LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 4: SOLUTIONS.

My name is Vasily Krylov. If you have any questions or comments, please feel free to ask me by email (krvas@mit.edu) or during my office hours (Thursday 5 p.m. - 7 p.m., Room 2-361).

1. Problem 1

Find bases and dimensions for the four subspaces ($C(A^T), N(A), C(A), N(A^T)$) associated with the following matrix A:

(a)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

Solution of the part (a).

Proof. The echelon form of A is

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

We conclude that $C(A^T)$ has dimension equal to 1 and is generated by the vector $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ i.e.

$$C(A^T) = \Big\{ \begin{bmatrix} a \\ 2a \\ 4a \end{bmatrix} \mid a \in \mathbb{R} \Big\}.$$

Space $N(A)$ consists of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$x_1 + 2x_2 + 4x_3 = 0.$$

We have $\dim N(A) = 3 - \operatorname{rk} A = 2$ and one possible choice of a basis of $N(A)$ is

$$\Big\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \Big\}.$$

Let us now describe vector spaces $C(A)$, $N(A^T)$. The echelon form of A^T is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We conclude that $C(A)$ has dimension equal to 1 and is generated by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ i.e.

$$C(A) = \Big\{ \begin{bmatrix} a \\ 2a \end{bmatrix} \mid a \in \mathbb{R} \Big\}.$$

Space $N(A^T)$ consists of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$x_1 + 2x_2 = 0.$$

We have $\dim N(A^T) = 2 - \operatorname{rk} A = 1$ and the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \in N(A^T)$ forms a basis of $N(A^T)$. $\square$

Solution of the part (b).

Proof. The echelon form of A is

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}.$$

We conclude that $C(A^T)$ has dimension equal to 2 and has a basis consisting of vectors $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ i.e.

$$C(A^T) = \Big\{ \begin{bmatrix} a \\ b \\ 4a \end{bmatrix} \mid a, b \in \mathbb{R} \Big\}.$$

Space $N(A)$ consists of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$x_1 + 4x_3 = 0,\ x_2 = 0.$$

We have $\dim N(A) = 3 - \operatorname{rk} A = 1$ and one possible choice of a basis of $N(A)$ is

$$\Big\{ \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \Big\}.$$

Let us now describe vector spaces $C(A)$, $N(A^T)$. The echelon form of A^T is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We conclude that $C(A)$ has dimension equal to 2 and has a basis consisting of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ i.e.

$$C(A) = \mathbb{R}^2.$$

Space $N(A^T)$ consists of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$x_1 = x_2 = 0.$$

so $N(A^T) = \{0\}$. □

2. Problem 2

(a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces?

(b) If a 3 by 4 matrix has rank 3, what are the column space $(C(A))$ and the left nullspace $(N(A^T))$?

Solution of the part (a).

Proof. We have $\dim C(A^T) = \dim C(A) = 5$. We conclude that $\dim N(A) = 9 - 5 = 4$ and $\dim N(A^T) = 7 - 5 = 2$. □

Solution of the part (b).

Proof. We have $\dim C(A) = 3$ and $C(A) \subset \mathbb{R}^3$ so we must have $C(A) = \mathbb{R}^3$. Recall that $\dim N(A^T) = 3 - 3 = 0$ so $N(A^T) = \{0\}$. □

3. Problem 3

For which numbers c and d do this matrix have rank **2**:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}?$$

Solution: for $c = 0$ and $d = 2$.

Proof. Consider four cases:

Case 1: $c \neq 0$, $d \neq 0$.

The echelon form of A is:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2/c & 2/c \\ 0 & 0 & 0 & 1 & 2/d \end{bmatrix}$$

and this matrix has a rank equal to 3.

Case 2: $c = 0$, $d \neq 0$.

The echelon form of A is:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & (2/d) - 1 \end{bmatrix}$$

and this matrix has a rank equal to 2 iff $(2/d) - 1 = 0$ i.e. $d = 2$.

Case 3: $c \neq 0$, $d = 0$.

The echelon form of A is:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2/c & 2/c \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and this matrix has a rank equal to 3.

Case 4: $c = 0$, $d = 0$.

The echelon form of A is:

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and this matrix has rank equal to 3.

□

4. Problem 4

Suppose A is a symmetric matrix ($A = A^T$).

If $Ax = 0$, $Az = 5z$, which subspaces contain these "eigenvectors" x and z? Show that x and z are perpendicular.

Solution.

Proof. Let us first of all note that $N(A) = N(A^T)$, $C(A) = C(A^T)$ (use that A is symmetric).

$Ax = 0$ implies that $x \in N(A) = N(A^T)$.

$Az = 5z$ implies that $5z \in C(A) = C(A^T)$ so $z \in C(A) = C(A^T)$.

Recall that $N(A)$ is perpendicular to $C(A^T)$. It follows that $x \perp z$ (since $x \in N(A)$, $z \in C(A) = C(A^T)$).

□