MIT 18.06 Exam 2 Solutions, Spring 2023

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Recitation: ____________________

Problem 1 (12+10+12=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 3×3 symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

1(a) Write down a basis for the nullspace of A using **only** vectors whose entries are 0, $+1$, and -1.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Solution: These are two possible basis vectors for the null space whose entries are only 0, $+1$, and -1. The signs of either of these two vectors could be flipped.

1(b) If you are allowed to add a column and row to make A a 4×4 matrix, how large can you make the rank of the new matrix?

largest possible rank = 3

Reason: The reduced row echelon form of the new 4×4 matrix can have at most 3 pivots, so the maximum rank is 3, not 4! Adding a column can increase the rank by one and adding a row can increase the rank by one. The largest possible rank increase is from rank 1 to rank 3.

1(c) True or false (Write T or F in the blank space provided).

__ T __ All vectors in the nullspace of A are in the nullspace of $A^T A$.

Reason: This is true because if $Ax = 0$, then $A^T Ax = A^T(Ax) =$

$A(0) = 0$.

__ F __ The rank of $A^T A$ is smaller than the rank of A^T.

Reason: The rank of $A^T A$ and A^T are equal. In this case, it follows immediately from a direct calculation ($A^T A = 6A$). It is also true for general matrices: $A^T A$, A, and A^T always have the same number of linearly independent rows and columns, i.e, the same rank.

__ T __ The orthogonal projector onto the column space of A has rank 1. **Reason:** True because the orthogonal projector onto the column

space projects every vector onto the one-dimensional column space of A. So its column space is also one-dimensional and the projector has rank 1.

__ F __ The orthogonal projector onto the nullspace of A has rank 1. **Reason:** False because the orthogonal projector onto the nullspace

space projects every vector onto the two-dimensional nullspace of A. So its column space is two-dimensional and the projector has rank 2.

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Problem 2 (10+10+8+8=36 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 4×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

2(a) Use elimination to reduce A to the reduced row echelon form $R = \begin{pmatrix} I & F \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

$$R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution: Subtracting the first row from the third produces a zero row and subtracting twice the first row from the fourth row produces the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

Adding the second row to the fourth row triangularizes A and subtracting the second row from the first row completes the transformation to reduced row echelon form. The identity in the first two columns identifies the pivot columns and the third column corresponds to one free variable.

2(b) Write down a basis for the column space of A and a basis for the row space of A.

$$\text{column space basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{row space basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Solution: The first two columns of A, corresponding to the pivot columns of R, are a basis for the column space, and the nonzero rows of R are a basis for the row space.

2(c) Using the reduced row echelon form R from part (a), compute a basis for the nullspace of A.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Solution: The last column of R corresponds to the free variable x_3. We can rewrite the equations from the first two rows of the nullspace equation $Rx = 0$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We set $x_3 = 1$ and solve for the pivot variables: $x_1 = 1$ and $x_2 = -1$. Our basis vector is $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$. Recall that R and A have the same row space and the same nullspace, so this basis for the nullspace of R is also a basis for the nullspace of A.

2(d) Use the column space basis vectors in part (b) to calculate the orthogonal projection of $b = \begin{pmatrix} 0 & 3 & 3 & 0 \end{pmatrix}^T$ onto the column space of A.

$$Pb = \frac{1}{2} \begin{pmatrix} 3 \\ 6 \\ 3 \\ 0 \end{pmatrix}$$

Solution: Since the columns of C are independent by construction, we can compute the orthogonal projection using the usual formula $Pb = C(C^TC)^{-1}C^Tb$. The normal equations $C^TCx = C^Tb$ are

$$\begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

We can solve with either elimination or the formula for the inverse of a 2×2 matrix. The solution entries $x_1 = -3/2$ and $x_2 = 3$ tell us what combination of the two columns of C give us Pb.

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Problem 3 (6+6+6+6+6=30 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Given two column vectors $x = (1, 0, 1)^T$ and $y = (1, 1, 0)^T$, consider the following three 3×3 matrices (I is the 3×3 identity matrix):

$$A = I - \frac{xx^T}{x^T x}, \qquad B = I - \frac{yy^T}{y^T y}, \qquad C = I - \frac{yx^T}{x^T y}.$$

Put a **T** next to each matrix for which the statement is true and an **F** next to each matrix for which the statement is false. Provide a **reason** for parts (a) and (c) in the space provided.

3(a) The vector $y = (1, 1, 0)^T$ is in the nullspace of this matrix.

__ F __ A **Reason:** We have that $Ay = y - x\frac{x^T y}{x^T x} \neq 0$, since x and y are independent.

__ T __ B **Reason:** We calculate directly that $By = y - y\frac{y^T y}{y^T y} = 0$.

__ T __ C **Reason:** We calculate directly that $Cy = y - y\frac{x^T y}{x^T y} = 0$.

3(b) The column space of this matrix is orthogonal to the vector $x = (1, 0, 1)^T$.

__ T __ A

__ F __ B

__ T __ C

Solution: If x is orthogonal to the columns of M, then $M^T x$=0. We calculate that $A^T x = x - x\frac{x^T x}{x^T x} = 0$ and $C^T x = x - x\frac{y^T x}{x^T y} = 0$ (since $x^T y = y^T x$). On the other hand, we have that $B^T x = x - y\frac{y^T x}{y^T y} \neq 0$ because x and y are independent.

3(c) The column space of this matrix has dimension 2.

___ T ___ A

___ T ___ B

___ T ___ C

Solution: The rank one matrices $\frac{xx^T}{x^Tx}$ and $\frac{yy^T}{y^Ty}$ are orthogonal projections onto the lines spanned by, respectively, x and y in $\mathbb{R}^3$. The matrices A and B are the orthogonal projections onto the orthogonal complement of these two lines, respectively, so their column spaces are both two dimensional. What about C? Any vector orthogonal to the column space of C must satisfy $0 = C^Tv = v - x\frac{y^Tv}{x^Ty}$: in other words, vectors orthogonal to the column space of C are multiples of x. So the dimension of the column space of C is also $3 - 1 = 2$.

3(d) This matrix is symmetric (recall that M is symmetric if $M = M^T$).

___ T ___ A

___ T ___ B

___ F ___ C

Solution: We can calculate directly that $A^T = (I - \frac{xx^T}{x^Tx})^T = I - \frac{(xx^T)^T}{x^Tx} = I - \frac{xx^T}{x^Tx} = A$, and the same calculation holds for B (with x replaced by y). Alternatively, we know that orthogonal projection matrices (like A and B) are symmetric matrices. For C, we can check that $C^T = (I - \frac{yx^T}{x^Ty})^T = I - \frac{(yx^T)^T}{x^Ty} = I - \frac{xy^T}{x^Ty} \neq I - \frac{yx^T}{x^Ty} = C$.

3(e) This matrix is its own square (i.e., a matrix that satisfies $M^2 = M$).

___ T ___ A

___ T ___ B

___ T ___ C

Solution: Since both A and B are orthogonal projection matrices, we know that $A^2 = A$ and $B^2 = B$. For C, we calculate that $R^2 = \frac{yx^T}{x^Ty}\frac{yx^T}{x^Ty} = \frac{yx^T}{x^Ty} = R$, so that

$$C^2 = (I - R)^2 = (I - R)I - (I - R)R = I - R - R + R^2 = I - R.$$

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