MIT 18.06 Practice Exam 3 Solutions, Spring 2023

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NOTE: This practice exam is a bit longer and more computationally intensive than an in-class exam. It is inteneded as a study-guide. If you understand the concepts and can carry out the computations for each problem, you will be in an excellent position to succeed on exam 3!

Problem 1:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 4×3 matrix with integer entries

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ -2 & 0 & -3 \\ -2 & -2 & -1 \end{pmatrix}$$

1(a) Apply Gram-Schmidt to the columns of A to compute an orthonormal basis for the column space of A.

$$\text{orthonormal basis} = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Solution: The first basis vector is the first column of A normalized to have unit length, so that

$$q_1 = \frac{1}{4}\begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

To get the second orthonormal basis vector, we compute how much of the second column of A is in the direction of q_1 using the dot product:

$$q_1^T a_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{2}(4) = 2.$$

This is how much of q_1 we need to subtract from the second column a_2 to make it orthogonal to q_1,

$$v_2 = a_2 - (q_1^T a_2)q_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix} - (2)\,\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

To get an orthonormal vector, we need to orthonormalize, so we divide by the length of v_2 and get

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Finally, we make the third column, a_3, orthogonal to q_1 and q_2. We calculate the length of the component of a_3 along q_1,

$$q_1^T a_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -3 \\ -1 \end{pmatrix} = \frac{1}{2}(8) = 4,$$

and then the length of the component of a_3 along q_2,

$$q_2^T a_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -3 \\ -1 \end{pmatrix} = \frac{1}{2}(-4) = -2.$$

Subtracting off these components from a_3 leads to the vector

$$v_3 = \begin{pmatrix} 1 \\ 3 \\ -3 \\ -1 \end{pmatrix} - (4)\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} - (-2)\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It turns out that a_3 did not have any component perpendicular to both q_1 and q_2! In other words, a_3 was a combination of q_1 and q_2, and therefore, of the columns a_1 and a_2. So A has rank $= 2$ and q_1 and q_2 provide an orthonormal basis for the column space of A.

1(b) Add a third and fourth column to make the following matrix an orthogonal matrix (Recall that an orthogonal matrix is a square matrix with orthonormal columns). In what fundamental subspace of A must these new third and fourth columns lie?

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

Solution: First, notice that there is more than one way to choose the new third and fourth columns of Q to make it orthogonal. A nice choice is to pick the third and fourth columns to fill out the 4×4 Hadamard matrix: an orthogonal matrix with entries ± 1. Alternatively (and more systematically), we can pick two new (arbitrary) linearly independent vectors with four components and continue the Gram-Schmidt process, (subtracting off components along q_1, q_2, and so on) to get an orthonormal basis for the whole 4-dimensional space. The new columns are orthogonal to the column space of A, so they must lie in its orthogonal complement, the null space of A^T.

1(c) Write down a 4×3 upper triangular matrix R so that $A = QR$, where Q is the orthogonal matrix from part (b). (HINT: the last two rows of the matrix R should be all zero.)

$$R = \begin{pmatrix} 4 & 2 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: First note that all the columns of A are combinations of the first two columns of q_1 and q_2, since these are the basis for its column space. This means that only the first two rows of R are nonzero. The combinations of q_1 and q_2 that we need to recover the column of A are given by the dot products and normalization factors we computed in part (a) during Gram-Schmidt. The lengths of a_1 and v_1 go on the diagonal: from above we find that $\|a_1\| = 4$ and $\|v_1\| = 2$. The off-diagonal entry in the $(i, j)^{\text{th}}$ position is the dot product $q_i^T a_j$, which (gathering the dot product calculations from part (a)) yields the matrix R written above.

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Problem 2:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Follow the steps in 2(a)-(c) to solve the forward difference equation $u_{k+1} - u_k = Au_k$ with

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{and} \quad u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2(a) Compute all three eigenvalues of the matrix A.

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{1}{2}, \qquad \lambda_3 = -\frac{1}{2}.$$

Solution: We start by computing $\det(A - \lambda I)$ and then look for its zeros. To simplify, we use the linearity of the determinant in each column to pull out the three factors of $1/2$, so that

$$\det\begin{pmatrix} 1/2-\lambda & 1/2 & 0 \\ 1/2 & -\lambda & -1/2 \\ 0 & -1/2 & 1/2-\lambda \end{pmatrix} = \frac{1}{2^3}\det\begin{pmatrix} 1-2\lambda & 1 & 0 \\ 1 & -2\lambda & -1 \\ 0 & -1 & 1-2\lambda \end{pmatrix}.$$

We calculate the determinant directly from the "big formula" in chapter 5.2 of the textbook (add up all products of diagonal entries and subtract all products of anti-diagonal entries). The zero pattern leaves only three nonzero terms:

$$\det\begin{pmatrix} 1-2\lambda & 1 & 0 \\ 1 & -2\lambda & -1 \\ 0 & -1 & 1-2\lambda \end{pmatrix} = (1-2\lambda)^2(-2\lambda) - (1-2\lambda) - (1-2\lambda).$$

After pulling out the common factor $1-2\lambda$ and simplifying, we are left with the equation

$$\det(A - \lambda I) = (1-2\lambda)(4\lambda^2 - 2\lambda - 2) = 0.$$

The first factor tells us that one zero occurs at $\lambda = 1/2$. Factoring the remaining quadratic polynomial as

$$(4\lambda^2 - 2\lambda - 2) = (2\lambda - 2)(2\lambda + 1),$$

reveals that the other two zeros occur at $\lambda = -1/2$ and $\lambda = 1$. Here are our three eigenvalues. Note that the $\lambda_1 + \lambda_2 + \lambda_2 = 1 = \text{trace}(A)$ as it should! That is a good check that our calculations are correct. Also note that we could have also computed $\det(A - \lambda I)$ using a cofactor expansion along the first row in this problem (with slightly messier algebra).

2(b) Compute the 3 linearly independent eigenvectors of A associated with the eigenvalues in part (a). Normalize them to have unit length.

$$x_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \qquad x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad x_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Solution: We compute the eigenvector x_1 by solving the nullspace equation $(A - \lambda_1 I)x_1 = 0$. After subtracting $\lambda_1 = 1$ from each diagonal entry of A and pulling out the factor of $1/2$ to simplify,

$$A - \lambda_1 I = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

We can compute the nullspace by elimination (disregarding the scalar factor $1/2$ since it does not change the nullspace):

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenvector associated with $\lambda_1 = 1$ is $x_1 = (-1, -1, 1)^T/\sqrt{3}$ after normalizing for unit length. Similarly, subtracting $\lambda_2 = 1/2$ from the diagonal entries of A, we have

$$A - \lambda_2 I = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Again, we compute the nullspace by elimination. The first and third rows are multiples so we eliminate the third row. Then, we permute the first and second rows to put the pivot in the first column on the diagonal. Subtracting the second row from the first finishes the reduction:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvector associated with $\lambda_2 = 1/2$ is $x_2 = (1, 0, 1)^T/\sqrt{2}$ after normalizing for unit length. Finally, subtracting $\lambda_3 = -1/2$ from the diagonal entries of A, we have

$$A - \lambda_3 I = \frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Computing the nullspace by elimination one last time, we compute that

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \implies \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \implies \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenvector associated with $\lambda_3 = -1/2$ is $x_3 = (-1, 2, 1)^T/\sqrt{6}$ after normalizing for unit length.

2(c) Fill in the diagonal entries of the middle matrix below to solve for u_{k+1} in terms of u_0 using the eigenvalues and eigenvectors of A.

$$u_{k+1} = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2^{k+1} & & \\ & (\frac{3}{2})^{k+1} & \\ & & (\frac{1}{2})^{k+1} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solution: Solving the difference equation for u_{k+1} in terms of u_0 yields

$$u_{k+1} = u_k + Au_k = (I + A)u_k = (I + A)^{k+1}u_0.$$

The eigenvalue decomposition of $I + A$ is related to the eigendecomposition of A by

$$I + A = I + X\Lambda X^T = XX^T + X\Lambda X^T = X(I + \Lambda)X^T.$$

In other words, the eigenvectors of $I + A$ are the eigenvectors of A, and the eigenvalues of $I + A$ are the eigenvalues of A shifted by one: $1 + 1 = 2$, $1/2 + 1 = 3/2$, and $-1/2 + 1 = 1/2$. Raising $I + A$ to the power $k + 1$, we arrive at

$$(I + A)^{k+1} = X(I + \Lambda)^{k+1}X^T.$$

A diagonal matrix raised to the $(k+1)^{\text{th}}$ power means raising each entry to the $(k+1)^{\text{th}}$ power. Therefore, to complete the solution formula linking u_{k+1} to u_0, we need to fill in the powers of 2^{k+1}, $(3/2)^{k+1}$, and $(1/2)^{k+1}$ in the diagonal entries of the middle matrix.

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Problem 3:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following three 3×3 structured matrices:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

3(a) Write down the determinant and trace of each matrix.

Solution: The determinant of a triangular matrix is the product of its diagonal entries, so $|A| = 6$. The matrix B is a rotation matrix: it leaves the e_1 direction unchanged and rotates by $90°$ in the e_1-e_2 plane. Therefore, $|B| = 1$ (alternatively, cofactor expansion is quick to calculate here as well). The matrix C has linearly dependent columns so it has $|C| = 0$. The trace is the sum of the diagonal elements: we have $\operatorname{tr}(A) = 6$, $\operatorname{tr}(B) = 1 + \sqrt{2}$, and $\operatorname{tr}(C) = 14$.

3(b) Write down the eigenvalues of each matrix.

Solution: Since A is triangular, the eigenvalues of A are the diagonal elements 1, 2, and 3. The matrix C is rank-one since all columns are a multiples of the first:

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}.$$

The eigenvalues of a rank-one matrix uv^T are $0, 0$, and $v^T u$, so the nonzero eigenvalue of C is 14. We can calculate the eigenvalues of the matrix B fairly quickly from a cofactor expansion of $|B - \lambda I|$. We have that

$$|B - \lambda I| = (1 - \lambda)\left((\frac{1}{\sqrt{2}} - \lambda)^2 + \frac{1}{2} \right) = (1 - \lambda)\left(\lambda^2 - \sqrt{2}\lambda + 1 \right).$$

The eigenvalues are therefore 1 and, with some help from the quadratic formula,

$$\lambda_{\pm} = \frac{1}{2}(\sqrt{2} \pm \sqrt{-2}) = \frac{1}{\sqrt{2}}(1 \pm i).$$

3(c) Explain why $(C + I)^{-1}$ is invertible and write down its trace (without calculating the inverse explicitly).

Solution: The matrix $C+I$ has the eigenvalues of C shifted to the right by 1, i.e., eigenvalues 1, 1, and 15. It is invertible because all three eigenvalues are nonzero (note that this means its determinant is also nonzero). The eigenvalues of $(C + I)^{-1}$ are the reciprocals of the eigenvalues of $C + I$, that is, they are 1, 1, and $1/15$. The trace is equal to the sum of the eigenvalues, therefore,

$$\mathrm{tr}((C + I)^{-1}) = 1 + 1 + \frac{1}{15} \approx 2.0667.$$

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