Review of Chapter 2—Square n by n matrices A

1. 3 possibilities for $A\boldsymbol{x} = \boldsymbol{b}$

(a) A has rank n $\quad A^{-1}$ exists $\Leftrightarrow$ always $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

(b) A has rank $< n \Leftrightarrow A\boldsymbol{x} = \boldsymbol{b}$ has no solutions for most $\boldsymbol{b}$

(c) A has rank $< n \Leftrightarrow A\boldsymbol{x} = \boldsymbol{b}$ has ∞ solutions if $\boldsymbol{b}$ is in $\mathbf{C}(A)$

2. Computational steps when A has full rank n

(a) Elimination matrix E_{ij} subtracts a multiple of row j from row $i > j$

(b) Steps $E_{21}, E_{31}, \ldots, E_{n1}$ produce zeros in column 1 below pivot $u_{11} = a_{11}$

(c) Steps $E_{32}, \ldots, E_{n2}$ produce zeros in column 2 below new pivot u_{22}

(d) Lower triangular E (product of $E_{ij}, i > j$) produces $EA = U =$ upper triangular

(e) Lower triangular $L = E^{-1} =$ product of E_{ij}^{-1} in reverse order $j = n-1, \ldots, 1$

(f) $\boldsymbol{A} = \boldsymbol{LU} =$ (**lower triangular**) (**upper triangular**) (**2 proofs**)

3. Row exchanges $\rightarrow$ Permutation matrices $\boldsymbol{P}$

(a) P has the rows of I in any of the $n!$ possible orders

(b) P is **even** or **odd**: product of even or odd number of simple row exchanges

(c) Use exchanges to get nonzero pivots (and larger pivots). Now $\boldsymbol{PA} = \boldsymbol{LU}$

4. Inverse matrix $\quad A^{-1}A = I$ and $AA^{-1} = I$ and $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

(a) **Invertible matrix** $\Leftrightarrow n$ independent columns (and rows)

(b) $A\boldsymbol{x} = \boldsymbol{b}$ has 1 solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ for every $\boldsymbol{b}$: $\boldsymbol{A^{-1}}$ **is a slow way to** $\boldsymbol{x}$!

(c) A and B are invertible n by $n \Rightarrow \boldsymbol{(AB)^{-1} = B^{-1}A^{-1}}$

(d) Elimination on A has n pivots $\neq 0$ (possibly after row exchanges)

(e) Determinant of $A = \pm$ product of the n pivots (not zero!)

5. By hand: Add $\boldsymbol{b}$ as column $n+1$; elimination gives $U\boldsymbol{x} = \boldsymbol{c}$; backsubstitution gives $\boldsymbol{x}$.
$A\boldsymbol{x} = \boldsymbol{b}$ is solved by $\boldsymbol{x} = \boldsymbol{b} \backslash \boldsymbol{A} =$ backslash in MATLAB
Operation count for L and U: $\frac{1}{3}n^3$ multiply-subtract steps
Operation count for $\boldsymbol{x}$: n^2 steps for each right side $\boldsymbol{b}$

6. Transpose matrix $\quad (A^{\mathrm{T}})_{ij} = A_{ji}$ (n by m) and $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$
Symmetric matrix $\quad S^{\mathrm{T}} = S$. Note $S = A^{\mathrm{T}}A$ is always symmetric!