LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 1: SOLUTIONS.

My name is Vasily Krylov. If you have any questions or comments about these solutions, please feel free to ask me by email (krvas@mit.edu) or during my office hours (Thursday 4.30 p.m. - 6.30 p.m. Room 2-341).

1. Problem 1

(a) Are columns of the matrix $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ linearly independent?

(b) How about columns of the matrix $\begin{bmatrix} 5 & 11 & 14 \\ 8 & 10 & 11 \\ 9 & 11 & 12 \end{bmatrix}$?

(c) How could you decide if the vectors $(1,1,1)$, $(-1,1,-1)$ and (a,b,c) are linearly independent or dependent?

Solution of part (a): we claim that $(1,-1)$, $(2,0)$ are linearly independent.

Proof. Assume that there are numbers $a, b \in \mathbb{R}$ such that

$$a \cdot (1,1) + b \cdot (2,0) = (0,0).$$

Our goal is to show that $a = b = 0$. Note that $a \cdot (1,1) + b \cdot (2,0) = (a+2b, a)$ so we conclude that

$$(a+2b, a) = (0,0)$$

that is equivalent to

$$\begin{cases} a + 2b = 0 \\ a = 0 \end{cases} \Leftrightarrow a = b = 0.$$

□

Solution of part (b): we claim that the vectors $(5,8,9)$, $(11,10,11)$ and $(14,11,12)$ are linearly dependent.

Proof. Our goal is to find a, b, c such that

$$a \cdot (5,8,9) + b \cdot (11,10,11) + c \cdot (14,11,12) = (0,0,0).$$

So we need to find a nonzero solution of the following system of linear equations:

$$\begin{cases} 5a + 11b + 14c = 0 \\ 8a + 10b + 11c = 0 \\ 9a + 11b + 12c = 0 \end{cases} \Leftrightarrow \begin{cases} 5a + 11b + 14c = 0 \\ 8a + 10b + 11c = 0 \\ 4a - 2c = 0 \end{cases}$$

We conclude that $c = 2a$. Substituting this in our system of equations we get

$$\begin{cases} 33a + 11b = 0 \\ 30a + 10b = 0 \end{cases} \Leftrightarrow b = -3a.$$

We conclude that every vector of the form $(a, -3a, 2a)$ is a solution of our system of equations. It is also easy to check that

$$(5, 8, 9) - 3(11, 10, 11) + 2(14, 11, 12) = (0, 0, 0).$$

□

Solution of part (c): we claim that the vectors $(1, 1, 1)$, $(-1, 1, -1)$ and (a, b, c) are linearly dependent iff $a = c$.

Proof. Note that the vectors $(1, 1, 1)$, $(-1, 1, -1)$ are linearly independent. It follows (do you see, how?) that vectors $(1, 1, 1)$, $(-1, 1, -1)$, (a, b, c) are linearly dependent iff there exist $k, l \in \mathbb{R}$ such that

$$(a, b, c) = k \cdot (1, 1, 1) + l \cdot (-1, 1, -1).$$

So we get

$$(a, b, c) = (k - l, k + l, k - l).$$

So we must have $a = c$.

Note now that if $a = c$ then we have

$$(a, b, a) = \frac{a + b}{2}(1, 1, 1) + \frac{b - a}{2}(-1, 1, -1).$$

□

2. Problem 2

Describe geometrically (line, plane, or all $\mathbb{R}^3$) all linear combinentions of

(a) $(1, 2, 3)$ and $(3, 6, 9)$.

Solution of part (a): we claim that vectors $(1, 2, 3)$, $(3, 6, 9)$ generate the line $\ell = \{(c, 2c, 3c) \mid c \in \mathbb{R}\} \subset \mathbb{R}^3$.

Proof. It is clear that $(1, 2, 3) \in \ell$, it is also clear that $(3, 6, 9) = 3 \cdot (1, 2, 3) \in \ell$. It remains to show that every element of ℓ can be obtained as a linear combination of $(1, 2, 3)$, $(3, 6, 9)$, this is clear since $(c, 2c, 3c) = c \cdot (1, 2, 3)$. □

(b) $(2, 0, 0)$, $(0, 2, 2)$ and $(2, 2, 3)$.

Solution of part (b): we claim that vectors $(2, 0, 0)$, $(0, 2, 2)$ and $(2, 2, 3)$ generate the whole $\mathbb{R}^3$.

Proof. Let $V \subset \mathbb{R}^3$ be the vector subspace of $\mathbb{R}^3$ generated by $(2, 0, 0)$, $(0, 2, 2)$ and $(2, 2, 3)$. Our goal is to show that $V = \mathbb{R}^3$. Indeed note first that

$$(0, 0, 1) = (2, 2, 3) - (2, 0, 0) - (0, 2, 2)$$

so $(0, 0, 1) \in V$. It follows that

$$(0, 2, 0) = (0, 2, 2) - 2 \cdot (0, 0, 1) \in V$$

so $(0, 1, 0) \in V$. Dividing $(2, 0, 0)$ by 2 we conclude that $(1, 0, 0) \in V$.

It follows that $(1,0,0),(0,1,0),(0,0,1) \in V$. It remains to note that for every $a,b,c \in \mathbb{R}$ we have

$$(a,b,c) = a \cdot (1,0,0) + b \cdot (0,1,0) + c \cdot (0,0,1) \in V.$$

□

3. Problem 3

True or false (give a reason if true or find a counterexample if false):

(a) If $\mathbf{u} = (1,1)$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ (i.e. $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$), then $\mathbf{v}$ is parallel to $\mathbf{w}$ (i.e. $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent).

(b) If $\mathbf{u} = (1,1,1)$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$ (i.e. $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$), then $\mathbf{v}$ is parallel to $\mathbf{w}$ (i.e. $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent).

(c) If $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is perpendicular to $\mathbf{v} + 2\mathbf{w}$.

(d)* If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors then $\|\mathbf{u} - \mathbf{v}\|^2 = 2$.

Solution of part (a): True.

Proof. Let $\mathbf{v} = (a_1, b_1)$, $\mathbf{w} = (a_2, b_2)$ be coordinates of our vectors. Conditions

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$$

are equivalent to

$$\begin{cases} a_1 + b_1 = 0 \\ a_2 + b_2 = 0. \end{cases}$$

It follows that $b_1 = -a_1$, $b_2 = -a_2$ i.e.

$$\mathbf{v} = a_1(1,-1),\ \mathbf{w} = a_2(1,-1)$$

are linearly dependent. □

Solution of part (b): False. Counterexample is

$$\mathbf{v} = (1,-1,0),\ \mathbf{w} = (1,1,-2).$$

Proof. □

Solution of part (c): True.

Proof. We have

$$\mathbf{u}\cdot(\mathbf{v} + 2\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + 2\cdot\mathbf{u}\cdot\mathbf{w} = 0 + 2\cdot 0 = 0.$$

□

Solution of part (d): True.

Proof. We have

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + (-\mathbf{v})\cdot(-\mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 1 + 1 = 2.$$

□

4. Problem 4

(a) If three corners of a parallelogram are $(1,1)$, $(4,2)$ and $(1,3)$, what are the possible fourth corners? Draw those parallelograms.

Solution of part (a): there are three possibilities for the fourth corner, $(4,4)$**,** $(-2,2)$ **or** $(4,0)$**.**

Proof. Let (a,b) be coordinates of the fourth corner. There are three possibilities: (a,b) is opposite to $(1,1)$ to $(4,2)$ or to $(1,3)$.

Recall now that in general if we have some parallelogram $(c_1,d_1),(c_2,d_2),(c_3,d_3),(c_4,d_4)$ with edges $[(c_1,d_1),(c_2,d_2)]$, $[(c_2,d_2),(c_4,d_4)]$, $[(c_4,d_4),(c_3,d_3)]$, $[(c_1,d_1),(c_3,d_3)]$ then we have

$$(c_1+c_3, d_1+d_3) = (c_2+c_4, d_2+d_4). \tag{4.1}$$

To see that, we can shift all coordinates by (c_1,d_1) i.e. assume that $(c_1,d_1)=(0,0)$. Now the claim follows from the fact that the vector (c_4,d_4) should be the sum of the vectors (c_2,d_2), (c_3,d_3).

Let us return to the problem.

If (a,b) is opposite to $(1,1)$ then from (4.1) we conclude that

$$(a+1, b+1) = (5,5)$$

so

$$(a,b) = (4,4).$$

If (a,b) is opposite to $(4,2)$ then from (4.1) we conclude that

$$(a+4, b+2) = (2,4)$$

so

$$(a,b) = (-2,2).$$

If (a,b) is opposite to $(1,3)$ then from (4.1) we conclude that

$$(a+1, b+3) = (5,3)$$

so

$$(a,b) = (4,0).$$

□

(b) If four corners of a unit cube are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, what are the possibility for other four corners? Draw. Find coordinates of its center. How many edges this cube has? Describe one face and find the coordinates of its center. Describe all faces.

Solution of part (b) (sketch)

The only possibility for other four corners are $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$.

If we shift the cube by vector $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ then it will be a cube with corners $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ and its center will be $(0,0,0)$ (do you understand, why?). We conclude that the center of the original cube has coordinates $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

To find the number of edges note that every corner of the cube lies on 3 edges. We have 8 corners. It follows that the number of edges if $\frac{8\cdot 3}{2}=12$ (do you understand, why?).

An example of a face is the face that contains corners $(1, 0, 1), (0, 1, 1), (0, 0, 1), (1, 1, 1)$. Its center is $(\frac{1}{2}, \frac{1}{2}, 1)$.

There are six faces, they are given by setting one of coordinates being equal to 0 or to 1.

5. Problem 5*

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows, then it also has dependent columns.

Hint: prove that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows iff $ad - bc = 0$.