MIT 18.06 Practice Exam 2, Spring 2023

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Recitation: ______________________________

Problem 1:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 3×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

1(a) Use elementary row operations to reduce A to the reduced row echelon form $R = \begin{pmatrix} I & F \end{pmatrix}$.

$$R = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

1(b) Use the reduced row echelon form R to write down a basis for the column space and row space of A.

$$\left\{ \qquad\qquad \right\} \qquad \left\{ \qquad\qquad \right\}$$

1(c) Use the reduced row echelon form R to write down a basis for the nullspace of A.

$$\left\{ \qquad\qquad\qquad \right\}$$

1(d) Write down the general solution to $Ax = b$, when $b = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}^T$.

$$x =$$

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Problem 2:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Follow the steps in 2(a)-(c) to find the parabola $b = C + Dt + Et^2$ that is closest to the four points $(t_1, t_2, t_3, t_4)^T = (-1, 0, 1, 2)^T$ and $(b_1, b_2, b_3, b_4)^T = (0, -1, 0, 3)^T$.

2(a) Write down the 4×3 coefficient matrix A and right-hand side b associated with the 4 equations $b_k = C + Dt_k + Et_k^2$ (for $k = 1, 2, 3, 4$) for the 3 unknowns, C, D, and E.

$$A = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \\ \end{pmatrix}, \qquad b = \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix}$$

2(b) Compute the 3×3 matrix $M = A^T A$ and the 3×1 vector $c = A^T b$. Is the matrix M invertible? Why?

$$M = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \end{pmatrix}, \qquad c = \begin{pmatrix} \quad \\ \\ \\ \end{pmatrix}$$

2(c) Use elimination to solve the normal equations $Mx = c$ for the coefficients of the best fit parabola, $x = \begin{pmatrix} C & D & E \end{pmatrix}^T$.

$$x = \begin{pmatrix} \quad \\ \\ \\ \end{pmatrix}$$

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Problem 3:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Given two column vectors $x = (1, 1, 0)^T$ and $y = (0, 1, 1)^T$, consider the following two 3×3 *oblique projection* matrices (I is the 3×3 identity matrix):

$$N = \frac{xy^T}{y^T x}, \qquad M = I - \frac{xy^T}{y^T x}.$$

3(a) What are the dimensions of the four fundamental subspaces of N? Write down one nonzero vector in each subspace.

3(b) What are the dimensions of the four fundamental subspaces of M? Write down one nonzero vector in each subspace.

3(c) Use the four fundamental subspaces to explain why NM and MN are the zero matrix.

3(d) Are either of N or M an orthogonal projection matrix? Why or why not? (Recall that an orthogonal projection matrix P satisfies $P^2 = P$ and $P^T = P$.)

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