18.06 FINAL EXAM. IN 1993.
TO POST

Part I

1. Find all possible values for the determinant of the given type of 3×3 real matrix.

[2] (a) A matrix with independent columns.

$\det(A) \neq 0$

[2] (b) A matrix with $A^2 = A$.

$\det A = \det A^2 = (\det A)^2 \to \det(A) = \pm 1$ or $\det(A) = 0$

[2] (c) A matrix with pivots 1, 2 and 3.

$\det(A) = \pm 1(2)(3) = \pm 6$

[2] (d) ~~A Markov matrix.~~

[3] 2. Why is there no orthogonal matrix Q such that $Q^{-1}AQ = \Lambda$ if

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}?$$

If there were, then A would be symmetric, which it is not.

[2] 3. The left null space $\mathcal{N}(A^T)$ of a matrix A is spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. The set of solutions of the equation $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is (circle one)

the empty set, a point, a line, a plane, a three-dimensional hyperplane in $\mathbb{R}^4$, all of $\mathbb{R}^4$.

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \subset N(A^T) \perp C(A)$ so $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has no solutions

[4] 4. Apply the Gram-Schmidt algorithm to the columns of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ to obtain three orthonormal vectors.

$a_1 \; a_2 \; a_3$

$$q_1 = a_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{q}_2 = a_2 - q_1 \underbrace{(q_1^T a_2)}_{=1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 = \tilde{q}_2 / \|\tilde{q}_2\| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{q}_3 = a_3 - q_1 \underbrace{(q_1^T a_3)}_{=2} - q_2 \underbrace{(q_2^T a_3)}_{=1/\sqrt{2}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

[3] 5. For what values of a and b is the quadratic form $ax^2 + 2xy + by^2 = [x \; y] \underbrace{\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$ positive definite?

$\hookrightarrow \lambda_1, \lambda_2 > 0$
eigenvals

$\det(A) = ab - 1 = \lambda_1 \lambda_2 \rightarrow \lambda_1, \lambda_2$ same sign when $\det A > 0$
$\mathrm{tr}(A) = a + b = \lambda_1 + \lambda_2 \rightarrow \lambda_1, \lambda_2 > 0$ when $\det A > 0$, $\mathrm{tr}(A) > 0$

When both $\mathrm{tr}(A) > 0$
and $\det(A) > 0$, then $\lambda_1, \lambda_2 > 0$
and quadratic form is pos. def.

6. Suppose A is an m by n matrix, with independent columns.

[2] • What can you deduce about the relation of m and n?

fewer cols than rows, so $m \geq n$

[2] • What can you deduce about the set of solutions to $A\mathbf{x} = 0$?

only $x=0$ solves $Ax=0$

[2] • For which m and n are there nonzero solutions to $A^T\mathbf{y} = 0$?

If $m > n$, then $A^Ty = 0$ has nonzero solns.

[2] • Give two properties of the matrix A^TA (other than the fact that it is square).

A^TA is <u>symmetric</u> & <u>positive definite</u>

[2] 7. Give an example of a matrix with exactly two zero eigenvalues and no zero entries.

$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has 1 lin. indep.

eigenvector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = 0, 0$.

[3] 8. Consider the square matrix $A = \begin{bmatrix} 7 & -3 \\ 9 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$. Find the solution to the differential equation $\frac{du(t)}{dt} = Au(t)$ with initial condition $u(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$u(t) = e^{At} u(0) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\Rightarrow c_1 = 0, c_2 = 1$

$$u(t) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (0) e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1) e^{-2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{-2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

[3] 9. Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \end{bmatrix}$.

$a_1 \quad a_2 \quad a_3$

Need a 4th vector orthogonal to a_1, a_2, a_3.

all zero in 3rd component

$$\Rightarrow a_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \perp \text{span}\{a_1, a_2, a_3\}$$

b/c $a_4^T a_j = 0 \quad j = 1, 2, 3.$

10. Are the following statements true or false? You get 2 points for a correct answer and -2 points for an incorrect answer.

T F (a) If $M^{-1}AM = B$, then A and B must have the same eigenvectors.

They have same eigenvalues

Note: eigenvector u of $B \Rightarrow Bu = \lambda u$

then $(M^{-1}AM)u = \lambda u \Rightarrow A(Mu) = \lambda(Mu)$

$\Rightarrow Mu$ is eigenvector of A

T F (b) The matrices A and A^T always have the same rank.

T F (c) There is a matrix with column space is spanned by the vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$, and with row space spanned by vectors $\begin{bmatrix}1\\4\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$.

lin. dep.

lin indep.

but col rank = row rank

so impossible

T F (d) If a square matrix has a repeated eigenvalue, it cannot be diagonalizable.

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ has $\lambda_1 = 1 = \lambda_2 = 1$

and is diagonal(izable)

T F (e) The set of vectors in $\mathbb{R}^3$ with integer (whole number) components is a subspace of $\mathbb{R}^3$.

Vector $c\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ doesn't have

integer entries unless c is an integer.

Part II

1. Let A be the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

[3] (a) Find a factorization $A = LU$, where L is a lower triangular matrix, and U is in echelon form.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 - \frac{R_2}{3}]{} \underbrace{\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 1/3 & 1 \end{bmatrix}$$

$A = LU$ (check work) ✓

[4] (b) Find the general solution of $Ax = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

↳ last column of A

$$\Rightarrow A \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{particular soln}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow Ax = 0 \Rightarrow LUx = 0 \Rightarrow Ux = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = -x_4$

$x_1 = -x_3$

$$x = \overset{\text{Particular soln}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} + \underbrace{c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Soln's to } Ax = 0}$$

[4] 1. (c) The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the column space of A if a, b and c satisfy what linear conditions?

Need $Ax = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $x \in \mathbb{R}^4$

$$\Rightarrow (LU)x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow Ux = \underbrace{L^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{y}$$

need (last entry of y) $= 0$ b/c last row of U is zeros (no pivot).

$$Ly = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$y_1 = a,\ y_2 = b - 2y_1 = b - 2a,\ y_3 = c - y_1 - 1/3\, y_2 = c - a - \frac{1}{3}(b - 2a)$$

$$y_3 = 0 \iff c = a + \frac{1}{3}(b - 2a) = \frac{1}{3}(a + b)$$

[4] 2. The vector space $\mathbb{R}^2$ has bases $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \mathbf{w}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$. Write the matrix for the identity linear transformation I from the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$X = x_1 V_1 + x_2 V_2 = c_1 w_1 + c_2 w_2$$

↑ coordinates in $\{v_1, v_2\}$ basis ↑ coordinates in $\{w_1, w_2\}$ basis

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

matrix for coordinate trans.

$$= \frac{1}{9-8}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$.

[3] (a) Find a matrix Q with orthonormal columns and an upper triangular matrix R such that $A = QR$.

$$Q = [q_1 \; q_2] \qquad q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \tilde{q}_2 = \underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{a_2} - \underbrace{\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix}}_{q_1 q_1^T}\underbrace{\begin{bmatrix}0\\1\\2\end{bmatrix}}_{a_2}$$

$$q_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} \qquad = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$R = \begin{bmatrix}\sqrt{2} & \sqrt{2} \\ & \sqrt{3}\end{bmatrix}$$

$r_{ij} = q_i^T a_j, \; i \geq j$

[3] (b) Find the closest vector to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in the column space of A.

y

$$y = QQ^T\begin{bmatrix}1\\1\\1\end{bmatrix} = \underbrace{\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\sqrt{2} \\ 1/\sqrt{3}\end{bmatrix}}_{Q^Tb} = \begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-1/3\\1/3\\1/3\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\4/3\end{bmatrix}$$

QQ^T: orthogonal projection onto col(A)

[3] (c) If **v** and **w** are any two linearly independent vectors, find a nonzero linear combination that is perpendicular to **v**.

$$w - \frac{(v^T w)}{v^T v} v$$ is perpendicular to v

[3] (d) Compute the matrix P which projects onto the column space of A.

$$P = QQ^T = q_1 q_1^T + q_2 q_2^T = \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}\begin{bmatrix}-1 & 1 & 1\end{bmatrix}$$

$$= \frac{3}{6}\begin{bmatrix}1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1\end{bmatrix} + \frac{2}{6}\begin{bmatrix}1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5\end{bmatrix}$$

4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the Euclidean space $\mathbb{R}^n$, and let A be the square matrix $\mathbf{u}\mathbf{v}^T$.

[3] (a) Describe the row space and nullspace of A in terms of $\mathbf{u}$ and $\mathbf{v}$.

row space of A = span$\{v\}$

nullspace of A = vectors $\perp$ v (all x s.t. $v^Tx = 0$)

[2] (b) Show that $\mathbf{u}$ is an eigenvector of A, and find the corresponding eigenvalue.

$$Au = uv^Tu = \underbrace{(v^Tu)}_{=\lambda}u$$

[2] (c) What condition must be satisfied by $\mathbf{u}$ and $\mathbf{v}$ for A to be skew-symmetric ($A = -A^T$)?

$$A = -A^T \iff uv^T = -vu^T$$

$i = j \Rightarrow 0 = A_{ii} = u_i v_i$ — either $u_i = 0$ or $v_i = 0$ for every $i = 1, 2, \dots, n$

$i \neq j \Rightarrow u_i v_j = -v_i u_j$ (one of these is zero) $\rightarrow u_i v_j = 0$ for every $i, j = 1, 2, \dots, n$ $\Rightarrow$ u or v $= 0$ and $A = 0$

[2] (d) What condition must be satisfied by $\mathbf{u}$ and $\mathbf{v}$ so that $A^2 = A$?

$$A^2 = (uv^T)(uv^T) = u\underbrace{(v^Tu)}_{\text{need } v^Tu = 1}v^T$$

5. Let A_n be the $n \times n$ matrix with entries $a_{ij} = \begin{cases} -2, & i = j-1, \\ 1, & i = j, \\ 1, & i = j+1, \\ 0, & \text{other entries.} \end{cases}$ For example,

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

[4] (a) Let $d_n = \det(A_n)$. Find numbers a and b such that for $n = 3, 4, 5, \ldots,$

$$d_n = a d_{n-1} + b d_{n-2}.$$

First notice that $A_n = \begin{bmatrix} 1 & -2 \; 0 \cdots \\ 1 & \\ 0 & A_{n-1} \\ \vdots & \end{bmatrix}$ $n \geq 3$

So Cofactor $C_{11} = \det A_{n-1}$

and Cofactor $C_{21} = (-1)\det\begin{bmatrix} -2 & 0 \cdots \\ \vdots & A_{n-2} \end{bmatrix} = +2 \det A_{n-2}$

Then $\det A_n = (1) C_{11} + (1) C_{21} = \det A_{n-1} + 2 \det A_{n-2}$

$a = 1$ $\quad b = 2$

[1] (b) What is d_4?

$d_1 = 1, d_2 = 1 + 2 = 3, d_3 = 3 + 2(1) = 5, \; d_4 = 5 + 2(3) = 11$

[4] (c) Write the matrix A such that $\begin{bmatrix} d_{n+1} \\ d_n \end{bmatrix} = A \begin{bmatrix} d_n \\ d_{n-1} \end{bmatrix}$, and calculate its eigenvalues and eigenvectors.

$d_{n+1} = d_n + 2 d_{n-1}, \; d_n = d_n$

$$\Rightarrow \begin{bmatrix} d_{n+1} \\ d_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_n \\ d_{n-1} \end{bmatrix}$$

$\det\begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda) - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

$\lambda_1 = 2, \lambda_2 = -1$ $\quad \underline{\lambda_1 = 2}$ $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{u_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \underline{\lambda_2 = -1}$ $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{u_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \rightarrow u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

[2] 5. (d) Find the number λ such that d_n/λ^n tends to a non-zero, finite limit as n tends to infinity.

$$\begin{bmatrix} d_{n+1} \\ d_n \end{bmatrix} = A^{n-1}\begin{bmatrix} d_2 \\ d_1 \end{bmatrix} = A^{n-1}\left[\underbrace{c_1}_{\text{coeffs}} u_1 + c_2 u_2\right]$$

$$= c_1\lambda_1^{n-1}u_1 + c_2\lambda_2^{n-1}u_2$$

$$\Rightarrow d_n = c_1 2^{n-1} - c_2(-1)^{n-1}$$

$\lambda = 2$

$$= c_1 2^{n-1} \pm c_2 \Rightarrow \frac{d_n}{2^n} = \frac{c_1}{2} \pm \frac{c_2}{2^n} \to \frac{c_1}{2} \text{ as } n \to \infty.$$

6. (a) If a, b, c are real numbers, find the eigenvalues and null space of the skew-symmetric

[5] matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = -\lambda^3 + cab - bca$$

$$-\lambda b^2 - \lambda c^2 - \lambda a^2 = 0$$

$$-\lambda^3 - \lambda(a^2+b^2+c^2) = -\lambda(\lambda^2 + a^2+b^2+c^2) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = \sqrt{-(a^2+b^2+c^2)}, \quad \lambda_3 = -\sqrt{-(a^2+b^2+c^2)}$$

$$= i\sqrt{a^2+b^2+c^2} \qquad = -i\sqrt{a^2+b^2+c^2}$$

$$\text{null space} = \text{span}\left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\}$$

[2] (b) Explain why the matrix A in part (a) of this problem cannot be orthogonal.

orthogonal matrix has $Q^{-1} = Q^T$ so is invertible, but A in part (a) is not invertible b/c it has a nontrivial nullspace.