unique linear map $T: V \to U$ with $T(v_1) = u_1$

$\vdots$

$T(v_n) = u_n$

i.e T is defined by it's action on a basis.

$v \in V \to v = a_1 v_1 + . + a_n v_n$

$$\begin{aligned} \text{so } T(v) &= T(a_1 v_1 + \ldots + a_n v_n) \\ &= a_1 T(v_1) + .. + a_n T(v_n) \end{aligned}$$

$\therefore$ T is linear!

Ex. find the unique linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T(1,2) = (2,3)$; $T(0,1) = (1,4)$

Given: $T(v_1)$ and $T(v_2)$ when $B = \{v_1, v_2\}$ is a basis of $\mathbb{R}^2$

$v_1 = (1,2)$, $v_2 = (0,1)$

Find: $T(x,y)$ i.e w.r.t standard basis

method 1: linear combination into matrix : row reduce.

method 2: ~~find P at~~

$$\begin{tikzcd} B \arrow[r, bend left, "P"] & S \arrow[l, bend left, "P^{-1}"] \end{tikzcd}$$

(i) $P = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{array} \to P^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$$[(a,b)]_S = [a, -2a+b]_B$$

$$P^{-1}\begin{bmatrix} a \\ b \end{bmatrix}_S = \begin{bmatrix} a \\ -2 + B \end{bmatrix}_B$$

So $T(a,b) = T\left[\, a(1,2) + (-2a+b)(0,1) \right]$

$= aT(1,2) + (-2a+b)\,T(0,1)$

$= a(2,3) + (-2a+b)(1,4)$

$= (b, -5a+4b)$

$\therefore \quad T(a,b) = (b, -5a+4b)$

Example : Find the unique linear map : $T : R^3 \to R^2$

so that $T(1,1,1) = (1,0)$

$T(1,1,0) = (2,-1)$

$T(1,0,0) = (4,3)$

Given $T(v_1), T(v_2), T(v_3)$ when $B = \{v_1, v_2, v_3\}$ is a basis of R^3

$v_1 = (1,1,1)$

$v_2 = (1,1,0)$

$v_3 = (1,0,0)$

Find : $T(x,y,z) =$

Sol'n :

$$\begin{tikzcd} B \arrow[r, bend left, "P"] & S \arrow[l, bend left, "P^{-1}"] \end{tikzcd}$$

(i) $P = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ v_1 & v_2 & v_3 \end{array}\right] \to P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

to do 3x3 inverse $\left[A \mid I \right]$

row reduce until $\left[I \mid A^{-1} \right]$

(ii) $P^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_S = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}_B$

(from $[(a,b,c)]_S = c v_1 + (b-c) v_2 + (a-b) v_3$)

So $T(a,b,c) = T(c v_1 + (b-c) v_2 + (a-b) v_3)$

$= c\,T(v_1) + (b-c)\,T(v_2) + (a-b)\,T(v_3)$

b/c T is linear...

$T(a,b,c) = c(1,0) + (b-c)(2,-1) + (a-b)(4,3)$

$= (c + 2b - 2c + 4a - 4b, -b + c + 3a - 3b)$

$T(a,b,c) = (4a - 2b - c, 3a - 4b + c)$ //

Range (Image) of a Linear Map

V, U ~~same~~ v.s. same scalars

A linear map $T: V \to U$ has range (image)

~~$T(v) = S_p \{T(v_1, \dots, v_n\}$~~

$T(V) = S_p \{T(v_1), \dots T(v_n)\}$ where

$V = S_p \{v_1, \dots, v_n\}$

So linear maps preserve linear combinations

Ex. Find a basis for the range of $T: \mathbb{R}^4 \to \mathbb{R}^3$

where ⊛ $T(x,y,z,t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$

Sol'n: T is defined by it's action on a basis (of $\mathbb{R}^4$)

Pick, say S.

$$\left.\begin{aligned} T(1,0,0,0) &= (1,2,3) \\ T(0,1,0,0) &= (-1,-2,-3) \\ T(0,0,1,0) &= (1,3,4) \\ T(0,0,0,1) &= (1,4,5) \end{aligned}\right\} \text{given by ⊛}$$

use col space method to get a basis for $T(\mathbb{R}^4)$

$$\begin{array}{cccc} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cccc|cccc} c_1 & & c_2 & & c_1 & & c_2 & \end{array} \longrightarrow \text{base of col space}$$

$\downarrow$

basis for $T(\mathbb{R}^4)$

Conclusion: A basis for $T(\mathbb{R}^4)$ is $\{(1,2,3), (1,3,4)\}$ //

Q: Does a linear map preserve linear independence?

A: No! eg. previous example!

The rank(T) of a linear map is dim (T(V))

previous ex. Rank (T) = 2

Kernel of Linear Maps

$\ker(T) = \{v \in V \mid T(v) = 0\}$

Ex. Find the basis for the kernel of $T: R^4 \to R^3$ where $T(x,y,z,t) = ⊛$ (prev. page)

Sol'n: need to solve $T(x,y,z,t) = (0,0,0)$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

matrix from prev ↑

row reduce to
$$\begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & \text{free} & & \text{free} \end{array} \end{array}$$

$$z = -2t$$
$$\Rightarrow x = y - z - t$$
$$= y + t$$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} y+t \\ y \\ -2t \\ t \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

concl: basis for $\ker(T)$ is $\{(1,1,0,0), (1,0,-2,1)\}$ //

$\text{null}(T) = \dim(\ker(T))$