

**LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 12:
EXERCISES.**

There are a lot of problems in this recitation sheet. Please feel free to choose those that are of the most interest to you.

1. PROBLEM 1 (**$A\mathbf{x} = \mathbf{b}$**)

For

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

solve the equation $A\mathbf{x} = \mathbf{b}$.

2. PROBLEM 2 (**ORTHOGONAL PROJECTION MATRICES AND ORTHOGONALIZATION**)

(a) Compute the orthogonal projection matrix P onto the plane $V \subset \mathbb{R}^3$, consisting of $V = \{(x, y, z) \mid x + y + 2z = 0\}$.

(b) Find an orthonormal basis q_1, q_2 of V , extend it to an orthonormal basis q_1, q_2, q_3 of the whole \mathbb{R}^3 . Which of the four fundamental spaces of P contains q_3 ?

3. PROBLEM 3 (**DETERMINANT**)

(a) Are the vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ independent or dependent?

(b) Find the determinant of the following matrix

$$A = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 7 & -2 \\ -1 & 4 & 1 \end{bmatrix}$$

using.

(i) The definition of the determinant. Recall that:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}.$$

(ii) The cofactors of one of the rows.

(c) Find the determinant of the following matrix

$$B = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 1 & 0 & 5 & -5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}.$$

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(d) Find the determinant of the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 3 \\ 4 & 10 & 14 & 6 \\ 3 & 4 & 2 & 7 \end{bmatrix}.$$

Hint: look at this matrix!

4. PROBLEM 4 (EIGENVALUES)

(a) Suppose the real column vectors q_1 and q_2 and q_3 are orthonormal. Show that the matrix $q_1 q_1^T + 2q_2 q_2^T + 5q_3 q_3^T$ has eigenvalues $\lambda = 1, 2, 5$.

(b) Find the eigenvalues of the matrix:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Eigenvalues of a 4×4 matrix D are given as $2, 3, 0, -1$. What is the rank of D ? What is the value of $\det D$? How about the trace of D ? Trace of D^2 ?

5. PROBLEM 5 (DETERMINANT + EIGENVALUES)

The symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \text{ and } H^2 = 4I.$$

(a) What is the determinant of H ?

(b) What are the eigenvalues of H ?

Hint: use $H^2 = 4I$ and the trace of H .

(c) What are the singular values of H (σ_i 's)?

6. PROBLEM 6 (SVD)

Find SVD decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$