2.2 Fixed-Point Iteration

Burden, Richard L.; Faires, J. Douglas (1985). "2.2 Fixed-Point Iteration". Numerical Analysis (3rd ed.). PWS Publishers. ISBN 0-87150-857-5..

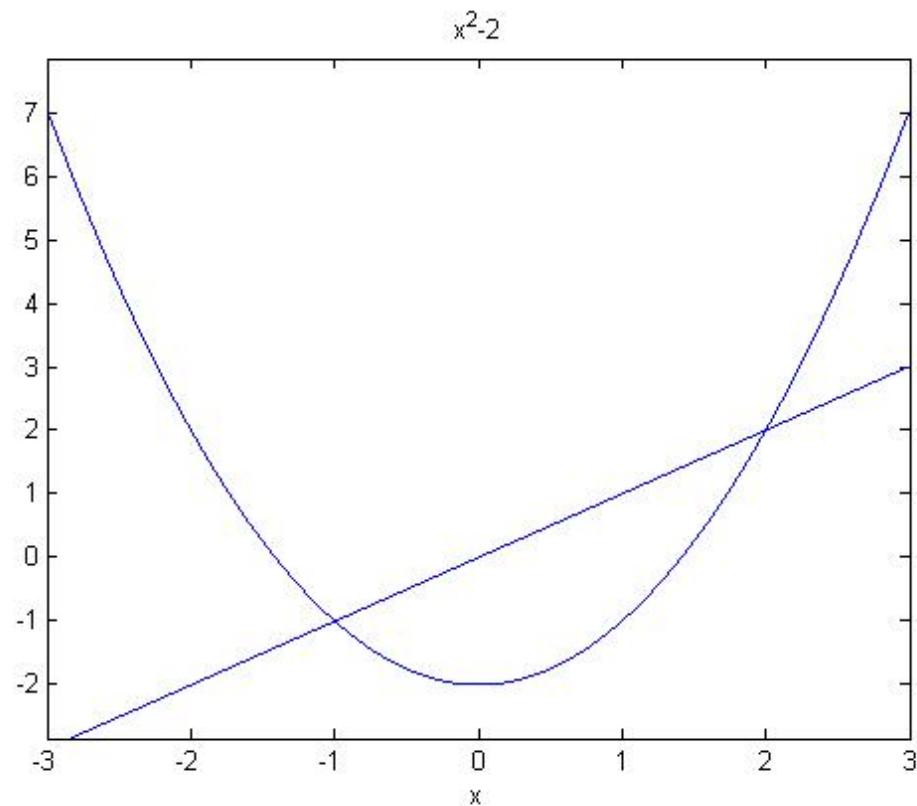**Definition 2.2.** The number p is a **fixed point** for a given function $g(x)$ if $g(p) = p$.

Geometric interpretation of **fixed point.**

➢ Consider the graph of function $g(x)$, and the graph of equation $y = x$.

➢ If they intersect, what are the coordinates of the intersection point?

Example 2.2.1.

Determine the fixed points of the function $g(x) = x^2 - 2$.

Connection between fixed-point problem and root-finding problem

1. Given a root-finding problem, i.e., to solve $f(x) = 0$. Suppose a root is p, so that $f(p) = 0$.

   There are many ways to define $g(x)$ with fixed-point at p.

   For example, $g(x) = x - f(x)$,
   $$g(x) = x + 3f(x),$$

   ...

2. If $g(x)$ has a fixed-point at p, then $f(x)$ defined by $f(x) = x - g(x)$ has a zero at p.

Sufficient conditions for existence and uniqueness of a fix point

Theorem 2.3. Existence and Uniqueness Theorem

a. If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one **fixed-point** in $[a, b]$

b. If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a, b),$$

then there is **exactly one fixed-point** in $[a, b]$.

Note:

1. $g \in C[a, b] \rightarrow g$ is continuous in $[a, b]$
2. $g(x) \in [a, b] \rightarrow$ range of g is in $[a, b]$

Example 2. Show $g(x) = \dfrac{x^2 - 1}{3}$ has a unique fixed point on [-1, 1].

Example 3. Show that **Theorem 2.3** does not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval [0, 1], even through a unique fixed point on this interval does exist.

Solution: $g'(x) = -3^{-x} \ln(3)$.

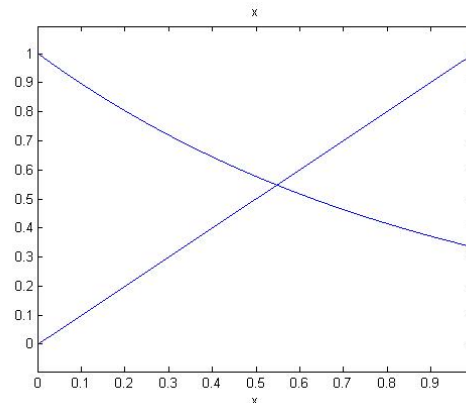$g'(x) < 0$ on [0,1]. So g is strictly decreasing on [0,1].

$g(1) = \frac{1}{3} \leq g(x) \leq g(0) = 1$, for $0 \leq x \leq 1$.

Part a of **Theorem 2.3** ensures there is at least one fixed point.

Since $|g'(0.01)| = |-3^{-0.01} \ln(3)| \approx 1.0866$,

$|g'(x)| \nleq 1$ on (0,1).

Since Part b of **Theorem 2.3** is NOT satisfied, **Theorem 2.3** can not determine uniqueness.

Graphs of 3^{-x} and $y = x$:

Fixed-Point Iteration Algorithm

- Choose an initial approximation p_0, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$.
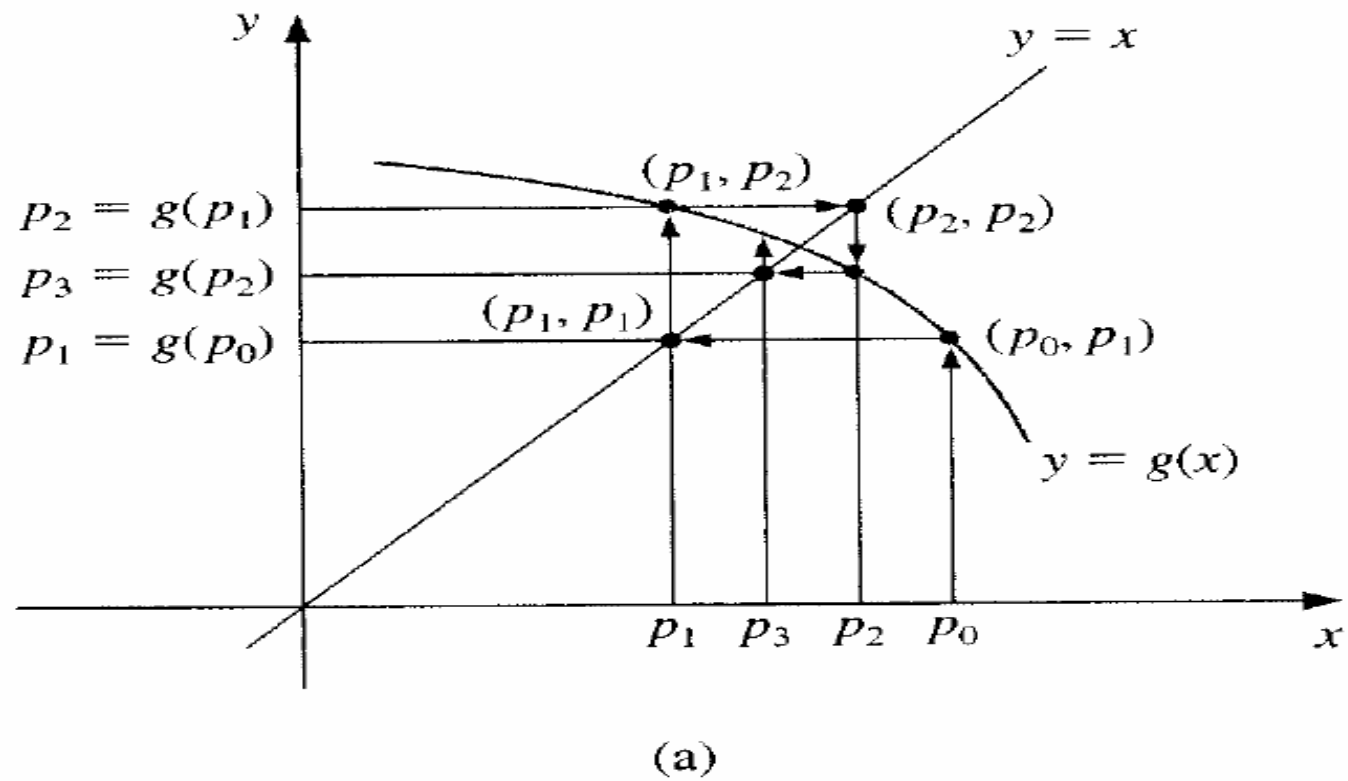- If the sequence converges to p, then

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g\left(\lim_{n \to \infty} p_{n-1}\right) = g(p)$$

A Fixed-Point Problem

Determine the fixed point of the function $g(x) = \cos(x)$ for $x \in [-0.1, 1.8]$.

Remark: See also the Matlab code.

The Algorithm

(a)

INPUT　　　**p0**; tolerance **TOL**; maximum number of iteration **N0**.

OUTPUT　solution **p** or message of failure

STEP1　　Set i = 1.　　　　　　　　　　　　　// init. counter

STEP2　　While i ≤ N0 do Steps 3-6

　　　　　STEP3　　Set **p** = g(**p0**).

　　　　　STEP4　　If |**p**-**p0**| < **TOL**　then

　　　　　　　　　　　OUTPUT(p);　　　　// successfully found the solution

　　　　　　　　　　　STOP.

　　　　　STEP5　　Set i = i + 1.

　　　　　STEP6　　Set **p0** = **p**.　　　　// update **p0**

STEP7　　OUTPUT("The method failed after **N0** iterations");

　　　STOP.

Convergence

Fixed-Point Theorem 2.4

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with
$$|g'(x)| \leq k, \qquad \text{for all } x \in (a, b)$$
Then, for any number p_0 in $[a, b]$, the sequence defined by
$$p_n = g(p_{n-1})$$
converges to the unique fixed point p in $[a, b]$.

Corollary 2.5

If g satisfies the above hypotheses, then bounds for the error involved using p_n to approximating p are given by
$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$
$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0|$$

Illustration Equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1,2]. Use algebraic manipulation to obtain fixed-point iteration function g to solve this root-finding problem.