MIT 18.06 Exam 2, Spring 2023
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Your name: ______________________________
(*printed*)

Student ID: ______________________________

Recitation: ______________________________

Problem 1 (12+10+12=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 3×3 symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

1(a) Write down a basis for the nullspace of A using **only** vectors whose entries are 0, +1, and −1.

$$\text{nullspace basis} = \left\{ \qquad\qquad\qquad \right\}$$

1(b) If you are allowed to add a column and row to make A a 4×4 matrix, how large can you make the rank of the new matrix?

largest possible rank =

Reason:

1(c) True or false (Write **T** or **F** in the blank space provided) with a **reason**.

________ All vectors in the nullspace of A are in the nullspace of $A^T A$.

Reason:

________ The rank of $A^T A$ is smaller than the rank of A^T.

Reason:

________ The orthogonal projector onto the column space of A has rank 1.

Reason:

________ The orthogonal projector onto the nullspace of A has rank 1.

Reason:

(blank page for your work if you need it)

Problem 2 (10+10+8+8=36 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 4×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

2(a) Use elimination to reduce A to its reduced row echelon form $R = \begin{pmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

$$R = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \end{pmatrix}$$

2(b) Write down a basis for the column space of A and a basis for the row space of A.

column space basis $= \left\{ \qquad \right\}$, row space basis $= \left\{ \qquad \right\}$

2(c) Using the reduced row echelon form R from part (a), compute a basis for the nullspace of A.

$$\text{nullspace basis} = \left\{ \qquad \qquad \right\}$$

2(d) Use the column space basis vectors in part (b) to calculate the orthogonal projection of $b = \begin{pmatrix} 0 & 3 & 3 & 0 \end{pmatrix}^T$ onto the column space of A.

$$Pb = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

(blank page for your work if you need it)

Problem 3 (6+6+6+6+6=30 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Given two column vectors $x = (1, 0, 1)^T$ and $y = (1, 1, 0)^T$, consider the following three 3×3 matrices (I is the 3×3 identity matrix):

$$A = I - \frac{xx^T}{x^Tx}, \qquad B = I - \frac{yy^T}{y^Ty}, \qquad C = I - \frac{yx^T}{x^Ty}.$$

Put a **T** next to each matrix for which the statement is true and an **F** next to each matrix for which the statement is false. Provide a **reason** for parts (a) and (c) in the space provided.

3(a) The vector $y = (1, 1, 0)^T$ is in the nullspace of this matrix.

______ A **Reason:**

______ B **Reason:**

______ C **Reason:**

3(b) The column space of this matrix is orthogonal to the vector $x = (1, 0, 1)^T$.

______ A

______ B

______ C

3(c) The column space of this matrix has dimension 2.

______ A **Reason:**

______ B **Reason:**

______ C **Reason:**

3(d) This matrix is symmetric (recall that M is symmetric if $M = M^T$).

______ A

______ B

______ C

3(e) This matrix is its own square (i.e., a matrix that satisfies $M^2 = M$).

______ A

______ B

______ C

(blank page for your work if you need it)