LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 11: SOLUTIONS.

1. Problem 1

Find CR decomposition for a matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 7 \\ 1 & 3 & 5 \end{bmatrix}.$$

Solution:

The first two columns c_1, c_2 of A are linearly independent, the third one c_3 is equal to $2c_1 + c_2$. We conclude that:

$$C = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

2. Problem 2

A and B are symmetric across the diagonal. Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

Remark 2.1. *Recall that LDU factorization is the factorization into lower triangular matrix L with 1's on the diagonal, diagonal matrix D and upper triangular matrix U with 1's on the diagonal.*

Solution:

We have

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \right) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU.$$

We see that $L = U^T$.

For the matrix B we have

$$\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}B=\begin{bmatrix}1&4&0\\0&-4&4\\0&0&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-4&0\\0&0&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix}.$$

We conclude that

$$B=\begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-4&0\\0&0&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\4&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-4&0\\0&0&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix}.$$

We see that $L=U^T$.

Note now that if A is any symmetric matrix (i.e. $A^T=A$) and $A=LDU$ is its LDU decomposition, then $L=U^T$. To see that, note that $A^T=U^TDL^T$ is the LDU decomposition of A^T. From the uniqueness of LDU decompositions (together with $A=A^T$) it follows that $L=U^T$.

3. Problem 3

Find the height C of the best *horizontal line* to fit $\mathbf{b}=(0,8,8,20)$. An exact fit would solve the unsolvable equations $C=0, C=8, C=8, C=20$. Find the 4 by 1 matrix A in these equations and solve $A^TA\hat{x}=A^T\mathbf{b}$. Draw the horizontal line at height $\hat{x}=C$ and the four errors in $\mathbf{e}$.

Solution:

The unsolvable equations $C=0, C=8, C=8, C=20$ are equivalent to $\begin{bmatrix}C\\C\\C\\C\end{bmatrix}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}$ i.e. that

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}C\end{bmatrix}=\begin{bmatrix}0\\8\\8\\20\end{bmatrix}.$$

We see that $A=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $A^TA=4$ and $A^T\mathbf{b}=36$. It remains to solve the equation $4C=36$. We see that $C=9$ so $A\begin{bmatrix}C\end{bmatrix}=\begin{bmatrix}9\\9\\9\\9\end{bmatrix}$.

The error $\mathbf{e}$ is equal to

$$\mathbf{e} = \mathbf{b} - A\left[C\right] = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} -9 \\ -1 \\ -1 \\ 11 \end{bmatrix}.$$

4. Problem 4

(a) Find orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ such that $\mathbf{q}_1$, $\mathbf{q}_2$ span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Solution:

We use the Gram-Schmidt orthogonalization process. Start from $x_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

We get

$$q_1 = \frac{x_1}{||x_1||} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \; q_2' = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + 3q_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \; q_2 = \frac{q_2'}{||q_2'||} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

To find q_3, note that we can apply Gram-Schmidt to the basis q_1, q_2, x_3, where $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (note that $\det(q_1 \; q_2 \; x_3) \neq 0$ so q_1, q_2, x_3 indeed form a basis of $\mathbb{R}^3$).

We have

$$q_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{3}q_1 - \frac{2}{3}q_2 = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{1}{9} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \; q_3 = \frac{q_3'}{||q_3'||} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Which of the four fundamental spaces contains $\mathbf{q}_3$?
Recall that $C(A)^{\perp} = N(A^T)$ so $q_3 \in N(A^T)$.
Solution:
(c) Solve $A\mathbf{x} = (1, 2, 7)$ by least squares.
Solution:

We need to solve the equation $A^T A\hat{x} = A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 \\ 27 \end{bmatrix}$.

Let us write a QR-decomposition of the matrix A: it follows from (a) that we have

$$A = QR = \left(\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}\right)\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}.$$

We have $A^T A = R^T Q^T Q R = R^T R = \begin{bmatrix} 3 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}$.

So we are solving the equation:

$$\begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix} \hat{x} = \begin{bmatrix} -9 \\ 27 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

We have

$$\hat{x} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$