Multiplicity of a root

Let $f^{(0)}(x)$ denote $f(x)$ & $f^{(i)}(x)$ denote the i^{th} derivative of f evaluated at $x \in \mathbb{R}$ $i = 1, 2, 3, \ldots$

← there exists an $r \in R$

If $\exists r \in \mathbb{R}$ satisfying $f^{(0)}(r) = f^{(1)}(r) = \ldots\ldots = f^{(m-1)}(r) = 0$

$m \geq 1$

$m \in \mathbb{Z}$ ← set of integers

and

$$f^{(m)}(r) \neq 0$$

then r is a root of $f^{(0)}(x)$ with multiplicity $\underline{\underline{m}}$.

If $r \in \mathbb{R}$ is a zero of $f(x)$ having multiplicity m then we can write $f(x) = (x-r)^m g(x)$ where $g(r) \neq 0$

Example 1

For $f(x) = f_1(x) = x^2 - 4$

$= (x+2)(x-2)$

There are two zeros $r_1 = 2$ & $r_2 = -2$

$$f^{(0)}(x) = x^2 - 4$$

$$f^{(0)}(r_1) = f^{(0)}(r_2) = 0$$

$f^{(1)}(x) = 2x$ $\quad \Rightarrow m = 1$

$f^{(1)}(r_1) \neq f^{(1)}(r_2) \neq 0$

Therefore -2 & 2 are roots of $f(x)$ with multiplicity 1.
Also called simple roots or isolated roots.

If $m=1$, the zero is a "simple" or "isolated" root of $f(x)$

Example 2

$$f(x) = f_2(x) = x^2+4x+4 = (x+2)^2$$

$r_1 = -2 \Rightarrow f_2(r_1) = 0 \Rightarrow -2$ is a root of $f_2(x)$

$f^{(0)}(r_1) = 0$

$f^{(1)}(x) = 2x+4$

$f^{(1)}(-2) = -4+4 = 0 \Rightarrow M=2$

$f^{(2)}(x) = 2x$

$f^{(2)}(r_1) \neq 0$

$\Rightarrow r=-2$ is a root of $f_2(x)$ with a multiplicity of 2

Example 3

For $f(x) = f_3(x) = e^x + \log_e(x)$?

Example 4

$f_4(x) = x^3 + 4x^2 + 4x = x(x+2)^2$

$\Rightarrow$ using the second def of multiplicity where $f(x) = (x-r)^m g(x)$

we have 2 zeros $r_1 = 0$ & $r_2 = -2$

assume $(x+2)^2 = g(x)$

$\Rightarrow f_4(x) = (x-0)^1 g(x) \Rightarrow r_1 = 0$ is a root of multiplicity 1

assume $x = g(x)$

$\Rightarrow f_4(x) = (x+2)^2 g(x) \Rightarrow r_1 = -2$ is a root of multiplicity 2

Rootfinding Methods

- One way to characterize methods is whether or not they require derivatives

- Another way to characterize the methods is whether or not they sequentially reduce the interval in which a root is suspected.

Derivative Free Enclosure Methods

- Some times called braketing methods

- Start with two pts $x = a$ & $x = b$, satisfying $f(a)f(b) < 0$. ($a < b$)
- If $f \in C^0[a,b]$
  ↳ func values are continuous in the interval $[a,b]$

  then IVT tells us that atleast one root exists in $[a,b]$

→

- in some way find a pt b/w $[a,b]$, perhaps evaluate the func at the pt or only the sign. Then we replace either a or b by that pt denping on the sign making sure $f(a)f(b)<0$. This way you reduce the interval of uncertainty.

Bolzano's Method

- derivative free
- enclosure
- only ~~requireds~~ $\text{sgn}[f(x)]$ → only requires sign of $f(x)$
- maintains $[a,b]$ with $f(a)f(b)<0$
- This method is often called the Bisection Method or mid point rule

$\boxed{\text{Review taylor polynomials}}$

Given values $x=a_0$ and $x=b_0$ satisfying $a_0<b_0$ & $f(a_0)f(b_0)<0$

Assumptions

$f \in C^0[a,b]$

~~Agl~~ Algorithm

At iteration $k=0,1,\ldots..$ compute $m_k = \dfrac{a_k+b_k}{2}$ (the interval midpoint)

$$\text{set } [a_{k+1}, b_{k+1}] \leftarrow \begin{cases} [m_k, b_k] & \text{if } f(m_k)f(b_k)<0 \\ [a_k, m_k] & \text{if } f(a_k)f(m_k)<0 \end{cases}$$

Terminate if appropriate, otherwise $k \leftarrow k+1$ & repeat

Note that

m_k = x intercept of line $y(x)$ joining

$(a_k, \operatorname{sgn}[f(a_k)])$ and $(b_k, \operatorname{sgn}[f(b_k)])$