LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 8: EXERCISES.

1. Problem 1

Find the eigenvalues and eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \; B = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

*Is there any relationship between eigenvalues/eigenvectors of A and B?

2. Problem 2

Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \; A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

*Is there any relationship between eigenvalues/eigenvectors of A and A^{-1}?

3. Problem 3

What do you do to the equation $A\mathbf{x} = \lambda\mathbf{x}$, in order to prove (a), (b), and (c)?
(a) $\lambda + 1$ is an eigenvalue of $A + I$.
(b) λ^{-1} is an eigenvalue of A^{-1}.
(c) λ^2 is an eigenvalue of A^2.

4. Problem 4

Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

$$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

5. Problem 5

From the unit vector $\mathbf{u} = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = \mathbf{u}\mathbf{u}^T$. This matrix has $P^2 = P$ because $\mathbf{u}^T\mathbf{u} = 1$.
(a) $P\mathbf{u} = \mathbf{u}$ comes from $(\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(-)$. Then $\lambda = 1$.
(b) If $\mathbf{v}$ is perpendicular to $\mathbf{u}$ show that $P\mathbf{v} = 0$. Then $\lambda = 0$.
(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

6. Problem 6*

Let $\mathbf{u}$, $\mathbf{v}$ be some vectors. Show that $\mathbf{u}$ is an eigenvector of the rank one 2×2 matrix $A = \mathbf{u}\mathbf{v}^T$. Find both eigenvalues of A.

7. Problem 7*

Find a 2 by 2 orthogonal (rotation) matrix (other than I) with $A^3 = I$. Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What are the trace and determinant of A?