January 26, 2016 8:34 AM

System Stability

$$\frac{Y(s)}{R(s)} = \frac{N(s)}{\Delta(s)}$$

System poles are given by roots (zeros) of $\Delta(s)$

Note: for stability, we only care if poles are in LHP vs RHP, not exact locations

Routh-Hurwitz Criteria

Step 1

Expand $\Delta(s) = 0$ into the form

$$\Delta(s) = a_n s^n + \dots + a_1 s + a_0$$

Step 2

Apply the Hurwitz Test - Examine the signs of all the coefficients of $a_n, \dots, a_0$

If the coefficients are not all the same sign, or if some are missing, then there will be roots of $\Delta(s)$ in the RHP

If all the coefficients are non-zero and the same sign the system may still be unstable

Ex

$$\Delta(s) = s^3 + \frac{1}{2}s^2 + \frac{7}{2}s + 4$$

$$= \underbrace{(s+1)}_{\text{LHP}} \underbrace{\left(s - \frac{1}{4} + j\frac{\sqrt{63}}{4}\right)\left(s - \frac{1}{4} - j\frac{\sqrt{63}}{4}\right)}_{\text{RHP}}$$

$\therefore$ This system is unstable

Step 3

Use the Routh Criterion

3a) Construct the Routh array

For $\Delta(s) = a_n s^n + \dots + a_1 s + a_0$

Form:

$$\begin{array}{c|cccc l}
s^n & a_n & a_{n-2} & & a_1 & \leftarrow \text{row } 0 \\
s^{n-1} & a_{n-1} & a_{n-3} & & a_0 & \leftarrow \text{row } 1 \\
s^{n-2} & \delta_{2,1} & \delta_{2,2} & \delta_{2,3} & \cdots & \leftarrow \text{row } 2 \\
\vdots & \delta_{3,1} & \delta_{3,2} & \delta_{3,3} & \cdots & \\
s^1 & & & & & \\
s^0 & & & & &
\end{array}$$

Pivot Column

Note: $\delta_{i,j}$ Row Col

Where

$$\delta_{i,j} = \frac{-\left(\delta_{i-2,1}\,\delta_{i-1,j+1} - \delta_{i-1,1} \cdot \delta_{i-2,j+1}\right)}{\delta_{i-1,1}}$$

3b) Apply the Routh criterion:

The characteristic polynomial $\Delta(s)=0$ has roots in the RHP or on the $j\omega$ axis if there are any zeros or sign changes in the pivot column of the Routh array

<u>Ex</u>

$\Delta(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0, \quad a_i > 0, \; i = 1,2,3,4$

This passes the Hurwitz test, so construct the Ruth array

s^3	a_3	a_1	0
s^2	a_2	a_0	0
s^1	δ_{21}	$\delta_{2,2}=0$	
s^0	$\delta_{3,1}=a_0$		

$\delta_{2,1} = \dfrac{a_1 a_2 - a_3 a_0}{a_2}$

$\delta_{2,2} = 0$

Examine the pivot column $a_3 > 0, \; a_2 > 0, \; a_0 > 0$ so for stability:

$$\delta_{2,1} > 1$$

a.k.a.:

$$a_2 a_1 - a_3 a_0 > 0$$

is required

<u>Ex</u>

$\Delta(s) = a_2 s^2 + a_1 s + a_0 \quad$ where $\quad a_i > 0, \; i = 0,1,2$

s^2	a_2	a_0
s^1	a_1	0
s^0	a_0	

<u>Ex</u>

$\Delta(s) = s^4 + 10s^3 + s^2 + 15s + 3$

This passes the Hurwitz test, so form the Ruth array

s^4	1	1	3
s^3	10	15	0
s^2	$-1/2$	3	
s^1	75		
s^0	3		

$$\delta_{31} = \frac{(-1/2)(15) - (3)(10)}{(-1/2)} = \frac{-7.5 - 30}{-1/2} = 75$$

The pivot column has 2 sign changes so the system is unstable.

The actual roots of $\Delta(s)$ are:

-0.1976

-10.05

$+0.1218 \pm j1.2234$

More Information from the Ruth array:

1) The # of roots of $\Delta(s)$ in the RHP equals the number

of sign changes in the pivot column

ii) A row of zeros in the Routh array implies roots located symmetrically about the origin. These roots can be found from the auxiliary equation:

s^n			
$\vdots$			
s^2	a	b	0
s^1	0	0	0
s^0			

Take the auxiliary equation from here

$$\Delta_a(s) = a s^2 + b s^0 = 0$$

$$= a s^2 + b$$

Roots of $\Delta_a(s) = 0$

are $s_{1,2} = \pm \sqrt{\frac{-b}{a}}$

$$= \pm j \sqrt{\frac{b}{a}}$$