LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 9: SOLUTIONS.

1. Problem 1

For the complex number $z = 1 - i$, find $\overline{z}$ and $r = |z|$ and the angle θ.
Solution:
We have
$$\overline{z} = 1 + i,\ r = |z| = \sqrt{1+1} = \sqrt{2}.$$
It follows that
$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = \cos\theta + i\sin\theta$$
so $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$ i.e.
$$\theta = -\frac{\pi}{4}.$$

2. Problem 2

Find the eigenvalues and eigenvectors of the Hermitian matrix
$$S = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$$
(This is a problem from PSet 7, so I will not post the complete solution)

3. Problem 3

If $\overline{Q}^T Q = 1$ (unitary matrix = complex orthogonal) and $Q\mathbf{x} = \lambda\mathbf{x}$, show that $|\lambda| = 1$.
Hint: look at $|Q\mathbf{x}|^2 = Q\mathbf{x} \cdot Q\mathbf{x} = (\overline{Q}\overline{x})^T Q\mathbf{x}$.
(This is a problem from PSet 7, so I will not post the complete solution)

4. Problem 4

(a) Verify Euler's great formula $e^{i\theta} = \cos\theta + i\sin\theta$ using these first terms for
$$e^{i\theta} \text{ is aproximately } 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{6}(i\theta)^3,$$
$$\cos\theta \text{ is aproximately } 1 - \frac{1}{2}\theta^2,\ \sin\theta \text{ is aproximately } \theta - \frac{1}{6}\theta^3.$$
Solution:
It is easy to see that:
$$1 - \frac{1}{2}\theta^2 + i(\theta - \frac{1}{6}\theta^3) = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{6}(i\theta)^3.$$
(b) Find $\cos 2\theta$ and $\sin 2\theta$ using Euler's great formula and $(e^{i\theta})(e^{i\theta}) = (e^{2i\theta})$.
Solution:

We have

$\cos 2\theta + i\sin 2\theta = e^{2i\theta} = e^{i\theta}\cdot e^{i\theta} = (\cos\theta + i\sin\theta)^2 = (\cos\theta)^2 - (\sin\theta)^2 + i\cdot(2\cos\theta\sin\theta)$

so

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2,\ \sin 2\theta = 2\cos\theta\sin\theta$$

as desired.

5. Problem 5

(a) Find the matrix F_3 with orthogonal columns = eigenvectors of

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution:

We see that $\det(P - \lambda\,\mathrm{I}) = -\lambda^3 + 1$ so eigenvalues of P are solutions of the equation $\lambda^3 = 1$. Every solution of the equation $\lambda^n = 1$ has the form $e^{\frac{2\pi ik}{n}} = \cos\frac{2\pi ik}{n} + i\sin\frac{2\pi ik}{n}$ for some $k = 0, 1, \ldots, n-1$. For $n = 3$ we get solutions:

$$\lambda_0 = 1,\ \lambda_1 = \cos\frac{2\pi i}{3} + i\sin\frac{2\pi i}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \lambda_2 = \cos\frac{4\pi i}{3} + i\sin\frac{4\pi i}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Eigenvector with eigenvalue 1 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Let us find eigenvector with eigenvalue $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. We need to solve the equation

$$\begin{bmatrix} \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 \\ 1 & 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We modify our matrix subtract first row times $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ from the third row and get new matrix:

$$\begin{bmatrix} \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 \\ 0 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Now we subtract second row times $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and get matrix:

$$\begin{bmatrix} \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 & 0 \\ 0 & \frac{1}{2} + i\frac{\sqrt{3}}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ should satisfy

$$(\frac{1}{2} + i\frac{\sqrt{3}}{2})x_1 + x_2 = 0,\ (\frac{1}{2} + i\frac{\sqrt{3}}{2})x_2 + x_3 = 0$$

so we have a solution $x_1 = 1$, $x_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $x_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

We conclude that an eigenvector with eigenvalue λ_2 is

$$\begin{bmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Similarly, one can show that an eigenvector with eigenvalue λ_1 is

$$\begin{bmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{bmatrix}$$

(b) Write P as $F_3 \Lambda F_3^{-1}$ for some diagonal matrix Λ.

Solution:

Form part (a) we see that:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} + i\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}^{-1}$$

6. Problem 6

If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the — and — roots of 1.

Solution:

If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the $64/2 = 32$ and $64 \cdot 2 = 128$ roots of 1.