MIT 18.06 Practice Exam 3 Solutions, Spring 2023

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Recitation: ______________________________

NOTE: This practice exam is a bit longer and more computationally intensive than an in-class exam. It is inteneded as a study-guide. If you understand the concepts and can carry out the computations for each problem, you will be in an excellent position to succeed on exam 3!

Problem 1:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 4×3 matrix with integer entries

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ -2 & 0 & -3 \\ -2 & -2 & -1 \end{pmatrix}$$

1(a) Apply Gram-Schmidt to the columns of A to compute an orthonormal basis for the column space of A.

$$\text{orthonormal basis} = \left\{ \qquad\qquad\qquad \right\}$$

1(b) Add a third and fourth column to make the following matrix an orthogonal matrix (Recall that an orthogonal matrix is a square matrix with orthonormal columns). In what fundamental subspace of A must these new third and fourth columns lie?

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ -1 & 1 & & \\ -1 & -1 & & \end{pmatrix}$$

1(c) Write down a 4×3 upper triangular matrix R so that $A = QR$, where Q is the orthogonal matrix from part (b). (HINT: the last two rows of the matrix R should be all zero.)

$$R = \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix}$$

(blank page for your work if you need it)

Problem 2:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Follow the steps in 2(a)-(c) to solve the forward difference equation $u_{k+1} - u_k = Au_k$ with

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{and} \quad u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2(a) Compute all three eigenvalues of the matrix A.

$$\lambda_1 = \qquad\qquad \lambda_2 = \qquad\qquad \lambda_3 =$$

2(b) Compute the 3 orthogonal eigenvectors of A associated with the eigenvalues in part (a). Normalize them to have unit length.

$$x_1 = \qquad\qquad x_2 = \qquad\qquad x_3 =$$

2(c) Fill in the diagonal entries of the middle matrix below to solve for u_{k+1} in terms of u_0 using the eigenvalues and eigenvectors of A.

$$u_{k+1} = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \qquad & \qquad & \qquad \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(blank page for your work if you need it)

Problem 3:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following three 3×3 structured matrices:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

3(a) Write down the determinant and trace of each matrix.

3(b) Write down the eigenvalues of each matrix.

3(c) Explain why $(C + I)^{-1}$ is invertible and write down its trace (without calculating the inverse explicitly).

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