LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 7: SOLUTIONS.

1. Problem 1

If a 3 by 3 matrix has $\det A = -1$, find:
(a) $\det(2A)$
(b) $\det(-A)$
(c) $\det(A^2)$
(d) $\det(A^{-1})$.
Solution.
(a) We have $2A = (2I) \cdot A$ so $\det(2A) = \det(2I) \det A = 8 \det A = -8$.
(b) We have $(-A) = (-I) \cdot A$ so $\det(-A) = \det(-I) \det A = (-1) \cdot (-1) = 1$.
(c) We have $\det(A^2) = \det A \det A = 1$.
(d) We have $1 = \det(I) = \det(A \cdot (A^{-1})) = \det A \det(A^{-1})$ so $\det(A^{-1}) = (\det A)^{-1} = -1$.

2. Problem 2

Compute determinants and inverses of these matrices:
(a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
(b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
Solution
(a) We have $\det A = 1$. We have $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. It follows that

$$A^{-1} = \frac{C^T}{1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) We have $\det A = 1 + 1 - 1 = 1$. We have $C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. It follows that

$$A^{-1} = \frac{C^T}{1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(c) First and third columns of A are equal so A is not invertible, hence, $\det A = 0$.

3. Problem 3

Find the determinants of a rank one matrix A and a skew-symmetric matrix B.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}.$$

Solution.

(a) We have $A = \begin{bmatrix} 1 & -4 & 5 \\ 1 & -8 & 10 \\ 3 & -12 & 15 \end{bmatrix}$, it has linearly dependent columns so $\det A = 0$.

(b) We have $\det B = 12 - 12 = 0$.

4. Problem 4

If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

Solution.

Recall that $\det A$ is a linear combination of cofactors so if all of them are zero then $\det A$ is also zero.

For a matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ we have $C_{11} = C_{12} = C_{21} = C_{22} = 1 \neq 0$ but $\det A = 0$.

5. Problem 5*

Suppose $\det A = 1$ and you know all the cofactors of A. How can you find A?

Solution.

We have $A^{-1} = \frac{C^T}{\det A} = C^T$. It follows that

$$A = (C^T)^{-1} = (C^{-1})^T. \tag{5.1}$$

Note now that $\det(C^T) = \det C = \det(A^{-1}) = 1$. We conclude from (5.1) (use that $C^{-1} = A^T$) that the matrix A is the matrix of cofactors of the matrix C.