

- 1 The 4 by 6 matrix A has all 2's below the diagonal and elsewhere all 1's:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

- (a) By elimination factor A into L (4 by 4) times U (4 by 6).
(b) Find the rank of A and a basis for its nullspace (the special solutions would be good).

- 2 Suppose you know that the 3 by 4 matrix A has the vector $\mathbf{s} = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the *rank* of A and the complete solution to $A\mathbf{x} = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A ?

- 3 The following matrix is a *projection matrix*:

$$P = \frac{1}{21} \begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}.$$

- (a) What subspace does P project onto?
- (b) What is the *distance* from that subspace to $\mathbf{b} = (1, 1, 1)$?
- (c) What are the three eigenvalues of P ? Is P diagonalizable?

- 4 (a) Suppose the product of A and B is the zero matrix: $AB = 0$. Then the (1) space of A contains the (2) space of B . Also the (3) space of B contains the (4) space of A . Those blank words are

(1) _____ (2) _____ (3) _____ (4) _____

- (b) Suppose that matrix A is 5 by 7 with rank r , and B is 7 by 9 of rank s . What are the dimensions of spaces (1) and (2)? From the fact that space (1) contains space (2), what do you learn about $r + s$?

5 Suppose the 4 by 2 matrix Q has orthonormal columns.

- (a) Find the least squares solution \hat{x} to $Qx = b$.
- (b) Explain why QQ^T is not positive definite.
- (c) What are the (nonzero) singular values of Q , and why?

6 Let S be the subspace of \mathbf{R}^3 spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$.

- (a) Find an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2$ for S by Gram-Schmidt.
- (b) Write down the 3 by 3 matrix P which projects vectors perpendicularly onto S .
- (c) Show how the properties of P (what are they?) lead to the conclusion that $P\mathbf{b}$ is orthogonal to $\mathbf{b} - P\mathbf{b}$.

- 7 (a) If v_1, v_2, v_3 form a basis for \mathbf{R}^3 then the matrix with those three columns is _____.
- (b) If v_1, v_2, v_3, v_4 span \mathbf{R}^3 , give all possible ranks for the matrix with those four columns. _____.
- (c) If q_1, q_2, q_3 form an orthonormal basis for \mathbf{R}^3 , and T is the transformation that projects every vector v onto the plane of q_1 and q_2 , what is the matrix for T in this basis? Explain.

- 8 Suppose the n by n matrix A_n has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then the determinant of A_n for $n > 4$.

9 There are six 3 by 3 permutation matrices P .

- (a) What numbers can be the *determinant* of P ? What numbers can be *pivots*?
- (b) What numbers can be the *trace* of P ? What *four numbers* can be eigenvalues of P ?

- 10 Suppose A is a 4 by 4 upper triangular matrix with 1, 2, 3, 4 on its main diagonal. (You could put all 1's above the diagonal.)
- (a) For $A - 3I$, which columns have pivots? Which components of the eigenvector \mathbf{x}_3 (the special solution in the nullspace) are definitely zero?
- (b) Using part (a), show that the eigenvector matrix S is also upper triangular.