Problem 1. Power Law Degree Distributions

Consider the power law distribution $p(k) = Ak^{-\gamma}$, with support $k = 1$ to $k \to \infty$. In the steps below, you can either treat the k's using a continuum approximation or you can treat the k's as discrete.

Part (a)

Show that we must have $\gamma > 1$ for this to be a properly defined probability distribution function. Recall a pdf must have two properties: 1) $p(k) \geq 0$ for all k, and 2) it must be normalized.

To normalize we set the infinite sum equal to zero,

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} Ak^{-\gamma} \\
&\approx \int_{k=1}^{\infty} Ak^{-\gamma} dk = \left(\frac{A}{1-\gamma}\right) k^{-\gamma+1}\bigg|_{k=1}^{\infty}
\end{aligned}$$

which requires $\gamma > 1$ to allow A to be finite. As A must be finite to normalize the power law, $\gamma > 1$.

Part (b)

Solve for the normalization constant A. Assuming $\gamma > 1$,

$$\begin{aligned}
1 &= \left(\frac{A}{1-\gamma}\right) k^{-\gamma+1}\bigg|_{k=1}^{\infty} \\
&= \left(\frac{A}{1-\gamma}\right)\left(\infty^{-\gamma+1} - 1\right) \\
&= \frac{A}{\gamma - 1} \implies \boxed{A = \gamma - 1}
\end{aligned}$$

Part (c)

Show that if $1 < \gamma \leq 2$, the average value $\langle k \rangle$ diverges.

$$\begin{aligned}
\langle k \rangle &= \int_{k=1}^{\infty} (\gamma - 1) k^{-\gamma+1} dk \\
&= -\frac{(\gamma-1)}{(\gamma-2)} k^{-\gamma+2}\bigg|_{k=1}^{\infty} \\
&= -\frac{(\gamma-1)}{(\gamma-2)} \left(\infty^{-\gamma+2} - 1\right).
\end{aligned}$$

The limit at ∞ blows up for $1 < \gamma < 2$, and the denominator of the leading fraction, $(\gamma - 2)$, blows up at $\gamma = 2$. For $\gamma > 2$ we have

$$\begin{aligned}\langle k\rangle &= -\frac{(\gamma-1)}{(\gamma-2)}\left(\infty^{-\gamma+2}-1\right).\\ &= \frac{(\gamma-1)}{(\gamma-2)}\end{aligned}$$

Part (d)

Show that if $2 < \gamma \leq 3$, the average is finite, but the standard deviation, diverges.

Similarly to the above,

$$\begin{aligned}\langle k^2\rangle &= \int_{k=1}^{\infty}(\gamma-1)k^{-\gamma+2}dk\\ &= -\frac{(\gamma-1)}{(\gamma-3)}k^{-\gamma+3}\bigg|_{k=1}^{\infty}\\ &= -\frac{(\gamma-1)}{(\gamma-3)}\left(\infty^{-\gamma+3}-1\right).\end{aligned}$$

The limit at ∞ blows up for $2 < \gamma < 3$, and the denominator of the leading fraction, $(\gamma-3)$, blows up at $\gamma = 3$. For $\gamma > 3$ we have

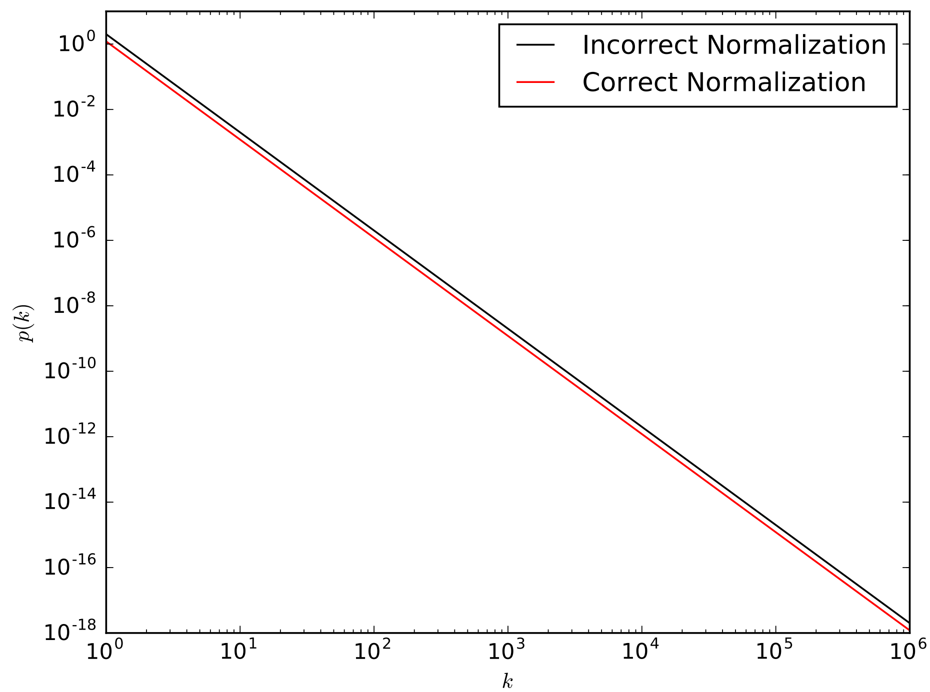
$$\begin{aligned}\langle k^2\rangle &= -\frac{(\gamma-1)}{(\gamma-3)}\left(\infty^{-\gamma+3}-1\right).\\ &= \frac{(\gamma-1)}{(\gamma-3)}\end{aligned}$$

The standard deviation, however, still diverges for $\gamma \leq 3$, as

$$\begin{aligned}\sigma^2 &= \langle k^2\rangle - \langle k\rangle^2\\ &= \infty.\end{aligned}$$

Part (e)

Plot $p(k) = Ak^{\gamma}$, for $k = 1$ to $k = 100,000$ for $\gamma = 3$, and properly normalize.



Part (f)

In a finite network with N nodes, what is the largest possible value of degree, k_{max}, that can ever be observed? So can we ever have $\langle k \rangle \to \infty$ in a finite network?

The largest possible value of degree for any node in a finite network with N nodes is N, so can never observe $\langle k \rangle \to \infty$ in a finite network.