ECS 253 / MAE 253, Network Theory
Spring 2016
Problem Set # 3, Solutions

Problem 1: Resolution limit of modularity function

(a) For the intuitive division $C = n$; $l_c = m(m-1)/2 = (L-n)/n$ and $d_c = m(m-1)+2 = 2L/n$ for all c. Therefore

$$Q_{\text{single}} = n\frac{L/n - 1}{L} - n\left(\frac{2L/n}{2L}\right)^2 = 1 - \frac{n}{L} - \frac{1}{n}. \tag{1}$$

If we group two neighboring cliques together, we get $C = n/2$; $l_c = m(m-1)+1 = 2(L-n)/n + 1 = 2L/n - 1$ and $d_c = 2m(m-1)+4 = 4L/n$ for all c. Therefore

$$Q_{\text{pairs}} = \frac{n}{2}\frac{2L/n - 1}{L} - \frac{n}{2}\left(\frac{4L/n}{2L}\right)^2 = 1 - \frac{n}{2L} - \frac{2}{n}. \tag{2}$$

(b) The correct division is found if $Q_{\text{single}} > Q_{\text{pairs}}$:

$$1 - \frac{n}{L} - \frac{1}{n} > 1 - \frac{n}{2L} - \frac{2}{n}, \tag{3}$$
$$2L > n^2. \tag{4}$$

(c) Modularity favors community divisions where the number of communities is $\sim \sqrt{L}$, thus may fail to uncover a more intuitive division.

Problem 2: User versus system optimal

(a) In case of selfish drivers, taking any road takes the same time T:

$$x = T, \tag{5}$$
$$\frac{1}{2}y + \frac{1}{2} = T, \tag{6}$$
$$\frac{1}{3}z + \frac{1}{3} = T. \tag{7}$$

From here we get

$$x = T, \tag{8}$$
$$y = \frac{T - 1/2}{1/2} = 2T - 1, \tag{9}$$
$$z = 3T - 1. \tag{10}$$

Substituting into the normalization condition:

$$1 = T + 2T - 1 + 3T - 1 \quad \rightarrow \quad T = \frac{1}{2} \tag{11}$$

Substituting back we get $x = 1/2$, $y = 0$ and $z = 1/2$. Both the worst case driving and the average driving time is equal to $T = 1/2$.

(b) The average travel time is given by

$$T(x, y, z) = xT_1(x) + yT_2(y) + zT_3(z) = x^2 + \frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{3}z^2 + \frac{1}{3}z, \tag{12}$$

we have to minimize this function with the constraint $x + y + z = 1$. For this we use the Lagrange function

$$L(x, y, z, \lambda) = x^2 + \frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{3}z^2 + \frac{1}{3}z - \lambda(x + y + z - 1). \tag{13}$$

The extremal value is found by solving the equations $\nabla L = 0$:

$$2x = \lambda \tag{14}$$
$$y + \frac{1}{2} = \lambda \tag{15}$$
$$\frac{2}{3}z + \frac{1}{3} = \lambda \tag{16}$$
$$x + y + z = 1. \tag{17}$$

From the first three equations we get:

$$x = \frac{\lambda}{2} \tag{18}$$
$$y = \lambda - \frac{1}{2} \tag{19}$$
$$\frac{2}{3}z + \frac{1}{3} = \frac{3\lambda - 1}{2}. \tag{20}$$

Substituting this into the constraint:

$$1 = \frac{\lambda}{2} + \lambda - \frac{1}{2} + \frac{3\lambda - 1}{2}, \tag{21}$$

providing $\lambda = 2/3$. Substituting back we obtain $x = 1/3$, $y = 1/6$ and $z = 1/2$. We find $0 \leq x, y, z \leq 1$; therefore the solution corresponds to a valid extremal value. The average travel time is:

$$T = \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{6}\left(\frac{1}{2}\frac{1}{6} + \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3}\frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3}\frac{1}{3} + \frac{1}{6}\frac{7}{12} + \frac{1}{2}\frac{1}{2} = \frac{11}{24} \approx 0.458, \tag{22}$$

which is smaller than the selfish average; therefore the extremum is a minimum. The worst case travel time corresponds to Road 2: $T_2 = 7/12$.

Problem 3: User versus system optimal

(a) In case of selfish drivers, taking any road takes the same time T:

$$x = T, \tag{23}$$
$$ay + b = T, \tag{24}$$
$$x + y = 1. \tag{25}$$

From here we get

$$x_s = a(1 - x_s) + b \quad \rightarrow \quad x_s = \frac{a+b}{1+a}. \tag{26}$$

The average and the worst case selfish travel time is also $T_s = \frac{a+b}{1+a}$. All traffic travels on Road 1 if

$$x_s = \frac{a+b}{1+a} \geq 0 \quad \rightarrow \quad b \geq 1. \tag{27}$$

(b) The average travel time is given by

$$T(x, y) = xT_1(x) + yT_2(y), \tag{28}$$

using the constraint we get

$$T(x) = x^2 + (1 - x)[a(1 - x) + b]. \tag{29}$$

The place of the extremum is given by

$$0 = 2x_g + 2ax_g - 2a - b \quad \rightarrow \quad x_g = \frac{2a+b}{2(1+a)}. \tag{30}$$

The global optimum of the average travel time is

$$T_g = \left(\frac{a+b/2}{1+a}\right)^2 + a\left(1 - \frac{a+b/2}{1+a}\right)^2 + b\left(1 - \frac{a+b/2}{1+a}\right). \tag{31}$$

All traffic travels on Road 1 if

$$x_g = \frac{2a+b}{2(1+a)} \geq 1 \quad \rightarrow \quad b \geq 2. \tag{32}$$

(c)

$$T_s - T_g = \frac{a+b}{a+1} - \left(\frac{a+b/2}{1+a}\right)^2 - a\left(1 - \frac{a+b/2}{1+a}\right)^2 - b\left(1 - \frac{a+b/2}{1+a}\right) \tag{33}$$

$$= \frac{(a^2 + ab + a + b) - (a^2 + b^2/4 + ab) - (a + ab^2/4 - ab) - (ab - ab^2/2 + b - b^2/2)}{(a+1)^2} \tag{34}$$

$$= \frac{b^2}{4(a+1)} \tag{35}$$

$$> 0 \tag{36}$$