

Problem 1.

In an industrial application a part is to turn 20 degrees about a rod shown below (in the direction indicated) and move down six inches along the same rod.

- Please determine a single 4x4 matrix transformation that can be used to compute the new coordinates of an arbitrary point on the part.
- Using the results of part a, please determine the new world coordinates of a point whose original coordinates were X, Y, Z.

Given

$$P = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix},$$

$$w = \frac{P - O}{\|P - O\|} = \begin{bmatrix} 0 \\ -.8 \\ .6 \end{bmatrix}$$

$$\begin{aligned} \underline{\underline{\Theta}} &= \underline{\underline{I}} + \underline{\underline{w}} \sin \phi + \underline{\underline{w}}^2 (1 - \cos \phi) \\ &= \begin{bmatrix} (w_x^2 - 1)(1 - \cos \theta) + 1 & w_x w_y (1 - \cos \theta) - w_z \sin \theta & w_x w_z (1 - \cos \theta) + w_y \sin \theta \\ w_x w_y (1 - \cos \theta) + w_z \sin \theta & (w_y^2 - 1)(1 - \cos \theta) + 1 & w_y w_z (1 - \cos \theta) - w_x \sin \theta \\ w_x w_z (1 - \cos \theta) - w_y \sin \theta & w_y w_z (1 - \cos \theta) + w_x \sin \theta & (w_z^2 - 1)(1 - \cos \theta) + 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 \\ 0.2052 & 0.9782 & -0.0289 \\ 0.2736 & -0.0289 & 0.9614 \end{bmatrix} \\ d &= (\underline{\underline{I}} - \underline{\underline{\Theta}}) \underline{\underline{P}} + \phi \underline{\underline{w}} \\ &= \begin{bmatrix} \phi w_x - P_x(w_x^2 - 1)(1 - \cos \theta) + P_y(-w_x w_y (1 - \cos \theta) + w_z \sin \theta) - P_z(w_x w_z (1 - \cos \theta) + w_y \sin \theta) \\ \phi w_y - P_x(w_x w_y (1 - \cos \theta) + w_z \sin \theta) - P_y(w_y^2 - 1)(1 - \cos \theta) + P_z(-w_y w_z (1 - \cos \theta) + w_x \sin \theta) \\ \phi w_z + P_x(-w_x w_z (1 - \cos \theta) + w_y \sin \theta) - P_y(w_y w_z (1 - \cos \theta) + w_x \sin \theta) - P_z(w_z^2 - 1)(1 - \cos \theta) \end{bmatrix} \\ &= \begin{bmatrix} 1.6416 \\ -4.6263 \\ 3.8315 \end{bmatrix} \\ T &= \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 & 1.6416 \\ 0.2052 & 0.9782 & -0.0289 & -4.6263 \\ 0.2736 & -0.0289 & 0.9614 & 3.8315 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$T \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9396X - 0.2052Y - 0.2736Z + 1.6416 \\ 0.2052X + 0.9782Y - 0.0289Z - 4.6263 \\ 0.2736X - 0.0289Y + 0.9614Z + 3.8315 \\ 1 \end{bmatrix}$$

Problem 2.

Complete the derivation of the Denavit-Hartenberg (D-H) Transformation from what was done in the classroom using the joint and Shape matrices based on the diagram and the derivation started in the class on 2-1-2018.

$$\begin{aligned} \underline{T}_{h-,h+} &= \underline{I} \Phi_h \underline{S}_{h+1,h}^{-1} \\ \Phi_h &= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 & 0 & 0 \\ \sin \phi_h^1 & \cos \phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \underline{\Theta} &= \underline{I} + \underline{\tilde{w}} \sin \phi + \underline{\tilde{w}}^2 (1 - \cos \phi) \\ \underline{\tilde{w}} &= \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \end{aligned}$$

To solve for $\underline{S}_{h+1,h}^{-1}$, we define the transformation:

Fixed coordinate system: $(uvw)_{h'}$

Moving coordinate system: $(uvw)_{h+1}$

Screw axis: $u'_h, w = (1, 0, 0)$

$\underline{P} = \underline{0}$

$s = a_h, u$ direction

$\phi = \alpha_h$

$$\begin{aligned}
\underline{\underline{\Theta}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin \alpha_h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} (1 - \cos \alpha_h) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_h & -\sin \alpha_h \\ 0 & \sin \alpha_h & \cos \alpha_h \end{bmatrix} \\
S_{h+,h}^{-1} &= \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos \alpha_h & -\sin \alpha_h & 0 \\ 0 & \sin \alpha_h & \cos \alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\underline{\underline{T}}_{h-,h+} &= \underline{\underline{I}} \Phi_h \underline{\underline{S}}_{h+1,h}^{-1} \\
&= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 & 0 & 0 \\ \sin \phi_h^1 & \cos \phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos \alpha_h & -\sin \alpha_h & 0 \\ 0 & \sin \alpha_h & \cos \alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos \phi_h^1 & -\sin \phi_h^1 \cos \alpha_h & \sin \alpha_h \sin \phi_h^1 & a_h \cos \phi_h^1 \\ \sin \phi_h^1 & \cos \alpha_h \cos \phi_h^1 & -\sin \alpha_h \cos \phi_h^1 & a_h \sin \phi_h^1 \\ 0 & \sin \alpha_h & \cos \alpha_h & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Problem 3.

Consider the robot manipulator shown below.

- Determine the D-H parameters for the robot and the D-H transformation for each joint.
- Derive the kinematic equations for the coordinates of a point at the tip of the last link (XYZ) in terms of the joint variables.
- Determine the inverse kinematic solution.

	1	2	3
a_i	0	0	0
α_i	0	-90	0
θ_i	θ	90	0
s_i	h	0	r

$$\underline{T}_{12} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{T}_{23} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{T}_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \underline{x}_1 &= \underline{T}_{12} \underline{T}_{23} \underline{T}_{34} \underline{x}_4 \\ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -\sin \theta & 0 & -\cos \theta & -r \cos \theta \\ \cos \theta & 0 & -\sin \theta & -r \sin \theta \\ 0 & -1 & 0 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -X \sin \theta - Z \cos \theta - r \cos \theta \\ X \cos \theta - Z \sin \theta - r \sin \theta \\ -Y + h \\ 1 \end{bmatrix} \\ \underline{x}_4 &= \left(\underline{T}_{12} \underline{T}_{23} \underline{T}_{34} \right)^{-1} = \underline{T}_{41} \underline{x}_1 \\ \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} &= \begin{bmatrix} -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & -1 & h \\ -\cos \theta & -\sin \theta & 0 & -r \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 \sin \theta + y_1 \cos \theta \\ -z_1 + h \\ -x_1 \cos \theta - y_1 \sin \theta - r \\ 1 \end{bmatrix} \end{aligned}$$