LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 5: SOLUTIONS.

My name is Vasily Krylov. If you have any questions or comments, please feel free to ask me by email (krvas@mit.edu) or during my office hours (Thursday, 5 p.m. - 7 p.m., Room 2-361).

1. Problem 1

Find the projection matrix P onto the plane $2x - y + z = 0$.
Solution.

Proof. Vectors $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis of the plane $2x - y + z = 0$.

We can consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$, $C(A)$ is exactly our plane $2x - y + z = 0$.

It remains to compute

$$P = A(A^TA)^{-1}A^T.$$

We have

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix},\ A^TA = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

Using the general formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ we see that

$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 5/6 \end{bmatrix}.$$

We have

$$(A^TA)^{-1}A^T = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 5/6 & 1/6 \end{bmatrix}$$

so

$$P = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 5/6 & 1/6 \\ -1/3 & 1/6 & 5/6 \end{bmatrix}.$$

□

2. Problem 2

Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$, solve the "normal equation" $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ for $\mathbf{b} = (1, 1, 1)$.

Solution.

Proof. We have

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \ A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \ A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

It follows that

$$(A^T A)^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

We have $A^T\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we conclude that

$$\hat{\mathbf{x}} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

□

3. Problem 3

Consider points with coordinates $(t_1, b_1) = (0, 0)$, $(t_2, b_2) = (1, 8)$, $(t_3, b_3) = (3, 8)$, $(t_4, b_4) = (4, 20)$ (see Figure 4.8 at page 172 of the textbook).

(a) Find C, D defining the line $C + Dt$ that is closest to these four points. Do this by setting up and solving the normal equations:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

(recall that $\hat{\mathbf{x}} = (C, D)$ and $\mathbf{b} = (0, 8, 8, 20)$).

(b) For the best straight line $C + Dt$ (as in part (a)), find its four heights p_1, p_2, p_3, p_4 and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

Recall that if $\mathbf{p} = (p_1, p_2, p_3, p_4)$ then $\mathbf{p} = A\hat{\mathbf{x}}$ and if $\mathbf{e} = (e_1, e_2, e_3, e_4)$ then $\mathbf{e} = \mathbf{b} - \mathbf{p}$.

Solution of part (a).

Proof. We have $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and our goal is to solve the equation

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

We have

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}.$$

We have

$$(A^TA)^{-1} = \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}.$$

We have

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

We finally conclude that

$$\begin{bmatrix} C \\ D \end{bmatrix} = \hat{\mathbf{x}} = \frac{1}{40}\begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 36 \\ 112 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 13 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 28 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

□

Solution of part (b).

Proof. We have $\mathbf{p} = A\hat{\mathbf{x}}$ so

$$\mathbf{p} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}.$$

We get

$$\mathbf{e} = \mathbf{b} - \mathbf{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix},\ E = \|\mathbf{e}\| = 44.$$

□

4. Problem 4

If $P^2 = P$ show that $(I-P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ——- .

Solution.

Proof. We have

$$(I-P)^2 = (I-P)(I-P) = I^2 - I\cdot P - P\cdot I + P^2 = I - 2P + P = I - P.$$

Recall now that matrix P has the following properties:

(1) $P^2 = P$,
(2) $Px = x$ for $x \in C(A)$,
(3) $Px = 0$ for $x \in C(A)^\perp = N(A^T)$,
and P is uniquely determined by these properties.

We see from (2) that $(I-P)x = 0$ for $x \in C(A)$. From (3) we see that $(I-P)x = x$ for $x \in N(A^T)$. We also know that $(I-P)^2 = (I-P)$.

We conclude that $I - P$ projects onto the $C(A)^\perp = N(A^T)$. □