MIT 18.06 Exam 3, Spring 2023

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Your name: Solutions!
(*printed*)

Student ID: ______

Recitation: ______

Problem 1 (12 + 10 + 12 = 34 points):
Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

1(a) Find the eigenvalues λ_1, λ_2 and eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ of A:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

<u>Eigenvalues</u>

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix}$$
$$= 8 - 6\lambda + \lambda^2 - 3$$
$$= \lambda^2 - 6\lambda + 5$$
$$= (\lambda - 1)(\lambda - 5)$$
$$\lambda_1 = 1, \ \lambda_2 = 5$$

<u>Eiguvectors</u>

$\underline{\lambda_1 = 1}$
$$A - I = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$
$\rightarrow x_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ solves $(A-I)x_1 = 0$

$\underline{\lambda_2 = 5}$
$$A - 5 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$$
$\rightarrow x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ solves $(A-5I)x_2 = 0$

$\lambda_1 = 1$	$\lambda_2 = 5$	$\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$	$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1(b) Diagonalize $A = X\Lambda X^{-1}$ by finding those three matrices X and Λ and X^{-1}

$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$X^{-1} = \frac{1}{3(1) - 1(-1)} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \qquad X^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}}$$

1(c) Express the vector $\boldsymbol{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ as a combination of the eigenvectors x_1 and x_2 of A.

Then express the vector $A^4 u$ as a combination of those eigenvectors of A

$$u = c_1 x_1 + c_2 x_2 = \underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{X} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = X^{-1} u = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^4 u = A^4(c_1 x_1 + c_2 x_2) = c_1 \lambda_1^4 x_1 + c_2 \lambda_2^4 x_2$$

$$= 1^4 x_1 + 5^4 x_2$$

$$\boldsymbol{u} = X_1 + X_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A^4\boldsymbol{u} = x_1 + 5^4 x_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 5^4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(blank page for your work if you need it)

Problem 2 (10 + 10 = 20 points):
Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

2(a) Find the determinant of this permutation matrix P and **explain your reasoning!**

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

determinant of P = -1

b/c P is three row permutations away from the identity matrix. Each permutation switches the sign of det $I = 1$.

Alternate: cofactor formula or "big formula"

2(b) Find the cofactor of A_{11} and the determinant of A

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

C_{11} = cofactor of $A_{11} = \det(A_{\underbrace{2:4,2:4}})$

$= \underline{1}$

2nd col/row through 4th col/row

$\det(A) = 2C_{11} - 1\cancel{C_{12}}$ $=0$

via cofactor formula

$= 2(1)$

$= 2$

Cofactor of A_{11} = 1

Determinant of A = 2

(blank page for your work if you need it)

Problem 3 (12 + 12 = 24 points) :

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

An n by n real matrix Q is an orthogonal matrix if $Q^{\mathrm{T}}Q = I$.

3(a) Show that the length of x = the length of Qx for every real vector x

<u>lengths</u>

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

$$\|Qx\| = \sqrt{(Qx)^T(Qx)}$$

$$\|Qx\| = \sqrt{(Qx)^T(Qx)} = \sqrt{x^T \underbrace{Q^T Q}_{=I} x}$$

$$= \sqrt{x^T x} = \|x\|$$

3(b) Show that the determinant of Q is 1 or -1

Recall $$Q^T Q = I$$

Product & Transpose identity for determinant.

$$\det AB = \det A \det B$$

square A, B

$$\det A^T = \det A$$

square

$$\boxed{\begin{aligned} Q^T Q = I \quad &\Rightarrow \quad \det Q^T Q = \det I = 1 \\ 1 = \det Q^T Q &= \det Q^T \det Q \\ &= \det Q \det Q \\ &= (\det Q)^2 \\ \Rightarrow \det Q &= \pm 1 \end{aligned}}$$

Problem 4 (12 + 10 = 22 points):

Record your final answer in the allotted spaces. You may use the remaining space for your calculations.

4(a) What are the possible eigenvalues of a $\underbrace{\text{symmetric}}_{P^T = P}$ $\underbrace{\text{projection}}_{P^2 = P}$ matrix P and why ?

- Note that $\mathcal{C}(P) = \mathcal{C}(P^T) \perp N(P)$, so every x can be written $x = \underbrace{v}_{\text{in } \mathcal{C}(P)} + \overbrace{w}^{\text{in } N(P)}$

- If v is in $\mathcal{C}(P)$ then v can be written as a combo of P's columns, $v = Pu$ for some u. In particular $Pv = P(Pu) = P^2 u = Pu = v$.

If $Px = \lambda x$, then

$$P(v + w) = \overbrace{\cancel{P}v}^{= v \text{ since } v \text{ is in } \mathcal{C}(P)} + \overbrace{\cancel{P}w}^{= 0 \text{ since } w \text{ is in } N(P)}$$
$$= (1)v + (0)w$$

So the only way x can be an eigvec is ~~if~~ $v = \underline{0}$ (then, $\lambda = 0$) or if $w = \underline{0}$ (then, $\lambda = 1$).

4(b) If the vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n$ are orthonormal and $\boldsymbol{v} = c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + \cdots + c_n\boldsymbol{q}_n$, find a formula for the last coefficient c_n.

$$v = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{Q} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Q^{-1} v = Q^T v$$

$$= \begin{bmatrix} q_1^T v \\ q_2^T v \\ \vdots \\ q_n^T v \end{bmatrix}$$

$\boldsymbol{c_n}$ **= equals** $q_n^T v$