Problem 1.

In an industrial application a part is to turn 20 degrees about a rod shown below (in the direction indicated) and move down six inches along the same rod.

(a) Please determine a single 4x4 matrix transformation that can be used to compute the new coordinates of an arbitrary point on the part.

(b) Using the results of part a, please determine the new world coordinates of a point whose original coordinates were X, Y, Z.

Given

$$P = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} O = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix},$$

$$w = \frac{P - O}{\|P - O\|} = \begin{bmatrix} 0 \\ -.8 \\ .6 \end{bmatrix}$$

$$\begin{aligned}
\underline{\underline{\Theta}} &= \underline{\underline{I}} + \underline{\underline{\widetilde{w}}} \sin\phi + \underline{\underline{\widetilde{w}}}^2 (1 - \cos\phi) \\
&= \begin{bmatrix} (w_x^2 - 1)(1-\cos\theta) + 1 & w_x w_y (1-\cos\theta) - w_z \sin\theta & w_x w_z (1-\cos\theta) + w_y \sin\theta \\ w_x w_y (1-\cos\theta) + w_z \sin\theta & (w_y^2 - 1)(1-\cos\theta) + 1 & w_y w_z (1-\cos\theta) - w_x \sin\theta \\ w_x w_z (1-\cos\theta) - w_y \sin\theta & w_y w_z (1-\cos\theta) + w_x \sin\theta & (w_z^2 - 1)(1-\cos\theta) + 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 \\ 0.2052 & 0.9782 & -0.0289 \\ 0.2736 & -0.0289 & 0.9614 \end{bmatrix} \\
d &= \left(\underline{\underline{I}} - \underline{\underline{\Theta}}\right) \underline{P} + \phi \underline{w} \\
&= \begin{bmatrix} \phi w_x - P_x(w_x^2 - 1)(1-\cos\theta) + P_y(-w_x w_y(1-\cos\theta) + w_z \sin\theta) - Pz(w_x w_z(1-\cos\theta) + w_y \sin\theta) \\ \phi w_y - P_x(w_x w_y(1-\cos\theta) + w_z \sin\theta) - P_y(w_y^2 - 1)(1-\cos\theta) + Pz(-w_y w_z(1-\cos\theta) + w_x \sin\theta) \\ \phi w_z + P_x(-w_x w_z(1-\cos\theta) + w_y \sin\theta) - P_y(w_y w_z(1-\cos\theta) + w_x \sin\theta) - Pz(w_z^2 - 1)(1-\cos\theta) \end{bmatrix} \\
&= \begin{bmatrix} 1.6416 \\ -4.6263 \\ 3.8315 \end{bmatrix}
\end{aligned}$$

$$T = \begin{bmatrix} 0.9396 & -0.2052 & -0.2736 & 1.6416 \\ 0.2052 & 0.9782 & -0.0289 & -4.6263 \\ 0.2736 & -0.0289 & 0.9614 & 3.8315 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9396X - 0.2052Y - 0.2736Z + 1.6416 \\ 0.2052X + 0.9782Y - 0.0289Z - 4.6263 \\ 0.2736X - 0.0289Y + 0.9614Z + 3.8315 \\ 1 \end{bmatrix}$$

Problem 2.

Complete the derivation of the Denavit-Hartenberg (D-H) Transformation from what was done in the classroom using the joint and Shape matrices based on the diagram and the derivation started in the class on 2-1-2018.

$$\underline{\underline{T}}_{h-,h+} = \underline{\underline{I}}\Phi_h \underline{\underline{S}}^{-1}_{h+1,h}$$

$$\Phi_h = \begin{bmatrix} \cos\phi_h^1 & -\sin\phi_h^1 & 0 & 0 \\ \sin\phi_h^1 & \cos\phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{\Theta}} = \underline{\underline{I}} + \underline{\underline{\widetilde{w}}}\sin\phi + \underline{\underline{\widetilde{w}}}^2\left(1 - \cos\phi\right)$$

$$\underline{\underline{\widetilde{w}}} = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_z \\ -w_y & w_x & 0 \end{bmatrix}$$

To solve for $\underline{\underline{S}}^{-1}_{h+1,h}$, we define the transformation:

Fixed coordinate system: $(uvw)_{h'}$
Moving coordinate system: $(uvw)_{h+1}$
Screw axis: $u'_h, w = (1, 0, 0)$
$\underline{P} = \underline{0}$
$s = a_h$, u direction
$\phi = \alpha_h$

$$
\underline{\underline{\Theta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin\alpha_h + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} (1-\cos\alpha_h)
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_h & -\sin\alpha_h \\ 0 & \sin\alpha_h & \cos\alpha_h \end{bmatrix}
$$

$$
S_{h+,h}^{-1} = \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos\alpha_h & -\sin\alpha_h & 0 \\ 0 & \sin\alpha_h & \cos\alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\underline{\underline{T}}_{h-,h+} = \underline{\underline{I}}\Phi_h \underline{\underline{S}}_{h+1,h}^{-1}
$$

$$
= \begin{bmatrix} \cos\phi_h^1 & -\sin\phi_h^1 & 0 & 0 \\ \sin\phi_h^1 & \cos\phi_h^1 & 0 & 0 \\ 0 & 0 & 1 & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_h \\ 0 & \cos\alpha_h & -\sin\alpha_h & 0 \\ 0 & \sin\alpha_h & \cos\alpha_h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\phi_h^1 & -\sin\phi_h^1 \cos\alpha_h & \sin\alpha_h \sin\phi_h^1 & a_h \cos\phi_h^1 \\ \sin\phi_h^1 & \cos\alpha_h \cos\phi_h^1 & -\sin\alpha_h \cos\phi_h^1 & a_h \sin\phi_h^1 \\ 0 & \sin(\alpha_h) & \cos(\alpha_h) & \phi_h^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Problem 3.

Consider the robot manipulator shown below.

(a) Determine the D-H parameters for the robot and the D-H transformation for each joint.

(b) Derive the kinematic equations for the coordinates of a point at the tip of the last link (XYZ) in terms of the joint variables.

(c) Determine the inverse kinematic solution.

	1	2	3
a_i	0	0	r
α_i	0	180^o	0
θ_i	θ	0	180^o
s_i	0	h	0

$$\underline{\underline{T}}_{12} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{\underline{T}}_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \underline{\underline{T}}_{34} = \begin{bmatrix} -1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{x}_1 = \underline{\underline{T}}_{12}\underline{\underline{T}}_{23}\underline{\underline{T}}_{34}\underline{x}_4$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & -\sin\theta & 0 & r\cos\theta \\ -\sin\theta & \cos\theta & 0 & r\sin\theta \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -X\cos\theta - Y\sin\theta + r\cos\theta \\ -X\sin\theta + Y\cos\theta + r\sin\theta \\ Z - h \\ 1 \end{bmatrix}$$

Which gives us three equations. The third row is immediately solvable,

$$\boxed{Z = h}$$

Setting rows one and two to zero, and then squaring yields

$$(-X + r)^2 \cos^2\theta = Y^2 \sin^2\theta$$
$$(-X + r)^2 \sin^2\theta = Y^2 \cos^2\theta$$

Adding these and taking advantage of trignometric identities

$$(-X + r)^2 \left(\cos^2\theta + \sin^2\theta\right) = Y^2 \left(\cos^2\theta + \sin^2\theta\right)$$
$$Y = -X + r$$

Substituting this back in row 2 yields

$$0 = -X\sin\theta + (-X + r)\cos\theta + r\sin\theta$$
$$0 = -X(\sin\theta + \cos\theta) + r(\cos\theta + \sin\theta)$$
$$\boxed{X = r}$$

Finally, plugging this result into row 1 yields

$$0 = -r\cos\theta - Y\sin\theta + r\cos\theta$$
$$Y = \frac{1}{\sin\theta}$$
$$\boxed{Y = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}}$$