LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 12: SOLUTIONS.

There are a lot of problems in this recitation sheet. Please feel free to choose those that are of the most interest to you.

1. Problem 1 ($A\mathbf{x} = \mathbf{b}$)

For

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

solve the equation $A\mathbf{x} = \mathbf{b}$.

Solution

We look at:

$$\begin{bmatrix} 1 & 3 & 3 & , & 1 \\ 2 & 6 & 9 & , & 5 \\ -1 & -3 & 3 & , & 5 \end{bmatrix}$$

The echelon form is:

$$\begin{bmatrix} 1 & 3 & 3 & , & 1 \\ 0 & 0 & 3 & , & 3 \\ 0 & 0 & 0 & , & 0 \end{bmatrix}.$$

Nullspace is generated by $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and a particular solution is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

So the solution is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \ c \in \mathbb{R}.$$

2. Problem 2 (Orthogonal projection matrices and orthogonalization)

(a) Compute the orthogonal projection matrix P onto the plane $V \subset \mathbb{R}^3$, consisting of $V = \{(x, y, z) \,|\, x + y + 2z = 0\}$.

Solution

$V = C(A)$ for a matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Recall that $P = A(A^TA)^{-1}A^T$. We have

$$A^TA = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}, \ (A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

It follows that

$$P = \frac{1}{6}\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}.$$

(b) Find an orthonormal basis q_1, q_2 of V, extend it to an orthonormal basis q_1, q_2, q_3 of the whole $\mathbb{R}^3$. Which of the four fundamental spaces of P contains q_3?

Solution

We apply Gram Schmidt orthogonalization process to the basis $v_1 = (1, -1, 0)$, $v_2 = (0, 2, -1)$, $v_3 = (0, 0, 1)$.

We have $q_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{2}}(1, -1, 0)$, $q_2' = v_2 - \mathrm{proj}_{q_1} v_2 = (0, 2, -1) + \sqrt{2}(q_1) = (1, 1, -1)$, $q_2 = \frac{q_2'}{||q_2'||} = \frac{1}{\sqrt{3}}(1, 1, -1)$, $q_3' = v_3 - \mathrm{proj}_{q_1} v_3 - \mathrm{proj}_{q_2} v_3 = \frac{1}{3}(1, 1, 2)$, $q_3 = \frac{1}{\sqrt{6}}(1, 1, 2)$.

Vector q_3 lies in $N(P)$.

Remark 2.1. Note that $V \cdot (1, 1, 2) = 0$ from the definition of V so we actually do not need to do anything to compute q_3.

3. Problem 3 (Determinant)

(a) Are the vectors $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ independent or dependent?

Solution We have

$$\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0 + 1 + 1 - 0 - 0 - 0 = 2$$

so these vectors are linearly independent.

(b) Find the determinant of the following matrix

$$A = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 7 & -2 \\ -1 & 4 & 1 \end{bmatrix}$$

using.

(i) The definition of the determinant. Recall that:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}.$$

Solution

We have

$$\det A = 14 - 10 + 0 - 0 + 21 + 16 = 41.$$

(ii) The cofactors of one of the rows.

Solution

We use cofactors for the second row:

$$\det A = 7 \cdot 5 + 2 \cdot 3 = 41.$$

(c) Find the determinant of the following matrix

$$B = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 1 & 0 & 5 & -5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}.$$

We use cofactors for the third row:

$$\det B = 5 \cdot (-7) = -35.$$

(d) Find the determinant of the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 3 \\ 4 & 10 & 14 & 6 \\ 3 & 4 & 2 & 7 \end{bmatrix}.$$

Solution
Second and third rows of this matrix are linearly dependent so $\det C = 0$.

4. Problem 4 (Eigenvalues)

(a) Suppose the real column vectors q_1 and q_2 and q_3 are orthonormal. Show that the matrix $A = q_1q_1^T + 2q_2q_2^T + 5q_3q_3^T$ has eigenvalues $\lambda = 1, 2, 5$.
Solution
We have

$$(q_1q_1^T + 2q_2q_2^T + 5q_3q_3^T)q_1 = q_1(q_1^Tq_1) + 2q_2(q_2^Tq_1) + 5q_3(q_3^Tq_1) = q_1.$$

Similarly, we have $Aq_2 = 2q_2$ and $Aq_3 = 5q_3$.
(c) Find the eigenvalues of the matrix:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We have (cofacrors of the first row):

$$\det(B - \lambda I) = (-\lambda) \cdot (-\lambda)^3 - 1 \cdot 1 = \lambda^4 - 1.$$

Solutions of the equation $\lambda^4 = 1$ are

$$\lambda = i, -i, -1, 1.$$

(d) Eigenvalues of a 4×4 matrix D are given as $2, 3, 0, -1$. What is the rank of D? What is the value of $\det D$? How about the trace of D? Trace of D^2?
Rank of D is equal to 3, $\det D = 2 \cdot 3 \cdot 0 \cdot (-1) = 0$, $\operatorname{tr} D = 2 + 3 + 0 - 1 = 4$, $\operatorname{tr} D^2 = 2^2 + 3^2 + 0^2 + (-1)^2 = 14$.

5. Problem 5 (Determinant + Eigenvalues)

The symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \text{ and } H^2 = 4I.$$

(a) What is the determinant of H?

Solution We subtract the first row from the other three rows and get the matrix (with the same determinant): $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix}$. Determinant of this matrix is equal to $1 \cdot (16) = 16$.

(b) What are the eigenvalues of H?

Solution

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be eigenvalues of H. We know that

$$H^2 = 4I$$

so

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = 4$$

so eigenvalues are ± 2. Recall also that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \operatorname{tr} H = 0$ so we must have $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = \lambda_4 = -2$.

(c) What are the singular values of H (σ_i's)?

Solution

Recall that the singular values are square roots of eigenvalues of $H^T H$. We have $H^T H = H^2 = 4I$ so $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sqrt{4} = 2$.

6. Problem 6 (SVD)

Find SVD decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We have

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\lambda_1 = 2$, $\lambda_2 = 1$ and $\sigma_1 = \sqrt{2}$, $\sigma_2 = 1$. We also see that $u_1 = (1, 0)$ and $u_2 = (0, 1)$.

We have

$$A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Eigenvector of $A^T A$ with eigenvalue 2 is $(1, 0, -1)$ so $v_1 = \frac{1}{\sqrt{2}}(1, 0, -1)$ (recall that we must have $Av_1 = \sqrt{2} u_1$). Eigenvector with eigenvalue 1 is $(0, 1, 0)$ so $v_2 = (0, 1, 0)$ (recall that we must have $Av_2 = u_2$). Nullspace is generated by $\frac{1}{\sqrt{2}}(1, 0, 1)$.

So the SVD decomposition is:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$