MIT 18.06 Exam 1 Solutions, Spring 2023
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Your name: ______________________________
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Recitation: ______________________________

Problem 1 (12+10+12=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

1(a) Compute the factorization $A = CR$ of the 4×4 matrix A. As usual, work from left to right until C contains a full set of linearly independent columns of A.

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix} = CR.$$

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

1(b) True or false (Write T or F in the blank space provided).

__ T __ The row space of A is a 2D plane.

__ F __For every 4×1 vector b, $Ax = b$ has a unique solution.

__ T __The column space and row space of A are the same.

1(c) Compute the **first** column of the matrix $B^{-1}A$:

$$B^{-1}A = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}}_{B^{-1}} \underbrace{\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix}}_{A}$$

HINT: Solve $Bx = a_1$,where a_1 is the first column of A, so that $x = B^{-1}a_1 =$ (first column of $B^{-1}A$).

$$(\text{first column of } B^{-1}A) = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

Solution: We solve the triangular system $Bx = a_1$ by backward substitution: $x_4 = 1$, $x_3 = 0 + x_4 = 1$, $x_2 = 1 + x_3 = 2$, and $x_1 = 0 + x_2 = 2$.

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Problem 2 (18+14+2=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

The bottom row of a 3×3 matrix A depends on a positive parameter p:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & p & p \end{pmatrix}$$

Compute the factorization $A = LU$ in the following steps.

2(a) Write down elimination matrices E_{21}, E_{31}, and E_{32} that introduce zeros in the $(2,1)$, $(3,1)$, and $(3,2)$ entries so that $E_{32}E_{31}E_{21}A = U$ is upper triangular. The elimination matrices may depend on the parameter p.

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \qquad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & p & 1 \end{pmatrix}$$

Solution: To eliminate the $(2,1)$ entry of A, we subtract $2\times$(row 1) from (row 2). To eliminate the $(3,1)$ entry of $E_{21}A$, we subtract $2\times$(row 1) from (row 3). Finally, to eliminate the $(3,2)$ entry of $E_{31}E_{21}A$, we add $p\times$(row 2) to (row 3). The sequence of transformations is

$$E_{21}A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 2 & p & p \end{pmatrix} \implies E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & p & p-2 \end{pmatrix}$$

$$\implies E_{32}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -p-2 \end{pmatrix}$$

Write down the lower and upper triangular factors L and U that multiply to make $A = LU$. The triangular factors may depend on the parameter p.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -p & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -p-2 \end{pmatrix}$$

Solution: The multipliers -2, -2, and $+p$ from elimination go directly into the corresponding entries of L with opposite signs. The triangular form produced by elimination is exactly U.

If p is allowed to be negative, A is not invertible when $p =$ ___ -2 ___.

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Problem 3 (15+15+2=32 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

This upper triangular matrix A is not invertible:

$$A = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

Show in two ways that A has no inverse.

3(a) Find a nontrivial combination of the columns that produces the zero vector, that is, find numbers x_1, x_2, x_3, and x_4 (not all equal to zero) such that

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ 6 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$x_1 = -1$ $\qquad$ $x_2 = 2$ $\qquad$ $x_3 = -1$ $\qquad$ $x_4 = 0$

Solution: Any multiple of $x = \begin{pmatrix} -1 & 2 & -1 & 0 \end{pmatrix}^T$ solves $Ax = 0$. In fact, these are also the *only* solutions of $Ax = 0$ for this A (stay tuned for chapter 3!).

Find a column vector $b = (b_1, b_2, b_3, b_4)^T$ so that $Ax = b$ has no solution $x = (x_1, x_2, x_3, x_4)^T$.

$b_1 = 0$ $\qquad$ $b_2 = 0$ $\qquad$ $b_3 = 1$ $\qquad$ $b_4 = 1$

Solution: If $Ax = b$ has a solution, the last 2 entries of b must be a multiple of the tuple $\begin{pmatrix} 8 & 9 \end{pmatrix}^T$ so that b is in the column space of A. Any other b leads to *no solutions*.

In general, an upper triangular matrix is invertible exactly when its ______ diagonal ______ entries are ______ nonzero ______.

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