Hwk 3 Solutions

5.6. Consider a robot manipulator as shown in Figure P5.6. The kinematic structure of this robotic arm is very similar to that of the Stanford manipulator studied in example 5.6 except that it has an offset (a) between the base and the shoulder (the first two) joint axes. For this robotic arm derive and solve the kinematic posture equations using shape and joint matrices.

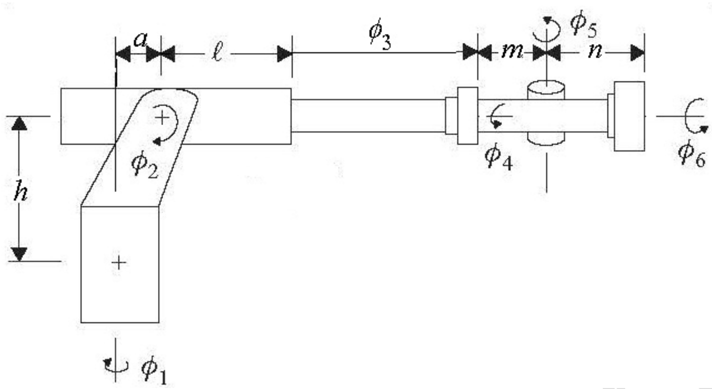
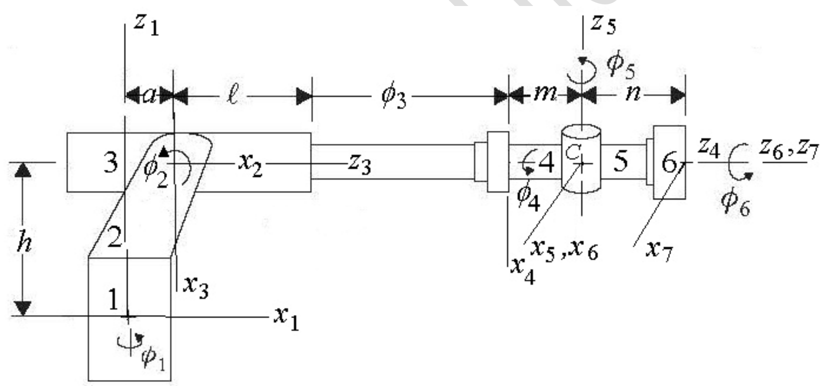


Figure P5.6

Using the axes and symbols shown in the following figure,



we can establish the following shape matrices by inspection:

$$S_{11} = I,\ S_{21} = \begin{bmatrix} 1 & 0 & 0 & -a \\ 0 & 0 & 1 & -h \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ S_{22} = I,\ S_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ S_{33} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \ell \\ 0 & 0 & 0 & 1 \end{bmatrix},\ S_{43} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_{44} = I,\ \ S_{54} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -m \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ S_{55} = I,\ \ S_{65} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ S_{66} = I, S_{67} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using these and Eq. (5.4) with the proper joint types, we form

$$T_{12} = S_{11}\Phi_1 S_{21}^{-1} = \begin{bmatrix} \cos\phi_1 & 0 & \sin\phi_1 & a\cos\phi_1 \\ \sin\phi_1 & 0 & -\cos\phi_1 & a\sin\phi_1 \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{23} = S_{22}\Phi_2 S_{32}^{-1} = \begin{bmatrix} \cos\phi_2 & 0 & -\sin\phi_2 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{34} = S_{33}\Phi_3 S_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \ell+\phi_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{45} = S_{44}\Phi_4 S_{54}^{-1} = \begin{bmatrix} \cos\phi_4 & 0 & \sin\phi_4 & 0 \\ \sin\phi_4 & 0 & -\cos\phi_4 & 0 \\ 0 & 1 & 0 & m \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{56} = S_{55}\Phi_5 S_{65}^{-1} = \begin{bmatrix} \cos\phi_5 & 0 & -\sin\phi_5 & 0 \\ \sin\phi_5 & 0 & \cos\phi_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{67} = S_{66}\Phi_6 S_{76}^{-1} = \begin{bmatrix} \cos\phi_6 & -\sin\phi_6 & 0 & 0 \\ \sin\phi_6 & \cos\phi_6 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, following section 5.7, we find the coordinates of the wrist center point *C* as

$$R_C = T_{17}\begin{bmatrix} 0 \\ 0 \\ -n \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

and, since the values of T_{17} are given as the input command, the values of *X*, *Y*, *Z* become known.

Now, Eq. (5.18) can be used in concept, and from the above we find

$$T_{14} = T_{12}T_{23}T_{34} = \begin{bmatrix} \cos\phi_1\cos\phi_2 & -\sin\phi_1 & -\cos\phi_1\sin\phi_2 & a\cos\phi_1 - (\ell+\phi_3)\cos\phi_1\sin\phi_2 \\ \sin\phi_1\cos\phi_2 & \cos\phi_1 & -\sin\phi_1\sin\phi_2 & a\sin\phi_1 - (\ell+\phi_3)\sin\phi_1\sin\phi_2 \\ \sin\phi_2 & 0 & \cos\phi_2 & h + (\ell+\phi_3)\cos\phi_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, the wrist point *C* is the point where $r_{C_4} = [0, 0, m, 1]^t$ for which Eq. (5.18) gives

$$R_C = T_{14}\begin{bmatrix} 0 \\ 0 \\ m \\ 1 \end{bmatrix} = \begin{bmatrix} a\cos\phi_1 - (\ell+\phi_3+m)\cos\phi_1\sin\phi_2 \\ a\sin\phi_1 - (\ell+\phi_3+m)\sin\phi_1\sin\phi_2 \\ h + (\ell+\phi_3+m)\cos\phi_2 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

where *X*, *Y*, *Z* are the requested target point coordinates, found above, to be reached by point *C*.

These equations can now be rearranged to read

$$
\begin{aligned}
(\ell+\phi_3+m)\cos\phi_2 &= Z-h \\
\left[a-(\ell+\phi_3+m)\sin\phi_2\right]\sin\phi_1 &= \left[a-(Z-h)\tan\phi_2\right]\sin\phi_1 = Y \qquad (a)\\
\left[a-(\ell+\phi_3+m)\sin\phi_2\right]\cos\phi_1 &= \left[a-(Z-h)\tan\phi_2\right]\cos\phi_1 = X
\end{aligned}
$$

From these we get $\tan\phi_1 = Y/X$ and, by use of tangent of half-angle formula this gives

$$\phi_1 = 2\tan^{-1}\left[\frac{\sqrt{X^2+Y^2}-X}{Y}\right] \qquad \underline{Ans}.$$

Notice that this half-angle approach gives the result in the proper quadrant and is not double-valued. The same result can be achieved in software by use of the atan2 function.

Returning to Eqs. (a) once ϕ_1 is known we can write

$$\tan\phi_2 = \frac{a-\left(X/\cos\phi_1\right)}{Z-h} = \frac{a-\left(Y/\sin\phi_1\right)}{Z-h} = \frac{1}{S}$$

Again using the half-angle approach to assure the proper quadrant, the solution to this equation is

$$\phi_2 = 2\tan^{-1}\left(\sqrt{S^2+1}-S\right) \qquad \underline{Ans}.$$

Finally, from the first of Eqs. (a)

$$\phi_3 = \frac{Z-h}{\cos\phi_2} - (\ell+m) \qquad \underline{Ans}.$$

Now that values are known for ϕ_1, ϕ_2, and ϕ_3, the entire matrix T_{14} can be evaluated numerically. Also, since the entire matrix T_{17} is given, the transformation $T_{47} = T_{14}^{-1}T_{17}$ can be found numerically. Also, from the forms of the transformations shown above, we find

$$T_{47} = T_{45}T_{56}T_{67} = \begin{bmatrix} \cos\phi_4\cos\phi_5\cos\phi_6 - \sin\phi_4\sin\phi_6 & -\cos\phi_4\cos\phi_5\sin\phi_6 - \sin\phi_4\cos\phi_6 & -\cos\phi_4\sin\phi_5 & -n\cos\phi_4\sin\phi_5 \\ \sin\phi_4\cos\phi_5\cos\phi_6 + \cos\phi_4\sin\phi_6 & -\sin\phi_4\cos\phi_5\sin\phi_6 + \cos\phi_4\cos\phi_6 & -\sin\phi_4\sin\phi_5 & -n\sin\phi_4\sin\phi_5 \\ \sin\phi_5\cos\phi_6 & -\sin\phi_5\sin\phi_6 & \cos\phi_5 & m+n\cos\phi_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and, since we now know numeric values for T_{47}, we can find from row 3, column 3 that

$$\phi_5 = \cos^{-1} T_{47}(3,3) \qquad \underline{Ans}.$$

and then, from column 3,

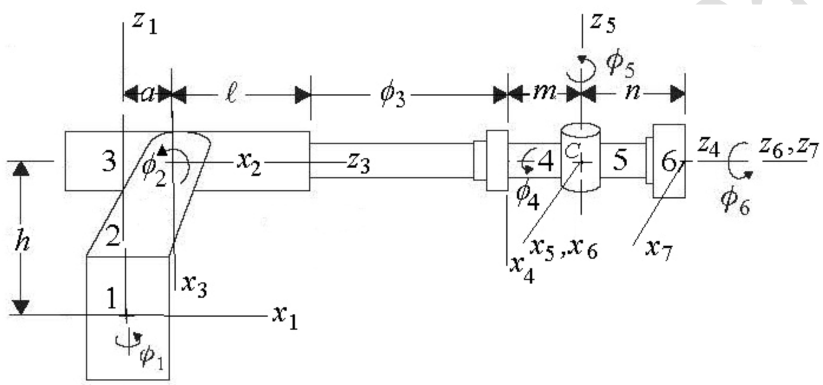
$$\phi_4 = \tan^{-1}\frac{T_{47}(2,3)}{T_{47}(1,3)} \qquad \underline{Ans}.$$

Notice that, if the elements of column 4, rows 1 and 2, are not in agreement with these results, it is necessary to reverse the sign of ϕ_5 and to reevaluate ϕ_4. Finally, from row 3,

$$\phi_6 = \tan^{-1}\frac{-T_{47}(3,2)}{T_{47}(3,1)} \qquad \underline{Ans}.$$

5.7. For the robot manipulator of problem 5.6, derive the kinematic position equations using Denavit-Hartengberg transformation matrices and find the solution to these equations using the partitioning method of section 5.7.

Using the axes and symbols shown in the following figure,



we can establish the following D-H parameters:

$$R_1\begin{vmatrix} a \\ 90^\circ \\ \phi_1 \\ h \end{vmatrix} R_2\begin{vmatrix} 0 \\ -90^\circ \\ \phi_2 \\ 0 \end{vmatrix} P_3\begin{vmatrix} 0 \\ 0 \\ 0 \\ \ell+\phi_3 \end{vmatrix} R_4\begin{vmatrix} 0 \\ 90^\circ \\ \phi_4 \\ m \end{vmatrix} R_5\begin{vmatrix} 0 \\ -90^\circ \\ \phi_5 \\ 0 \end{vmatrix} R_6\begin{vmatrix} 0 \\ 0 \\ \phi_6 \\ n \end{vmatrix}$$

Notice that the figure is drawn with $\phi_2 = -90^\circ$ and $\phi_4 = -90^\circ$.

Using these parameters and Eq. (5.8), we form

$$\mathbb{T}_{12} = \begin{bmatrix} \cos\phi_1 & 0 & \sin\phi_1 & a\cos\phi_1 \\ \sin\phi_1 & 0 & -\cos\phi_1 & a\sin\phi_1 \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbb{T}_{23} = \begin{bmatrix} \cos\phi_2 & 0 & -\sin\phi_2 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbb{T}_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \ell+\phi_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbb{T}_{45} = \begin{bmatrix} \cos\phi_4 & 0 & \sin\phi_4 & 0 \\ \sin\phi_4 & 0 & -\cos\phi_4 & 0 \\ 0 & 1 & 0 & m \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$
\mathbb{T}_{56} = \begin{bmatrix} \cos\phi_5 & 0 & -\sin\phi_5 & 0 \\ \sin\phi_5 & 0 & \cos\phi_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbb{T}_{67} = \begin{bmatrix} \cos\phi_6 & -\sin\phi_6 & 0 & 0 \\ \sin\phi_6 & \cos\phi_6 & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Next, following section 5.7, we can find the requested coordinates of the wrist center point C as

$$
R_C = T_{17} \begin{bmatrix} 0 \\ 0 \\ -n \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}
$$

and, since the values of T_{17} are given as the input command, the values of X, Y, Z become known.

Now, Eq. (5.18) can be used in concept, and from the above transformations we find

$$
T_{14} = \mathbb{T}_{12}\mathbb{T}_{23}\mathbb{T}_{34} = \begin{bmatrix} \cos\phi_1\cos\phi_2 & -\sin\phi_1 & -\cos\phi_1\sin\phi_2 & a\cos\phi_1 - (\ell+\phi_3)\cos\phi_1\sin\phi_2 \\ \sin\phi_1\cos\phi_2 & \cos\phi_1 & -\sin\phi_1\sin\phi_2 & a\sin\phi_1 - (\ell+\phi_3)\sin\phi_1\sin\phi_2 \\ \sin\phi_2 & 0 & \cos\phi_2 & h + (\ell+\phi_3)\cos\phi_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

However, the wrist point C is the point where $r_{C_4} = [0, 0, m, 1]^t$ for which Eq. (5.18) gives

$$
R_C = T_{14} \begin{bmatrix} 0 \\ 0 \\ m \\ 1 \end{bmatrix} = \begin{bmatrix} a\cos\phi_1 - (\ell+\phi_3+m)\cos\phi_1\sin\phi_2 \\ a\sin\phi_1 - (\ell+\phi_3+m)\sin\phi_1\sin\phi_2 \\ h + (\ell+\phi_3+m)\cos\phi_2 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}
$$

where X, Y, Z are the requested target point coordinates found above to be reached by point C.

These equations can now be rearranged to read

$$
\begin{aligned}
(\ell+\phi_3+m)\cos\phi_2 &= Z - h \\
\left[a - (\ell+\phi_3+m)\sin\phi_2\right]\sin\phi_1 &= \left[a - (Z-h)\tan\phi_2\right]\sin\phi_1 = Y \\
\left[a - (\ell+\phi_3+m)\sin\phi_2\right]\cos\phi_1 &= \left[a - (Z-h)\tan\phi_2\right]\cos\phi_1 = X
\end{aligned} \qquad (a)
$$

From these we can get $\tan\phi_1 = Y/X$ and, by use of tangent of half-angle formulae this gives

$$\phi_1 = 2\tan^{-1}\left[\frac{\sqrt{X^2+Y^2}-X}{Y}\right] \qquad \underline{Ans}.$$

Notice that this half-angle approach gives the result in the proper quadrant and is not double-valued. The same result can be achieved in software by use of the `atan2` function.

Returning to Eqs. (a) once ϕ_1 is known we can write

$$\tan\phi_2 = \frac{a-\left(X/\cos\phi_1\right)}{Z-h} = \frac{a-\left(Y/\sin\phi_1\right)}{Z-h} = \frac{1}{S}$$

Again using the half-angle approach to assure the proper quadrant, the solution to this equation is

$$\phi_2 = 2\tan^{-1}\left(\sqrt{S^2+1}-S\right) \qquad \underline{Ans}.$$

Finally, from the first of Eqs. (a)

$$\phi_3 = \frac{Z-h}{\cos\phi_2} - (\ell + m) \qquad \underline{Ans}.$$

Now that values are known for ϕ_1, ϕ_2, and ϕ_3, the entire matrix T_{14} can be evaluated numerically. Also, since the entire matrix T_{17} is given as the input command, the transformation $T_{47} = T_{14}^{-1}T_{17}$ can be found numerically. Also, from the forms of the transformations shown above, we find

$$T_{47} = \mathbb{T}_{45}\mathbb{T}_{56}\mathbb{T}_{67} = \begin{bmatrix} \cos\phi_4\cos\phi_5\cos\phi_6 - \sin\phi_4\sin\phi_6 & -\cos\phi_4\cos\phi_5\sin\phi_6 - \sin\phi_4\cos\phi_6 & -\cos\phi_4\sin\phi_5 & -n\cos\phi_4\sin\phi_5 \\ \sin\phi_4\cos\phi_5\cos\phi_6 + \cos\phi_4\sin\phi_6 & -\sin\phi_4\cos\phi_5\sin\phi_6 + \cos\phi_4\cos\phi_6 & -\sin\phi_4\sin\phi_5 & -n\sin\phi_4\sin\phi_5 \\ \sin\phi_5\cos\phi_6 & -\sin\phi_5\sin\phi_6 & \cos\phi_5 & m+n\cos\phi_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and, since we now know numeric values for T_{47}, we find from row 3 column 3 that

$$\phi_5 = \cos^{-1} T_{47}(3,3) \qquad \underline{Ans}.$$

and then, from column 3,

$$\phi_4 = \tan^{-1}\frac{T_{47}(2,3)}{T_{47}(1,3)} \qquad \underline{Ans}.$$

Notice that, if the signs of column 4, rows 1 and 2, are not in agreement with these results, it is necessary to reverse the sign of ϕ_5. Finally, from row 3,

$$\phi_6 = \tan^{-1}\frac{-T_{47}(3,2)}{T_{47}(3,1)} \qquad \underline{Ans}.$$