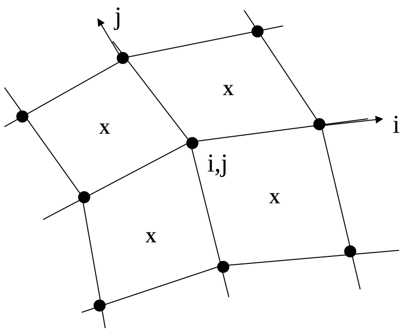
Lecture 5 – Numerical Techniques

- **There are several tricks used in finite-volume schemes that make the creation of first and second difference and averaging operators more straightforward**
- **These finite-volume operations stay within the confines of a given block and do not require obtaining information outside of the block**
- **All of the operators "accumulate" cell contributions at the nodes**

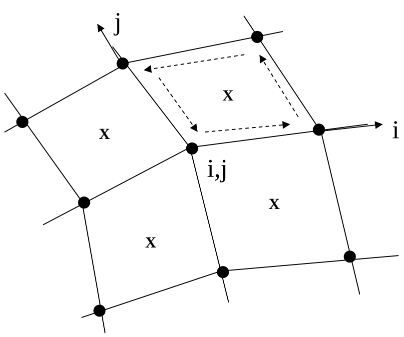
First-Derivative Operators

- **Finite-difference derivative operators have influence from the 5 nodes along the i- and j-directions (corner node influence is omitted)**
- **Finite-volume derivative operators will have influence from all 9 nodes surrounding any given node (in 2D)**



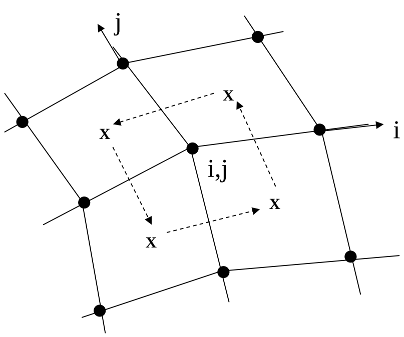
Cell-Centered Scheme First-Derivative Operator

- **Cell-centered schemes integrate around the "primary" control volumes made up from the nodes**
- **Physical variables are stored at the cell centers whereas coordinates are stored at the nodes**
- **First-derivatives at cell-centers are found using Green's integrations around the primary control volumes**



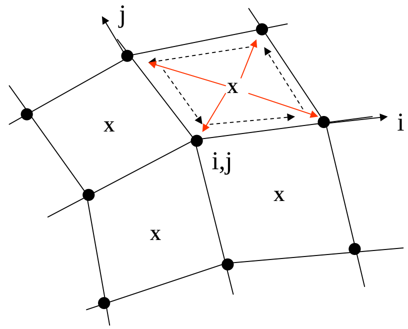
Node-Centered Scheme First-Derivative Operator

- **Node-centered schemes could integrate around the "secondary" control volumes made up from the nodes**
- **Physical variables and coordinates are both stored at the nodes**
- **First-derivatives at nodes are found using Green's integrations around the secondary control volumes**



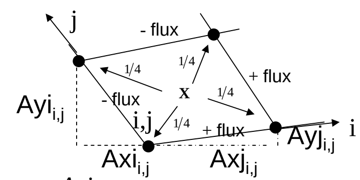
Alternate Node-Centered Scheme First-Derivative Operator

- Node-centered schemes could also be found from integrations around the “primary” control volumes made up from the nodes followed by subsequent distributions to the nodes
- The equivalent of the node-centered integration around the node requires that ¼ of the cell-centered first derivative value be distributed to the nodes making up the primary control volume



- Note that this operator stays within the block of cells and never requires information outside of it
- Essentially an averaging operator

Node-Centered Scheme First-Derivative Operator



- Trapezoidal counter-clockwise integration to get first x-derivative at cell-center using Gauss’s theorem: $\frac{\partial\phi}{\partial x} = \frac{1}{Vol}\oint_{CV}\phi\, dy \quad \frac{\partial\phi}{\partial y} = -\frac{1}{Vol}\oint_{CV}\phi\, dx$

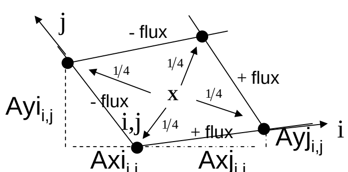
$$
\begin{aligned}
Ayi_{i,j} &= y_{i,j+1} - y_{i,j}\\
Axi_{i,j} &= x_{i,j+1} - x_{i,j}\\
Ayj_{i,j} &= y_{i+1,j} - y_{i,j}\\
Axj_{i,j} &= x_{i+1,j} - x_{i,j}
\end{aligned}
$$

$$
\frac{\partial T}{\partial x} = \frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\left[\sum_{faces} T_{\substack{average\\ of\ face}}\,\Delta y_{\substack{axross\ face\ in\\ counter-clockwise\\ direction}}\right]
$$

$$
\left.\frac{\partial T}{\partial x}\right|_{i+\frac{1}{2},\,j+\frac{1}{2}} = \frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\begin{bmatrix}\left(T_{i+1,j}+T_{i+1,j+1}\right)Ayi_{i+1,j} - \left(T_{i,j}+T_{i,j+1}\right)Ayi_{i,j}\\ -\left(T_{i,j+1}+T_{i+1,j+1}\right)Ayj_{i,j+1} + \left(T_{i,j}+T_{i+1,j}\right)Ayj_{i,j}\end{bmatrix}
$$

$$
= \frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\begin{bmatrix}axialfluxi_{i+1} - axialfluxi_{i}\\ -axialfluxj_{j+1} + axialfluxj_{j}\end{bmatrix}
$$

Node-Centered Scheme First-Derivative Operator



- Likewise, trapezoidal counter-clockwise integration to get first y-derivative at cell-center using Gauss’s theorem:

$$
\begin{aligned}
Ayi_{i,j} &= y_{i,j+1} - y_{i,j}\\
Axi_{i,j} &= x_{i,j+1} - x_{i,j}\\
Ayj_{i,j} &= y_{i+1,j} - y_{i,j}\\
Axj_{i,j} &= x_{i+1,j} - x_{i,j}
\end{aligned}
$$

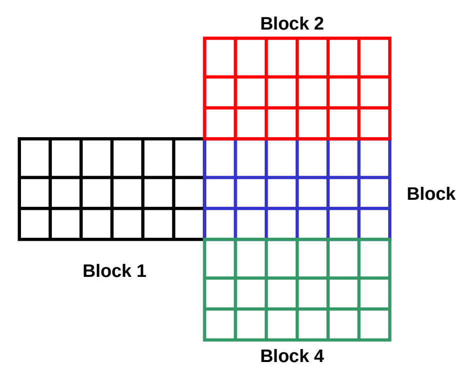
$$
\frac{\partial T}{\partial y} = -\frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\left[\sum_{faces} T_{\substack{average\\ of\ face}}\,\Delta x_{\substack{axross\ face\ in\\ counter-clockwise\\ direction}}\right]
$$

$$
\left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},\,j+\frac{1}{2}} = -\frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\begin{bmatrix}\left(T_{i+1,j}+T_{i+1,j+1}\right)Axi_{i+1,j} - \left(T_{i,j}+T_{i,j+1}\right)Axi_{i,j}\\ -\left(T_{i,j+1}+T_{i+1,j+1}\right)Axj_{i,j+1} + \left(T_{i,j}+T_{i+1,j}\right)Axj_{i,j}\end{bmatrix}
$$

$$
= -\frac{1}{2Vol_{i+\frac{1}{2},j+\frac{1}{2}}}\begin{bmatrix}tangentialfluxi_{i+1} - tangentialfluxi_{i}\\ -tangentialfluxj_{j+1} + tangentialfluxj_{j}\end{bmatrix}
$$

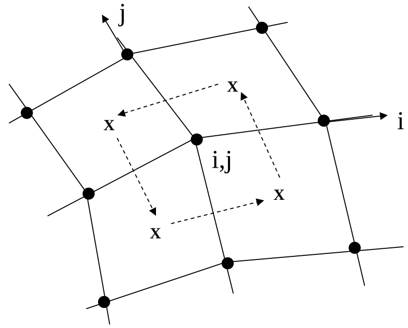
Node-Centered Scheme First-Derivative Operator

- IF additional blocks are point-matched along the edges, then the first derivatives can be obtained by accumulating the contributions from the adjacent blocks (gather-add operation)
- At physical boundaries, Dirichlet or Neumann boundary conditions often preclude the need for derivatives to be determined using finite volume operators



Node-Centered Scheme Second-Derivative Operator

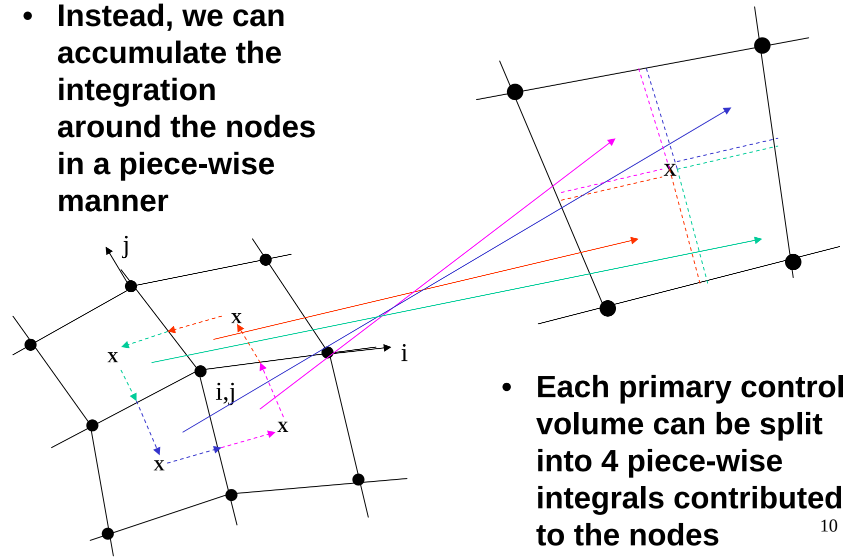
- Node centered second derivative operators can be found more directly by using Green’s theorem and integrating the first-derivatives around the secondary control volume
- This gives the second derivatives directly at the nodes
- Note, however that this could require information outside of a block when performing this operation along the block edges



- We can reformulate this operator to avoid this problem

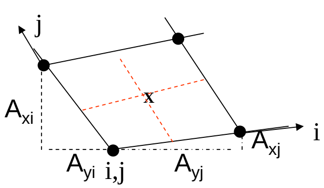
Alternate Distributive Scheme Second-Derivative Operator

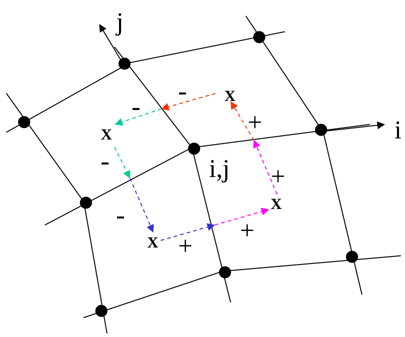
- Instead, we can accumulate the integration around the nodes in a piece-wise manner



- Each primary control volume can be split into 4 piece-wise integrals contributed to the nodes

Alternate Distributive Scheme Second-Derivative Operator





- Areas used in calculation of fluxes:

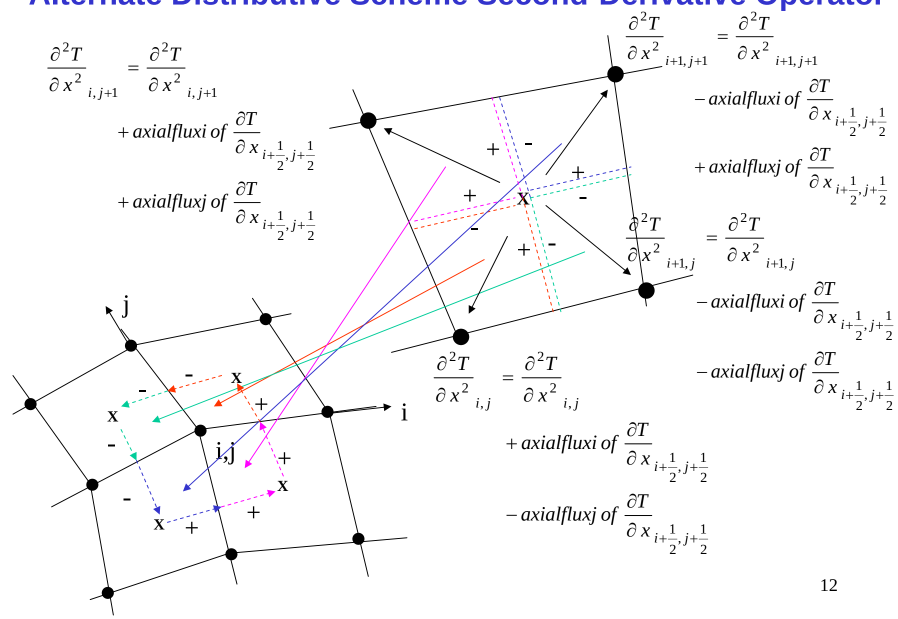
$$Ayi_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{Ayi_{i+1,j} + Ayi_{i,j}}{4}$$

$$Axi_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{Axi_{i+1,j} + Axi_{i,j}}{4}$$

$$Ayj_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{Ayj_{i,j} + Ayj_{i,j+1}}{4}$$

$$Axj_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{Axj_{i,j} + Axj_{i,j+1}}{4}$$

Alternate Distributive Scheme Second-Derivative Operator



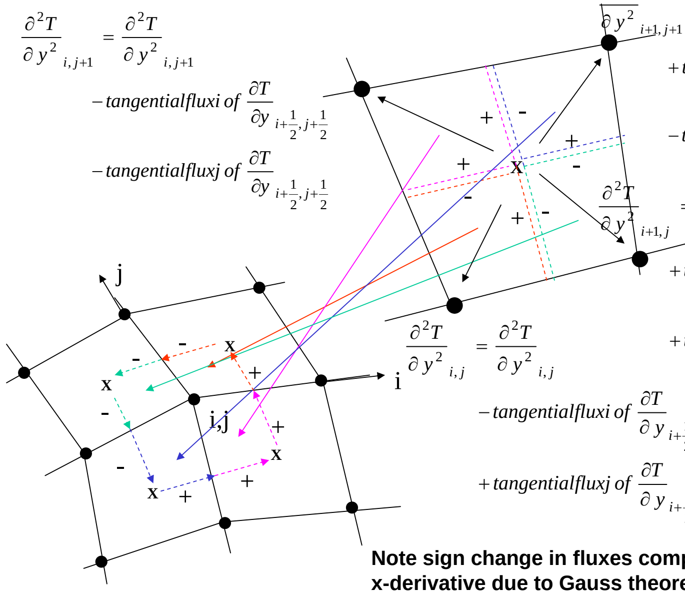
Alternate Distributive Scheme Second-Derivative Operator

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j+1} = \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j+1} - tangentialfluxi\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}} - tangentialfluxj\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i+1,j+1} = \left.\frac{\partial^2 T}{\partial y^2}\right|_{i+1,j+1} + tangentialfluxi\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}} - tangentialfluxj\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i+1,j} = \left.\frac{\partial^2 T}{\partial y^2}\right|_{i+1,j} + tangentialfluxi\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}} + tangentialfluxj\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} = \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} - tangentialfluxi\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}} + tangentialfluxj\ of\ \left.\frac{\partial T}{\partial y}\right|_{i+\frac{1}{2},j+\frac{1}{2}}$$



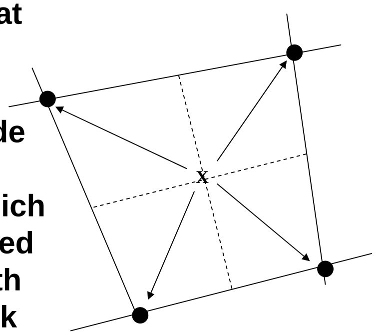
Note sign change in fluxes compared to x-derivative due to Gauss theorem

Node-Centered Scheme Second-Derivative Operators

- **The advantages of this accumulation operator are**
  - Both the first and second derivatives at the interior nodes can be found in one loop across the primary control volume cells
  - the second-derivatives at the edges where adjoining blocks exist can be again found with a simple gather-add operation

Averaging Operator

- **We often need to find the average value of quantities at the nodes from the cell-centered values**
- **We can find the average node values using a similar distribution technique in which we distribute the cell-centered value to the nodes along with a "hit" index that keeps track of the number of contributions**
- **In a second loop, we then divide the total accumulated contribution at the node by the number of "hits" to obtain the average**



- **The advantage of this approach is it allows for any number of cells around an edge or corner node**