MIT 18.06 Exam 1, Spring 2023

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Your name: ______________________________
(*printed*)

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Recitation: ______________________________

Problem 1 (12+10+12=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

1(a) Compute the factorization $A = CR$ of the 4×4 matrix A. As usual, work from left to right until C contains a full set of linearly independent columns of A.

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix} = CR.$$

$$C = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \end{pmatrix} \qquad R = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \end{pmatrix}$$

1(b) True or false (Write T or F in the blank space provided).

______ The row space of A is a 2D plane.

______ For every 4×1 vector b, $Ax = b$ has a unique solution.

______ The column space and row space of A are the same.

1(c) Compute the **first** column of the matrix $B^{-1}A$:

$$B^{-1}A = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}}_{B^{-1}} \underbrace{\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & -2 & 0 \end{pmatrix}}_{A}$$

HINT: Solve $Bx = a_1$,where a_1 is the first column of A, so that $x = B^{-1}a_1 =$ (first column of $B^{-1}A$).

$$(\text{first column of } B^{-1}A) = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

(blank page for your work if you need it)

Problem 2 (18+14+2=34 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

The bottom row of a 3×3 matrix A depends on a positive parameter p:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 2 & p & p \end{pmatrix}$$

Compute the factorization $A = LU$ in the following steps.

2(a) Write down elimination matrices E_{21}, E_{31}, and E_{32} that introduce zeros in the $(2,1)$, $(3,1)$, and $(3,2)$ entries so that $E_{32}E_{31}E_{21}A = U$ is upper triangular. The elimination matrices may depend on the parameter p.

$$E_{21} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad E_{31} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad E_{32} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

2(b) Write down the lower and upper triangular factors L and U that multiply to make $A = LU$. The triangular factors may depend on the parameter p.

$$L = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad U = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

2(c) If p is allowed to be negative, A is not invertible when $p =$ ________.

(blank page for your work if you need it)

Problem 3 (15+15+2=32 points):

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

This upper triangular matrix A is not invertible:

$$A = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

Show in two ways that A has no inverse.

3(a) Find a nontrivial combination of the columns that produces the zero vector, that is, find numbers x_1, x_2, x_3, and x_4 (not all equal to zero) such that

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ 6 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$x_1 =$ $\qquad$ $x_2 =$ $\qquad$ $x_3 =$ $\qquad$ $x_4 =$

3(b) Find a column vector $b = (b_1, b_2, b_3, b_4)^T$ so that $Ax = b$ has no solution $x = (x_1, x_2, x_3, x_4)^T$.

$b_1 =$ $\qquad$ $b_2 =$ $\qquad$ $b_3 =$ $\qquad$ $b_4 =$

3(c) In general, an upper triangular matrix is invertible exactly when its ____________ entries are ____________.

(blank page for your work if you need it)