ECS 253 / MAE 253, Network Theory
Spring 2016
Problem Set # 1, Solutions

Problem 1: The Erdős-Rényi random graph – analyzing the phase transition

(a) For each j that not equals i: (i) The probability of that i is not linked to j is $1-p$. (ii) The probability that j is not a part of giant component is u and the probability that i is linked to j is p. Since these two events are independent, the probability that i is linked to j and j is not a part of giant component is pu. (i) and (ii) are mutually exclusive therefore the probability of (i) or (ii) is $(1-p+pu)$. There are $N-1$ choice for j and all of them must satisfy (i) or (ii) and they are independent too. Thus:

$$u = (1-p+pu)^{N-1}. \tag{1}$$

(b)

$$\begin{aligned}
\ln u &= \ln(1-p+pu)^{N-1} \\
&= (N-1)\ln(1-p+pu) \\
&= (N-1)\ln(1-p(1-u)) \\
&= (N-1)\ln(1-\frac{\langle k\rangle}{N-1}(1-u)) \\
&\approx (N-1)\frac{-\langle k\rangle}{N-1}(1-u) \\
&= -\langle k\rangle(1-u).
\end{aligned} \tag{2}$$

(c)

$$\begin{aligned}
u &= e^{-\langle k\rangle S} \\
(1-u) = S &= 1-e^{-\langle k\rangle S}
\end{aligned} \tag{3}$$

(d) See figure below.
(e) Take derivative of both side w.r.t S we have:

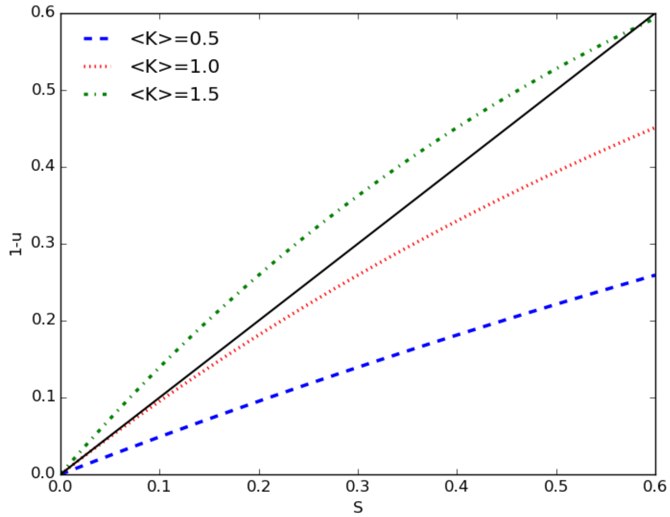
$$1 = \langle k\rangle e^{-\langle k\rangle S} \tag{4}$$

Plug $S=0$ into it we get:

$$k_c = \langle k\rangle = 1 \tag{5}$$

(f)

$$\begin{aligned}
I_G &= N(1-p)^{N_G} \\
&\approx N(1-p)^N \\
&= N(1-\frac{Np}{N})^N \\
&\approx Ne^{-Np}
\end{aligned} \tag{6}$$



Setting $I_G = 1$ gives:

$$1 = Ne^{-Np}$$

Therefore:

$$e{-}Np = \frac{1}{N}$$

$$p = \frac{\ln N}{N}$$

Problem 2: The Erdős-Rényi random graph – cluster size distribution

(a) During the edge adding process, c_k can be changed in two ways. The first one is joining two smaller components and form a new component of size k. The other one is adding edges to a component of size k and this will decrease the number of components of size k.

$$\frac{dc_k}{dt} = \Big(\sum_{i+j=k} \frac{1}{2}(ic_i)(jc_j)\Big) - kc_k$$

(b) Let $k = 1$, we have:

$$\frac{dc_1}{dt} = -c_1$$

Solve this differential equation we have:

$$c_1 = e^{-t}$$

Similarly, we can get:

$$c_2 = \frac{1}{2}te^{-2t}$$

$$c_3 = \frac{1}{2}t^2e^{-3t}$$

(c) We probably would guess that:

$$c_k = \alpha_k t^{k-1} e^{-kt}$$

(d) Using Stirling's formula to approximate factorial:

$$k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$$

When $t = 1$:

$$\begin{aligned} c_k &= \frac{k^{k-2}}{k!}e^{-k} \\ &\approx \frac{1}{\sqrt{2\pi k}}k^{-k}e^k k^{k-2}e^{-k} \\ &= \frac{1}{\sqrt{2\pi k}}k^{-2} \\ &= \frac{1}{\sqrt{2\pi}}k^{-2.5} \end{aligned} \tag{7}$$