LINEAR ALGEBRA. VASILY KRYLOV. RECITATION 3: EXERCISES.

My name is Vasily Krylov. If you have any questions or comments, please feel free to ask me by email (krvas@mit.edu) or during my office hours (Thursday 5p.m. - 7 p.m. Room 2-361).

1. Problem 1

(a) Does the set of all 3×3 matrices of rank 3 form a vector space?
(b) Does the set of all 3×3 matrices of rank $\leqslant 2$ form a vector space?
Solution of the part (a): NO.

Proof. Zero matrix does not lie in the set of 3×3 matrices of rank 3. $\square$

Solution of the part (b): NO.

Proof. Let $\mathbf{S}$ be the set of 3×3 matrices of rank 3. Take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

. We have $A, B \in \mathbf{S}$ but $A + B = \mathrm{Id} \notin \mathbf{S}$. $\square$

2. Problem 2

Let $\mathbf{M}$ be the vector space of all 2×2 matrices.
(a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
(b) If a subspace of $\mathbf{M}$ does contain A and B, must it contain the identity matrix?
Solution of the part (a): take the space $V = \left\{ \begin{bmatrix} b & 2b \\ 0 & b \end{bmatrix} \mid b \in \mathbb{R} \right\}$.
Solution of the part (b): YES.

Proof. Let $V \subset \mathbf{M}$ be a subspace that contains A, B. Note that

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A - 2B.$$

We conclude that $\mathrm{Id} = A - 2B \in V$. $\square$

3. Problem 3

For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Solution of the part (a): for $b_1 \in \mathbb{R}$, $b_2 = 2b_1$, $b_3 = -b_1$.

The set of possible $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ coincides with the vector space $C(A)$ that is generated by columns of $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}$. Note that

$$\begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

so the space $C(A)$ is generated by the first column of the matrix A. We conclude that $C(A) = \Big\{ b \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \mid b \in \mathbb{R} \Big\} = \Big\{ \begin{bmatrix} b \\ 2b \\ -b \end{bmatrix} \mid b \in \mathbb{R} \Big\}$.

Solution of the part (b): for $b_1, b_2 \in \mathbb{R}$ and $b_3 = -b_1$.

Our goal is to describe the vector space $C(A)$ for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}$.

Set $\mathbf{v}_1 := \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 := \begin{bmatrix} 4 \\ 9 \\ -4 \end{bmatrix}$. Note that $\mathbf{v}_2 - 4\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so $\mathbf{v}_3 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in C(A)$. Note also that $\mathbf{v}_1 - 2\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $\mathbf{v}_4 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in C(A)$. Every vector of $C(A)$ is a linear combination of $\mathbf{v}_3, \mathbf{v}_4$ i.e. has the form

$$p \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + q \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} q \\ p \\ -q \end{bmatrix}, \ p, q \in \mathbb{R}.$$

So we must have $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} q \\ p \\ -q \end{bmatrix}$ for some $p, q \in \mathbb{R}$ i.e. $b_1 = q = -b_3$ and $b_2 = p$ is arbitrary.

4. Problem 4

Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$.

Solution: $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$**.**

Proof. We will find this matrix in form

$$A = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{bmatrix}.$$

Our goal is to find $a, b, c \in \mathbb{R}$ such that $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$. We have

$$A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+a \\ 1+b \\ c \end{bmatrix}.$$

The equation $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ is equivalent to the system of equations

$$\begin{cases} 1+a=0 \\ 1+b=0 \\ c=0 \end{cases}$$

that has the unique solution $a = b = -1$, $c = 0$. We conclude that the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ works. $\square$