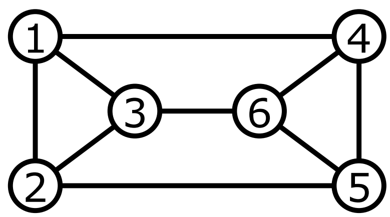
ECS 253 / MAE 253, Network Theory and Applications
Spring 2016
Common Problem Set # 3, Due May 12

1 Problem 1: Modularity matrix

You would probably want to use a software package in order to compute the eigenvalues and eigenvectors for the matrices involved in this problem. Note that the methods shown below will work only for splitting the network into two parts. You cannot use the same method to further split the network.

1.1 Bisection of a binary undirected network



1. Consider the graph given above. Give the adjacency matrix ($\mathbf{A}$) of this graph.

2. Let m denote the total number of edges in this graph. Construct a matrix $\mathbf{B}$ such that

$$B_{ij} = A_{ij} - \frac{k_i k_j}{(2m)}. \tag{1}$$

If we define a column vector $\mathbf{k}$ such that the ith component is the degree of node i, we can express $\vec{B}$ as

$$\mathbf{B} = \mathbf{A} - \frac{\mathbf{k} * \mathbf{k}^T}{2m}. \tag{2}$$

Note that $\mathbf{B}$ is called the modularity matrix.

3. Obtain the eigenvalues of $\mathbf{B}$ and list them.

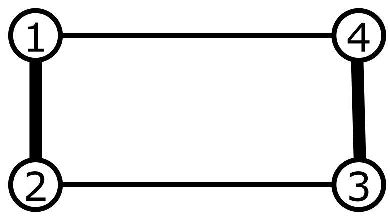
4. What is the eigenvector (say $\vec{v}$) corresponding to the largest eigenvalue?

5. Look at the sign of each component of $\vec{v}$. If $v_i > 0$ assign node i to community 1 else assign node i to community 2. List the nodes that belong to community 1 and nodes that belong to community 2. Is the assignment reasonable?

6. Extra Credit: Why does this work?

1.2 Bisection of a weighted undirected network

For this section we will use an approach very similar to the one described above but slightly modify it to handle a weighted network. In the graph below the thicker edges have a weight of 5 and the other edges have a weight of 1.



1. Consider the graph given above. Give the weighted adjacency matrix ($\mathbf{A}$) of this graph.

2. Let m denote the total weight of edges in this graph. You can obtain m by summing all the entries in $\mathbf{A}$ and dividing it by 2.

3. Construct a matrix $\mathbf{B}$ such that
$$B_{ij} = A_{ij} - \frac{k_i k_j}{(2m)}. \tag{3}$$
If we define a column vector $\mathbf{k}$ such that the ith component is the total weight associated with node i (i.e, sum the weight of all edges that involve node i), we can express $\mathbf{B}$ as
$$\mathbf{B} = \mathbf{A} - \frac{\mathbf{k} * \mathbf{k}^T}{2m}. \tag{4}$$

4. Obtain the eigenvalues of $\mathbf{B}$ and list them.

5. What is the eigenvector (say $\vec{v}$) corresponding to the largest eigenvalue?

6. Look at the sign of each component of $\vec{v}$. If $v_i > 0$ assign node i to community 1 else assign node i to community 2. List the nodes that belong to each of the two communities? Is the assignment reasonable?

2 Pigou's congestion example

Recall Pigou's example discussed in class, where there are two roads that connect a source, s, and destination, t. The roads have different travel costs. Fraction x_1 of the traffic flow on route 1, and the remainder x_2 on route 2. Here consider the following scenario.

- The first road has "infinite" capacity but is slow and requires 1 hour travel time, $T_1 = 1$.
- The second road always requires at least 15 mins, which then increases as a function of traffic density, $T_2 = 0.25 + 0.75x_2$.

If drivers act in a "selfish" manner – the user optimal scenario – all the traffic will flow on the second path, as one is never worse off. Worst case scenario for path 2, both paths take one hour. So no one is incentivized to change their behavior.

1. Assume user optimal behavior, and calculate τ the expected travel time per car.
2. If instead we could control the flows, we could minimize the expected travel time. Using the expression in part (a), calculate the optimal allocation of flows $\bar{x_1}$ and $\bar{x_2}$ that minimize the expected travel time per car.
3. What is τ_m, the expected travel time when the flow is optimized?