

**Problem 1.**

In an industrial application a part is to turn 20 degrees about a rod shown below (in the direction indicated) and move down six inches along the same rod.

- Please determine a single 4x4 matrix transformation that can be used to compute the new coordinates of an arbitrary point on the part.
- Using the results of part a, please determine the new world coordinates of a point whose original coordinates were X, Y, Z.

Given

$$P = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix},$$

$$w = \frac{P - O}{\|P - O\|} = \begin{bmatrix} 0 \\ -.8 \\ .6 \end{bmatrix}$$

$$\underline{\underline{\Theta}} = \underline{\underline{I}} + \underline{\underline{w}} \sin \phi + \underline{\underline{w}}^2 (1 - \cos \phi)$$

$$= \begin{bmatrix} (w_x^2 - 1)(1 - \cos \theta) + 1 & w_x w_y (1 - \cos \theta) - w_z \sin \theta & w_x w_z (1 - \cos \theta) + w_y \sin \theta \\ w_x w_y (1 - \cos \theta) + w_z \sin \theta & (w_y^2 - 1)(1 - \cos \theta) + 1 & -w_x \sin \theta + w_y w_z (1 - \cos \theta) \\ w_x w_z (1 - \cos \theta) - w_y \sin \theta & w_x \sin \theta + w_y w_z (1 - \cos \theta) & (w_z^2 - 1)(1 - \cos \theta) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 \\ 0.2052 & 0.9782 & -0.0289 \\ 0.2736 & -0.0289 & 0.9614 \end{bmatrix}$$

$$d = (\underline{\underline{I}} - \underline{\underline{\Theta}}) \underline{\underline{P}} + \phi \underline{\underline{w}}$$

$$= \begin{bmatrix} \phi w_x - P_x(w_x^2 - 1)(1 - \cos \theta) + P_y(-w_x w_y (1 - \cos \theta) + w_z \sin \theta) + P_z(-w_x w_z (1 - \cos \theta) - w_y \sin \theta) \\ \phi w_y + P_x(-w_x w_y (1 - \cos \theta) - w_z \sin \theta) - P_y(w_y^2 - 1)(1 - \cos \theta) + P_z(w_x \sin \theta - w_y w_z (1 - \cos \theta)) \\ \phi w_z + P_x(-w_x w_z (1 - \cos \theta) + w_y \sin \theta) + P_y(-w_x \sin \theta - w_y w_z (1 - \cos \theta)) - P_z(w_z^2 - 1)(1 - \cos \theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1.6416 \\ -4.6263 \\ 3.8315 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 & 1.6416 \\ 0.2052 & 0.9782 & -0.0289 & -4.6263 \\ 0.2736 & -0.0289 & 0.9614 & 3.8315 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9396X - 0.2052Y - 0.2736Z + 1.6416 \\ 0.2052X + 0.9782Y - 0.0289Z - 4.6263 \\ 0.2736X - 0.0289Y + 0.9614Z + 3.8315 \\ 1 \end{bmatrix}$$

## Problem 2.

Complete the derivation of the Denavit-Hartenberg (D-H) Transformation from what was done in the classroom using the joint and Shape matrices based on the diagram and the derivation started in the class on 2-1-2018.

$$\begin{aligned} \underline{T}_{h-,h+} &= \underline{I} \Phi_h \underline{S}_{h+1,h}^{-1} \\ \Phi_h &= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 & 0 & 0 \\ \sin \phi_h^1 & \cos \phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \underline{\Theta} &= \underline{I} + \underline{\tilde{w}} \sin \phi + \underline{\tilde{w}}^2 (1 - \cos \phi) \\ \underline{\tilde{w}} &= \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \end{aligned}$$

To solve for  $\underline{S}_{h+1,h}^{-1}$ , we define the transformation:

Fixed coordinate system:  $(uvw)_{h'}$

Moving coordinate system:  $(uvw)_{h+1}$

Screw axis:  $u'_h, w = (1, 0, 0)$

$\underline{P} = \underline{0}$

$s = a_h, u$  direction

$\phi = \alpha_h$

$$\begin{aligned}
\underline{\underline{\Theta}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin \alpha_h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} (1 - \cos \alpha_h) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_h & -\sin \alpha_h \\ 0 & \sin \alpha_h & \cos \alpha_h \end{bmatrix} \\
S_{h+,h}^{-1} &= \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos \alpha_h & -\sin \alpha_h & 0 \\ 0 & \sin \alpha_h & \cos \alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_{\underline{\underline{h-}},h+} &= I \Phi_h S_{\underline{\underline{h+1}},h}^{-1} \\
&= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 & 0 & 0 \\ \sin \phi_h^1 & \cos \phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos \alpha_h & -\sin \alpha_h & 0 \\ 0 & \sin \alpha_h & \cos \alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 \cos \alpha_h & \sin \alpha_h \sin \phi_h^1 & a_h \cos \phi_h^1 \\ \sin \phi_h^1 & \cos \alpha_h \cos \phi_h^1 & -\sin \alpha_h \cos \phi_h^1 & a_h \sin \phi_h^1 \\ 0 & \sin(\alpha_h) & \cos(\alpha_h) & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

### Problem 3.

Consider the robot manipulator shown below.

- Determine the D-H parameters for the robot and the D-H transformation for each joint.
- Derive the kinematic equations for the coordinates of a point at the tip of the last link (XYZ) in terms of the joint variables.
- Determine the inverse kinematic solution.

	1	2	3
$a_i$	0	0	$r$
$\alpha_i$	0	$180^\circ$	0
$\theta_i$	$\theta$	0	$180^\circ$
$s_i$	0	$h$	0

$$\underline{T}_{12} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{T}_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{T}_{34} = \begin{bmatrix} -1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \underline{x}_1 &= \underline{T}_{12} \underline{T}_{23} \underline{T}_{34} \underline{x}_4 \\ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -\cos \theta & -\sin \theta & 0 & r \cos \theta \\ -\sin \theta & \cos \theta & 0 & r \sin \theta \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} -X \cos \theta - Y \sin \theta + r \cos \theta \\ -X \sin \theta + Y \cos \theta + r \sin \theta \\ Z - h \\ 1 \end{bmatrix} \end{aligned}$$

Which gives us three equations. The third row is immediately solvable,

$$\boxed{Z = h}$$

Setting rows one and two to zero, and then squaring yields

$$\begin{aligned} (-X + r)^2 \cos^2 \theta &= Y^2 \sin^2 \theta \\ (-X + r)^2 \sin^2 \theta &= Y^2 \cos^2 \theta \end{aligned}$$

Adding these and taking advantage of trigonometric identities

$$\begin{aligned} (-X + r)^2 (\cos^2 \theta + \sin^2 \theta) &= Y^2 (\cos^2 \theta + \sin^2 \theta) \\ Y &= -X + r \end{aligned}$$

Substituting this back in row 2 yields

$$\begin{aligned} 0 &= -X \sin \theta + (-X + r) \cos \theta + r \sin \theta \\ 0 &= -X (\sin \theta + \cos \theta) + r (\cos \theta + \sin \theta) \end{aligned}$$

$$\boxed{X = r}$$

Finally, plugging this result into row 1 yields

$$0 = -r \cos \theta - Y \sin \theta + r \cos \theta$$

$$Y = \frac{1}{\sin \theta}$$

$$\boxed{Y = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}}$$