Examples

1) Zero vector space $V = \{\underline{0}\}$
over R, say scalars

$\underline{0} + \underline{0} = \underline{0}$

$k\underline{0} = \underline{0}$

smallest possible vector space (V.S.)

Need $\underline{0} \in V$ at least $\Rightarrow$ Null space cannot be a vector space

2) F^n - n-tuples of scalars (R $\overset{eg}{\text{or}}$ C)
R^n or C^n

$$\underbrace{(a_1, \ldots\ldots, a_n)}_{u} \underbrace{+}_{+} \underbrace{(b_1, \ldots, b_n)}_{v}$$

$= (a_1 \underset{\llcorner\lrcorner}{+} b_1, \ldots\ldots, a_n \underset{\llcorner\lrcorner}{+} b_n)$ Coordinate wise vector sum

uses scalar sum (already known) to define vector sum (new)

$ku = k(a_1, \ldots\ldots, a_n) = (ka_1, \ldots\ldots, \underbrace{ka_n})$

↳ scalar multiple (vector space operation)

↳ multiplication of real numbers (scalars)

3) $u = (\ldots\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots\ldots)$
$v = (\ldots\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\ldots)$ } Discrete signal space

$u + v = (\ldots, u_{-2}+v_{-2}, u_{-1}+v_{-1}, u_0+v_0, u_1+V_1, u_2+V_2, \ldots\ldots)$

→

$ku = (\ldots, ku_{-2}, ku_{-1}, ku_0, ku_1, ku_2, \ldots)$

coordinate wise defns

4) P Polynomial Space

$p \in P$ vector is a polynomial, any degree

Q what is $p+q$ for $p+q \in P$

Say $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k$
$q(t) = b_0 + b_1 t + \ldots + b_l t^l$

Define $p+q \in P$ using a "test point" t and real (scalar) arithmetic

$(p+q)(t) = a_0 + a_1 t + \ldots a_k t^k + b_0 + b_1 t + \ldots + b_l t^l$

scalars

test point

polynomial vector

$(cp)(t) = c(a_0 t + \ldots + a_k t^k)$

scalar arithmetic

new polynomial

test pt.

5) Matrix space M_{ij} vectors $A \in M_{ij}$ are $i \times j$ matrices

Use coordinate-wise defn for $A+B$ & kA

M_{ij} "look exactly like" R^{ij}.
isomorphic

6) Function spaces

X any set

$\{f : X \rightarrow F\} = F$

function from X to scalars F

$f, g \in F$

scalar sum

$(f+g)(t) = f(t) + g(t)$

vector sum defined

test pt in X

scalars

X can be any set, often R^n or C^n

$f : X \rightarrow R$ is a real valued function

Axioms $\Rightarrow$ useful, familiar results

a) $u_1 + u_2 + \dots + u_n$ is unambiguous (repeated application of S2)

b) $\underline{0}$ is unique

Pf: Suppose $\underline{0}, \underline{0}'$ are both zero vectors.

Any $u \in V$

$\Rightarrow u + \underline{0} = u$ -(1) & $u + \underline{0}' = u$ -(2) (By S3)

Put $u = \underline{0}'$ in (1) & $u = \underline{0}$ in (2)

$\Rightarrow \underline{0}' + \underline{0} = \underline{0}'$ & $\underline{0} + \underline{0}' = \underline{0}$

$\underline{0}' + \underline{0} = \underline{0} + \underline{0}'$ (By S1)

$\Rightarrow \underline{0}' = \underline{0}$

$\Rightarrow$ zero vector is unique

c) $-\underline{u}$ is unique

Pf: Suppose u', u'' are inverses of u

$u'' = u'' + \underline{0}$ (S3) $= u'' + (u' + u)$

u' is inverse of u

$= u'' + (u + u')$ by (S1)

cause assumption

$= (u'' + u) + u'$ (S2) $= \underline{0} + u' = u'$

$\Rightarrow u'' = u'$ $\Rightarrow$ $-u$ is unique, only 1 inverse

Allows substraction of vectors meaning fully

$u - v = u + (-v)$ unique!