MIT 18.06 Final Exam, Spring 2023

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Recitation: ______________________________

Problem 1 (4+4+4+2=14 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Consider the rank-one matrix $A = u_1 u_2^T$ where u_1 and u_2 form an orthonormal basis for $\mathbb{R}^2$.

1(a) Write down a unit vector w ($\|w\| = 1$) that makes the length $\|Aw\|$ as large as possible. What is the maximum length of Aw?

Solution: We have $Aw = u_1 u_2^T w = u_1 \cos\theta$, where θ is the angle between w and u_1, so the length $\|Aw\|$ is largest when w is a unit vector in the direction of u_2 (or $-u_2$). The maximum length is $\|u_1 u_2^T u_2\| = \|u_1\| = 1$.

1(b) Write down a unit vector w ($\|w\| = 1$) that makes the length $\|Aw\|$ as small as possible. What is the minimum length of Aw?

Solution: Similarly, the length $\|Aw\|$ is smallest when w is a unit vector that is perpendicular to u_2, which means it is in the direction of u_1 (or $-u_1$). The maximum length is $\|u_1 u_2^T u_1\| = \|(0)u_1\| = 0$.

1(c) What are the eigenvalues and eigenvectors of A?

Solution: A has a single eigenvector u_1 with eigenvalue $\lambda_1 = 0$. A also has $\lambda_2 = 0$, but no second linearly independent eigenvector, because any eigenvector is either a multiple of u_1 or orthogonal to u_2: in this case, these are the same one-dimensional subspace spanned by u_1.

1(d) Is A diagonalizable? Why or why not?

Solution: A has only a single linearly independent eigenvector u_1 so it is not diagonalizable (the eigenvector matrix is not invertible).

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Problem 2 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the 3×2 matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2(a) Use Gram-Schmidt orthogonalization to compute an orthonormal basis, q_1 and q_2, for the column-space of A.

$$\text{orthonormal basis} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

2(b) Find a third orthonormal vector, q_3, that spans the nullspace of A^T.

$$q_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

2(c) Find a 3×2 upper triangular matrix R that satisfies $A = QR$, where the columns of Q are q_1, q_2, and q_3 from parts (a) and (b).

$$R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

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Problem 3 (4+4+4+2=14 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the paraboloid $z = a + bx^2 + cy^2$ that best fits these four data points in format (x, y, z): $(1, 1, 3)$, $(0, 1, 1)$, $(1, 0, 1)$, $(0, 0, 0)$.

3(a) Write down four linear equations that the unknowns a, b, and c must satisfy if all four measurements lie on the paraboloid.

Solution: Each data point provides an equation: $3 = a+b(1)^2+c(1)^2 = a+b+c$, $1 = a + b(0)^2 + c(1)^2 = a + c$, $1 = a + b(1)^2 + c(0)^2 = a + b$, and $0 = a + b(0)^2 + c(0)^2 = a$. In matrix form (could write matrix form here or in part (b) - either is okay), this is $Ax = b$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

3(b) Write down the normal equations for the least-squares solution $\hat{x} = (\hat{a}, \hat{b}, \hat{c})^T$ to the overdetermined system of equations in part (a).

Solution: The normal equations are $A^T Ax = A^T b$, which is the linear system

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$$

3(c) Solve the normal equations in part (b) to find the least-squares solution $\hat{x} = (\hat{a}, \hat{b}, \hat{c})^T$.

Solution: After reduction to upper triangular form via elementary row operations, we have

$$\begin{pmatrix} 4 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 3/2 \\ 3/2 \end{pmatrix}.$$

Solving this upper triangular system by back-substitution, we find that $\hat{x} = \begin{pmatrix} -1/4 & 3/2 & 3/2 \end{pmatrix}^T$.

3(d) Does the data lie on the surface of a paraboloid? Explain.

Solution: The data does not lie exactly on the surface of a paraboloid, because the system of equations in part (a) has no solution. One can check that the residual $A\hat{x} - b = \begin{pmatrix} -1 & 1 & 1 & -1 \end{pmatrix}^T / 4$.

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Problem 4 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Suppose that $Ax = b$ has the general solution $x = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T + c \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}^T$ when the right-hand side is $\begin{pmatrix} 2 & 4 & 6 \end{pmatrix}^T$, where c can be any real number.

4(a) What are the dimensions of the four fundamental subspaces of A? Explain.

Solution: The dimension of the nullspace is 1 because their is one free parameter in the general solution. A is 3×3 because its solution and the right-hand side have three components. Therefore, the column space has dimension $3 - 1 = 2$. The row space then also has dimension 2 and the nullspace of A^T has dimension 1.

4(b) Write down the reduced row echelon form of A (this is the R factor in $A = CR$ except with extra zero rows so R and A have the same dimensions).

Solution: Since the row space of A has dimension 2, R will have 2 nonzero rows. Since the column space of A is 2 dimensional, the 2×2 identity is embedded in those 2 rows. Finally, since the vector $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}^T$ is a basis for the nullspace of A, we know $A \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, so the second column of A is twice the first column of A with opposite sign, while the first and third columns are linearly independent. Therefore, the reduced row echelon form of A must be

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

4(c) Find the first two columns of A. Can you determine the third? Explain.

Solution: The particular solution $x = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ tells us that the second column of A is $A \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 & 6 \end{pmatrix}^T$. The first column is half the second column with opposite sign, so the first column is $-\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$. All that we know about the third column is that it is not a combination of the first two columns, so we cannot determine it uniquely.

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Problem 5 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Suppose the vectors q_1, q_2, q_3 form an orthonormal basis for $\mathbb{R}^3$ and the matrix A satisfies $Aq_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, $Aq_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, and $Aq_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.

5(a) Write the matrix A explicitly in terms of the vectors q_1, q_2, q_3.

Solution: The three equations tell us that matrix $AQ = I$, where the columns of Q are q_1, q_2, q_3. In other words, $A = Q^{-1}$, and since Q is an orthogonal matrix, we have $A = Q^{-1} = Q^T$.

5(b) Write down all possibilities for $\det A$.

Solution: Since $A = Q^T$ is also orthogonal with real entries, $\det A = \pm 1$.

5(c) Put an X next to **each correct completion**. The eigenvalues of A must

_______ (i) be real numbers.

_______ (ii) be positive real numbers.

_______ (iii) be imaginary numbers.

This one is right! _______ (iv) have absolute value $|\lambda| = 1$.

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Problem 6 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Consider the coupled system of linear difference equations: $x_{k+1} = 2x_k - y_k$ and $y_{k+1} = 2y_k - x_k$, subject to initial condition $u_1 = \begin{pmatrix} x_1 & y_1 \end{pmatrix}^T$.

6(a) Identify two linearly independent initial conditions with unit length for which the solution $u_k = \begin{pmatrix} x_k & y_k \end{pmatrix}^T$ never changes direction in $\mathbb{R}^2$ at any step $k \geq 1$.

Solution: These are the eigenvectors of the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, wich are $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ (or any scalar multiples thereof).

6(b) After many iterations (in the limit $k \to \infty$), what direction does the solution starting from $u_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ tend toward? Give your answer as a unit vector that points in the correct direction.

Solution: They tend toward the direction of the dominant eigenvector $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T / \sqrt{2}$ which has eigenvalue $\lambda_2 = 3 > \lambda_1 = 1$.

6(c) Put an X next to the correct answer. For the initial condition in part (b), the solution u_k

This one is right! ________ (i) becomes exponentially large as $k \to \infty$.

________ (ii) becomes exponentially small as $k \to \infty$.

________ (iii) oscillates with a fixed amplitude as $k \to \infty$.

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Problem 7 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

7(a) Use elementary row transformations to make A into an upper triangular matrix U.

Solution: Subtracting the first row from the second yields a new second row $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$. Interchanging second and third rows brings A to upper triangular form

$$U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

7(b) Use your results from part (a) to compute $\det A$ (don't compute cofactors).

Solution: The determinant is the product of the pivots (diagonal entries) in U with a minus sign because of the permutation: $\det A = -(1)(1)(1) = -1$.

7(c) If the column vector $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is added to the last column ,$\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$, of A, what is the new determinant? Explain your reasoning.

Solution: By multilinearity, the determinant of the new matrix is

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The first term is $\det A$ and the second term is zero because the last column is a combination of the first two columns (second column minus first column). So we still have determinant equal to -1.

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Problem 8 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the 3×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

8(a) Compute the postive-definite matrix $M = A^T A$.

Solution: The matrix is $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

8(b) Compute the singular values and singular vectors (unit vectors u_1, u_2 and v_1, v_2) of A.

Solution: The singular values and right singular vectors are related to the eigenvalues and eigenvectors of M. We have $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$, while $v_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T / \sqrt{2}$ and $v_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. The left singular vectors are then $u_1 = Av_1/\sigma_1 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}^T / \sqrt{6}$ and $u_2 = Av_2/\sigma_2 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T / \sqrt{2}$.

8(c) Select an orthonormal basis for the column space of A from among the singular vectors computed in part (b).

Solution: The left singular vectors u_1 and u_2 from part (b) are an orthonormal basis for the column space of A.

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