# SSRMS

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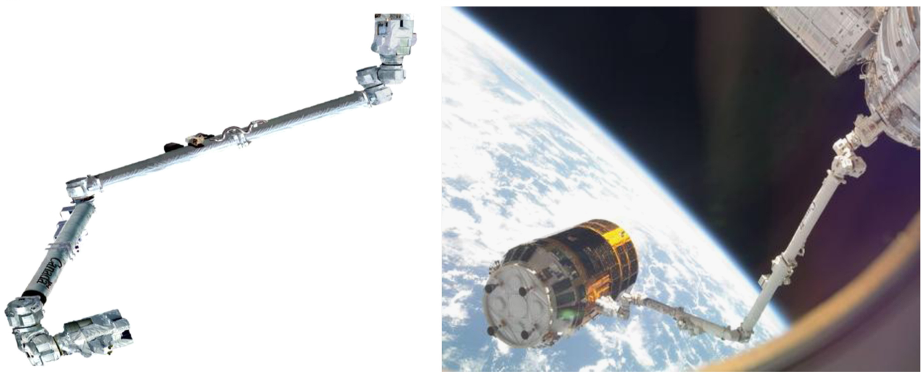
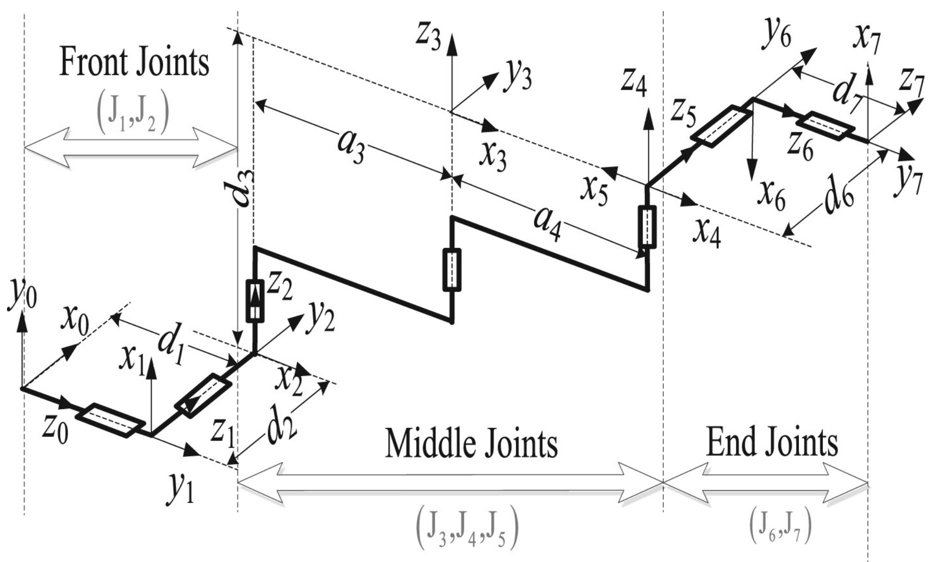
# 1 Introduction

a b






We're going to play with a shoulder roll locked SSRMS.

| $i$ | $\theta_i$ | $\alpha_i$ | $a_i$ | $d_i$ |
|---|---|---|---|---|
| 1 | 90 | 90 | 0 | $d_1$ |
| 2 | 90 | 90 | 0 | $d_2$ |
| 3 | 0 | 0 | $a_3$ | $d_3$ |
| 4 | 0 | 0 | $a_4$ | 0 |
| 5 | 180 | 90 | 0 | 0 |
| 6 | -90 | 90 | 0 | $d_6$ |
| 7 | 180 | 90 | 0 | $d_7$ |

Table 1: The Denavit-Hartenberg parameters for the SSRMS. These parameters are the joint angle, $\theta$, the link twist angle, $\alpha$, the link length, $a$, and the joint offset, $d$. These $\theta_i$s give the initial or "zero-displacement" configuration, but each $\theta_i$ is modeled as an individual variable below.

# 2 Finite Kinematic Analysis

## 2.1 Denavit-Hartenberg Parameters

The resulting matrices are therefore

$$T_{01} = \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{12} = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) & 0 \\ \sin(\theta_2) & 0 & -\cos(\theta_2) & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3\cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3\sin(\theta_3) \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{34} = \begin{bmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & a_4\cos(\theta_4) \\ \sin(\theta_4) & \cos(\theta_4) & 0 & a_4\sin(\theta_4) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{45} = \begin{bmatrix} \cos(\theta_5) & 0 & \sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & -\cos(\theta_5) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{56} = \begin{bmatrix} \cos(\theta_6) & 0 & \sin(\theta_6) & 0 \\ \sin(\theta_6) & 0 & -\cos(\theta_6) & 0 \\ 0 & 1 & 0 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{67} = \begin{bmatrix} \cos(\theta_7) & 0 & \sin(\theta_7) & 0 \\ \sin(\theta_7) & 0 & -\cos(\theta_7) & 0 \\ 0 & 1 & 0 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And multiplying all of these together yields

$$
\begin{aligned}
T_{07} =& T_{01}T_{12}T_{23}T_{34}T_{45}T_{56}T_{67} \qquad (1)\\
T_{07}[1,1] =& \left(-s_1c_{345} + s_{345}c_1c_2\right)s_7 + \left(s_1s_{345}c_6 + s_2s_6c_1 + c_1c_2c_6c_{345}\right)c_7\\
T_{07}[1,2] =& s_1s_6s_{345} - s_2c_1c_6 + s_6c_1c_2c_{345}\\
T_{07}[1,3] =& \left(s_1c_{345} - s_{345}c_1c_2\right)c_7 + \left(s_1s_{345}c_6 + s_2s_6c_1 + c_1c_2c_6c_{345}\right)s_7\\
T_{07}[1,4] =& a_3s_1s_3 + a_3c_1c_2c_3 + a_4s_1s_{34} + a_4c_1c_2c_{34} + d_2s_1 + d_3s_2c_1 - d_6s_1c_{345} + d_6s_{345}c_1c_2\\
&+ d_7s_1s_6s_{345} - d_7s_2c_1c_6 + d_7s_6c_1c_2c_{345}\\
T_{07}[2,1] =& \left(s_1s_{345}c_2 + c_1c_{345}\right)s_7 + \left(s_1s_2s_6 + s_1c_2c_6c_{345} - s_{345}c_1c_6\right)c_7\\
T_{07}[2,2] =& -s_1s_2c_6 + s_1s_6c_2c_{345} - s_6s_{345}c_1\\
T_{07}[2,3] =& -\left(s_1s_{345}c_2 + c_1c_{345}\right)c_7 + \left(s_1s_2s_6 + s_1c_2c_6c_{345} - s_{345}c_1c_6\right)s_7\\
T_{07}[2,4] =& a_3s_1c_2c_3 - a_3s_3c_1 + a_4s_1c_2c_{34} - a_4s_{34}c_1 - d_2c_1 + d_3s_1s_2 + d_6s_1s_{345}c_2 + d_6c_1c_{345}\\
&- d_7s_1s_2c_6 + d_7s_1s_6c_2c_{345} - d_7s_6s_{345}c_1\\
T_{07}[3,1] =& \left(s_2c_6c_{345} - s_6c_2\right)c_7 + s_2s_7s_{345}\\
T_{07}[3,2] =& s_2s_6c_{345} + c_2c_6\\
T_{07}[3,3] =& \left(s_2c_6c_{345} - s_6c_2\right)s_7 - s_2s_{345}c_7\\
T_{07}[3,4] =& a_3s_2c_3 + a_4s_2c_{34} + d_1 - d_3c_2 + d_6s_2s_{345} + d_7s_2s_6c_{345} + d_7c_2c_6\\
T_{07}[4,1] =& 0\\
T_{07}[4,2] =& 0\\
T_{07}[4,3] =& 0\\
T_{07}[4,4] =& 1
\end{aligned}
$$

## 2.2 Joint/Shape Matrices

We can similarly use joint and shape matrices to arrive at these $T$ matrices. All of the joints of the SSRMS are revolute, and can be modeled with the joint matrix of

$$
\Phi_h\left(\phi_h\right) = \begin{bmatrix} \cos\phi_h & -\sin\phi_h & 0 & 0\\ \sin\phi_h & \cos\phi_h & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
T_{i,i+1} &= S_{i,j}\Phi_j S_{i+1,j}^{-1} \\
T_{12} &= S_{1A}\Phi_A S_{2A}^{-1} \\
T_{23} &= S_{2B}\Phi_B S_{3B}^{-1} \\
T_{34} &= S_{3C}\Phi_C S_{4C}^{-1} \\
T_{45} &= S_{4D}\Phi_D S_{5D}^{-1} \\
T_{56} &= S_{5E}\Phi_E S_{6E}^{-1} \\
T_{67} &= S_{6F}\Phi_F S_{7F}^{-1} \\
T_{78} &= S_{7G}\Phi_G S_{8G}^{-1}
\end{aligned}
$$

For joints $\Phi_A, \Phi_B, \Phi_C, \Phi_D, \Phi_E, \Phi_F$, and $\Phi_G$, we also define two shape matrices.

$$T_{01} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{1A} = I, S_{2A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{2B} = I, S_{3B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{3C} = I, S_{4C} = \begin{bmatrix} 1 & 0 & 0 & -a_3 \\ 0 & 0 & 1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{4D} = I, S_{5D} = \begin{bmatrix} 1 & 0 & 0 & -a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{5E} = I, S_{6E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{6F} = I, S_{7F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{7G} = I, S_{8G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -d_7 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## 2.3 Inverse Kinematics Solution

In general we can define

$$\begin{aligned} T_{07} &= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= T_{01}T_{12}T_{23}T_{34}T_{45}T_{56}T_{67} \end{aligned}$$

Premultiplying both sides by $T_{01}^{-1}$ yields,

$$T_{01}^{-1}T_{07} = T_{12}T_{23}T_{34}T_{45}T_{56}T_{67}$$

Equating each element $(i, j)$ on both the left and right hand sides yields:

$$n_x c_1 + n_y s_1 = (s_2 s_6 + c_2 c_6 c_{345})\, c_7 + s_7 s_{345} c_2 \tag{2}$$
$$o_x c_1 + o_y s_1 = -s_2 c_6 + s_6 c_2 c_{345} \tag{3}$$
$$a_x c_1 + a_y s_1 = (s_2 s_6 + c_2 c_6 c_{345})\, s_7 - s_{345} c_2 c_7 \tag{4}$$
$$p_x c_1 + p_y s_1 = a_3 c_2 c_3 + a_4 c_2 c_{34} + d_3 s_2 + d_6 s_{345} c_2 - d_7 s_2 c_6 + d_7 s_6 c_2 c_{345} \tag{5}$$
$$n_z = (s_2 c_6 c_{345} - s_6 c_2)\, c_7 + s_2 s_7 s_{345} \tag{6}$$
$$o_z = s_2 s_6 c_{345} + c_2 c_6 \tag{7}$$
$$a_z = (s_2 c_6 c_{345} - s_6 c_2)\, s_7 - s_2 s_{345} c_7 \tag{8}$$
$$-d_1 + p_z = a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6 \tag{9}$$
$$n_x s_1 - n_y c_1 = -s_7 c_{345} + s_{345} c_6 c_7 \tag{10}$$
$$o_x s_1 - o_y c_1 = s_6 s_{345} \tag{11}$$
$$a_x s_1 - a_y c_1 = s_7 s_{345} c_6 + c_7 c_{345} \tag{12}$$
$$p_x s_1 - p_y c_1 = a_3 s_3 + a_4 s_{34} + d_2 - d_6 c_{345} + d_7 s_6 s_{345} \tag{13}$$
$$0 = 0 \tag{14}$$
$$0 = 0 \tag{15}$$
$$0 = 0 \tag{16}$$
$$1 = 1 \tag{17}$$

where we have defined $s_i = \sin i, c_i = \cos i, s_{ij} = \sin(i + j), c_{ij} = \cos(i + j), s_{ijk} = \sin(i + j + k)$ and $c_{ijk} = \cos(i + j + k)$. Manipulating the equations, we take $(Eq.\ 2)\, s_2 - (Eq.\ 6)\, c_2$ and simplify, producing

$$\left(n_x c_1 + n_y s_1\right) s_2 - n_z c_2 = s_6 c_7 \tag{18}$$

Similarly, we can do $(Eq.\ 5)\, s_2 - (Eq.\ 9)\, c_2$ and simplify, which results in

$$\left(p_x c_1 + p_y s_1\right) s_2 - (-d_1 + p_z)\, c_2 = d_3 - c_6 d_7 \tag{19}$$

We can also subtract $(Eq.\ 7)\, c_2 - (Eq.\ 3)\, s_2$

$$o_z c_2 - \left(o_x c_1 + o_y s_1\right) s_2 = c_6 \tag{20}$$

Finally, we can also subtract $(Eq.\ 4)\, s_2 - (Eq.\ 8)\, c_2$

$$\left(a_x c_1 + a_y s_1\right) s_2 - a_z c_2 = s_6 s_7 \tag{21}$$

Rearranging Equations 19 and 20 to be equal to $c_6$ and equating the two yields

$$-d_3 = \Big((o_x d_7 - p_x)\, c_1 + \left(o_y d_7 - p_y\right) s_1\Big)\, s_2 + (-o_z d_7 - d_1 + p_z)\, c_2 \tag{22}$$

Locking the shoulder roll angle to a known angle, $\boxed{\theta_1 = \beta}$, we can solve for $\theta_2$,

$$\boxed{\theta_2 = \mathrm{SHOULDER} \cdot \mathrm{acos}\left(\frac{d_3}{\sqrt{h_1^2 + q_1^2}}\right) + \mathrm{atan2}(q_1, h_1)} \tag{23}$$

where

$$h_1 = (-o_z d_7 - d_1 + p_z) \tag{24}$$

$$q_1 = \Big((o_x d_7 - p_x)\, c_\beta + \big(o_y d_7 - p_y\big)\, s_\beta\Big) \tag{25}$$

With $\theta_1$ and $\theta_2$ now known, $\theta_6$ can be solved using Equation 20,

$$\boxed{\theta_6 = \mathrm{WRIST} \cdot \mathrm{acos}\Big(o_z c_2 - \big(o_x c_1 + o_y s_1\big)\, s_2\Big)} \tag{26}$$

And we can then combine Equations 18 and 21, yielding

$$\boxed{\theta_7 = \mathrm{atan2}\left(\frac{\big(n_x c_1 + n_y s_1\big)\, s_2 - n_z c_2}{s_6}, \frac{\big(a_x c_1 + a_y s_1\big)\, s_2 - a_z c_2}{s_6}\right)} \tag{27}$$

With the shoulder and wrist joints resolved, we can now solve for the middle joints. We now take

$$\left(T_{12}^{-1}\right)(T_{17})\left(T_{67}^{-1}\right)\left(T_{56}^{-1}\right) = (T_{23})\,(T_{34})\,(T_{45})$$

Taking the left and right hand side $(1,4)$ and $(2,4)$ elements from the resulting matrix yields

$$\begin{aligned} a_3 c_3 + a_4 c_{34} = {} & d_6 \Big(a_z s_2 + c_2 \big(a_x c_1 + a_y s_1\big)\Big)\, c_7 \\ & - d_6 \Big(n_z s_2 + c_2 \big(n_x c_1 + n_y s_1\big)\Big)\, s_7 \\ & - d_7 \Big(o_z s_2 + c_2 \big(o_x c_1 + o_y s_1\big)\Big) \\ & + (-d_1 + p_z)\, s_2 + c_2 \big(p_x c_1 + p_y s_1\big) \end{aligned} \tag{28}$$

$$\begin{aligned} a_3 s_3 + a_4 s_{34} = {} & -d_2 + d_6 \big(a_x s_1 - a_y c_1\big)\, c_7 - d_6 \big(n_x s_1 - n_y c_1\big)\, s_7 \\ & - d_7 \big(o_x s_1 - o_y c_1\big) + p_x s_1 - p_y c_1 \end{aligned} \tag{29}$$

$\theta_4$ is then solved by combining the above two equations, resulting in

$$\boxed{\theta_4 = \mathrm{ELBOW} \cdot \mathrm{acos}\left(\frac{X^2 + Y^2 - a_3^2 - a_4^2}{2 a_3 a_4}\right)} \tag{30}$$

where

$$X = d_6\left(\left(a_z s_2 + c_2\left(a_x c_1 + a_y s_1\right)\right)c_7 - \left(n_z s_2 + c_2\left(n_x c_1 + n_y s_1\right)\right)s_7\right)$$
$$- d_7\left(o_z s_2 + c_2\left(o_x c_1 + o_y s_1\right)\right) + \left(-d_1 + p_z\right)s_2 + c_2\left(p_x c_1 + p_y s_1\right)$$
$$Y = -d_2 + d_6\left(a_x s_1 - a_y c_1\right)c_7 - d_6\left(n_x s_1 - n_y c_1\right)s_7 - d_7\left(o_x s_1 - o_y c_1\right) + p_x s_1 - p_y c_1$$

Substituting the solution into $\theta_4$ and Equations 28 and 29 and combining yields

$$\boxed{\theta_3 = \text{atan2}\left(Y\left(a_3 + a_4 c_4\right) - X a_4 s_4, X\left(a_3 + a_4 c_4\right) + Y a_4 s_4\right)}$$

Subtracting $(Eq.\ 12)c_7$ and $(Eq.\ 10)s_7$ yields

$$c_{345} = \left(a_x s_1 - a_y c_1\right)c_7 - \left(n_x s_1 - n_y c_1\right)s_7$$

And from Equation 11 we have

$$s_{345} = \frac{o_x s_1 - o_y c_1}{s_6}$$

which we can combine to solve for the last joint

$$\theta_5 = \left(\theta_3 + \theta_4 + \theta_5\right) - \left(\theta_3 + \theta_4\right)$$
$$\boxed{\theta_5 = \text{atan2}\left(s_{345}, c_{345}\right) - \left(\theta_3 + \theta_4\right)}$$

## 2.4 Numerical Example

For practical purposes, the link length and offset values can be set to

$$a_3 = 2.30, a_4 = 2.30, d_1 = 0.65, d_2 = 0.30$$
$$d_3 = 0.90, d_6 = 0.30, d_7 = 0.65$$

As an example, plugging in these values and the initial angles given in Table 1 into Equation 1 yields

$$T_{07} = \begin{bmatrix} 0 & 0 & 1 & 0.6 \\ 1 & 0 & 0 & 0.9 \\ 0 & 1 & 0 & 5.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

| $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ | $\theta_7$ |
|---|---|---|---|---|---|---|
| 60.000 | -20.268 | 64.074 | 79.722 | -149.770 | 138.205 | -77.426 |
| 60.000 | -20.268 | 58.153 | 99.444 | 16.428 | -138.205 | 102.573 |
| 60.000 | -20.268 | 143.797 | -79.722 | -70.048 | 138.205 | -77.426 |
| 60.000 | -109.087 | 35.576 | 79.140 | -119.938 | 49.659 | -85.275 |
| 60.000 | -20.268 | 157.598 | -99.444 | 115.872 | -138.205 | 102.573 |
| 60.000 | -109.087 | 23.189 | 100.025 | 51.565 | -49.659 | 94.724 |
| 60.000 | -109.087 | 114.717 | -79.140 | -40.797 | 49.659 | -85.275 |
| 60.000 | -109.087 | 123.214 | -100.025 | 151.590 | -49.659 | 94.724 |

Table 2: The eight possible configurations for locking the first joint.

# 3 Differential Kinematic Analysis

## 3.1 Method 1: Kinematic Jacobian

Where $\hat{z}_i$ is taken from the last column of $T_{1i}$, and can be defined

$$T_{1i} = \left[\begin{array}{ccc:c} & \underline{\underline{\Theta}}_i & & a_i \\ & \cdots & & \cdots \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{\underline{\Theta}}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}$$

$$\hat{z}_i = \left( \prod_{i=1}^{n} \underline{\underline{\Theta}}_i \right) z_i$$

and $\vec{r}_i$ is defined

$$\vec{r}_i = \sum_{i=1}^{n} \vec{a}_i$$

With these definitions, we can find the Jacobian via

$$\dot{\vec{P}} = \sum_{i=1}^{n} \left(\hat{z}_i \times \vec{r}_i\right) \dot{\theta}_i$$

$$\vec{w} = \sum_{i=1}^{n} \dot{\theta}_i \hat{z}_i$$

$$\underline{\underline{J}} \dot{q} = \begin{bmatrix} \underline{\dot{P}} \\ \underline{\vec{w}} \end{bmatrix}$$

$$\begin{bmatrix} \hat{z}_1 \times \vec{r}_1 & \hat{z}_2 \times \vec{r}_2 & \cdots & \hat{z}_7 \times \vec{r}_7 \\ \hat{z}_1 & \hat{z}_2 & \cdots & \hat{z}_7 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_7 \end{bmatrix} = \begin{bmatrix} \underline{\dot{P}}_{EE} \\ \underline{w}_{EE} \end{bmatrix}$$

$$
\begin{aligned}
J[1,1] &= -a_3 s_1 c_2 c_3 + a_3 s_3 c_1 - a_4 s_1 c_2 c_{34} + a_4 s_{34} c_1 + d_2 c_1 - d_3 s_1 s_2 \\
&\quad - d_6 s_1 s_{345} c_2 - d_6 c_1 c_{345} + d_7 s_1 s_2 c_6 - d_7 s_1 s_6 c_2 c_{345} + d_7 s_6 s_{345} c_1 \\
J[2,1] &= a_3 s_1 s_3 + a_3 c_1 c_2 c_3 + a_4 s_1 s_{34} + a_4 c_1 c_2 c_{34} + d_2 s_1 + d_3 s_2 c_1 \\
&\quad - d_6 s_1 c_{345} + d_6 s_{345} c_1 c_2 + d_7 s_1 s_6 s_{345} - d_7 s_2 c_1 c_6 + d_7 s_6 c_1 c_2 c_{345} \\
J[3,1] &= 0 \\
J[4,1] &= 0 \\
J[5,1] &= 0 \\
J[6,1] &= 1
\end{aligned}
$$

$$
\begin{aligned}
J[1,2] &= -\left(a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6\right) c_1 \\
J[2,2] &= -\left(a_3 s_2 c_3 + a_4 s_2 c_{34} - d_3 c_2 + d_6 s_2 s_{345} + d_7 s_2 s_6 c_{345} + d_7 c_2 c_6\right) s_1 \\
J[3,2] &= a_3 c_2 c_3 + a_4 c_2 c_{34} + d_3 s_2 + d_6 s_{345} c_2 - d_7 s_2 c_6 + d_7 s_6 c_2 c_{345} \\
J[4,2] &= s_1 \\
J[5,2] &= -c_1 \\
J[6,2] &= 0
\end{aligned}
$$

$$
\begin{aligned}
J[1,3] &= a_3 s_1 c_3 - a_3 s_3 c_1 c_2 + a_4 s_1 c_{34} - a_4 s_{34} c_1 c_2 + d_6 s_1 s_{345} \\
&\quad + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2 \\
J[2,3] &= -a_3 s_1 s_3 c_2 - a_3 c_1 c_3 - a_4 s_1 s_{34} c_2 - a_4 c_1 c_{34} + d_6 s_1 c_2 c_{345} \\
&\quad - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345} \\
J[3,3] &= \left(-a_3 s_3 - a_4 s_{34} + d_6 c_{345} - d_7 s_6 s_{345}\right) s_2 \\
J[4,3] &= s_2 c_1 \\
J[5,3] &= s_1 s_2 \\
J[6,3] &= -c_2
\end{aligned}
$$

$$J[1,4] = a_4 s_1 c_{34} - a_4 s_{34} c_1 c_2 + d_6 s_1 s_{345} + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2$$
$$J[2,4] = -a_4 s_1 s_{34} c_2 - a_4 c_1 c_{34} + d_6 s_1 c_2 c_{345} - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345}$$
$$J[3,4] = \left(-a_4 s_{34} + d_6 c_{345} - d_7 s_6 s_{345}\right) s_2$$
$$J[4,4] = s_2 c_1$$
$$J[5,4] = s_1 s_2$$
$$J[6,4] = -c_2$$

$$J[1,5] = d_6 s_1 s_{345} + d_6 c_1 c_2 c_{345} + d_7 s_1 s_6 c_{345} - d_7 s_6 s_{345} c_1 c_2$$
$$J[2,5] = d_6 s_1 c_2 c_{345} - d_6 s_{345} c_1 - d_7 s_1 s_6 s_{345} c_2 - d_7 s_6 c_1 c_{345}$$
$$J[3,5] = \left(d_6 c_{345} - d_7 s_6 s_{345}\right) s_2$$
$$J[4,5] = s_2 c_1$$
$$J[5,5] = s_1 s_2$$
$$J[6,5] = -c_2$$

$$J[1,6] = d_7 \left(s_1 s_{345} c_6 + s_2 s_6 c_1 + c_1 c_2 c_6 c_{345}\right)$$
$$J[2,6] = d_7 \left(s_1 s_2 s_6 + s_1 c_2 c_6 c_{345} - s_{345} c_1 c_6\right)$$
$$J[3,6] = d_7 \left(s_2 c_6 c_{345} - s_6 c_2\right)$$
$$J[4,6] = -s_1 c_{345} + s_{345} c_1 c_2$$
$$J[5,6] = s_1 s_{345} c_2 + c_1 c_{345}$$
$$J[6,6] = s_2 s_{345}$$

$$J[1,7] = 0$$
$$J[2,7] = 0$$
$$J[3,7] = 0$$
$$J[4,7] = \left(s_1 s_{345} + c_1 c_2 c_{345}\right) s_6 - s_2 c_1 c_6$$
$$J[5,7] = \left(s_1 c_2 c_{345} - s_{345} c_1\right) s_6 - s_1 s_2 c_6$$
$$J[6,7] = s_2 s_6 c_{345} + c_2 c_6$$

### 3.2 Method 2: Geometric Jacobian

We first form our $D_i$ matrices from

$$D_i = T_{0i} Q_i T_{0i}^{-1}$$

where, as all our joints are revolute,

$$Q = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Selecting elements from these $D_i$ matrices, we form the Jacobian via

$$J = \begin{bmatrix} {}^0D_{14} & {}^1D_{14} & {}^2D_{14} & {}^3D_{14} & {}^4D_{14} & {}^5D_{14} & {}^6D_{14} \\ {}^0D_{24} & {}^1D_{24} & {}^2D_{24} & {}^3D_{24} & {}^4D_{24} & {}^5D_{24} & {}^6D_{24} \\ {}^0D_{34} & {}^1D_{34} & {}^2D_{34} & {}^3D_{34} & {}^4D_{34} & {}^5D_{34} & {}^6D_{34} \\ {}^0D_{32} & {}^1D_{32} & {}^2D_{32} & {}^3D_{32} & {}^4D_{32} & {}^5D_{32} & {}^6D_{32} \\ {}^0D_{13} & {}^1D_{13} & {}^2D_{13} & {}^3D_{13} & {}^4D_{13} & {}^5D_{13} & {}^6D_{13} \\ {}^0D_{21} & {}^1D_{21} & {}^2D_{21} & {}^3D_{21} & {}^4D_{21} & {}^5D_{21} & {}^6D_{21} \end{bmatrix}$$

Resulting in

$$\begin{aligned} J[1,1] &= 0 \\ J[2,1] &= 0 \\ J[3,1] &= 0 \\ J[4,1] &= 0 \\ J[5,1] &= 0 \\ J[6,1] &= 1 \\ \\ J[1,2] &= d_1 c_1 \\ J[2,2] &= d_1 s_1 \\ J[3,2] &= 0 \\ J[4,2] &= s_1 \\ J[5,2] &= -c_1 \\ J[6,2] &= 0 \\ \\ J[1,3] &= -d_1 s_1 s_2 + d_2 c_1 c_2 \\ J[2,3] &= d_1 s_2 c_1 + d_2 s_1 c_2 \\ J[3,3] &= d_2 s_2 \\ J[4,3] &= s_2 c_1 \\ J[5,3] &= s_1 s_2 \\ J[6,3] &= -c_2 \end{aligned}$$

$$J[1,4] = -a_3 s_1 c_3 + a_3 s_3 c_1 c_2 - d_1 s_1 s_2 + d_2 c_1 c_2$$
$$J[2,4] = a_3 s_1 s_3 c_2 + a_3 c_1 c_3 + d_1 s_2 c_1 + d_2 s_1 c_2$$
$$J[3,4] = (a_3 s_3 + d_2)\, s_2$$
$$J[4,4] = s_2 c_1$$
$$J[5,4] = s_1 s_2$$
$$J[6,4] = -c_2$$

$$J[1,5] = -a_3 s_1 c_3 + a_3 s_3 c_1 c_2 - a_4 s_1 c_{34} + a_4 s_{34} c_1 c_2 - d_1 s_1 s_2 + d_2 c_1 c_2$$
$$J[2,5] = a_3 s_1 s_3 c_2 + a_3 c_1 c_3 + a_4 s_1 s_{34} c_2 + a_4 c_1 c_{34} + d_1 s_2 c_1 + d_2 s_1 c_2$$
$$J[3,5] = (a_3 s_3 + a_4 s_{34} + d_2)\, s_2$$
$$J[4,5] = s_2 c_1$$
$$J[5,5] = s_1 s_2$$
$$J[6,5] = -c_2$$

$$J[1,6] = -(d_1 c_2 - d_3)(s_1 s_{345} + c_1 c_2 c_{345}) - (a_3 c_{45} + a_4 c_5 + d_1 s_2 c_{345} + d_2 s_{345})\, s_2 c_1$$
$$J[2,6] = -(d_1 c_2 - d_3)(s_1 c_2 c_{345} - s_{345} c_1) - (a_3 c_{45} + a_4 c_5 + d_1 s_2 c_{345} + d_2 s_{345})\, s_1 s_2$$
$$J[3,6] = a_3 c_2 c_{45} + a_4 c_2 c_5 + d_2 s_{345} c_2 + d_3 s_2 c_{345}$$
$$J[4,6] = -s_1 c_{345} + s_{345} c_1 c_2$$
$$J[5,6] = s_1 s_{345} c_2 + c_1 c_{345}$$
$$J[6,6] = s_2 s_{345}$$

$$\begin{aligned}
J[1,7] = {} & \big((s_1 s_{345} + c_1 c_2 c_{345})\, c_6 + s_2 s_6 c_1\big)(a_3 s_{45} + a_4 s_5 + d_1 s_2 s_{345} - d_2 c_{345} + d_6) \\
& + (s_1 c_{345} - s_{345} c_1 c_2)(a_3 c_6 c_{45} + a_4 c_5 c_6 + d_1 s_2 c_6 c_{345} - d_1 s_6 c_2 + d_2 s_{345} c_6 + d_3 s_6) \\
J[2,7] = {} & \big((s_1 c_2 c_{345} - s_{345} c_1)\, c_6 + s_1 s_2 s_6\big)(a_3 s_{45} + a_4 s_5 + d_1 s_2 s_{345} - d_2 c_{345} + d_6) \\
& - (s_1 s_{345} c_2 + c_1 c_{345})(a_3 c_6 c_{45} + a_4 c_5 c_6 + d_1 s_2 c_6 c_{345} - d_1 s_6 c_2 + d_2 s_{345} c_6 + d_3 s_6) \\
J[3,7] = {} & (s_2 c_6 c_{345} - s_6 c_2)(a_3 s_{45} + a_4 s_5 + d_1 s_2 s_{345} - d_2 c_{345} + d_6) - \\
& (a_3 c_6 c_{45} + a_4 c_5 c_6 + d_1 s_2 c_6 c_{345} - d_1 s_6 c_2 + d_2 s_{345} c_6 + d_3 s_6)\, s_2 s_{345} \\
J[4,7] = {} & s_1 s_6 s_{345} - s_2 c_1 c_6 + s_6 c_1 c_2 c_{345} \\
J[5,7] = {} & -s_1 s_2 c_6 + s_1 s_6 c_2 c_{345} - s_6 s_{345} c_1 \\
J[6,7] = {} & s_2 s_6 c_{345} + c_2 c_6
\end{aligned}$$

## 3.3 Velocity Equation

I should write this?

# 4 Conclusions

## References

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