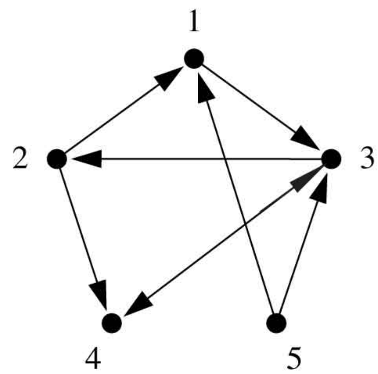
ECS 253 / MAE 253, Network Theory
Spring 2016
Problem Set # 1, Solutions

Problem 1: Adjacency matrix of a simple network



(a) Let $A_{ij} = 1$ in the following matrix denote the presence of a link from node j to node i and $A_{ij} = 0$ denote the absence of a link. The adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

(b) In order to obtain the steady state distribution we normalize the adjacency matrix given above to obtain the transition matrix T. Entry T_{ij} denotes the probability of going from node j to node i in the next time step.

$$T = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

The steady state probability corresponds to the eigenvector of T with eigenvalue 1. Using a standard software package we find that the eigenvector that corresponds to the eigenvalue of 1 is

$$\pi = \begin{pmatrix} 0.182574 \\ 0.365148 \\ 0.730296 \\ 0.547722 \\ 0 \end{pmatrix} \tag{3}$$

Note that the above eigenvector is normalized such that the sum of the square of the elements is equal to 1. Renormalizing such that the sum of the terms is 1 we obtain

$$\pi = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.4 \\ 0.3 \\ 0 \end{pmatrix} \tag{4}$$

(c) For any undirected graph, the steady state probability of finding a random walker is directly proportional to the degree of the node. Therefore,

$$\pi = \frac{1}{14} \begin{pmatrix} 3 \\ 3 \\ 4 \\ 2 \\ 2 \end{pmatrix} \tag{5}$$

Problem 2: Rate equations for uniform attachment

a) Let $n_{k,t}$ denote the expected number of nodes of degree k at time t. We need to consider nodes of degree $k > 1$ and $k = 1$ separately:

$$\begin{aligned} k > 1; \quad n_{k,t+1} &= n_{k,t} + \frac{1}{t} n_{k-1,t} - \frac{1}{t} n_{k,t}. \\ k = 1; \quad n_{1,t+1} &= n_{1,t} + 1 - \frac{1}{t} n_{1,t}. \end{aligned} \tag{6}$$

b) Assumption 1: $p_{k,t} = \frac{n_{k,t}}{n_t} = \frac{n_{k,t}}{t}$, thus $n_{k,t} = p_{k,t} \cdot t$.

Writing Eqns (6) in terms using $n_{k,t} = p_k \cdot t$:

$$k > 1 : \quad p_k \cdot (t+1) = p_k \cdot t + \frac{1}{t} p_{k-1} \cdot t - \frac{1}{t} p_k \cdot t. \tag{7}$$

$$k = 1; \quad p_1 \cdot (t+1) = p_1 \cdot t + 1 - \frac{1}{t} p_1 \cdot t. \tag{8}$$

c) Assumption 2: **steady state** $p_{k,t} \to p_k$.

Solving Eqn (7) gives the recurrence:

$$p_k = \frac{1}{2} p_{k-1} \tag{9}$$

d) Solving Eqn (8) we find $p_1 = 1/2$. Thus $p_2 = \frac{1}{2} p_1 = \left(\frac{1}{2}\right)^2$, $p_3 = \frac{1}{2} p_2 = \left(\frac{1}{2}\right)^3$, etc. So in general

$$p_j = \left(\frac{1}{2}\right)^j \tag{10}$$

This is *not* a power law. It is a **geometric distribution** (the discrete analog of an exponential distribution).