$y = \arg\max_{\hat{y}=\{0,1\}} P(\hat{y}|x)$

To prove, decision boundary is linear when $\Sigma_1 = \Sigma_2$

Decision boundary $\delta(x) = \frac{P(Y=2|X=x)}{P(Y=1|X=x)}$

If $\delta(x) > 1$, x belongs to class 2
$\delta(x) < 1$, x belongs to class 1

Taking log

$\log(\delta(x)) = \log P(Y=2|X=x) - \log P(Y=1|X=x)$

Using Bayes' rule

$$P(Y=j|X=x) = \frac{P(X=x|Y=j)\,P(Y=j)}{P(X=x)}$$

and denoting $\pi_j = P(Y=j)$

$$\log(\delta(x)) = \log(P(X=x|Y=2) + \log(\pi_2) - \log(P(X=x) - \log P(X=x|Y=1) - \log(\pi_1) + \log(P(X=x))$$

$$= \log(P(X=x|Y=2)) + \log(\pi_2) - \log(P(X=x|Y=1)) - \log(\pi_1)$$

We know $P(X=x|Y=y) = \frac{1}{\sqrt{(2\pi)^d |\Sigma_y|}} \exp\left(\frac{-(x-\mu_y)\Sigma_y^{-1}(x-\mu_y)'}{2}\right)$

$$\log \delta(x) = \log \frac{P(X=x \mid Y=2)}{P(X=x \mid Y=1)} + \log \frac{\pi_2}{\pi_1}$$

$$= \log \sqrt{\frac{|\Sigma_1|}{|\Sigma_2|}} + \frac{-(x-\mu_2)\Sigma_2^{-1}(x-\mu_2)^T}{2} + \frac{(x-\mu_1)\Sigma_1^{-1}(x-\mu_1)^T}{2} + \log \frac{\pi_2}{\pi_1}$$

When $\Sigma_1 = \Sigma_2 = \Sigma$

$$\log \delta(x) = 0 + \big(\cancel{-x\Sigma_2^{-1}x} + x\Sigma_2^{-1}\mu_2' + \mu_2\Sigma_2^{-1}x' - \mu_2\Sigma_2^{-1}\mu_2' + \cancel{x\Sigma_1^{-1}x'} - \mu_1\Sigma_1^{-1}x' - x\Sigma_1^{-1}\mu_1 - \mu_1\Sigma_1^{-1}\mu_1'\big)/2 + \log \pi_2/\pi_1$$

$$= \big(x\Sigma^{-1}\mu_2' + \mu_2\Sigma^{-1}x' - \cancel{\mu_2\Sigma\mu_2'} - \mu_1\Sigma^{-1}x' - x\Sigma^{-1}\mu_1\big)/2 + C$$

where $C = \log \pi_2/\pi_1 + \mu_1\Sigma^{-1}\mu_1 + \mu_2\Sigma^{-1}\mu_2'$

f is a constant term

$$\boxed{\log \delta(x) = \Sigma^{-1}x^T(\mu_2 - \mu_1) + C}$$

thus it can be expressed in terms of x making the boundary linear when $\Sigma_1 = \Sigma_2$

When $\Sigma_1 \neq \Sigma_2$,
$\log \delta(x)$ term will have the terms $x\Sigma_2^{-1}x'$ and $x\Sigma_1^{-1}x'$, making the boundary linear.

When $\Sigma_1 \neq \Sigma_2$

$$\log \delta(x) = \frac{x \Sigma_1^{-1} x' - x \Sigma_2^{-1} x''}{2} + \Sigma_2^{-1} x' \mu_2$$
$$- \Sigma_1^{-1} x' \mu_1 + C$$

Thus the decision boundary is quadratic

$\cancel{x \Sigma}$