LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 8: SOLUTIONS.

1. Problem 1

Find the eigenvalues and eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \; B = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

Solution

We have

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$$

so

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16+20}}{2} = \{-1, 5\}.$$

Now

$$A - \lambda_1 I = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

annihilates vector $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

We also have

$$A - \lambda_2 I = A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

annihilates vector $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So eigenvalues of A are $\{-1, 5\}$ and eigenvectors are $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Remark 1.1. Note that if $\mathbf{v}$ is an eigenvector of some matrix A, then $k\mathbf{v}$ is also an eigenvector of A (with the same eigenvalue), here k is any nonzero number.

Let us now deal with matrix $A + I$. It follows from Problem 3 below that $A + I$ has the same eigenvectors as A, and their eigenvalues are $-1 + 1 = 0$ and $5 + 1 = 6$, respectively.

2. Problem 2

Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \; A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

Solution

We have

$$\det(A - \lambda I) = (-\lambda)(1 - \lambda) - 2 = \lambda^2 - \lambda - 2$$

has roots $-1, 2$. So eigenvalues of A are -1 and 2.

We have

$$A + I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

annihilates vector $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

We have

$$A - 2I = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

annihilates vector $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So eigenvalues of A are $\{-1, 2\}$ and eigenvectors are $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Let us now deal with matrix A^{-1}. It follows from Problem 3 below that A^{-1} has the same eigenvectors as A, and their eigenvalues are $(-1)^{-1} = -1$, $2^{-1} = 1/2$ respectively.

3. Problem 3

What do you do to the equation $A\mathbf{x} = \lambda\mathbf{x}$, in order to prove (a), (b), and (c)?

(a) $\lambda + 1$ is an eigenvalue of $A + I$.

Solution We have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

It follows that

$$(A + I)\mathbf{x} = A\mathbf{x} + I\mathbf{x} = \lambda\mathbf{x} + \mathbf{x} = (\lambda + 1)\mathbf{x}$$

so $\mathbf{x}$ is an eigenvector of $A + I$ with eigenvalue $\lambda + 1$.

(b) λ^{-1} is an eigenvalue of A^{-1}.

Solution

We have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

It follows that

$$A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x})$$

so

$$\mathbf{x} = \lambda A^{-1}(\mathbf{x})$$

so

$$A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$$

so $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

(c) λ^2 is an eigenvalue of A^2.

Solution

We have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

It follows that

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A(\mathbf{x}) = \lambda \cdot \lambda\mathbf{x} = \lambda^2\mathbf{x}$$

so $\mathbf{x}$ is an eigenvector of A^2 with eigenvalue λ^2.

Remark 3.1. More generally it is true that if $A\mathbf{x} = \lambda\mathbf{x}$ then $A^n\mathbf{x} = \lambda^n\mathbf{x}$.

4. Problem 4

Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

$$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

Solution

Recall that if A is a 2 by 2 matrix with eigenvalues λ_1, λ_2 then

$$\lambda_1\lambda_2 = \det A,$$

$$\lambda_1 + \lambda_2 = \operatorname{tr} A,$$

where $\operatorname{tr} A$ is the sum of diagonal terms of A.

In our situation let A be $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$. Recall that eigenvalues of A are $4, 7$ so we must have

$$-a = \det A = 4 \cdot 7 = 28$$

and

$$b = \operatorname{tr} A = 4 + 7 = 11.$$

We conclude that $a = -28$, $b = 11$ so

$$A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}.$$

Now to find B write $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, our goal is to find p, q, r.

Again we have

$$p = \det B = 1 \cdot 2 \cdot 3 = 6$$

and

$$r = \operatorname{tr} B = 1 + 2 + 3 = 6.$$

It remains to find q. Recall that (by Problem 3 above) eigenvalues of B^2 are $1^2 = 1$, $2^2 = 4$, $3^2 = 9$. It follows that

$$\operatorname{tr} B^2 = 1 + 4 + 9 = 14.$$

Note note that

$$B^2 = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p + qr & q + r^2 \end{bmatrix}$$

so $\operatorname{tr} B^2 = 2q + r^2 = 2q + 36$.

It follows that

$$2q + 36 = 14$$

so $q = -11$.

We conclude that

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

5. Problem 5

From the unit vector $\mathbf{u} = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = \mathbf{u}\mathbf{u}^T$. This matrix has $P^2 = P$ because $\mathbf{u}^T\mathbf{u} = 1$.

(a) $P\mathbf{u} = \mathbf{u}$ comes from $(\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(-)$. Then $\lambda = 1$.

Solution

We have

$$(\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(\mathbf{u}^T\mathbf{u}) = \mathbf{u} \cdot 1 = \mathbf{u}.$$

(b) If $\mathbf{v}$ is perpendicular to $\mathbf{u}$ show that $P\mathbf{v} = 0$. Then $\lambda = 0$.

Solution

We have

$$(\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{u}(\mathbf{u}^T\mathbf{v}) = \mathbf{u}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot 0 = 0.$$

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

Solution

We know from (b) that every nonzero element of $\mathbf{u}^\perp$ (subspace of $\mathbb{R}^4$ consisting of vectors $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{u} = 0$) will be an eigenvector of P with eigenvalue 0. The subspace $\mathbf{u}^\perp \subset \mathbb{R}^4$ consists of vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that

$$\frac{1}{6}x_1 + \frac{1}{6}x_2 + \frac{3}{6}x_3 + \frac{5}{6}x_4 = 0$$

that is equivalent to

$$x_1 + x_2 + 3x_3 + 5x_4 = 0.$$

This vector space has a basis consisting of vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ -3 \end{bmatrix}.$$

6. Problem 6*

Let $\mathbf{u}$, $\mathbf{v}$ be some vectors. Show that $\mathbf{u}$ is an eigenvector of the rank one 2×2 matrix $A = \mathbf{u}\mathbf{v}^T$. Find both eigenvalues of A.

Solution We have

$$A\mathbf{u} = (\mathbf{u}\mathbf{v}^T)\mathbf{u} = \mathbf{u}(\mathbf{v}^T\mathbf{u}) = (\mathbf{v}^T\mathbf{u}) \cdot \mathbf{u}$$

so $\mathbf{u}$ is indeed an eigenvector with eigenvalue $\mathbf{v}^T\mathbf{u} = \mathbf{v} \cdot \mathbf{u}$.

Matrix A has rank 1, so one of its eigenvalues must be equal to 0. We see that eigenvalues of A are $\{0, \mathbf{v} \cdot \mathbf{u}\}$ (eigenvector with eigenvalue 0 is any element of $\mathbf{v}^\perp$).

Remark 6.1. One should be a bit careful if $\mathbf{v} \cdot \mathbf{u} = 0$. In this case, one can show that $A^2 = 0$, so the only eigenvalue that it has is equal to 0.

7. Problem 7*

Find a 2 by 2 orthogonal (rotation) matrix (other than I) with $A^3 = I$. Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What are the trace and determinant of A?

Solution

We will find matrix A in the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in [0, 2\pi]$. Note that

$$A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix}.$$

The condition $A^3 = I$ implies that $\cos 3\theta = 1$, $\sin 3\theta = 0$. It follows that $\theta = \frac{2\pi}{3}$ works and our matrix A is equal to:

$$A = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix}.$$