21/01/2016 312 Lecture 7

I-2 (finally) Linear Maps

V, U vectorspace some scalars F
$f: V \to U$, a function i.e. $f(v) \in U$, $v \in V$
is a <u>linear map</u> if it preserves the vectorspace operations,
i.e.,

$$\left.\begin{matrix} v, w \in V \\ k \in F \end{matrix}\right\} \quad \underbrace{f(v+w)}_{\text{in } V} = \underbrace{f(v)+f(w)}_{\text{in } U}$$

$$f(\underbrace{kv}_{\text{in } V}) = \underbrace{k f(v)}_{\text{in } U}$$

<u>Other terminology</u>
$f: R^n \to R^m$ Linear Transformation
$f: V \to V$ Operator (Linear)

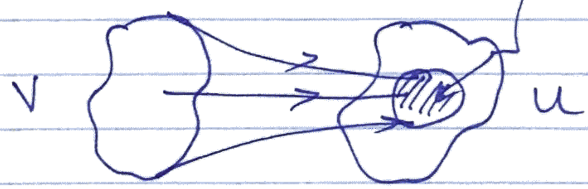
<u>Ex.</u> $T: R^2 \to R^2$, where $T(x,y) = (x+y, x)$

$T(x+x'+y+y') = (x+x'+y+y', x+x')$ defn of T
$= (x+y, x) + (x'+y', x')$
$= T(x,y) + T(x',y')$

Since $T(kx, ky) = k\,T(x,y)$

<u>Image</u> of a linear map (or <u>range</u>)

$T: V \to U$ $\quad$ $Im(T)$ <u>or</u> $T(V)$



$T(V)$ or $Im(T) = \{u \in U \mid u = T(v),$ some $v \in V\}$

Q: is $T(V)$ a subspace of U

Pick $u, u' \in T(V)$ & $k \in F$

$\Rightarrow u = T(v)$ some $v \in V$

$u' = T(v')$ some $v' \in V$

$u + u' = T(v) + T(v') = T(v + v')$ $T \to$ linear.

$u + u' \in T(V)$

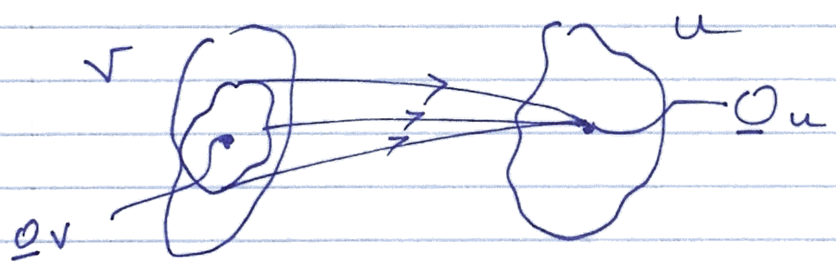
$ku = k(T(v))$

$= T(kv)$ (linear map)

$\Rightarrow ku \in T(V)$ ✓ <u>Yes</u>

$T(V) < U$

<u>Kernal</u> of a linear map



$T(\underline{0}_v) = T(0v) = 0\,T(v) = 0_u = \underline{0}_u$

cause T is linear

The <u>kernal</u> of T

$ker(T) = \{v \in V \mid T(v) = \underline{0}_u\}$

We know $\underline{0}_v \in ker\,T$ always

~~Q: is~~

Q: Is the kernal(T) a subspase of V?

Pick $v, v' \in \ker(T)$, $k \in F$

check

$$T(v+v') \overset{T \text{ linear}}{=} T(v) + T(v') = \underline{0}_v + \underline{0}_v = \underline{0}$$

$v+v' \in \ker(T)$

$$T(kv) = kT(v) = k\underline{0}_v = \underline{0}_v$$

$kv \in \ker(T)$

Yes $\boxed{k(T) < V}$

Some examples of Linear Maps

The zero map $T_0 : V \to U$
where $T_0(v) = \underline{0}_u$ all $v \in V$
$\ker(T_0) = V$
$\mathrm{Img}(T_0) = \{\underline{0}_u\}$

Identify operator $T_I : V \to V$
where $T_I(v) = v$ all $v \in V$
$\ker(T_I) = \{\underline{0}_v\}$
$\mathrm{Im}(V) = V$

A projection operator

$T : \mathbb{R}^n \to W < \mathbb{R}^n$

eg $T(x,y,z) = (x,y,0)$ projection on to the x-y plane

dilation, contraction operator ✓
reflection operator on $\mathbb{R}^n$

eg $T(x,y,z) = (-x,y,z)$, reflection in the x-z plane

rotation $T: R^2 \to R^2$

$T(x,y) = (\cos\theta\, x - \sin\theta\, y,\ \sin\theta\, x + \cos\theta\, y)$

<u>Linear maps of function spaces</u>

$T: P_n \to P_{n+1}$

$T(p)$ is defined by $[T(p)](t) = t \cdot p(t)$

cont differentiable funcs $f: R \to R$

$D: C^{1}(R) \to C^{0}(R)$

cont funcs $R \to R$

The differentiation map where

$D(f) = f'$

functions

derivative f

---

test pls $\to [D(f)](x) = f'(x) = \frac{df}{dx}(x)$

Linear since $(f+g)' = f' + g'$

& $(kf)' = kf'$

Q: what is $\ker(D)$?

scalar

$\ker(D) = \{ f: R \to R \mid f(x) = k \}$

$J: C^0(R) \to C^1(R)$

The integration map, where

$[J(f)](x) = \int_0^x f(x')\, dx'$

(test pt x)

Non Examples

① $T: M_{n,n} \to R$

where $T(A) = \det(A)$

not linear b/c

$\det(A+B) \neq \det A + \det B$

② $T: R^3 \to R$ where

$T(x,y,z) = x^2+y^2+z^2$

$$\underbrace{T(kx, ky, kz)}_{T(kv)} = k^2(x^2+y^2+z^2) = k^2 T(v)$$

× Not linear.

③ Translation Operator

$T: R^2 \to R^2$

$T(x,y) = (x+2, y+5)$

$T(0,0) = (2,5) \neq (0,0)$

× not linear

Composition of Maps

$T_1 : V \to U$, $T_2 : U \to W$

linear Maps

The map $T_1 \circ T_2 : V \to W$ is the composition (map) of T_1 & T_2

where $(T_1 \circ T_2)(v) = T_2(T_1(v))$

Matrix transformations

A is an $n \times n$ matrix

Defn T_A : $T_A(v) = A \cdot v$

$m \times n$ $\quad$ $n \times 1$

$m \times 1$

Composition of matrix transform

- A is $m \times p$
- B is $p \times n$

Define $T_A : R^p \to R^m$, $T_A(v) = Av$
$T_B : R^n \to R^p$, $T_B(u) = Bu$

$T_A \circ T_B : R^n \to R^m$

$(T_A \circ T_B)(u) = A(Bu)$

i.e. $T_A \circ T_B = T_{AB}$

matrix product corresponds to composition of linear transform

$$\begin{tikzcd} R^n \arrow[r, "T_B"] \arrow[rr, bend right, "T_A \circ T_B = T_{AB}"'] & R^p \arrow[r, "T_A"] & R^m \end{tikzcd}$$

Connection b/w matrix transformation & matrix spaces

A $m \times n$ matrix

$T_A : R^n \to R^m$

The null space of A is the $\ker(T_A)$

$\text{null}(A) = \{ v \in R^n \mid Av = 0_m \}$

i.e. $T_A(v) = 0_m$

Link to : soln of homogenous linear system $(m \times n)$ with coeff matrix A

Also : $\boxed{\text{colsp}(A) = \text{Im}(T_A)}$

Q: Both same vector space? Yes R^m

Look at

$\text{Im}(T_A) = \{ u \in R^m \mid u = T_A(v), \text{ some } v \in R^n \}$

but $T_A(v) = Av$

To show any vector of form Av is a linear combination of the cols of A

How Ex (first)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

A v matrix product

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= x_1 C_1 + x_2 C_2 + x_3 C_3$$

i.e. a linear combination of cols of A