# MIT 18.06 Final Exam, Spring 2023
Gilbert Strang and Andrew Horning

**Your name:** ______________________
**(*printed*)**

**Student ID:** ______________________

**Recitation:** ______________________

**Problem 1 (4+4+4+2=14 points):**

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Consider the rank-one matrix $A = u_1 u_2^T$ where $u_1$ and $u_2$ form an orthonormal basis for $\mathbb{R}^2$.

**1(a)** Write down a unit vector $w$ ($\|w\| = 1$) that makes the length $\|Aw\|$ as large as possible. What is the maximum length of $Aw$?

**1(b)** Write down a unit vector $w$ ($\|w\| = 1$) that makes the length $\|Aw\|$ as small as possible. What is the minimum length of $Aw$?

**1(c)** What are the eigenvalues and eigenvectors of $A$?

**1(d)** Is $A$ diagonalizable? Why or why not?

*(blank page for your work if you need it)*

## Problem 2 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the $3 \times 2$ matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

**2(a)** Use Gram-Schmidt orthogonalization to compute an orthonormal basis, $q_1$ and $q_2$, for the column-space of $A$.

**2(b)** Find a third orthonormal vector, $q_3$, that spans the nullspace of $A^T$.

**2(c)** Find a $3 \times 2$ upper triangular matrix $R$ that satisfies $A = QR$, where the columns of $Q$ are $q_1$, $q_2$, and $q_3$ from parts (a) and (b).

*(blank page for your work if you need it)*

## Problem 3 (4+4+4+2=14 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the paraboloid $z = a + bx^2 + cy^2$ that best fits these four data points in format $(x, y, z)$: $(1, 1, 3)$, $(0, 1, 1)$, $(1, 0, 1)$, $(0, 0, 0)$.

**3(a)** Write down four linear equations that the unknowns $a$, $b$, and $c$ must satisfy if all four measurements lie on the paraboloid.

**3(b)** Write down the normal equations for the least-squares solution $\hat{x} = (\hat{a}, \hat{b}, \hat{c})^T$ to the overdetermined system of equations in part (a).

**3(c)** Solve the normal equations in part (b) to find the least-squares solution $\hat{x} = (\hat{a}, \hat{b}, \hat{c})^T$.

**3(d)** Does the data lie exactly on the surface of a paraboloid? Explain.

*(blank page for your work if you need it)*

## Problem 4 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Suppose that $Ax = b$ has the general solution $x = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T + c \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}^T$ when the right-hand side is $\begin{pmatrix} 2 & 4 & 6 \end{pmatrix}^T$, where $c$ can be any real number.

**4(a)** What are the dimensions of the four fundamental subspaces of $A$? Explain.

**4(b)** Write down the reduced row echelon form of $A$ (this is the $R$ factor in $A = CR$ except with extra zero rows so $R$ and $A$ have the same dimensions).

**4(c)** Find the first two columns of $A$. Can you determine the third? Explain.

*(blank page for your work if you need it)*

## Problem 5 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Suppose the vectors $q_1, q_2, q_3$ form an orthonormal basis for $\mathbb{R}^3$ and the matrix $A$ satisfies $Aq_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, $Aq_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, and $Aq_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.

**5(a)** Write the matrix $A$ explicitly in terms of the vectors $q_1, q_2, q_3$.

**5(b)** Write down all possibilities for $\det A$.

**5(c)** Put an X next to **each correct completion**. The eigenvalues of $A$ must

________ (i) be real numbers.

________ (ii) be positive real numbers.

________ (iii) be imaginary numbers.

________ (iv) have absolute value $|\lambda| = 1$.

*(blank page for your work if you need it)*

## Problem 6 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

Consider the coupled system of linear difference equations: $x_{k+1} = 2x_k - y_k$ and $y_{k+1} = 2y_k - x_k$, subject to initial condition $u_1 = \begin{pmatrix} x_1 & y_1 \end{pmatrix}^T$.

**6(a)** Identify two linearly independent initial conditions with unit length for which the solution $u_k = \begin{pmatrix} x_k & y_k \end{pmatrix}^T$ never changes direction in $\mathbb{R}^2$ at any step $k \geq 1$.

**6(b)** After many iterations (in the limit $k \to \infty$), what direction does the solution starting from $u_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ tend toward? Give your answer as a unit vector that points in the correct direction.

**6(c)** Put an X next to the correct answer. For the initial condition in part (b), the solution $u_k$

________ (i) becomes exponentially large as $k \to \infty$.

________ (ii) becomes exponentially small as $k \to \infty$.

________ (iii) oscillates with a fixed amplitude as $k \to \infty$.

*(blank page for your work if you need it)*

## Problem 7 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the $3 \times 3$ matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

**7(a)** Use elementary row transformations to make $A$ into an upper triangular matrix $U$.

**7(b)** Use your results from part (a) to compute $\det A$ (don't compute cofactors).

**7(c)** If the column vector $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is added to the last column ,$\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$, of $A$, what is the new determinant? Explain your reasoning.

*(blank page for your work if you need it)*

## Problem 8 (4+4+4=12 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

This problem is about the $3 \times 2$ matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

**8(a)** Compute the postive-definite matrix $M = A^T A$.

**8(b)** Compute the singular values and singular vectors (unit vectors $u_1, u_2$ and $v_1, v_2$) of $A$.

**8(c)** Select an orthonormal basis for the column space of $A$ from among the singular vectors computed in part (b).

*(blank page for your work if you need it)*