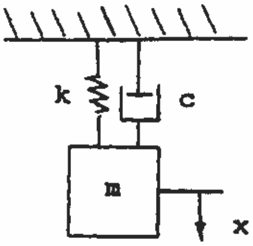
UNIVERSITY OF CALIFORNIA
Dept. of Mechanical and Aeronautical Engineering

MAE - 275

EIGENANALYSIS OF DYNAMIC SYSTEMS

Interpreting Eigenvectors

Consider a spring-mass-damper system where c, k, and m are such that the equation of motion is $\ddot{x} + 1.414\dot{x} + x = 0$



Now with $x_1 = x$ and $x_2 = \dot{x}$ the state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 1.414x_2 \tag{1}$$

$$\text{or} \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Now assume a solution to Eqs. 1 of the form:

$$\begin{aligned} x_1(t) &= a_i e^{st} \\ x_2(t) &= a_j e^{st} \end{aligned} \tag{2}$$

Differentiating Eqs. 2 gives

$$\begin{aligned} \dot{x}_1(t) &= s a_i e^{st} \\ \dot{x}_2(t) &= s a_j e^{st} \end{aligned} \tag{3}$$

Substituting Eqs. 2 and 3 into Eqs. 1 gives

$$s\begin{Bmatrix} a_i \\ a_j \end{Bmatrix} e^{st} = \mathbf{A}\begin{Bmatrix} a_i \\ a_j \end{Bmatrix} e^{st}$$

or

$$[\mathbf{A} - s\mathbf{I}]\begin{Bmatrix} a_i \\ a_j \end{Bmatrix} = 0 \qquad (4)$$

Equation 4 will have a non-trivial solution if and only if the determinant of the coefficient matrix equals zero. That is

$$|s\mathbf{I} - \mathbf{A}| = 0$$

This system has two eigenvalues or characteristic roots obtained from the above equation. Here

$$\begin{vmatrix} s & -1 \\ 1 & (s+1.414) \end{vmatrix} = 0 \quad or \quad s^2+1.414s+1 = 0$$

The eigenvalues or characteristic roots are

$$s_1 = 0.707(-1+j) = \sigma+j\omega$$

$$s_2 = 0.707(-1-j) = \sigma-j\omega$$

Now eigenvectors are defined for this system as vectors $\mathbf{x}$ which satisfy

$$(\mathbf{A} - s_i\mathbf{I})\mathbf{x} = 0 \qquad (5)$$

Now recall that a solution to the homogeneous (no forcing function) Eqs. 1 can be written

$$\begin{aligned} x_1(t) &= a_1e^{s_1t} + a_2e^{s_2t} \\ x_2(t) &= a_3e^{s_1t} + a_4e^{s_2t} \end{aligned} \qquad (6)$$

It can also be shown that if s_1 and s_2 are complex conjugates, i.e. $s_2 = s_1^*$, then $a_2 = a_1^*$ and $a_4 = a_3^*$.

Now consider just that part of the solutions of Eqs. 1 associated with the root s_1:

$$\begin{aligned} x_1(t) &= a_1e^{s_1t} \\ x_2(t) &= a_3e^{s_1t} \end{aligned} \qquad (7)$$

differentiation both sides yields

$$\dot{x}_1(t) = s_1 a_1 e^{s_1 t}$$

$$\dot{x}_2(t) = s_1 a_3 e^{s_1 t} \tag{8}$$

Now consider the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9}$$

and substituting Eq. 7 and 8 into Eq. 9 gives

$$s_1 \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} e^{s_1} t = \mathbf{A} \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} e^{s_1} t$$

or

$$(\mathbf{A} - s_1 \mathbf{I}) \begin{Bmatrix} a_1 \\ a_3 \end{Bmatrix} = 0$$

But this is identical to Eq. 5 which defined the <u>eigenvectors</u>. Thus, a_1 and a_3 are the elements of the eigenvector associated with the characteristic root or eigenvalue $s = s_1$. Likewise, a_2 and a_4 are the elements of the eigenvector associated with the characteristic root or eigenvalue $s = s_2$. Now let

$$a_1 = c+dj \quad a_2 = c-dj$$

$$a_3 = g+hj \quad a_4 = g-hj$$

where $j=\sqrt{-1}$. Equations 6 can now be written

$$x_1(t) = 2\sqrt{c^2+d^2}e^{\sigma t}[\cos(\omega t+\psi_1)]$$

$$x_2(t) = 2\sqrt{g^2+h^2}e^{\sigma t}[\cos(\omega t+\psi_2)]$$

where

$$\psi_1 = \tan^{-1}\left[\frac{d}{c}\right]$$

$$\psi_2 = \tan^{-1}\left[\frac{h}{g}\right]$$

Now at t = 0

$$x_1(t) = a_1 e^{(\sigma + j\omega)t} + a_2 e^{(\sigma - j\omega)t}$$

$$= e^{\sigma t}\left[a_1 e^{j\omega t} + a_2 e^{-j\omega t}\right]$$

but $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$e^{-j\omega t} \quad \cos\omega t - j\sin\omega t$

$$x_1(t) = e^{\sigma t}\left[a_1(\cos\omega t + j\sin\omega t) + a_2(\cos\omega t - j\sin\omega t)\right]$$

$$= e^{\sigma t}\left[(a_1 + a_2)\cos\omega t + (a_1 - a_2)j\sin\omega t\right]$$

but $a_1 = c + dj$

$a_2 = c - dj$

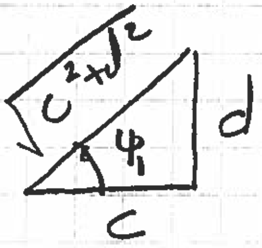
$a_1 + a_2 = 2c$

$a_1 - a_2 = 2dj$

$$\therefore \quad x_1(t) = 2\boxed{e^{\sigma t}}\left[c\cos\omega t - d\sin\omega t\right]$$

dynamic stability!

$$= 2\sqrt{c^2+d^2}\, e^{\sigma t}\Big[\underbrace{\frac{c}{\sqrt{c^2+d^2}}}_{\cos\varphi_1}\cos\omega t - \underbrace{\frac{d}{\sqrt{c^2+d^2}}}_{\sin\varphi_1}\sin\omega t\Big]$$



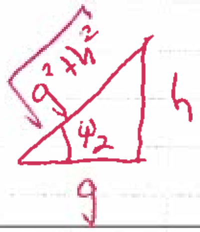
$$\varphi_1 = \tan^{-1}\frac{d}{c}$$

$$x_1(t) = 2\sqrt{c^2+d^2}\,e^{\sigma t}\cos(\omega t+\varphi_1)$$

likewise

$$x_2(t) = 2\sqrt{g^2+h^2}\,e^{\sigma t}\cos(\omega t+\varphi_2)$$

$$\varphi_2 = \tan^{-1}\left[\frac{h}{g}\right]$$



@ $t=0$

$$\left.\begin{aligned} x_1(0) &= 2\sqrt{c^2+d^2}\cos\varphi_1 = 2c \\ x_2(0) &= 2\sqrt{g^2+h^2}\cos\varphi_2 = \boxed{2g} \end{aligned}\right\}$$

2 x real part of eigenvector

initial conditions!

in general I.C.'s that will result in only one mode of motion

∴ real part of eigenvector associated with any ch. root will provide I.C, so that only the mode associated with that eigenvalue is excited.

$$x_1(0) = 2\sqrt{c^2+d^2}\cos(\psi_1) = 2[\mathit{real\ part\ of}\ a_1]$$

$$x_2(0) = 2\sqrt{g^2+h^2}\cos(\psi_2) = 2[\mathit{real\ part\ of}\ a_3]$$

and for $t \neq 0$, $x_1(t)$ can be considered the real part of a complex number whose magnitude is $2e^{\sigma t}$ times the magnitude of the eigenvector element a_1 and whose phase angle is $\omega t+\psi_1$. Adding ωt to ψ_1 is synonymous with considering the eigenvector element to be rotating in a counterclockwise direction in the complex plane with angular velocity ω. A similar statement can be made for $x_2(t)$. Often in the study of dynamic systems, the rotating eigenvector element is referred to as a phasor.

If the phasor we draw is associated with the eigenvalue with a negative imaginary part, then we have to interpret the phasor as rotating in a clockwise direction. To avoid any confusion when selecting and plotting phasors, we will always choose the eigenvector associated with the eigenvalue with the positive imaginary part. Now the factor of "2" is of no consequence in this analysis, since we are only interested in comparing the relative magnitudes and phase angles (ψ) of the phasors.

Calculation the Eigenvectors

Writing Eq. 5 for the spring-mass-damper system

$$\begin{aligned} s\overline{x_1} - \overline{x_2} &= 0 \\ \overline{x_1} + (s+1.414)\overline{x_2} &= 0 \end{aligned} \qquad (10)$$

Now the zero value of the determinant of the coefficient matrix which defines the eigenvalues guarantees a non-trivial solution to Eq. 10 which will involve $\overline{x_1}$ as a function of $\overline{x_2}$ or vice-versa. This simply means that either $\overline{x_2}$ or $\overline{x_1}$ is arbitrary. Let us choose $\overline{x_1} = 1.0$ as the arbitrary phasor and solve either of Eqs. 7 for $\overline{x_2}$. (You will get the same answer with either one). We do this, of course, for the eigenvalue with the positive imaginary part:

$$\overline{x_2} = \frac{-1}{(-0.707+0.707j+1.414)} = 0.707(-1+j)$$

Summarizing our phasors:

$$\overline{x_1} \equiv a_1 = 1.0$$

$$\overline{x_2} \equiv a_3 = 0.707(-1+j)$$

The phasor diagram and the interpretation in terms of time functions is shown on the following page. This diagram summarizes the single natural mode of motion or natural mode for this system.

Summary

Consider a system described by a set of 'n' first-order linear differential equations where the dependent variables are $x_i(t)$, i = 1,2,...n.

1.) Substitute $x_i(t) = x_i(0)e^{st}$ into the differential equations

2.) Form the determinant of the coefficient matrix and set it equal to zero. The resulting equation is called the characteristic equation.

3.) Values of 's' which satisfy the characteristic equation are called characteristic roots or eigenvalues. The system is dynamically stable if and only if all these characteristics roots lie in the left half of the complex plane.

4.) There is a fundamental or natural mode of motion associated with

a.) each pair of complex conjugate characteristic roots

$$s_{1,2} = \sigma \pm j\omega$$

b.) each single real root

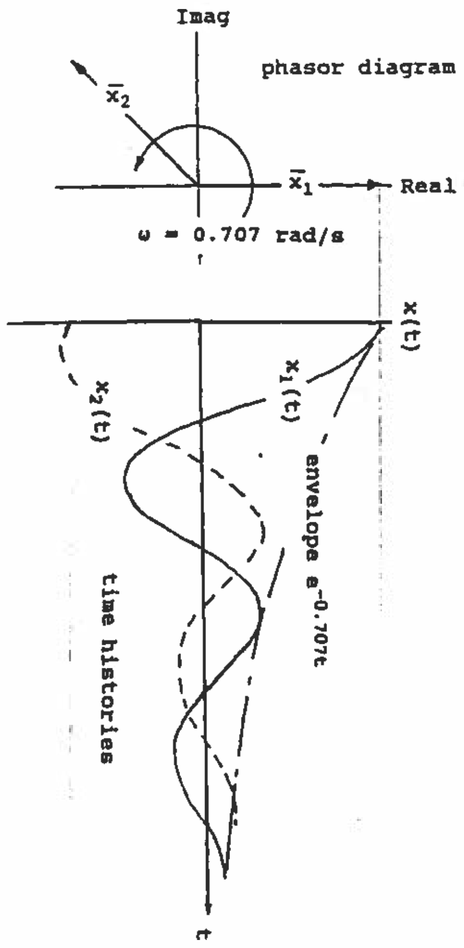
$$s = \sigma$$

5.) Each mode of motion contains 'n' variables.

6.) In each mode, at any time t_1, the system response $x_i(t_1)$ can be thought of as the real part of a phasor. The phasor is one element in a complex vector called and eigenvector. As time progresses, the phasor can be thought of as rotating with an angular velocity ω rad/s (counter-clockwise if the eigenvector is associated with a characteristic root with a positive imaginary part, clockwise if the eigenvector is associated with a characteristic root with a negative imaginary part). In addition to rotating, the magnitude of the phasor is proportional to $e^{\sigma t}$, where σ is the real part of the characteristic root associated with the mode in question.

7.) For each mode, one can draw a phasor diagram containing 'n' such rotating phasors. By convention, the eigenvectors are selected as those associated with the characteristic roots with positive imaginary parts.

8.) For any linear dynamic system described by 'n' first-order differential equations and 'p' natural modes, one will have 'p' phasor diagrams, each with 'n' phasors describing the mode shapes.


Imag
phasor diagram
$\bar{x}_2$
$\bar{x}_1$
Real
ω = 0.707 rad/s
x(t)
$x_1(t)$
$x_2(t)$
envelope $e^{-0.707t}$
time histories
t

MAE 275

Longitudinal Eigenanalysis for Boeing 747

On the following page are the longitudinal stability derivatives for the Boeing 747 aircraft in cruise flight condition (configuration 7). Derivatives not shown can be considered to be negligible. In this flight condition, $u_0 = 830$ ft/sec, $\theta_0 = 0$ rad.

1.) Using MATLAB, complete an eigenanalysis of the longitudinal dynamics of the aircraft at this flight condition. This should include

a.) Defining the state equations (**A**,**B**,**C** and **D** matrices). In defining, **C** and **D**, assume Δu and $\Delta\theta$ are the "outputs" of interest.

b.) Determining the characteristic roots (eigenvalues) and eigenvectors and identifying the modes of motion

c.) Plot of eigenvectors (phasor diagrams) for each mode with sketches of time responses

2.) Again using MATLAB determine $\Delta\theta(t)$ and $\Delta u(t)$ if an initial condition of $\Delta\dot{\theta} = \Delta q = 5\,\text{deg/sec}\,(0.08726\ \text{rad.sec})$ is specified. Plot $500[\Delta\theta(t)]$ and $\Delta u(t)$ vs time using different time scales: one for $0 < t < 10$ sec and one for $0 < t < 1500$ sec. Which of these corresponds to the short-period mode and which to the phugoid mode?

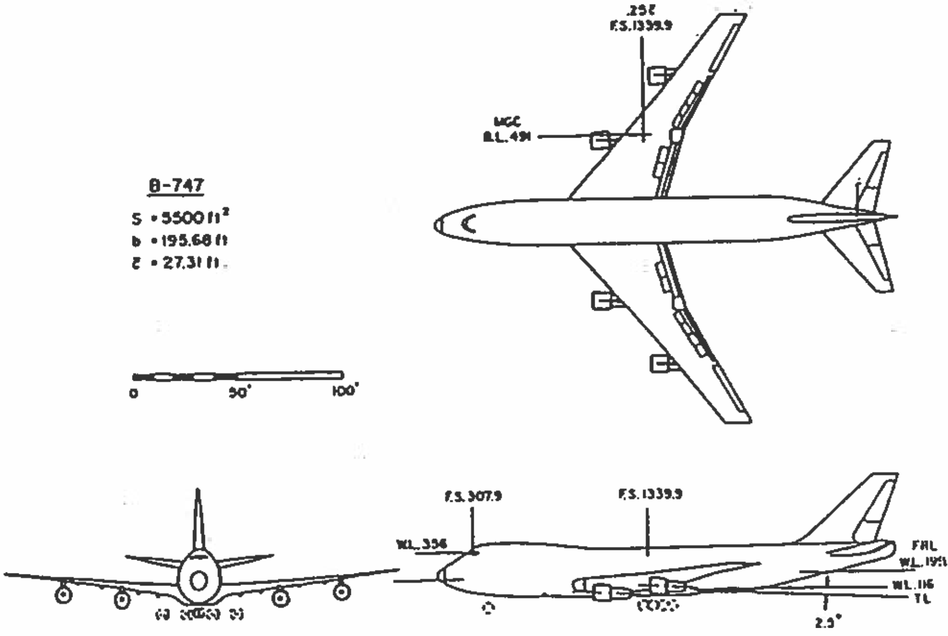


TABLE IX-4

B-747 LONGITUDINAL DIMENSIONAL DERIVATIVES

(BODY AXIS SYSTEM)

Flight Condition
alt = 29 000 ft
Mach No = .8

F/C #	1	2	3	4	5	6	7	8	9	10
H	SL	SL	SL	SL	20 K	20 K	20 K	40 K	40 K	40 K
M	.198	.249	.450	.650	.500	.650	.800	.700	.800	.900
XU * = X_u	-.0209	-.0108	-.00499	-.00777	-.00247	-.00280	-.00643	.00187	-.00276	-.0200
ZU * = Z_u	-.202	-.150	-.0807	-.126	-.0679	-.0832	-.0941	-.0696	-.0650	-.0424
MU * = M_u	.000117	.000181	.000146	-.000199	.000247	.885E-4	-.000222	.000259	.000193	-.623E-4
XW = X_w	.122	.106	.0743	.0345	.0782	.0482	.0253	.0263	.0389	.0159
ZW = Z_w	-.512	-.613	-.736	-.963	-.433	-.539	-.624	-.292	-.317	-.401
MW = M_w	-.00177	-.00193	-.00262	-.00239	-.00170	-.00190	-.00153	-.00101	-.00105	-.00190
ZWD	.0334	.0338	.0297	.0293	.0157	.0156	.0144	.00704	.00556	.00614
ZQ	-6.22	-7.58	-10.4	-12.8	-6.39	-8.09	-9.99	-4.32	-5.16	-6.71
MWD	-.000246	-.000240	-.000221	-.000228	-.000125	-.000155	-.000212	-.905E-4	-.000116	-.000160
MQ	-.357	-.437	-.699	-.925	-.421	-.535	-.659	-.284	-.339	-.401
XDE = X_{δ_e}	.959	.971	1.18	0.	2.02	1.15	0.	1.93	1.44	.781
ZDE	-6.42	-9.73	-21.8	-32.4	-16.9	-26.4	-32.7	-15.1	-17.9	-18.6
MDE	-.378	-.574	-1.40	-2.07	-1.09	-1.69	-2.09	-.970	-1.16	-1.22
XDTH	.570E-4	.570E-4	.505E-4	.505E-4	.505E-4	.505E-4	.505E-4	.505E-4	.505E-4	.505E-4
ZDTH	-.249E-5	-.249E-5	-.220E-5	-.220E-5	-.220E-5	-.220E-5	-.220E-5	-.220E-5	-.220E-5	-.220E-5
MDTH	.310E-6	.310E-6	.302E-6	.302E-6	.302E-6	.302E-6	.302E-6	.302E-6	.302E-6	.302E-6

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State-Space Form of Linearized Aircraft Equations of Motion

longitudinal equations

$$\Delta\dot{u} = X_u\Delta u + X_w\Delta w - g\cos\theta_0\Delta\theta + \sum_{i=1}^{n}\cancel{X_{\delta_i}\Delta\delta_i}$$

$$\Delta\dot{w} = \frac{Z_u}{1-Z_{\dot{w}}}\Delta u + \frac{Z_w}{1-Z_{\dot{w}}}\Delta w + \frac{Z_q+u_0}{1-Z_{\dot{w}}}\Delta q - \frac{g\sin\theta_0}{1-Z_{\dot{w}}}\Delta\theta + \frac{1}{1-Z_{\dot{w}}}\cancel{\sum_{i=1}^{n}Z_{\delta_i}\Delta\delta_i}$$

$$\Delta\dot{q} = \left[M_u + \frac{M_{\dot{w}}Z_u}{1-Z_{\dot{w}}}\right]\Delta u + \left[M_w + \frac{M_{\dot{w}}Z_w}{1-Z_{\dot{w}}}\right]\Delta w + \left[M_q + \frac{M_{\dot{w}}(Z_q+u_0)}{1-Z_{\dot{w}}}\right]\Delta q - \left[\frac{M_{\dot{w}}g\sin\theta_0}{1-Z_{\dot{w}}}\right]\Delta\theta + \frac{M_{\dot{w}}}{1-Z_{\dot{w}}}\cancel{\sum_{i=1}^{n}Z_{\delta_i}\Delta\delta_i} + \cancel{\sum_{i=1}^{n}M_{\delta_i}\Delta\delta_i}$$

$$\Delta\dot{\theta} = \Delta q$$

$$\dot{x} = (u_0+\;u)\cos\theta_0 + \Delta w\sin\theta_0 - u_0\Delta\theta\sin\theta_0$$

$$\dot{z} = -(u_0+\;u)\sin\theta_0 + \Delta w\cos\theta_0 - \Delta u_0\Delta\theta\cos\theta_0$$

$$\Delta\dot{u} = -.00643\,\Delta u + .0253\,\Delta w + 0\,\Delta q - 32.2\,\Delta\theta$$

WITH $1 - Z_{\dot{w}} = 1 - .0144 = .986$

$$\Delta\dot{w} = -\frac{.0991}{.986}\Delta u - \frac{.624}{.986}\Delta w + \frac{(830-9.98)}{.986}\Delta q - 0\;\theta$$

WITH $\dfrac{M_{\dot{w}}}{1-Z_{\dot{w}}} = \dfrac{-.000212}{.986} = -2.15\cdot10^{-4}$

$$\Delta\dot{q} = (-.000222 - 2.15\cdot10^{-4}\cdot(-.0991))\Delta u +$$

$$(-.00153 - 2.15\cdot10^{-4}(-.624))\Delta w +$$

$$(-.668 - 2.15\cdot10^{-4}(-9.98+830))\Delta q - 0$$

$$\Delta\dot{\theta} = \Delta q$$

$\underline{\sigma}$

$$\Delta\dot{u} = -6.93\cdot10^{-3}\Delta u + 2.53\cdot10^{-2}\Delta w + 0\Delta q - 32.2\Delta\theta$$

$$\Delta\dot{w} = -9.54\cdot10^{-2}\Delta u - .633\Delta w + 832\Delta q - 0\Delta\theta$$

$$\Delta\dot{q} = -2.02\cdot10^{-4}\Delta u - 1.4\cdot10^{-3}\Delta w - .844\Delta q - 0\Delta\theta$$

$$\Delta\dot{\theta} = \Delta q$$

$$\underline{A} = \begin{bmatrix} -6.93\cdot10^{-3} & 2.53\cdot10^{-2} & 0 & -32.2 \\ -9.54\cdot10^{-2} & -.633 & 832 & 0 \\ -2.02\cdot10^{-4} & -1.4\cdot10^{-3} & -.844 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

```
» A=[-6.43e-3  2.53e-2  0 -32.2; -9.54e-2  -.633  832  0; -2.02e-4  -1.4e-3  -.844 0;

A =

   -0.0064    0.0253         0  -32.2000
   -0.0954   -0.6330  832.0000         0
   -0.0002   -0.0014   -0.8440         0
         0         0    1.0000         0

» format short e
» A

A =

 -6.4300e-003  2.5300e-002            0 -3.2200e+001
 -9.5400e-002 -6.3300e-001  8.3200e+002            0
 -2.0200e-004 -1.4000e-003 -8.4400e-001            0
            0            0  1.0000e+000            0

» B=[0;0;0;0]

3 =

     0
     0
     0
     0

› C=[1 0 0 0;0 0 0 1]

: =

     1     0     0     0
     0     0     0     1

· d=[0;0]

! =

     0
     0
```

SHORT PERIOD EIGENVECTOR

PHUGOID EIGENVECTOR

```
» [v,d]=eig(A)

v =

  Columns 1 through 3

u -1.1306e-002 +4.3023e-003i -1.1306e-002 -4.3023e-003i -2.6902e-001 -9.5216e-001i
α -9.9817e-001 -5.9224e-002i -9.9817e-001 +5.9224e-002i  4.0088e-002 +1.3933e-001i
q  2.0195e-004 -1.2820e-003i  2.0195e-004 +1.2820e-003i -2.1556e-006 -3.2154e-006i
θ -8.9769e-004 +4.2883e-004i -8.9769e-004 -4.2883e-004i -2.3361e-004 +2.9071e-004i

  Column 4

 -2.6902e-001 +9.5216e-001i
  4.0088e-002 -1.3933e-001i
 -2.1556e-006 +3.2154e-006i
 -2.3361e-004 -2.9071e-004i


d =

  Columns 1 through 3

 -7.3862e-001 +1.0752e+000i                           0                           0
                          0 -7.3862e-001 -1.0752e+000i                           0
                          0                           0 -3.1000e-003 +9.9062e-003i
                          0                           0                           0

  Column 4

                          0
                          0
                          0
 -3.1000e-003 -9.9062e-003i

»
```

SHORT PERIOD

$\omega = 1.08$ rad/sec

$\sigma = -.738$

$\omega = 9.91 \cdot 10^{-3}$ rad/sec

$\sigma = -3.1 \cdot 10^{-3}$

PHUGOID

SHORT PERIOD EIGENVECTORS (NORMALIZED)

$$\frac{\Delta u}{u_0} : \frac{-1.13\cdot 10^{-2} + 4.30\cdot 10^{-3} i}{830} = -1.36\cdot 10^{-5} + 5.18\cdot 10^{-6} i$$

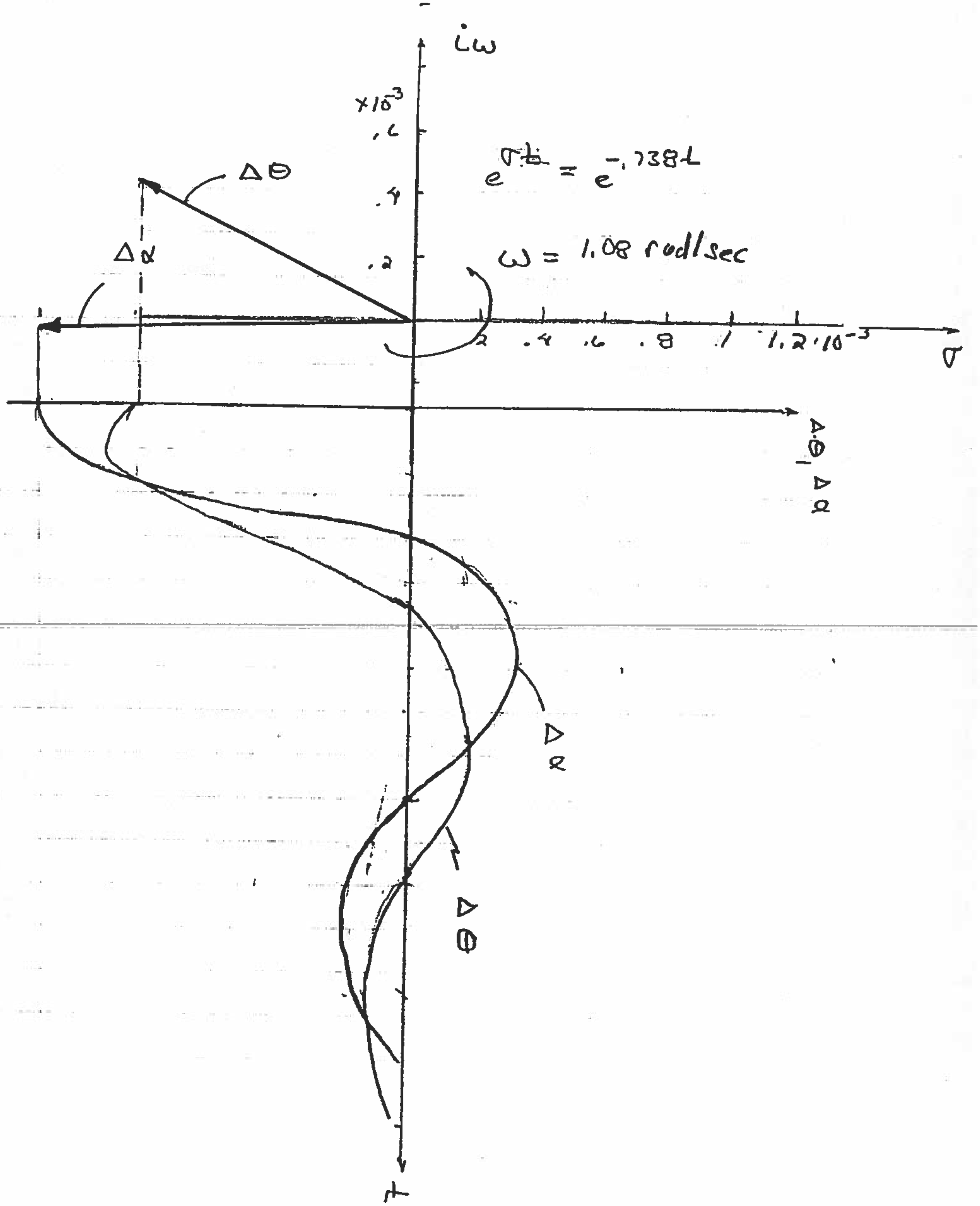
$$\frac{\Delta w}{u_0} : \frac{-9.98\cdot 10^{-1} - 5.92\cdot 10^{-2} i}{830} = -1.2\cdot 10^{-3} - 7.13\cdot 10^{-5} i$$

$$\frac{(\Delta q)\bar{c}}{2u_0} : .0165(2.02\cdot 10^{-4} - 1.28\cdot 10^{-3} i) = 3.33\cdot 10^{-6} - 2.12\cdot 10^{-5} i$$

$$\Delta\theta : -8.97\cdot 10^{-4} + 4.28\cdot 10^{-4} i$$

DOMINANT NORMALIZED EIGENVECTORS:

$$\boxed{\begin{aligned} &\frac{\Delta w}{u_0} = \Delta\alpha : -1.2\cdot 10^{-3} - .00713\cdot 10^{-3} i \\ &\Delta\theta : -.897\cdot 10^{-3} + .428\cdot 10^{-3} i \end{aligned}}$$


iω
×10⁻³
.6
.4
.2
ΔΘ
Δα
e^{σt} = e^{-.738t}
ω = 1.08 rad/sec
.2
.4
.6
.8
1
1.2×10⁻³
σ
ΔΘ, Δα
Δα
ΔΘ
t

PHUGOID EIGENVECTORS (NORMALIZED)

$$\frac{\Delta u}{u_0} : \frac{-.269-.953i}{830} = -3.24\cdot10^{-4} - 1.15\cdot10^{-3}i$$

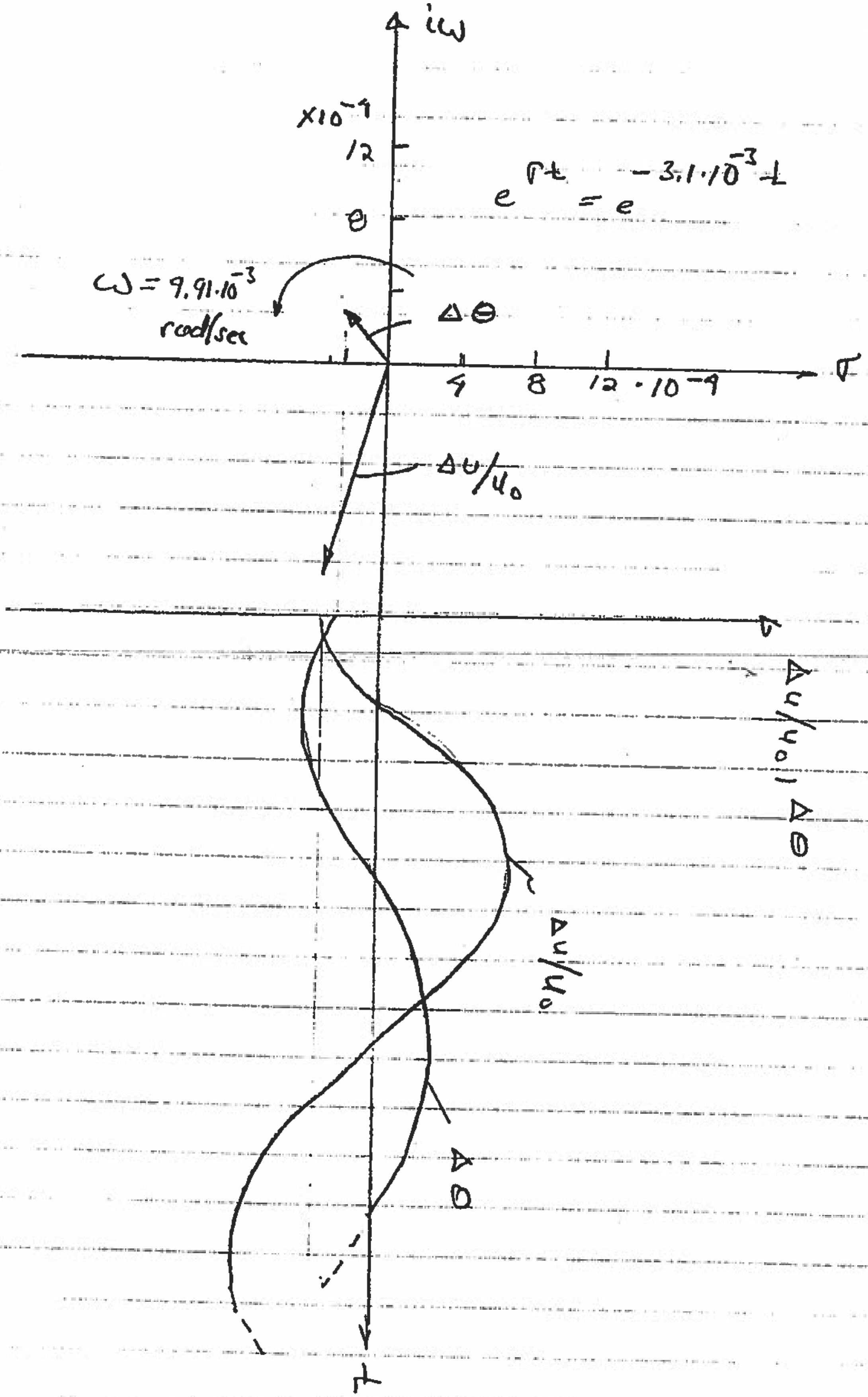
$$\frac{\Delta w}{u_0} : \frac{4\cdot10^{-2}+.139}{830} = 4.82\cdot10^{-6} + 1.675\cdot10^{-4}i$$

$$\frac{\Delta q\,\bar{c}}{2u_0} = .0165(-2.16\cdot10^{-6} - 3.215\cdot10^{-6}i) = -3.47\cdot10^{-8} - 5.28\cdot10^{-8}i$$

$$\Delta\theta : -2.34\cdot10^{-4} + 2.91\cdot10^{-4}i$$

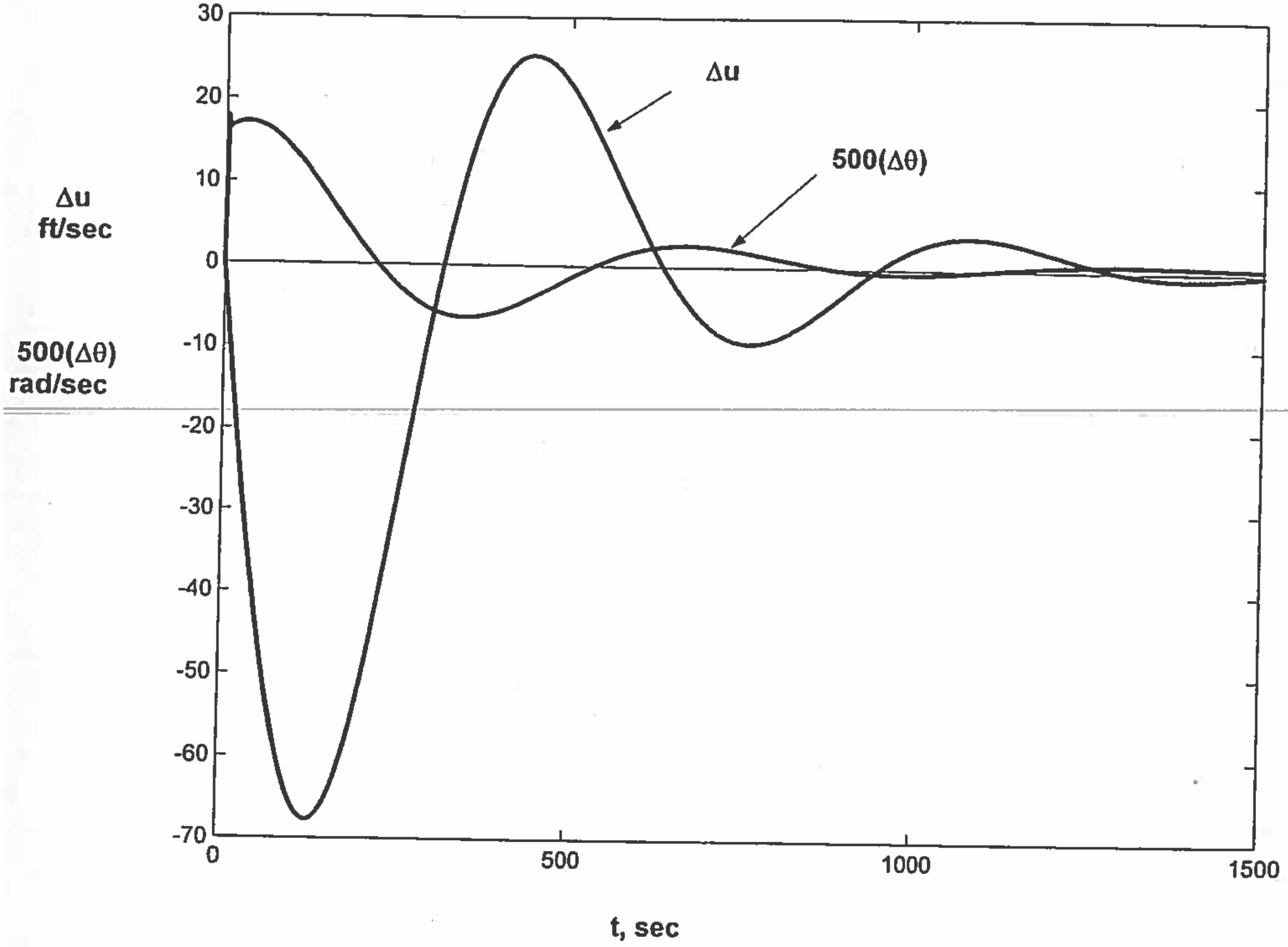
DOMINANT NORMALIZED EIGENVECTORS:

$$\boxed{\begin{array}{l} \frac{\Delta u}{u_0} : -3.24\cdot10^{-4} - 11.5\cdot10^{-4}i \\ \Delta\theta : -2.34\cdot10^{-4} + 2.91\cdot10^{-4}i \end{array}}$$

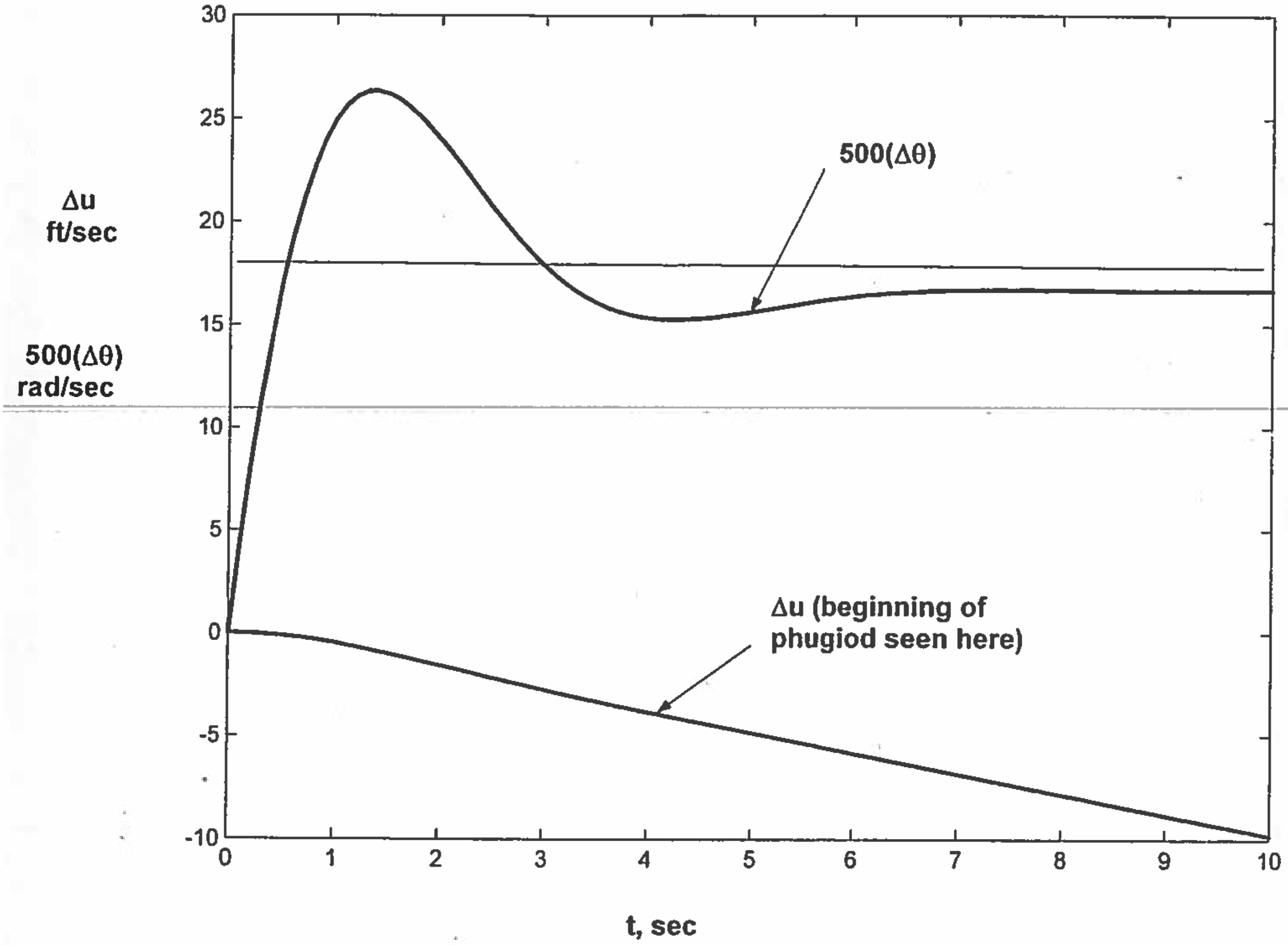

iω
x10^-4
12
8
ω = 9.91·10^-3
rad/sec
Δθ
e^σt = e^-3.1·10^-3 t
4
8
12 ·10^-4
σ
Δu/u₀
t
Δu/u₀ | Δθ
Δu/u₀
Δθ
t

```
»
» B747=ss(A,B,C,D);
» [y,t]=initial(B747,[0, 0, .08726, 0],1500);
» plot(t,y(:,1),t,500*y(:,2))
» [y,t]=initial(B747,[0, 0, .08726, 0],10);
» plot(t,y(:,1),t,500*y(:,2))
```

PHUGOID



SHORT PERIOD



Exciting Separate Modes

Boeing 747 longitudinal

```
A =

   -0.0064    0.0253         0  -32.2000
   -0.0954   -0.6330  832.0000         0
   -0.0002   -0.0014   -0.8440         0
         0         0    1.0000         0

>> B

B =

   -0.0253
    0.6330
    0.0014
         0

>> C
```

1/832 ~ changing w to α

```
C =

    1.0000         0         0         0
         0    0.0012         0         0
         0         0         0    1.0000

>> D

D =

     0
     0
     0

>> [v,d]=eig(A)

v =

   0.0110 - 0.0050i   0.0110 + 0.0050i   0.9894             0.9894
   0.9999             0.9999            -0.1450 + 0.0007i  -0.1450 - 0.0007i
  -0.0001 + 0.0013i  -0.0001 - 0.0013i   0.0000 - 0.0000i   0.0000 + 0.0000i
   0.0009 - 0.0005i   0.0009 + 0.0005i  -0.0002 - 0.0003i  -0.0002 + 0.0003i


d =

  -0.7386 + 1.0752i        0                  0                  0
        0            -0.7386 - 1.0752i        0                  0
        0                  0            -0.0031 + 0.0099i        0
        0                  0                  0            -0.0031 - 0.0099i

>>
```

```
>> i1=v(:,1)

i1 =

   0.0110 - 0.0050i
   0.9999
  -0.0001 + 0.0013i
   0.0009 - 0.0005i
```

1st column of eigenvector matrix $\underline{v}$

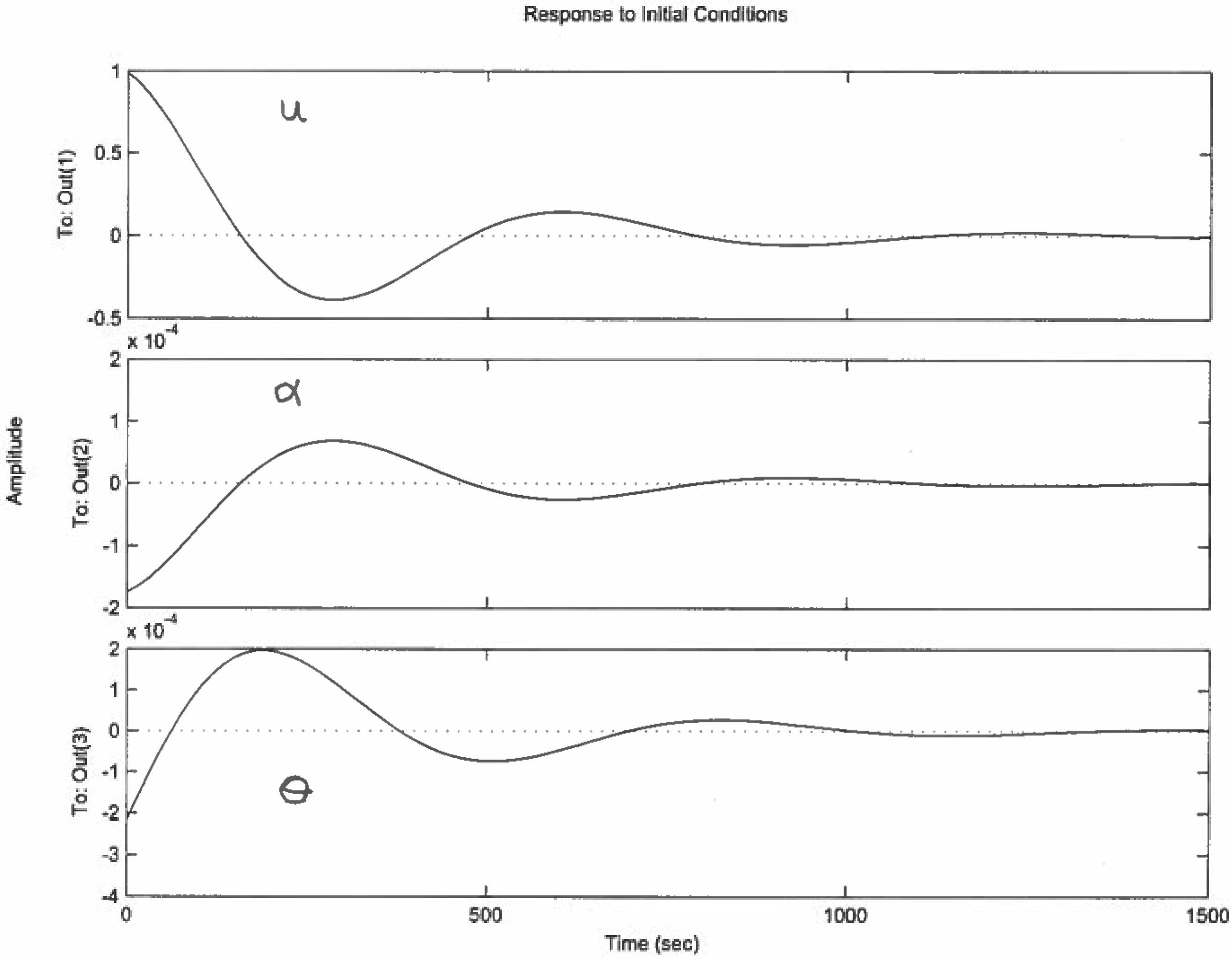
```
>> i2=v(:,3)

i2 =

   0.9894
  -0.1450 + 0.0007i
   0.0000 - 0.0000i
  -0.0002 - 0.0003i
```

3rd colum of eigenvector matrix $\underline{v}$

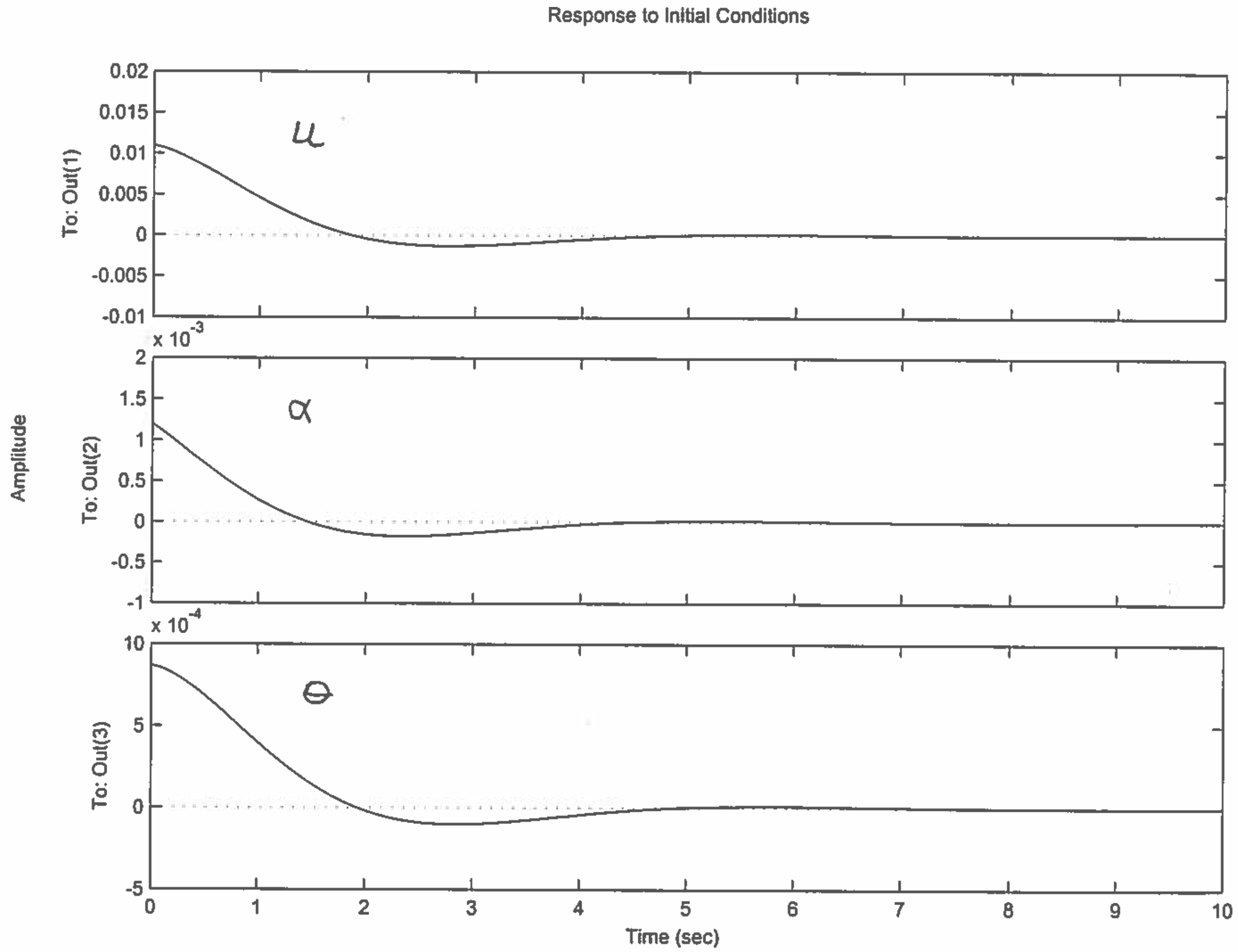
```
>> i1=real(i1)

i1 =

    0.0110
    0.9999
   -0.0001
    0.0009

>> i2=real(i2)

i2 =

    0.9894
   -0.1450
    0.0000
   -0.0002

>> initial(A,B,C,D,i1,10);
>> initial(A,B,C,D,i2,1500);
>>
```

```
>> damp(A)

        Eigenvalue                  Damping     Freq. (rad/s)

  0.00e+000                        -1.00e+000     0.00e+000
 -3.10e-003 + 9.91e-003i            2.99e-001     1.04e-002
 -3.10e-003 - 9.91e-003i            2.99e-001     1.04e-002
 -7.39e-001 + 1.08e+000i            5.66e-001     1.30e+000
 -7.39e-001 - 1.08e+000i            5.66e-001     1.30e+000

>>
```

phugoid


Response to Initial Conditions
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
u
α
θ
x 10^-4
x 10^-4
1
0.5
0
-0.5
2
1
0
-1
-2
2
1
0
-1
-2
-3
-4
0
500
1000
1500
Time (sec)

short-period



```
A =

   -0.0064    0.0253         0  -32.2000         0
   -0.0954   -0.6330  832.0000         0         0
   -0.0002   -0.0014   -0.8440         0         0
         0         0    1.0000         0         0
         0   -1.0000         0  832.0000         0

>> B

B =

   -0.0253
    0.6330
    0.0014
         0
         0

>> C

C =

    1.0000         0         0         0         0
         0    0.0012         0         0         0
         0         0         0    1.0000         0
         0         0         0         0    1.0000

>> D

D =

     0
     0
     0
     0

>> [v,d]=eig(A)

v =

  Columns 1 through 4

        0                  -0.0103 + 0.0047i  -0.0103 - 0.0047i  -0.0363 - 0.0171i
        0                  -0.9370            -0.9370             0.0053 + 0.0025i
        0                   0.0001 - 0.0012i   0.0001 + 0.0012i  -0.0000 - 0.0000i
        0                  -0.0008 + 0.0005i  -0.0008 - 0.0005i   0.0000 + 0.0000i
   1.0000                   0.1250 - 0.3260i   0.1250 + 0.3260i   0.9992

  Column 5

  -0.0363 + 0.0171i
   0.0053 - 0.0025i
  -0.0000 + 0.0000i
   0.0000 - 0.0000i
   0.9992
```

added h as
new state

$\dot{h} = -w + U_0 \theta$

```
d =

  Columns 1 through 4

        0                   0                   0                   0
        0         -0.7386 + 1.0752i             0                   0
        0                   0         -0.7386 - 1.0752i             0
        0                   0                   0         -0.0031 + 0.0099i
        0                   0                   0                   0

  Column 5

        0
        0
        0
        0
  -0.0031 - 0.0099i
```

```
>> i1=v(:,2)

i1 =

  -0.0103 + 0.0047i
  -0.9370
   0.0001 - 0.0012i
  -0.0008 + 0.0005i
   0.1250 - 0.3260i
```

2nd column of new eigenvector matrix $\underline{v}$

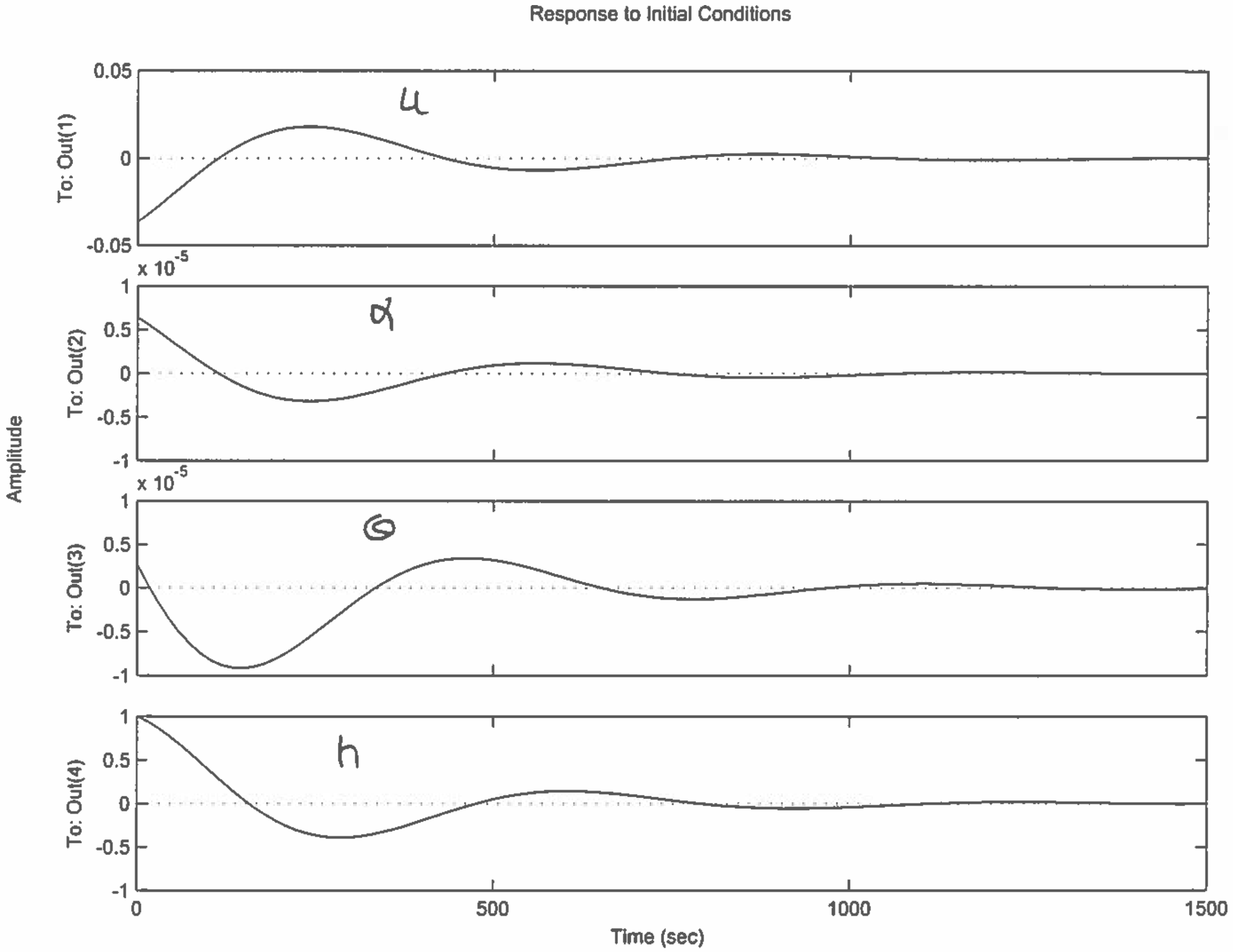
```
>> i2=v(:,4)

i2 =

  -0.0363 - 0.0171i
   0.0053 + 0.0025i
  -0.0000 - 0.0000i
   0.0000 + 0.0000i
   0.9992
```

4 th column of new eigenvector matrix $\underline{v}$

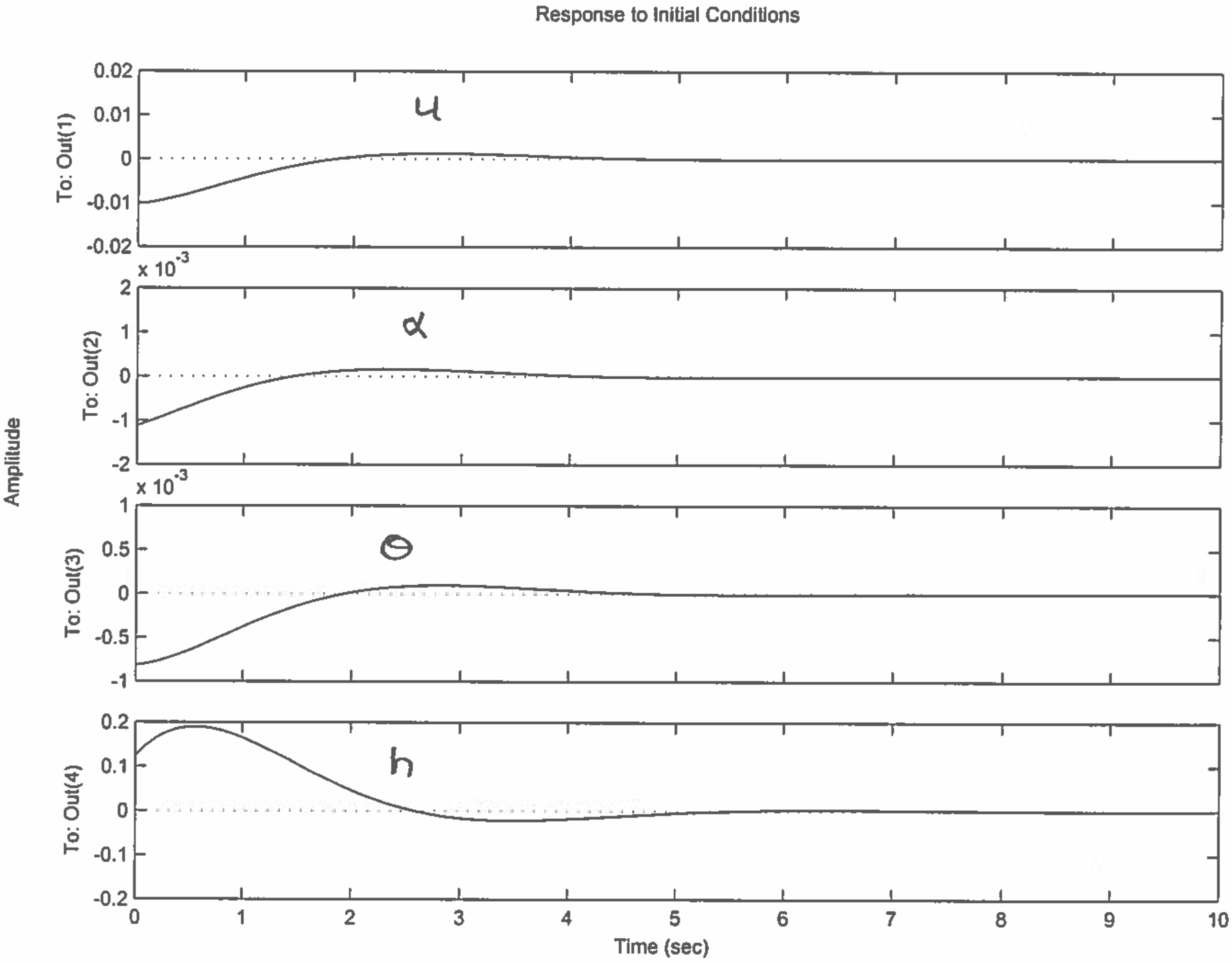
```
>> i1=real(i1)

i1 =

   -0.0103
   -0.9370
    0.0001
   -0.0008
    0.1250

>> i2=real(i2)

i2 =

   -0.0363
    0.0053
   -0.0000
    0.0000
    0.9992

>> initial(A,B,C,D,i1,10);
>> initial(A,B,C,D,i2,1500);
>>
```

Phugoid (with h(t))

(note new IC's)



short-period (with h(t))
(note new I.C's)



MAE 275

Lateral/Directional Eigenanalysis for Boeing 747

On the following page are the lateral-directional stability derivatives for the Boeing 747 in a flight condition at sea-level (configuration 2 on the enclosed sheet). Here the aircraft trim velocity is 278 ft/sec. Derivatives not shown can be assumed negligible. Notice that the primed" stability derivatives for the lateral-directional mode are already calculated for you. See the handout "State-Space Form of Linearized Aircraft Equations of Motion".

1.) Using MATLAB, complete an eigenanalyis of the lateral-directional dynamics of the aircraft at this flight condition. This should include:

a.) Defining the state equations (A,B,C and D) matrices. In defining **C** and **D**, assume $\Delta\phi$, $\Delta\psi$, and $\Delta\beta$ are the "outputs" of interest. (Recall that $\Delta\beta = \Delta v/u_0$).

b.) Determining the characteristic roots (eigenvalues) and eigenvectors and identifying the modes of motion.

c.) Plot of eigenvectors (phasor diagrams) for each mode with sketches of time responses. Normalize the eigenvectors by $1/u_0$ for linear velocities and by $b/(2u_0)$ for angular velocties. Here b = wing span = 195.68 ft.

2.) Again using MATLAB, determine $\Delta\phi$, $\Delta\psi$, and $\Delta\beta$ if an intial condition of $\Delta\beta =$ 5 deg (.08726 rad/sec) is specified. Use a time axis of $0 \le t \le 20$ sec.

3.) Now consider that a control input is to be applied.

a.) Return to your **B** matrix, and allow a rudder input, $\Delta\delta_r = 10$ deg. (0.17452 rad). Plot the $\Delta\beta$ response for 20 sec. (You will use the MATLAB command "step" for this).

TABLE IX-8

B-747 LATERAL-DIRECTIONAL DIMENSIONAL DERIVATIVES

(STABILITY AXIS SYSTEM)

F/C ▸		1	2	3	4	5	6	7	8	9	10
H		SL	SL	SL	SL	20 K	20 K	20 K	40 K	40 K	40 K
M		.198	.249	.450	.650	.500	.650	.800	.700	.800	.900
YV	Y_v	-.0890	-.0997	-.143	-.197	-.0822	-.104	-.120	-.0488	-.0558	-.0606
YB	Y_β	-19.7	-27.8	-71.7	-143.	-42.6	-70.4	-99.4	-33.1	-43.2	-52.8
LB'	L'_β	-1.33	-1.63	-3.19	-5.45	-2.05	-2.96	-4.12	-1.45	-3.05	-1.32
NB'	N'_β	.168	.247	.810	1.82	.419	.923	1.62	.404	.598	.971
LP'	L'_p	-.975	-1.10	-1.12	-1.47	-.652	-.804	-.974	-.404	-.465	-.459
NP'	N'_p	-.166	-.125	-.0706	-.0214	-.0701	-.0531	-.0157	-.0366	-.0316	.00284
LR'	L'_r	.327	.198	.379	.256	.376	.317	.292	.312	.388	.280
NR'	N'_r	-.217	-.229	-.246	-.344	-.140	-.193	-.232	-.0963	-.115	-.141
Y*CA		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
L'CA		.227	.318	.229	.372	.128	.210	.310	.0964	.143	.186
N'DA		.0264	.0300	.0285	.0371	.0177	.0199	.0127	.00875	.00775	-.00611
Y*CR	$Y_{\delta r}$	.0148	.0182	.0226	.0213	.0131	.0142	.0124	.00777	.00729	.00464
L'CR	$L'_{\delta r}$	.0636	.110	.254	.318	.148	.211	.183	.115	.153	.100
N'CR	$N'_{\delta r}$	-.151	-.233	-.614	-.970	-.391	-.616	-.922	-.331	-.475	-.442

$\theta_0 = 0$

Hess

EAE-129

lateral equations

$$\Delta\dot{v} = Y_v\Delta v + Y_p\Delta p + [Y_r - u_0]\Delta r + g\cos\theta_0\Delta\phi + \sum_{i=1}^{n} Y_{\delta_i}\Delta\delta_i$$

$$\Delta\dot{p} = L_v'\Delta v + L_p'\Delta p + L_r'\Delta r + \sum_{i=1}^{n} L_{\delta_i}'\Delta\delta_i$$

$$\Delta\dot{r} = N_v'\Delta v + N_p'\Delta p + N_r'\Delta r + \sum_{i=1}^{n} N_{\delta_i}'\Delta\delta_i$$

$$\Delta\dot{\phi} = \Delta p + \Delta r\tan\theta_0$$

$$\Delta\dot{\psi} = \Delta r\sec\theta_0$$

$$\Delta\dot{v} = -.0997\Delta v + 0\Delta p + [0-278]\Delta r + 32.2\Delta\phi + 0\Delta\psi$$

$$\Delta\dot{p} = \frac{1}{278}(-1.63)\Delta v - 1.1\Delta p + .198\Delta r + 0\Delta\phi + 0\Delta\psi$$

$$\Delta\dot{r} = \frac{1}{278}(.247)\Delta v - .125\Delta p - .229\Delta r + 0\Delta\phi + 0\Delta\psi$$

$$\Delta\dot{\phi} = 0\Delta v + (1)\Delta p + 0\Delta r + 0\Delta\phi + 0\Delta\psi$$

$$\Delta\dot{\psi} = 0\Delta v + 0\Delta p + (1)\Delta r + 0\Delta\phi + 0\Delta\psi$$

$$[A] = \begin{bmatrix} -.0997 & 0 & -278 & 32.2 & 0 \\ -5.86\cdot 10^{-3} & -1.1 & +.198 & 0 & 0 \\ 8.88\cdot 10^{-4} & -.125 & -.229 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ for part (1)}$$

$$B = \begin{bmatrix} +.0182 \\ .110 \\ -.233 \\ 0 \\ 0 \end{bmatrix} \text{ for part (3)}$$

$$C = \begin{bmatrix} 1/278 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

```
» [v,d]=eig(A)

v =

  Columns 1 through 3
```

rolling convergence eigenvector

dutch-roll eigenvector

v
p
r
ϕ
ψ

```
               0      -9.9948e-001       9.1393e-001 -4.0577e-001i
               0      -2.5472e-002      -2.6973e-003 +3.6683e-003i
               0      -2.1176e-003      -6.4702e-004 -1.8066e-003i
               0       1.9393e-002       5.3330e-003 +3.5933e-003i
  1.0000e+000          1.6122e-003      -2.5117e-003 +1.0179e-003i

  Columns 4 through 5

  9.1393e-001 +4.0577e-001i -9.8749e-001
 -2.6973e-003 -3.6683e-003i  4.0249e-003
 -6.4702e-004 +1.8066e-003i -8.0242e-003
  5.3330e-003 -3.5933e-003i -7.0586e-002
 -2.5117e-003 -1.0179e-003i  1.4072e-001
```

spiral mode eigenvector

```
d =

  Columns 1 through 3
```

"nuisance mode"

```
             0                    0                 0
             0         -1.3135e+000                 0
             0                    0      -2.9102e-002 +7.0746e-001i
             0                    0                 0
             0                    0                 0
```

roll convergence mode

```
  Columns 4 through 5

             0                    0
             0                    0
             0                    0
 -2.9102e-002 -7.0746e-001i       0
             0         -5.7021e-002
```

Dutch roll mode

spiral mode

Rolling Convergence Mode

$\frac{\Delta u}{u_0} = \Delta\beta : \quad -\frac{1}{278} = -3,6 \cdot 10^{-3}$

$\frac{\Delta p b}{2 u_0} : \quad -8,96 \cdot 10^{-3}$

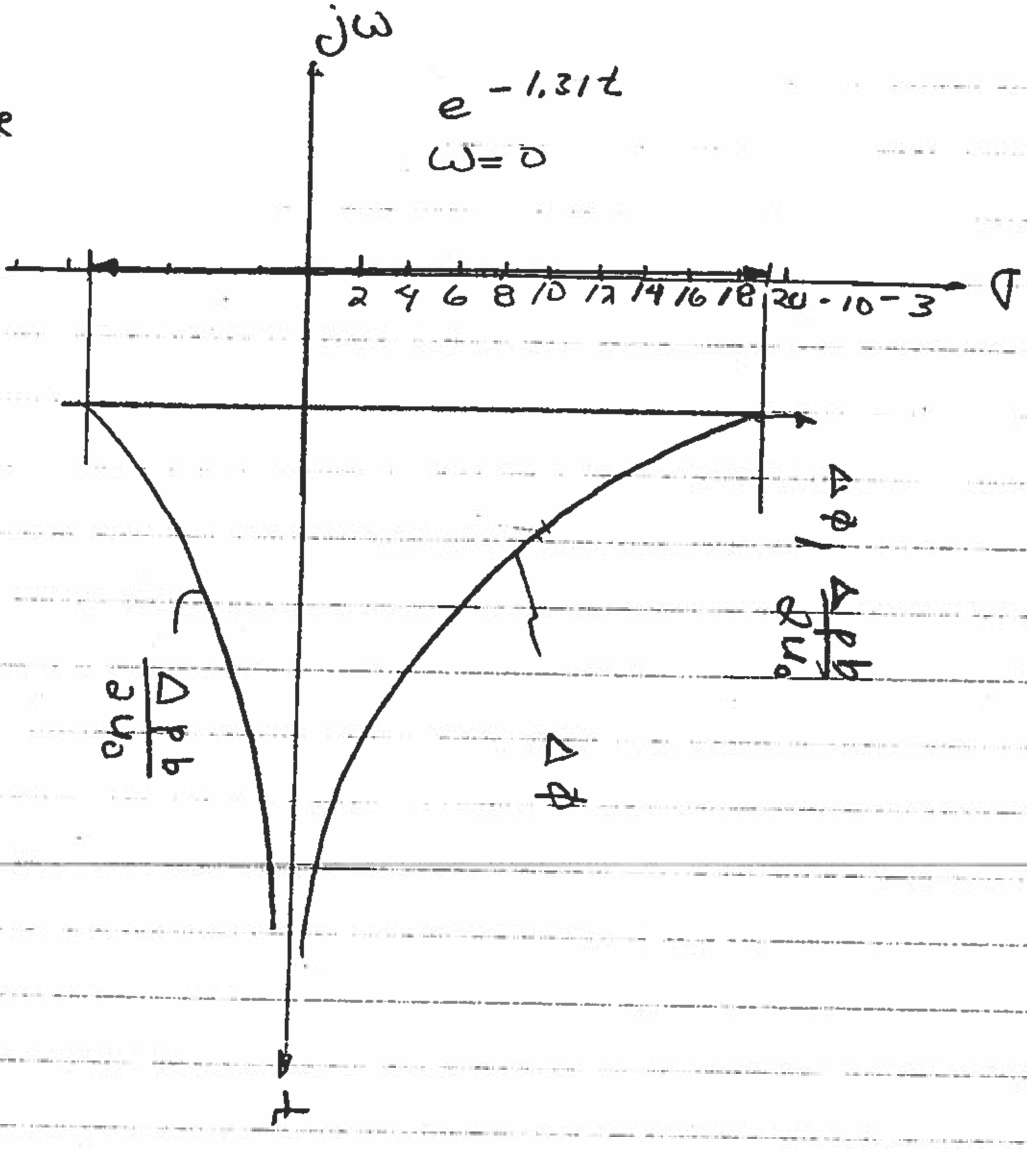
$\frac{\Delta r b}{2 u_0} = \quad -7,95 \cdot 10^{-4}$

$\Delta \phi = \quad 1,94 \cdot 10^{-2}$

$\Delta \varphi = \quad 1,61 \cdot 10^{-3}$

small in comparison

Rolling
Convergence



Dutch-roll mode

$\frac{\Delta v}{u_0} = \Delta\beta$: $\frac{1}{278}(+.914 - .406i) = +3.29 \cdot 10^{-3} - 1.46 \cdot 10^{-3} i$

$\frac{\Delta pb}{2u_0}$: $.352(-2.7 \cdot 10^{-3} + 3.67 \cdot 10^{-3} i) = -.95 \cdot 10^{-3} + 1.29 \cdot 10^{-3} i$

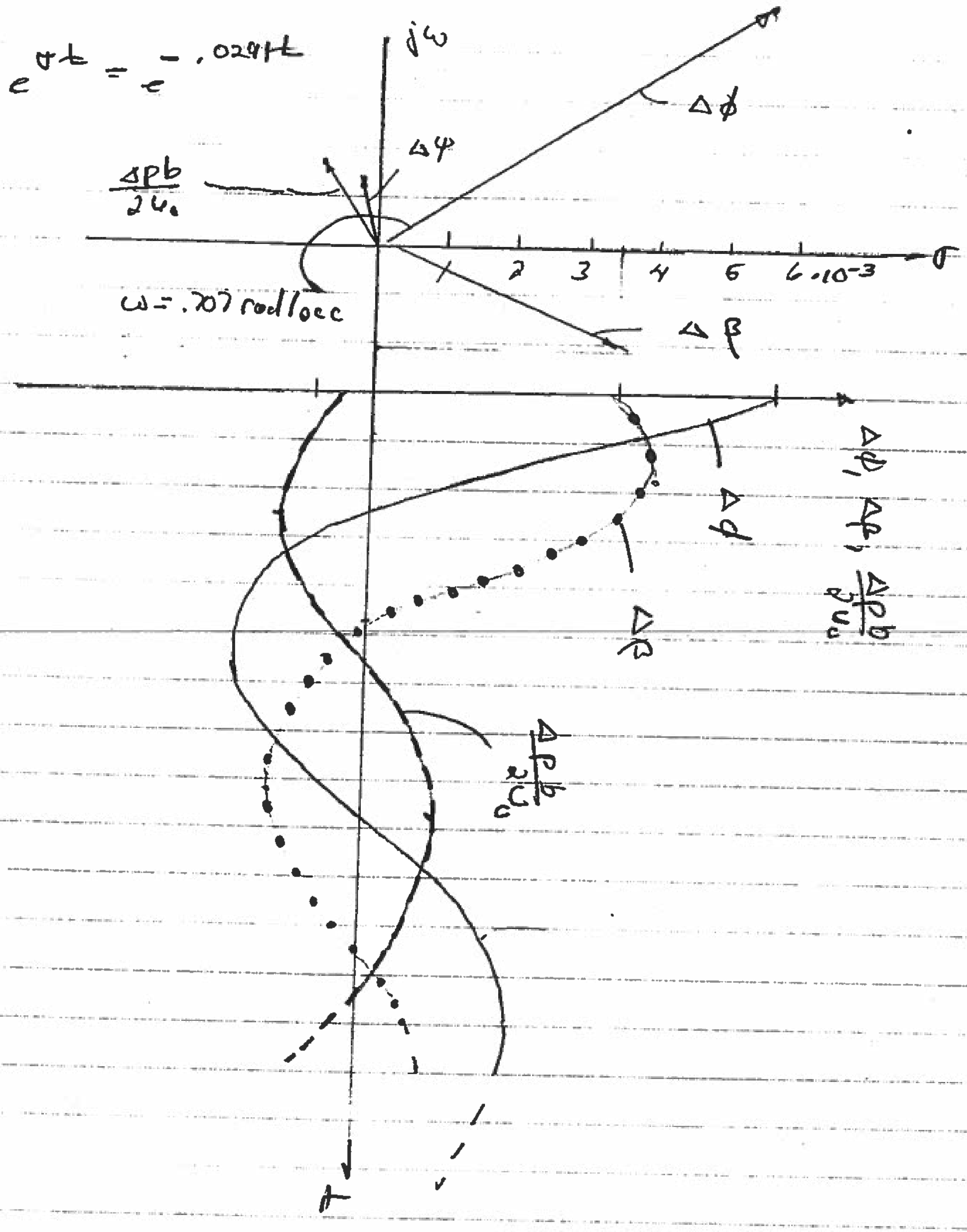
$\frac{\Delta rb}{2u_0}$: $.352(-6.47 \cdot 10^{-4} - 1.81 \cdot 10^{-3} i) = -2.28 \cdot 10^{-4} - 6.37 \cdot 10^{-4} i$

$\Delta\phi$: $5.33 \cdot 10^{-3} + 3.54 \cdot 10^{-3} i$

$\Delta\psi$: $-.251 \cdot 10^{-3} + 1.02 \cdot 10^{-3} i$

All but $\frac{\Delta rb}{2u_0}$ should be included

Dutch roll mode



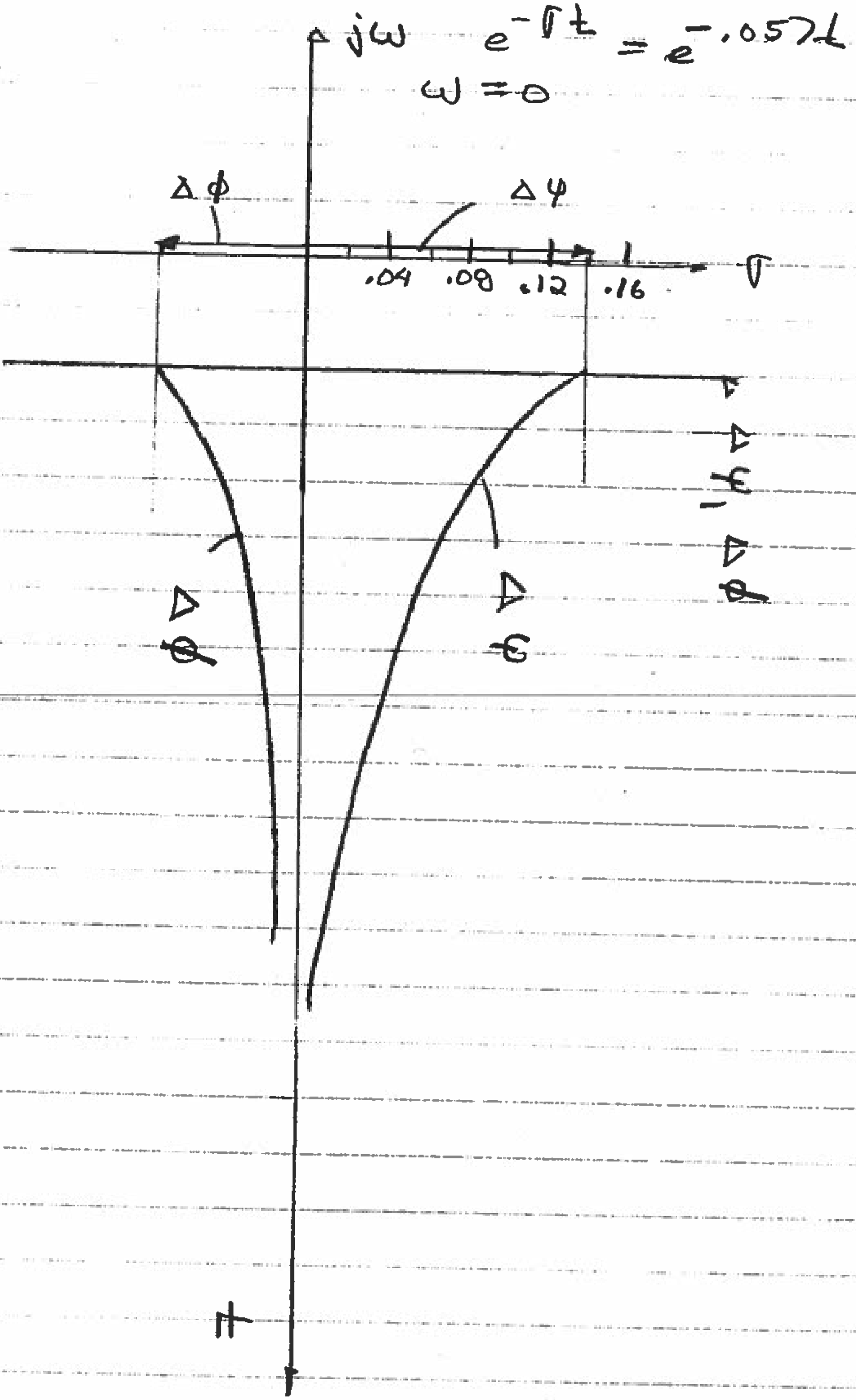
Spiral Mode

$$\left.\begin{array}{l}\frac{\Delta v}{u_0} = d\beta : \frac{1}{278}(-.987) = -3.56\cdot10^{-3} \\[2ex] \frac{\Delta p b}{2u_0} : .352(4.025\cdot10^{-3}) = 1.42\cdot10^{-3} \\[2ex] \frac{\Delta r b}{2u_0} : .352(-8.024\cdot10^{-3}) = -2.9\cdot10^{-3}\end{array}\right\} \text{small in comparison}$$

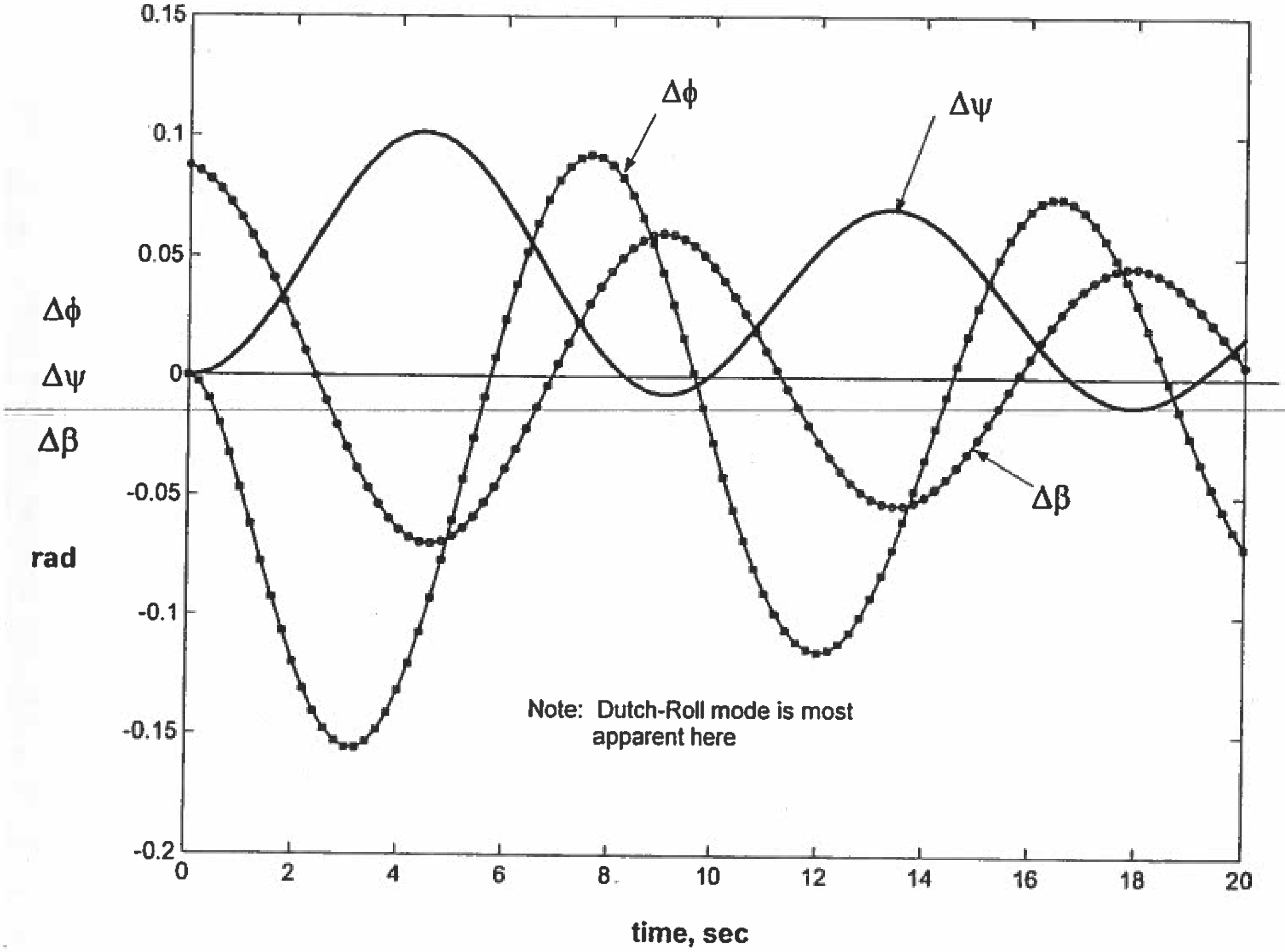
$\Delta\phi$: $-7.058\cdot10^{-2}$

$\Delta\psi$: $.1407$

Spiral Mode



```
» B747=ss(A,B̄,C,D);
» [y,t]=initial(B747,[24.25,0,0,0,0],20);
» plot(t,y)
»
```

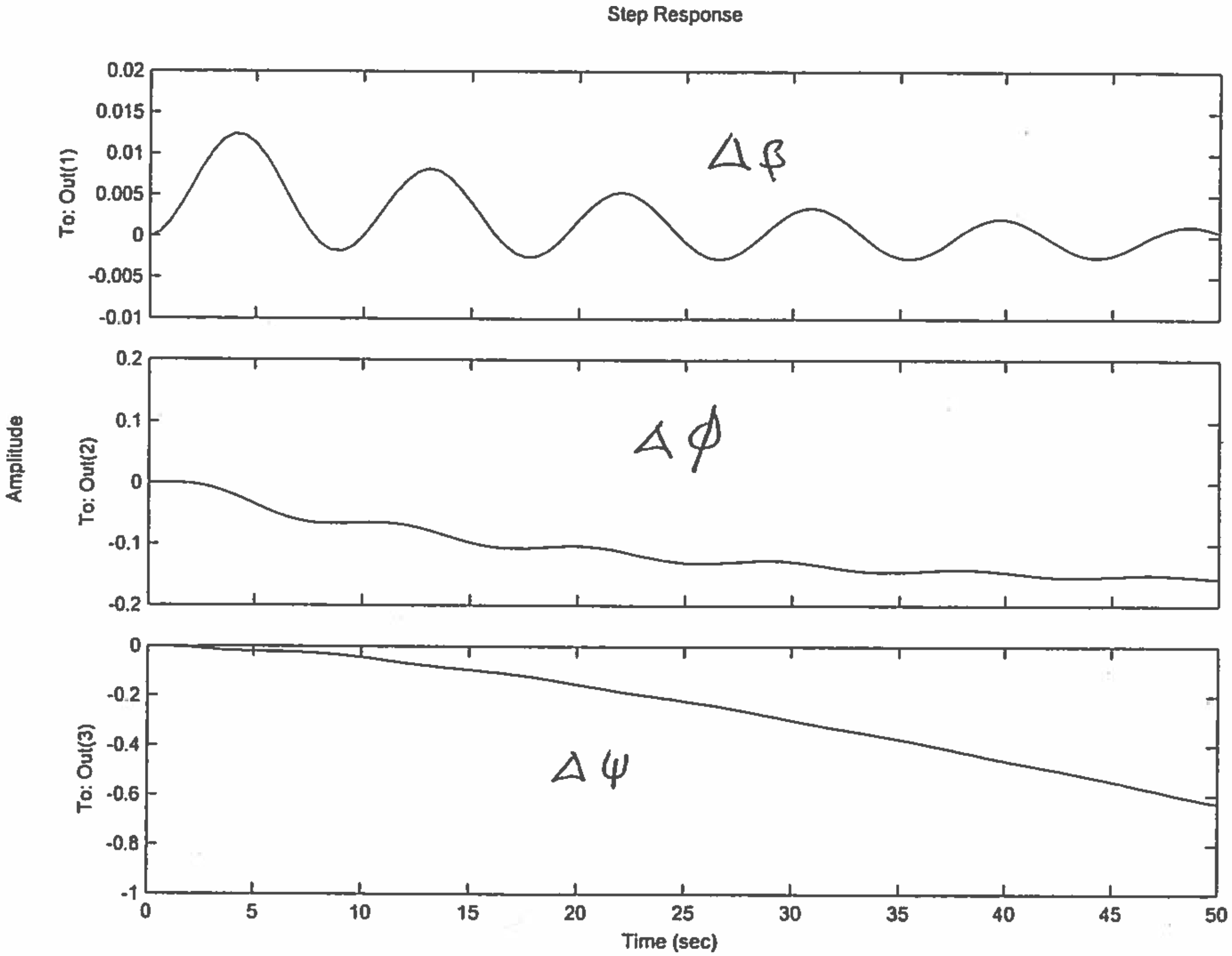


```
>> A=[-.0997 0 -278 32.2 0;-5.86e-3 -1.1 .198 0 0;8.88e-4 -.125 -.229 0 0; 0 1 0 0 0;0 0 1  ↙
0 0]

A =

   -0.0997         0 -278.0000   32.2000         0
   -0.0059   -1.1000    0.1980         0         0
    0.0009   -0.1250   -0.2290         0         0
         0    1.0000         0         0         0
         0         0    1.0000         0         0



>> B=[.0182;.110;-.233;0;0] ← for rudder input

B =

    0.0182
    0.1100
   -0.2330
         0
         0

>> C=[1/278 0 0 0 0;0 0 0 1 0;0 0 0 0 1]

C =

    0.0036         0         0         0         0
         0         0         0    1.0000         0
         0         0         0         0    1.0000

>> D=[0;0;0];
>>

>> sys=ss(A,B,C,D);
>> step(sys,50)
>>
```

```
>> B=(10/57.3)*B

B =

   3.1763e-003
   1.9197e-002
  -4.0663e-002
             0
             0

>> sys=ss(A,B,C,D);
>>
```

$\Delta\delta_r = 10°$

$\left(\frac{10}{573}\right)$ rad



For $\Delta\delta_r = 57.3°$!!