LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 10: EXERCISES.

1. Problem 1

a) Find A^TA and AA^T and the singular vectors v_1, v_2, u_1, u_2 for A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution

We have

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Eigenvalues of A^TA are solutions of the equation $(2-\lambda)(1-\lambda)-1 = 0$ i.e. $\lambda^2-3\lambda+1 = 0$ so eigenvalues are $\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}$.

We conclude that

$$\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2},\ \sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{1-\sqrt{5}}{2}.$$

The eigenvector v_1 lies in the nullspace of $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}$ so is collinear to the vector $\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$. It follows that

$$v_1 = \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

The vector v_2 lies in the nullspace of $\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}$ so is collinear to the vector $\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$. It follows that

$$v_2 = \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

b) Find (and check) the SVD decomposition:

$$A = U\Sigma V^T.$$

Recall that matrices U, V should be orthogonal and the matrix Σ is diagonal.

Solution

The answer can be extracted from (a) (using the equations $Av_1 = \sigma_1 u_1$, $Av_2 = \sigma_2 u_2$).

2. Problem 3

Find the SVD factors U and Σ and V^T for

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We have $AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$. We have

$$\det(AA^T - \lambda I) = \lambda^2 - 6\lambda + 4$$

so

$$\lambda_1 = 3 + \sqrt{5},\ \lambda_2 = 3 - \sqrt{5}.$$

It follows that

$$\sigma_1 = \frac{\sqrt{2}}{2}(\sqrt{5}+1),\ \sigma_2 = \frac{\sqrt{2}}{2}(\sqrt{5}-1).$$

We also see that

$$u_1 = \frac{1}{\sqrt{10+2\sqrt{5}}} \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix},\ u_2 = \frac{1}{\sqrt{10+2\sqrt{5}}} \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix}.$$

One can compute v_1, v_2 similarly (note that we already know λ_1, λ_2).

3. Problem 4

(a) For this rectangular matrix find v_1, v_2, v_3 and u_1, u_2 and σ_1, σ_2:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution

We have $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, it has eigenvalues

$$\lambda_1 = 3,\ \lambda_2 = 1,\ \lambda_3 = 0.$$

It follows that

$$\sigma_1 = \sqrt{3},\ \sigma_2 = 1.$$

We see that the eigenvectors of A^TA are

$$v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Using that $Av_i = \sigma_i u_i$ we get $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) Write the SVD for A as $U\Sigma V^T = (2 \times 2)(2 \times 3)(3 \times 3)$.

Solution

Clear from part (a).