LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 2: SOLUTIONS.

My name is Vasily Krylov. If you have any questions or comments about these solutions, please feel free to ask me by email (krvas@mit.edu) or during my office hours (Thursday 5p.m. - 7 p.m. Room 2-361).

1. Problem 1

Solve equation $A\mathbf{v} = \mathbf{b}$ for the following A, $\mathbf{b}$ (use elimination):

(a) $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ 20 \\ 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} -2 & -1 & 1 \\ 4 & 2 & -1 \\ 0 & 5 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$.

Solution of part (a): $\mathbf{v} = (2, 1, 1)$.

Proof. We have $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$.

Now $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix}$.

It remains to solve the system of equations

$$\begin{cases} 2x + 3y + z = 8 \\ y + 3z = 4 \\ 8z = 8. \end{cases}$$

We conclude that the solution is

$$\mathbf{v} = (x, y, z) = (2, 1, 1).$$

□

Solution of part (b): $\mathbf{v} = (1, 2, 3)$.

Proof. We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 4 & 2 & -1 \\ 0 & 5 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 5 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

Now

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}-2&-1&1\\0&0&1\\0&5&-2\end{bmatrix}=\begin{bmatrix}-2&-1&1\\0&5&-2\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}-1\\3\\4\end{bmatrix}=\begin{bmatrix}-1\\4\\3\end{bmatrix}.$$

It remains to solve the system of equations

$$\begin{cases}-2x-y+z=-1\\5y-2z=4\\z=3\end{cases}$$

We conclude that

$$\mathbf{v}=(x,y,z)=(1,2,3).$$

□

2. Problem 2

Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{cases}x+by=0\\x-2y-z=0\\y+z=0\end{cases}$$

Solution: $b=-2$ leads to a row exchange, $b=-1$ leads to a missing pivot, in that case solutions are of the form $(a,a,-a)$, $a\in\mathbb{R}$.

Proof. Consider the corresponding matrix $\begin{bmatrix}1&b&0\\1&-2&-1\\0&1&1\end{bmatrix}$.

We have

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&b&0\\1&-2&-1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&b&0\\0&-2-b&-1\\0&1&1\end{bmatrix}.$$

We see that $b=-2$ leads to a row exchange, in this case we have

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&-2&0\\0&0&-1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&-2&0\\0&1&-1\\0&0&-1\end{bmatrix}$$

and the only solution of our equation is $x=y=z=0$.

Let us now find b that leads to a missing pivot. We can assume that $b=-2$ so can multiply by $\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{1}{2+b}&1\end{bmatrix}$:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{1}{2+b}&1\end{bmatrix}\begin{bmatrix}1&b&0\\0&-2-b&-1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&b&0\\0&-2-b&-1\\0&0&1-\frac{1}{2+b}\end{bmatrix}.$$

We see that missing pivot appears iff $1 - \frac{1}{2+b} = 0$ i.e. $b = -1$.

Substituting $b = -1$ we see that It remains to find any solution of the equation

$$\begin{cases} x - y = 0 \\ -y - z = 0 \end{cases}$$

One solution is $x = y = 1$, $z = -1$. Any other solution has the form $(a, a, -a)$ for some $a \in \mathbb{R}$. □

3. Problem 3

Find LU decomposition for the following matrices.

(a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}$.

Solution of part (a)**.**

Proof. We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

We conclude that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix},\ U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

□

Solution of part (b)**.**

We have

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{bmatrix}.$$

Now

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}.$$

We conclude that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$$

so

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}.$$

We conclude that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},\ U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}.$$