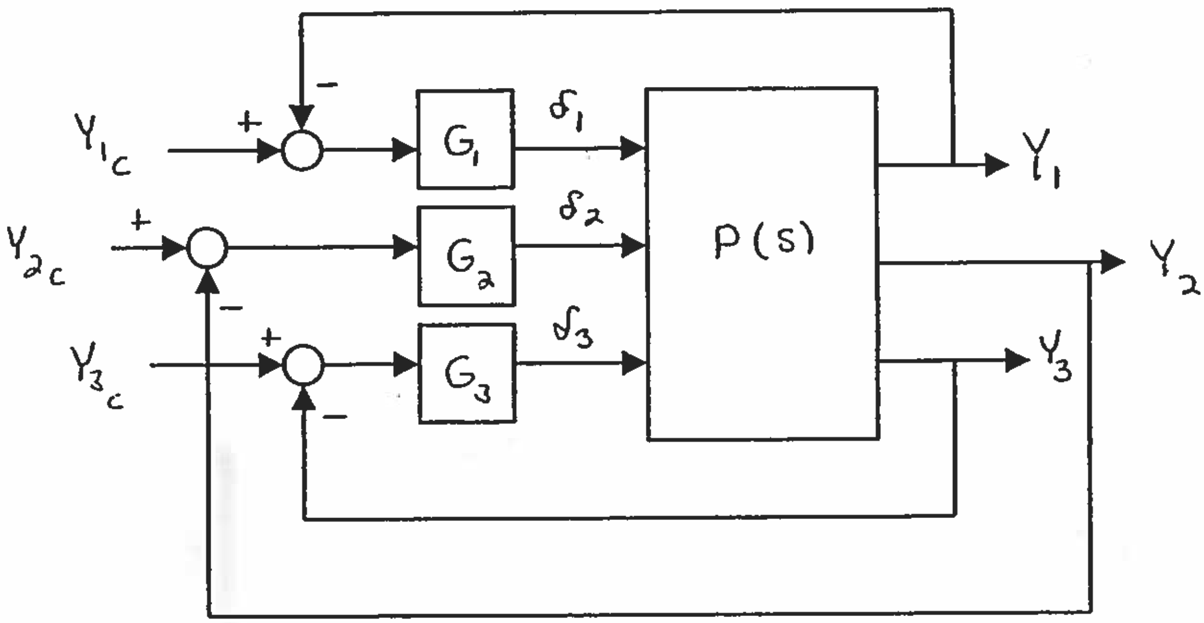
MAE - 275

Coupling Numerators for a 3 x 3 Control System

added pages



$$\dot{x} = Ax + B\delta$$
$$y = Cx + D\delta$$
$$P(s) = [C(sI - A)^{-1}B + D]_{3x3}$$

$$\left.\frac{Y_1}{\delta_1}\right|_{Y_2 \to \delta_2;\, Y_3 \to \delta_3} = \frac{N_{\delta_1}^{Y_1} + G_2 N_{\delta_1\delta_2}^{Y_1Y_2} + G_3 N_{\delta_1\delta_3}^{Y_1Y_3} + G_2 G_3 N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3}}{\Delta + G_2 N_{\delta_2}^{Y_2} + G_3 N_{\delta_3}^{Y_3} + G_2 G_3 N_{\delta_2\delta_3}^{Y_2Y_3}}$$

Rules for a 3 by 3 system:

1.) The effective denominator is equal to

   a.) The open-loop denominator

   b.) plus the sum of all the remaining compensator transfer functions, each one multiplied by the appropriate type 0 coupling numerator[1]

   c.) plus the sum of all the remaining compensator transfer functions taken two at a time, each pair multiplied by the appropriate type 1 coupling numerator[2]

2.) The effective numerator is equal to

   a.) the open-loop numerator (type 0 coupling numerator)

   b.) plus the sum of all the remaining compensator transfer functions, each one multiplied by the appropriate type 1 coupling numerator[3]

   c.) plus the usm of all the compensator transfer functions taken two at a time, each pair multiplied by the appropriate type 2 coupling numerator [4]

---

1 The appropriate type 0 coupling numerator is that associated with the multiplying compensator

2 The appropriate type 1 coupling numerator is that associated with the pair of multiplying compensators

3 The appropriate type 1 coupling numerator is that associated with the input-output pair on the left hand side of the equation and that associated with the multiplying compensator

4 The appropirate type 2 coupling numerator is that associated with the input-output pair on the left hand side of the equation and that associated with the pair of multiplying compensators

$$Y(s) = P(s)\delta(s)$$
$$P^{-1}(s)Y(s) = I\delta(s)$$

$$\text{Let } E(s) = P^{-1}(s) = \frac{\left[P_{adj}(s)\right]^T}{\det P(s)} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Then

$$E(s)Y(s) = I\,\delta(s)$$

and

$$N_{\delta_1}^{Y_1} = \begin{vmatrix} 1 & e_{12} & e_{13} \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{vmatrix} \quad N_{\delta_1\delta_2}^{Y_1Y_2} = \begin{vmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & e_{33} \end{vmatrix} \quad N_{\delta_2\delta_3}^{Y_2Y_3} = \begin{vmatrix} e_{11} & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & 0 & 1 \end{vmatrix}$$

$$N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

With loops 2 and 3 "tightly constrained" with high bandwidth feedback loops, i.e. with

$$|G_2(s)| >> 1 \quad \text{and} \quad |G_3(s)| >> 1$$

$$\left.\frac{Y_1}{\delta_1}\right|_{Y_2\to\delta_2;\,Y_3\to\delta_3} \approx \frac{G_2G_3N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3}}{G_2G_3N_{\delta_2\delta_3}^{Y_2Y_3}} = \frac{1}{N_{\delta_2\delta_3}^{Y_2Y_3}} = \frac{1}{e_{11}} = \frac{\det P(s)}{\left(\left[P_{adj}\right]^T\right)_{11}}$$

Values of s which make $\det P(s) = 0$ are called *transmission zeros* of P(s). For non-square systems, transmission zeros are values of s which cause the matrix P(s) to lose rank.

AFTI-F-16

Transmission Zeros, Coupling Numerators and Input-Output Pairing in Square Control System Architectures

R. A. Hess[1]
Dept. of Mechanical and Aeronautical Engineering
University of California
Davis, CA 95616

Introduction

The use of coupling numerators as a analysis/synthesis tool for frequency-domain control system design is well established.[1] Of particular interest is their use in approximating "effective" plant dynamics between a particular plant input and output pair when remaining response variables are assumed to be tightly constrained by feedback.[2]

To begin this discussion, consider the state space formulation for a linear system.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

The plant transfer function matrix P(s) can be introduced as

$$y(s) = [C(sI - A)^{-1}B + D]u(s) = P(s)u(s) \tag{2}$$

An alternative means of describing system input/output behavior in the Laplace domain can be given as

$$E(s)y(s) = u(s) = Iu(s) \tag{3}$$

or

$$y(s) = E^{-1}(s)u(s) \tag{4}$$

Assuming $P^{-1}(s)$ exists, Eqs. 2 and 3 yield

$$E(s) = P^{-1}(s) = \frac{[P_{adj}]^T}{\det P(s)} \tag{5}$$

Coupling Numerators

Equation 3 leads to the definition of coupling numerators as described in Ref. 1. In Ref. 1, the vector y(s) represents the transformed state variable vector, whereas here it represents the transformed output vector. The matrix algebra remains unchanged however, and this formulation leads to a particularly simple form for the coupling

---

[1] Professor and Vice Chairman, Associate Fellow AIAA

numerators. For example, a coupling numerator of order 1 between two response variables $y_1(s)$ and $y_2(s)$ and two control inputs $u_1(s)$ and $u_2(s)$ in a 2 x 2 system can be given simply as

$$N^{y_1 y_2}_{u_1 u_2}(s) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.0 \tag{6}$$

Note that the coupling numerator of Eq. 6 has no polynomials in its elements and is considerably simpler than those encountered when the system description of Ref. 1 is used. For the purposes of exposition, the following discussion and example will concentrate on 2 x 2 systems. It should be emphasized that extension to n x n square systems is straightforward.

Sequential Loop Closure in a Square System Architecture

Consider the 2 x 2 square control system architecture of Fig. 1. The matrix K(s) can be a static control distribution matrix or contain dynamic elements that serve as pre-compenators, e.g., to approximately decouple the plant.[3] At this juncture, K(s) is the identity matrix and $u_1(s) \equiv v_1(s)$, $u_2(s) \equiv v_2(s)$. Here two linear compensation elements $G_1(s)$ and $G_2(s)$ are shown, i.e., the compensation matrix is diagonal. It will be assumed that the design in question is a sequential loop-closure procedure and that $G_1(s)$ has been obtained by loop shaping with the second loop open. Now in order to design $G_2(s)$, a model of the effective plant with $G_1(s)$ in place is sought. Reference 1 shows that

$$\left.\frac{y_2}{u_2}(s)\right|_{y_1 \to u_1} = \frac{y_2}{u_2}(s) \cdot \frac{1 + G_1(s)\dfrac{N^{y_2 y_1}_{u_2 u_1}}{N^{y_2}_{u_2}}}{1 + G_1(s)\dfrac{y_1}{u_1}(s)} \tag{7}$$

where the notation on the left hand side of Eq. 7 means that the first loop has been closed with a feedback of y_1 through the compensation G_1 to provide u_1. Now, for "tightly constrained" control in the first control loop, consider $|G_1(s)| >> 1$. Thus

$$\left.\frac{y_2}{u_2}(s)\right|_{y_1 \to u_1} \cong \frac{y_2}{u_2}(s) \cdot \frac{G_1(s)\dfrac{N^{y_2 y_1}_{u_2 u_1}}{N^{y_2}_{u_2}}}{G_1(s)\dfrac{y_1}{u_1}(s)} = \frac{y_2}{u_2}(s) \cdot \frac{\dfrac{N^{y_2 y_1}_{u_2 u_1}}{N^{y_2}_{u_2}}}{\dfrac{y_1}{u_1}(s)} \tag{8}$$

Note that terms such as

$$\frac{y_1}{u_1}(s) = P_{11}(s) \text{ and } \frac{y_2}{u_2}(s) = P_{22}(s), \text{ etc} \quad (9)$$

Following the format of Ref. 1, the coupling numerator of order 0 is obtained as

$$N^{y_2}_{u_2} = \begin{vmatrix} e_{11}(s) & 0 \\ e_{21}(s) & 1 \end{vmatrix} = e_{11}(s) \quad (10)$$

But from Eq. 5,

$$e_{11}(s) = \frac{P_{22}(s)}{\det P(s)} \quad (11)$$

Thus, with Eqs. 6-11, Eq. 8 becomes

$$\left.\frac{y_2}{u_2}(s)\right|_{y_1 \to u_1} \cong \frac{P_{22}(s)}{P_{11}(s)} \frac{\det P(s)}{P_{22}(s)} = \frac{\det P(s)}{P_{11}(s)} \quad (12)$$

Transmission zeros of the square plant of Fig. 1 are defined as zeros of det P(s).[4] Equation 12 demonstrates that transmission zeros of a plant will appear as zeros of "effective" plant transfer functions when previous loop(s) are tightly constrained. Here "tightly constrained" means that previous loops are closed with arbitrarily high bandwidths. This, in turn, means that if there are transmission zeros in the right half plane (RHP), the bandwidth of subsequent loop closures will be limited to values considerably smaller than the magnitude of these RHP zeros. While the importance of transmission zeros has been emphasized in state-space control system synthesis techniques, their role in classical, frequency-domain designs is not often discussed.

The square architecture of Fig. 1 with K(s) as the identity matrix implies plant input-output pairing, i.e. $u_1(s)$ is used to control $y_1(s)$ and $u_2(s)$ is used to control $y_2(s)$. It can be easily shown that the use of a matrix K(s) differing from the identity matrix will still result in transmission zeros appearing in effective plant transfer functions. From Fig. 1 one sees new pseudo-controls $v_1(s)$ and $v_2(s)$ serving as inputs to the K(s). The pseudo-controls are then distributed to $u_1(s)$ and $u_2(s)$ through K(s). One may now define a new plant matrix P′(s) as

$$P'(s) = P(s)K(s) \quad (13)$$

Again assume that the first loop is closed as before with a new $G_1(s)$ with $|G_1(s)| >> 1$. The effective plant for the second loop closure can be obtained from Eq. 12 as

$$\left.\frac{y_2}{v_2}(s)\right|_{y_1 \to v_1} \cong \frac{\det P'(s)}{P'_{11}(s)} = \frac{\det P(s) \cdot \det K(s)}{P'_{11}(s)} \quad (14)$$

Equation 14 clearly indicates that, with the first loop tightly constrained, the second loop will again exhibit the transmission zeros of the plant P(s).

In many applications, the number of plant inputs exceeds the number of plant outputs. In such cases, both P(s) and K(s) are rectangular. The approach just outlined remains valid. One merely replaces P(s) with P(s) K(s), and considers the K(s) of Fig. 1 as the identity matrix. Note however, that the result on the far right hand side of Eq. 14 is no longer valid since the determinants of P(s) and K(s) are no longer defined. The pertinent transmission zeros would be those of P(s)K(s).

Input-Output Pairing

The coupling numerator approach can be used to provide guidance in selecting input-output pairs for square architectures. Assume that it is desired to change the pairing of input-output variables and redraw Fig. 1 to allow $u_1(s)$ to control $y_2(s)$ and $u_2(s)$ to control $y_1(s)$. Similar to Eqs. 7 and 8, the effective plant transfer function with $y_i(s)$ tightly constrained by $u_j(s)$ $(i \neq j)$ would be,

$$\left.\frac{y_i}{u_j}(s)\right|_{y_j \to u_i} \cong \frac{y_i}{u_j}(s)\cdot\frac{G_i(s)\dfrac{N^{y_2 y_1}_{u_1 u_2}}{N^{y_i}_{u_j}}}{G_i(s)\dfrac{y_j}{u_i}(s)} = \frac{y_i}{u_j}(s)\cdot\frac{\dfrac{N^{y_2 y_1}_{u_1 u_2}}{N^{y_i}_{u_j}}}{\dfrac{y_j}{u_i}(s)} = -\frac{y_i}{u_j}\cdot\frac{\dfrac{N^{y_2 y_1}_{u_2 u_1}}{N^{y_i}_{u_j}}}{\dfrac{y_j}{u_i}} \tag{15}$$

and, finally,

$$\left.\frac{y_i}{u_j}(s)\right|_{y_j \to u_i} \cong -\frac{P_{ij}(s)\,\det P(s)}{P_{ji}(s)\left[-P_{ij}(s)\right]} = \frac{\det P(s)}{P_{ji}(s)} \tag{16}$$

As will be demonstrated in the example of the next section, the resulting transfer functions defined by Eqs. 7 and 16 can provide guidance as to the selection of acceptable input-output pairs. Such pairing is an important step in a number of control system synthesis techniques such as Quantitiative Feedback Theory (QFT)[5] and Sliding Mode Control (SMC) using feedback linearization.[6]

Example

A brief example is in order at this juncture. A lateral/directional stability and command augmentation system (SCAS) is to be designed for a model of the AFTI F-16 fighter aircraft. The model is taken from Ref. 7. The system state-space model is given below. The flight condition is Mach No. 0.9 at an altitude of 20,000 ft.

"shorthand" transfer function notation

$$\frac{3(4)}{(5)[.3, 10]} = \frac{3(s+4)}{(s+5)[s^2 + 2(.3)10s + 10^2]}$$

$$\begin{Bmatrix}\dot\phi\\ \dot\beta\\ \dot p\\ \dot r\end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0.0345 & -0.3436 & 0.0326 & -0.9976\\ 0 & -55.256 & -2.80 & 0.1457\\ 0 & 7.237 & -0.0232 & -0.3625\end{bmatrix}\begin{Bmatrix}\phi\\ \beta\\ p\\ r\end{Bmatrix} + \begin{bmatrix} 0 & 0\\ -.00014 & 0.0373\\ -51.05 & 10.395\\ -1.2501 & -5.808\end{bmatrix}\begin{Bmatrix}\delta_a\\ \delta_r\end{Bmatrix} \tag{17}$$

In Eq. 17, the angles and angular rates are in deg and deg/sec. Actuator dynamic have been ignored for simplicity. The system response variables are roll rate, p, and lateral acceleration of the center of gravity a_y. The corresponding output equations are

$$\begin{Bmatrix}p\\ a_y\end{Bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & -5.596 & 0.5309 & 0.0390\end{bmatrix}\cdot\begin{Bmatrix}\phi\\ \beta\\ p\\ r\end{Bmatrix} + \begin{bmatrix}0 & 0\\ -0.0227 & 0.6075\end{bmatrix}\cdot\begin{Bmatrix}\delta_a\\ \delta_r\end{Bmatrix} \tag{18}$$

In Eq. 18, the lateral acceleration is in ft/sec^2. Transmission zeros of this plant are located at $\boxed{s = 6.9014, -6.8757 \text{ and } 0.}$ The zero at the origin is related to the kinematic relation between roll angle and rate and is not of concern. Note that one transmission zero lies in the RHP.

An examination of both the elements of P(s) and the four possible input-output pairs (with tightly constrained variables in the remaining loop) can be made, with the latter obtained from Eqs. 8 and 16. Using shorthand notation§, the transfer functions are

$$\frac{p}{\delta_a} = \frac{-51.05[0.12, 2.949]}{(2.697)(0.0272)[0.131, 2.987]} \quad (a) \qquad \frac{p}{\delta_r} = \frac{10.395(5.071)(-4.644)}{(2.697)(0.0272)[0.131, 2.987]} \quad (b)$$

$$\frac{a_y}{\delta_a} = \frac{-0.2279(1199)(-.01269)[0.108, 2.949]}{(2.697)(0.272)[0.131, 2.987]} \quad (c) \qquad \frac{a_y}{\delta_r} = \frac{0.6073(13.2)(4.502)(-5.831)(.002362)}{(2.697)(0.2762)[0.131, 2.987]} \quad (d)$$

$$(19)$$

$$\left.\frac{p}{\delta_a}(s)\right|_{a_y\to\delta_r} \cong \frac{-50.66(0)(-6.9014)(6.8757)}{(13.2)(4.502)(-5.831)(0.002366)} \quad (a) \qquad \left.\frac{a_y}{\delta_r}(s)\right|_{p\to\delta_a} \cong \frac{0.6029(-6.9014)(6.8757)}{[0.12, 2.949]} \quad (b)$$

$$\left.\frac{a_y}{\delta_a}(s)\right|_{p\to\delta_r} \cong \frac{-1349.9(0)(-6.9014)(6.8757)}{(1199)(-.01269)[0.108, 2.949]} \quad (c) \qquad \left.\frac{p}{\delta_r}(s)\right|_{a_y\to\delta_a} \cong \frac{2.960(-6.9014)(6.8757)}{(5.071)(-4.644)} \quad (d)$$

$$(20)$$

---

§ $\dfrac{K(a)}{[\zeta, \omega_n]} = \dfrac{K(s+a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Equations 19 and 20 can provide insight into input-output pairing in a sequential loop-closure design. For example, if p is paired with δ_r and this loop is closed first, Eq. 19(b) indicates that maintenance of acceptable stability margins would allow a maximum crossover frequency no larger that 1–2 rad/sec due to the NMP zero at $s = 4.644$. It is doubtful that this would result in an acceptable bandwidth from the standpoint of handling qualities of a primary attitude axis. If a_y is paired with δ_a, and this loop is closed first, Eq. 19(c) indicates that only very small crossover frequencies could be employed.

If p is paired with δ_a and this loop is closed first, Eq. 19(a) indicates that arbitrarily large crossover frequencies could be employed. Equation 20(b) indicates that closing the a_y loop with δ_r next can be successful with a crossover frequency in the range of 1–2 rad/sec due to a NMP zero at $s = 6.9014$. If a_y is paired with δ_r and this loop is closed first, Eq. 19(d) indicates the loop would have a maximum crossover frequency in the range of 1-2 rad/sec due to the NPM zero at $s = 5.831$. Due to the limited bandwidth of this first closure, the use Eqs. 20(a) would not be justified here, as one could not assume that a_y is tightly constrained.

This exercise suggests a sequential loop-closure design procedure with the p and a_y variables paired with δ_a and δ_r, respectively, closing the p loop first. Had matrix K(s) in Fig. 1 not been the identity matrix, an analysis similar to that just completed could be undertaken with the transfer functions of Eq. 19 obtained from $P'(s) = P(s)K(s)$, and those of Eq. 20 obtained from relationships such as Eq. 14. In cases in which there are no NMP transmission zeros, pairing input-output variables and selecting loop closure sequences could be based upon the required compensation in each loop. The form of this compensation could be estimated from equations similar to Eqs. 19 and 20.

Figure 1 can serve as a diagram for the SCAS, now with $y_1(s) = p(s)$, $y_2(s) = a_y(s)$, $u_1(s) = \delta_a(s)$, $u_2(s) = \delta_r(s)$, and $K(s) = I$. The compensator $G_1(s)$ in Fig. 1 was obtained using standard loop-shaping techniques.[8] The result was a 5 rad/sec open-loop crossover frequency with

$$G_1(s) = \frac{0.074(3)(0.1)^2}{(0)^2(0.2)} \quad (21)$$

The effective plant for the second loop closure (now determined using the actual $G_1(s)$ of Eq. 21) is given by

$$\left.\frac{a_y}{\delta_r}\right|_{p \to \delta_a} = \frac{0.6075(15.71)(4.833)(-6.11)(1.147)}{[0.962, 3.385][0.131, 2.943]} \quad (22)$$

Because of the NMP zero in the numerator at s = 6.112, the maximum practical crossover frequency for this loop with acceptable stability margins is approximately 1 rad/sec. Again, a compensator was obtained using standard loop-shaping techniques and is given by

$$G_2(s) = \frac{-1.42[0.15, 3]}{(0)(50)} \quad (23)$$

The response of the aircraft to a unit doublet command in p with this SCAS is shown in Fig. 2. Similar command following and off-axis responses result when a unit doublet command in a_y is employed.

Conclusions

(1) The use of coupling numerators based upon output as opposed to state equations results in a simplification of the coupling numerator expressions.

(2) The use of these simplified numerators in square control system architectures with response variables assumed to be tightly constrained

(a) provides insight into the selection of appropriate pairings of plant inputs and outputs for square feedback designs, and

(b) demonstrates the effect of transmission zeros upon individual effective plant transfer function elements when other response variables are tightly constrained.

References

[1]McRuer, D., Ashkenas, I., and Graham, D., *Aircraft Dynamics and Automatic Control,* Princeton University Press, Princeton, NJ, 1973, Chap. 3.

[2]Heffley, R. K., "A Compilation and Analysis of Helicopter Handling Qualities Data, Vol. Two: Data Analysis, NASA CR-3144, March 1979.

[3]Catapang, D. R., Tischler, M. B., and Biezad, D. J., Robust Crossfeed Design for Hovering Rotorcraft," *International Journal of Control,* Vol.4, No., 1, 1994, pp. 161-180.

[4]Stengel, R. F., *Stochastic Optimal Control*, Wiley, New York, NY, 1986, p. 544.

[5]Horowitz, I., *Quantitative Feedback Theory,* QFT Publications, Boulder, CO, 1993.

[6]Wells, S. R., and Hess, R. A., "MIMO Sliding Mode Control for a Tailless Fighter Aircraft, An Alternative to Reconfigurable Architectures," AIAA Paper No. ----, 2002.

[7]Rivard, G. R., "Weighting Matrix Selection for QFT Designs," MS Thesis, Air Force Institute of Technology, AFIT/GE/ENG/89D-43, Wright-Patterson AFB, OH, Dec. 1989.

[8]Maciejowski, J. M., *Multivariable Feedback Design,* Addison-Wesley, Wokingham, UK, 1989, Chap. 1.

(14)

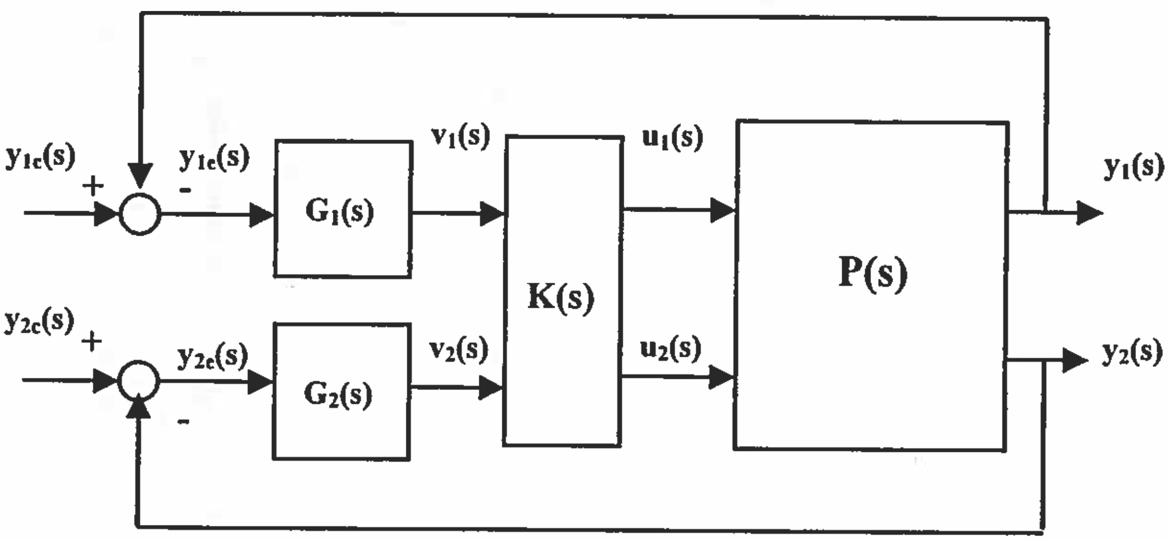


Figure 1 A 2 x 2 square control system architecture

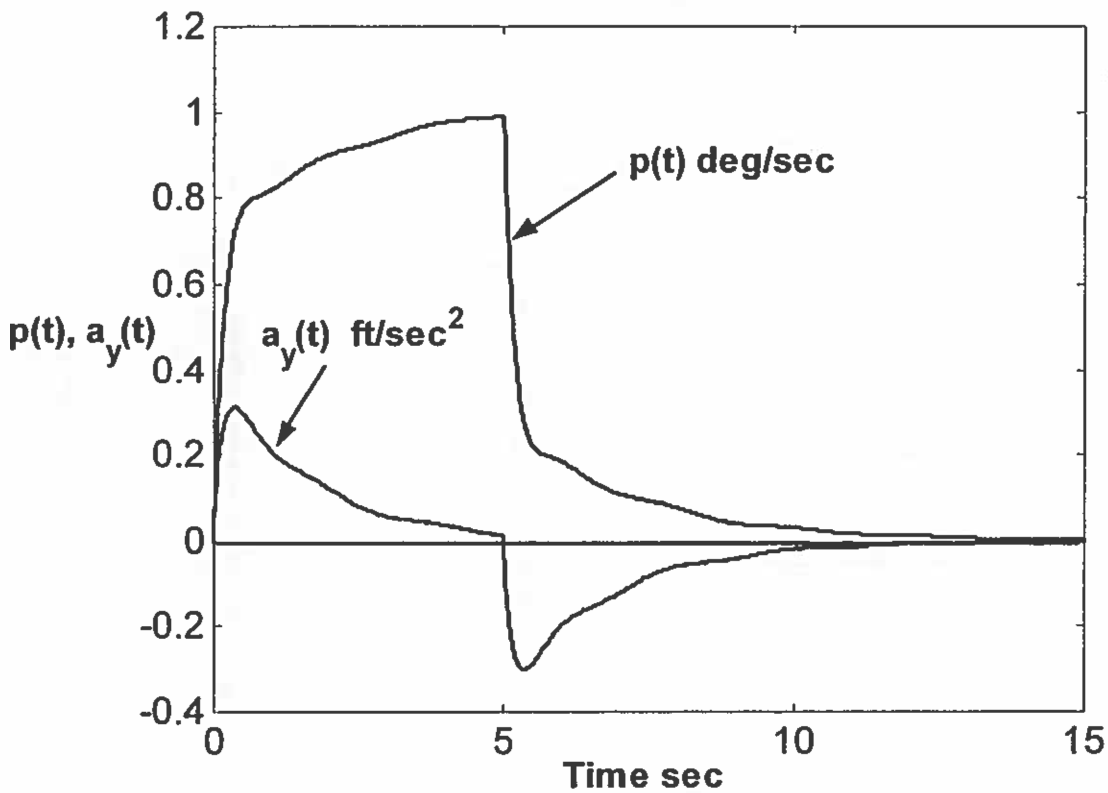
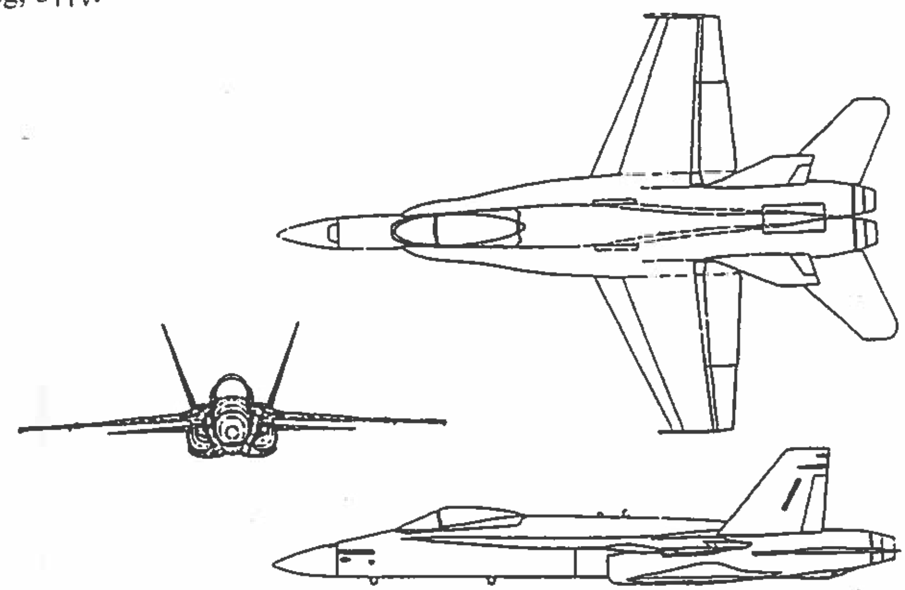


Fig. 2 Response of aircraft to unit doublet p command

MAE – 275

A Square-Effective Plant Design for Lateral Directional HARV Control System

For lateral-directional control, the vehicle has 5 control effectors: aileron, δ_A, differential horizontal tail, δ_{DT}, rudder, δR, roll thrust vectoring, δ_{RTV}, and yaw thrust vectoring, δ_{YTV}.



NASA HARV

Consider the simplified lateral-directional equations (Dutch-roll approximation) for this vehicle at a specified flight condition (Mach No. = 0.5, altitude = 20,000 ft). The simplified equations of motion are given below, with the vehicle A and B matrices for this flight condition.

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} Y_\beta & \sin(\alpha) & -\cos(\alpha) \\ L_\beta & L_p & L_r \\ N_\beta & N_p & N_r \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_{\delta DT} & Y_{\delta A} & Y_{\delta R} & Y_{\delta RTV} & Y_{\delta YTV} \\ L_{\delta DT} & L_{\delta A} & L_{\delta R} & L_{\delta RTV} & L_{\delta YTV} \\ N_{\delta DT} & N_{\delta A} & N_{\delta R} & N_{\delta RTV} & N_{\delta YTV} \end{bmatrix} \begin{bmatrix} \delta_{DT} \\ \delta_A \\ \delta_R \\ \delta_{RTV} \\ \delta_{YTV} \end{bmatrix}$$

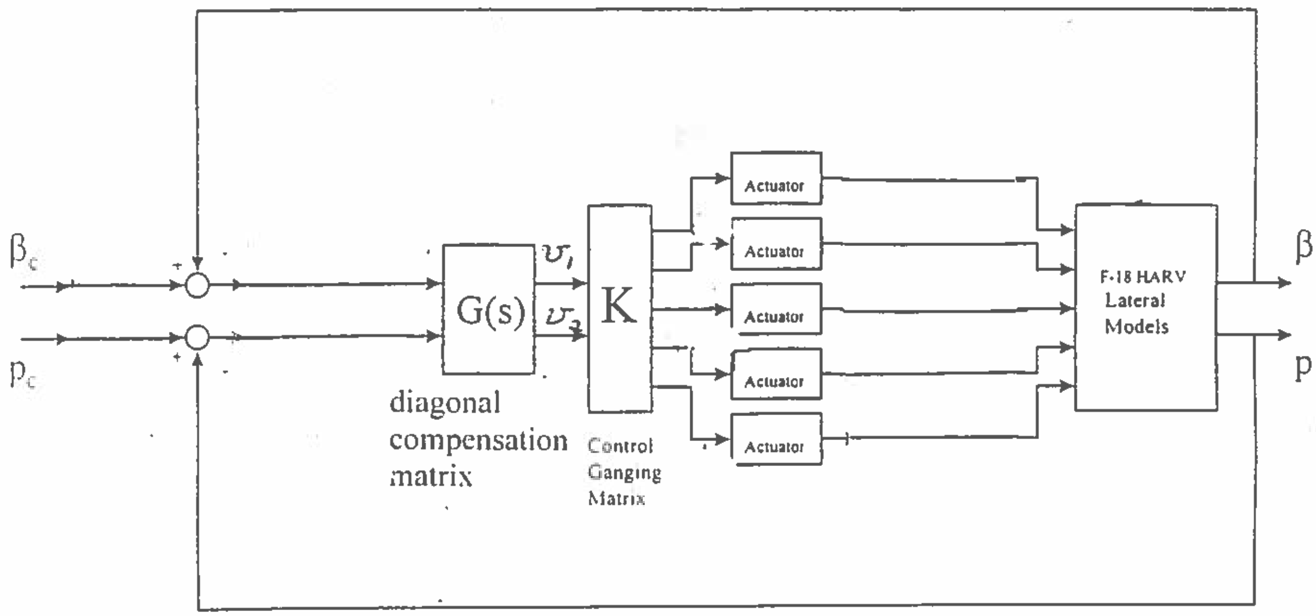
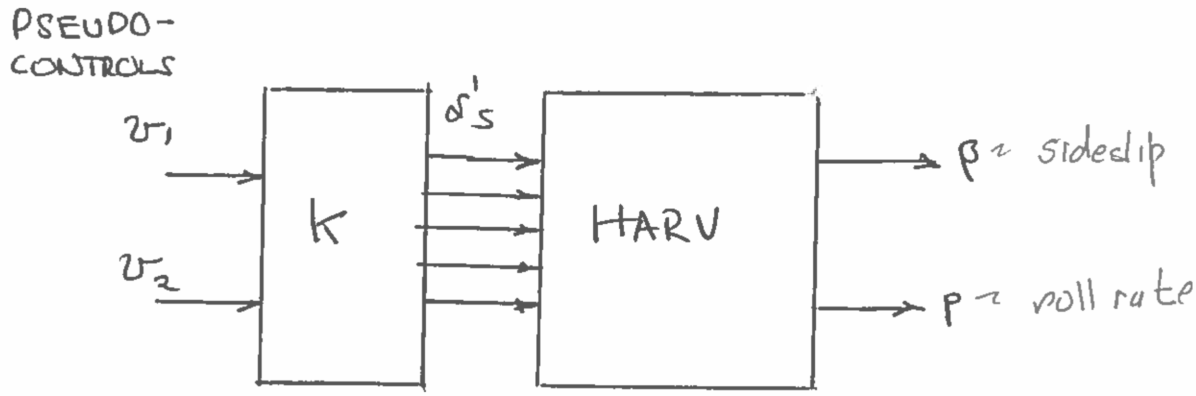


Fig. 1 Flight control system for design example

K =

0	0.6000
0	1.0000
1.0000	0
0	0.6000
0.6000	0


PSEUDO-
CONTROLS
v_1
v_2
K
δ's
HARV
β ~ sideslip
p ~ roll rate

```
» A

A =

  -1.3540e-001  9.0360e-002 -9.9490e-001
  -1.0370e+001 -1.4690e+000  5.1260e-001
   2.2891e+000 -1.4820e-002 -1.2770e-001
```

$$\begin{array}{l} \beta \\ p \\ r \end{array}$$

```
» B

B =
```

$$\begin{array}{ccccc} \delta_{DT} & \delta_A & \delta_R & \delta_{RTY} & \delta_{YTV} \end{array}$$

```
 -1.0910e-002 -5.6950e-003  1.5550e-002            0  4.8900e-003
  9.9300e+000  1.2120e+001  9.4160e-001  3.9770e-001  2.8170e-002
  2.7570e-001 -2.7970e-001 -7.4190e-001 -1.1750e-003 -3.8610e-001

» C

C =

     1     0     0
     0     1     0
```

$y_1 \equiv \beta$

$y_2 \equiv p$

```
» D

D =

     0     0     0     0     0
     0     0     0     0     0

» tzero(A,B,C,D)

ans =

   Empty matrix: 0-by-1

»
```

```
K =

            0   6.0000e-001
            0   1.0000e+000
  1.0000e+000             0
            0   6.0000e-001
  6.0000e-001   1.0000e+000

» [Pnum1,Pden]=ss2tf(A,B,C,D,1)

Pnum1 =

            0 -1.0910e-002   6.0556e-001 -1.3130e-001
            0  9.9300e+000   2.8670e+000  2.5652e+001


Pden =

  1.0000e+000  1.7321e+000   3.6258e+000  3.5385e+000

» [Pnum2,Pden]=ss2tf(A,B,C,D,2)

Pnum2 =

            0 -5.6950e-003   1.3643e+000  7.1327e-001
            0  1.2120e+001   3.1045e+000  2.4908e+001


Pden =

  1.0000e+000  1.7321e+000   3.6258e+000  3.5385e+000

»
```

```
» [Pnum3,Pden]=ss2tf(A,B,C,D,3)

Pnum3 =

             0   1.5550e-002   8.4803e-001   1.0777e+000
             0   9.4160e-001  -2.9382e-001  -5.5474e+000


Pden =

   1.0000e+000   1.7321e+000   3.6258e+000   3.5385e+000

» [Pnum4,Pden]=ss2tf(A,B,C,D,4)

Pnum4 =

             0  -8.8818e-016   3.7105e-002   1.2116e-002
             0   3.9770e-001   1.0403e-001   9.0040e-001


Pden =

   1.0000e+000   1.7321e+000   3.6258e+000   3.5385e+000


»
» [Pnum5,Pden]=ss2tf(A,B,C,D,5)

Pnum5 =

             0   4.8900e-003   3.9448e-001   5.4810e-001
             0   2.8170e-002  -2.4121e-001  -3.9463e+000


Pden =

   1.0000e+000   1.7321e+000   3.6258e+000   3.5385e+000

»
```

```
»
» p11=tf(Pnum1(1,:),Pden);
» p12=tf(Pnum2(1,:),Pden);
» p13=tf(Pnum3(1,:),Pden);
» p14=tf(Pnum4(1,:),Pden);
» p15=tf(Pnum5(1,:),Pden);
» p21=tf(Pnum1(2,:),Pden);
» p22=tf(Pnum2(2,:),Pden);
» p23=tf(Pnum3(2,:),Pden);
» p24=tf(Pnum4(2,:),Pden);
» p25=tf(Pnum5(2,:),Pden);
» zpk(p11)

Zero/pole/gain:
  -0.01091 (s-55.29) (s-0.2177)
----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{\beta}{u_1}$

```
» zpk(p12)

Zero/pole/gain:
 -0.005695 (s-240.1) (s+0.5217)
----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{\beta}{u_2}$

```
» zpk(p13)

Zero/pole/gain:
   0.01555 (s+53.23) (s+1.302)
----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{\beta}{u_3}$

```
» zpk(p14)

Zero/pole/gain:
-8.8818e-016 (s-4.178e013) (s+0.3265)
--------------------------------------
  (s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{\beta}{u_4}$

```
» zpk(p15)

Zero/pole/gain:
   0.00489 (s+79.26) (s+1.414)
----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{\beta}{u_5}$

```
» zpk(p21)
```

21.1

```
Zero/pole/gain:
    9.93 (s^2 + 0.2887s + 2.583)
-----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{P}{\delta_1}$

```
» zpk(p22)

Zero/pole/gain:
  12.12 (s^2 + 0.2561s + 2.055)
-----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{P}{\delta_2}$

```
» zpk(p23)

Zero/pole/gain:
   0.9416 (s-2.588) (s+2.276)
-----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{P}{\delta_3}$

```
» zpk(p24)

Zero/pole/gain:
 0.3977 (s^2 + 0.2616s + 2.264)
-----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{P}{\delta_4}$

```
» zpk(p25)

Zero/pole/gain:
   0.02817 (s-16.87) (s+8.305)
-----------------------------------
(s+1.188) (s^2 + 0.5444s + 2.979)
```

$= \frac{P}{\delta_5}$

```
» P=[p11 p12 p13 p14 p15;p21 p22 p23 p24 p25];
```

```
» PE=P*K;
```

~ P "effective"

```
» detPE=PE(1,1)*PE(2,2)-PE(1,2)*PE(2,1);
» detPE=minreal(detPE);
» detPE

Transfer function:
        0.3503 s + 17.67
-----------------------------------
s^3 + 1.732 s^2 + 3.626 s + 3.538

»
```

= det(PE)

» zpk(PE(1,1))

Zero/pole/gain:

$$\frac{0.018484\ (s+57.36)\ (s+1.327)\ \cancel{(s+1.188)}\ \cancel{(s^2 + 0.5444s + 2.979)}}{(s+1.188)^{\cancel{2}}\ (s^2 + 0.5444s + 2.979)^{\cancel{2}}} = \frac{\beta}{\sigma_1}$$

» zpk(PE(1,2))

Zero/pole/gain:

$$\frac{-0.007351\ (s-292.3)\ \cancel{(s+1.188)^3}\ (s+0.5538)\ \cancel{(s^2 + 0.5444s + 2.979)}^3}{(s+1.188)^{\cancel{4}}\ (s^2 + 0.5444s + 2.979)^{\cancel{4}}} = \frac{\beta}{\sigma_2}$$

» zpk(PE(2,1))

Zero/pole/gain:

$$\frac{0.9585\ (s-3.112)\ (s+2.654)\ \cancel{(s+1.188)}\ \cancel{(s^2 + 0.5444s + 2.979)}}{(s+1.188)^{\cancel{2}}\ (s^2 + 0.5444s + 2.979)^{\cancel{2}}} = \frac{\rho}{\sigma_1}$$

» zpk(PE(2,2))

Zero/pole/gain:

$$\frac{18.3448\ \cancel{(s+1.188)}^3\ (s^2 + 0.2533s + 2.011)\ \cancel{(s^2 + 0.5444s + 2.979)}^3}{(s+1.188)^{\cancel{4}}\ (s^2 + 0.5444s + 2.979)^{\cancel{4}}} = \frac{\rho}{\sigma_2}$$

»

```
» PEI11=PE(2,2)/detPE;
» PEI22=PE(1,1)/detPE;
» PEI12=-PE(2,1)/detPE;
» PEI21=-PE(1,2)/detPE;
» Y1V1=1/PEI11;
» Y2V2=1/PEI22;
» Y1V2=-1/PEI12;
» Y2V1=-1/PEI21;
» zpk(Y1V1)
```

$$\left.\frac{y_i}{u_j}(s)\right|_{y_j\to u_i} \cong \frac{y_i}{u_j}(s)\cdot\frac{G_i(s)\dfrac{N^{y_2y_1}_{u_1u_2}}{N^{y_i}_{u_j}}}{G_i(s)\dfrac{y_j}{u_i}(s)} = \frac{y_i}{u_j}(s)\cdot\frac{\dfrac{N^{y_2y_1}_{u_1u_2}}{N^{y_i}_{u_j}}}{\dfrac{y_j}{u_i}(s)} = -\frac{y_i}{u_j}\cdot\frac{\dfrac{N^{y_2y_1}_{u_2u_1}}{N^{y_1}_{u_j}}}{\dfrac{y_j}{u_i}} \qquad (15)$$

$$\left.\frac{y_i}{u_j}(s)\right|_{y_j\to u_i} \cong -\frac{P_{ij}(s)}{P_{ji}(s)}\,\frac{\det P(s)}{\left[-P_{ij}(s)\right]} = \frac{\det P(s)}{P_{ji}(s)} \qquad (16)$$

Zero/pole/gain:

$$\frac{0.019125\ (s+50.44)\ \cancel{(s+1.188)^3}\ \cancel{(s^2 + 0.5444s + 2.979)}^3}{\cancel{(s+1.188)}^3\ (s^2 + 0.2668s + 2.23)\ \cancel{(s^2 + 0.5444s + 2.979)}^3} = \left.\frac{\beta}{\upsilon_1}\right|_{p\to\upsilon_2}$$

```
» zpk(Y2V2)
```

Zero/pole/gain:

$$\frac{18.9514\ (s+50.44)\ \cancel{(s+1.188)^2}\ \cancel{(s^2 + 0.5444s + 2.979)}^2}{(s+57.36)\ (s+1.327)\ \cancel{(s+1.188)^2}\ \cancel{(s^2 + 0.5444s + 2.979)}^2} = \left.\frac{p}{\upsilon_2}\right|_{\beta\to\upsilon_1}$$

```
» zpk(Y1V2)
```

Zero/pole/gain:

$$\frac{0.36546\ (s+50.44)\ \cancel{(s+1.188)^2}\ \cancel{(s^2 + 0.5444s + 2.979)}^2}{(s-3.112)\ (s+2.654)\ \cancel{(s+1.188)^2}\ \cancel{(s^2 + 0.5444s + 2.979)}^2} = \left.\frac{\beta}{\upsilon_2}\right|_{p\to\upsilon_1}$$

```
» zpk(Y2V1)
```

Zero/pole/gain:

$$\frac{-28.6167\ (s+50.44)\ \cancel{(s+1.188)^3}\ \cancel{(s^2 + 0.5444s + 2.979)}^3}{(s-143.3)\ \cancel{(s+1.188)}^3\ (s+0.3658)\ \cancel{(s^2 + 0.5444s + 2.979)}^3} = \left.\frac{p}{\upsilon_1}\right|_{\beta\to\upsilon_2}$$

»

PAIRING: $\beta \to \upsilon_1$

$P \to \upsilon_2$

CLOSE $\beta \to \upsilon_1$ FIRST

Now solving input-output "pairing problem" and "loop-closure sequence"

1.) First, control β using v1? Looks OK from β/v1 transfer function (p. 20) and Bode plot (p.23).

2.) Then, control p loop using v2, with β-v1 loop closed? This looks good also from the transfer function (p. 25).

3.) How about controlling p with v1 first? NO, there is a non-minimum phase zero (s-3.112) in the transfer function (p. 20) that would limit the crossover frequency to values significantly below 3.112 rad/sec).

4.) How about closing β using v2 first? Looks OK from the transfer function (p. 20)…the nom-minimum phase zero (s-292.3) would not pose a problem.

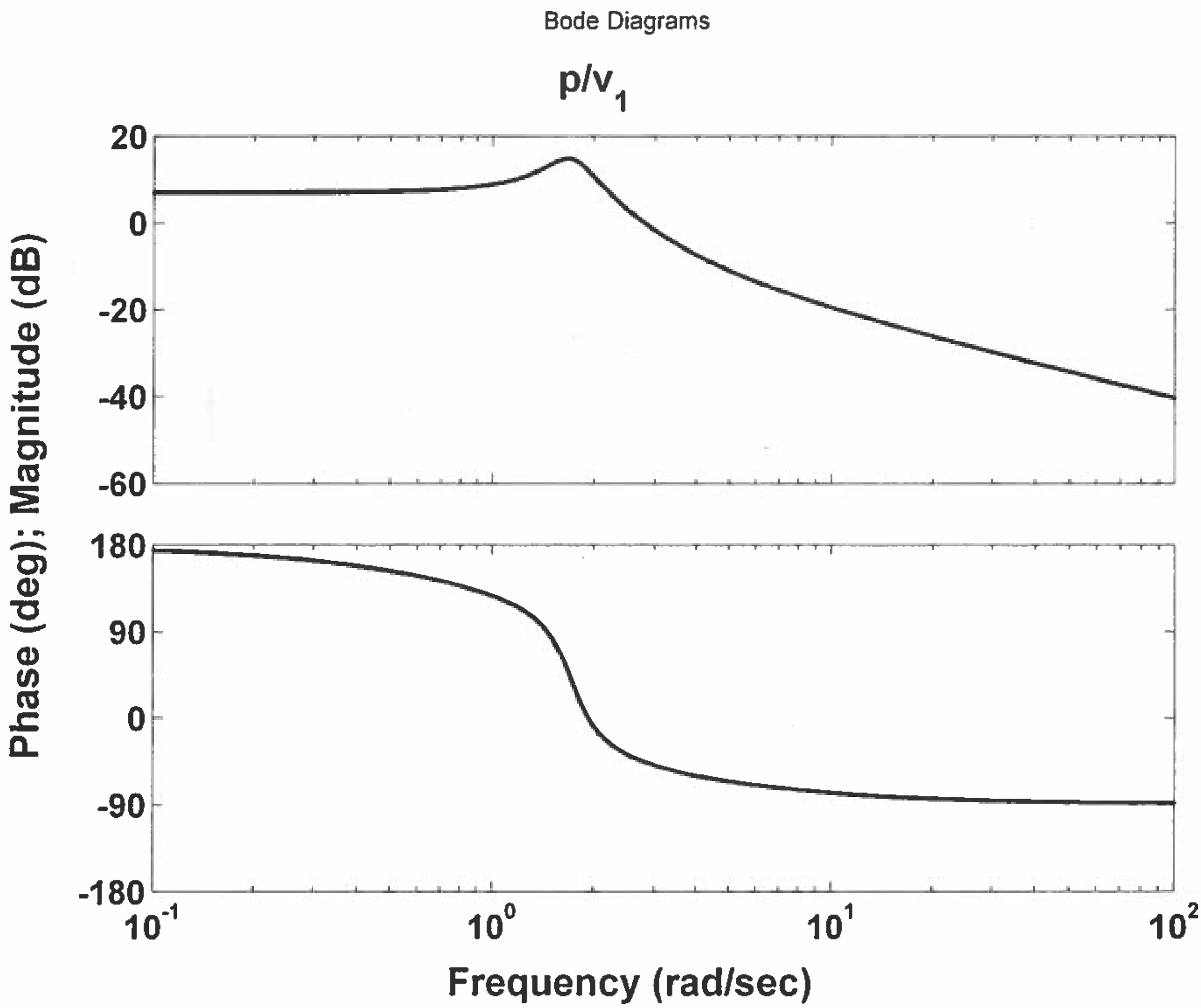
5.) But then, how about closing p using v1 with β-v2 loop closed? NO! Now a closed-loop unstable pole (s-293.2) in the transfer function (p. 25) results.

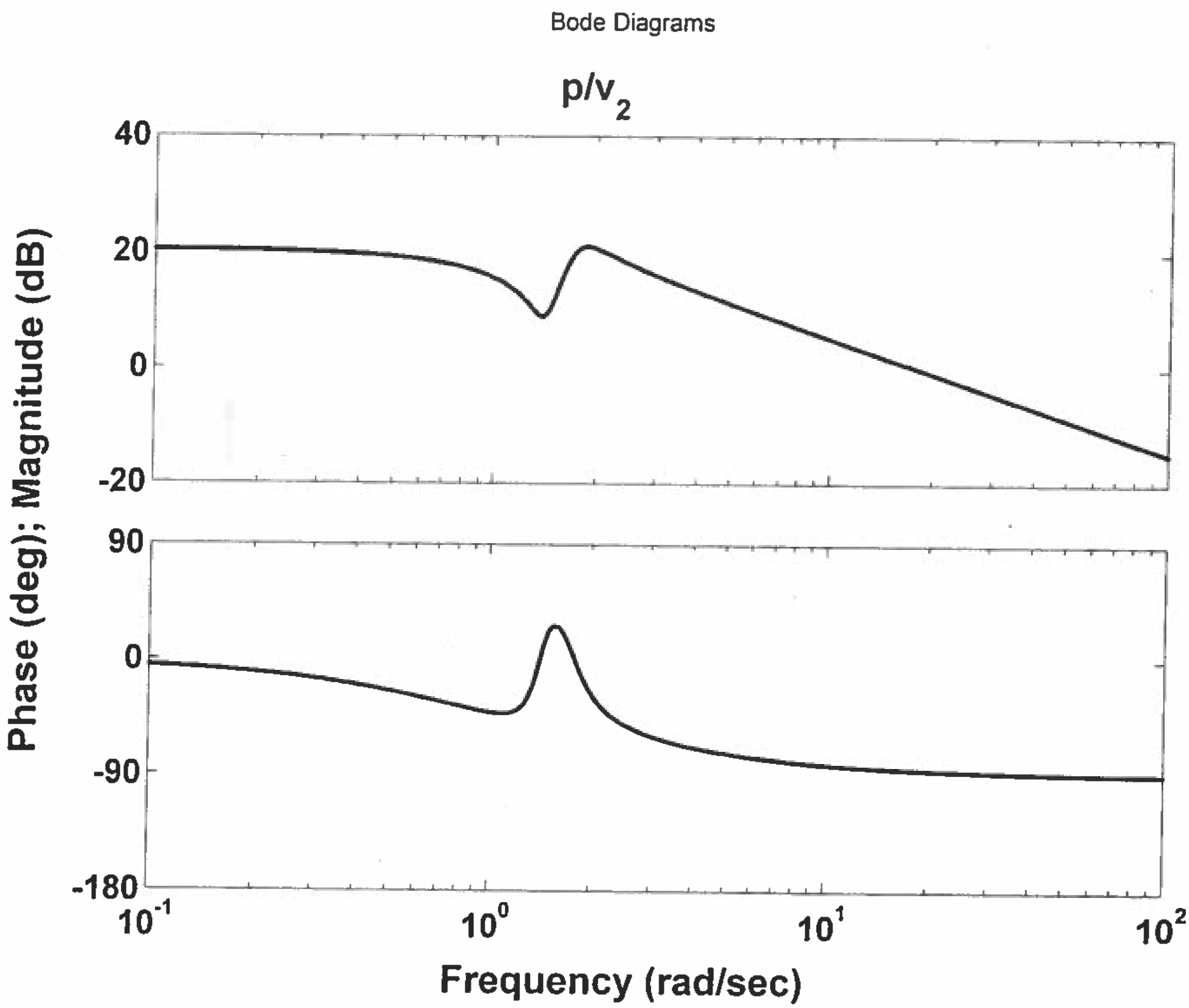
6.) What if we control p using v2 first? Not a good idea, since it would be hard to compensate as the p/v2 Bode plot (p. 22) indicates.

7.) Therefore, by a process of elimination, the designer should

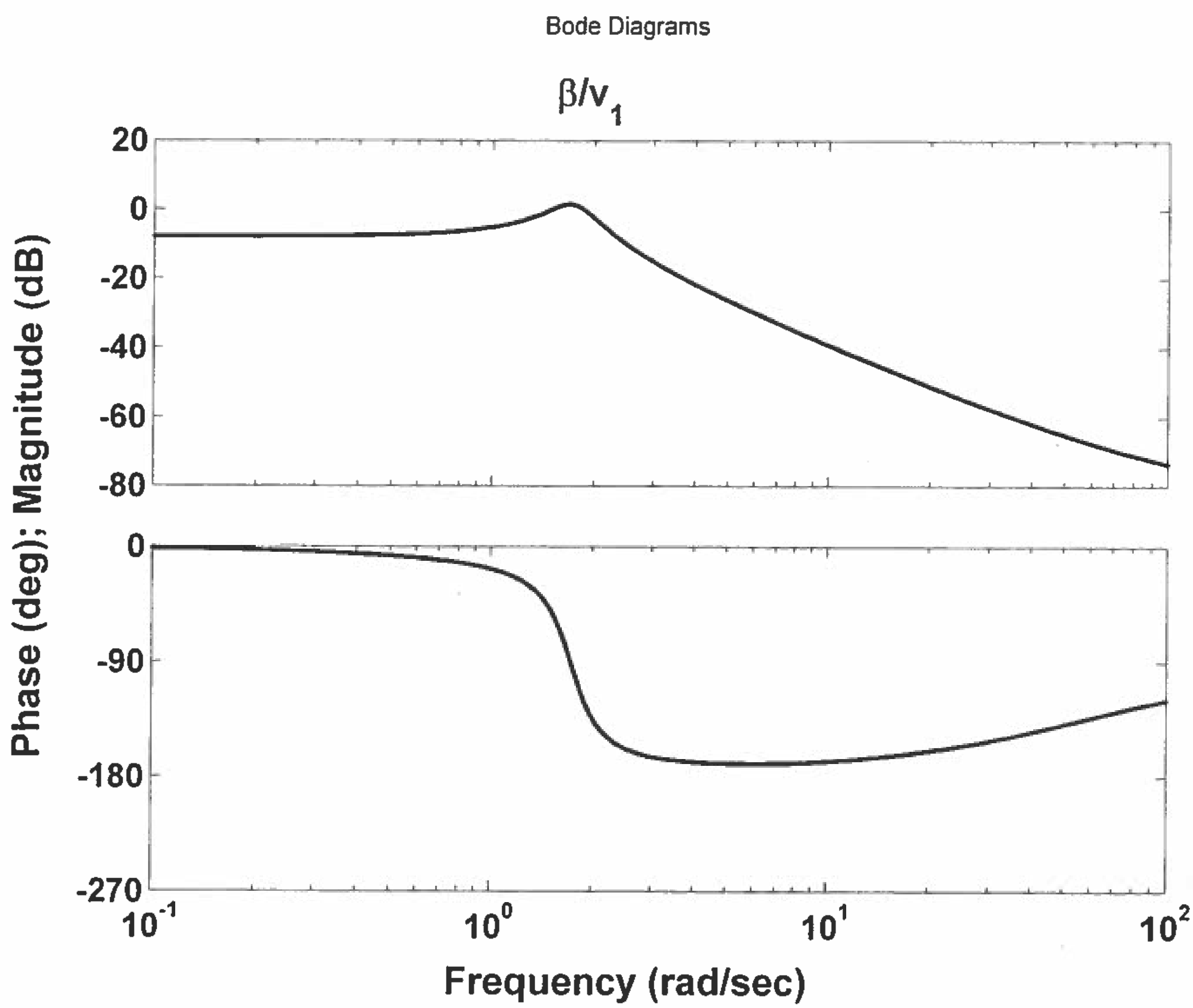
> **a.) First close β with v1, then**
>
> **b.) Close p with v2 with the β-v1 loop closed.**

The remaining pages take you through the loop shaping design and simulation.

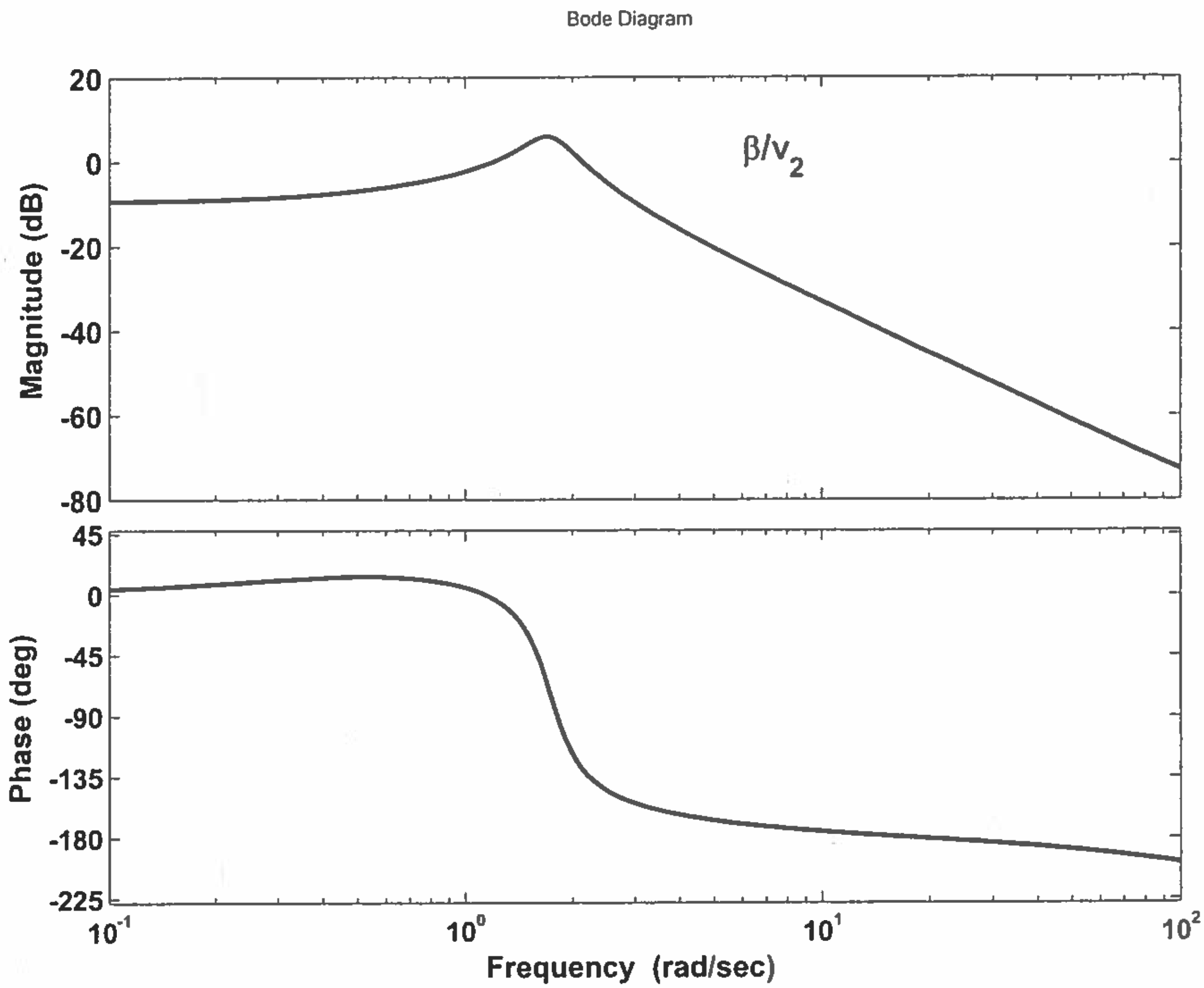

Bode Diagrams
p/v$_1$
Phase (deg); Magnitude (dB)
20
0
-20
-40
-60
180
90
0
-90
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)


Bode Diagrams
p/v_2
40
20
0
-20
90
0
-90
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
Phase (deg); Magnitude (dB)

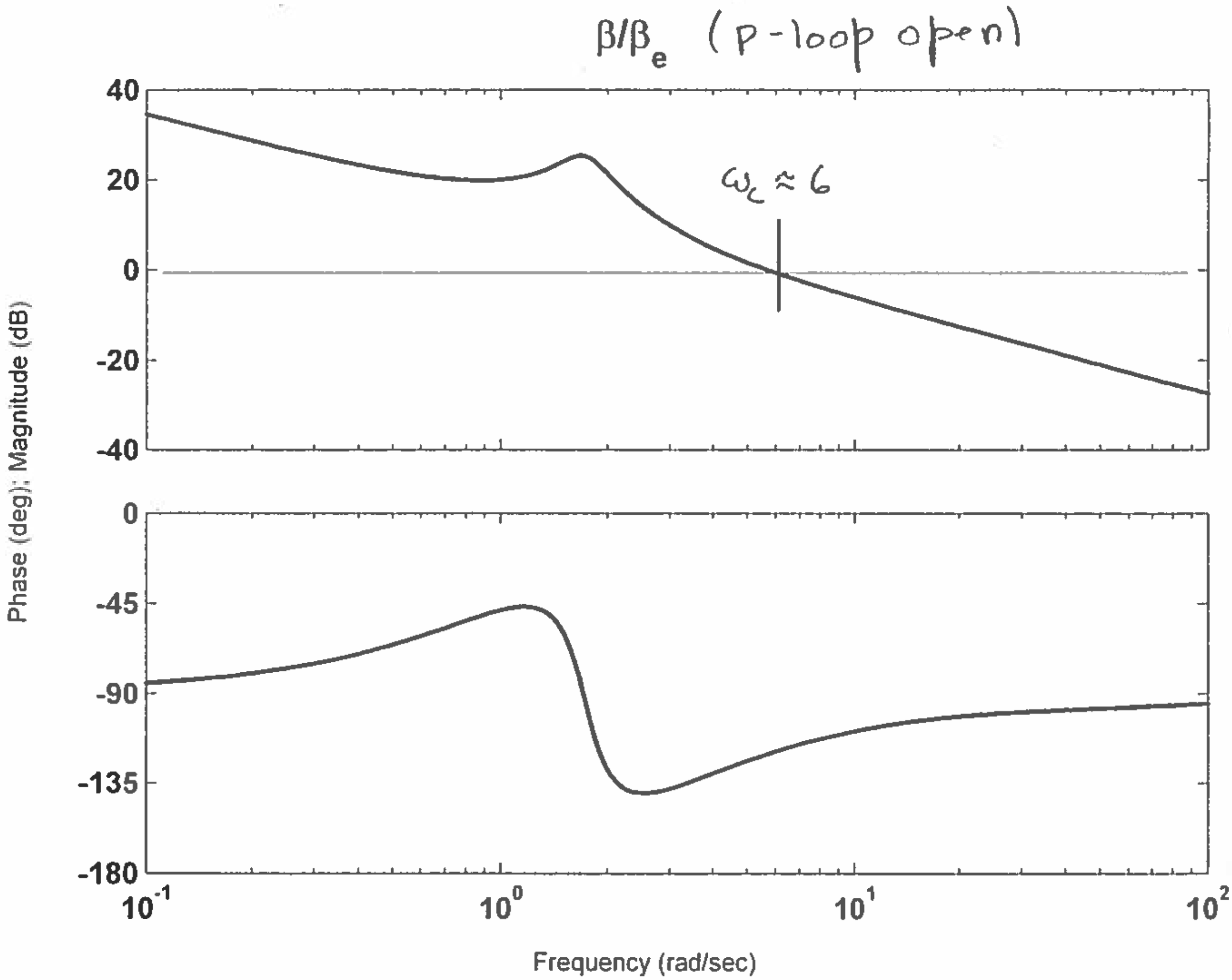
✓


Bode Diagrams
β/v1
20
0
-20
-40
-60
-80
0
-90
-180
-270
Phase (deg); Magnitude (dB)
10-1
100
101
102
Frequency (rad/sec)

26 ✓

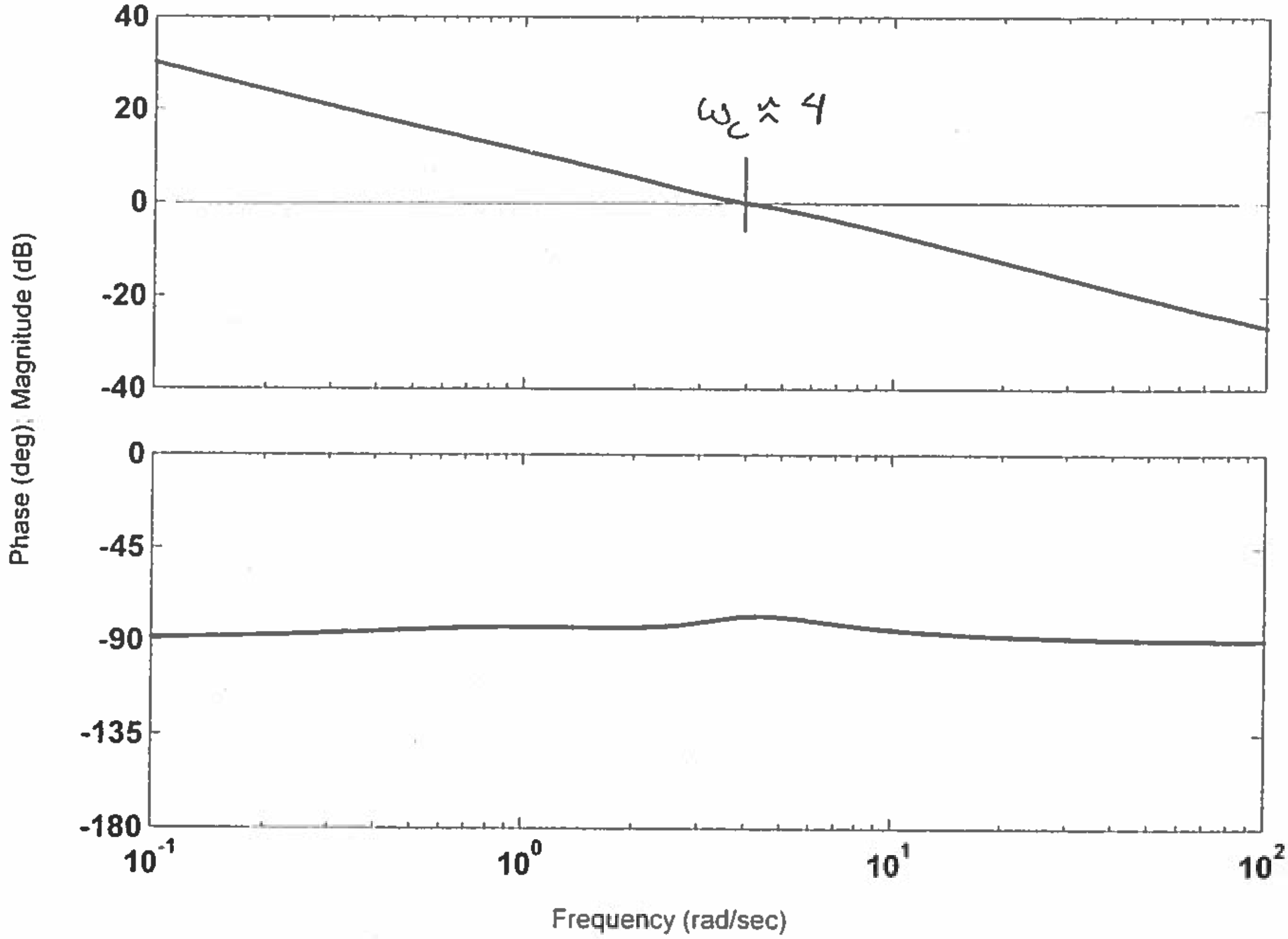


Bode Diagram
β/v2
Magnitude (dB)
20
0
-20
-40
-60
-80
Phase (deg)
45
0
-45
-90
-135
-180
-225
10-1
100
101
102
Frequency (rad/sec)



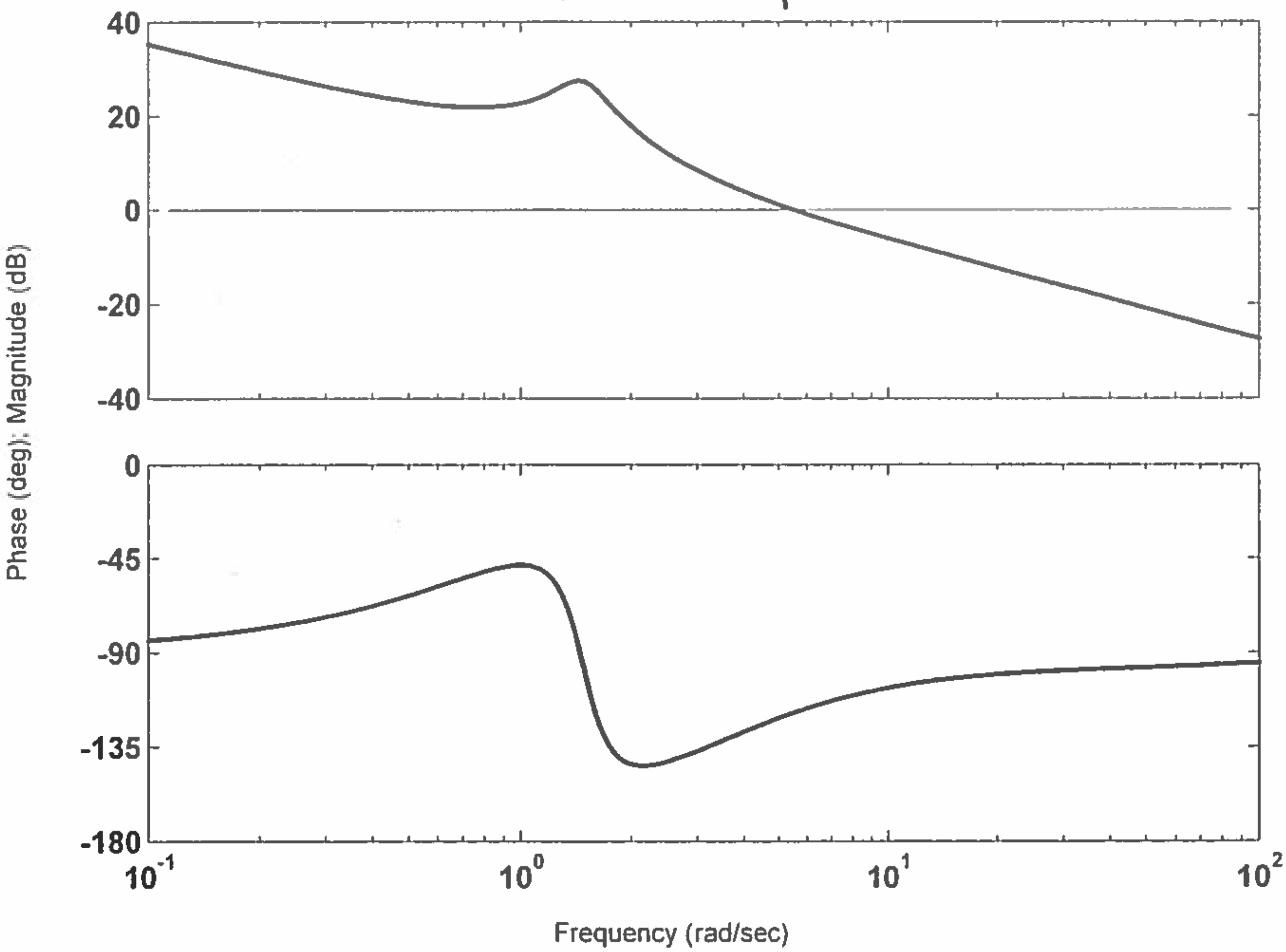
$$G_{c_\beta} = \frac{223.9(s+1.5)(s+2)}{s(s+50)}$$

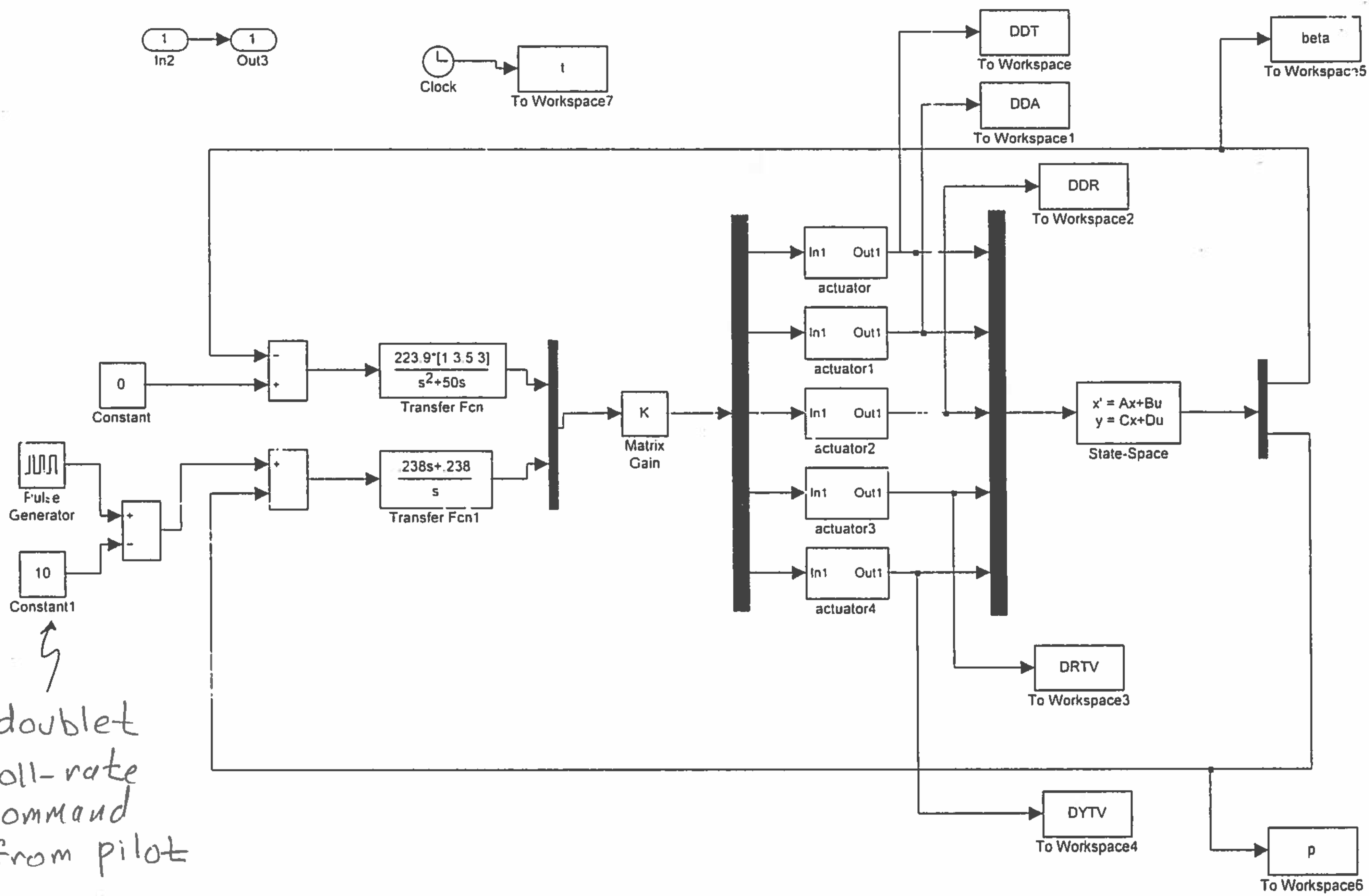
$$\left.\frac{p}{p_e}\right|_{\beta \to v, \text{ with } G_{c\beta}}$$



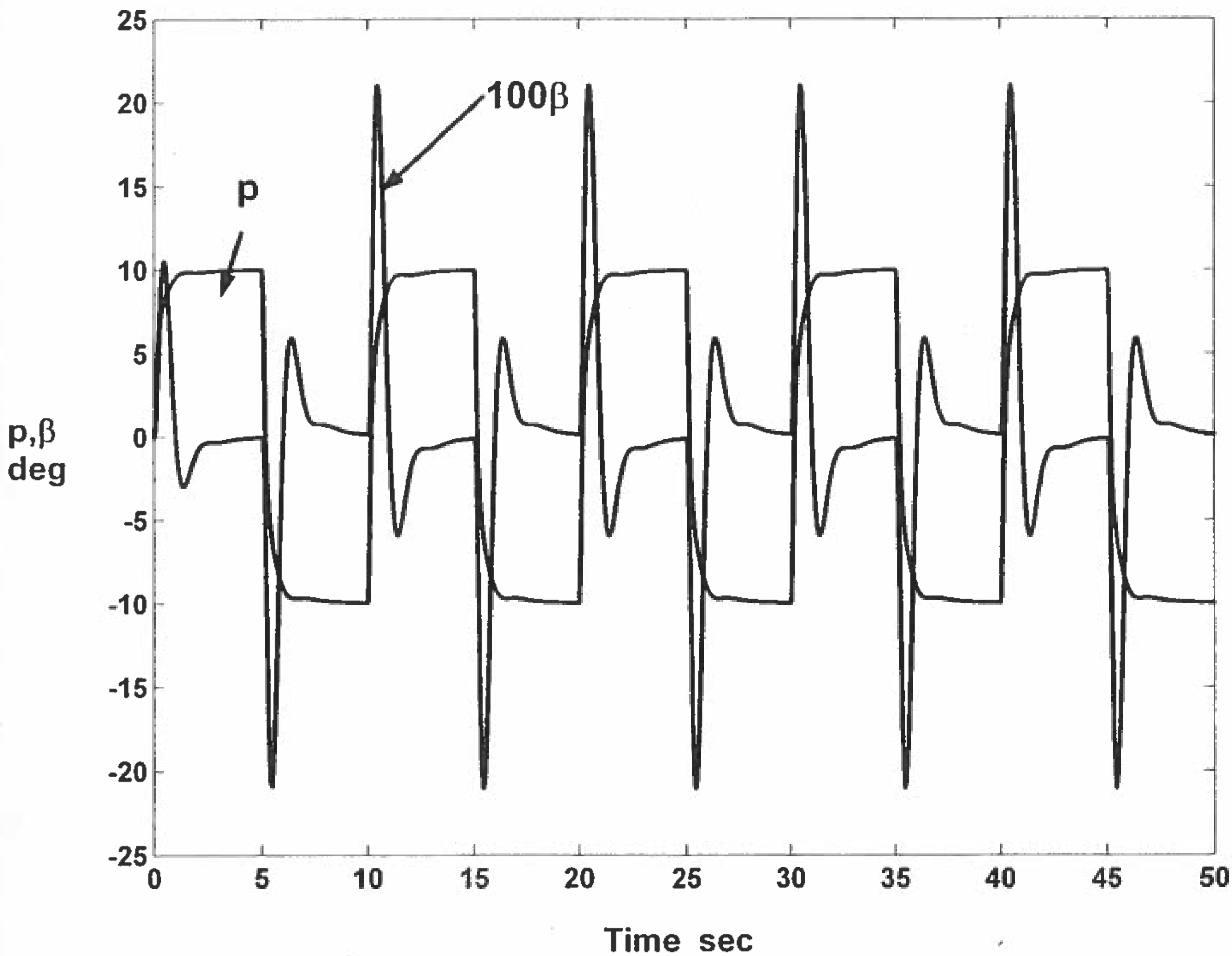
$$G_{cp} = \frac{0.254(s+1)}{s}$$

$\frac{\beta}{\beta_e}\Big|$ P – v_2 with Gcp

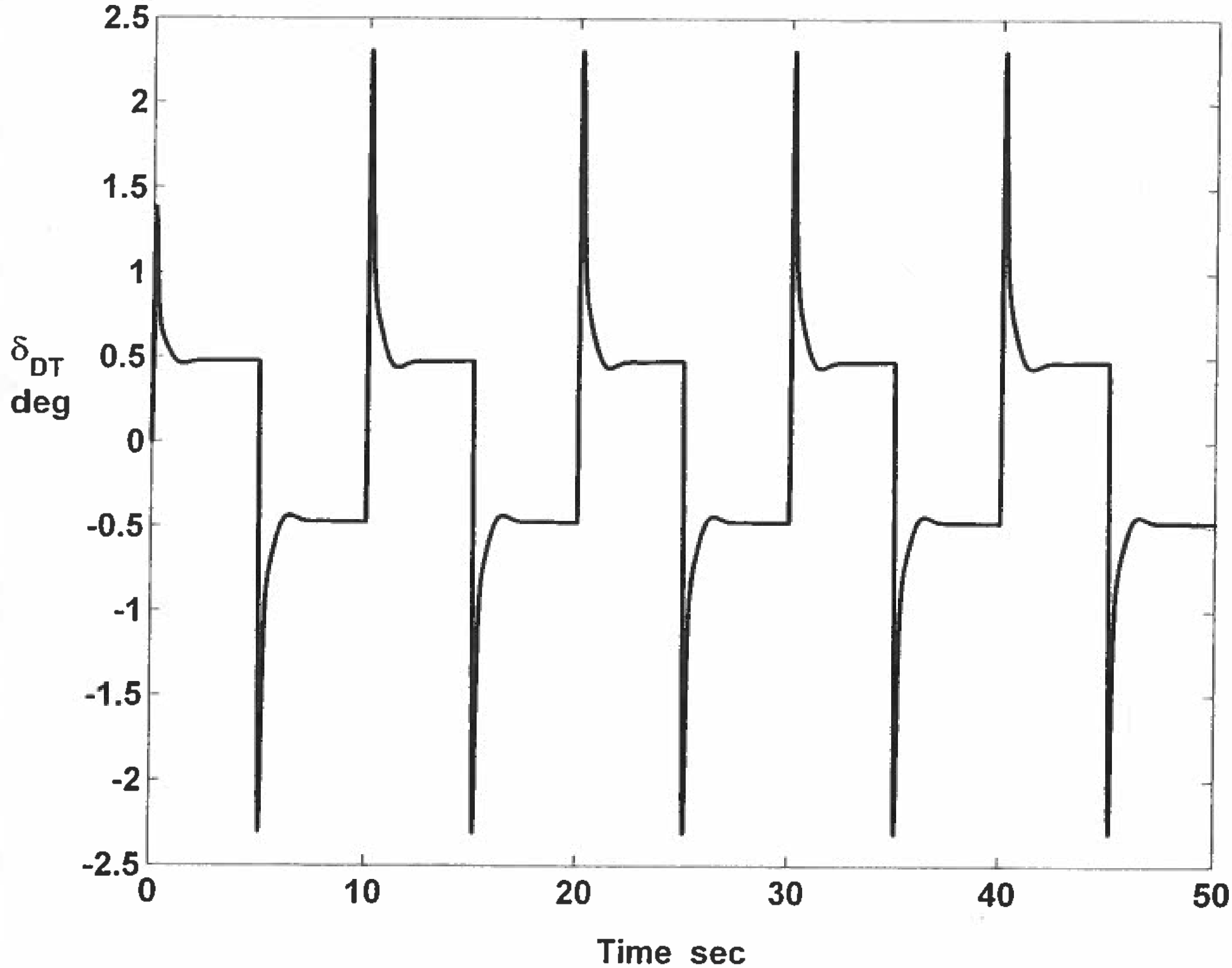


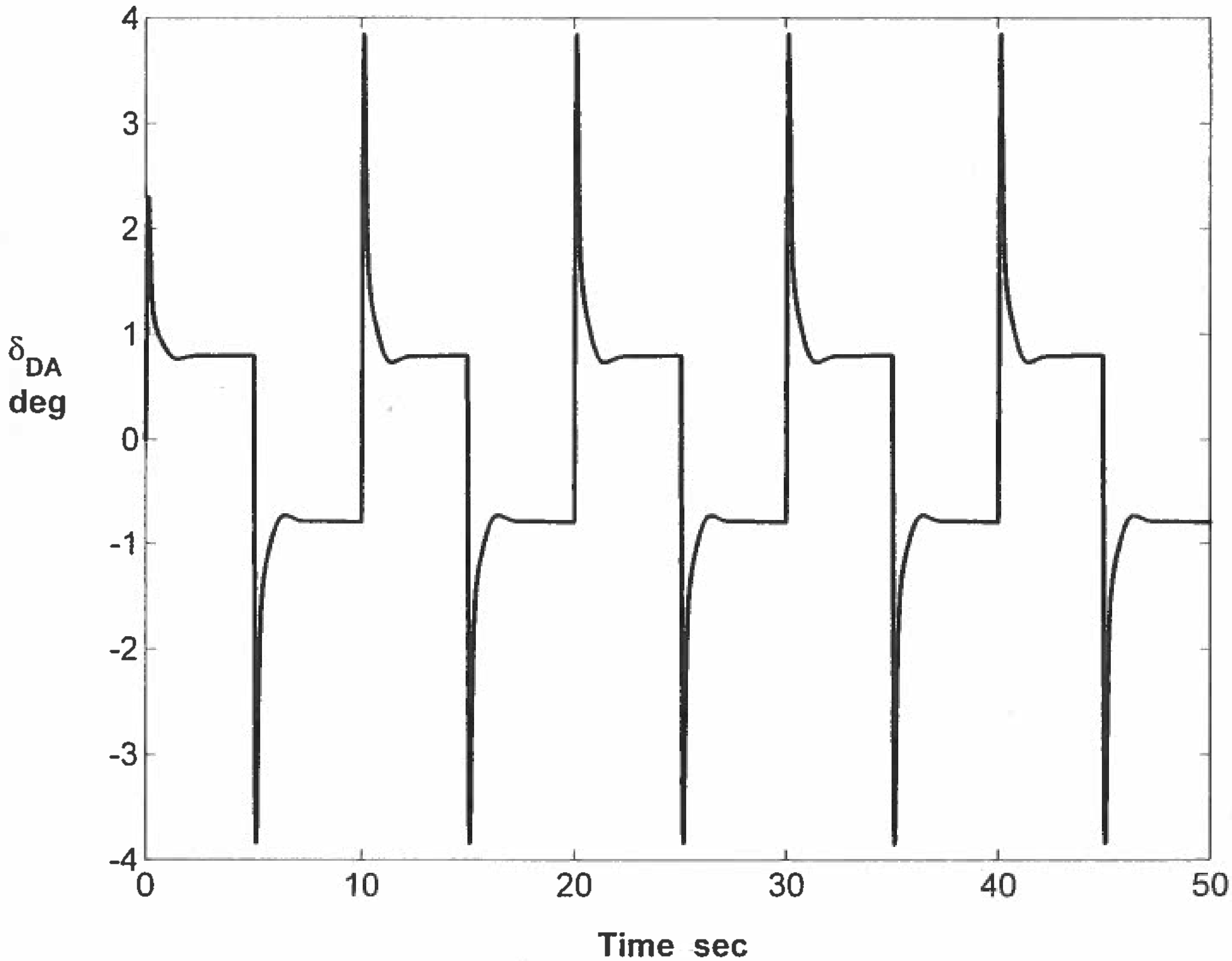


doublet
roll-rate
command
from pilot

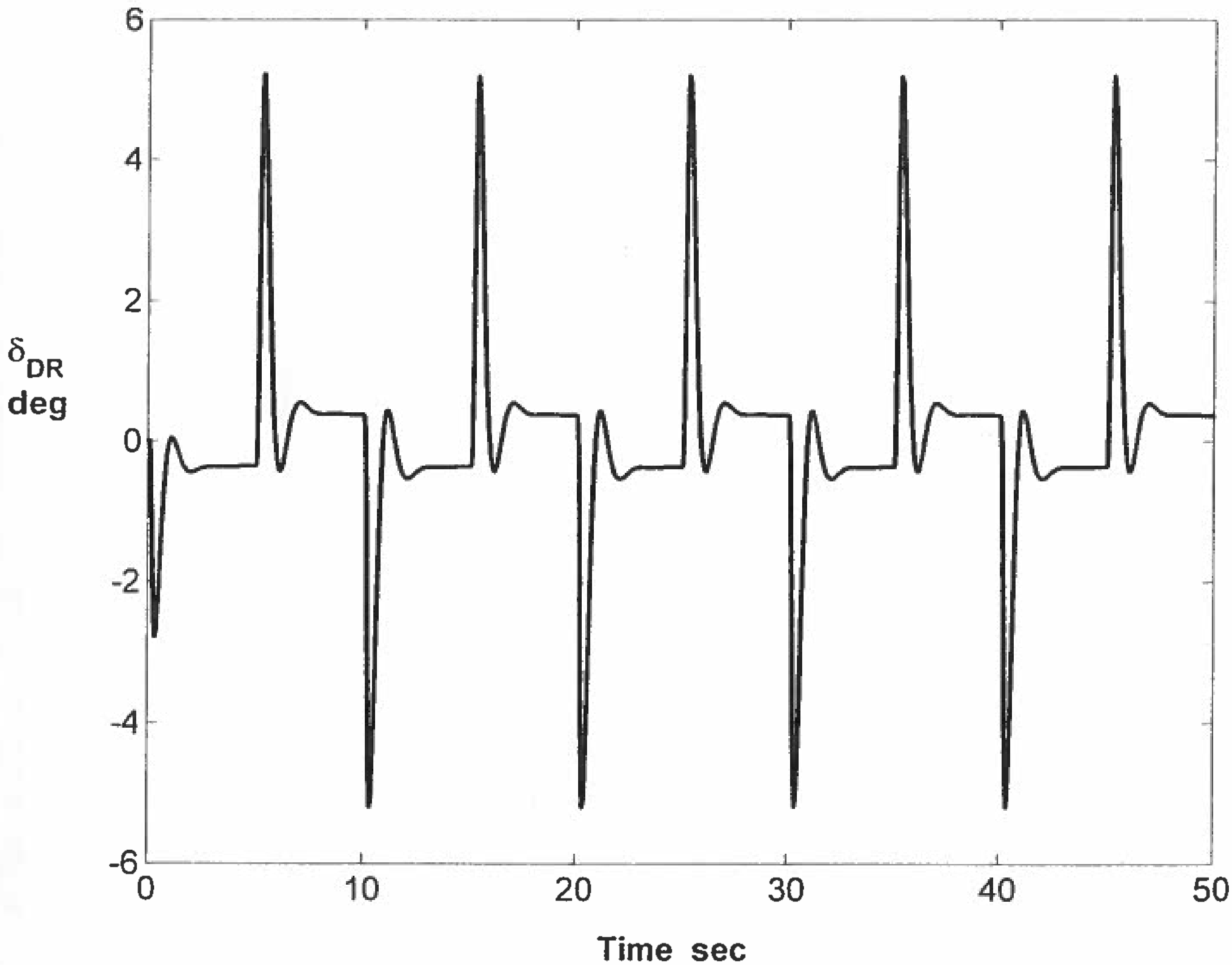


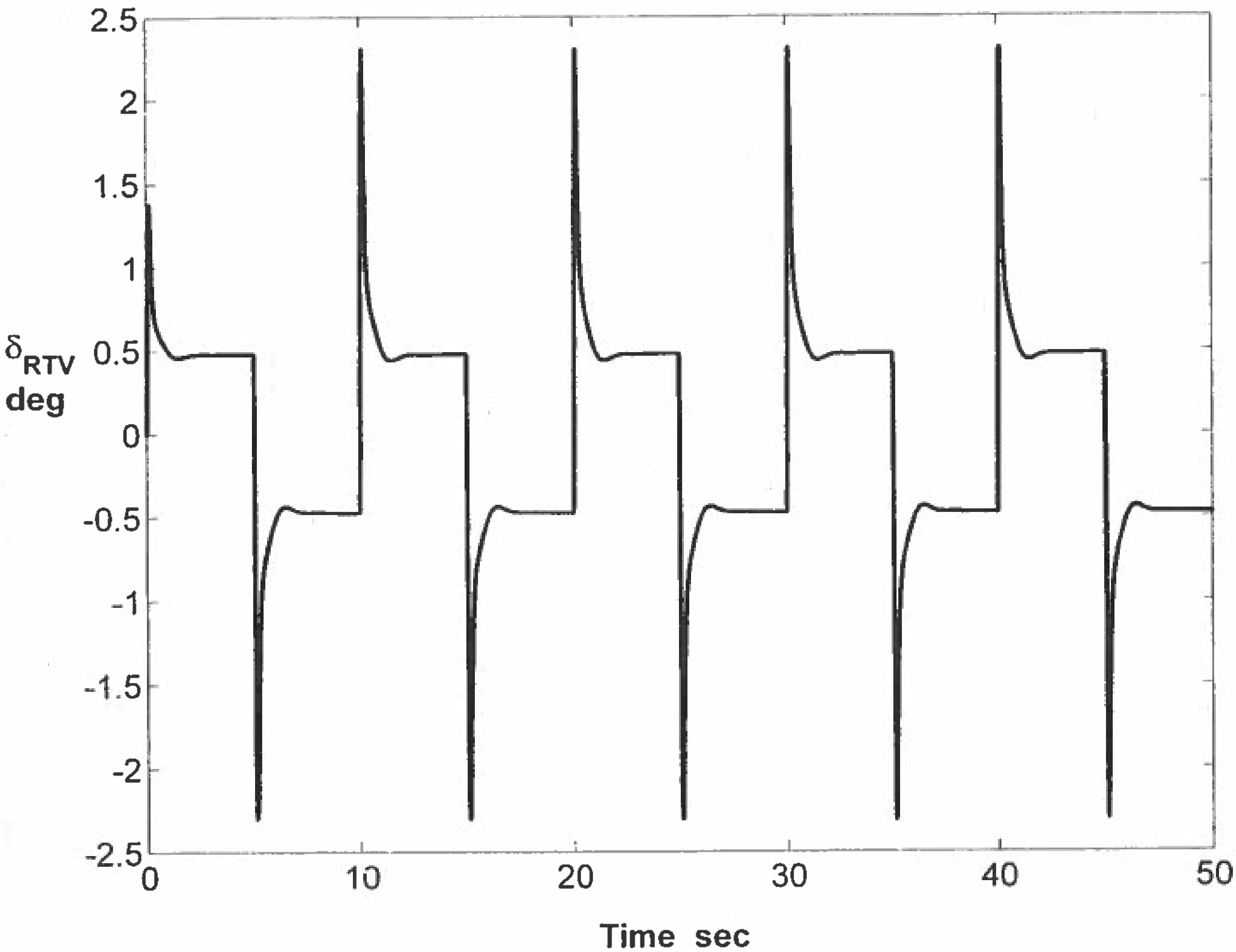
25
20
15
10
5
0
-5
-10
-15
-20
-25
p,β
deg
p
100β
0
5
10
15
20
25
30
35
40
45
50
Time sec

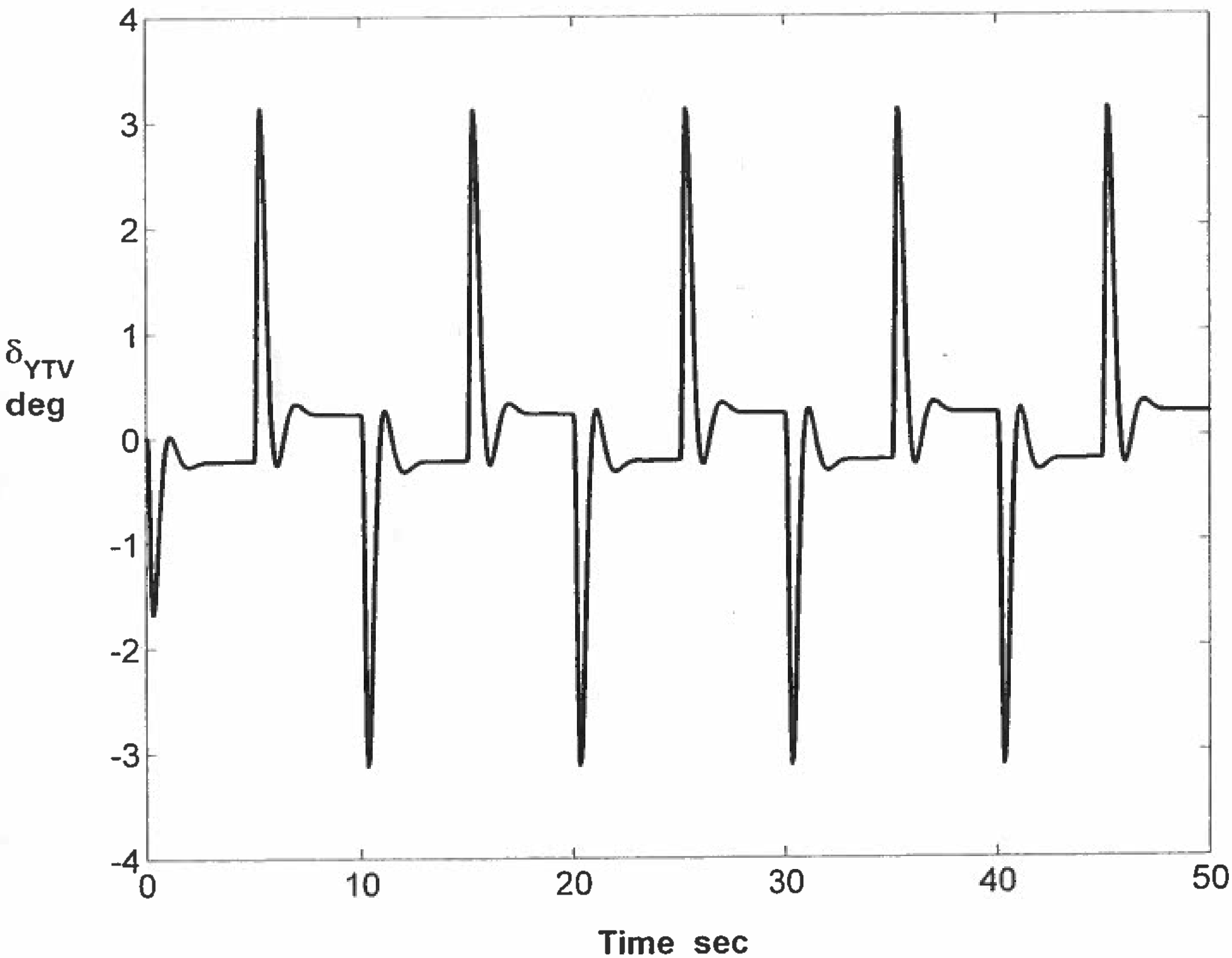

2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
δ_{DT} deg
0
10
20
30
40
50
Time sec



4
3
2
1
0
-1
-2
-3
-4
δ_{DA}
deg
0
10
20
30
40
50
Time sec


6
4
2
0
-2
-4
-6
δ_{DR}
deg
0
10
20
30
40
50
Time sec


2.5
2
1.5
1
δ_{RTV} deg
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
10
20
30
40
50
Time sec


4
3
2
1
δ_{YTV}
deg
0
-1
-2
-3
-4
0
10
20
30
40
50
Time sec