MAE - 275

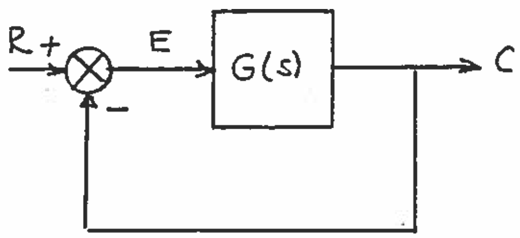
"PRIMARY RULE-OF-THUMB FOR FREQUENCY DOMAIN SYNTHESIS"

Consider the magnitude of the Bode diagram of the open-loop transfer function (call it G(s)) of a closed-loop system. On this diagram find, or create a fair stretch of frequency centered at the desired closed-loop bandwidth, ω_B, where $|G(j\omega)|$ has a -20 dB/dec slope. Adjust the open-loop gain (through the compensator) so that the open-loop crossover frequency, ω_c, equals the desired closed-loop bandwith. The crossover frequency, ω_c, is defined as that frequency where $|G(j\omega)| = 1.0$ (0 dB).

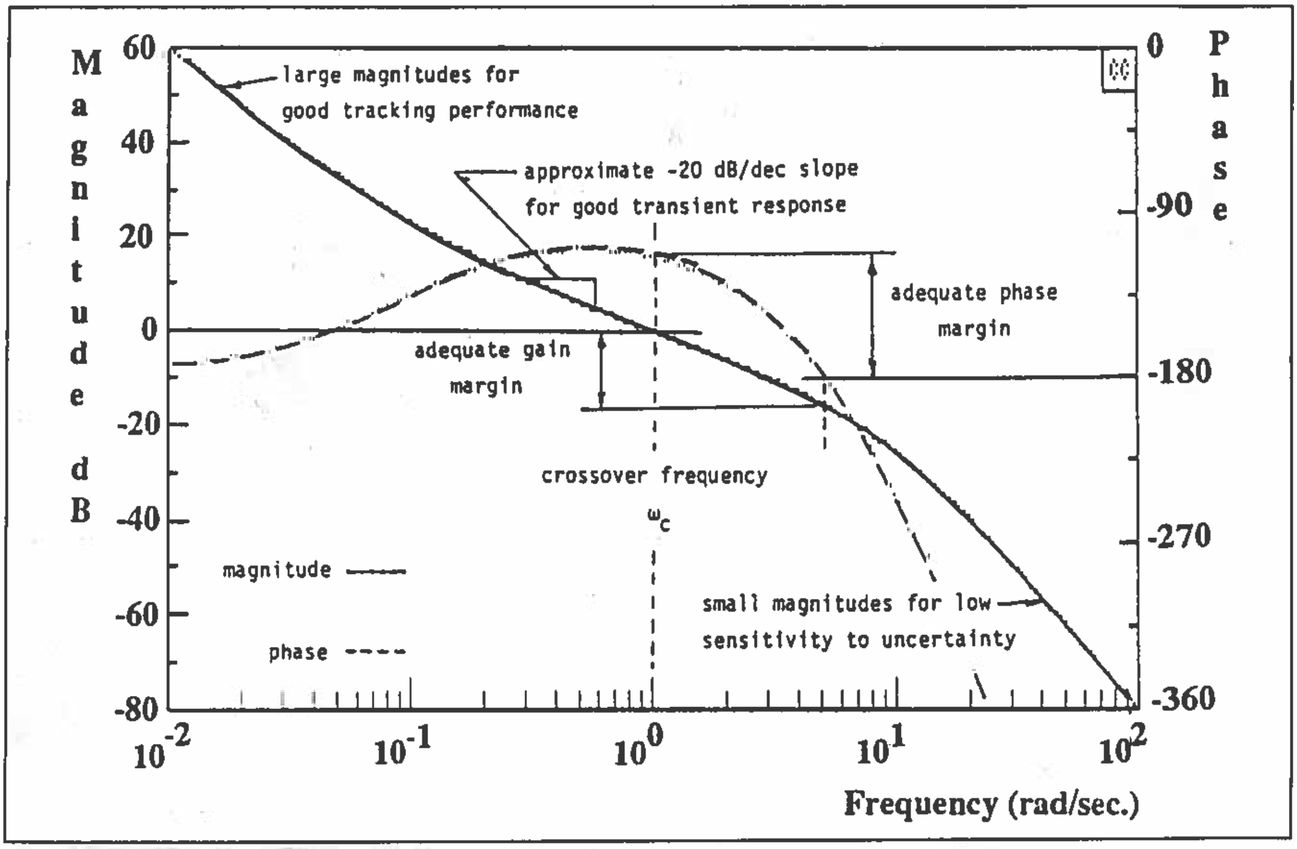
For frequencies well below ω_c, make $|G(j\omega)| > 1.0$, and for frequencies well above ω_c, make $|G(j\omega)| < 1.0$. This adjustment of G(s) is accomplished through selection of poles, zeros and gain of the compensator.

After this design procedure, you should always check the closed-loop stability using the Nyquist criterion (sketch the complete Nyquist diagram). This is especially important for non-minimum phase systems.

This rule of thumb will, in general, yield a closed-loop system with good transient responses, good disturbance rejection characteristics, and with a closed-loop bandwidth equal to or greater than ω_c.



G(jω)



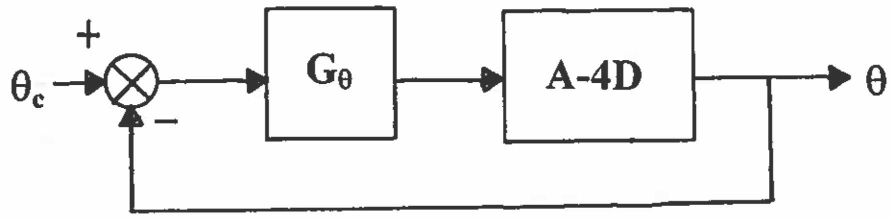
Magnitude dB
60
40
20
0
-20
-40
-60
-80
Phase
0
-90
-180
-270
-360
large magnitudes for good tracking performance
approximate -20 dB/dec slope for good transient response
adequate phase margin
adequate gain margin
crossover frequency
ωc
magnitude
phase
small magnitudes for low sensitivity to uncertainty
10-2
10-1
100
101
102
Frequency (rad/sec.)

Dept. of Mechanical and Aeronautical Engineering

MAE - 275

Sample Loop-Shaping Design

Design a pitch-attitude stability augmentation system for the A4-D aircraft for flight condition 5 in Appendix A of McRuer, Ashkenas and Graham. The feedback structure for your system to be simulated on Simulink is shown below:



A.) Meet or come as close as you can to meeting the following design criteria:

i. Closed loop-bandwidth for $\frac{\theta}{\theta_c}(s)$ > 2 rad/sec

ii. Damping ratios of all closed-loop oscillatory modes > 0.707

iii. Zero steady-state error to a step input θ_c = 5 degrees (remember the units in your model will be in radians!)

In demonstrating item (A. iii), use a Simulink simulation of the aircraft. To simulate an amplitude-limited elevator actuator, place a saturation element just downstream from your compensator with amplitude limits of ± 20/57.3 rad (± 20 degrees).

B.) Show the attitude and elevator responses that occur when the 5 degree step θ_c is applied. Is your elevator response reasonable?

```
A4 =

 -1.2700e-002 -5.9000e-003            0 -3.2200e+001            0
 -1.0100e-001 -8.1670e-001  6.3500e+002            0            0
 -3.0000e-004 -1.9500e-002 -1.4160e+000            0            0
            0            0  1.0000e+000            0            0
            0 -1.0000e+000            0  6.3400e+002            0

>> B4

B4 =

            0
 -5.6800e+001
 -1.9400e+001
            0
            0

>> C4
```

$u = \delta_e$

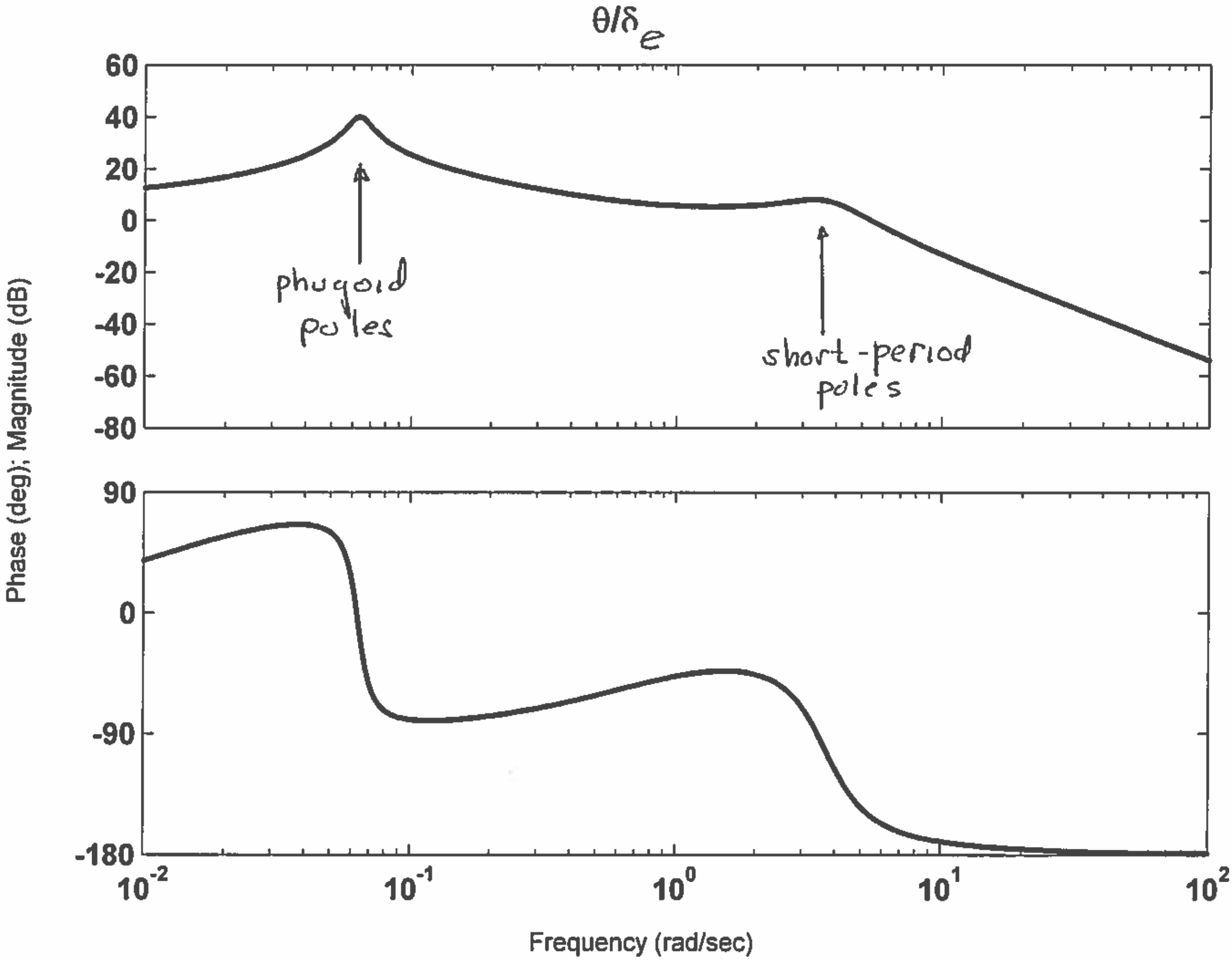
4

```
>> C41

C41 =

     0     0     0     1     0

>> D41

D41 =

     0     0     0     0

>> [num,den]=ss2tf(A4,B4,C41,D41,1);
>> gth=tf(num,den);
>> zpkg(gth)
??? Undefined function or method 'zpkg' for input arguments of type 'tf'.

>> zpk(gth)

Zero/pole/gain:
-1.7764e-015 s (s+1.092e016) (s+0.7604) (s+0.01191)
---------------------------------------------------
s (s^2 + 0.01117s + 0.004102) (s^2 + 2.234s + 13.54)
```

$$= \frac{\theta}{\delta_e}(s)$$

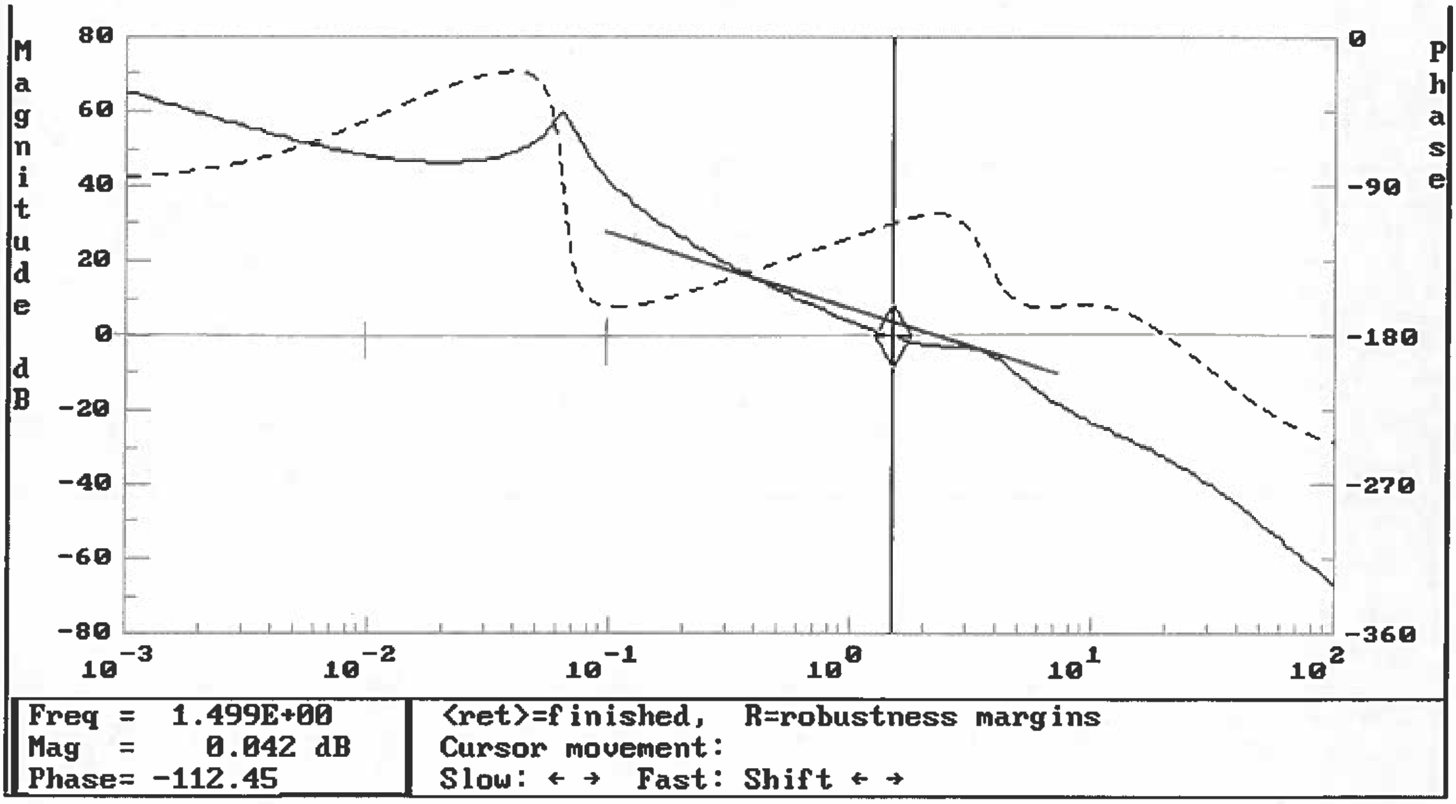
$$\frac{\theta}{\delta_t}(s) \triangleq \frac{-21.18(s+.7604)(s+.01191)}{\Delta(s)}$$

BARE - AIRFRAME

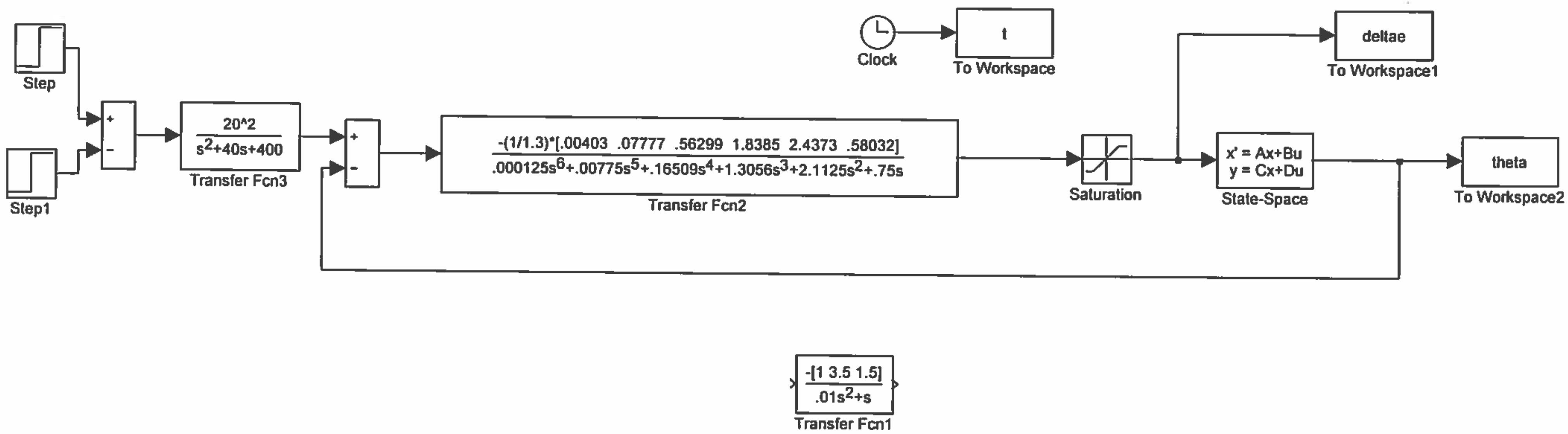


LOW-BANDWIDTH DESIGN

$$G_\theta = -.0031\frac{(s+.3)(s+4)^2(s+5)(s+6)}{s(s+.5)(s+1.5)(.05s+1)^3}$$

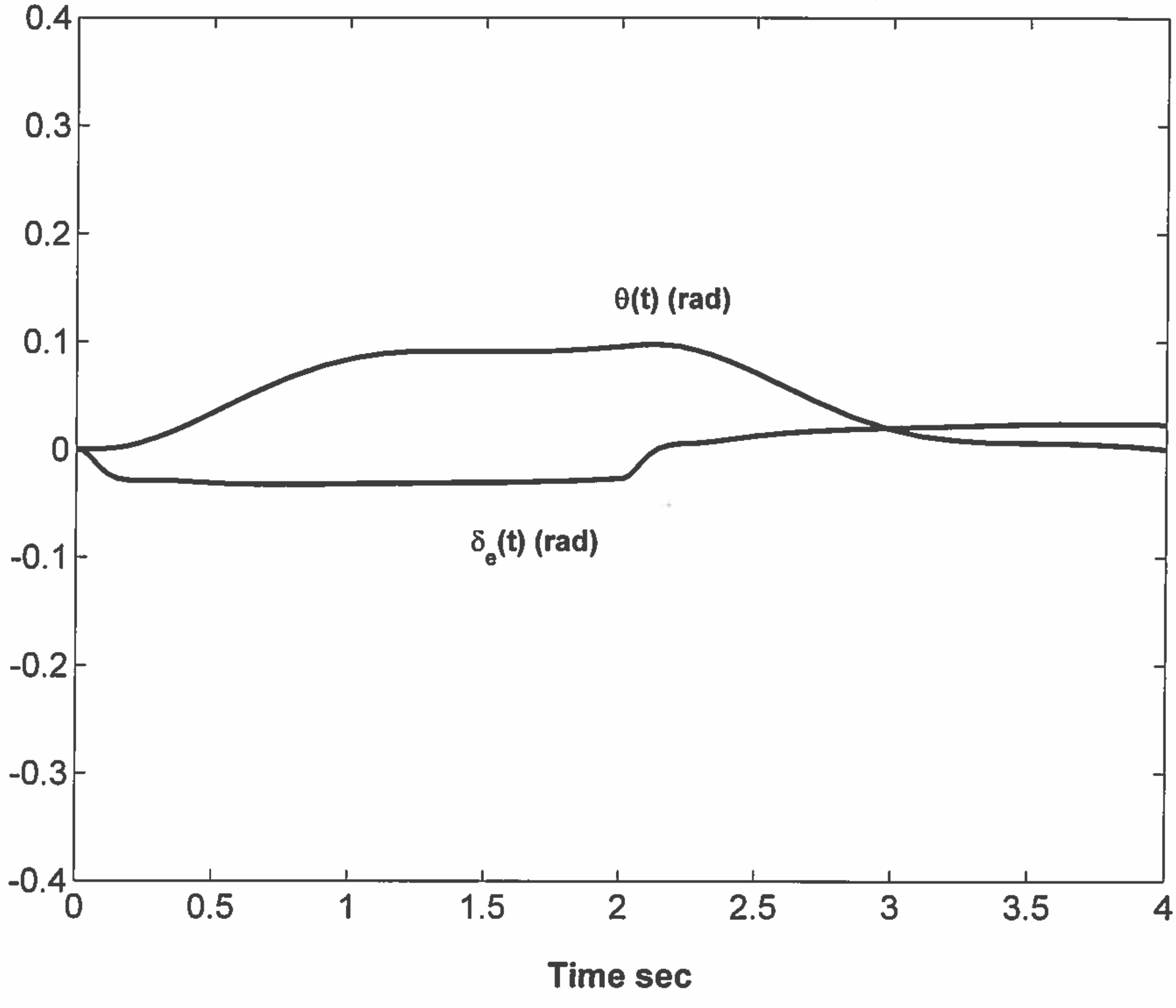


LOW BANDWIDTH

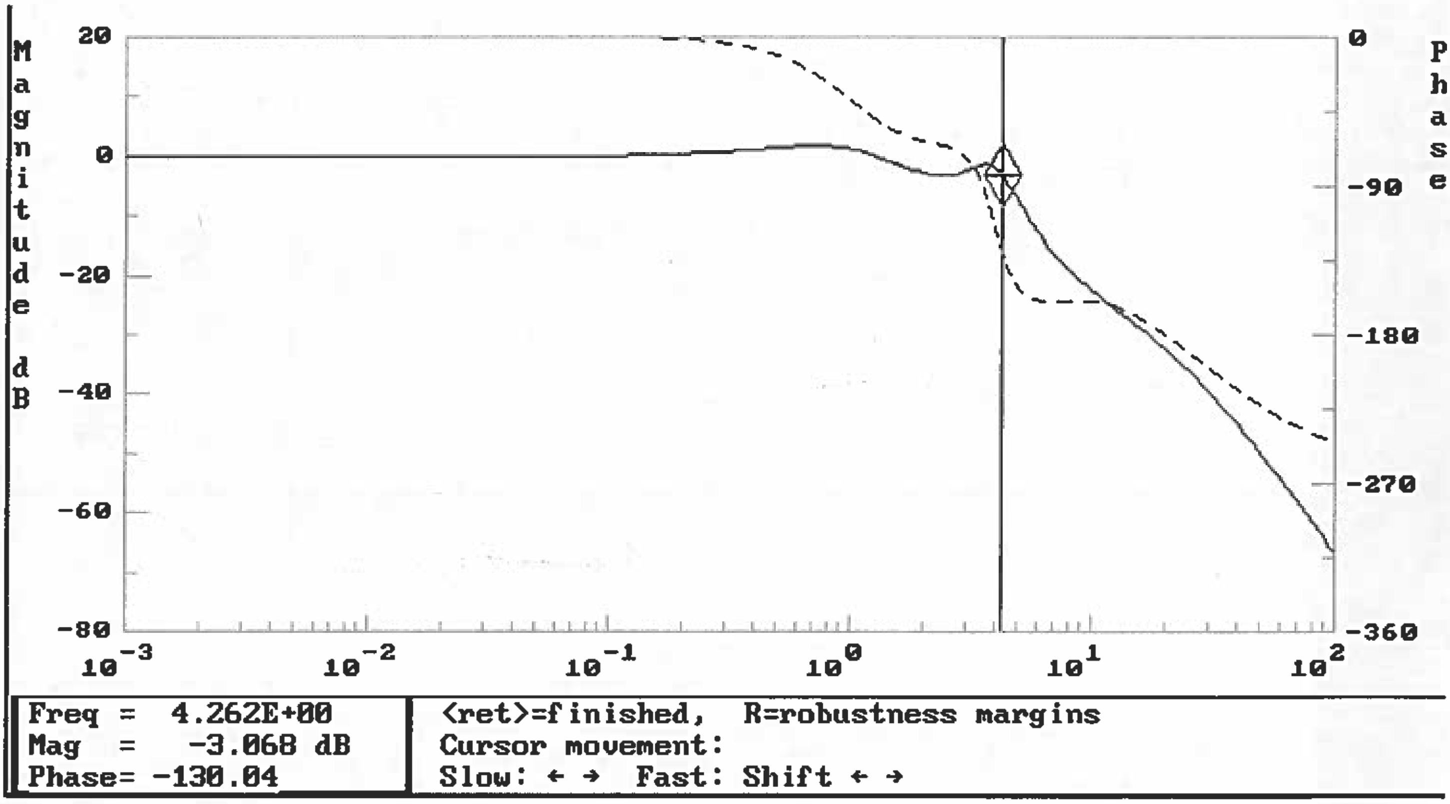


8

LOW BANDWIDTH

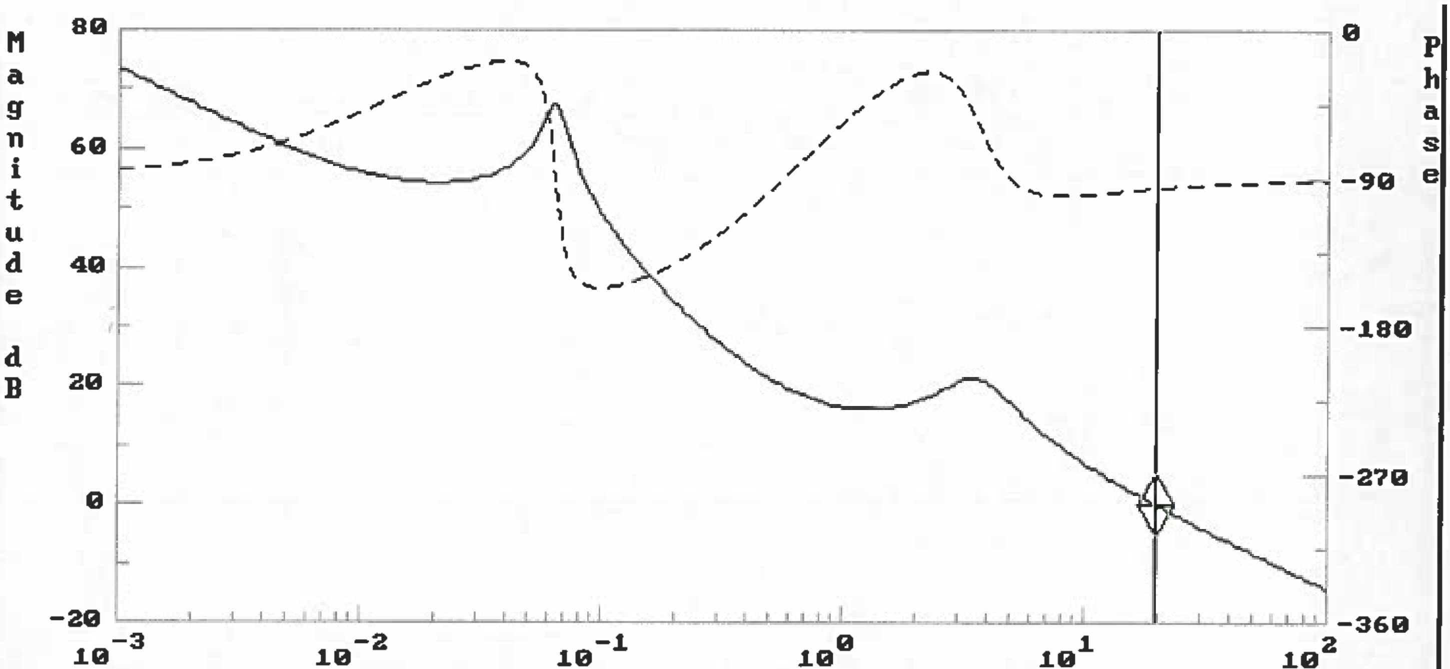


$\frac{\theta}{\theta_c}(s)$ LOW BANDWIDTH DESIGN



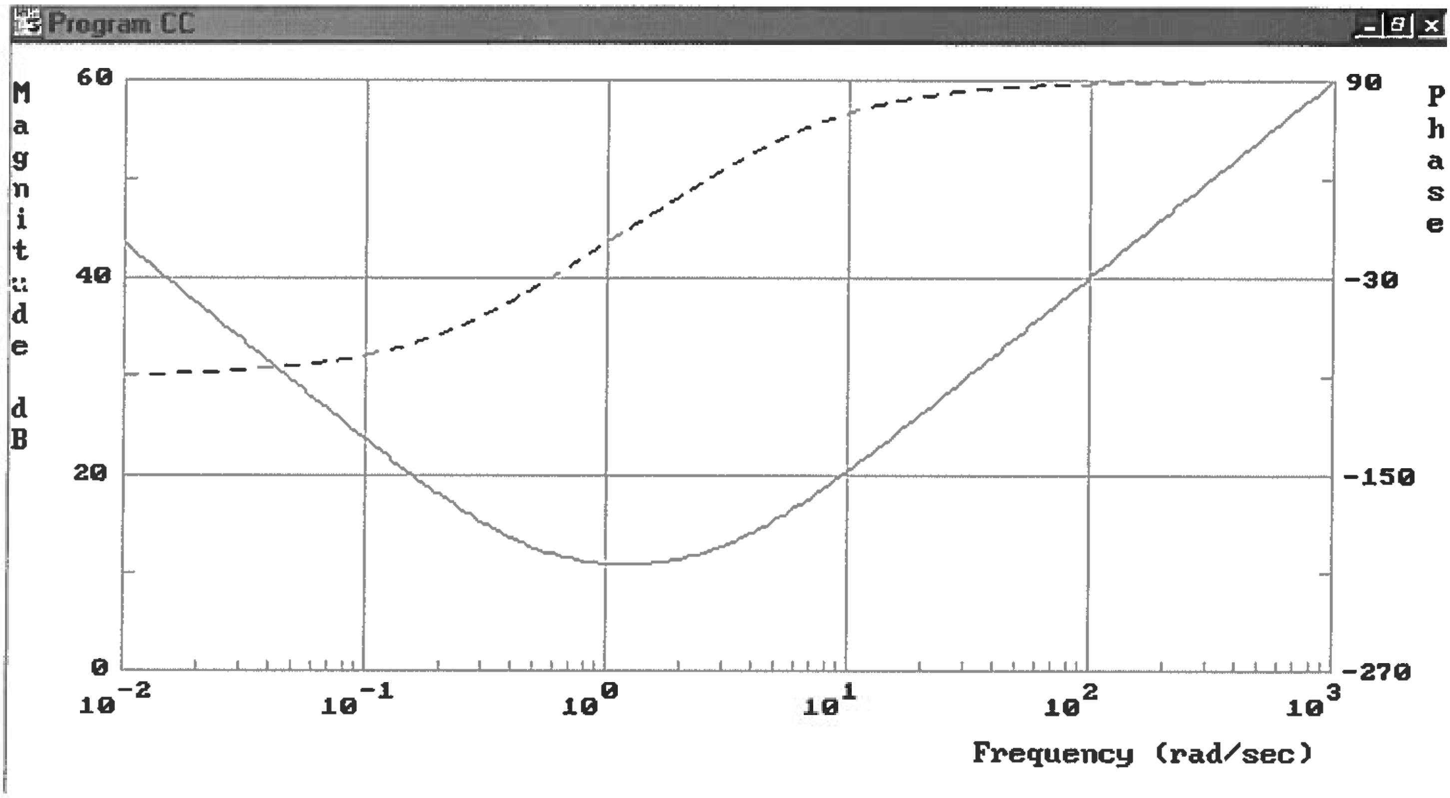
HIGH BANDWIDTH DESIGN

$$G_\theta = -\frac{(s+.5)(s+3)}{s}$$

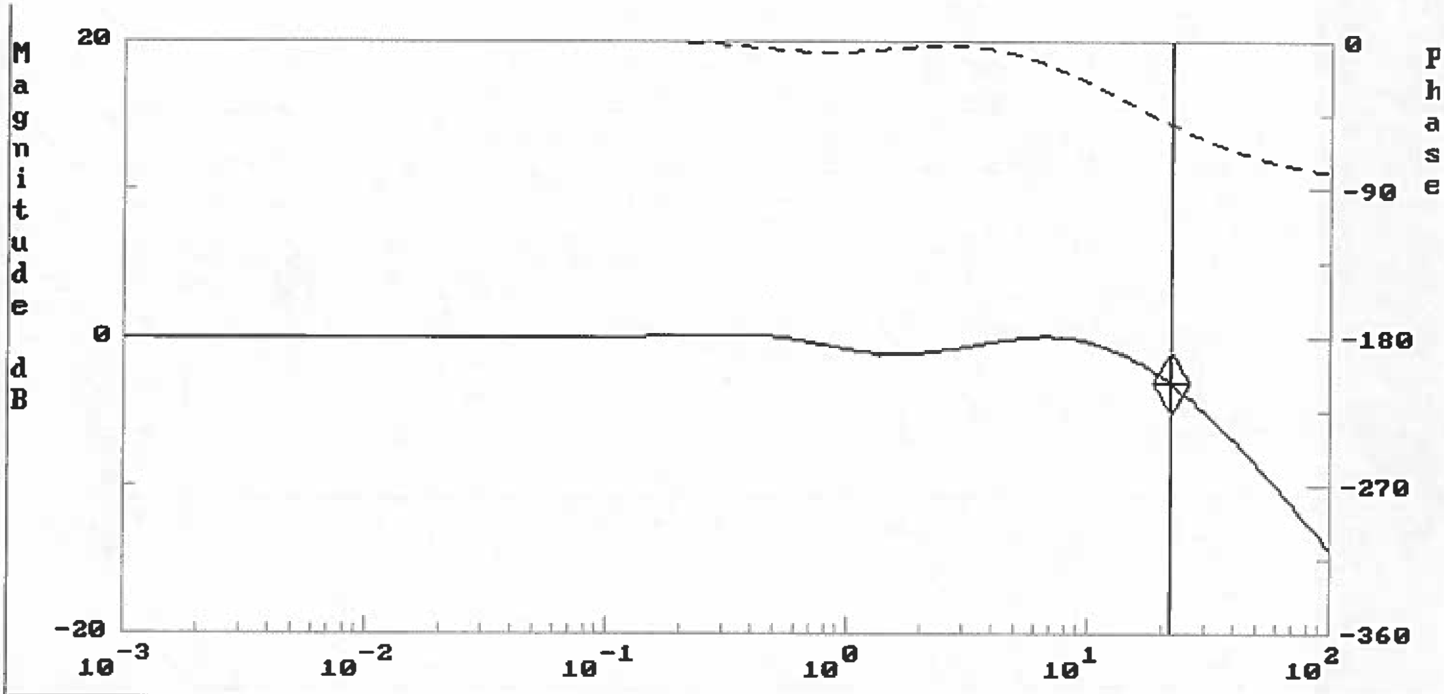


Freq = 1.978E+01 Mag = 0.180 dB Phase= -95.56	<ret>=finished, R=robustness margins Cursor movement: Slow: ← → Fast: Shift ← →

$$G_\theta = \frac{-(s+.5)(s+3)}{s}$$



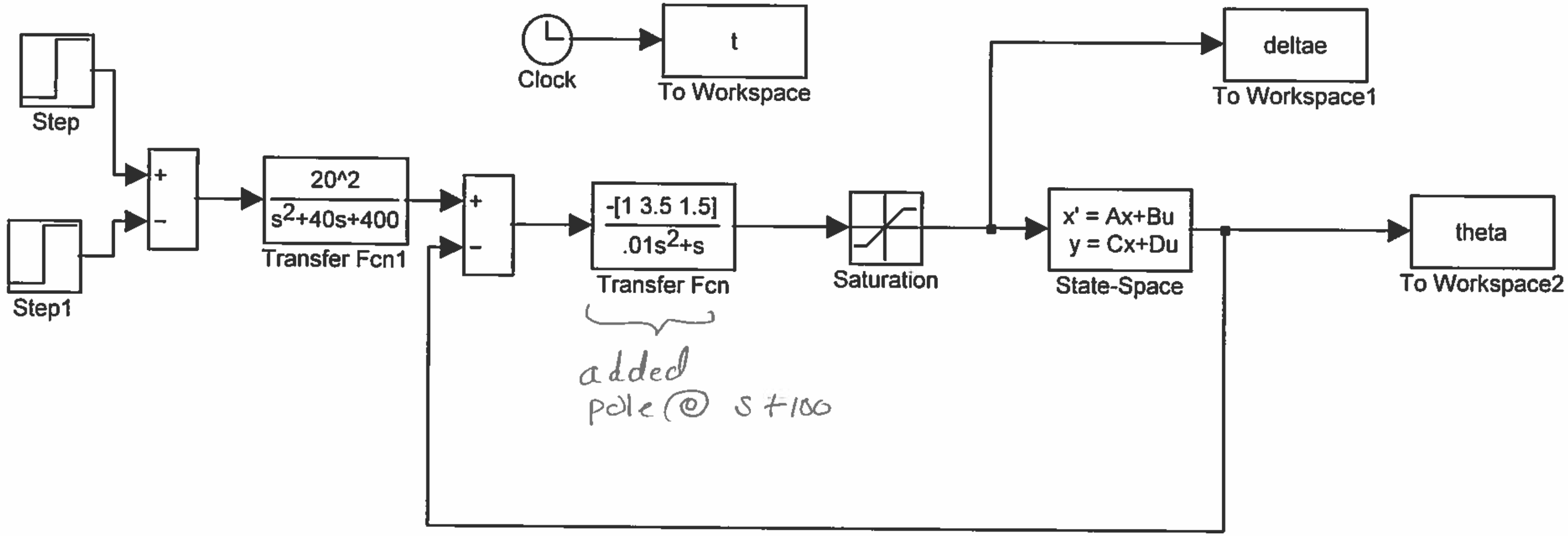
$\frac{\Theta}{\Theta_c}(s)$ HIGH BANDWIDTH DESIGN



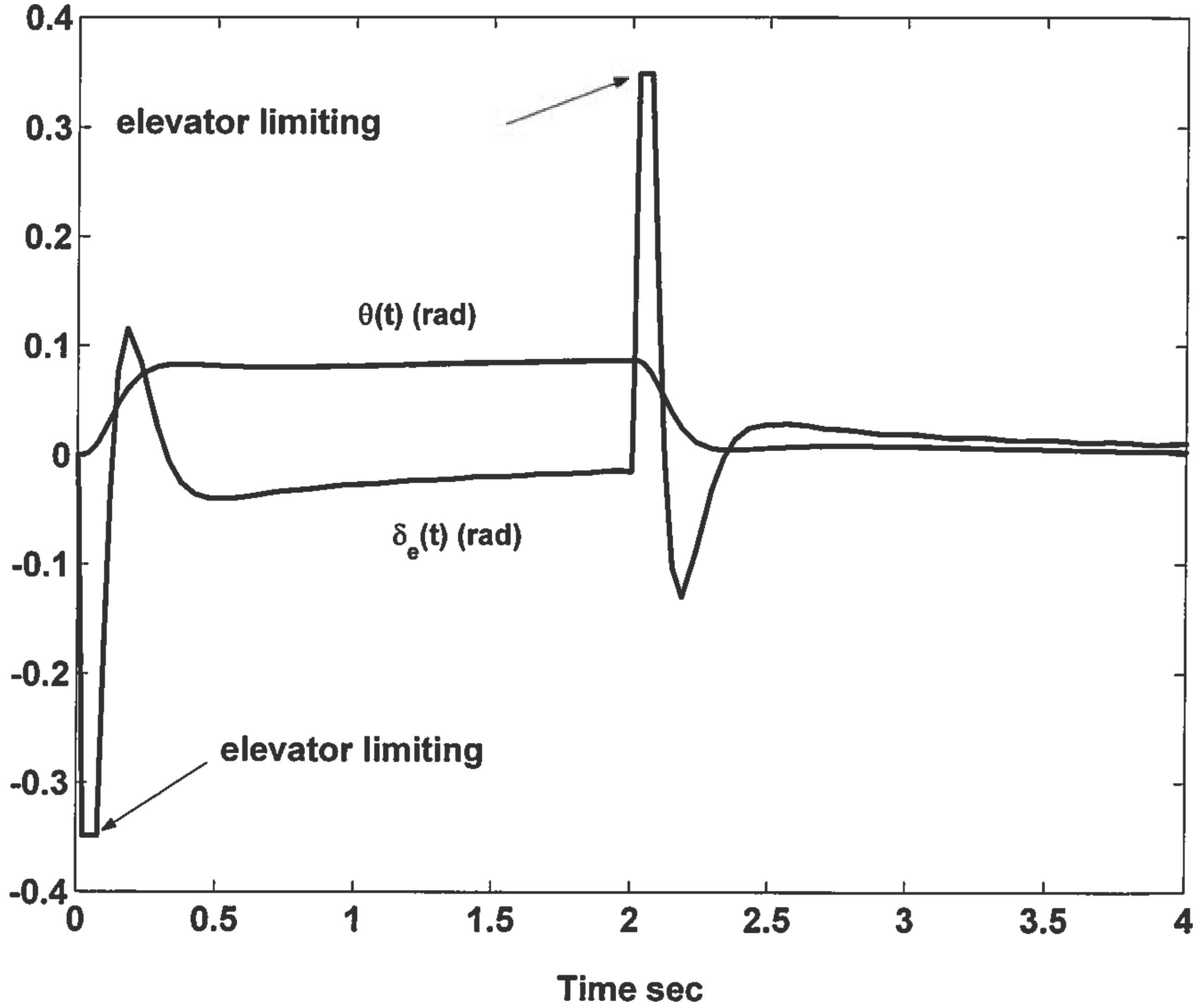
Freq = 2.201E+01
Mag = -3.038 dB
Phase= -50.45

<ret>=finished, R=robustness margins
Cursor movement:
Slow: ← → Fast: Shift ← →

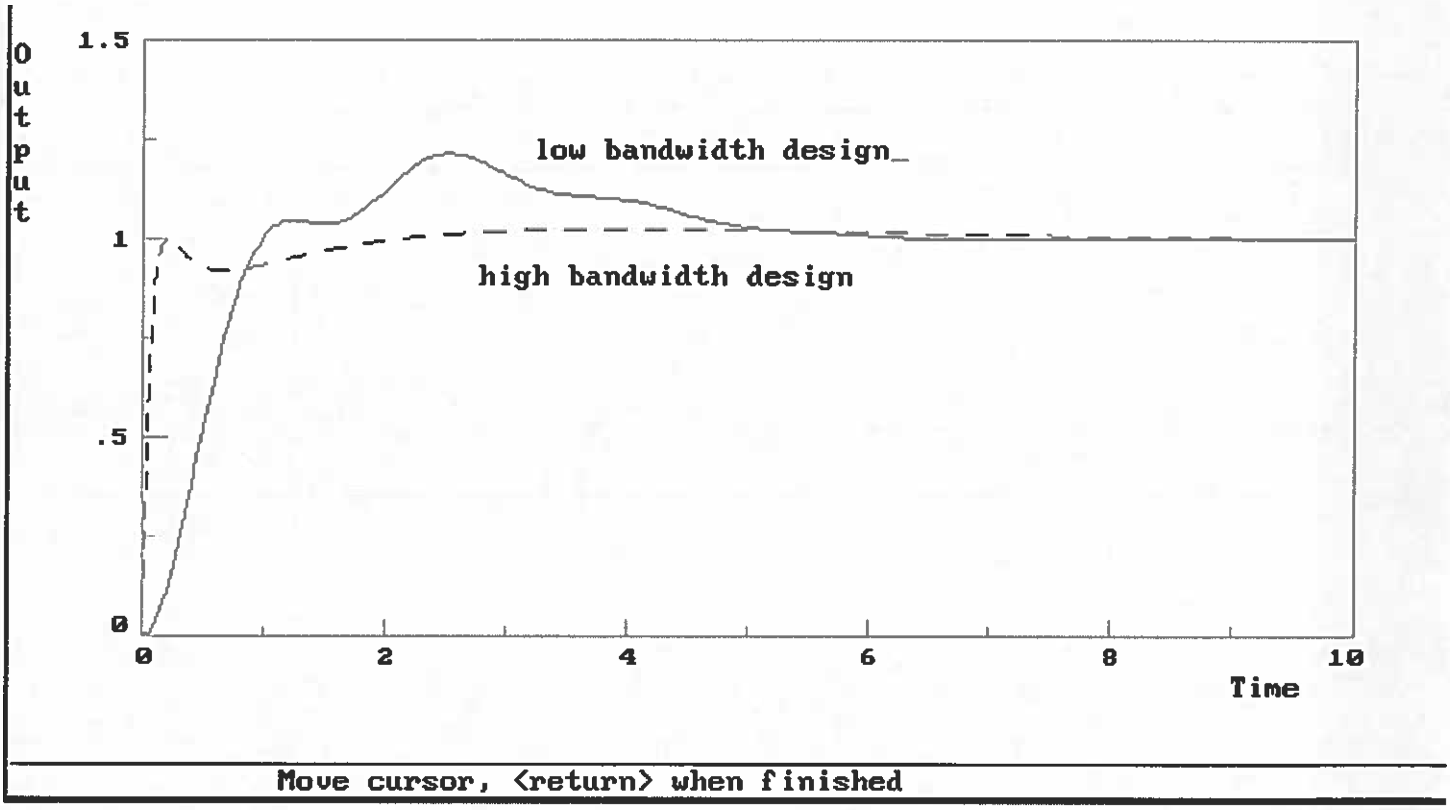
HIGH BANDWIDTH



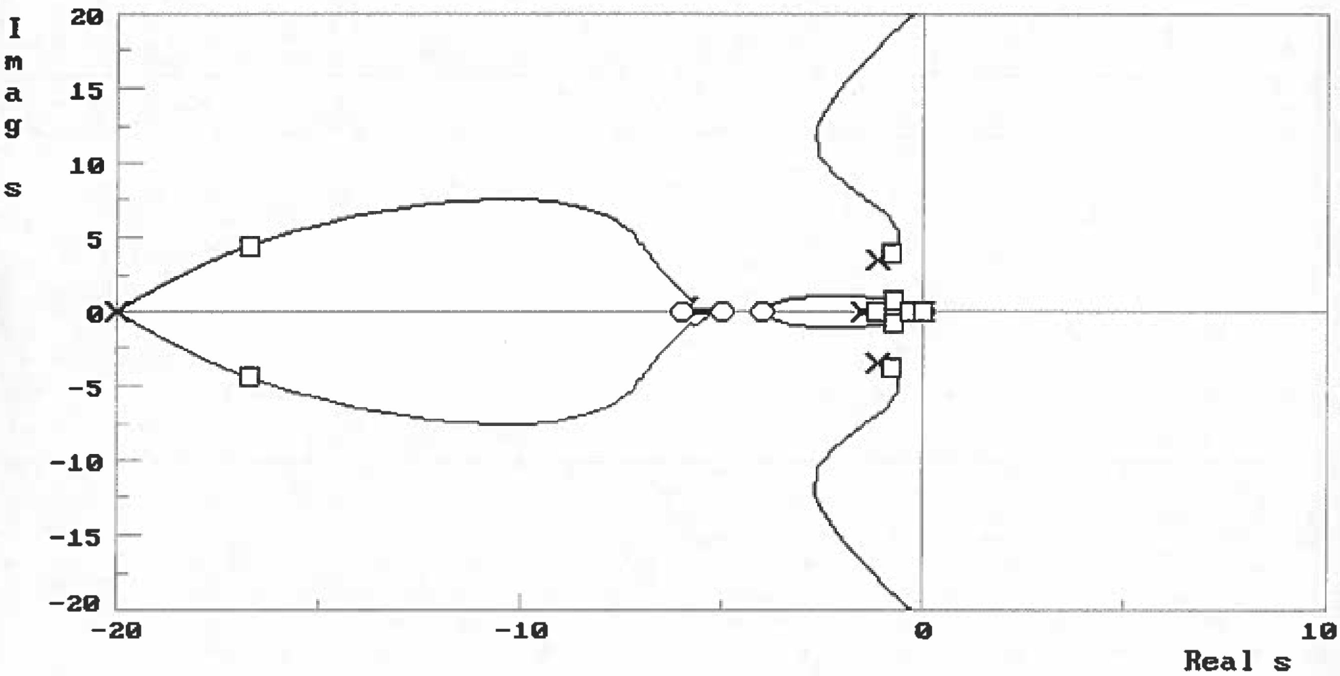
HIGH BANDWIDTH



UNIT STEP RESPONSES

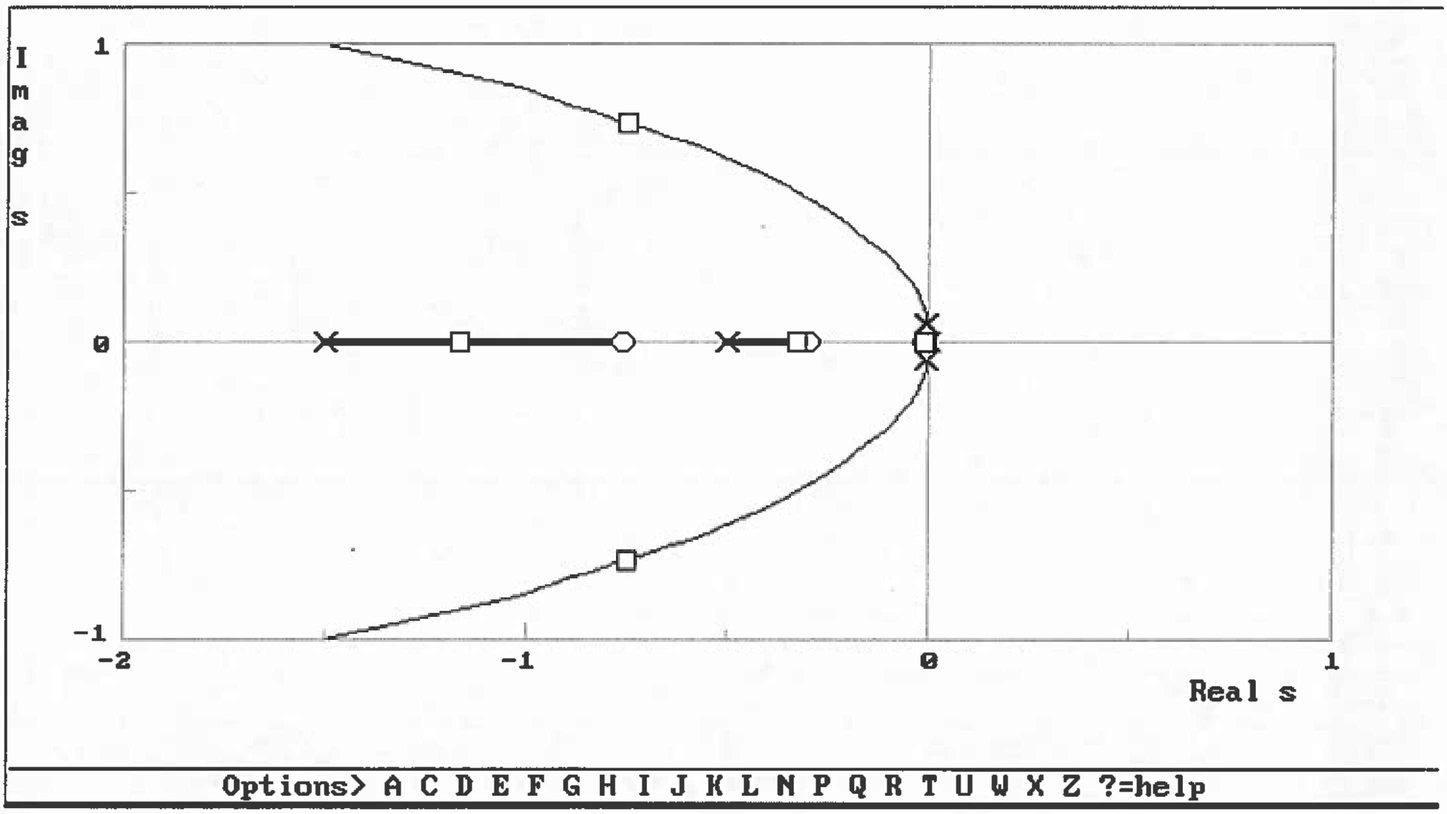


LOW BANDWIDTH ROOT LOCUS



Options> A C D E F G H I J K L N P Q R T U W X Z ?=help

LOW BANDWIDTH ROOT LOCUS
EXPANDED ORIGIN



MAE - 275

THEOREM OF CAUCHY

"Principle of the Argument"

CONSIDER A CLOSED CONTOUR Γ_S IN THE S-PLANE WHICH ENCIRCLES "Z" ZEROS AND "P" POLES OF F(S) BUT DOES NOT PASS <u>THROUGH</u> ANY POLES OR ZEROS OF F(S). CONSIDER TRAVERSING THE CONTOUR IN THE CLOCKWISE DIRECTION AND MAPPING EACH POINT TRAVERSED INTO THE F(S) PLANE, THUS DEFINING A CONTOUR Γ_F. Γ_F WILL ENCIRCLE THE ORIGIN OF THE F(S) PLANE "N = Z - P" TIMES IN THE CLOCKWISE DIRECTION.

HERE, A POINT IS SAID TO BE <u>ENCIRCLED</u> "N" TIMES IN THE CLOCKWISE DIRECTION IF A VECTOR DRAWN FROM THE POINT TO THE PATH MAKES "N" NET ROTATIONS IN THE CLOCKWISE DIRECTION AS THE PATH IS TRAVERSED.

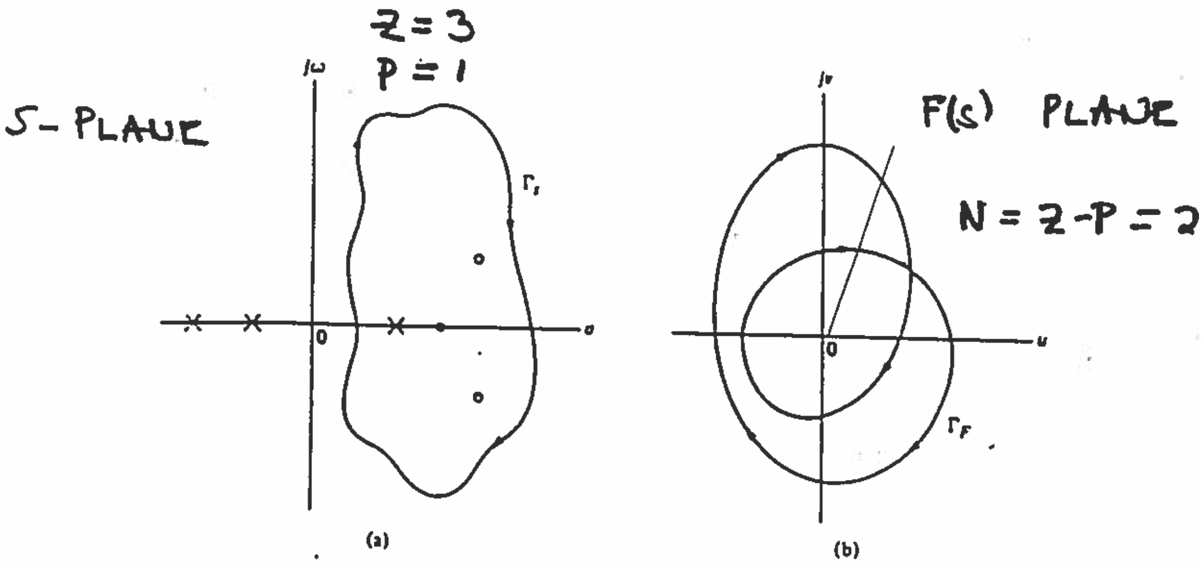


Fig. 8.5. Example of Cauchy's theorem.

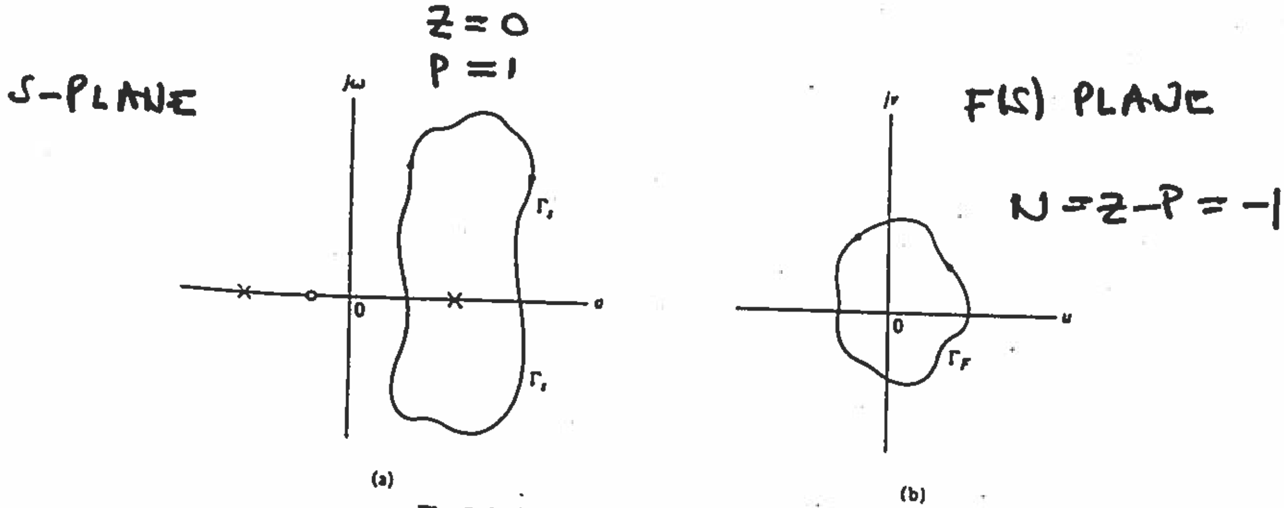
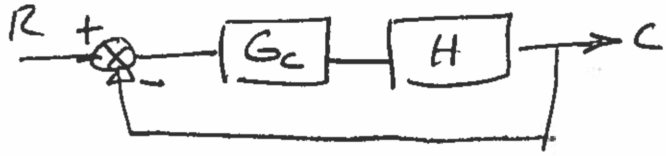


Fig. 8.6. Example of Cauchy's theorem.



$F(s) = 1 + G_cH(s)$ (numerator is characteristic polynomial)

Rather than count encirclements of origin in $F(s)$ plane, we count encirclements of $s=-1$ in $G_cH(s)$ plane.

$$N = Z - P$$

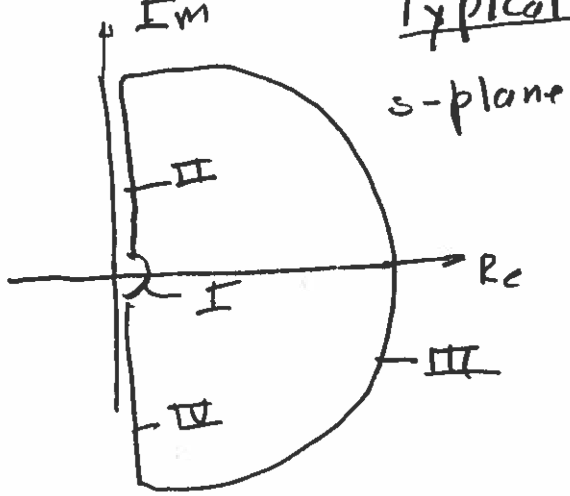
N = # clockwise encirclements

Z = # zeros of $1+G_cH(s)$ in right half plane (RHP)

P = # poles of $1+G_cH(s)$ " " " "

but poles of $1+G_cH(s)$ are poles of $G_cH(s)$

Typical Nyquist Contour



section I sidesteps any poles of $G_cH(s)$ at origin

All realistic open-loop T.F.'s (in s plane) contain more poles than zeros

$\therefore$ section III of Γ_s will always map into origin of G_cH plane

Mapping section I:

a) put G_cH into root locus form (coefficients of highest power of s in polynomials equal 1)

e.g. $G_cH = \underbrace{\dfrac{5(.1s+1)}{s(5s+1)}}_{\text{Bode form}} = \underbrace{\dfrac{0.1(s+10)}{s(s+0.2)}}_{\text{root locus form}}$

b.) $\dfrac{K}{s^n} \Rightarrow \varepsilon^{-n} e^{-j(n\phi + m\pi + p\pi)}$

ε = radius of infinitesimal circle $\ll 1.0$

$-\pi/2 < \phi < \pi/2$

n = # of poles of G_cH at origin

m = total # of real open loop poles / zeros in RHP

$p = 0$ if $K > 0$

$p = 1$ if $K < 0$

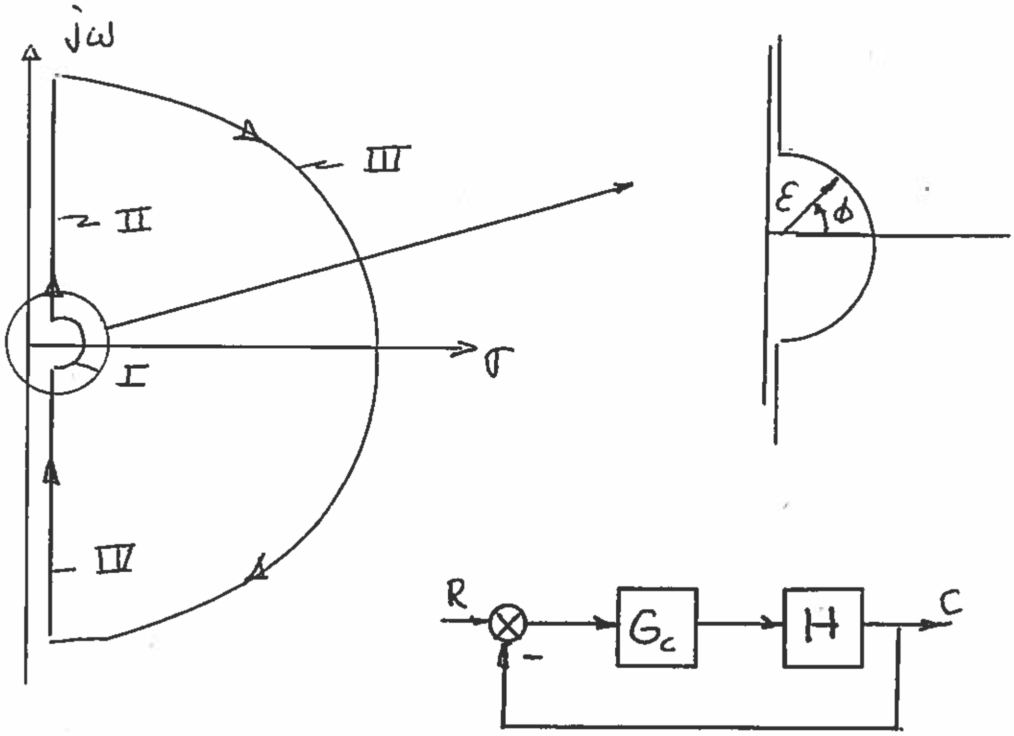
this leaves sectors II + IV but these are

just $|G_cH(j\omega)|$ $\angle G_cH(j\omega)$ } obtain from Bode plot

$|G_cH(-j\omega)|$ $\angle G_cH(-j\omega)$ } complex conjugate of above

Mapping the Nyquist Contour into the $G_cH(s)$ plane

1.) Mapping Section I in the Nyquist Contour onto $G_cP(s)$ plane



$G_cP(s)$ for section I where $s = \varepsilon^{j\phi}$ and $-\pi/2 < \phi < +\pi/2$ is given by

$$G_cH(s) = \varepsilon^{-n} e^{-j(n\cdot\phi + m\pi + p\pi)}$$

where ε = radius of section I

n= number of free s's in the denominator of $G_cH(s)$

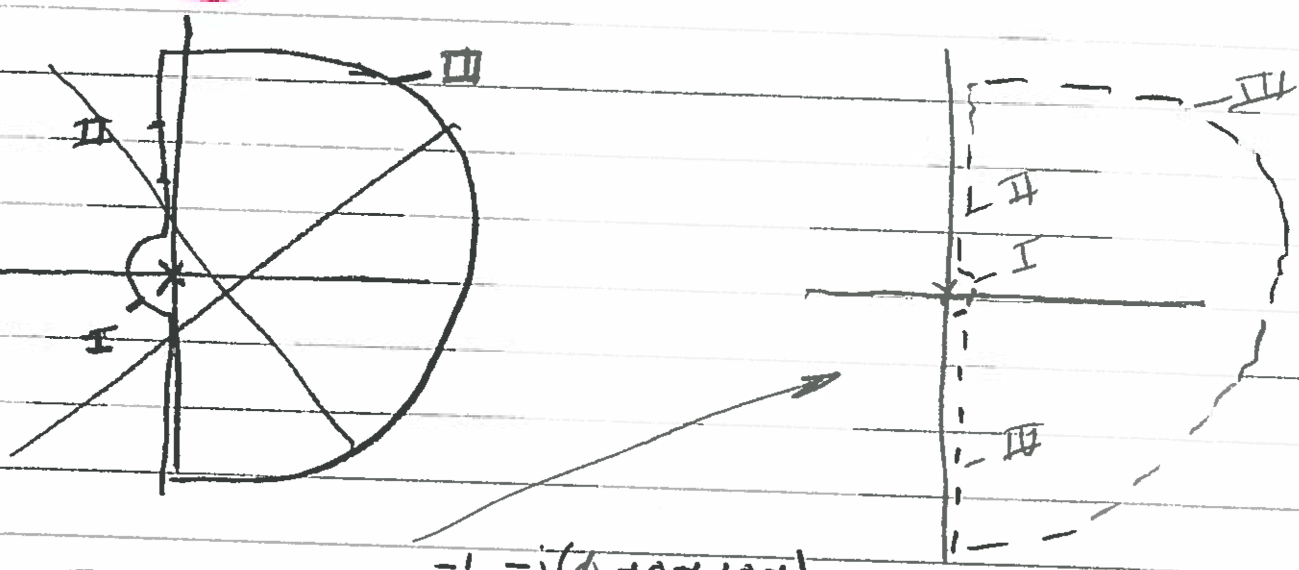
m= number of <u>real</u> poles and zeros of $G_cH(s)$ in right half of s-plane

p = 0 if the gain of $G_cH(s)$ is positive
  = 1 if the gain of $G_cH(s)$ is negative } $G_cH(s)$ in <u>root locus form</u>

2.) For sections II and IV, $G_cH(s) = G_cH(j\omega)$ and the Bode plot can be used to sketch the mapping.

3.) For the majority of applications, $G_cH(s)$ will have more poles than zeros which means section III maps into the origin of the $G_cH(s)$ plane.

$\frac{1}{s(s+1)}$ Example



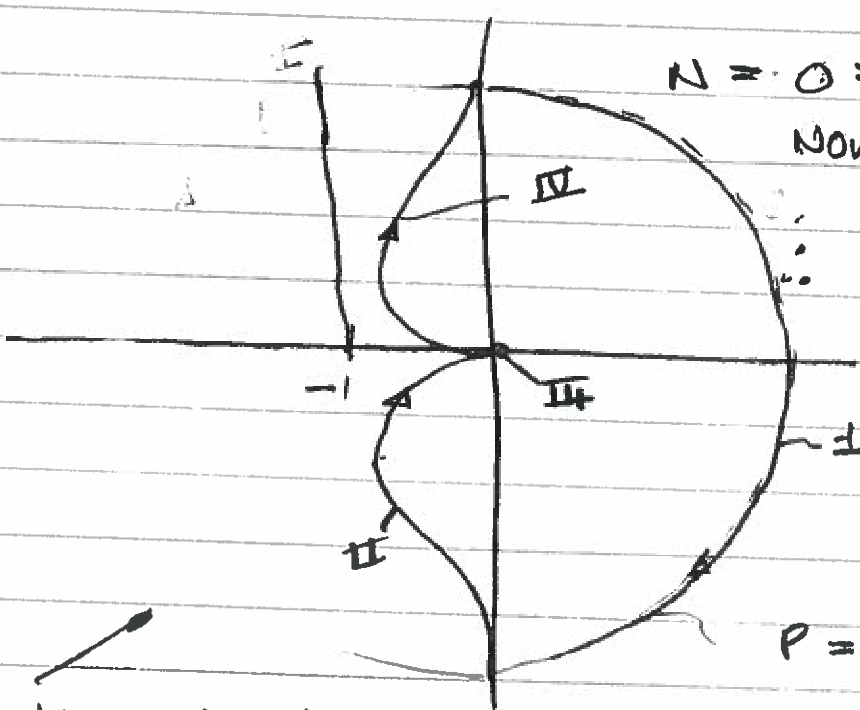
I: $n=1$ $\quad e^{-1} e^{-j(\phi + 0\pi + 0\pi)}$

$-\pi/2 \le \phi \le +\pi/2$

@ $\phi = -\pi/2 \rightarrow \pi/2$

@ $\phi = 0 \quad\quad 0$

@ $\phi = +\pi/2 \rightarrow -\pi/2$



$N = 0 = Z - P$

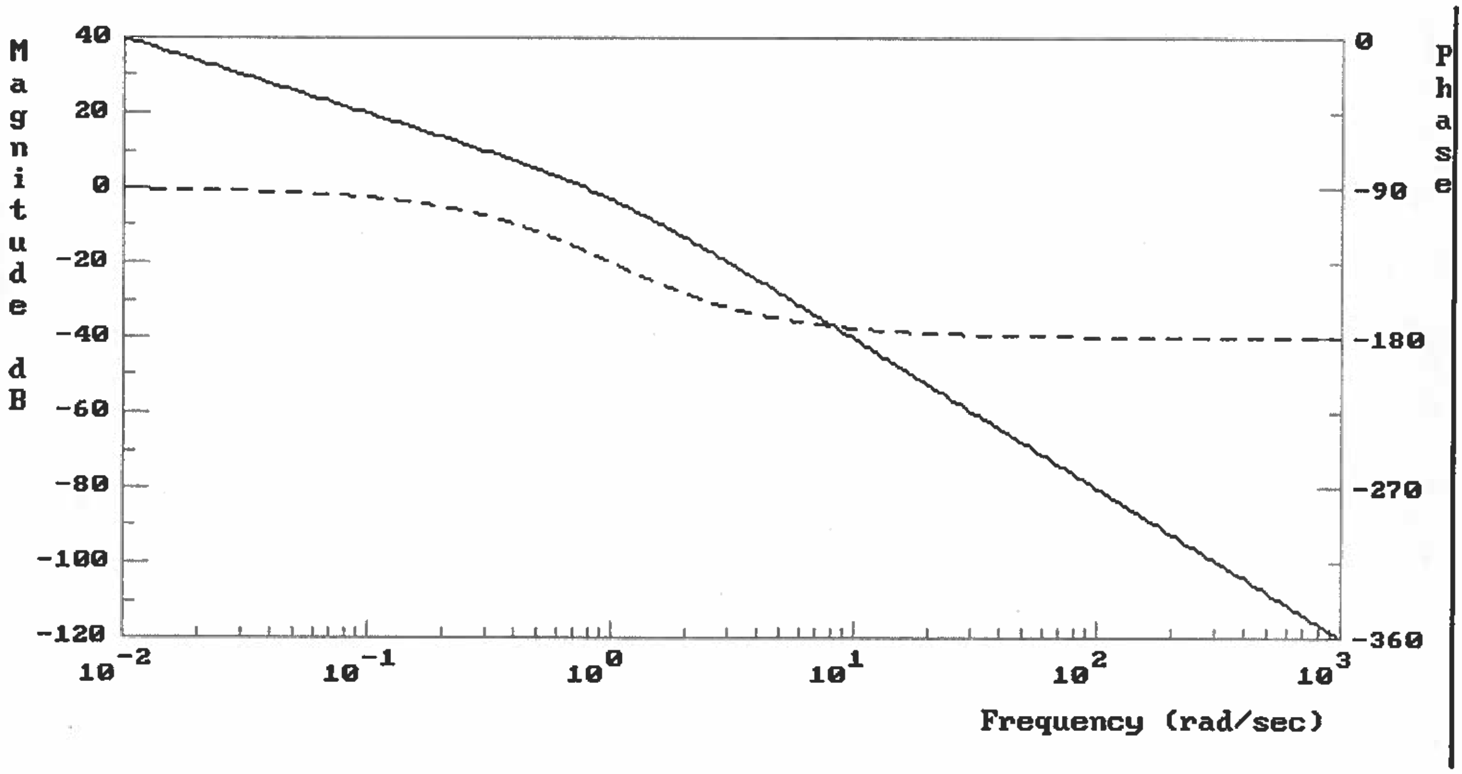
Now $P = 0$

$\therefore Z = 0$

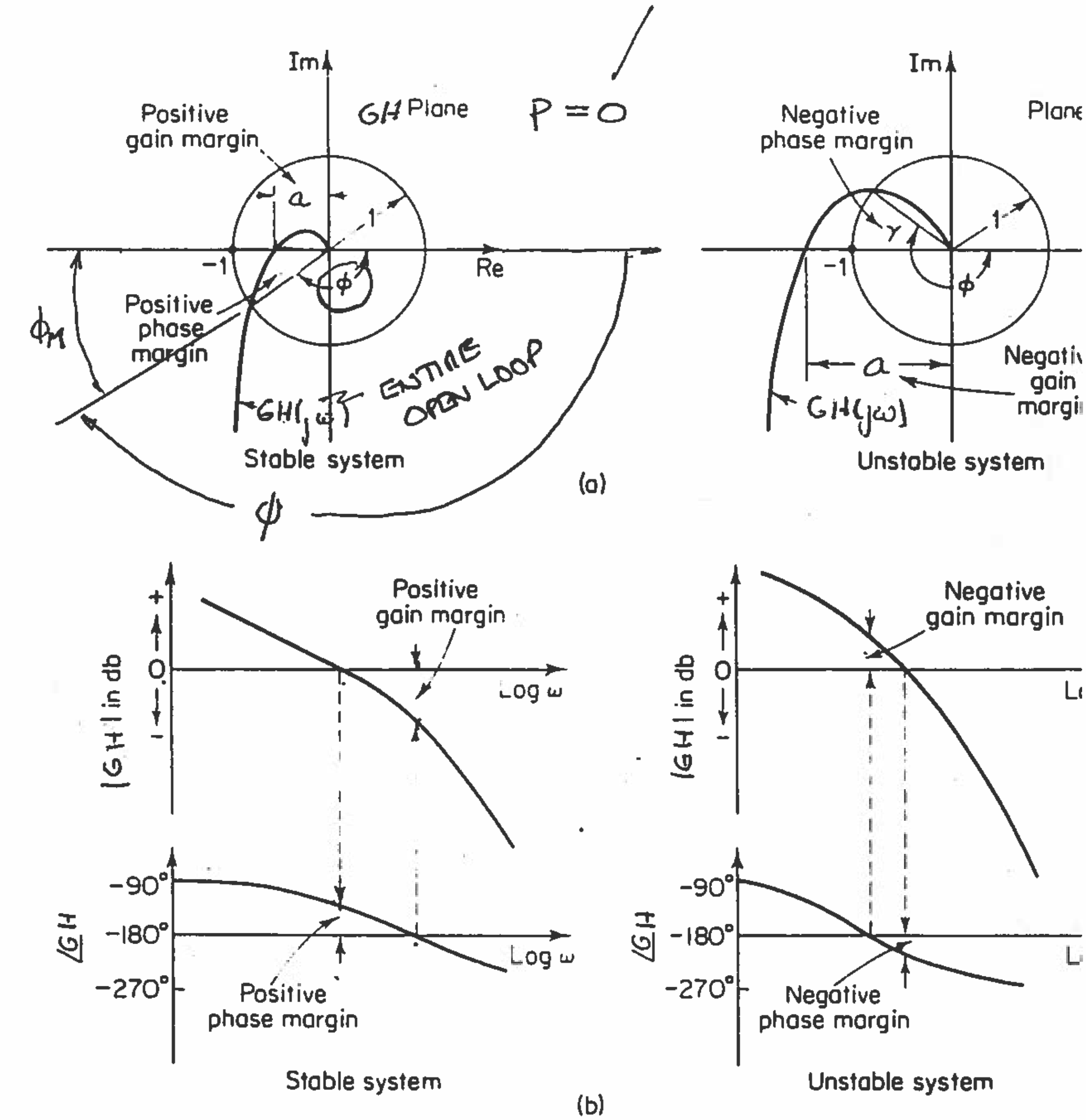
$P = 0$

can't destabilize by gain

$$\frac{1}{s(s+1)}$$



Magnitude dB
40
20
0
-20
-40
-60
-80
-100
-120
Phase
0
-90
-180
-270
-360
10^-2
10^-1
10^0
10^1
10^2
10^3
Frequency (rad/sec)



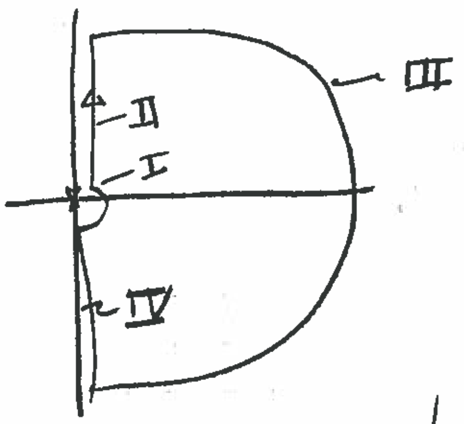
$$\phi_M = P.M = \phi + 180^\circ > 0 \text{ deg}$$

$$G.M = 20 \text{ Log} \left| \frac{1}{a} \right| > 0 \text{ dB}$$

GAIN MULTIPLYING FACTOR NECESSARY TO HAVE $|G(j\omega)| = 1$ at $\angle G(j\omega) = -180^\circ$

Example

$$G_cH(s) = \frac{k(s+1)}{s(s+2)} \qquad \underline{\underline{k=1}}$$



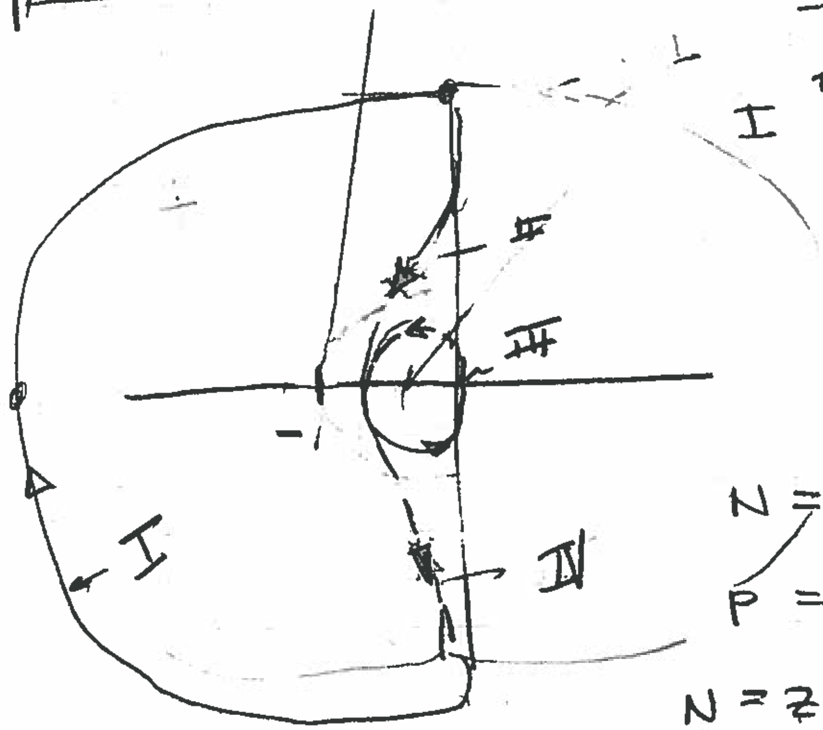
I $n=1$

$$G_cH(s) = \varepsilon^{-1} e^{-j(\phi + \pi + 0)}$$

ϕ	φ
$-\pi/2$	$-\pi/2$
$\pi/2$	$\pi/2$

I $G_cH(e^{j\phi}) = \frac{1}{\varepsilon} e^{-j(\phi+\pi)}$

$-\pi/2$ to $\pi/2$



$N = 1$

$P = 1$

$N = Z - P$

$1 = Z - 1$

$Z = 2$ ✓

$K = 10$ (move -1 pt inside loop)

$N = -1$

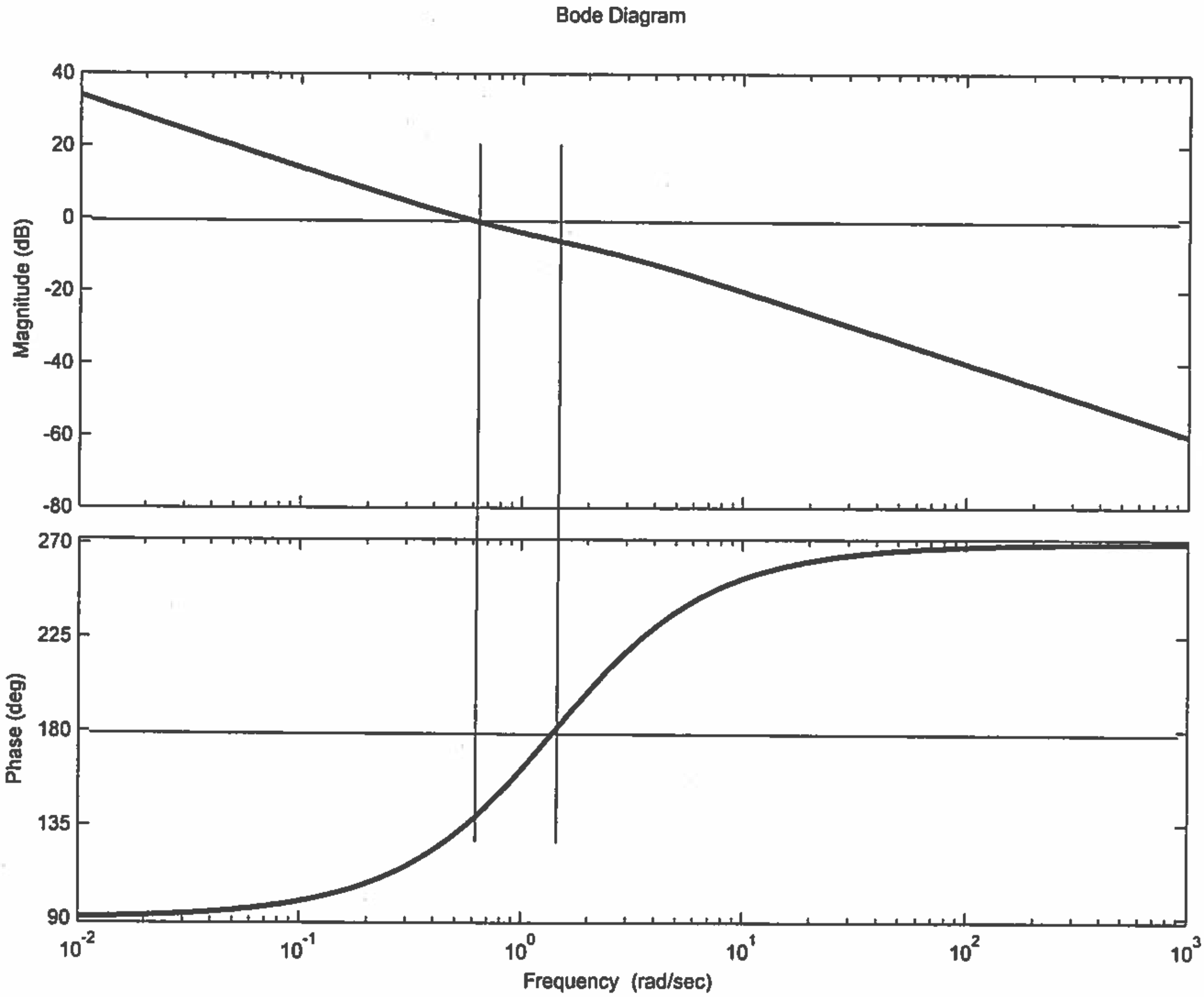
$P = 1$

$N = Z - P$

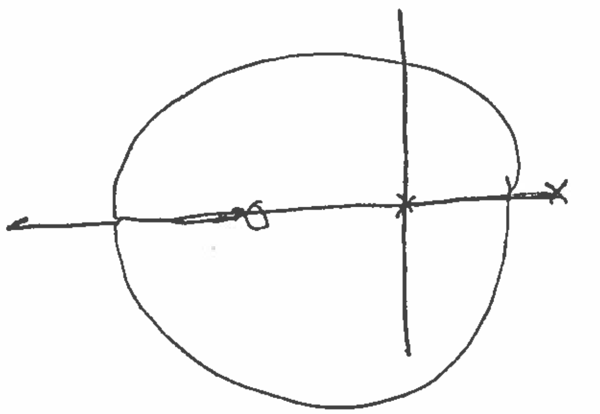
$-1 = Z - 1$

$Z = 0$ stable

$$\frac{k(s+1)}{s(s-2)} \quad k=1$$



Root Locus



Routh's

ch eqn = $s^2 + (k-2)s + k = 0$

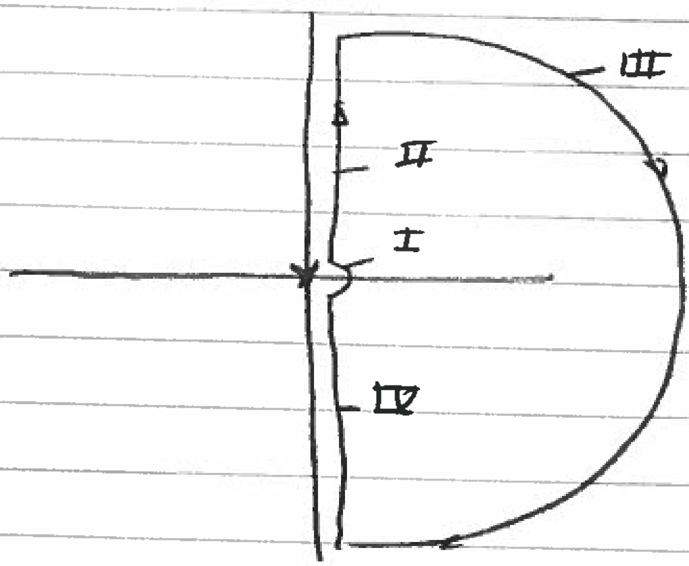
s^2	1	k
s^1	$k-2$	0
s^0	k	

$k > 2$

$k > 0$

Example

$$G_cH = \frac{-0.5(s-1)}{s(s+1)}$$

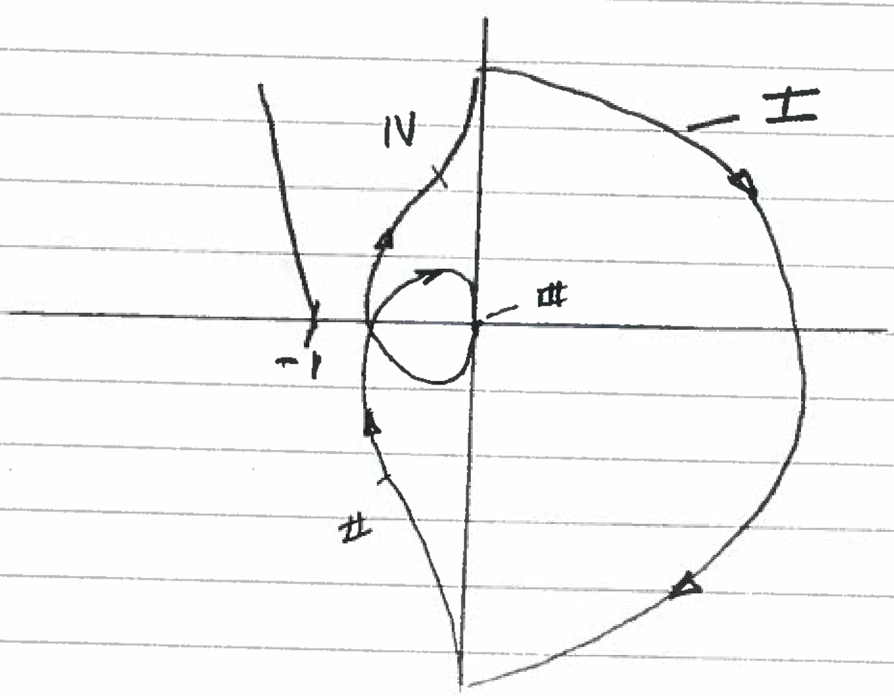


I: $n = 1 \quad \varepsilon^{-1} e^{-j(\phi + \pi + \pi)}$

$-\pi/2 \le \phi \le \pi/2$

@ $\phi = -\pi/2 \rightarrow -3\pi/2 \qquad \phi = \pi/2 \rightarrow -5\pi/2$

@ $\phi = 0 \qquad -2\pi$

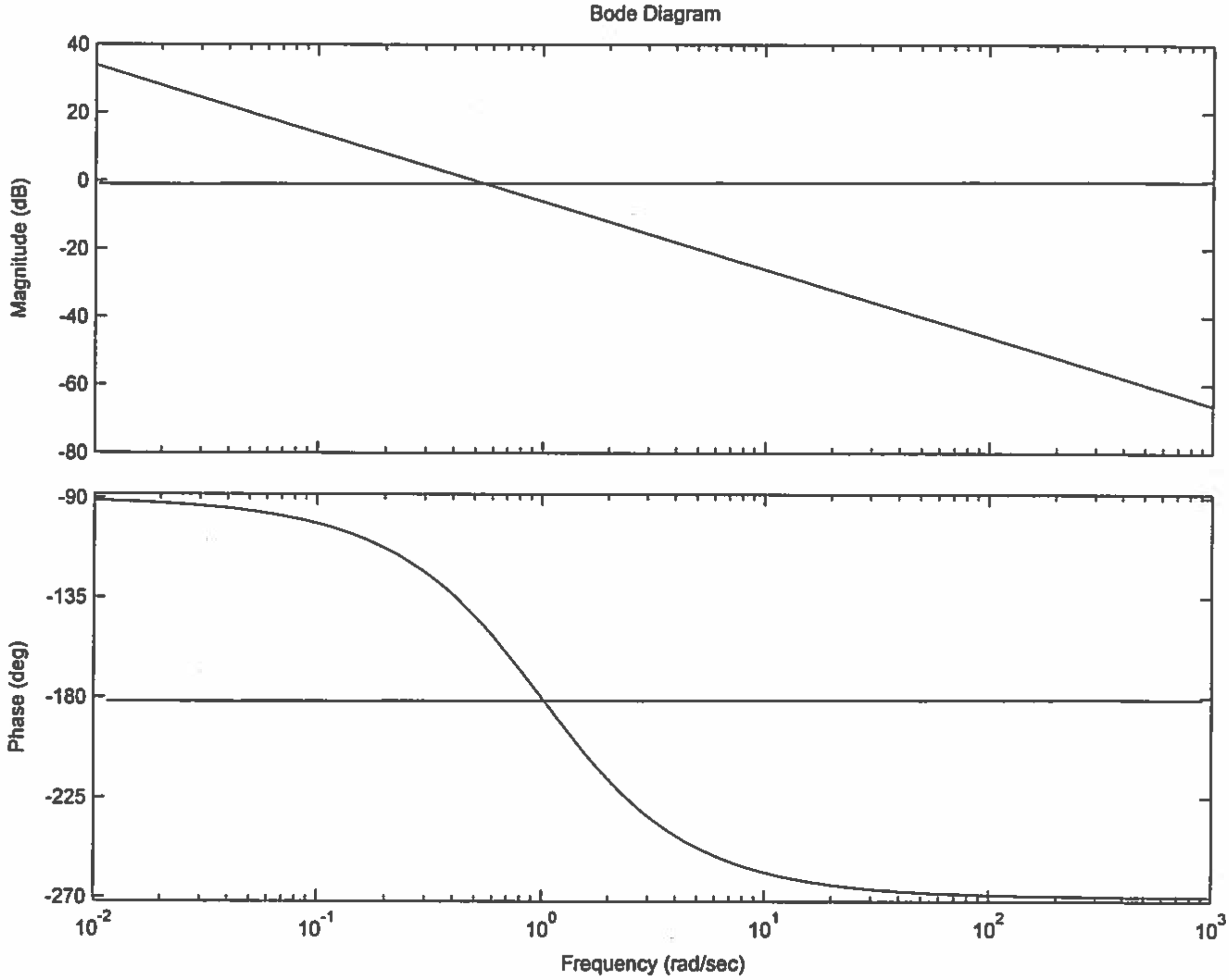


$N = 0 = Z - P$

$P = 0$

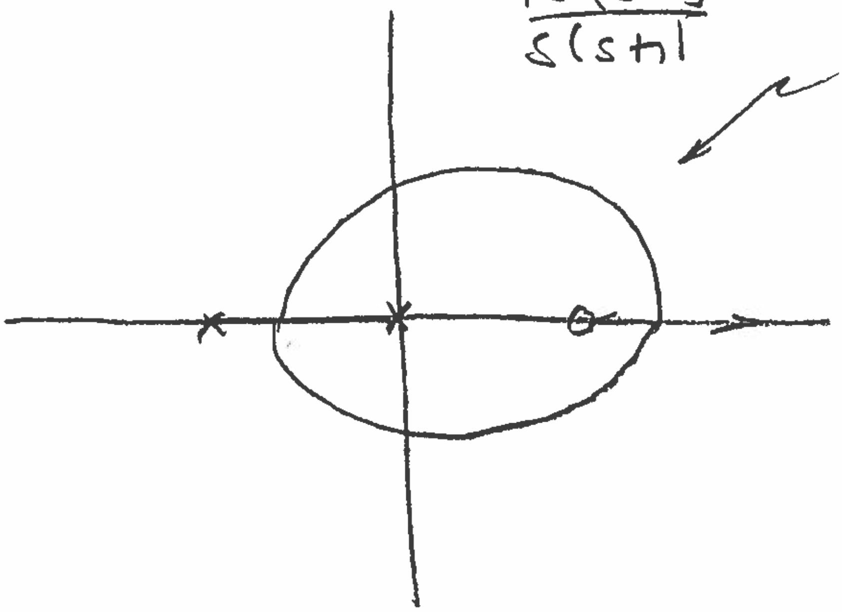
$\therefore \quad Z = 0$

$$G = \frac{-0{,}5(s-1)}{s(s+1)}$$



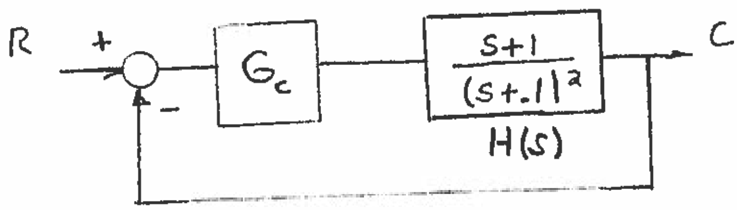
Root locus

$$\frac{-.5(s-1)}{s(s+1)}$$



MAE 275

A Simple Loop Shaping Example



Performance Requirements

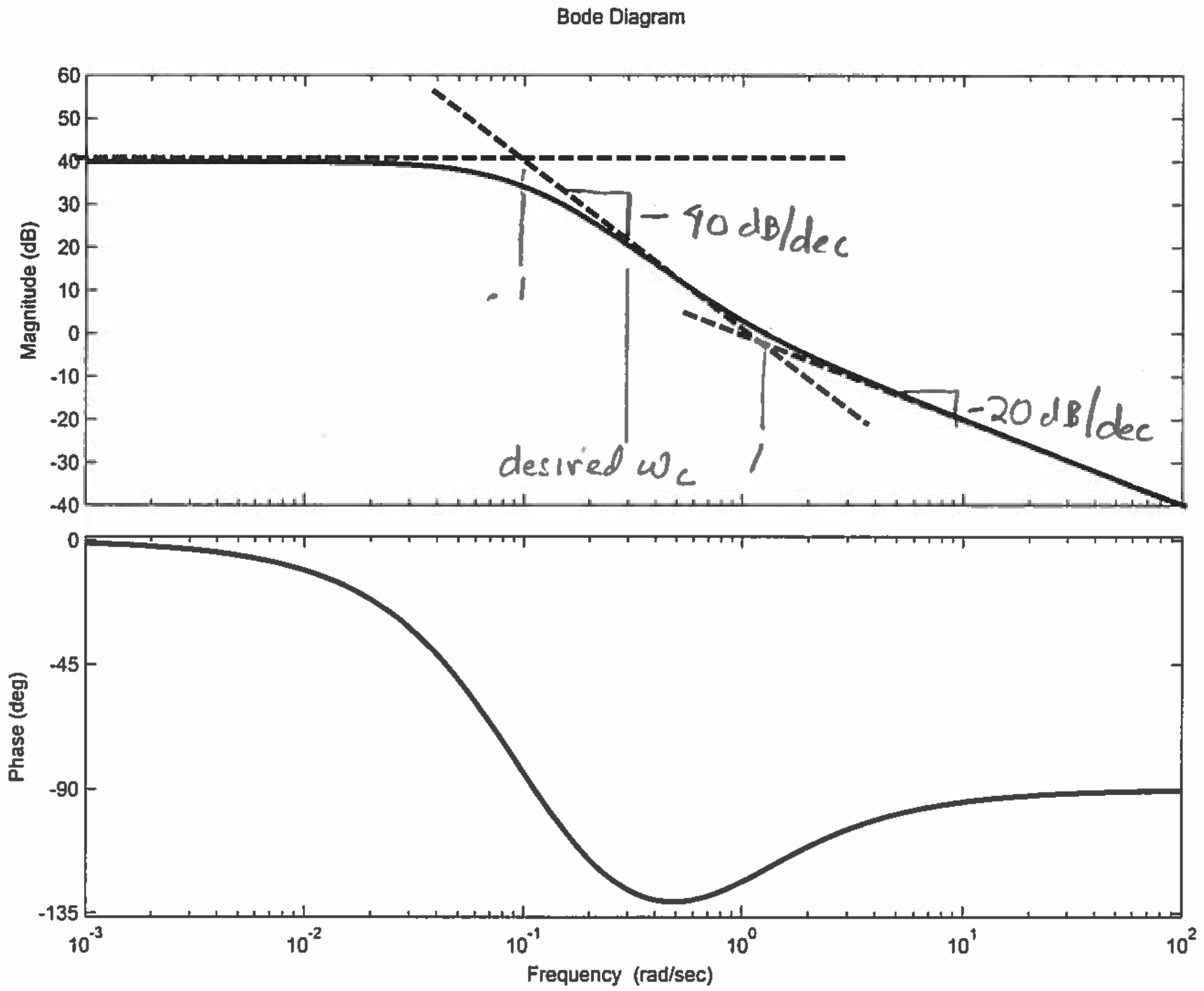
1.) Bandwith 0.3 rad/sec

2.) Gain margin > 20 dB

3.) Phase margin > 45 deg

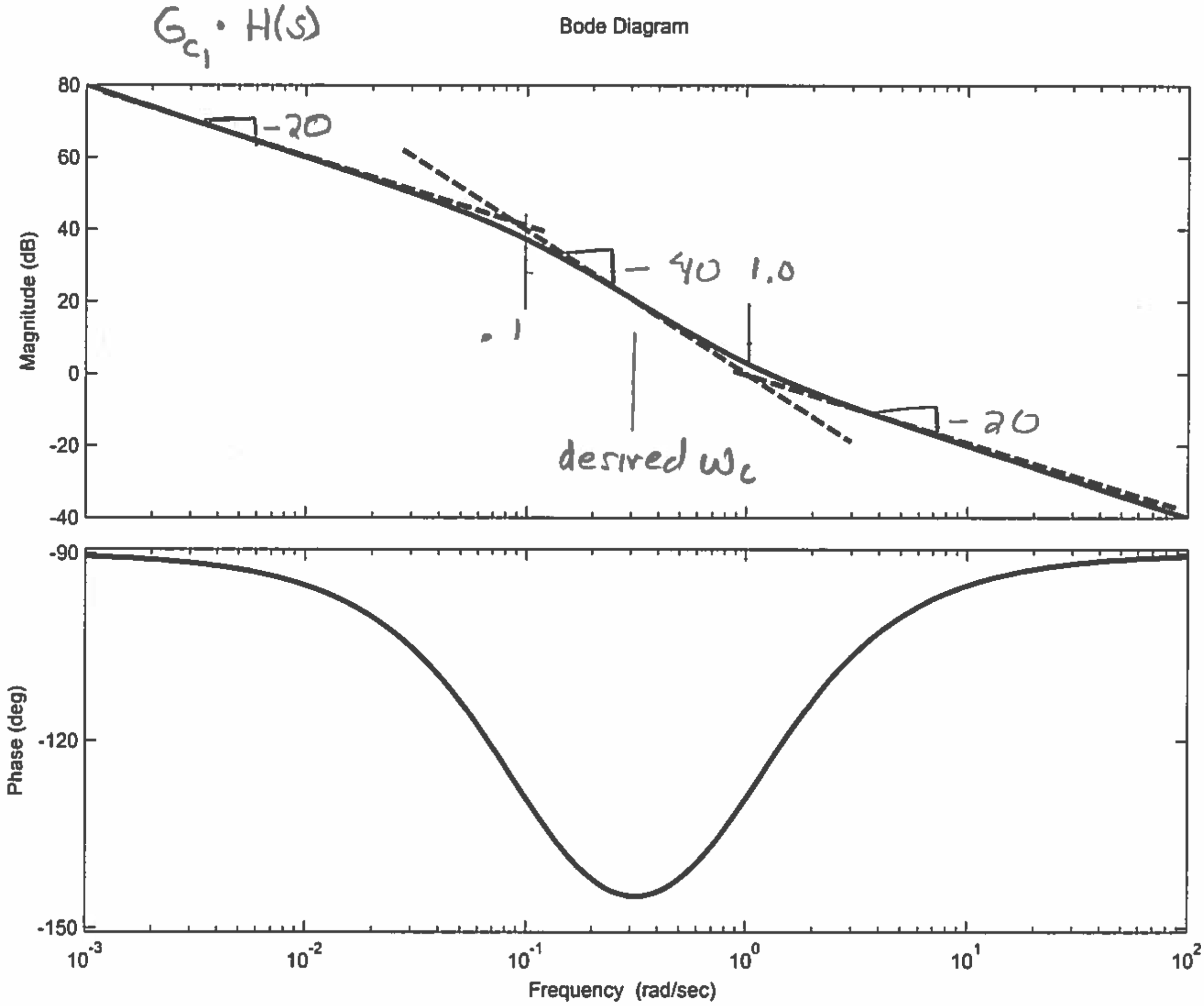
4.) Type 1 system (0 steady state error to a step input)

$$H(s) = \frac{(s+1)}{(s+.1)^2} \qquad \text{desired } \omega_c = 0.3 \text{ rad/sec}$$



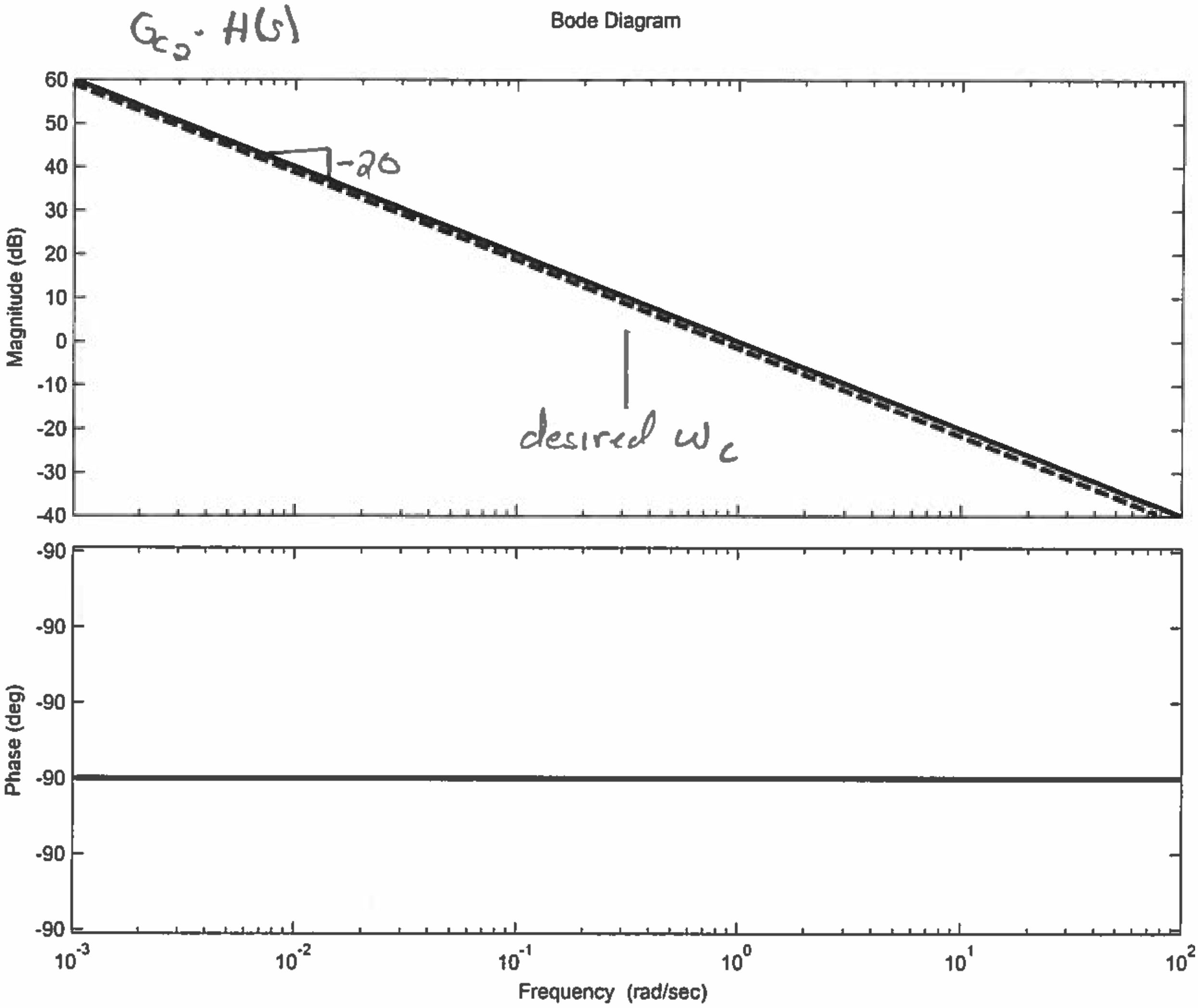
$$G_{c_1} = \frac{(s+.1)}{s}$$

$G_{c_1} \cdot H(s)$



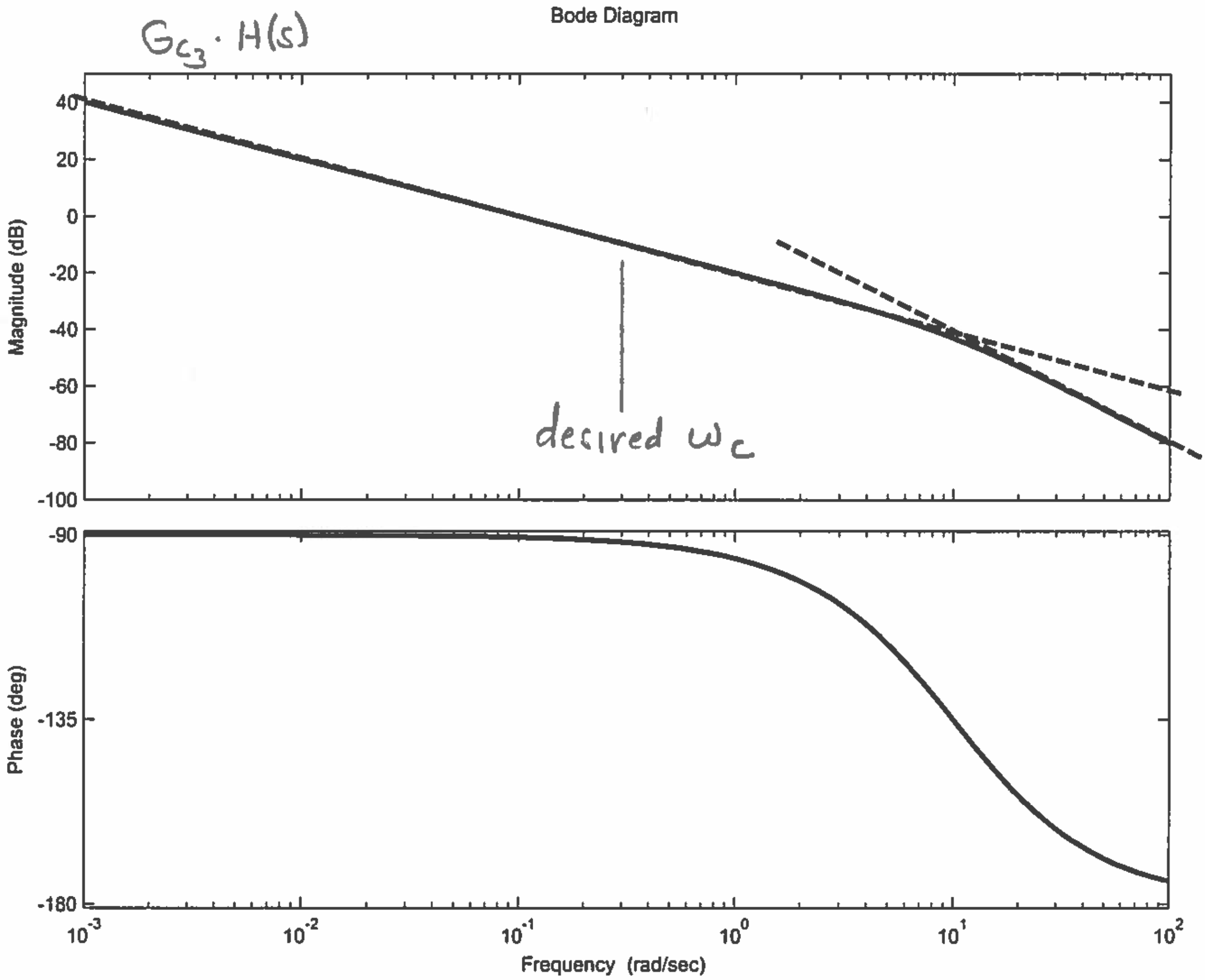
$$G_{c_2} = G_{c_1} \times \frac{(s+.1)}{(s+1)} = \frac{(s+.1)^2}{s(s+1)}$$

$G_{c_2} \cdot H(s)$

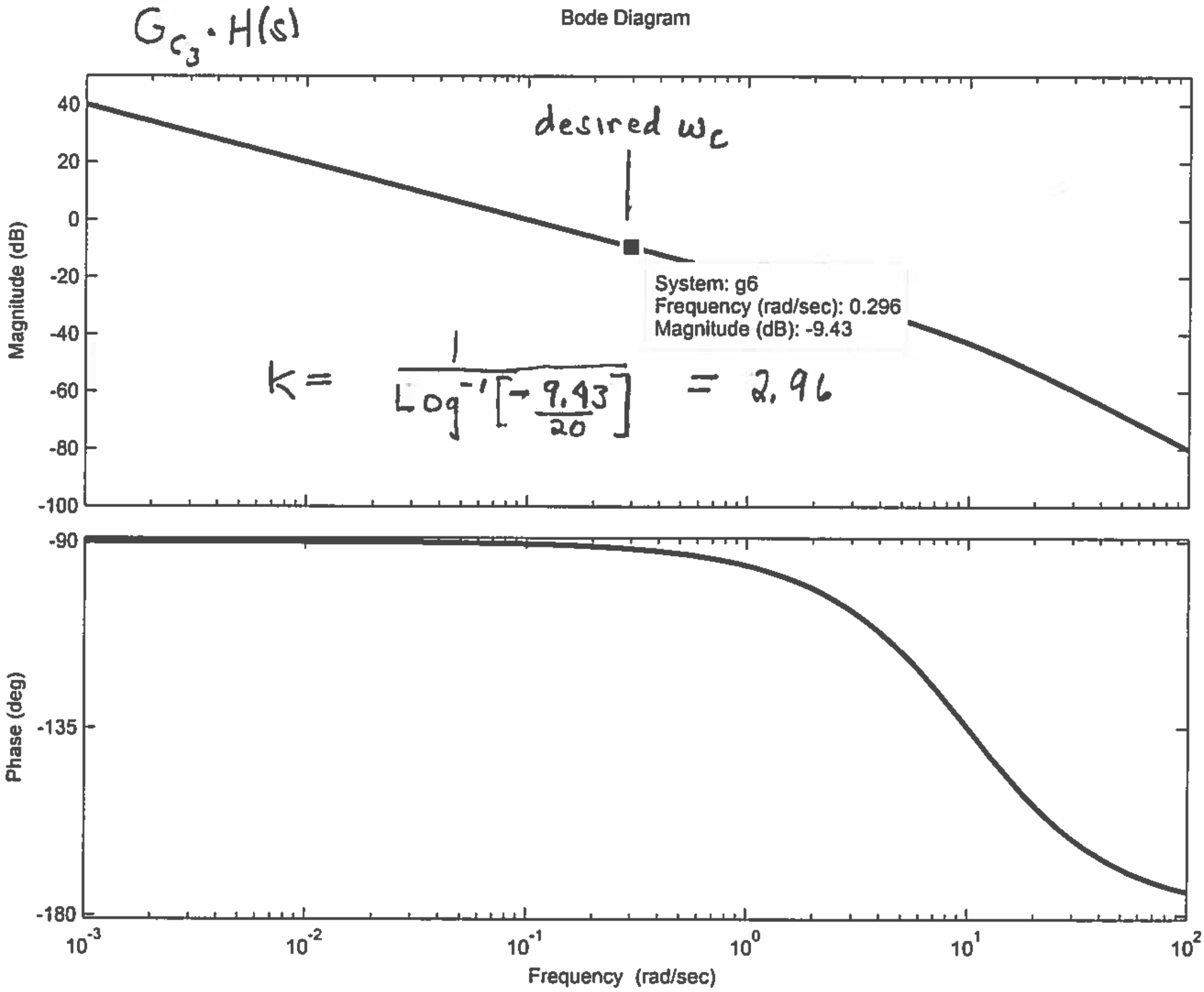


$$G_{c_3} = G_{c_2} \times \frac{1}{(s+10)} = \frac{(s+.1)^2}{(s)(s+1)(s+10)}$$

$G_{c_3} \cdot H(s)$

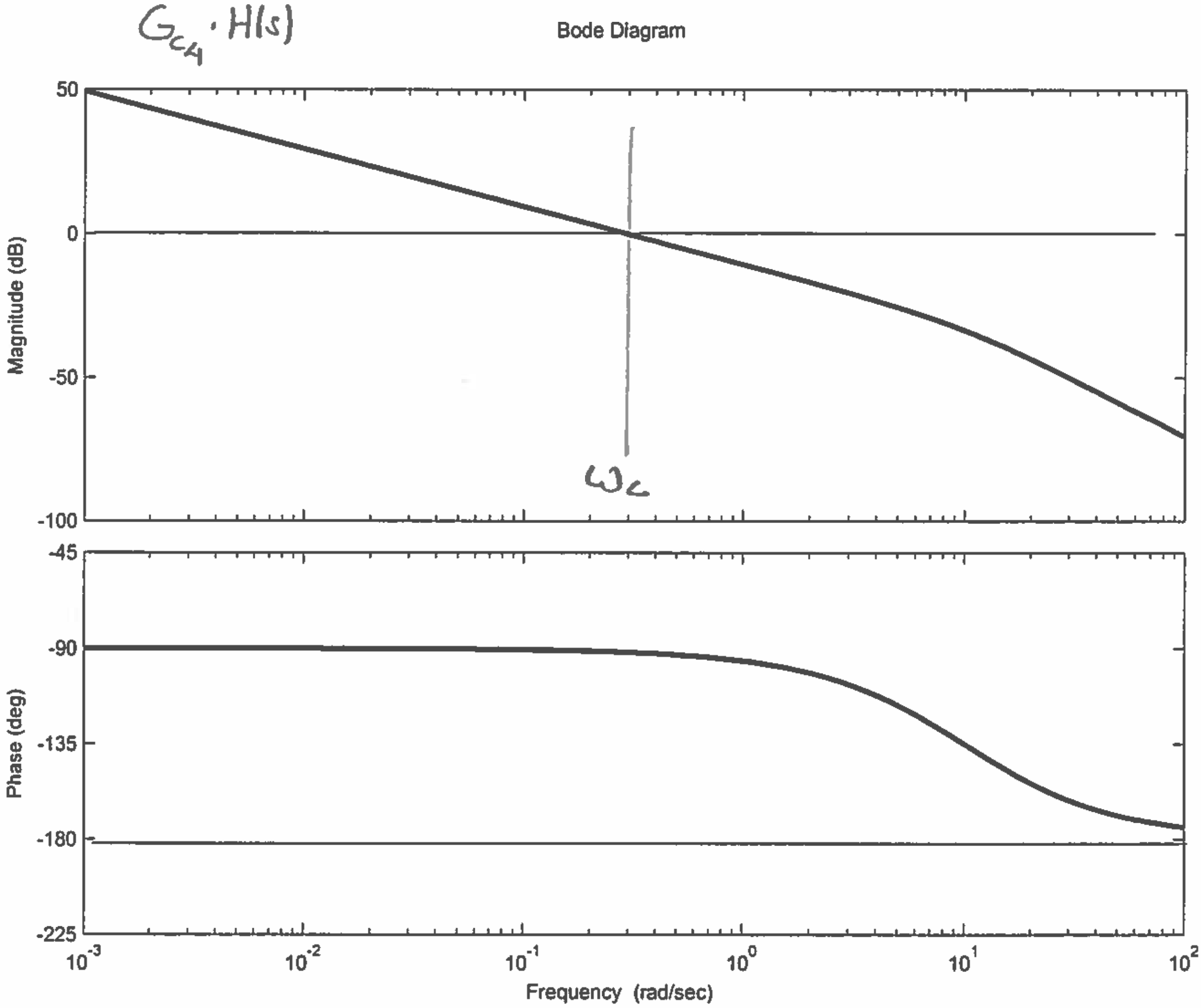


$G_{c_3} \cdot H(s)$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
desired ω_c
System: g6
Frequency (rad/sec): 0.296
Magnitude (dB): -9.43
$K = \dfrac{1}{\text{Log}^{-1}\left[-\frac{9.43}{20}\right]} = 2.96$

$$G_{c_4} = G_{c_3} * k = \frac{2.96\,(s+.1)^2}{s\,(s+1)(s+10)}$$

$G_{c_4} \cdot H(s)$



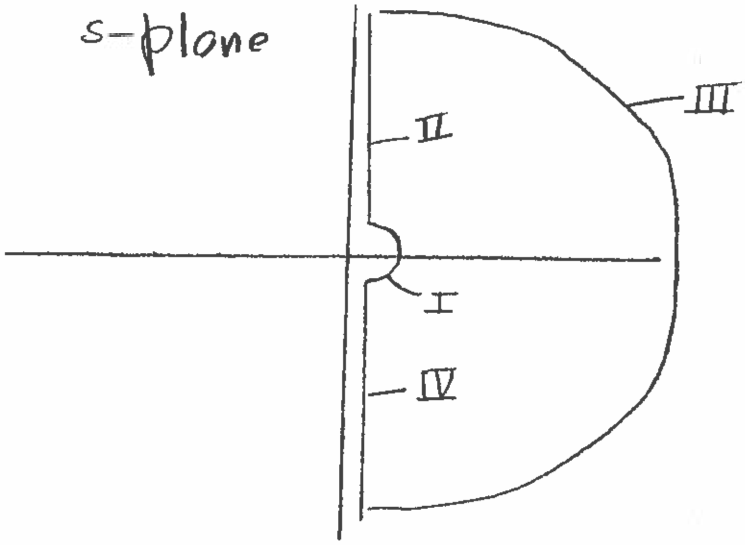

```
Transfer function:
  2.96 s^3 + 3.552 s^2 + 0.6216 s + 0.0296
-------------------------------------------
s^5 + 11.2 s^4 + 12.21 s^3 + 2.11 s^2 + 0.1 s

>> bode(g7,w)
>> zpk(g7)

Zero/pole/gain:
  2.96 (s+1) (s+0.1)^2
------------------------
s (s+10) (s+1) (s+0.1)^2
```

```
>> g7=minreal(g7)

Transfer function:
   2.96
----------
s^2 + 10 s
```

$n=1$

$m=0$

$p=0$



I: $\frac{1}{\varepsilon} e^{-j(\phi)}$

mapped phase

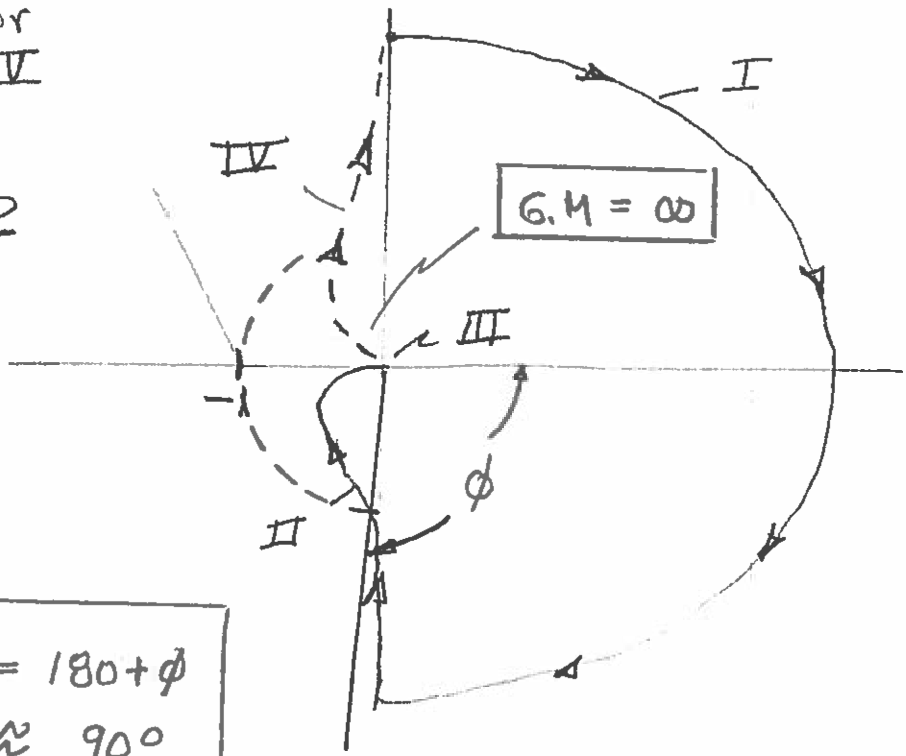
ϕ	$-\pi/2$	$\pi/2$
ϕ	0	0
ϕ	$\pi/2$	$-\pi/2$

Bode plot for sections II, IV

$N = 0 = Z - P$

but $P = 0$

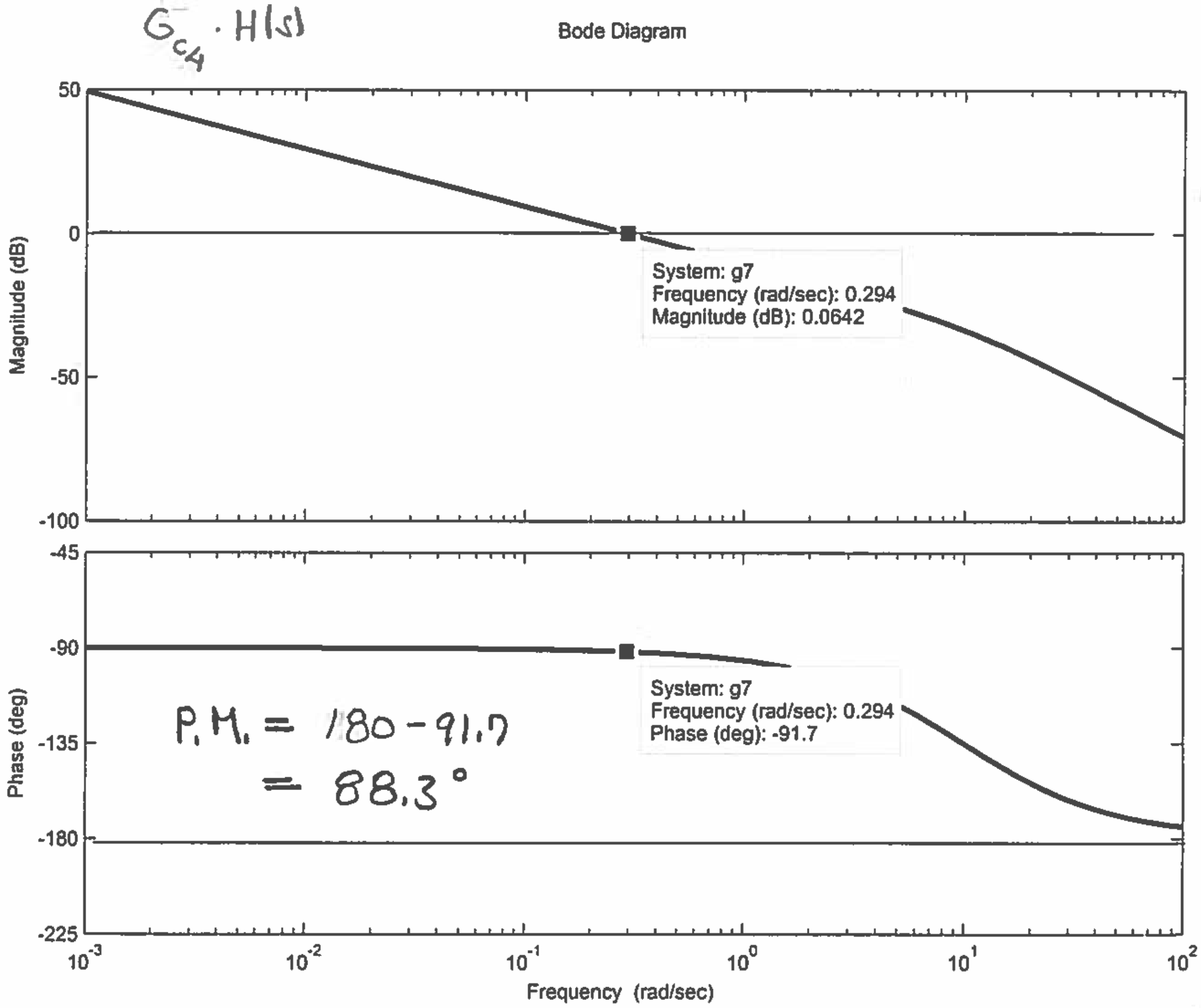
$\therefore Z = 0$



$PM = 180 + \phi$

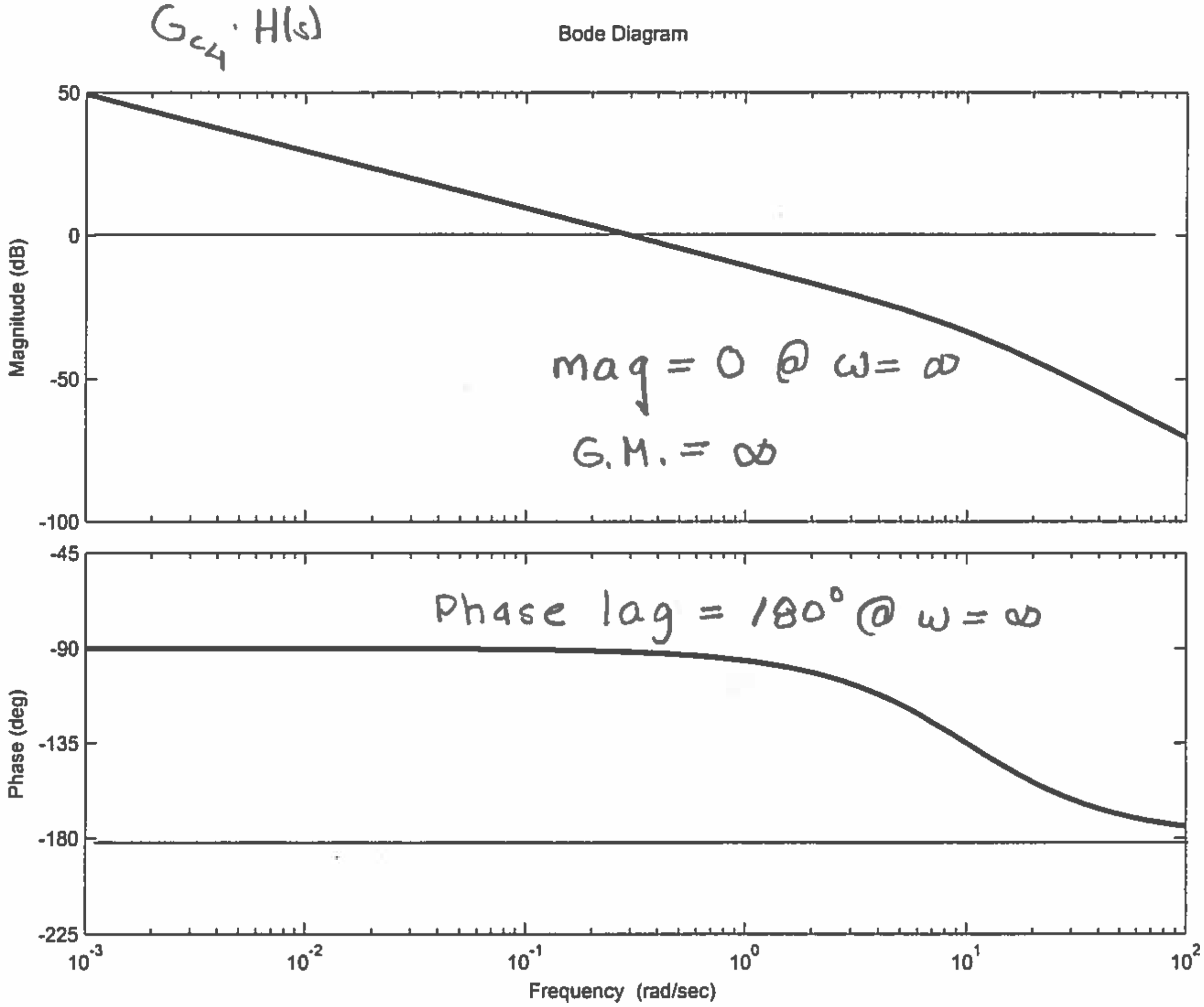
$\approx 90°$

$G_{c4} \cdot H(s)$



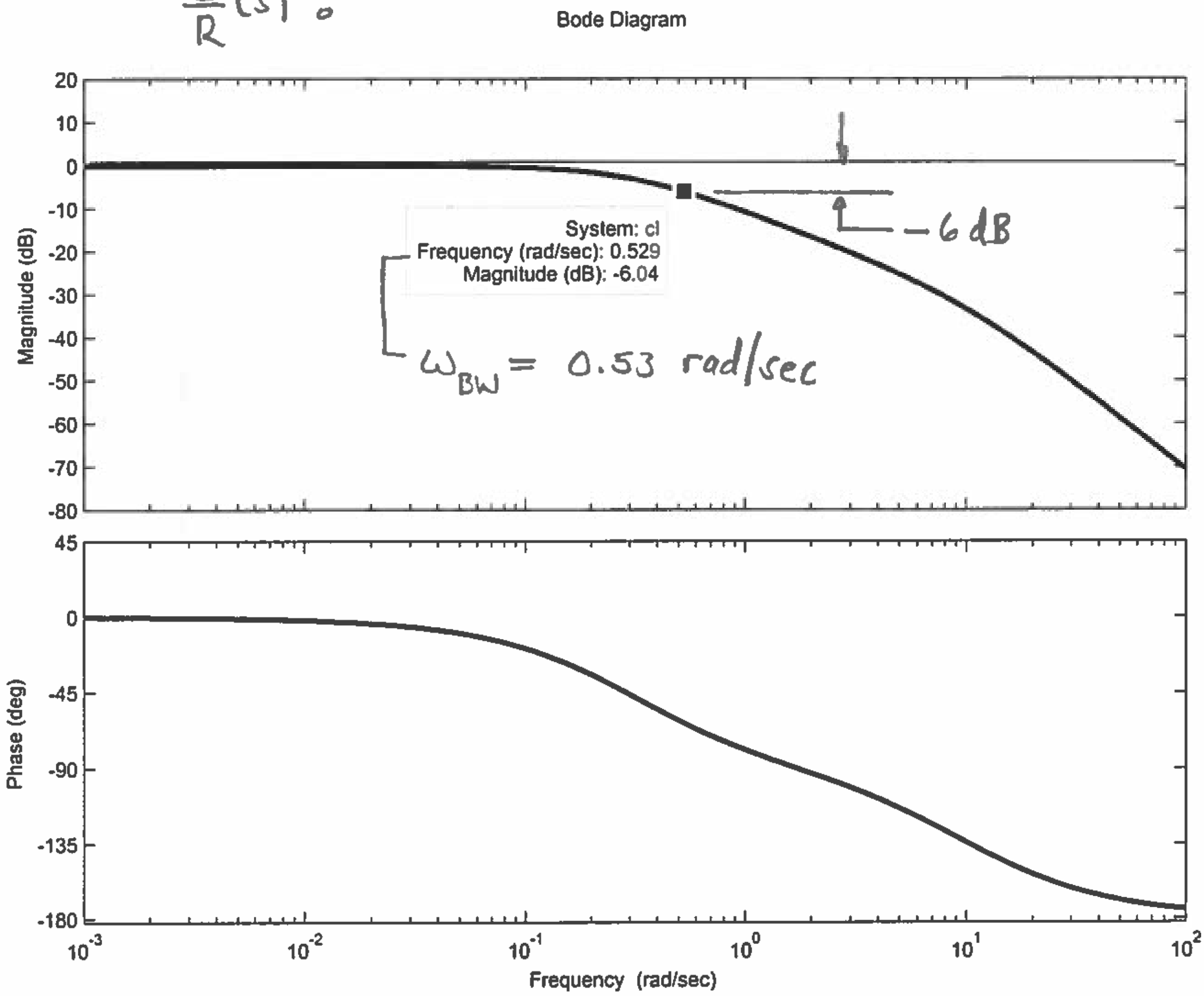
P.M. = 180 − 91.7
= 88.3°

$G_{c4} \cdot H(s)$


Bode Diagram
Magnitude (dB)
50
0
-50
-100
mag = 0 @ ω = ∞
G.M. = ∞
Phase (deg)
-45
-90
-135
-180
-225
Phase lag = 180° @ ω = ∞
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

closed-loop bandwidth using −6 dB criterion

$\frac{C}{R}(s)$:

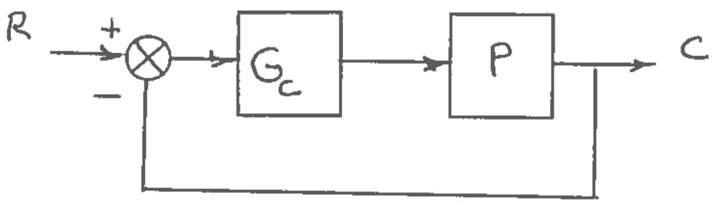


MAE – 275

Right Half Plane Poles and Zeros

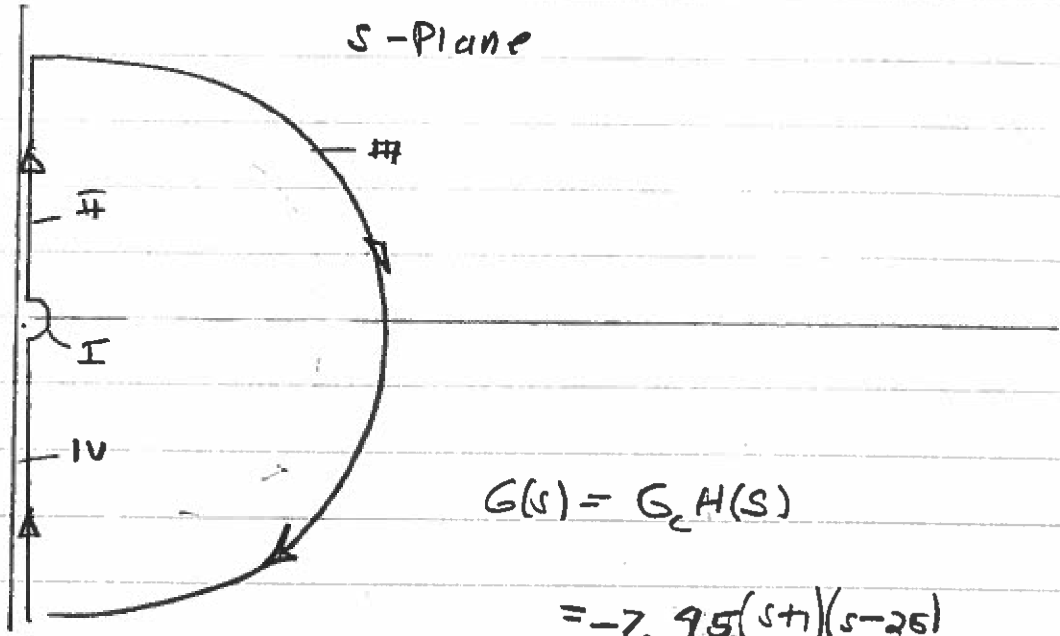
and

Loop Shaping



$$P(s) = \frac{-(s-25)}{s(s-0.5)}$$

$$G_c(s) = \frac{0.149(s+1)}{.02s+1} \quad \Big\} \text{ from loop-shaping}$$



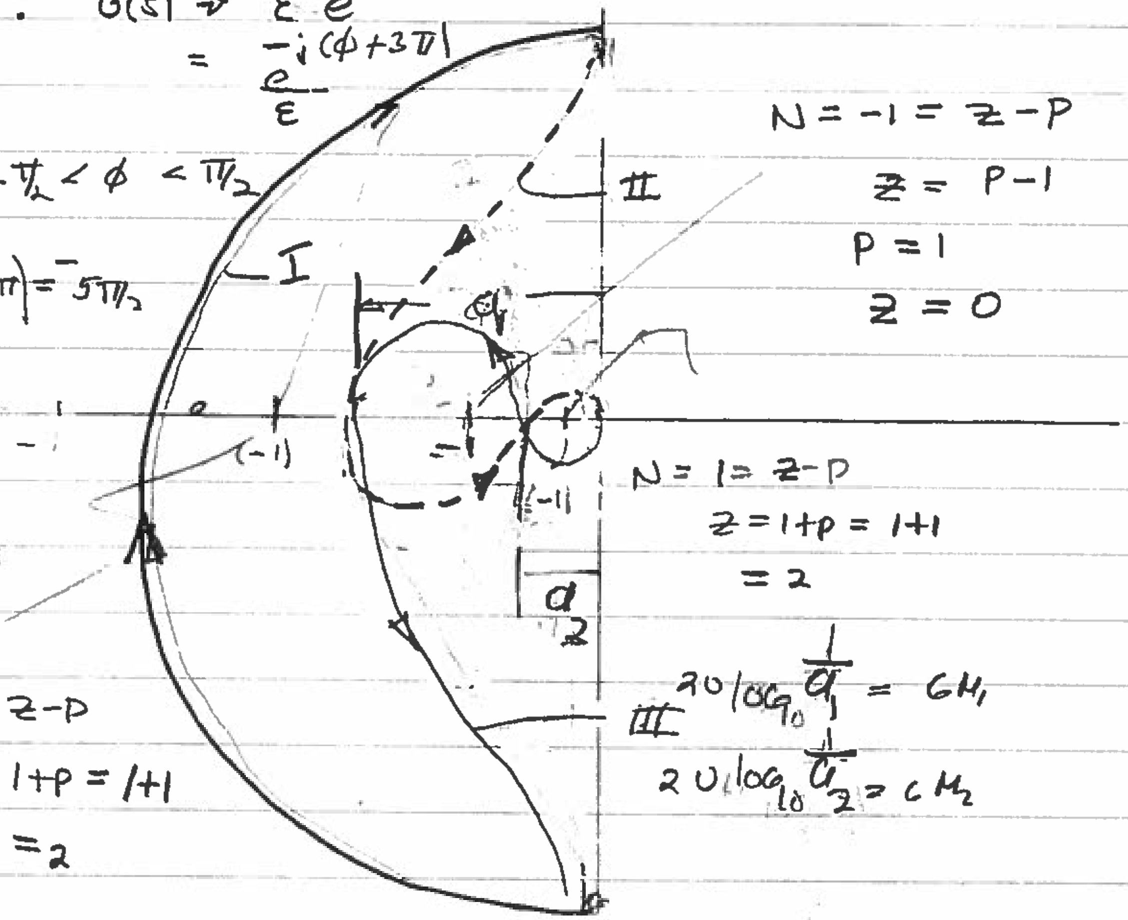
$$G(s) = G_c H(s)$$

$$= \frac{-7.95(s+1)(s-25)}{s(s-.5)(s+50)}$$

$$\mathrm{I}: \quad G(s) \Rightarrow \varepsilon^{-1} e^{-j(\phi+2\pi+\pi)} = \frac{e^{-j(\phi+3\pi)}}{\varepsilon}$$

$$-\pi/2 < \phi < \pi/2$$

$$(-\pi/2 + 3\pi) = -5\pi/2$$



$N = -1 = Z - P$

$Z = P - 1$

$P = 1$

$Z = 0$

$N = 1 = Z - P$

$Z = 1 + P = 1 + 1$

$= 2$

$20 \log_{10} \frac{1}{a_1} = GM_1$

$20 \log_{10} \frac{1}{a_2} = GM_2$

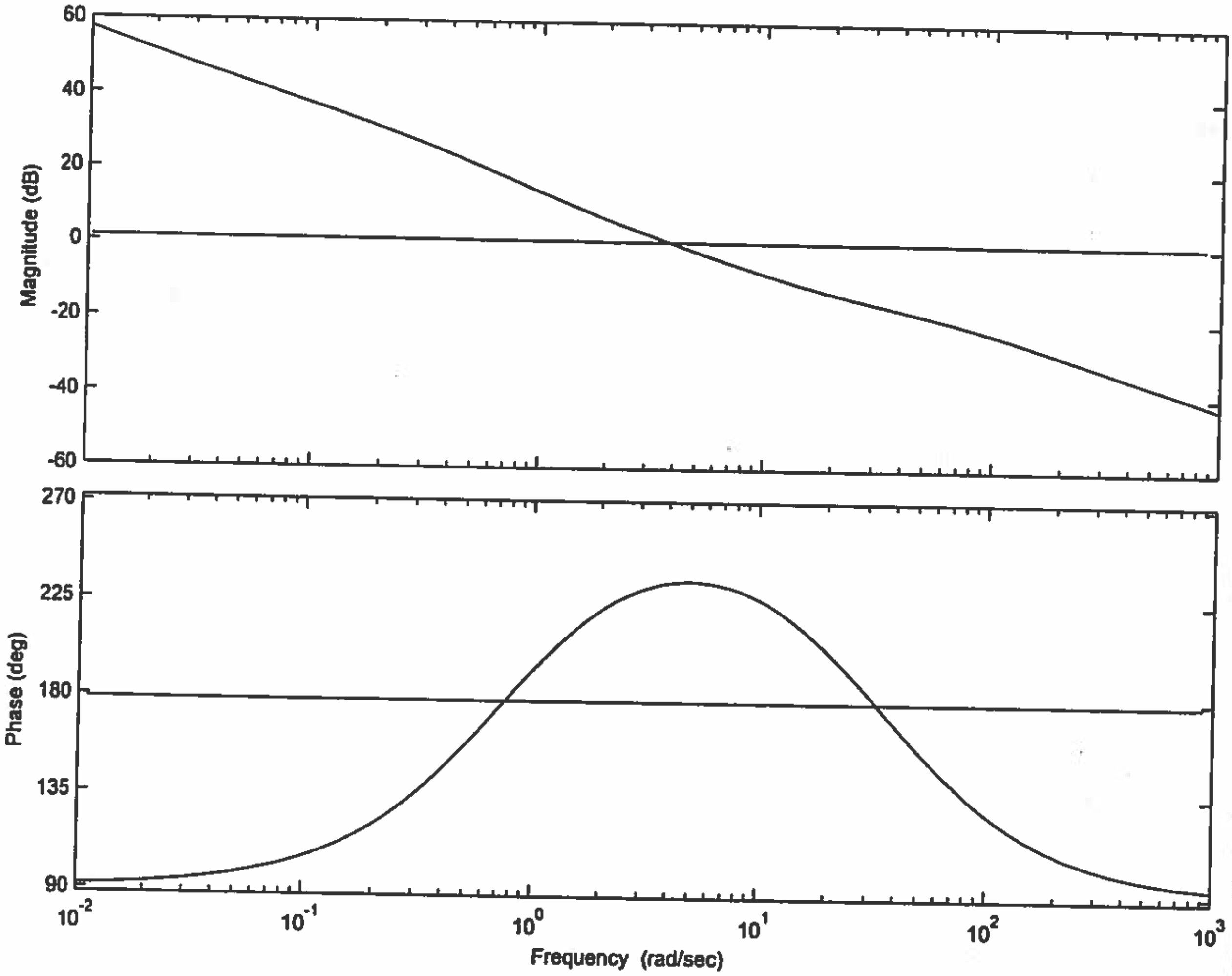
$N = 1 = Z - P$

$Z = 1 + P = 1 + 1$

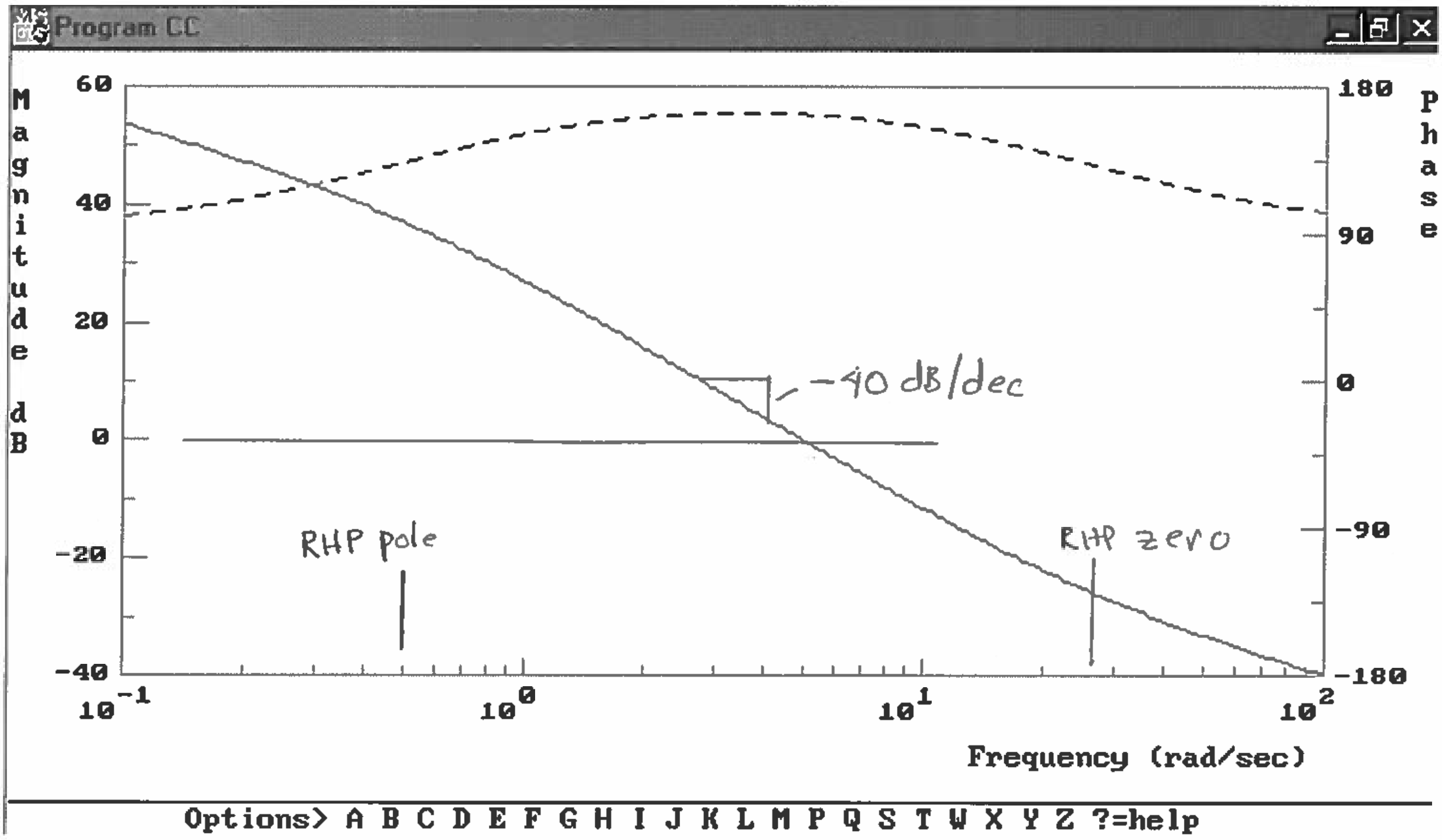
$= 2$

$$G_c = \frac{-7.45\,(s+1)(s-25)}{s(s-.5)(s+50)}$$

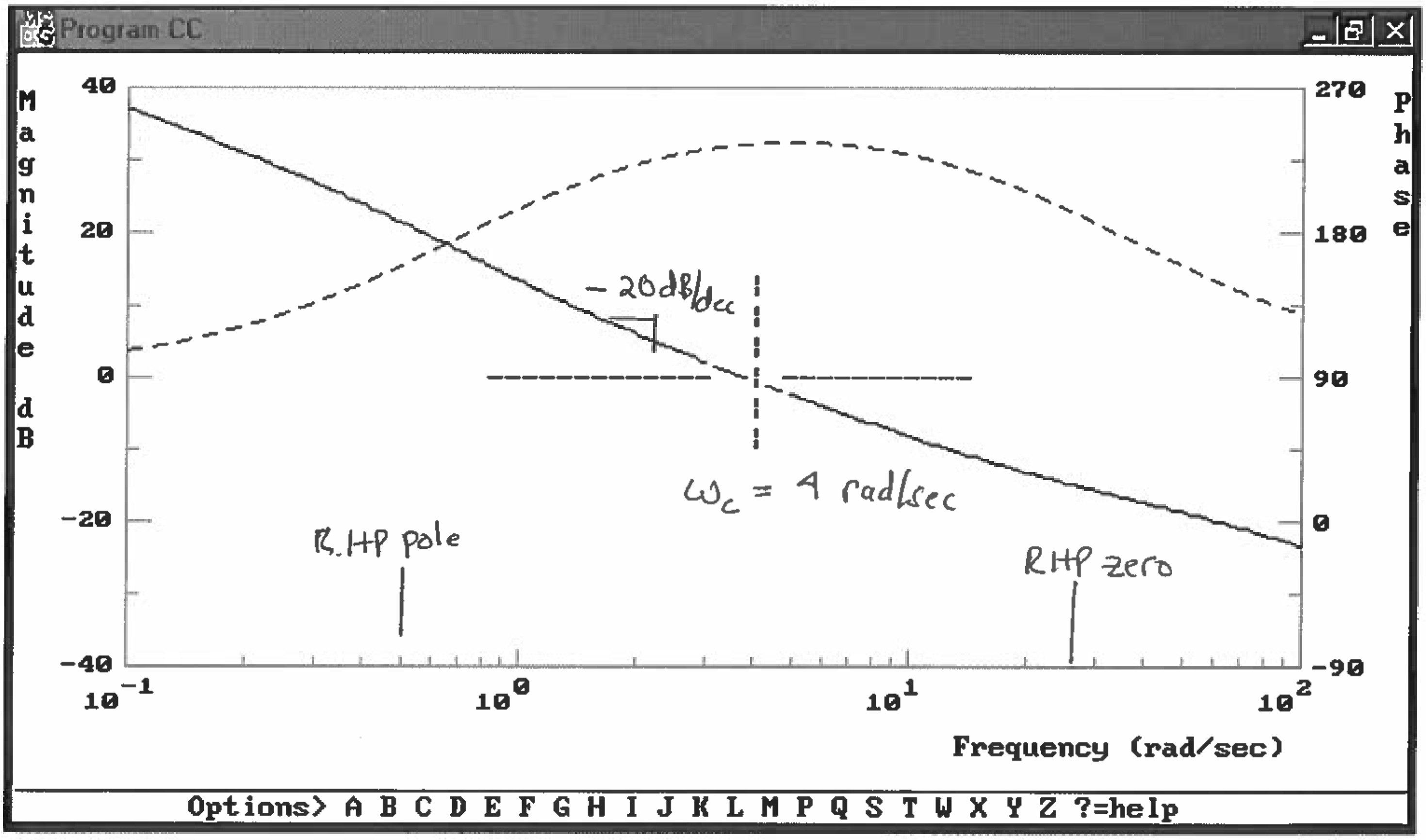
Bode Diagram

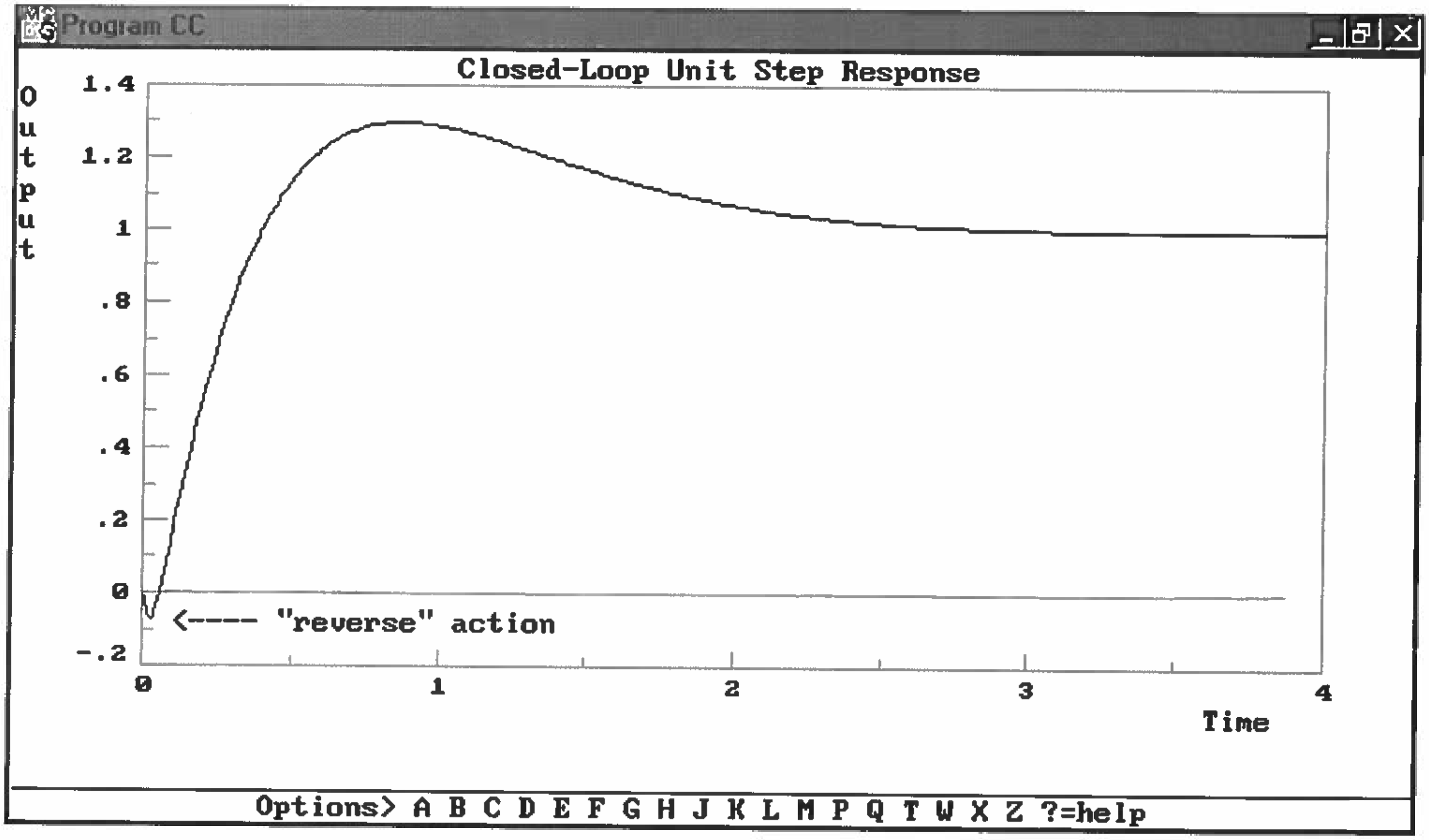


P(jω)

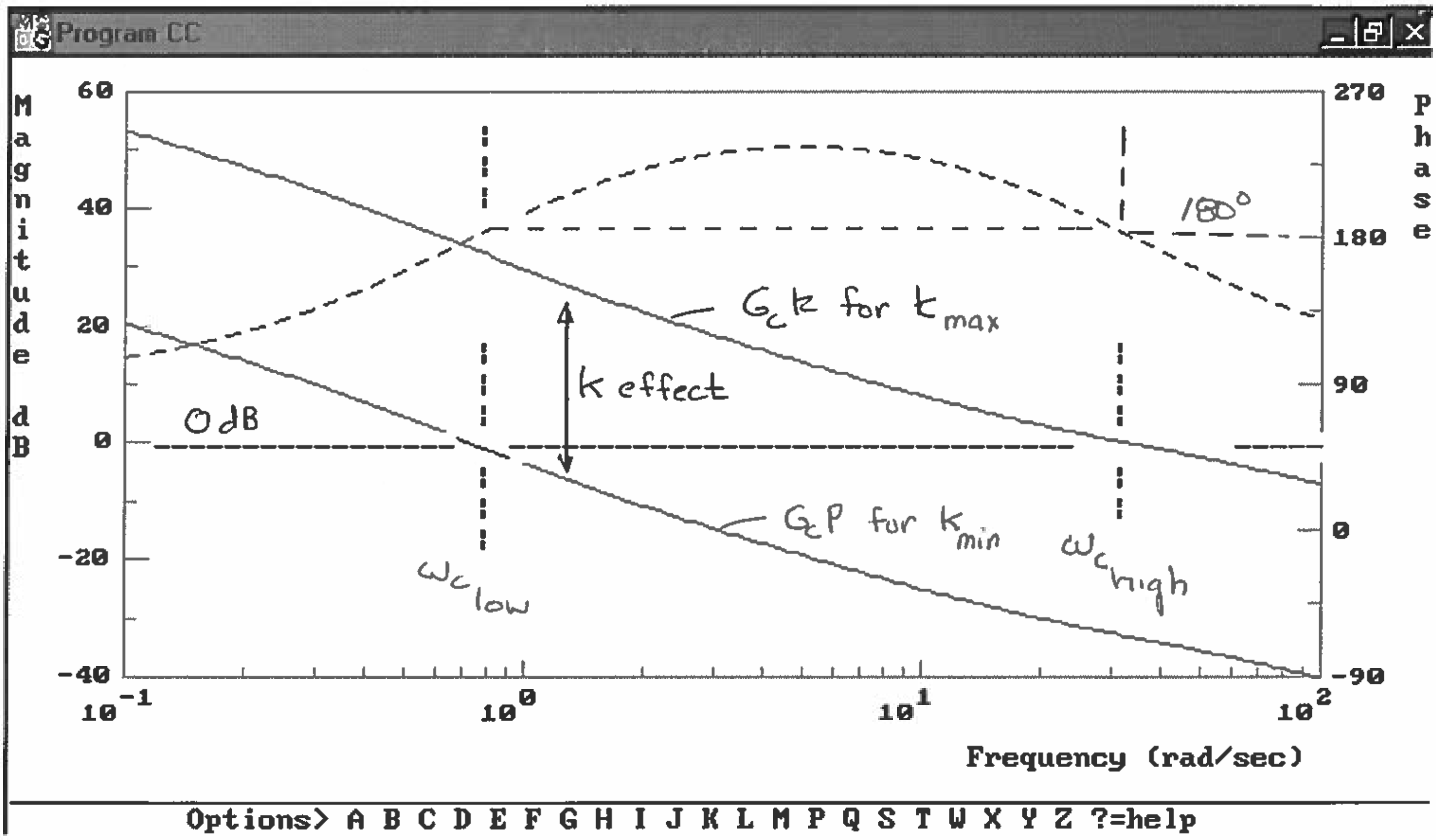

Program CC
Magnitude dB
Phase
60
40
20
0
-20
-40
180
90
0
-90
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
-40 dB/dec
RHP pole
RHP zero
Options> A B C D E F G H I J K L M P Q S T W X Y Z ?=help

$G_c(j\omega)\cdot P(j\omega)$

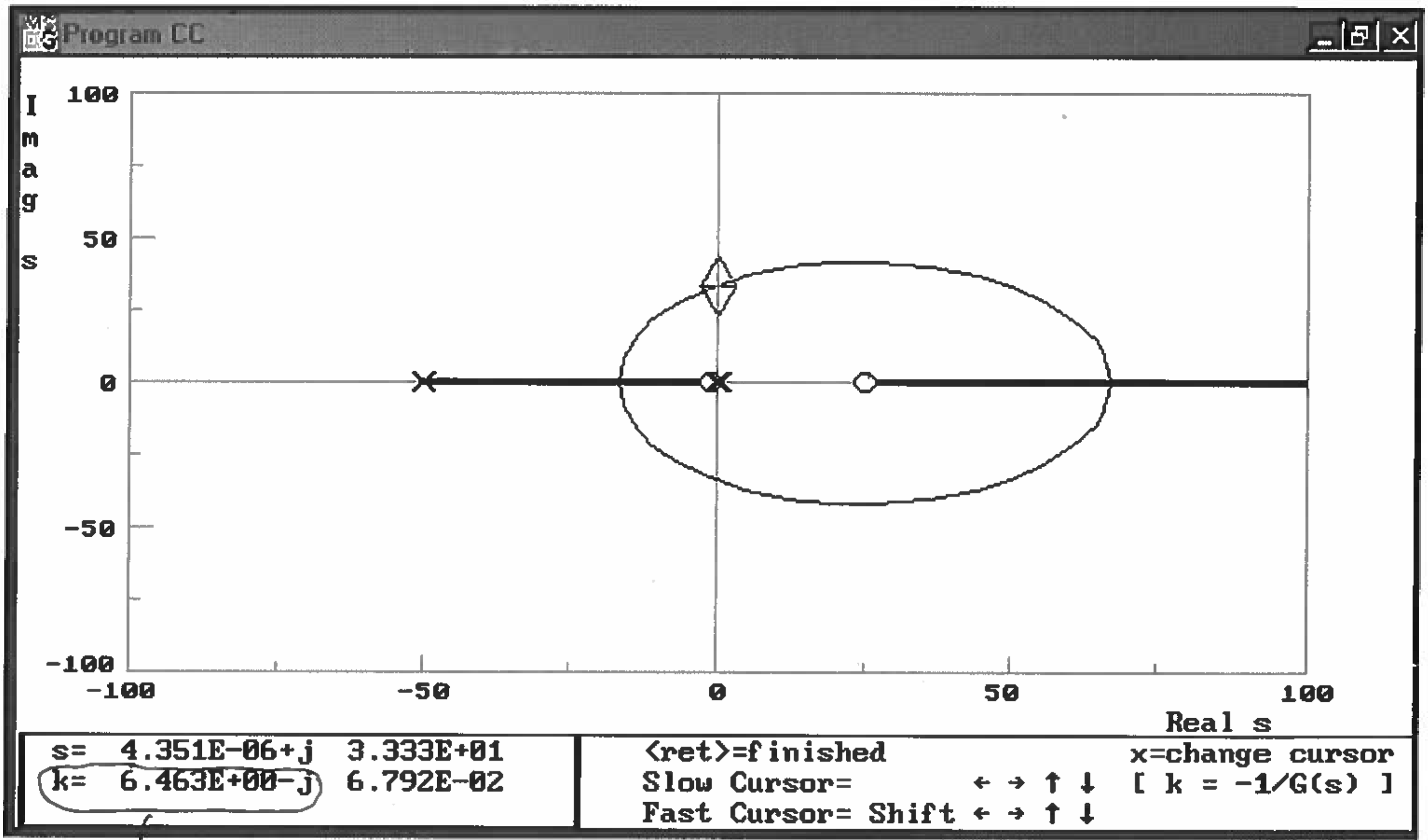



Program CC
Closed-Loop Unit Step Response
Output
1.4
1.2
1
.8
.6
.4
.2
0
-.2
<---- "reverse" action
0
1
2
3
4
Time
Options> A B C D E F G H J K L M P Q T W X Z ?=help

$G_c(j\omega) \cdot P(j\omega)$



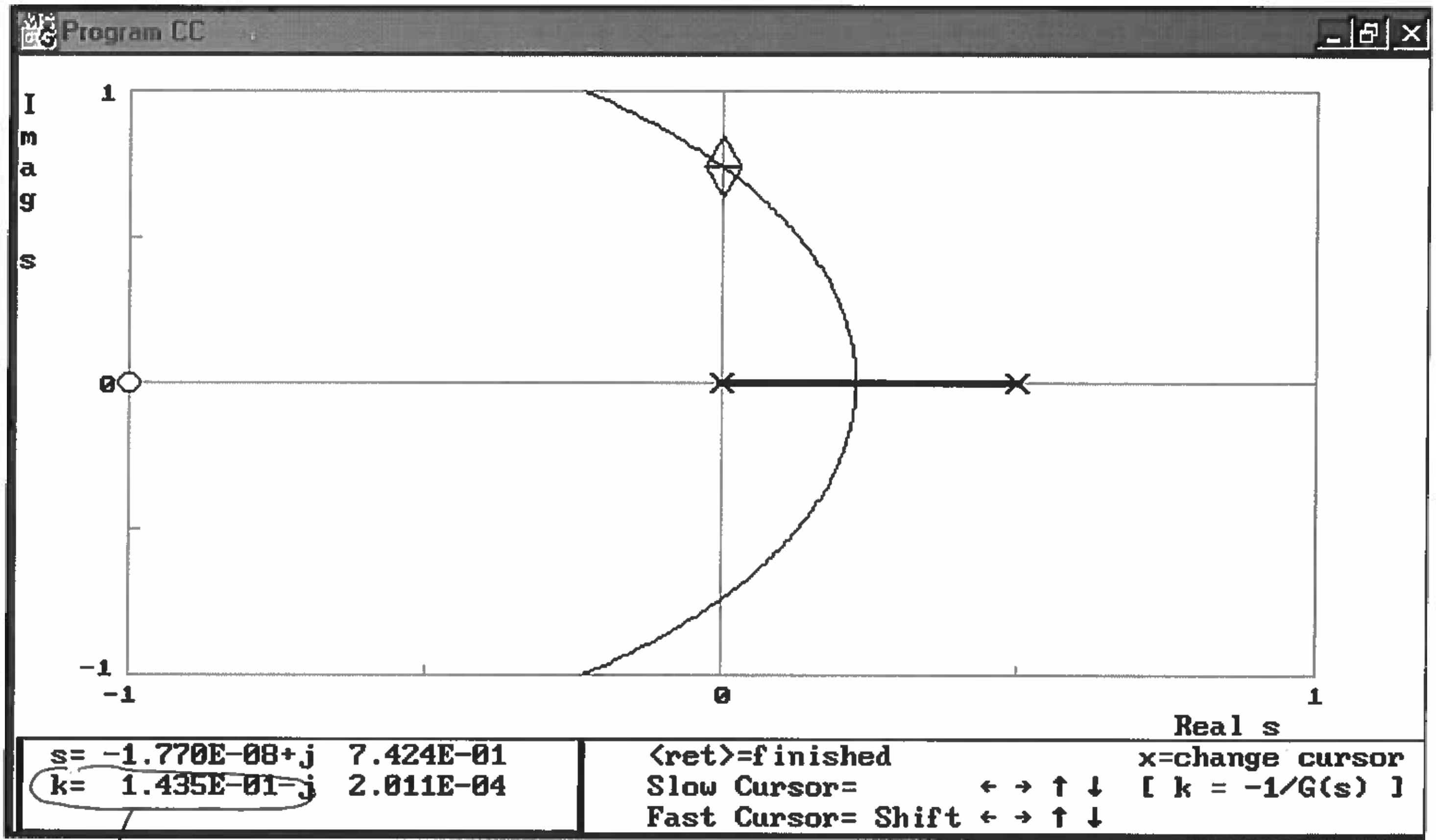
"Bottom Line" : ω_c **must be chosen larger than the magnitude of the RHP pole but smaller than the magnitude of the RHP zero.**



Maximum gain for stability, K_{max}

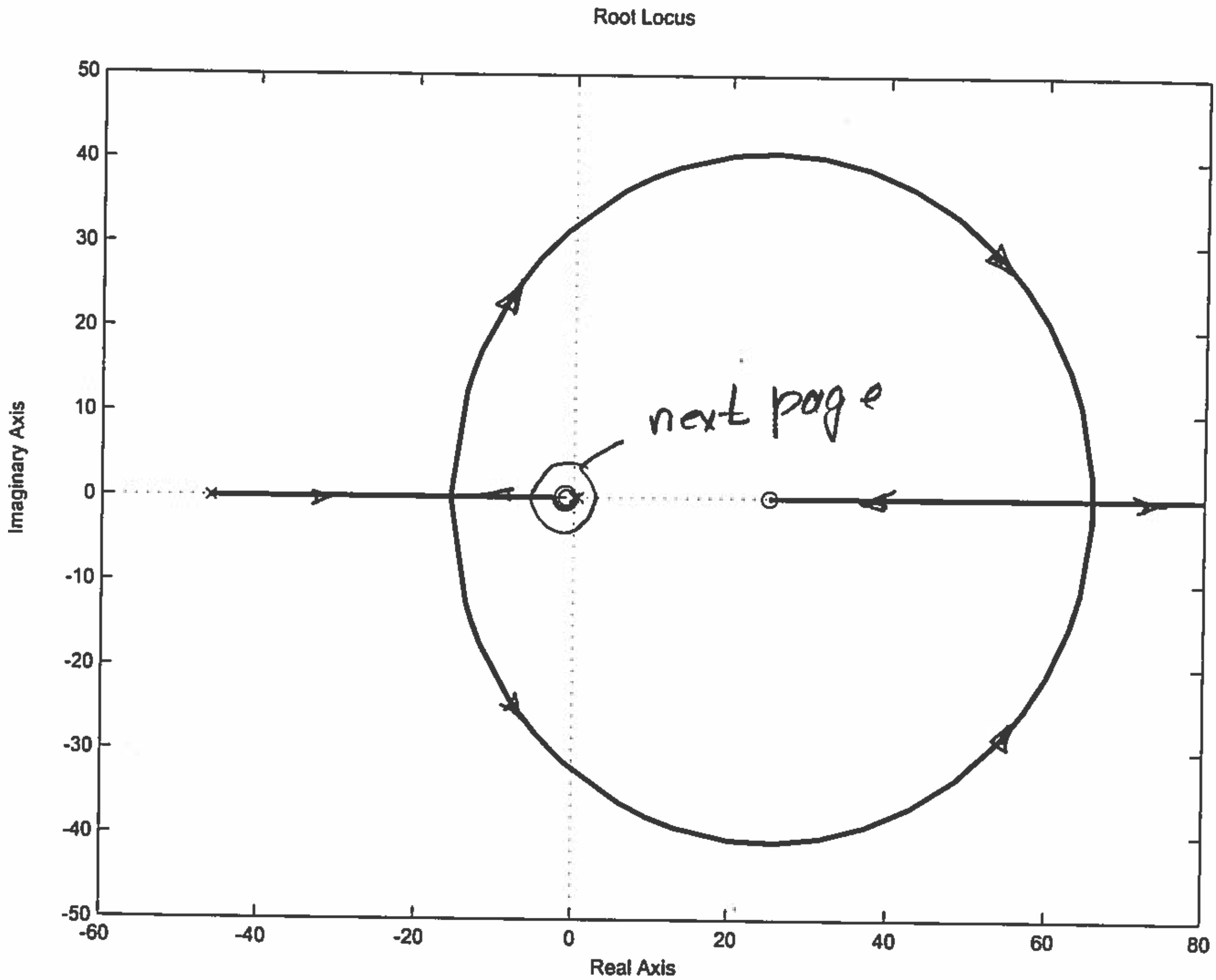
$$G_c = \frac{K \cdot (0.1991)(s+1)}{(.02s+1)}$$

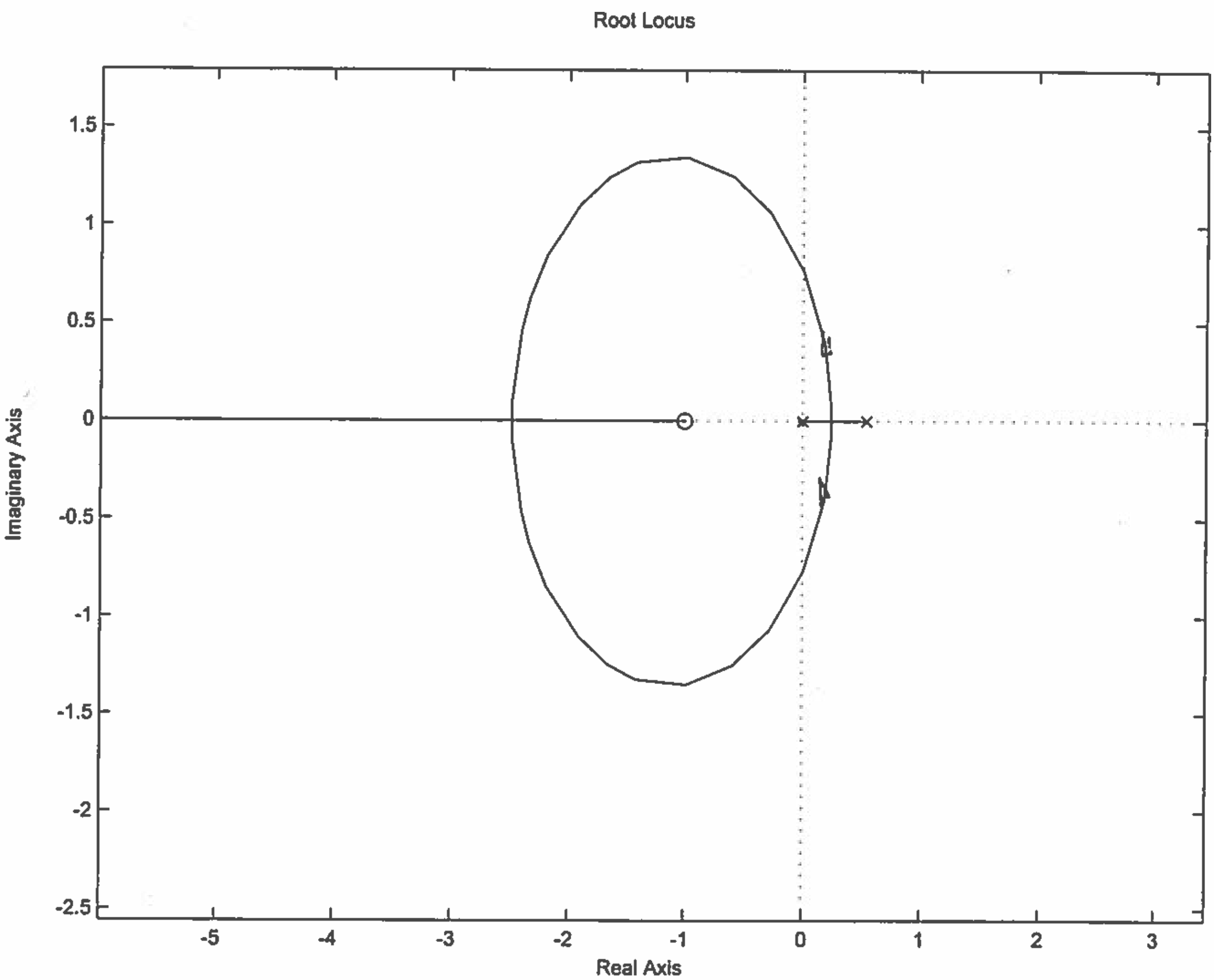
EXPANDED ORIGIN



minimum gain for stability, k_{min}

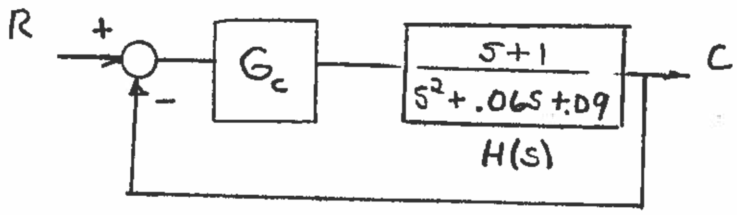
$$G_c = \frac{k(0.144)(s+1)}{(.02s+1)}$$


Root Locus
Imaginary Axis
Real Axis
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-40
-20
0
20
40
60
80
next page


Root Locus
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-5
-4
-3
-2
-1
0
1
2
3

MAE 275

A Second Loop Shaping Example



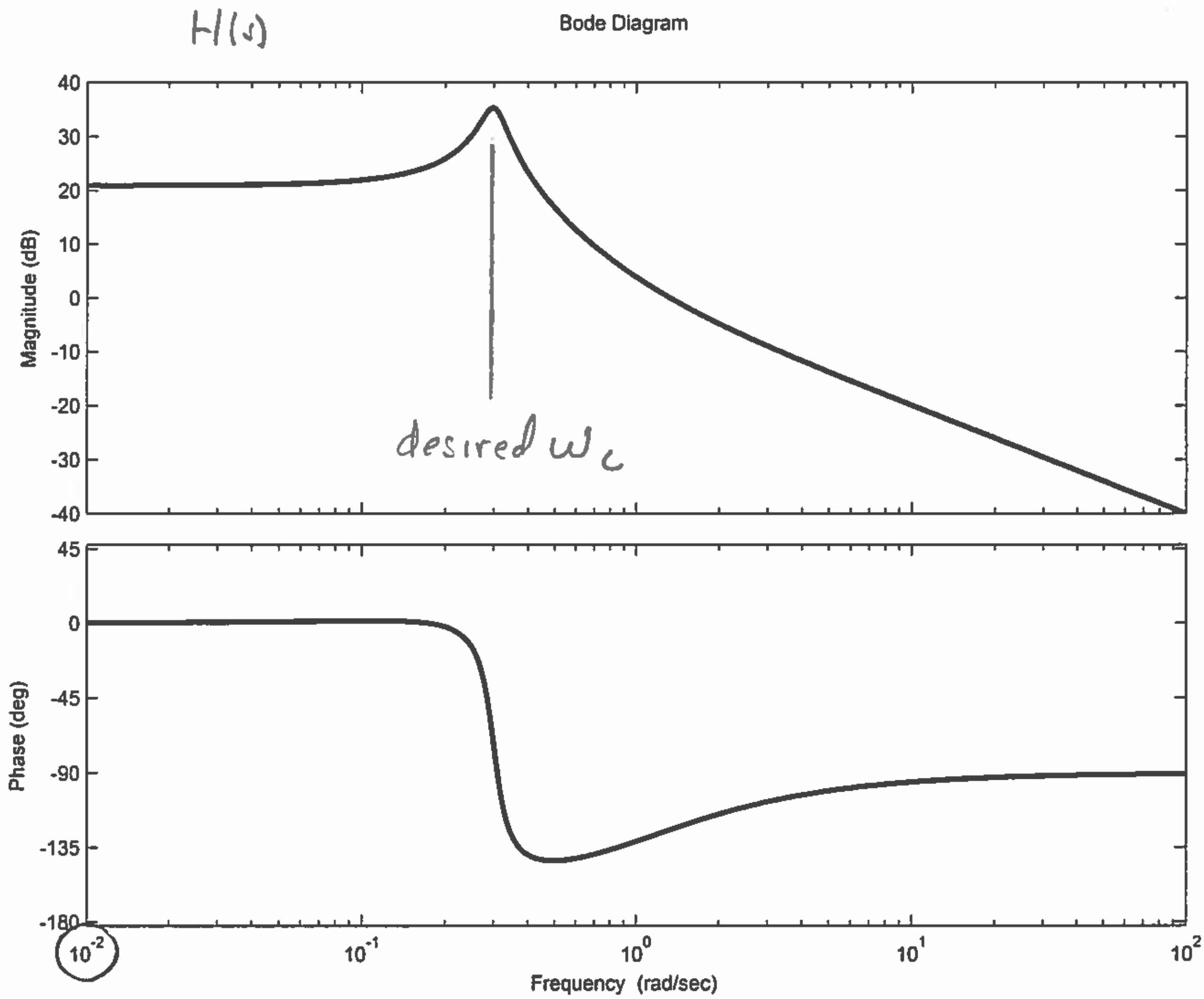
Performance Requirements

1.) Bandwith 0.3 rad/sec

2.) Gain margin > 20 dB

3.) Phase margin > 45 deg

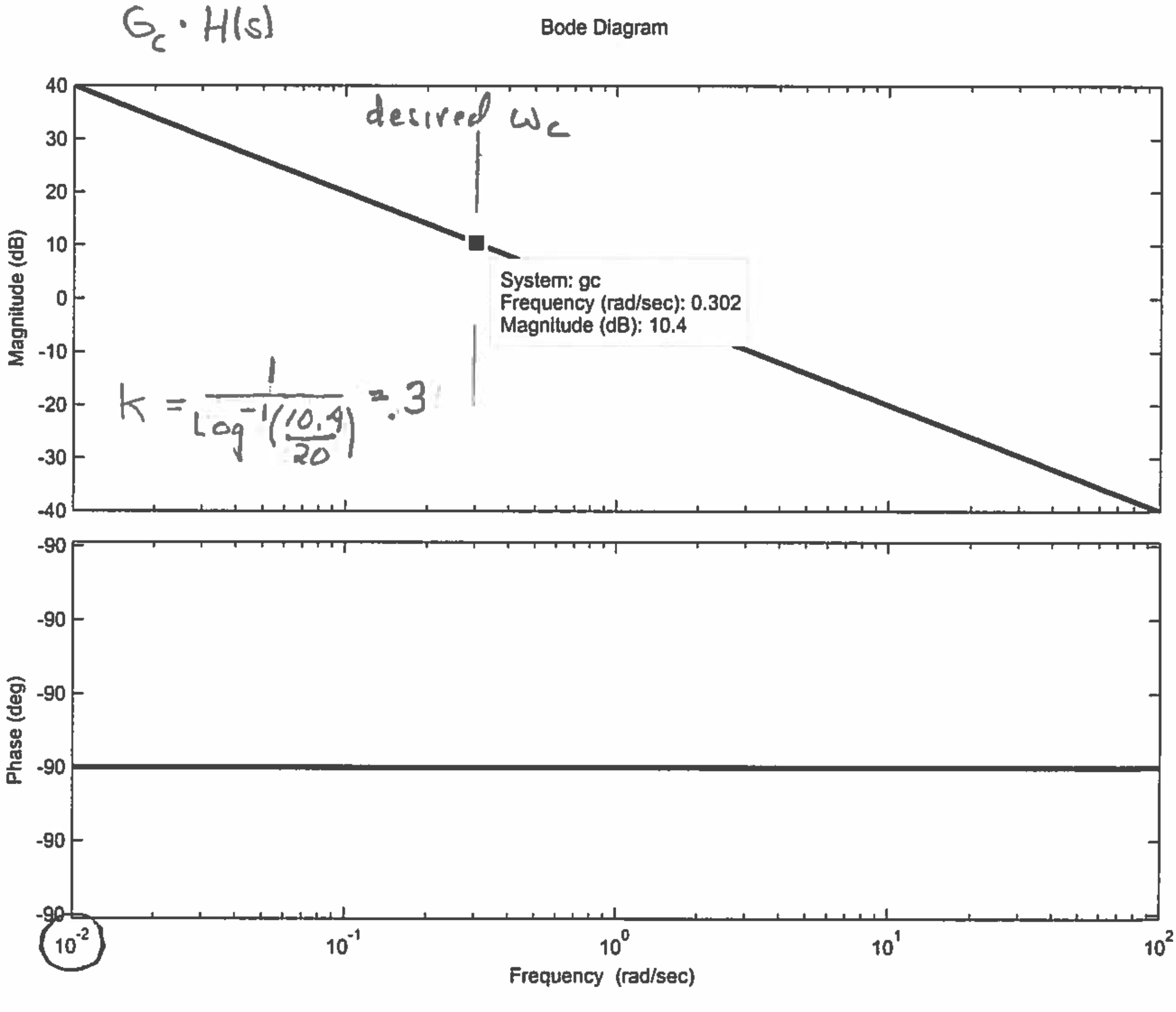
4.) Type 1 system (0 steady state error to a step input)


H(s)
Bode Diagram
40
30
20
10
0
-10
-20
-30
-40
Magnitude (dB)
desired ω_c
45
0
-45
-90
-135
-180
Phase (deg)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Exact Cancellation of lighty damped poles

$$G_c = \frac{(s^2 + .06s + .09)}{s(s+1)}$$

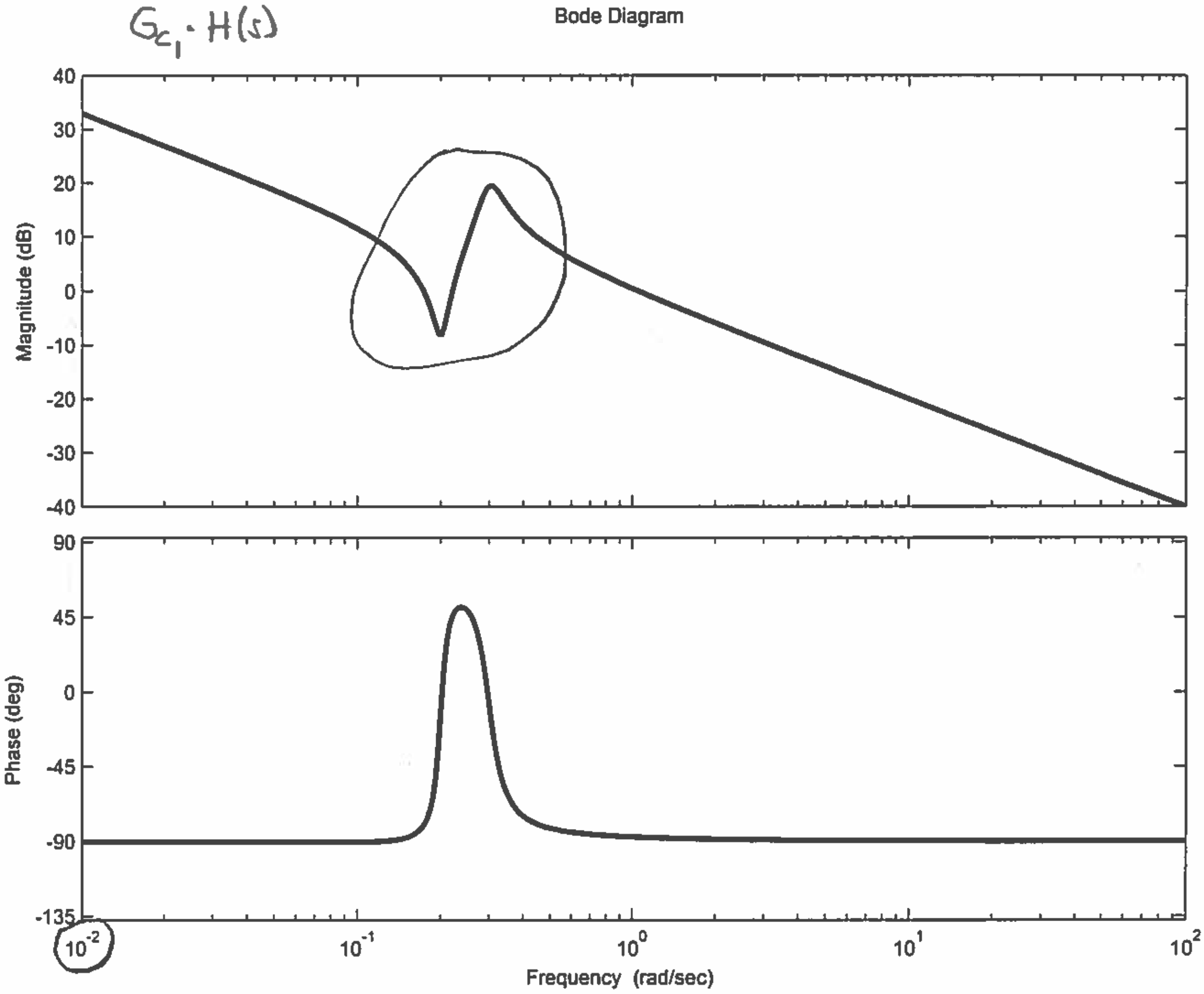
$G_c \cdot H(s)$



Effect of inexact cancellation of lightly damped poles

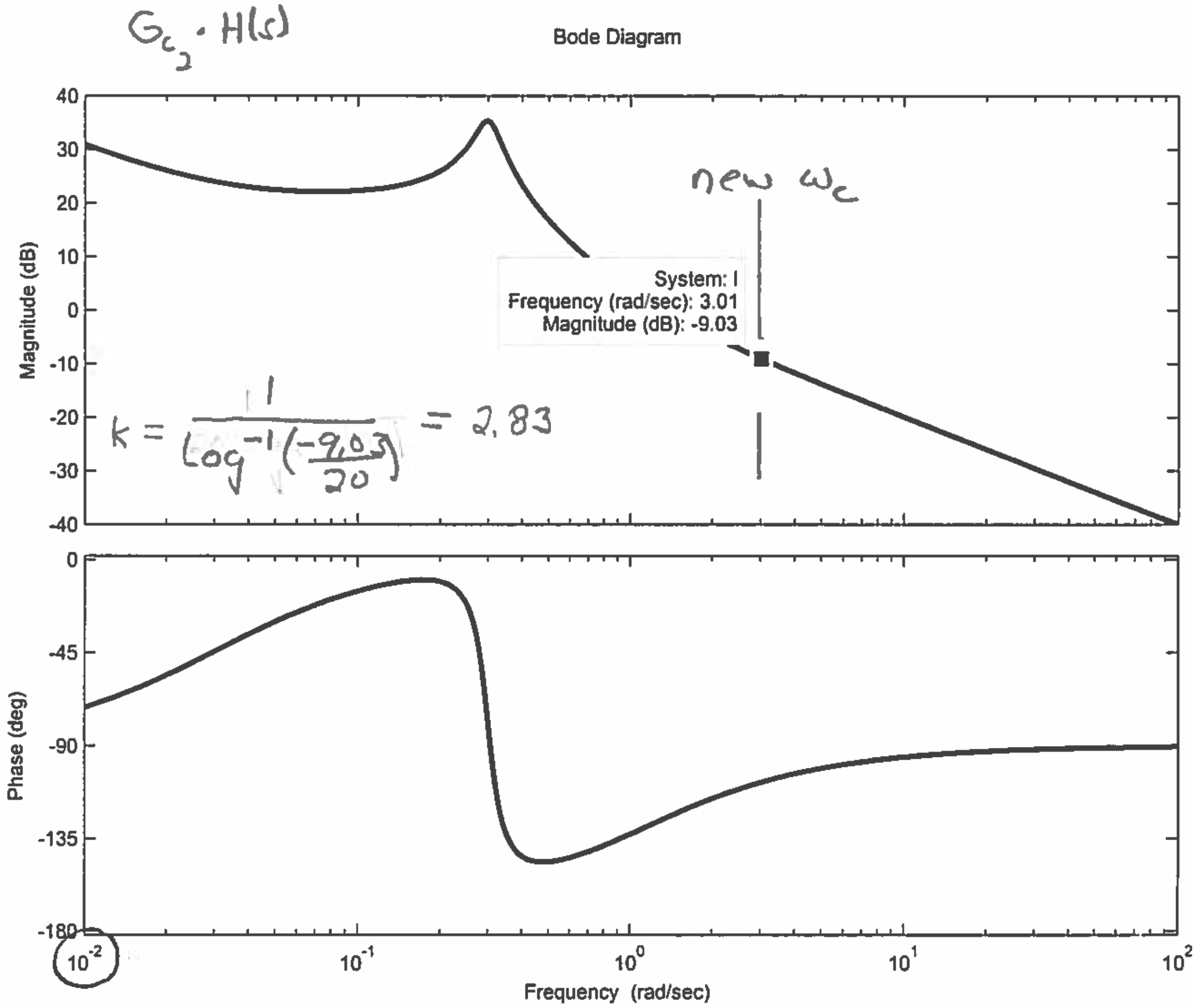
$$G_{c_1} = \frac{(s^2 + .02s + .04)}{s(s+1)}$$

$G_{c_1} \cdot H(s)$

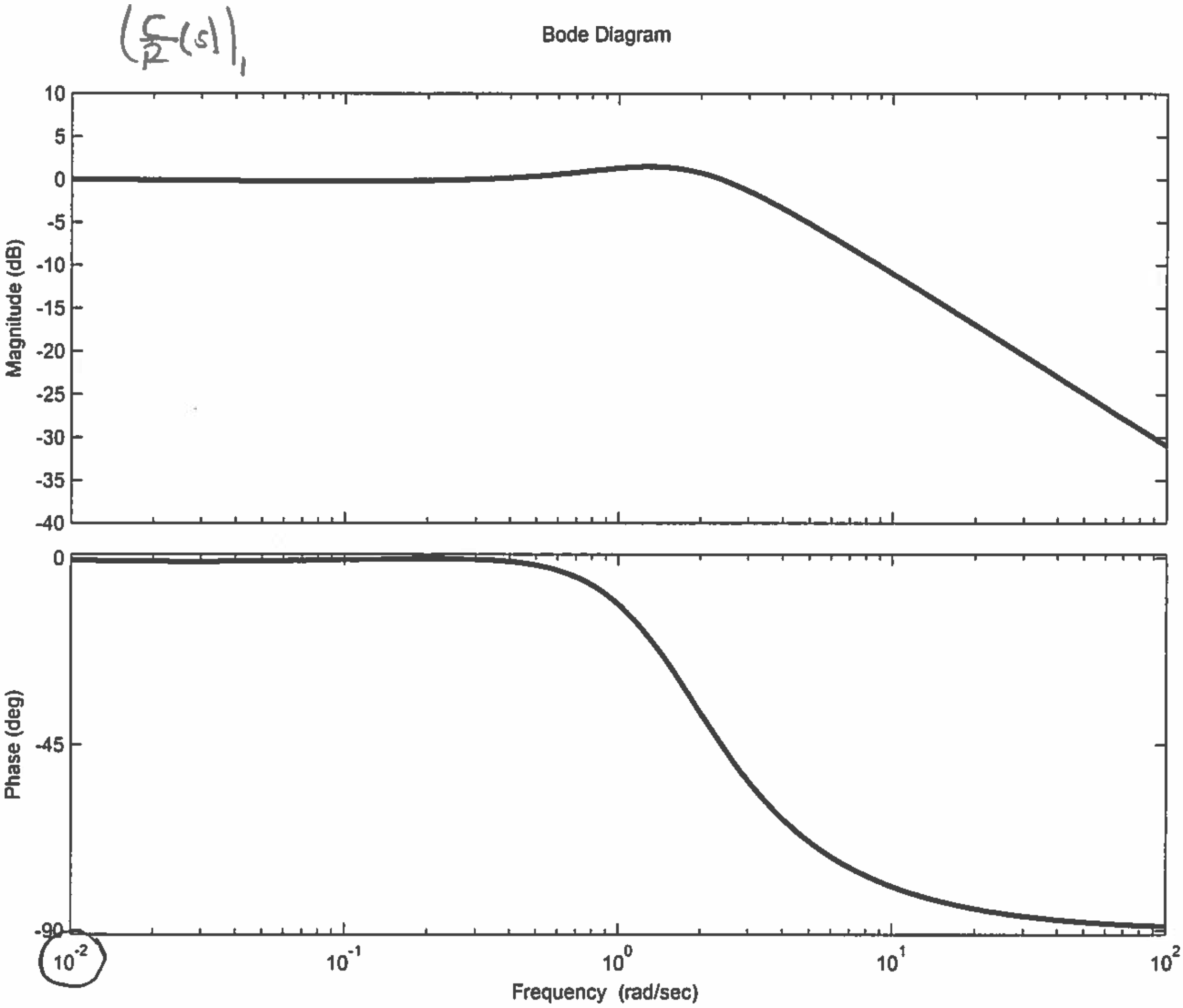


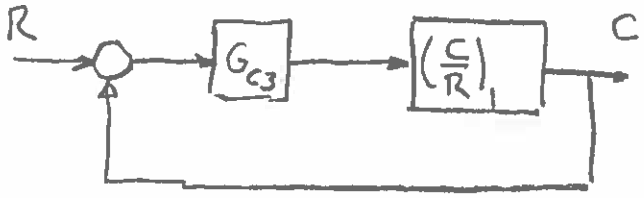
$$G_{c_2} = \frac{(s+.03)}{s} \times K$$

$G_{c_2} \cdot H(s)$



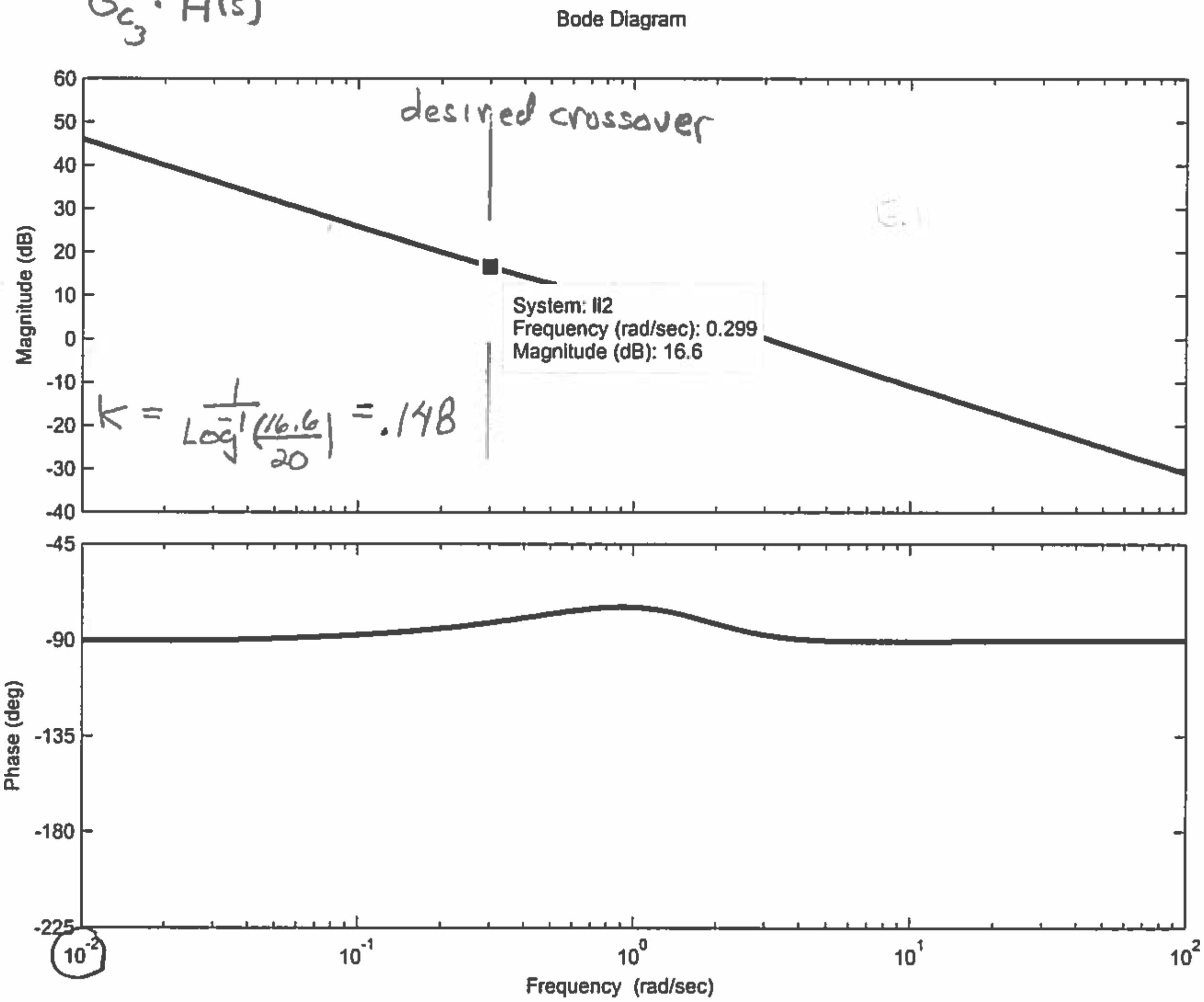
$\left(\frac{C}{R}(s)\right)_1$



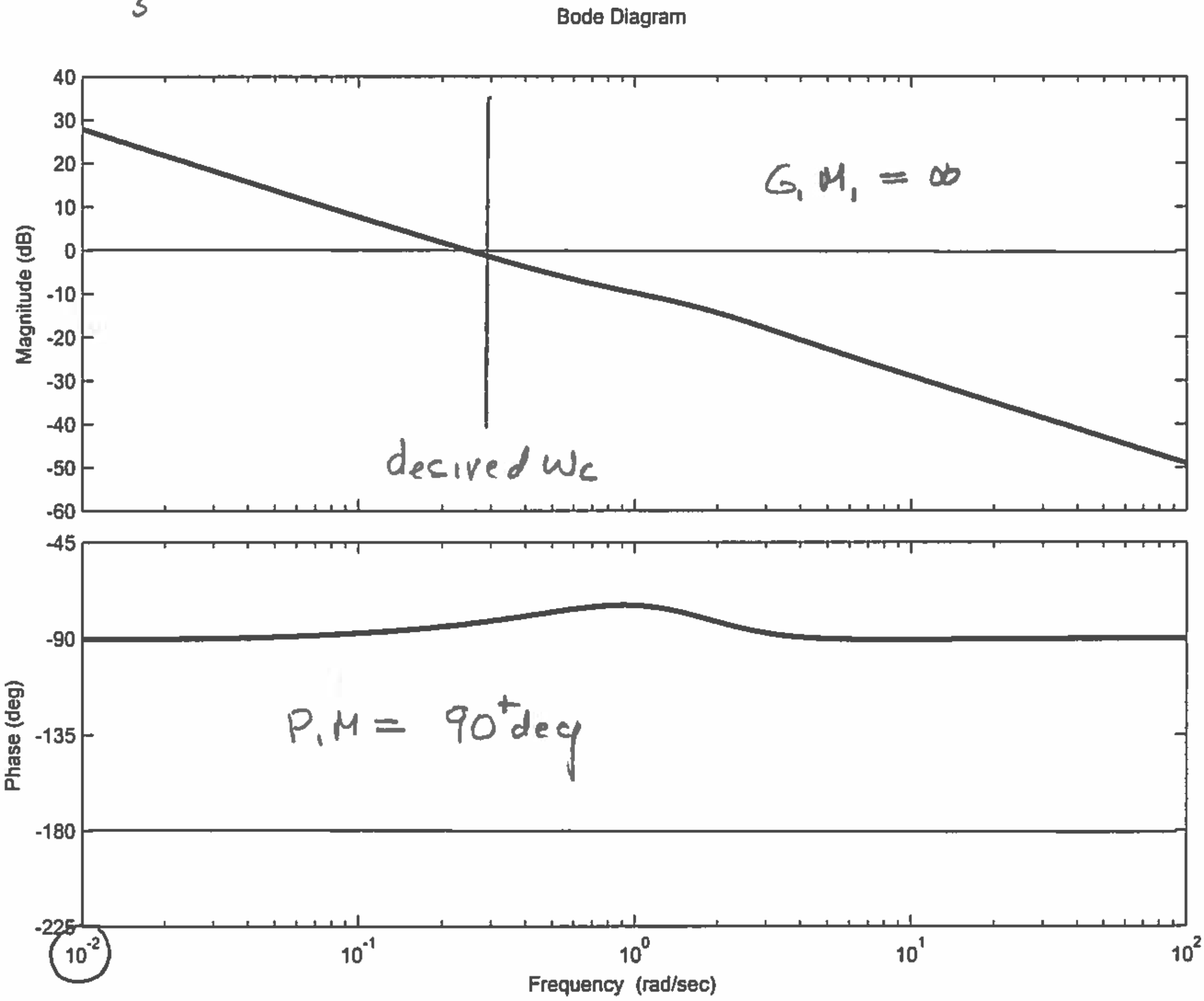


$$G_{c_3} = \frac{(s+2)}{s} \cdot K$$

$G_{c_3} \cdot H(s)$



$G_{c_3} \cdot H(s)$



Bode Diagram
Magnitude (dB)
40
30
20
10
0
-10
-20
-30
-40
-50
-60
G.M. = ∞
desired ωc
Phase (deg)
-45
-90
-135
-180
-225
P.M = 90⁺deg
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

closed-loop bandwidth using -6 dB criterion

$\frac{C}{R}(s)$