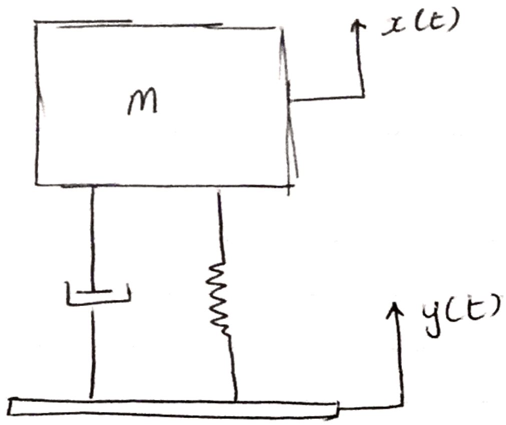
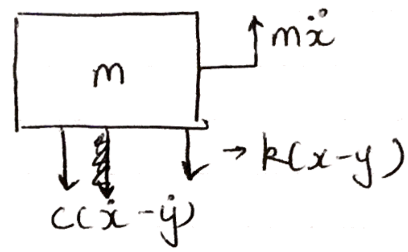
Midterm Exam: Wed, Feb 10, 4:30-6:30, Place: T.B.D

Example System



From Newton's 2nd Law
F.B.D



$$m\ddot{x} = -c(\dot{x}-\dot{y}) - k(x-y) \quad \text{or}$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad (*)$$

assume $y(t)$ is given, solve for $x(t)$.

Use La Place transforms

$$\mathcal{L}[f(t)] = F(s)$$

inverse LaPlace transform

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Laplace Transform Properties

1.) The Laplace variable s can be interpretted as a different operator.

$$\mathcal{L}\left[\frac{d}{dt}\right] = s$$

2) The reciprocal of s can be interpreted as an integral operator s.t.

$$\mathcal{L}\left[\int_0^t dt\right] = \frac{1}{s}$$

3) The laplace transform is a linear operator

$$\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

4) Derivatives

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$$

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\vdots$$

$$s^{0} f^{(n-1)}(0^-)$$

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) \underbrace{- s^{n-1} f(0^-) - s^{n-2} \dot{f}(0^-) \cdots\cdots - s^{0} f^{(n-1)}(0}_{\text{initial conditions}}$$

5) Integration

$$\mathcal{L}\left[\int_{0^-}^t f(t)\,dt\right] = \frac{1}{s} F(s)$$

6) Initial Value Theorem

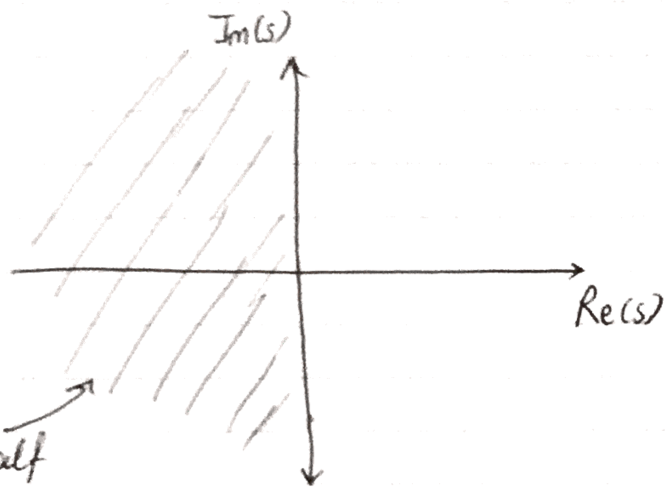
$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

7) Final value theorem

$$\lim_{t \to \infty} f(t) = \underbrace{\lim_{s \to 0} s F(s)}$$

↳ all the poles must be in the LHP (Left half plane)

<u>LHP</u>



left half plane
(does not include the imaginary axis)

Take the Laplace Transform of ⊛

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \quad ⊛$$

$$m\left(s^2 X(s) - s x_0 - v_0\right) + c\left(s X(s) - x_0\right) + k\left(X(s)\right)$$

$$= c\left(s Y(s) - y_0\right) + k\left(Y(s)\right)$$

$\rightarrow$

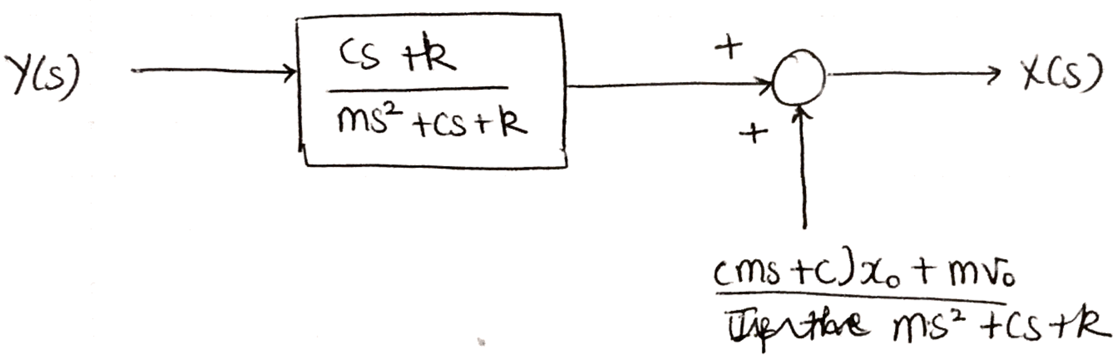
assume $y_0 = 0$

$$(ms^2 + cs + k)\,X(s) - (ms + c)\,x_0 - m v_0 = (cs + k)\,Y(s)$$

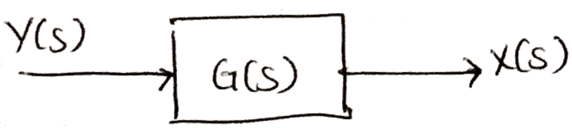
Solve for $X(s)$

$$X(s) = \underbrace{\left(\frac{cs + k}{ms^2 + cs + k}\right)Y(s)}_{\text{part of the solution due to the input}} + \underbrace{\frac{(ms + c)x_0 + m v_0}{ms^2 + cs + k}}_{\text{part of the solns that depend on the ICs}}$$

Block diagram



If the ICs are zero then this simplifies to



$$G(s) = \frac{cs + k}{ms^2 + cs + k}$$

$$X(s) = G(s)\,Y(s)$$

or, $$G(s) = \frac{X(s)}{Y(s)} = \frac{\text{output}}{\text{input}}$$

G(s) is the system transfer function

Solve this for $y(t) = 1$ (Heavyside step function)

$\Rightarrow Y(s) = \frac{1}{s}$

$\Rightarrow X(s) = \frac{cs+k}{s(ms^2+cs+k)}$

Let $m=1, c=3, k=2, x_0 = v_0 = 0$

$X(s) = \frac{3s+2}{s(s^2+3s+2)}$

Expand in partial fractions.

$X(s) = \frac{(3s+2)}{s(s+1)(s+2)} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{s+2}$

Take the inverse transform

$x(t) = \underbrace{H(t)}_{=1} + e^{-t} - 2e^{-2t}$ → system modes

recall $G(s) = \frac{3s+2}{s(s+1)(s+2)}$ poles: $\{0, -1, -2\}$

For complex poles
i.e.

$\frac{1}{(s+\sigma)^2+\omega^2} \xRightarrow{\mathcal{L}^{-1}} e^{-\sigma t}\sin\omega t$

or $\frac{s}{(s+\sigma)^2+\omega^2} \xRightarrow{\mathcal{L}^{-1}} e^{-\sigma t}\cos\omega t$