ECS 253 / MAE 253, Network Theory
Spring 2016
Problem Set # 3, Solutions

Problem 1: Modularity matrix

1.1 Bisection of a binary undirected network

1. The adjacency matrix is as following:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{1}$$

2. The modularity matrix is as following:

$$\begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix} \tag{2}$$

3. The eigenvalues of the modularity matrix are: 1,0,0,0,-2,-2.

4. The eigenvector corresponding to the largeset eigenvalue is:

$$\begin{bmatrix} -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \tag{3}$$

5. Nodes 1 2 3 belong to community 1 and nodes 4 5 6 belong to community 2.

   This assignment is reasonable.

6. By doing this we can approximately maximize the modularity. It's explained in chapter 11.8 of Newman's book.

1.2 Bisection of a weighted undirected network

1. The adjacency matrix is as following:

$$\begin{bmatrix} 0 & 5 & 0 & 1 \\ 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 5 & 0 \end{bmatrix} \tag{4}$$

2. m=12

3. The modularity matrix is as following:

$$\begin{bmatrix} -1.5 & 3.5 & -1.5 & -0.5 \\ 3.5 & -1.5 & -0.5 & -1.5 \\ -1.5 & -0.5 & -1.5 & 3.5 \\ -0.5 & -1.5 & 3.5 & -1.5 \end{bmatrix} \tag{5}$$

4. The eigenvalues of the modularity matrix are: 4,0,-4,-6.

5. The eigenvector corresponding to the largeset eigenvalue is:

$$\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix} \tag{6}$$

6. Nodes 1 2 belong to community 1 and nodes 3 4 belong to community 2.

Problem 2: Pigou's congestion example

2.1

Assume that a fraction of x traffic takes route 2. The average traffic time is:

$$\tau = (1 - x) * 1 + (0.25 + 0.75 * x) * x$$

$$\tau = 1 - x + 0.25x + 0.75 * x^2$$

$$\tau = 0.75x^2 - 0.75x + 1$$

For the user optimal time, we want their to be no incentive to deviate; it is pretty clear then that all users will (weakly) prefer to take the 2nd path – if any user took the 1st path, they could do no worse by taking the 2nd path.

Therefore, $\tau = 0.75(1)^2 - 0.75(1) + 1 = 1$.

2.2

$$\frac{d\tau}{dx} = 1.5x - 0.75$$

Setting the derivative equal to 0, we get the minimum value of $x = 0.5$. Since x was defined as the fraction of traffic taking route 2: $x_1 = 0.5, x_2 = 0.5$ is the optimal allocation.

2.3

Plugging in the values obtained in 2.2 into the formula for average traffic in 2.3, we get:

$$\bar{\tau}_{min} = 0.75 * (0.5)^2 - 0.75 * (0.5) + 1$$

$$= 0.1875 - 0.375 + 1$$

$$= 0.8125$$