27/09/16

Lecture 6

Recap

Newton Raphson Method (root finding)

motivation via 2-term taylor series

$$f(x) = \underbrace{f(x_k) + (x - x_k) f'(x_k)}_{=0}$$

$$\Rightarrow x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\boxed{x_{k+1} = g(x_k) \qquad k = 0, 1, \ldots} \rightarrow \text{Fixed pt iteration}$$

if

where

$$g(x) = x - \frac{f(x)}{f'(x)}$$

if $\{x_k\}_{k=0}^{\infty} \longrightarrow p$

$\Rightarrow p = g(p)$

If this sequence of fixed pt iterates does converge to a pt $x = p$ then p is called a "fixed-point of $g(x)$".

Ex Find roots of a function, $f(x) = 0$

$\underbrace{f(x) + x}_{\hookrightarrow g(x)} = x$ makes this a fixed pt function since $f(x) = 0$ pt we are trying to find.

$f(x) = x^3 + x - 1$

$x_{k+1} = g_i(x_k) \qquad i = 1, 2, 3 \qquad k = 0, 1, \ldots$

where

$g_1 = 1 - x^3 \qquad g_2 = \sqrt[3]{1 - x} \qquad g_3 = \dfrac{1 + 2x^3}{1 + 3x^2}$

How the fuck did we get that?

Suppose:

$x = p$ is a fixed point of $g_1(x)$

$\rightarrow$

$\Rightarrow p = g_1(p) = 1 - p^3$

$p^3 + p - 1 = 0$

if $x = p$ is a fixed point of $g_2(x)$ then

$p = g_2(p) = \sqrt[3]{1 - p}$

$p^3 = 1 - p$

$\Rightarrow p^3 + p - 1 = 0$

if $x = p$ is a fixed pt of $g_3(x)$ then

$p = g_3(p) = \frac{1 + 2p^3}{1 + 3p^2}$

$p + 3p^3 = 1 + 2p^3$

$\Rightarrow p^3 + 1 - 1 = 0$

An infinte number of fixed pt functions can be derived from manupilating $f(x)$.

$x_{k+1} = g_i(x_k)$ $\quad i = 1, 2, 3$
$\quad k = 0, 1, \ldots$

Assume that $x_0 = \frac{1}{2}$

For $i = 1$ & g_1

$\{x_k\}_{k=0}^{\infty}$ does not converge ~~but~~ (to a fixed pt) but insted oscillates b/w values 0 & 1

For $i=2$ & g_2

$\{x_k\}_{k=0}^{\infty}$ does converge to a fixed pt, $p = 0.68232780$ but relatively slowly.

p is a root of $f(x)$.

for $i=3$ & g_3

$\{x_k\}_{k=0}^{\infty}$ also converges to $x=p$ but relatively quickly.

↳ same p value as g_2

Cobweb Diagrams

On a graph we can plot two curves

$$\left.\begin{aligned} y &= g(x) \\ y &= x \end{aligned}\right\}$$ any intersection if it exists is a fixed pt of $g(x)$

You can graph the progression of the sequence $\{x_k\}_{k=0,1,\dots}$ on a cobweb diagram.

Convergence Theorem

if $g(x) \in C^1$ AND $x=p$ is a fixed point of $g(x)$ satisfying

$|g'(p)| < 1$

then there exists an interval of x around the pt p s.t. if x_0 lies in that interval then

$\{x_k\}_{k=0}^{\infty} \longrightarrow p$

↳ converges to

in this case the fixed pt p is called an "attracting fixed point" of $g(x)$.

In N-R we used

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad k = 0, 1, \ldots$$

$$= g(x_k)$$

Suppose N-R converges to an isolated root r of $f(x)=0$

Prove that r is an attracting fixed pt for $g(x) = x - \frac{f(x)}{f'(x)}$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

Since r is an isolated root of $f(x)=0$

$\Rightarrow f(r) = 0$

$f'(r) \neq 0$

$\therefore g'(r) = 0$

$|g'(r)| < 1$

$\Rightarrow r$ is an attracting fixed pt of g

Learn Taylor Series

Convergence of sequences

A sequence $\{x_k\}_{k=0,1,----}$ is called a "convergent sequence" if it has a limit, say L

$$\lim_{k\to\infty} \{x_k\} \longrightarrow L$$

$\Downarrow$

for any $\delta > 0$ there is an index $K(\delta)$ such that

$$|x_k - L| < \delta$$

for all $k > K(\delta)$

Absolute error in root finding

$e_k = r - x_k$ where r is a root of $f(x) = 0$

Order of convergence in Root finding

$$\lim_{k\to\infty} \frac{|e_k|}{|e_{k+1}|^{\eta}} = M \qquad M > 0, \quad \eta \geq 1$$

$$|e_k| = M|e_{k+1}|^{\eta}$$

$\eta = 1 \quad M \in (0,1) \rightarrow$ linear

$\eta = 2 \quad \rightarrow$ quadratic