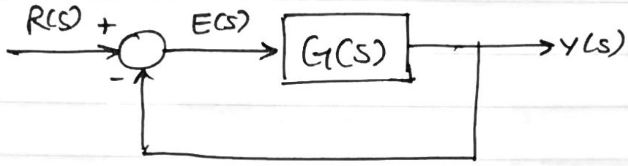
Jan 19, 2016

Time Domain Performances for second order systems:



Canonical form

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$\frac{Y}{R} = \frac{G}{1+G} = \frac{\omega_n^2}{s^2+2\zeta s\omega_n+\omega_n^2}$$

where $\omega_n \Rightarrow$ the natural frequency

$\zeta \Rightarrow$ the damping ratio

zeta

poles from :

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

are

$$s_{1,2} = -\zeta\omega_n \underbrace{\pm j\omega_n\sqrt{1-\zeta^2}}$$

form convinient for $\zeta \leq 1$

$\zeta < 1$ complex conjugate pairs

$\zeta = 1$ repeated roots

$\zeta > 1$ distinct real roots

$\rightarrow$

Introduce the damped Natural freq

$$\omega_d \triangleq \omega_n \sqrt{1-\zeta^2}$$

For a unit step input

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Expanding in partial fractions yields,

$$Y(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s^2+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

For a quiescent system → zero ICs

$$y(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\, e^{-\zeta\omega_n t}$$

$$= 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

1.) Steady State ~~Error~~ Value

$$y_{ss} = \lim_{t\to\infty} y(t) = 1$$

2.) Output Error:

$$E(s) = R(s) - Y(s)$$

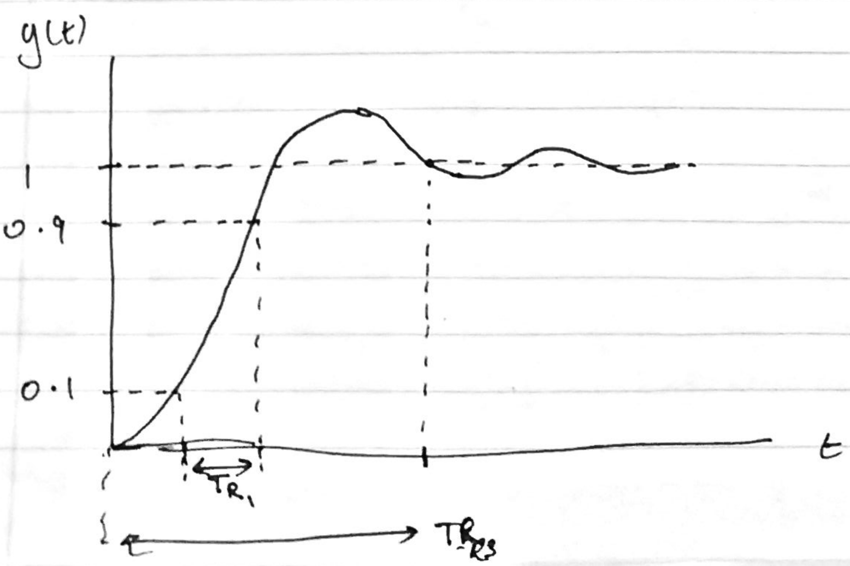
or

$$e(t) = r(t) - y(t)$$

$$r(t) = 1$$

$$e(t) = e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

$$e_{ss} = \lim_{t\to\infty} e(t) = 0$$



3) <u>Rise Time</u>

$$T_{R_1} \approx \frac{1.8}{\omega_n}$$

for T_{R_3}

$$y(0) = 0 \;, y_{ss} = 1$$

$$y(t) = 1 - \underbrace{e^{-\xi \omega_n t}}_{\neq 0} \underbrace{\overbrace{\left[\cos \omega_d t + \xi \omega_n / \omega_d \sin \omega_d t\right]}^{\text{when is this } = 0}}_{\text{Find the time } \cancel{T_R}\ T_{R_3} \text{ when this } = 0}$$

$$\cancel{\omega} \cos \omega_d T_{R_3} + \frac{\xi \omega_n}{\omega_d} \sin \omega_d T_{R_3} = 0$$

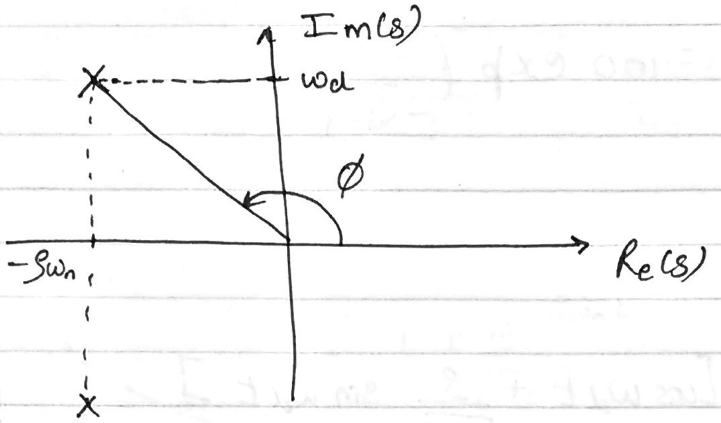
Solve for T_{R_3}.

Rearrange ~~the~~ to get

$$\frac{\sin \omega_d T_{R_3}}{\cos \omega_d T_{R_3}} = \frac{-\omega_d}{\xi \omega_n} = -\frac{\sqrt{1-\xi^2}}{\xi \cancel{\omega_n}}$$

$$\tan \omega_d T_{R_3} = \frac{-\omega_d}{\xi \omega_n}$$

$$\Rightarrow T_{R_3} = \frac{1}{\omega_d} \tan^{-1}\left(-\frac{\omega_d}{\xi \omega_n}\right) = \frac{1}{\omega_d} \tan^{-1} \phi$$



4) <u>Peak Time</u>

Occurs where $\left.\frac{dy}{dt}\right|_{T_p} = 0$

after differentiating $y(t)$ & simplifying we get

$$\left.\frac{dy}{dt}\right|_{T_p} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n T_p} \sin \omega_d T_p = 0$$

$$\therefore \quad T_p = \frac{\pi}{\omega_d}$$

5) <u>Maximum Overshoot</u>

$$M_p = y(T_p) - y_{ss}$$

$$= y(T_p) - 1$$

$$M_p = e^{-\zeta \omega_n \pi / \omega_d}$$

$$= e^{-\zeta \pi / \sqrt{1-\zeta^2}} = \exp\left(\frac{-\zeta \pi}{\sqrt{1-\zeta^2}} \right)$$

$$\text{Percent Overshoot} = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

6.) Settling Time : T_s

$$y(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

recall $\tan\phi = \frac{-\omega_d}{\zeta\omega_n} = -\frac{\sqrt{1-\zeta^2}}{\zeta}$

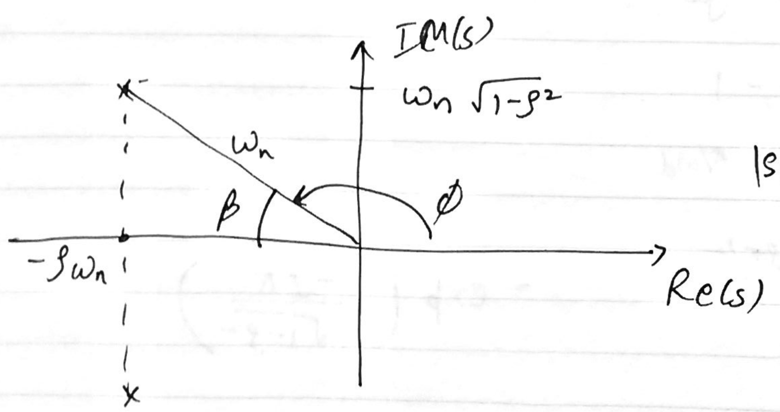
$$\therefore\ y(t) = 1 - e^{-\zeta\omega_n t}\left[\underbrace{\cos\omega_d t + (-\cot\phi)\sin\omega_d t}\right]$$

$$= \cos\omega_d t - \frac{\cos\phi}{\sin\phi}\sin\omega_d t$$

$$= \frac{1}{\sin\phi}\left(\cos\omega_d t\sin\phi - \sin\omega_d t\cos\phi\right)$$

$$= \frac{1}{\sin\phi}\left(-\sin(\omega_d t - \phi)\right)$$

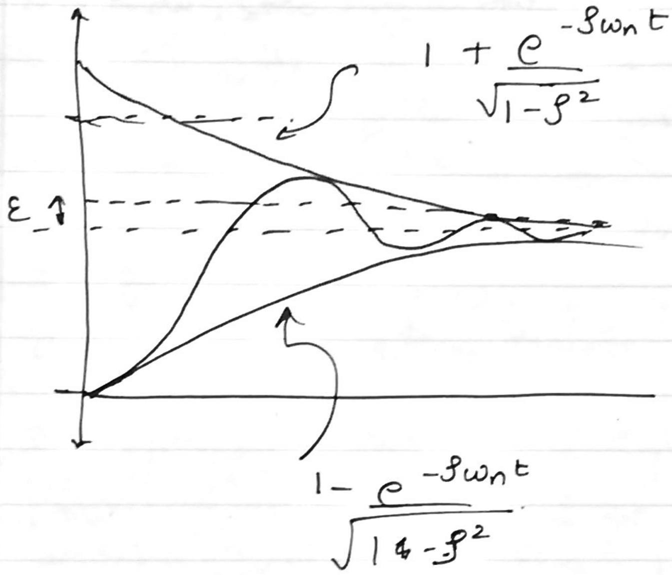
Recall



$|s_1, s_2| = \omega_n$

$\sin\phi = \sin\beta = \sqrt{1-\zeta^2}$

$$\therefore\ y(t) = 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \underbrace{\sin(\omega_d t - \phi)}_{\text{bounded on } (1,1)}$$



at T_s $\quad 1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = 1 \pm \varepsilon$

solve for T_s

$$T_s = -\frac{1}{\zeta\omega_d} \ln\left|\varepsilon\sqrt{1-\zeta^2}\right|$$