I-1 (Cont) Vector spaces vs Scalar spaces

Properties of vector spaces

axioms $\Rightarrow$ useful stuff

vector space over F ($\mathbb{R}$ or $\mathbb{C}$ usually!)

a) $u_1 + \dots + u_n$ is unambiguous

b) $\underline{0}$ is unique

Pf Assume $\underline{0}$ & $\underline{0}'$ are zero vectors

To show $\underline{0} = \underline{0}'$

$u + \underline{0} = u$ (1) any u (S3)

$u + \underline{0}' = u$ (2) any u (S3)

Put $u = \underline{0}'$ in (1)

$u = \underline{0}$ in (2)

$$\left.\begin{array}{l} \Rightarrow \underline{0}' + \underline{0} = \underline{0}' \\ \Rightarrow \underline{0} + \underline{0}' = \underline{0} \end{array}\right\} \text{By S1} \quad \underline{0} = \underline{0}'$$

c)

c) The additive inverse of a vector u is unique

Assume u' & u'' ~~and~~ are additive inverses for u

$\Rightarrow u + u' = \underline{0}$ $\qquad u + u'' = \underline{0}$ (S4)

$u'' = u'' + \underline{0}$ (S3)

$= u'' + (u + u')$ assumption

$= (u'' + u) + u'$ (S2) $\quad = (u + u'') + u'$ S1

$= \underline{0} + u'$ (assumption)

$\Rightarrow u'' = u'$ (S3)

d) $u + w = v + w \Rightarrow u = v$
cancellation of vectors allowed

Pf $(u+w) + (-w) = (v+w) + (-w)$

$u + (w + (-w)) = v + (w + (-w))$ (S2)

$u + \underline{0} = v + \underline{0}$? allowed? (S4)

Q Does $u = v \Rightarrow u + w = v + w$

- using only the axioms
- or do we need c) $\Rightarrow$ e)

e) subtraction of vector defined by:

$u - v = u + (-v)$ unambiguous b/c $-v$ is unique c)

f) $k\underline{0} = \underline{0}$

Pf: $k\underline{0} = k(\underline{0} + \underline{0})$ (S3)

$= k\underline{0} + k\underline{0}$ (M1)

$k\underline{0} + (-k\underline{0}) = (k\underline{0} + k\underline{0}) + (-k\underline{0})$

$\underline{0} = (k\underline{0} + (-k\underline{0})) + k\underline{0}$ S4

$\Rightarrow \underline{0} = \underline{0} + k\underline{0}$ S4

$\Rightarrow \underline{0} = k\underline{0}$ S3

g) $0u = \underline{0}$

Pf $0u = (0+0)u = 0u + 0u$ (M2)

scalar arith

i.e. $= 0u + 0u = 0u$

$0u + 0u + (-0u) = 0u + (-0u)$ (S4)

$0u + (0u + (-0u)) = \underline{0}$ (S4)

(S2)

$0u + \underline{0} = \underline{0}$ (S4)

$0u = \underline{0}$ (S3)

h) $k\underline{u} = \underline{0} \Rightarrow$ either $k = 0$ or $u = \underline{0}$

Pf $k\underline{u} = 0$ & $k \neq 0$ assume

To show $\Rightarrow u = \underline{0}$

$u = 1u$ (M4)

$= (1/k \cdot k)u$ Scalar arith with $k \neq 0$

$= \frac{1}{k}(ku)$ M3

$= \frac{1}{k}\underline{0}$ assumption

$= \underline{0}$ (f)

i) $-(ku) = -(k)u = k(-u)$

inv vector

scalar inverse

inverse

Pf $\underline{0} = k\underline{0}$ (f)

$= k(-u+u)$ S4

$= k(-u) + ku$ M1

$\Rightarrow k(-u) = -k(u)$ why By (c)

Next $\underline{0} = k\underline{0}$ f)

$\underline{0} = 0u$ (g)

$\underline{0} = (k-k)u$ scalar arith (some k)

$= ku + (-ku)$ M2

$-(ku) + \underline{0} = -(ku) + (ku + (-k)u)$

$-(ku) \cancel{+\underline{0}} = (-(ku) + ku) + (-k)u$ (S2)

$= \underline{0} + (-k)u$ (S4)

$= (-k)u$

i.e. $-ku = (-k)u$

Non - examples of v.s

$V = \{(x,y) \mid x,y \in \mathbb{R}\}$ set of vectors

$F = \mathbb{R}$ scalars

Can make this into different potential v.s's with using different rules for + & scalar multiple

Examples of bad definitions that don't give ~~vs~~ vector spaces

i) $(a,b) + (c,d) = (a+c, b+d)$

$k(a,b) = (ka, b)$

Check axioms &/or ~~p~~proven props

$0(a,b) = (0a, b) = (0,b) \neq (0,0)$ the zero vector

Not a vector space

ii) $(a,b) + (c,d) = (a,b)$

$k(a,b) = k(a, kb)$

Check s1

$(a,b) + (c,d) = (a,b)$

$(c,d) + (a,b) = (c,d)$ Not equal

Not a v.s.

iii) $(a,b)+(c,d) = (a,\cancel{b}c, b+d)$

$k(a,b) = (k^2 a, k^2 b)$

Check $(r+s)(a,b) \underset{\text{defn}}{=} ((r+s)^2 a, (r+s)^2 b)$

$= ((r^2+s^2+2rs)a, (r^2+s^2+2rs)b)$
Scalar arith

$= (r^2 a, r^2 b) + (s^2 a, s^2 b) + (2rsa + 2rsb)$

~~$= r(a,b) + s(a,b) + 2rs(a,b)$~~

~~M defn~~
$= r(a,b) + s(a,b) + \sqrt{2rs}(a,b)$
$\neq r(a,b) + s(a,b)$ in general

M2 fails

Not a vector space

iv) $(a,b)+(c,d) = (a+c, b+d)$

$k(a,b) = (ka, 0)$

$(0,0)$ is the zero vector
$1(a,b) = \cancel{1a}\ (1a, 0) = (a,0) \neq (0,0)$ in general.

M4 fails.

Subspaces

V v.s. over F

$W \subset V$ (W is a subset of vectors in V)

A subspace of V is a subset of vectors in V which is a v.s. using the same + & scalar multiple rules as V.

Axioms are all true in W
- inherited from V

(But need to check closure
i.e., $u, v \in W \Rightarrow u+v \in W$

$k \in F$ $u \in W \Rightarrow ku \in W$)

or combined

$$\left.\begin{array}{c} u, v \in W \\ k \in F \end{array}\right\} \quad u + kv \in W$$

Note this $\Rightarrow \underline{0} \in W$ automatically (put $k=0$)

Ex Euclidean subspaces of R^n

R^3
- $\{0\}$ — 0 subspace
- $\{ku, k \in R\}$ — u fixed — line through $(0,0,0)$
- $\{ku + lv, k, l \in R\}$ — u, v fixed. plane through $(0,0,0)$
- R^3 — full vector space

$W < V$ my notation for saying W is a subspace of V