LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 6: SOLUTIONS.

1. Problem 1

Are these pairs of vectors orthonormal or only orthogonal or only independent?

(a) $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$

Solution

The scalar product of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ is equal to $-1+0=-1\neq 0$ so these two vectors are only independent.

(b) $\begin{bmatrix}6\\8\end{bmatrix}$ and $\begin{bmatrix}4\\-3\end{bmatrix}$

Solution

The scalar product of $\begin{bmatrix}6\\8\end{bmatrix}$ and $\begin{bmatrix}4\\-3\end{bmatrix}$ is equal to $24-24=0$ so they are orthogonal. Note that the length of $\begin{bmatrix}6\\8\end{bmatrix}$ is equal to $36+64=100\neq 1$ and the length of $\begin{bmatrix}4\\-3\end{bmatrix}$ is $16+9=25\neq 1$ so they are not orthonormal.

(c) $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.

Solution

The scalar product of $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$ is equal to $-\cos\theta\sin\theta+\sin\theta\cos\theta=0$ so they are orthogonal. Moreover not that the length of both of our vectors equals to $(\cos\theta)^2+(\sin\theta)^2=1$ so they are orthonormal.

2. Problem 2

Give an example of each of the following:

(a) A matrix Q that has orthonormal columns but $QQ^T\neq I$.

Solution

Just take $Q=\begin{bmatrix}1\\0,\end{bmatrix}$

We have

$$QQ^T=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\neq I.$$

Remark 2.1. Note that if Q is a square matrix, then $QQ^T=I$.

(b) Two orthogonal vectors that are not linearly independent.

Solution

You can take $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. They are orthogonal but not linearly independent.

(c) An orthonormal basis for $\mathbb{R}^3$, including the vector $q_1 = (1, 1, 1)/\sqrt{3}$.

Solution Let us start from the basis q_1, v_2, v_3, where $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and apply the Gram-Schmidt algorithm to it.

Consider the vector q_2' that is equal to v_2 minus the projection of v_2 onto the line generated by q_1 i.e. $q_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \operatorname{proj}_{q_1} v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - (q_1^T v_2) q_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3}. \end{bmatrix}$ We get

$$q_2 = \frac{q_2'}{||q_2'||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

Now we have

$$q_3' = v_3 - \operatorname{proj}_{q_1} v_3 - \operatorname{proj}_{q_2} v_3 = v_3 - (q_1^T v_3) q_1 - (q_2^T v_3) q_2 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

We get

$$q_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

3. Problem 3

Find two orthogonal vectors in the plane $x + y + 2z = 0$. Make them orthonormal.

Solution

We can take $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. They are orthogonal. The corresponding orthonormal vectors are

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \ q_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

4. Problem 4

Find orthogonal vectors A, B, C by Gram-Schmidt from a, b, c:

$$a = (1, -1, 0, 0), \quad b = (0, 1, -1, 0), \quad c = (0, 0, 1, -1).$$

A, B, C and a, b, c are bases for the vectors perpendicular to $d = (1, 1, 1, 1)$.

We take $A = \frac{a}{||a||} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0)$. Now we take

$$B' = b - \operatorname{proj}_A b = b - (A^T b)A = b + (1/2, -1/2, 0, 0) = (1/2, 1/2, -1, 0)$$

and

$$B = \frac{B'}{||B'||} = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, 0).$$

Now

$$C' = c - \operatorname{proj}_A c - \operatorname{proj}_B c = c - (A^T c)A - (B^T c)B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1).$$

It follows that

$$C = \frac{C}{||C'||} = (\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}).$$