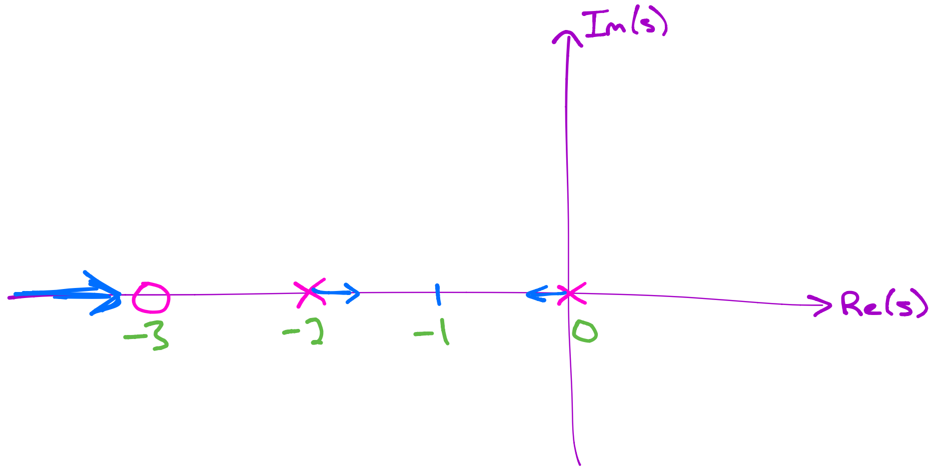
ex. $KGH(s) = \dfrac{\frac{4}{3}K(s+3)}{s(s+2)}$ O.L zeros: $\{-3\}$

O.L poles: $\{0, -2\}$



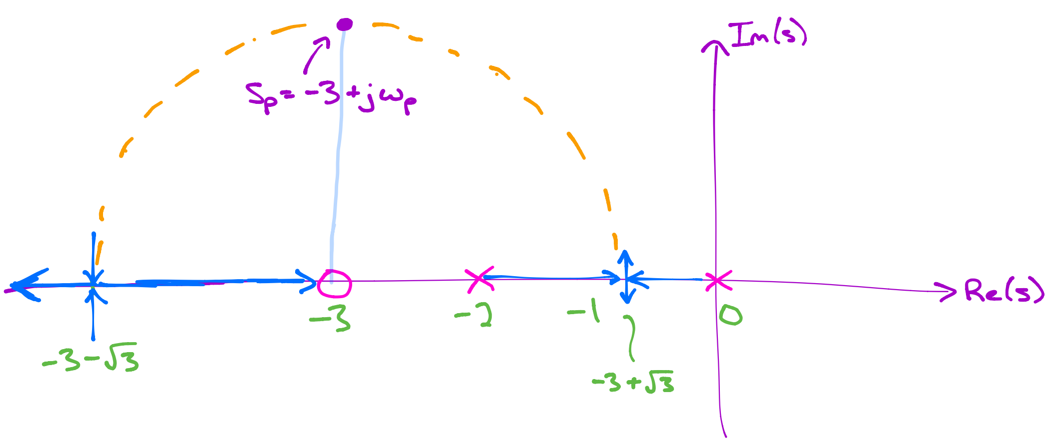
Find the breakaway & reentry points

$$GH(s) = \frac{\frac{4}{3}(s+3)}{s(s+2)} = \frac{\frac{4}{3}(s+3)}{s^2+2s}$$

$$\frac{dGH(s)}{ds} = 0 \longrightarrow \frac{\frac{4}{3}(s^2+2s) - \frac{4}{3}(s+3)(2s+2)}{(s^2+2s)^2} = 0$$

Numerator $\longrightarrow$ $\frac{4}{3}(s^2 - 2s - 2s^2 - 8s - 6) = 0$

$$s^2 + 6s + 6 = 0 \longrightarrow s_{1,2} = -3 \pm \sqrt{3}$$



Use the phase angle condition to find a few sample points off the real axis — to fill in the rest of the root locus

$$G(s)H(s) = \frac{\frac{4}{3}(s+3)}{s(s+2)}$$

can be any $n\pi$

$$\angle G(s)H(s) = \angle \tfrac{4}{3} + \angle (s+3) - \angle s - \angle (s+2) = -\pi$$

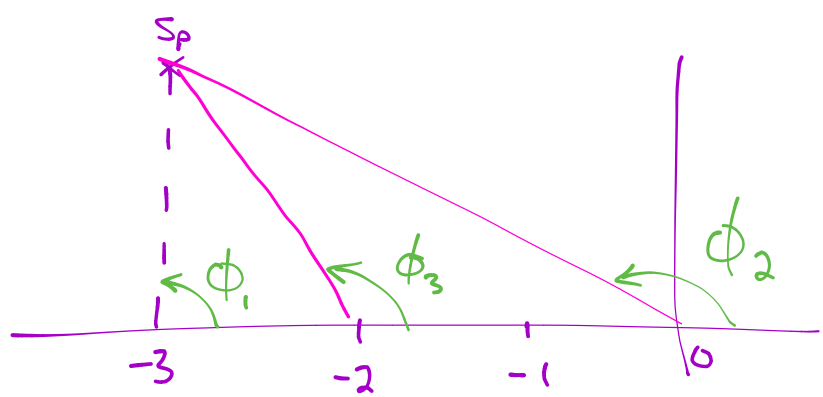
at $s = s_p$

$= s_p$

$$\angle G(s_p)H(s_p) = \angle \tfrac{4}{3} + \angle(-3+j\omega_p+3) - \angle(-3+j\omega_p) - \angle(-3+j\omega_p+2) = -\pi$$

$$= 0 + \underbrace{\angle(j\omega_p)}_{\phi_1} - \underbrace{\angle(-3+j\omega_p)}_{\phi_2} - \underbrace{\angle(-1+j\omega_p)}_{\phi_3} = -\pi$$

$$= 0 + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_p}{-3}\right) - \tan^{-1}\left(\frac{\omega_p}{-1}\right) = -\pi$$



Useful trig identity:

$$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$$

$$-\left(\tan^{-1}\left(\frac{\omega_p}{-3}\right) + \tan^{-1}\left(\frac{\omega_p}{-1}\right)\right) = \tan^{-1}\left(\frac{\frac{\omega_p}{-3} + \frac{\omega_p}{-1}}{1 - \frac{\omega_p^2}{3}}\right)$$

$$= \tan^{-1}\left(\frac{-\frac{4}{3}\omega_p}{1 - \frac{\omega_p^2}{3}}\right)$$

$$= \tan^{-1}\left(\frac{-4\omega_p}{3 - \omega_p^2}\right)$$

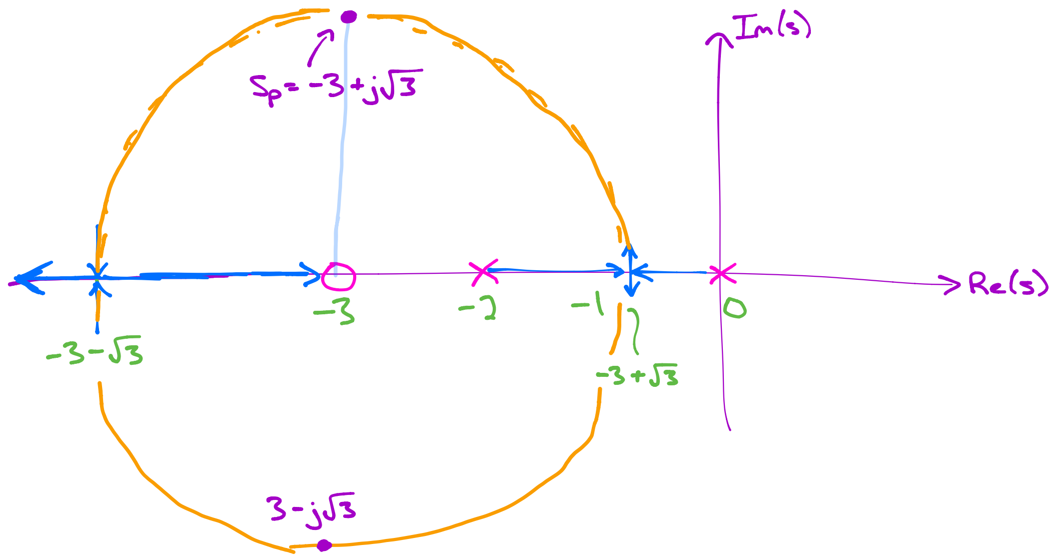
$$0 + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega_p}{-3}\right) - \tan^{-1}\left(\frac{\omega_p}{-1}\right) = -\pi$$

$$-\tan^{-1}\left(\frac{-4\omega_p}{3 - \omega_p^2}\right) = \frac{-3\pi}{2}$$

$$\tan^{-1}\left(\frac{-4\omega_p}{3 - \omega_p^2}\right) = \frac{3\pi}{2}$$

$$\frac{4\omega_p}{3 - \omega_p^2} = \infty$$

$$\hookrightarrow \omega_p^2 = 3 \rightarrow \omega_p = \pm\sqrt{3}$$



Rule 7

For large values of K, some branches will go to infinity after leaving the real axis.
They do so along straight lines called asymptotes.

let m = # O.L zeros
n = # O.L poles

$n - m$ = the pole/zero excess

Angles of asymptotes:

$$\Theta_k = \frac{(2k+1)\pi}{n-m} \quad , \; k = 0, 1, \dots, |n-m| - 1$$

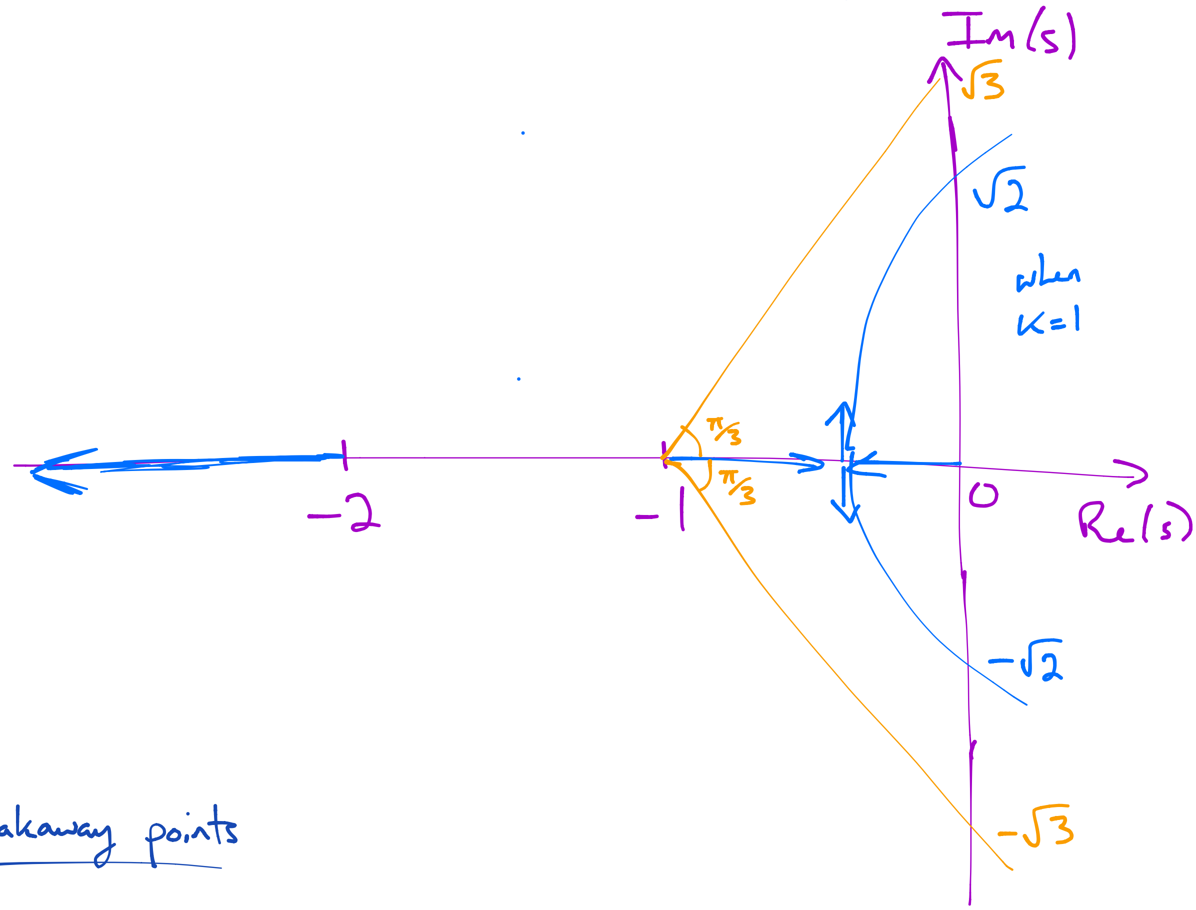
The origin of the asymptotes

i) is always on the real axis

ii) at $O.A = \dfrac{\Sigma \text{poles} - \Sigma \text{zeros}}{n-m}$

ex. $KGH = \dfrac{6K}{s(s+1)(s+2)}$ $O.A = \dfrac{(0-1-2)-(\ )}{3-0}$

$= -\frac{3}{3} = -1$



Breakaway points

$\dfrac{dK}{ds} = 0 \rightarrow K = \dfrac{s(s+1)(s+6)}{-6}$, $\dfrac{dK}{ds} = \dfrac{s^3 + 3s^2 + 2s}{6} = 0$

$s_{1,2} = -1 \pm \dfrac{1}{\sqrt{3}} = \begin{cases} -0.423 \\ -1.577 \end{cases}$

θ_k and O.A

$$\theta_k = \frac{(2k+1)}{3}, \quad k = 0, 1, 2$$

$$\theta_0 = \frac{\pi}{3}, \quad \theta_1 = \pi, \quad \theta_2 = \frac{5\pi}{3}$$

O.A $= -1$
(calc. above)

Find where asymptotes cross Im(s) axis

$$1 + KGH(s) = 0$$

$$1 + \frac{6K}{s(s+1)(s+2)} = 0 \longrightarrow s^3 + 3s^2 + 2s + 6K = 0$$

Use the Routh array

s^3	1	2
s^2	3	$6K$
s^1	$\frac{6-6K}{3}$	0
s^0	$6K$	

if $K=1 \longrightarrow$ row of zeros

auxiliary eqn:

$$3s^2 + 6(1)s^0 = 0$$

$$3s^2 + 6 = 0$$

$$s = \pm j\sqrt{2}$$