

Examples

- 1) Zero vector space $V = \{ \underline{0} \}$
over \mathbb{R} , say scalars

$$\underline{0} + \underline{0} = \underline{0}$$

$$k \underline{0} = \underline{0}$$

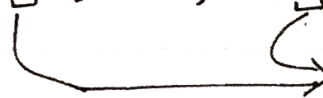
smallest possible vector space (V.S.)

* Need $\underline{0} \in V$ at least \Rightarrow Null space cannot be a vector space

- 2) F^n - n -tuples of scalars (\mathbb{R} or \mathbb{C})
 \mathbb{R}^n or \mathbb{C}^n

$$\underbrace{(a_1, \dots, a_n)}_u + \underbrace{(b_1, \dots, b_n)}_v$$

$$= (a_1 + b_1, \dots, a_n + b_n) \quad \text{Coordinate wise vector sum}$$

 uses scalar sum (already known) to define vector sum (new)

$$ku = k(a_1, \dots, a_n) = (ka_1, \dots, ka_n)$$

\hookrightarrow scalar multiple (vector space operation)

\hookrightarrow multiplication of real numbers (scalars)

- 3) $u = (\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots)$
 $v = (\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots)$ } Discrete signal space

$$u+v = (\dots, u_{-2}+v_{-2}, u_{-1}+v_{-1}, u_0+v_0, u_1+v_1, u_2+v_2, \dots)$$

\rightarrow

$$ku = (-----, ku_{-2}, ku_{-1}, ku_0, ku_1, ku_2, -----)$$

coordinate wise defns

4) P Polynomial Space

$p \in P$ vector is a polynomial, any degree

Q what is $p+q$ for $p+q \in P$

$$\text{Say } p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k$$

$$q(t) = b_0 + b_1 t + \dots + b_\ell t^\ell$$

Define $p+q \in P$ using a "test point" t and real (scalar) arithmetic

$$(p+q)(t) = a_0 + a_1 t + \dots + a_k t^k + b_0 + b_1 t + \dots + b_\ell t^\ell$$

Scalars

test point

polynomial vector

$$(cp)(t) = c(a_0 t + \dots + a_k t^k)$$

scalar arithmetic

new polynomial

test pt.

5) Matrix space M_{ij} vectors $A \in M_{ij}$ are $i \times j$ matrices

Use coordinate-wise defn for $A+B$ & kA

M_{ij} "look exactly like" R^{ij}
isomorphic

6) Function spaces

X any set

$\{f: X \rightarrow F\} = F$
↑ ↑ ↑
function from X to scalars F

$f, g \in F$

$(f+g)(t) = f(t) + g(t)$
↑ ↑ ↑ ↑
vector sum defined test pt in X scalars scalar sum

X can be any set, often R^n or C^n

$f: X \rightarrow R$ is a real valued function

Axioms \Rightarrow useful, familiar results

a) $u_1 + u_2 + \dots + u_n$ is unambiguous (repeated application of S2)

b) $\underline{0}$ is unique

Pf: Suppose $\underline{0}, \underline{0}'$ are both zero vectors.

Any $u \in V$

$$\Rightarrow u + \underline{0} = u - (1) \quad \& \quad u + \underline{0}' = u - (2) \quad (\text{By S3})$$

$$\text{Put } u = \underline{0}' \text{ in (1)} \quad \& \quad u = \underline{0} \text{ in (2)}$$

$$\Rightarrow \underline{0}' + \underline{0} = \underline{0}' \quad \& \quad \underline{0} + \underline{0}' = \underline{0}$$

$$\underline{0}' + \underline{0} = \underline{0} + \underline{0}' \quad (\text{By S1})$$

$$\Rightarrow \underline{0}' = \underline{0}$$

\Rightarrow Zero vector is unique

c) $-\underline{u}$ is unique

Pf: Suppose u', u'' are inverses of u

$$u'' = u'' + \underline{0} \quad (\text{S3}) = u'' + (u' + u)$$

$$= u'' + (u + u') \quad \text{by (S1)} \quad \begin{array}{l} \uparrow u' \text{ is inverse of } u \\ \text{cause assumption} \end{array}$$

$$= (u'' + u) + u' \quad (\text{S2}) = \underline{0} + u' = u'$$

$$\Rightarrow u'' = u' \Rightarrow -u \text{ is unique, only 1 inverse}$$

Allows subtraction of vectors meaningfully

$u - v = u + (-v)$ unique!