MAE-253: Homework 3

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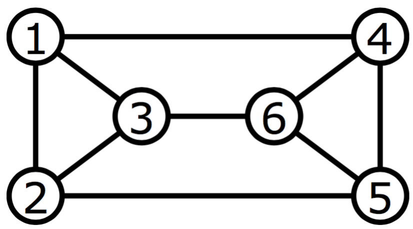
Problem 1: Modularity matrix

1.1 Bisection of a binary undirected network



1. Consider the graph given above. Give the adjacency matrix (A) of this graph.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

2. Let m denote the total number of edges in this graph. Construct a matrix B such that $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$.

$$B = \begin{bmatrix} -0.5 & 0.5 & 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}$$

3. Obtain the eigenvalues of B and list them.

$$\lambda = \left[1, -2, -2, 0, 0, 0\right]$$

4. What is the eigenvector (say v) corresponding to the largest eigenvalue?

$$v(\lambda = 1) = \left[0.408248, 0.408248, 0.408248, -0.408248, -0.408248, -0.408248\right]$$

5. Look at the sign of each component of v. If $v_i > 0$ assign node i to community 1 else assign node i to community 2. List the nodes that belong to community 1 and nodes that belong to community 2. Is the assignment reasonable?

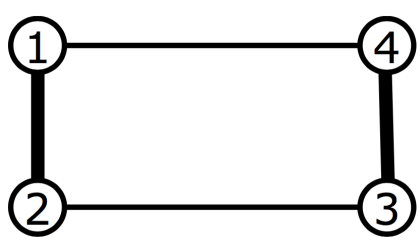
$$c = \left[1, 1, 1, 2, 2, 2\right]$$

Yes, this seems reasonable.

6. Extra Credit: Why does this work?

From Wikipedia, "Modularity is the fraction of the edges that fall within the given groups minus the expected such fraction if edges were distributed at random." Our choice for assigning nodes to communities in 5. maximizes $Q = \frac{1}{2m}\sum_{ij} s_i B_{ij} s_j$, where s is +1 for one community, and -1 for another community. In other words, we've chosen the groups that are the least likely to have randomly formed.

1.2 Bisection of a weighted undirected network



1. Consider the graph given above. Give the weighted adjacency matrix (A) of this graph.

$$A = \begin{bmatrix} 0 & 5 & 0 & 1 \\ 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 5 & 0 \end{bmatrix}$$

2. Let m denote the total weight of edges in this graph. You can obtain m by summing all the entries in A and dividing it by 2.

$$m = 12$$

3. Construct a matrix B such that $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$.

$$B = \begin{bmatrix} -1.5 & 3.5 & -1.5 & -0.5 \\ 3.5 & -1.5 & -0.5 & -1.5 \\ -1.5 & -0.5 & -1.5 & 3.5 \\ -0.5 & -1.5 & 3.5 & -1.5 \end{bmatrix}$$

4. Obtain the eigenvalues of B and list them.

$$\lambda = \left[4, 0, -6, -4\right]$$

5. What is the eigenvector (say v) corresponding to the largest eigenvalue?

$$v(\lambda = 4) = \left[0.5, 0.5, -0.5, -0.5\right]$$

6. Look at the sign of each component of v. If $v_i > 0$ assign node i to community 1 else assign node i to community 2. List the nodes that belong to each of the two communities? Is the assignment reasonable?

$$c = \left[1, 1, 2, 2\right]$$

This assignment is reasonable, as the nodes that are more highly linked are in communities.

Problem 2: Pigou's congestion example

Recall Pigou's example discussed in class, where there are two roads that connect a source, s, and destination, t. The roads have different travel costs. Fraction x_1 of the traffic flow on route 1, and the remainder x_2 on route 2. Here consider the following scenario.

- The first road has 'infinite' capacity but is slow and requires 1 hour travel time, $T_1 = 1$.
- The second road always requires at least 15 mins, which then increases as a function of traffic density, $T_2 = 0.25 + 0.75x_2$.

If drivers act in a 'selfish' manner - the user optimal scenario - all the traffic will flow on the second path, as one is never worse off. Worst case scenario for path 2, both paths take one hour. So no one is incentivized to change their behavior.

1. Assume user optimal behavior, and calculate τ the expected travel time per car.

$$\begin{aligned}
\tau &= x_1 T_1 + x_2 T_2 \\
&= x_1(1) + x_2(0.25 + 0.75x_2) \\
&= x_1 + 0.25x_2 + 0.75x_2^2
\end{aligned}$$

If $x_1 = 1 - x_2$, and $x_2 = x$, we have

$$\begin{aligned}
\tau &= 1 - x + 0.25x + 0.75x^2 \\
&= 1 - 0.75x + 0.75x^2
\end{aligned}$$

2. If instead we could control the flows, we could minimize the expected travel time. Using the expression in part (a), calculate the optimal allocation of flows $\bar{x}_1$ and $\bar{x}_2$ that minimize the expected travel time per car.

To minimize τ we take it's derivate and set it equal to zero,

$$\begin{aligned}
0 &= -0.75 + 1.5x \\
x &= \frac{1}{3} \implies x_1 = \frac{2}{3}, x_2 = \frac{1}{3}
\end{aligned}$$

3. What is τ_m, the expected travel time when the flow is optimized?

$$\begin{aligned}
\tau_m &= x_1 + 0.25x_2 + 0.75x_2^2 \\
&= \frac{2}{3} + 0.25\left(\frac{1}{3}\right) + 0.75\left(\frac{1}{3}\right)^2 \\
&= \frac{5}{6}
\end{aligned}$$