Hwk 1 Solutions

3.7 Measured data in millimeters for the position coordinates of three points of a moving body are known such that:

$$\boldsymbol{r}_1 = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{r}_3 = \begin{bmatrix} 0 \\ 125 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{R}_1(t_1) = \begin{bmatrix} 37.325 \\ -98.175 \\ 132.045 \\ 1 \end{bmatrix} \quad \boldsymbol{R}_2(t_1) = \begin{bmatrix} 71.800 \\ -118.925 \\ 161.725 \\ 1 \end{bmatrix} \quad \boldsymbol{R}_3(t_1) = \begin{bmatrix} 152.450 \\ -28.425 \\ 131.225 \\ 1 \end{bmatrix}$$

Find the (4×4) homogeneous transformation matrix for this displacement.

Using Eqs. (3.29) and its successor we calculate data for the two positions of an independent fourth point

$$\boldsymbol{r}_4 = \begin{bmatrix} 50\text{ mm} \\ 0 \\ -6\ 250\text{ mm} \\ 1 \end{bmatrix} \qquad \boldsymbol{R}_4(t_1) = \begin{bmatrix} -2\ 015.840\text{ mm} \\ 3\ 347.005\text{ mm} \\ 4\ 925.520\text{ mm} \\ 1 \end{bmatrix}$$

We now use Eq. (3.32) to find the T matrix

$$T = \begin{bmatrix} 37.325\text{ mm} & 71.800\text{ mm} & 152.450\text{ mm} & -2\ 015.840\text{ mm} \\ -98.175\text{ mm} & -118.925\text{ mm} & -28.425\text{ mm} & 3\ 347.005\text{ mm} \\ 132.045\text{ mm} & 161.725\text{ mm} & 131.225\text{ mm} & 4\ 925.520\text{ mm} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 50\text{ mm} & 0 & 0 & 50\text{ mm} \\ 0 & 0 & 125\text{ mm} & 0 \\ 0 & 0 & 0 & -6\ 250\text{ mm} \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$T = \begin{bmatrix} -0.68950 & 0.64520 & 0.32851 & 71.800\text{ mm} \\ 0.41500 & 0.72400 & -0.55123 & -118.925\text{ mm} \\ -0.59360 & -0.24400 & -0.76696 & 161.725\text{ mm} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{\textit{Ans}}.$$

3.13 Consider a robot end-effector with two coordinate systems attached to it as shown in Figure P3.17*a*. One coordinate system is attached to the end-effector with its origin at the wrist center point and the other is attached to the tip of the end-effector. The kinematic structure of the wrist is a spherical linkage and is illustrated in the Figure P3.17*b*.

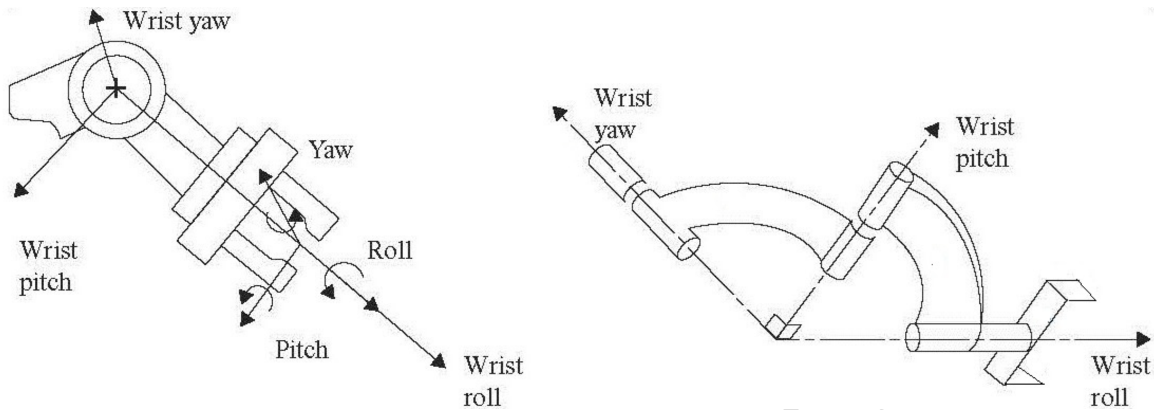
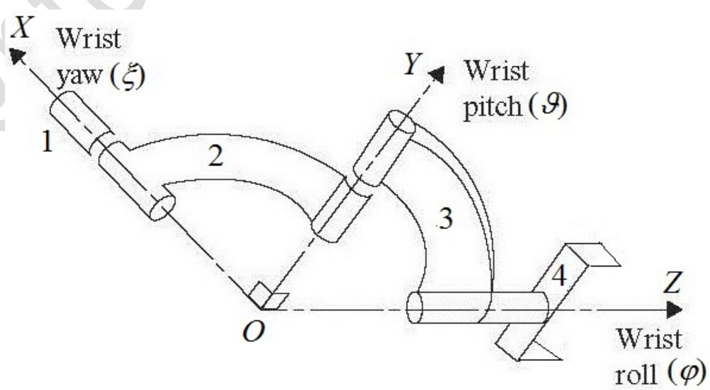


Figure P3.15 (*a*) (*b*)

Show that if we use the coordinate system attached to the end-effector at the wrist center point, the order in which we perform the roll, pitch, and yaw rotations is irrelevant; however, if we use the coordinate system attached to the end effector at its tip then the order does make a difference unless we are only concerned with differential or instantaneous rotations.

In order to permit discussion of this problem, the labels 1-4 are assigned to coordinate systems $x_1y_1z_1$ through $x_4y_4z_4$ and are attached to bodies 1 through 4, respectively, such that all are initially aligned and have their origins coincident at the wrist center point as shown here:



In such an arrangement of the coordiate systems, the roll, pitch, and yaw rotations of the end-effector coordinate system (coordinate system $x_4y_4z_4$) correspond to the joint motions of the spherical linkage described by bodies 1, 2, 3, and 4. The topology or kinematic architecture of this linkage is fixed and it is this kinematic structure that defines the roll, pitch, and yaw of the end-effector. In such a linkage the chronological order in which the joints are rotated does not matter in terms of the final posture of the end effector coordinate system. This can be proven analytically as follows:

Let us first consider rotating the joints starting in the order from left to right. The first rotation is by ξ about the yaw axis or the X_1 axis and can be written as $R_1 = T_{12} r_2$. This rotation brings the coordinate system $x_2 y_2 z_2$ into a new orientation with respect to the $X_1 Y_1 Z_1$ coordinate system while $x_3 y_3 z_3$ remains coincident with the new orientation of the $x_2 y_2 z_2$ system. Now, when we perform the pitch rotation, the coordiate system $x_3 y_3 z_3$ rotates by ϑ with respect to the new posture of the $x_2 y_2 z_2$ system and therefore we write: $r_2 = T_{23} r_3$. Similarly for the last rotation (roll by angle φ about $x_3 y_3 z_3$) we write: $r_3 = T_{34} r_4$. We note that each of these three rotations is performed with respect to the preceding coordinate system from a zero position coincident with the preceding coordinate system. Back substituting these equations for relative rotations we get $R_1 = T_{12} T_{23} T_{34} r_4$. The resultant yaw-pitch-roll (ξ, ϑ, φ) transformation is then given by:

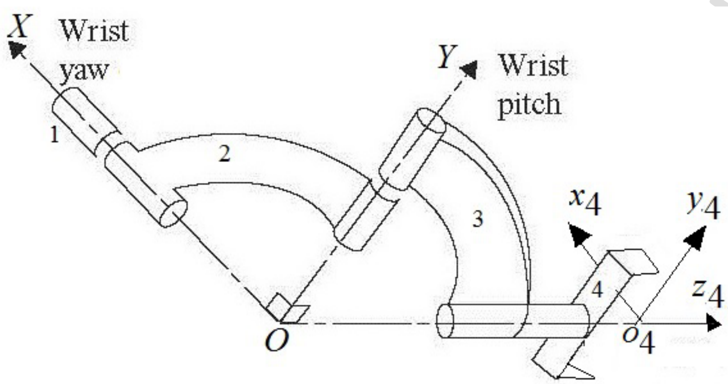
$$
\begin{aligned}
T_{14} &= T(\xi) T(\vartheta) T(\varphi) = T_{12} T_{23} T_{34} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi & 0 \\ 0 & \sin\xi & \cos\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\vartheta\cos\varphi & -\cos\vartheta\sin\varphi & \sin\vartheta & 0 \\ \sin\xi\sin\vartheta\cos\varphi + \cos\xi\sin\varphi & -\sin\xi\sin\vartheta\sin\varphi + \cos\xi\cos\varphi & -\sin\xi\cos\vartheta & 0 \\ -\cos\xi\sin\vartheta\cos\varphi + \sin\xi\sin\varphi & \cos\xi\sin\vartheta\sin\varphi + \sin\xi\cos\varphi & \cos\xi\cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{1}
$$

Now we change the order of performing the rotations starting in the order from right to left and we perform first a roll (φ'), then a pitch (ϑ'), and finally a yaw rotation (ξ') – all using the joint axes of the linkage as shown corresponding to these rotations. The initial roll rotation (φ') is shown by the figure to rotate body 4 with respect to body 3 starting from a reference position where these two coordinate systems are coincident. Therefore, the corresponding transformation is $r_3 = T_{34} r_4$. The second rotation, by the pitch angle (ϑ'), is shown by the figure to be a rotation of body 3 with respect to body 2 again starting from a reference position where thed coordinate systems of body 2 and body 3 are coincident. The corresponding transformation is $r_2 = T_{23} r_3$. Finally, the last rotation is a yaw rotation (ξ') and takes place about the yaw axis connecting body 1 to body 2; therefore, we still have $R_1 = T_{12} r_2$. Substituting these equations for relative rotations into one another, again we get:

$$R_1 = \Theta'_{12}\Theta'_{23}\Theta'_{34}r_4 \tag{2}$$

We see that this gives the same final transformation as in the first case. Thus, if we use the coordinate system attached to the end-effector at the wrist center point, the order in which we perform the roll, pitch, and yaw rotations is irrelevantfor the spherical wrist mechanism shown in the figure. *Q.E.D.*

If we now use the coordinate system attached to the end effector at its tip then the figure looks as follows:



In this case the yaw-pitch-roll rotations about the end effector coordinate system would not correspond to the yaw, pitch, and roll axes of the linkage of the wrist and it does not have any particular kinematic structure or topology. Each of the rotations would be a relative rotation represented by a correpsonding rotation matrix, but would take place about the o_4 origin which is translated by a distance d along Z_1 from O_1.

Taking the three rotations in yaw-pitch-roll order, a yaw rotation of ξ' about the x_4 axis gives a transformation of $R' = T'_{04}T\left(\xi'\right)r'_4$ or

$$R' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi' & -\sin\xi' & 0 \\ 0 & \sin\xi' & \cos\xi' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r'_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi' & -\sin\xi' & 0 \\ 0 & \sin\xi' & \cos\xi' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r'_4$$

When this is followed by a pitch rotation of ϑ' about the new y_4 axis, the transformation becomes $R' = T'_{04}T\left(\xi'\right)T\left(\vartheta'\right)r'_4$ or

$$R' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi' & -\sin\xi' & 0 \\ 0 & \sin\xi' & \cos\xi' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\vartheta' & 0 & \sin\vartheta' & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\vartheta' & 0 & \cos\theta' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4' = \begin{bmatrix} \cos\vartheta' & 0 & \sin\vartheta' & 0 \\ \sin\xi'\sin\vartheta' & \cos\xi' & -\sin\xi'\cos\vartheta' & 0 \\ -\cos\xi'\sin\vartheta' & \sin\xi' & \cos\xi'\cos\vartheta' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

Finally, when this is followed by a roll rotation of φ' about the displaced z_4 axis, the transformation becomes $R' = T_{04}'T(\xi')T(\vartheta')T(\varphi')r_4'$ or

$$R_4' = \begin{bmatrix} \cos\vartheta' & 0 & \sin\vartheta' & 0 \\ \sin\xi'\sin\vartheta' & \cos\xi' & -\sin\xi'\cos\vartheta' & 0 \\ -\cos\xi'\sin\vartheta' & \sin\xi' & \cos\xi'\cos\vartheta' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi' & -\sin\varphi' & 0 & 0 \\ \sin\varphi' & \cos\varphi' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

$$R' = \begin{bmatrix} \cos\vartheta'\cos\varphi' & -\cos\vartheta'\sin\varphi' & \sin\vartheta' & 0 \\ \sin\xi'\sin\vartheta'\cos\varphi' + \cos\xi'\sin\varphi' & -\sin\xi'\sin\vartheta'\sin\varphi' + \cos\xi'\cos\varphi' & -\sin\xi'\cos\vartheta' & 0 \\ -\cos\xi'\sin\vartheta'\cos\varphi' + \sin\xi'\sin\varphi' & \cos\xi'\sin\vartheta'\sin\varphi' + \sin\xi'\cos\varphi' & \cos\xi'\cos\vartheta' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4' \qquad (3)$$

However, if we now change the order of performing the rotations, and we perform first a roll (φ'), then a pitch (ϑ'), and finally a yaw rotation (ξ'). The initial roll rotation (φ') is shown by the figure to rotate the end effector about the z_4 axis. Therefore, we are required to write this transformation as

$$R' = T_{04}T(\varphi')r_4' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi' & -\sin\varphi' & 0 & 0 \\ \sin\varphi' & \cos\varphi' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4' = \begin{bmatrix} \cos\varphi' & -\sin\varphi' & 0 & 0 \\ \sin\varphi' & \cos\varphi' & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

The second rotation, by the pitch angle (ϑ'), is a rotation of body 4 by the angle ϑ' about the new y_4 axis. Therefore we must write

$$R' = T_{04}T(\varphi')T(\vartheta')r_4' = \begin{bmatrix} \cos\varphi' & -\sin\varphi' & 0 & 0 \\ \sin\varphi' & \cos\varphi' & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\vartheta' & 0 & \sin\vartheta' & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\vartheta' & 0 & \cos\vartheta' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'.$$

$$= \begin{bmatrix} \cos\varphi'\cos\vartheta' & -\sin\varphi' & \cos\varphi'\sin\vartheta' & 0 \\ \sin\varphi'\cos\vartheta' & \cos\varphi' & \sin\varphi'\sin\vartheta' & 0 \\ -\sin\vartheta' & 0 & \cos\vartheta' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

Finally, the last rotation is a yaw rotation (ξ') and takes place about the modified x_4 axis connecting body 4 to the end effector; therefore, we have

$$R' = \begin{bmatrix} \cos\varphi'\cos\vartheta' & -\sin\varphi' & \cos\varphi'\sin\vartheta' & 0 \\ \sin\varphi'\cos\vartheta' & \cos\varphi' & \sin\varphi'\sin\vartheta' & 0 \\ -\sin\vartheta' & 0 & \cos\vartheta' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi' & -\sin\xi' & 0 \\ 0 & \sin\xi' & \cos\xi' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

$$= \begin{bmatrix} \cos\varphi'\cos\vartheta' & -\sin\varphi'\cos\xi' + \cos\varphi'\sin\vartheta'\sin\xi' & \sin\varphi'\sin\xi' + \cos\varphi'\sin\vartheta'\cos\xi' & 0 \\ \sin\varphi'\cos\vartheta' & \cos\varphi'\cos\xi' + \sin\varphi'\sin\vartheta'\sin\xi' & -\cos\varphi'\sin\xi' + \sin\varphi'\sin\vartheta'\cos\xi' & 0 \\ -\sin\vartheta' & \cos\vartheta'\sin\xi' & \cos\vartheta'\cos\xi' & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4' \quad (4)$$

We see that this is not at all the same transformation as in the first case, (3); if we use the coordinate system attached to the end-effector at its tip, the order in which we perform the roll, pitch, and yaw rotations is not at all irrelevant.

Q.E.D.

However, if we consider only infinitesmal angles $\delta\eta$, then we can approximate $\cos\delta\eta \approx 1$ and $\sin\delta\eta \approx \delta\eta$. Making these substitutions and assuming that all quadratic and higher order terms in $\delta\eta$ are negligible, we find that both transformations (3) and (4) reduce to the form

$$R' = \begin{bmatrix} 1 & -\delta\varphi' & \delta\vartheta' & 0 \\ \delta\varphi' & 1 & -\delta\xi' & 0 \\ -\delta\vartheta' & \delta\xi' & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} r_4'$$

Since this is the same for both the (3) and (4) orders of angle changes, this example shows that the order is irrelevant when dealing with infinitesimal angle changes. *Q.E.D.*