SSRMS

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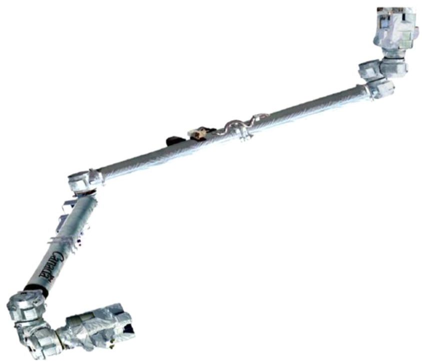
Contents

1 Introduction

a



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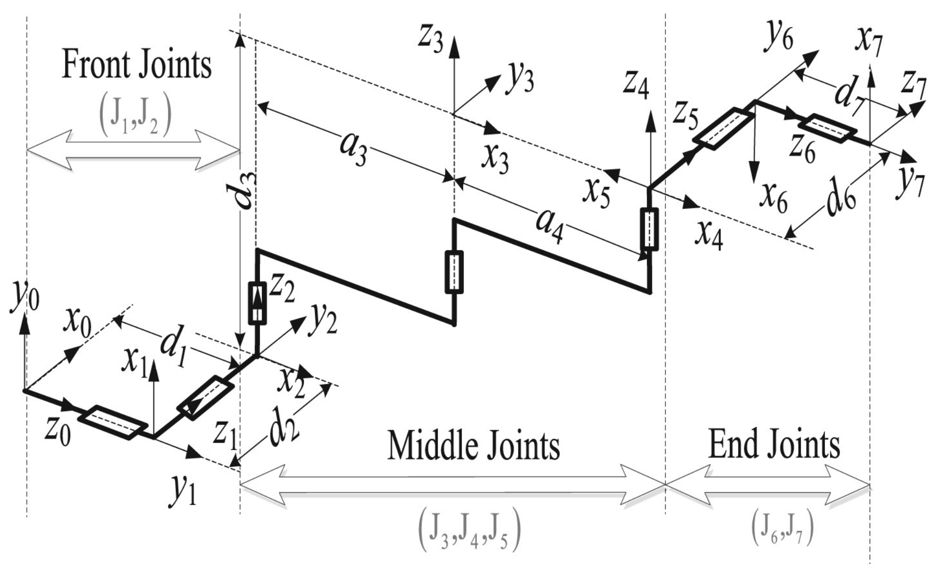

We're going to play with a shoulder roll locked SSRMS.

i	θ_i	α_i	a_i	d_i
1	90	90	0	d_1
2	90	90	0	d_2
3	0	0	a_3	d_3
4	0	0	a_4	0
5	180	90	0	0
6	-90	90	0	d_6
7	180	90	0	d_7

Table 1: The Denavit-Hartenberg parameters for the SSRMS. These parameters are the joint angle, θ, the link twist angle, α, the link length, a, and the joint offset, d. These θ_is give the initial or "zero-displacement" configuration, but each θ_i is modeled as an individual variable below.

2 Finite Kinematic Analysis

2.1 Denavit-Hartenberg Parameters

The resulting matrices are therefore

$$T_{01} = \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{12} = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & 0 \\ \sin(\theta_2) & 0 & -\cos(\theta_2) & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{34} = \begin{bmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & a_4\cos(\theta_4) \\ \sin(\theta_4) & \cos(\theta_4) & 0 & a_4\sin(\theta_4) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{45} = \begin{bmatrix} \cos(\theta_5) & 0 & \sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & -\cos(\theta_5) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{56} = \begin{bmatrix} \cos(\theta_6) & 0 & \sin(\theta_6) & 0 \\ \sin(\theta_6) & 0 & -\cos(\theta_6) & 0 \\ 0 & 1 & 0 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{67} = \begin{bmatrix} \cos(\theta_7) & 0 & \sin(\theta_7) & 0 \\ \sin(\theta_7) & 0 & -\cos(\theta_7) & 0 \\ 0 & 1 & 0 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And multiplying all of these together yields

$$
\begin{aligned}
T_{07} =& T_{01}T_{12}T_{23}T_{34}T_{45}T_{56}T_{67} \qquad (1)\\
T_{07}[1,1] =& \left(-s_1c_{345} + s_{345}c_1c_2\right)s_7 + \left(s_1s_{345}c_6 + s_2s_6c_1 + c_1c_2c_6c_{345}\right)c_7\\
T_{07}[1,2] =& s_1s_6s_{345} - s_2c_1c_6 + s_6c_1c_2c_{345}\\
T_{07}[1,3] =& \left(s_1c_{345} - s_{345}c_1c_2\right)c_7 + \left(s_1s_{345}c_6 + s_2s_6c_1 + c_1c_2c_6c_{345}\right)s_7\\
T_{07}[1,4] =& a_3s_1s_3 + a_3c_1c_2c_3 + a_4s_1s_{34} + a_4c_1c_2c_{34} + d_2s_1 + d_3s_2c_1 - d_6s_1c_{345} + d_6s_{345}c_1c_2\\
&+ d_7s_1s_6s_{345} - d_7s_2c_1c_6 + d_7s_6c_1c_2c_{345}\\
T_{07}[2,1] =& \left(s_1s_{345}c_2 + c_1c_{345}\right)s_7 + \left(s_1s_2s_6 + s_1c_2c_6c_{345} - s_{345}c_1c_6\right)c_7\\
T_{07}[2,2] =& -s_1s_2c_6 + s_1s_6c_2c_{345} - s_6s_{345}c_1\\
T_{07}[2,3] =& -\left(s_1s_{345}c_2 + c_1c_{345}\right)c_7 + \left(s_1s_2s_6 + s_1c_2c_6c_{345} - s_{345}c_1c_6\right)s_7\\
T_{07}[2,4] =& a_3s_1c_2c_3 - a_3s_3c_1 + a_4s_1c_2c_{34} - a_4s_{34}c_1 - d_2c_1 + d_3s_1s_2 + d_6s_1s_{345}c_2 + d_6c_1c_{345}\\
&- d_7s_1s_2c_6 + d_7s_1s_6c_2c_{345} - d_7s_6s_{345}c_1\\
T_{07}[3,1] =& \left(s_2c_6c_{345} - s_6c_2\right)c_7 + s_2s_7s_{345}\\
T_{07}[3,2] =& s_2s_6c_{345} + c_2c_6\\
T_{07}[3,3] =& \left(s_2c_6c_{345} - s_6c_2\right)s_7 - s_2s_{345}c_7\\
T_{07}[3,4] =& a_3s_2c_3 + a_4s_2c_{34} + d_1 - d_3c_2 + d_6s_2s_{345} + d_7s_2s_6c_{345} + d_7c_2c_6\\
T_{07}[4,1] =& 0\\
T_{07}[4,2] =& 0\\
T_{07}[4,3] =& 0\\
T_{07}[4,4] =& 1
\end{aligned}
$$

2.2 Joint/Shape Matrices

We can similarly use joint and shape matrices to arrive at these T matrices. All of the joints of the SSRMS are revolute, and can be modeled with the joint matrix of

$$
\Phi_h\left(\phi_h\right) = \begin{bmatrix} \cos\phi_h & -\sin\phi_h & 0 & 0\\ \sin\phi_h & \cos\phi_h & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
T_{i,i+1} &= S_{i,j}\Phi_j S_{i+1,j}^{-1} \\
T_{12} &= S_{1A}\Phi_A S_{2A}^{-1} \\
T_{23} &= S_{2B}\Phi_B S_{3B}^{-1} \\
T_{34} &= S_{3C}\Phi_C S_{4C}^{-1} \\
T_{45} &= S_{4D}\Phi_D S_{5D}^{-1} \\
T_{56} &= S_{5E}\Phi_E S_{6E}^{-1} \\
T_{67} &= S_{6F}\Phi_F S_{7F}^{-1} \\
T_{78} &= S_{7G}\Phi_G S_{8G}^{-1}
\end{aligned}
$$

For joints $\Phi_A, \Phi_B, \Phi_C, \Phi_D, \Phi_E, \Phi_F$, and Φ_G, we also define two shape matrices.

$$T_{01} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{1A} = I, S_{2A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{2B} = I, S_{3B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{3C} = I, S_{4C} = \begin{bmatrix} 1 & 0 & 0 & -a_3 \\ 0 & 0 & 1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{4D} = I, S_{5D} = \begin{bmatrix} 1 & 0 & 0 & -a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{5E} = I, S_{6E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{6F} = I, S_{7F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{7G} = I, S_{8G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_7 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3 Inverse Kinematics Solution

In general we can define

$$\begin{aligned} T_{07} &= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= T_{01}T_{12}T_{23}T_{34}T_{45}T_{56}T_{67} \end{aligned}$$

Premultiplying both sides by T_{01}^{-1} yields,

$$T_{01}^{-1}T_{07} = T_{12}T_{23}T_{34}T_{45}T_{56}T_{67}$$

Equating each element (i, j) on both the left and right hand sides yields:

$$\begin{aligned}
n_x c_1 + n_y s_1 &= (s_2 s_6 + c_2 c_6 c_{345})\, c_7 + s_7 s_{345} c_2 &(2)\\
o_x c_1 + o_y s_1 &= -s_2 c_6 + s_6 c_2 c_{345} &(3)\\
a_x c_1 + a_y s_1 &= (s_2 s_6 + c_2 c_6 c_{345})\, s_7 - s_{345} c_2 c_7 &(4)\\
p_x c_1 + p_y s_1 &= a_3 c_2 c_3 + a_4 c_2 c_{34} + d_3 s_2 + d_6 s_{345} c_2 - d_7 s_2 c_6 + d_7 s_6 c_2 c_{345} &(5)\\
n_z &= (s_2 c_6 c_{345} - s_6 c_2)\, c_7 + s_2 s_7 s_{345} &(6)\\
o_z &= s_2 s_6 c_{345} + c_2 c_6 &(7)\\
a_z &= (s_2 c_6 c_{345} - s_6 c_2)\, s_7 - s_2 s_{345} c_7 &(8)\\
-d_1 + p_z &= a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6 &(9)\\
n_x s_1 - n_y c_1 &= -s_7 c_{345} + s_{345} c_6 c_7 &(10)\\
o_x s_1 - o_y c_1 &= s_6 s_{345} &(11)\\
a_x s_1 - a_y c_1 &= s_7 s_{345} c_6 + c_7 c_{345} &(12)\\
p_x s_1 - p_y c_1 &= a_3 s_3 + a_4 s_{34} + d_2 - d_6 c_{345} + d_7 s_6 s_{345} &(13)\\
0 &= 0 &(14)\\
0 &= 0 &(15)\\
0 &= 0 &(16)\\
1 &= 1 &(17)
\end{aligned}$$

where we have defined $s_i = \sin i, c_i = \cos i, s_{ij} = \sin(i+j), c_{ij} = \cos(i+j), s_{ijk} = \sin(i+j+k)$ and $c_{ijk} = \cos(i+j+k)$. Manipulating the equations, we take $(Eq.\ 2)\, s_2 - (Eq.\ 6)\, c_2$ and simplify, producing

$$\left(n_x c_1 + n_y s_1\right) s_2 - n_z c_2 = s_6 c_7 \tag{18}$$

Similarly, we can do $(Eq.\ 5)\, s_2 - (Eq.\ 9)\, c_2$ and simplify, which results in

$$\left(p_x c_1 + p_y s_1\right) s_2 - (-d_1 + p_z)\, c_2 = d_3 - c_6 d_7 \tag{19}$$

We can also subtract $(Eq.\ 7)\, c_2 - (Eq.\ 3)\, s_2$

$$o_z c_2 - \left(o_x c_1 + o_y s_1\right) s_2 = c_6 \tag{20}$$

Finally, we can also subtract $(Eq.\ 4)\, s_2 - (Eq.\ 8)\, c_2$

$$\left(a_x c_1 + a_y s_1\right) s_2 - a_z c_2 = s_6 s_7 \tag{21}$$

Rearranging Equations 19 and 20 to be equal to c_6 and equating the two yields

$$-d_3 = \Big((o_x d_7 - p_x)\, c_1 + \left(o_y d_7 - p_y\right) s_1 \Big)\, s_2 + (-o_z d_7 - d_1 + p_z)\, c_2 \tag{22}$$

Locking the shoulder roll angle to a known angle, $\boxed{\theta_1 = \beta}$, we can solve for θ_2,

$$\boxed{\theta_2 = \text{SHOULDER} \cdot \text{acos}\left(\frac{d_3}{\sqrt{h_1^2 + q_1^2}}\right) + \text{atan2}(q_1, h_1)} \tag{23}$$

where

$$h_1 = (-o_z d_7 - d_1 + p_z) \tag{24}$$
$$q_1 = \Big((o_x d_7 - p_x)\, c_\beta + \big(o_y d_7 - p_y\big)\, s_\beta\Big) \tag{25}$$

With θ_1 and θ_2 now known, θ_6 can be solved using Equation 20,

$$\boxed{\theta_6 = \text{WRIST} \cdot \text{acos}\Big(o_z c_2 - \big(o_x c_1 + o_y s_1\big)\, s_2\Big)} \tag{26}$$

And we can then combine Equations 18 and 21, yielding

$$\boxed{\theta_7 = \text{atan2}\left(\frac{\big(n_x c_1 + n_y s_1\big)\, s_2 - n_z c_2}{s_6}, \frac{\big(a_x c_1 + a_y s_1\big)\, s_2 - a_z c_2}{s_6}\right)} \tag{27}$$

With the shoulder and wrist joints resolved, we can now solve for the middle joints. We now take

$$\left(T_{12}^{-1}\right)(T_{17})\left(T_{67}^{-1}\right)\left(T_{56}^{-1}\right) = (T_{23})\,(T_{34})\,(T_{45})$$

Taking the left and right hand side $(1,4)$ and $(2,4)$ elements from the resulting matrix yields

$$\begin{aligned}
a_3 c_3 + a_4 c_{34} = {} & d_6\Big(a_z s_2 + c_2\big(a_x c_1 + a_y s_1\big)\Big)\, c_7 \\
& - d_6\Big(n_z s_2 + c_2\big(n_x c_1 - n_y c_1\big)\Big)\, s_7 \\
& - d_7\Big(o_z s_2 + c_2\big(o_x c_1 + o_y s_1\big)\Big) \\
& + (-d_1 + p_z)\, s_2 + c_2\big(p_x c_1 + p_y s_1\big)
\end{aligned} \tag{28}$$

$$\begin{aligned}
a_3 s_3 + a_4 s_{34} = {} & -d_2 + d_6\big(a_x s_1 - a_y c_1\big)\, c_7 - d_6\big(n_x s_1 - n_y c_1\big)\, s_7 \\
& - d_7\big(o_x s_1 - o_y c_1\big) + p_x s_1 - p_y c_1
\end{aligned} \tag{29}$$

θ_4 is then solved by combining the above two equations, resulting in

$$\boxed{\theta_4 = \text{ELBOW} \cdot \text{acos}\left(\frac{X^2 + Y^2 - a_3^2 - a_4^2}{2a_3 a_4}\right)} \tag{30}$$

where

$$X = -d_7 \Big(o_z s_2 + c_2 \left(o_x c_1 + o_y s_1 \right) \Big) + \left(-d_1 + p_z \right) s_2 + c_2 \left(p_x c_1 + p_y s_1 \right)$$
$$Y = -d_2 + d_6 \left(a_x s_1 - a_y c_1 \right) c_7 - d_6 \left(n_x s_1 - n_y c_1 \right) s_7 - d_7 \left(o_x s_1 - o_y c_1 \right) + p_x s_1 - p_y c_1$$

Substituting the solution into θ_4 and Equations 28 and 29 and combining yields

$$\boxed{\theta_3 = \operatorname{atan2}\Big(Y \left(a_3 + a_4 c_4 \right) - X a_4 s_4, X \left(a_3 + a_4 c_4 \right) + Y a_4 s_4 \Big)}$$

Subtracting $(Eq.12)c_7$ and $(Eq.10)s_7$ yields

$$c_{345} = \left(a_x s_1 - a_y c_1 \right) c_7 - \left(n_x s_1 - n_y c_1 \right) s_7$$

And from Equation 11 we have

$$s_{345} = \frac{o_x s_1 - o_y c_1}{s_6}$$

which we can combine to solve for the last joint

$$\theta_5 = \left(\theta_3 + \theta_4 + \theta_5 \right) - \left(\theta_3 + \theta_4 \right)$$
$$\boxed{\theta_5 = \operatorname{atan2}\left(s_{345}, c_{345} \right) - \left(\theta_3 + \theta_4 \right)}$$

2.4 Numerical Example

For practical purposes, the link length and offset values can be set to

$$a_3 = 2.30, a_4 = 2.30, d_1 = 0.65, d_2 = 0.30$$
$$d_3 = 0.90, d_6 = 0.30, d_7 = 0.65$$

As an example, plugging in these values and the initial angles given in Table 1 into Equation 1 yields

$$T_{07} = \begin{bmatrix} 0 & 0 & 1 & 0.6 \\ 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 5.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Differential Kinematic Analysis

3.1 Method 1: Kinematic Jacobian

Where $\hat{z}_i$ is taken from the last column of T_{1i}, and can be defined

$$T_{1i} = \begin{bmatrix} & \underline{\underline{\Theta}}_i & & \vdots & a_i \\ & \cdots & & & \cdots \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\underline{\underline{\Theta}}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}$$

$$\hat{z}_i = \left(\prod_{i=1}^{n} \underline{\underline{\Theta}}_i \right) z_i$$

and $\vec{r}_i$ is defined

$$\vec{r}_i = \sum_{i=1}^{n} \vec{a}_i$$

With these definitions, we can find the Jacobian via

$$\dot{\vec{P}} = \sum_{i=1}^{n} \left(\hat{z}_i \times \vec{r}_i \right) \dot{\theta}_i$$

$$\vec{w} = \sum_{i=1}^{n} \dot{\theta}_i \hat{z}_i$$

$$\underline{\underline{J}} \dot{q} = \begin{bmatrix} \underline{\dot{P}} \\ \underline{\vec{w}} \end{bmatrix}$$

$$\begin{bmatrix} \hat{z}_1 \times \vec{r}_1 & \hat{z}_2 \times \vec{r}_2 & \cdots & \hat{z}_7 \times \vec{r}_7 \\ \hat{z}_1 & \hat{z}_2 & \cdots & \hat{z}_7 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_7 \end{bmatrix} = \begin{bmatrix} \underline{\dot{P}}_{EE} \\ \underline{w}_{EE} \end{bmatrix}$$

$$
\begin{aligned}
J[1,1] &= -a_3 s_1 c_2 c_3 + a_3 s_3 c_1 - a_4 s_1 c_2 c_{34} + a_4 s_{34} c_1 + d_2 c_1 - d_3 s_1 s_2 \\
&\quad - d_6 s_1 s_{345} c_2 - d_6 c_1 c_{345} + d_7 s_1 s_2 c_6 - d_7 s_1 s_6 c_2 c_{345} + d_7 s_6 s_{345} c_1 \\
J[2,1] &= a_3 s_1 s_3 + a_3 c_1 c_2 c_3 + a_4 s_1 s_{34} + a_4 c_1 c_2 c_{34} + d_2 s_1 + d_3 s_2 c_1 \\
&\quad - d_6 s_1 c_{345} + d_6 s_{345} c_1 c_2 + d_7 s_1 s_6 s_{345} - d_7 s_2 c_1 c_6 + d_7 s_6 c_1 c_2 c_{345} \\
J[3,1] &= 0 \\
J[4,1] &= 0 \\
J[5,1] &= 0 \\
J[6,1] &= 1
\end{aligned}
$$

$$
\begin{aligned}
J[1,2] &= -\left(a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6\right) c_1 \\
J[2,2] &= -\left(a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6\right) s_1 \\
J[3,2] &= a_3 c_2 c_3 + a_4 c_2 c_{34} + d_3 s_2 + d_6 s_{345} c_2 - d_7 s_2 c_6 + d_7 s_6 c_2 c_{345} \\
J[4,2] &= s_1 \\
J[5,2] &= -c_1 \\
J[6,2] &= 0
\end{aligned}
$$

$$
\begin{aligned}
J[1,3] &= a_3 s_1 c_3 - a_3 s_3 c_1 c_2 + a_4 s_1 c_{34} - a_4 s_{34} c_1 c_2 + d_6 s_1 s_{345} \\
&\quad + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2 \\
J[2,3] &= -a_3 s_1 s_3 c_2 - a_3 c_1 c_3 - a_4 s_1 s_{34} c_2 - a_4 c_1 c_{34} + d_6 s_1 c_2 c_{345} \\
&\quad - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345} \\
J[3,3] &= \left(-a_3 s_3 - a_4 s_{34} + d_6 c_{345} - d_7 s_6 s_{345}\right) s_2 \\
J[4,3] &= s_2 c_1 \\
J[5,3] &= s_1 s_2 \\
J[6,3] &= -c_2
\end{aligned}
$$

$J[1,4] = a_4 s_1 c_{34} - a_4 s_{34} c_1 c_2 + d_6 s_1 s_{345} + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2$
$J[2,4] = -a_4 s_1 s_{34} c_2 - a_4 c_1 c_{34} + d_6 s_1 c_2 c_{345} - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345}$
$J[3,4] = \left(-a_4 s_{34} + d_6 c_{345} - d_7 s_6 s_{345}\right) s_2$
$J[4,4] = s_2 c_1$
$J[5,4] = s_1 s_2$
$J[6,4] = -c_2$

$J[1,5] = d_6 s_1 s_{345} + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2$
$J[2,5] = d_6 s_1 c_2 c_{345} - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345}$
$J[3,5] = \left(d_6 c_{345} - d_7 s_6 s_{345}\right) s_2$
$J[4,5] = s_2 c_1$
$J[5,5] = s_1 s_2$
$J[6,5] = -c_2$

$J[1,6] = d_7 \left(s_1 s_{345} c_6 + s_2 s_6 c_1 + c_1 c_2 c_6 c_{345}\right)$
$J[2,6] = d_7 \left(s_1 s_2 s_6 + s_1 c_2 c_6 c_{345} - s_{345} c_1 c_6\right)$
$J[3,6] = d_7 \left(s_2 c_6 c_{345} - s_6 c_2\right)$
$J[4,6] = -s_1 c_{345} + s_{345} c_1 c_2$
$J[5,6] = s_1 s_{345} c_2 + c_1 c_{345}$
$J[6,6] = s_2 s_{345}$

$J[1,7] = 0$
$J[2,7] = 0$
$J[3,7] = 0$
$J[4,7] = \left(s_1 s_{345} + c_1 c_2 c_{345}\right) s_6 - s_2 c_1 c_6$
$J[5,7] = \left(s_1 c_2 c_{345} - s_{345} c_1\right) s_6 - s_1 s_2 c_6$
$J[6,7] = s_2 s_6 c_{345} + c_2 c_6$

3.2 Method 2: Geometric Jacobian

We first form our D_i matrices from

$$D_i = T_{0i} Q_i T_{0i}^{-1}$$

where, as all our joints are revolute,

$$Q = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Selecting elements from these D_i matrices, we form the Jacobian via

$$J = \begin{bmatrix} {}^0D_{14} & {}^1D_{14} & {}^2D_{14} & {}^3D_{14} & {}^4D_{14} & {}^5D_{14} & {}^6D_{14} \\ {}^0D_{24} & {}^1D_{24} & {}^2D_{24} & {}^3D_{24} & {}^4D_{24} & {}^5D_{24} & {}^6D_{24} \\ {}^0D_{34} & {}^1D_{34} & {}^2D_{34} & {}^3D_{34} & {}^4D_{34} & {}^5D_{34} & {}^6D_{34} \\ {}^0D_{32} & {}^1D_{32} & {}^2D_{32} & {}^3D_{32} & {}^4D_{32} & {}^5D_{32} & {}^6D_{32} \\ {}^0D_{13} & {}^1D_{13} & {}^2D_{13} & {}^3D_{13} & {}^4D_{13} & {}^5D_{13} & {}^6D_{13} \\ {}^0D_{21} & {}^1D_{21} & {}^2D_{21} & {}^3D_{21} & {}^4D_{21} & {}^5D_{21} & {}^6D_{21} \end{bmatrix}$$

Resulting in

$$
\begin{aligned}
J[1,1] &= d_1 c_1 \\
J[2,1] &= d_1 s_1 \\
J[3,1] &= 0 \\
J[4,1] &= s_1 \\
J[5,1] &= -c_1 \\
J[6,1] &= 0
\end{aligned}
$$

$$
\begin{aligned}
J[1,2] &= -d_1 s_1 s_2 + d_2 c_1 c_2 \\
J[2,2] &= d_1 s_2 c_1 + d_2 s_1 c_2 \\
J[3,2] &= d_2 s_2 \\
J[4,2] &= s_2 c_1 \\
J[5,2] &= s_1 s_2 \\
J[6,2] &= -c_2
\end{aligned}
$$

$$
\begin{aligned}
J[1,3] &= -a_3 s_1 c_3 + a_3 s_3 c_1 c_2 - d_1 s_1 s_2 + d_2 c_1 c_2 \\
J[2,3] &= a_3 s_1 s_3 c_2 + a_3 c_1 c_3 + d_1 s_2 c_1 + d_2 s_1 c_2 \\
J[3,3] &= (a_3 s_3 + d_2)\, s_2 \\
J[4,3] &= s_2 c_1 \\
J[5,3] &= s_1 s_2 \\
J[6,3] &= -c_2
\end{aligned}
$$

$$
\begin{aligned}
J[1,4] &= -a_3 s_1 c_3 + a_3 s_3 c_1 c_2 - a_4 s_1 c_{34} + a_4 s_{34} c_1 c_2 - d_1 s_1 s_2 + d_2 c_1 c_2 \\
J[2,4] &= a_3 s_1 s_3 c_2 + a_3 c_1 c_3 + a_4 s_1 s_{34} c_2 + a_4 c_1 c_{34} + d_1 s_2 c_1 + d_2 s_1 c_2 \\
J[3,4] &= (a_3 s_3 + a_4 s_{34} + d_2)\, s_2 \\
J[4,4] &= s_2 c_1 \\
J[5,4] &= s_1 s_2 \\
J[6,4] &= -c_2
\end{aligned}
$$

$$J[1,5] = -\left(d_1c_2 - d_3\right)\left(s_1s_{345} + c_1c_2c_{345}\right) - \left(a_3c_{45} + a_4c_5 + d_1s_2c_{345} + d_2s_{345}\right)s_2c_1$$
$$J[2,5] = -\left(d_1c_2 - d_3\right)\left(s_1c_2c_{345} - s_{345}c_1\right) - \left(a_3c_{45} + a_4c_5 + d_1s_2c_{345} + d_2s_{345}\right)s_1s_2$$
$$J[3,5] = a_3c_2c_{45} + a_4c_2c_5 + d_2s_{345}c_2 + d_3s_2c_{345}$$
$$J[4,5] = -s_1c_{345} + s_{345}c_1c_2$$
$$J[5,5] = s_1s_{345}c_2 + c_1c_{345}$$
$$J[6,5] = s_2s_{345}$$

$$\begin{aligned}
J[1,6] = {} & \big(\left(s_1s_{345} + c_1c_2c_{345}\right)c_6 + s_2s_6c_1\big)\left(a_3s_{45} + a_4s_5 + d_1s_2s_{345} - d_2c_{345} + d_6\right) \\
& + \left(s_1c_{345} - s_{345}c_1c_2\right)\left(a_3c_6c_{45} + a_4c_5c_6 + d_1s_2c_6c_{345} - d_1s_6c_2 + d_2s_{345}c_6 + d_3s_6\right)
\end{aligned}$$
$$\begin{aligned}
J[2,6] = {} & \big(\left(s_1c_2c_{345} - s_{345}c_1\right)c_6 + s_1s_2s_6\big)\left(a_3s_{45} + a_4s_5 + d_1s_2s_{345} - d_2c_{345} + d_6\right) \\
& - \left(s_1s_{345}c_2 + c_1c_{345}\right)\left(a_3c_6c_{45} + a_4c_5c_6 + d_1s_2c_6c_{345} - d_1s_6c_2 + d_2s_{345}c_6 + d_3s_6\right)
\end{aligned}$$
$$\begin{aligned}
J[3,6] = {} & \left(s_2c_6c_{345} - s_6c_2\right)\left(a_3s_{45} + a_4s_5 + d_1s_2s_{345} - d_2c_{345} + d_6\right) \\
& - \left(a_3c_6c_{45} + a_4c_5c_6 + d_1s_2c_6c_{345} - d_1s_6c_2 + d_2s_{345}c_6 + d_3s_6\right)s_2s_{345}
\end{aligned}$$
$$J[4,6] = s_1s_6s_{345} - s_2c_1c_6 + s_6c_1c_2c_{345}$$
$$J[5,6] = -s_1s_2c_6 + s_1s_6c_2c_{345} - s_6s_{345}c_1$$
$$J[6,6] = s_2s_6c_{345} + c_2c_6$$

$$\begin{aligned}
J[1,7] = {} & \Big(\big(\left(s_1s_{345} + c_1c_2c_{345}\right)c_6 + s_2s_6c_1\big)c_7 - \left(s_1c_{345} - s_{345}c_1c_2\right)s_7\Big) \\
& \left(a_3s_6c_{45} + a_4s_6c_5 + d_1s_2s_6c_{345} + d_1c_2c_6 + d_2s_6s_{345} - d_3c_6 + d_7\right) \\
& - \big(\left(s_1s_{345} + c_1c_2c_{345}\right)s_6 - s_2c_1c_6\big)\left(a_3s_7s_{45} + a_3c_6c_7c_{45} + a_4s_5s_7 + a_4c_5c_6c_7\right. \\
& \left. + d_1s_2s_7s_{345} + d_1s_2c_6c_7c_{345} - d_1s_6c_2c_7 - d_2s_7c_{345} + d_2s_{345}c_6c_7 + d_3s_6c_7 + d_6s_7\right)
\end{aligned}$$
$$\begin{aligned}
J[2,7] = {} & \Big(\big(\left(s_1c_2c_{345} - s_{345}c_1\right)c_6 + s_1s_2s_6\big)c_7 + \left(s_1s_{345}c_2 + c_1c_{345}\right)s_7\Big) \\
& \left(a_3s_6c_{45} + a_4s_6c_5 + d_1s_2s_6c_{345} + d_1c_2c_6 + d_2s_6s_{345} - d_3c_6 + d_7\right) \\
& - \big(\left(s_1c_2c_{345} - s_{345}c_1\right)s_6 - s_1s_2c_6\big)\left(a_3s_7s_{45} + a_3c_6c_7c_{45} + a_4s_5s_7 + a_4c_5c_6c_7\right. \\
& \left. + d_1s_2s_7s_{345} + d_1s_2c_6c_7c_{345} - d_1s_6c_2c_7 - d_2s_7c_{345} + d_2s_{345}c_6c_7 + d_3s_6c_7 + d_6s_7\right)
\end{aligned}$$
$$\begin{aligned}
J[3,7] = {} & \big(\left(s_2c_6c_{345} - s_6c_2\right)c_7 + s_2s_7s_{345}\big) \\
& \left(a_3s_6c_{45} + a_4s_6c_5 + d_1s_2s_6c_{345} + d_1c_2c_6 + d_2s_6s_{345} - d_3c_6 + d_7\right) \\
& - \left(s_2s_6c_{345} + c_2c_6\right)\left(a_3s_7s_{45} + a_3c_6c_7c_{45} + a_4s_5s_7 + a_4c_5c_6c_7\right. \\
& \left. + d_1s_2s_7s_{345} + d_1s_2c_6c_7c_{345} - d_1s_6c_2c_7 - d_2s_7c_{345} + d_2s_{345}c_6c_7 + d_3s_6c_7 + d_6s_7\right)
\end{aligned}$$
$$J[4,7] = s_1s_7s_{345}c_6 + s_1c_7c_{345} + s_2s_6s_7c_1 + s_7c_1c_2c_6c_{345} - s_{345}c_1c_2c_7$$
$$J[5,7] = s_1s_2s_6s_7 + s_1s_7c_2c_6c_{345} - s_1s_{345}c_2c_7 - s_7s_{345}c_1c_6 - c_1c_7c_{345}$$
$$J[6,7] = s_2s_7c_6c_{345} - s_2s_{345}c_7 - s_6s_7c_2$$

3.3 Velocity Equation

I should write this?

4 Conclusions

References

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