MAE-253: Homework 1

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Problem 1

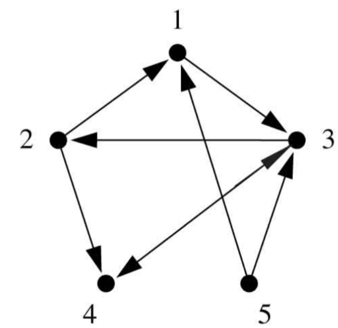

Figure 1: Simple network.

a) Consider the simple network shown above and write down its the adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Consider a random walk on this network. What is the steady-state probability of finding the walker on each node?

To find the stead-state probability of finding a random walker on any node, we first calculate the transition matrix by normalizing A column-wise.

$$T = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The steady-state probability is the eigenvector of T associated with the eigenvalue of 1,

$$\pi = \begin{bmatrix} -0.18257419 \\ -0.36514837 \\ -0.73029674 \\ -0.54772256 \\ 0 \end{bmatrix}.$$

Normalizing π, we have

$$\pi = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \\ 0.3 \\ 0 \end{bmatrix}.$$

c) What would be the steady-state probability of finding the walker on each node if the edges were instead undirected?

The steady-state probability of finding a walker on a node of an undirected graph is the degree of the node divided by the total number of edges,

$$\pi = \frac{1}{14}\begin{bmatrix}3\\3\\4\\2\\2\end{bmatrix}.$$

Problem 2

Consider a variant of the BA model that does not feature preferential attachment. We start with a single node at time $t = 1$. In each subsequent discrete time step, a new node is added with $m = 1$ links to existing nodes. The probability that a link arriving at time step $t+1$ connects to any existing node i is uniformly distributions and independent of i:

$$\pi_i = \frac{1}{t}.$$

Let $n_{k,t}$ denote the expected number of nodes of degree k at time t. For the steps below, proceed as in lecture.

a) Write the rate equation for $n_{k,t+1}$ in terms of the $n_{j,t}$'s. (Note you will need to equations, one for $k = 1$ and one for $k > 1$.)

$$k = 1; \qquad n_{1,t+1} = n_{1,t} + 1 - \frac{1}{t}n_{1,t}$$
$$k > 1; \qquad n_{k,t+1} = n_{k,t} + \frac{1}{t}n_{k-1,t} - \frac{1}{t}n_{k,t}$$

b) Converting from expected number of nodes to probabilities, $p_{k,t} = n_{k,t}/n_t$, rewrite the equations in part (a) in terms of the probabilities.

$$\begin{aligned} p_{k,t} = n_{k,t}/n_t \implies n_{k,t} =& p_{k,t}n_t \\ =& p_{k,t} \cdot t \end{aligned}$$

From a), we then have

$$k = 1; \qquad p_{1,t+1}(t+1) = p_{1,t} + 1 - \frac{1}{t}p_{1,t} \cdot t$$
$$k > 1; \qquad p_{k,t+1}(t+1) = p_{k,t} \cdot t + \frac{1}{t}p_{k-1,t} \cdot t - \frac{1}{t}p_{k,t} \cdot t$$

c) Assume steady-state, that $p_{k,t} = p_k$, and solve the recurrence relation to obtain p_k in terms of p_{k-1}.

$$
\begin{aligned}
k > 1; \qquad p_k(t+1) &= p_k \cdot t + \frac{1}{t} p_{k-1} \cdot t - \frac{1}{t} p_k \cdot t \\
p_k(t+1) &= p_k(t-1) + \frac{1}{t} p_{k-1} \\
p_k(t+1-t+1) &= \frac{1}{t} p_{k-1} \\
p_k &= \frac{1}{2} p_{k-1}
\end{aligned}
$$

d) Starting by solving for p_1 and recursing, derive the expression for the stationary degree distribution p_k.

$$
\begin{aligned}
k = 1; \qquad p_1(t+1) &= p_1 + 1 - \frac{1}{t} p_1 \cdot t \\
p_1(t+1) &= 1 \\
p_1 &= \frac{1}{2}
\end{aligned}
$$

Plugging this into our equation for p_k, we find the following relation:

$$p_k = \left(\frac{1}{2}\right)^k$$