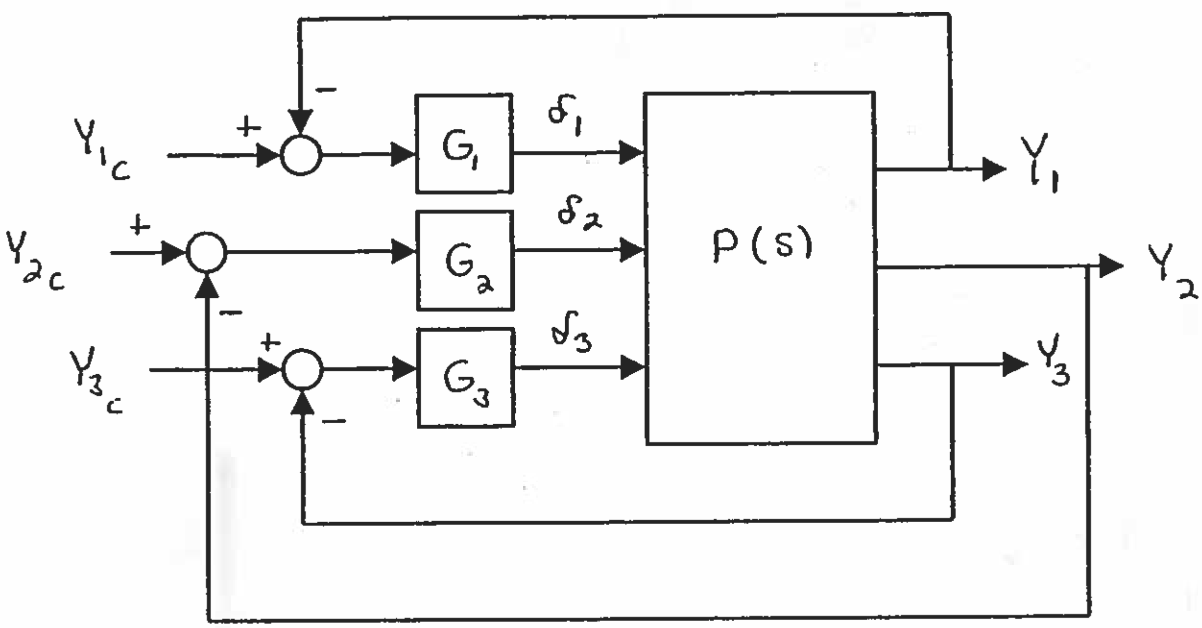
MAE - 275

Coupling Numerators for a 3 x 3 Control System



$$\dot{x} = Ax + B\delta$$
$$y = Cx + D\delta$$
$$P(s) = [C(sI - A)^{-1}B + D]_{3x3}$$

$$\left.\frac{Y_1}{\delta_1}\right|_{Y_2 \to \delta_2;\, Y_3 \to \delta_3} = \frac{N_{\delta_1}^{Y_1} + G_2 N_{\delta_1\delta_2}^{Y_1Y_2} + G_3 N_{\delta_1\delta_3}^{Y_1Y_3} + G_2 G_3 N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3}}{\Delta + G_2 N_{\delta_2}^{Y_2} + G_3 N_{\delta_3}^{Y_3} + G_2 G_3 N_{\delta_2\delta_3}^{Y_2Y_3}}$$

2

Rules for a 3 by 3 system:

1.) The effective denominator is equal to

   a.) The open-loop denominator

   b.) plus the sum of all the remaining compensator transfer functions, each one multiplied by the appropriate type 0 coupling numerator[1]

   c.) plus the sum of all the remaining compensator transfer functions taken two at a time, each pair multiplied by the appropriate type 1 coupling numerator[2]

2.) The effective numerator is equal to

   a.) the open-loop numerator (type 0 coupling numerator)

   b.) plus the sum of all the remaining compensator transfer functions, each one multiplied by the appropriate type 1 coupling numerator[3]

   c.) plus the usm of all the compensator transfer functions taken two at a time, each pair multiplied by the appropriate type 2 coupling numerator[4]

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1 The appropriate type 0 coupling numerator is that associated with the multiplying compensator

2 The appropriate type 1 coupling numerator is that associated with the pair of multiplying compensators

3 The appropriate type 1 coupling numerator is that associated with the input-output pair on the left hand side of the equation and that associated with the multiplying compensator

4 The appropirate type 2 coupling numerator is that associated with the input-output pair on the left hand side of the equation and that associated with the pair of multiplying compensators

$$Y(s) = P(s)\delta(s)$$

$$P^{-1}(s)Y(s) = I\delta(s)$$

Let $E(s) = P^{-1}(s) = \dfrac{[P_{adj}(s)]^T}{\det P(s)} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$

Then

$$E(s)Y(s) = I\,\delta(s)$$

and

$$N_{\delta_1}^{Y_1} = \begin{vmatrix} 1 & e_{12} & e_{13} \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{vmatrix} \quad N_{\delta_1\delta_2}^{Y_1Y_2} = \begin{vmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & e_{33} \end{vmatrix} \quad N_{\delta_2\delta_3}^{Y_2Y_3} = \begin{vmatrix} e_{11} & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & 0 & 1 \end{vmatrix}$$

$$N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

With loops 2 and 3 "tightly constrained" with high bandwidth feedback loops, i.e. with

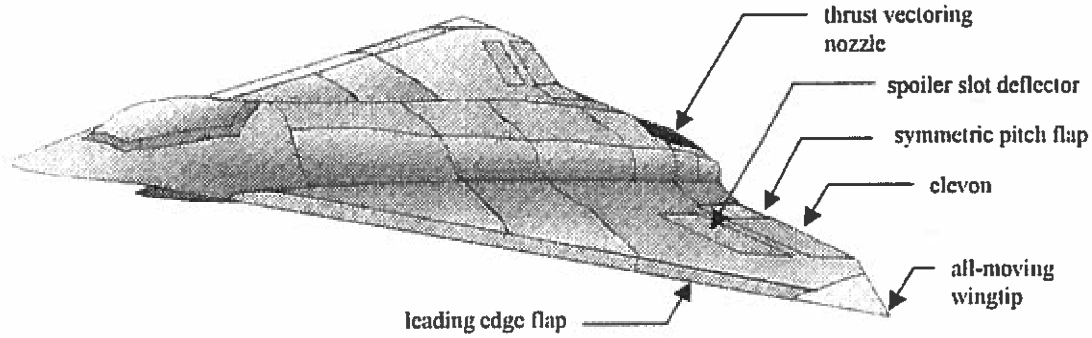
$$|G_2(s)| >> 1 \quad \text{and} \quad |G_3(s)| >> 1$$

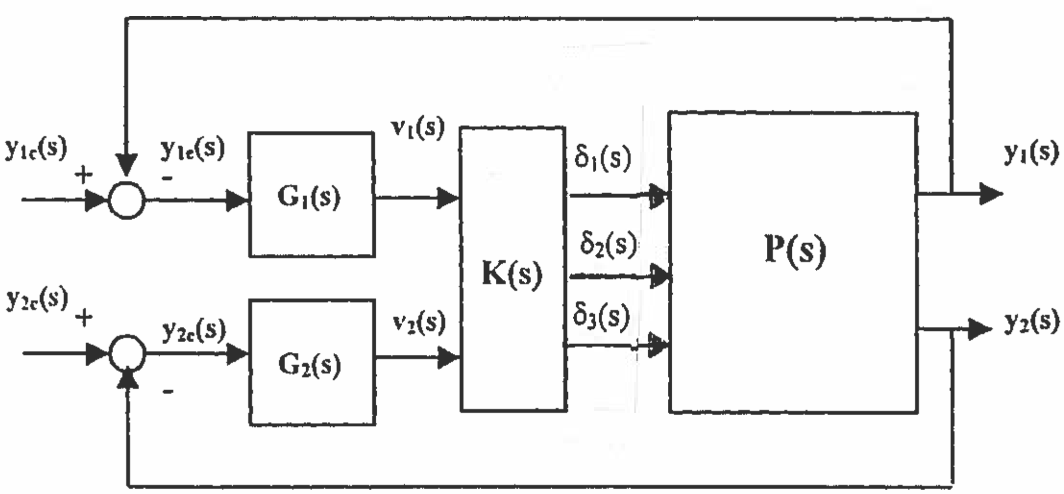
$$\left.\frac{Y_1}{\delta_1}\right|_{Y_2\to\delta_2;\,Y_3\to\delta_3} \approx \frac{G_2G_3N_{\delta_1\delta_2\delta_3}^{Y_1Y_2Y_3}}{G_2G_3N_{\delta_2\delta_3}^{Y_2Y_3}} = \frac{1}{N_{\delta_2\delta_3}^{Y_2Y_3}} = \frac{1}{e_{11}} = \frac{\det P(s)}{\left([P_{adj}]^T\right)_{11}}$$

Values of s which make $\det P(s) = 0$ are called *transmission zeros* of P(s). For non-square systems, transmission zeros are values of s which cause the matrix P(s) to lose rank.

Coupling Numerators Example

The aircraft shown below is called the Innovator Control Effector (ICE) vehicle. It has a number of novel control effectors. For this study, we will be interested in designing a lateral/directional flight control system ($G_1(s)$ and $G_2(s)$) as indicated in the block diagram below.





We will be interested in controlling $y_1(t)$ = yaw rate $r(t)$, $y_2(t)$ = roll rate $p(t)$ and using three of the many control effectors available. These three effectors are aileron, δ_1 = aileron deflection $\delta_a(t)$, $\delta_2(t)$ = differential wing tip deflection $\delta_{tip}(t)$, and and $\delta_3(t)$ = yaw thrust vectoring $\delta_{ytv}(t)$.

Assume a control distribution matrix given by

$$K = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \delta_1 = v_2 \qquad \delta_3 = v_1$$
$$\delta_2 = v_2$$

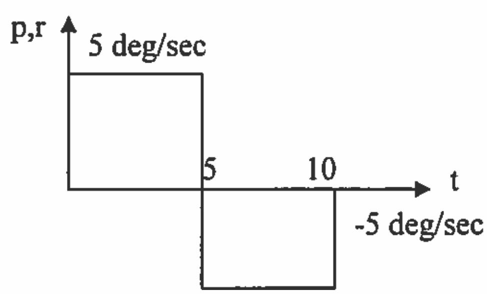
a.) Use the coupling numerator approach to do input-output pairing and to decide which loop to close first, The F-18 HARV example should offer some guidance in this approach.

b.) Using frequency-domain loop shaping, determine $G_1(s)$ and $G_2(s)$. The only design constraints are:

1.) A stable closed-loop vehicle

2.) At least a 5 rad/sec bandwidths in the p-loop and r-loops. Zero steady-state errors to step p(t) or r(t) commands.

3.) Simulate your vehicle and flight control system in Simulink. Determine the vehicle response (outputs and effector deflections) to the following pulsive inputs (**applied separately**). Before applying the inputs to the vehicle, filter each by passing them through a filter with transfer function $100/(s^2+14.14s+100)$.



The lateral/directional state-space equations are given on the following page for a flight condition of M = 0.3, Altitude = 15,000 ft (U_0 = 317.2 ft/sec). The state variables are, in order, v(t) (ft/sec), p(t) (deg/sec), r(t) (deg/sec) and ϕ(t) (deg). The control variables are, in order, δ_a(t) (deg), δ_{tip}(t) (deg), and δ_{ytv}(t) deg.

$$A = \begin{bmatrix} 0.01109 & 1.332 & -5.373 & 0.545 \\ -1.221 & -0.607 & 0.395 & 0 \\ -0.1262 & 0.00297 & 0.0206 & 0 \\ 0 & 1 & 0.24768 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -.009124 & -0.000248 & 0.124 \\ 2.412 & 0.537 & -0.0667 \\ 0.0377 & -0.0561 & -1.227 \\ 0 & 0 & 0 \end{bmatrix}$$

Note, these matrices are describing aircraft motion in a general body-fixed axis system, and not a stability axis system. This does not affect your work.

7

```
A =

   1.1090e-002  1.3320e+000 -5.3730e+000  5.4500e-001
  -1.2210e+000 -6.0700e-001  3.9500e-001            0
  -1.2620e-001  2.9700e-003  2.0600e-002            0
             0  1.0000e+000  2.4768e-001            0

>> B=[-.009124  -.000248  .124; 2.412 .537  -.0667; 0.0377 -.0561  -1.227; 0 0 0 ]

B =

  -9.1240e-003 -2.4800e-004  1.2400e-001
   2.4120e+000  5.3700e-001 -6.6700e-002
   3.7700e-002 -5.6100e-002 -1.2270e+000
             0            0            0

>> K=[ 0 1;0 1; 1 0]

K =

     0     1
     0     1
     1     0

>> B

B =

  -9.1240e-003 -2.4800e-004  1.2400e-001
   2.4120e+000  5.3700e-001 -6.6700e-002
   3.7700e-002 -5.6100e-002 -1.2270e+000
             0            0            0

>> BP=B*K

BP =

   1.2400e-001 -9.3720e-003
  -6.6700e-002  2.9490e+000
  -1.2270e+000 -1.8400e-002
             0            0
```

```
>> C=[0 0 1 0;0 1 0 0]

C =

     0     0     1     0       r
     0     1     0     0       p
```

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
>> tzero(A,BP,C,D)

ans =

  1.8735e-016
 -6.2863e-003

>> [num1,den]=ss2tf(A,BP,C,D,1)

num1 =

            0 -1.2270e+000 -7.4703e-001 -1.9860e+000 -8.1191e-001 - r to v1
            0 -6.6700e-002 -6.3396e-001 -8.0021e+000  2.0109e-001 - p to v1

den =

  1.0000e+000  5.7531e-001  9.2812e-001  2.8445e-001  2.4289e-002   Δ

>> [num2,den]=ss2tf(A,BP,C,D,2)

num2 =

            0 -1.8400e-002 -1.0235e-003 -5.2487e-001 -2.1507e-001 - r to v2
            0  2.9490e+000 -8.9279e-002 -2.1194e+000  5.3269e-002 - p to v2

den =

  1.0000e+000  5.7531e-001  9.2812e-001  2.8445e-001  2.4289e-002   Δ

>> p11=tf(num1(1,:),den)

Transfer function:
    -1.227 s^3 - 0.747 s^2 - 1.986 s - 0.8119
--------------------------------------------------
s^4 + 0.5753 s^3 + 0.9281 s^2 + 0.2845 s + 0.02429

>> p12=tf(num2(1,:),den)

Transfer function:
  -0.0184 s^3 - 0.001023 s^2 - 0.5249 s - 0.2151
--------------------------------------------------
s^4 + 0.5753 s^3 + 0.9281 s^2 + 0.2845 s + 0.02429

>> p21=tf(num1(2,:),den)

Transfer function:
    -0.0667 s^3 - 0.634 s^2 - 8.002 s + 0.2011
--------------------------------------------------
```

```
s^4 + 0.5753 s^3 + 0.9281 s^2 + 0.2845 s + 0.02429

>> p22=tf(num2(2,:),den)

Transfer function:
   2.949 s^3 - 0.08928 s^2 - 2.119 s + 0.05327
--------------------------------------------------
s^4 + 0.5753 s^3 + 0.9281 s^2 + 0.2845 s + 0.02429

>> zpk(p11)

Zero/pole/gain:
    -1.227 (s+0.4293) (s^2 + 0.1796s + 1.542)
-----------------------------------------------
(s^2 + 0.339s + 0.02968) (s^2 + 0.2363s + 0.8183)
```

$= \frac{r}{\sigma_1}$ } close first

```
>> zpk(p12)

Zero/pole/gain:
   -0.0184 (s+0.4077) (s^2 - 0.3521s + 28.67)
-----------------------------------------------
(s^2 + 0.339s + 0.02968) (s^2 + 0.2363s + 0.8183)
```

may be trouble

$= \frac{r}{\sigma_2}$

```
>> zpk(p21)

Zero/pole/gain:
     -0.0667 (s-0.02508) (s^2 + 9.53s + 120.2)
-----------------------------------------------
(s^2 + 0.339s + 0.02968) (s^2 + 0.2363s + 0.8183)
```

bad!

$= \frac{\rho}{\sigma_1}$

```
>> zpk(p22)

Zero/pole/gain:
     2.949 (s+0.8453) (s-0.8504) (s-0.02513)
-----------------------------------------------
(s^2 + 0.339s + 0.02968) (s^2 + 0.2363s + 0.8183)
```

bad

$= \frac{\rho}{\sigma_2}$

```
>> detP=p11*p22-p21*p12

Transfer function:

-3.62 s^14 - 6.27 s^13 - 13.71 s^12 - 15.46 s^11 - 16.6 s^10 - 12.54 s^9 - 7.726 s^8

        - 3.646 s^7 - 1.158 s^6 - 0.2334 s^5 - 0.02871 s^4 - 0.001994 s^3 - 6.333e-005 s^ ↙
2

                                                              - 3.261e-007 s + 1.999 ↙
e-019

------------------------------------------------------------------------------------ ↙
-----
s^16 + 2.301 s^15 + 5.698 s^14 + 8.307 s^13 + 11.03 s^12 + 11.12 s^11 + 9.623 s^10 + 6.8 ↙
s^9

        + 3.935 s^8 + 1.85 s^7 + 0.6687 s^6 + 0.1762 s^5 + 0.0327 s^4 + 0.004138 s^3

                                  + 0.0003396 s^2 + 1.631e-005 s + 3.481 ↙
```

```
e-007


>> detP=minreal(detP)

Transfer function:
               -3.62 s^2 - 0.02275 s
--------------------------------------------------
s^4 + 0.5753 s^3 + 0.9281 s^2 + 0.2845 s + 0.02429

>> zpk(detP)

Zero/pole/gain:
             -3.6197 s (s+0.006286)
-------------------------------------------------
(s^2 + 0.339s + 0.02968) (s^2 + 0.2363s + 0.8183)

>> PI11=p22/detP;
>> PI22=p11/detP;
```

\>> PI12=-~~p21/detP;~~ -p12/detP

\>> PI21=-~~p12/detP;~~ - p21/detP

```
>> Y1V1=1/PI11;
>> Y2V2=1/PI22;
>> Y1V2=-1/PI12;
>> Y2V1=-1/PI21;
>> Y1V1=minreal(Y1V1);
>> Y2V2=minreal(Y2V2);
>> Y1V2=minreal(Y1V2);
>> Y2V1=minreal(Y2V1);
>> zpk(Y1V1)

Zero/pole/gain:
     -1.2274 s (s+0.006286)
-----------------------------------
(s+0.8453) (s-0.8504) (s-0.02513)
```

$= \left.\frac{\rho}{v_1}\right|_{p \to v_2}$

```
>> zpk(Y2V2)

Zero/pole/gain:
       2.95 s (s+0.006286)
------------------------------------
(s+0.4293) (s^2 + 0.1796s + 1.542)
```

$= \left.\frac{\rho}{v_2}\right|_{r \to v_1}$ ✓ OK

\>> zpk(~~Y1V2~~)(Y2V1)

```
Zero/pole/gain:
     54.2676 s (s+0.006286)
-----------------------------------
(s-0.02508) (s^2 + 9.53s + 120.2)
```

$= \cancel{\left.\frac{r}{v_2}\right|_{p \to v_1}}$ $\left.\frac{\rho}{v_1}\right|_{r \to v_2}$

\>> zpk(~~Y2V1~~)(Y1V2)

```
Zero/pole/gain:
     196.7201 s (s+0.006286)
------------------------------------
(s+0.4077) (s^2 - 0.3521s + 28.67)
```

bad

$= \cancel{\left.\frac{\rho}{v_1}\right|_{r \to v_2}}$ $\left.\frac{r}{v_2}\right|_{p \to v_1}$

$$G_1(s) = \frac{-4(s+1)^2}{s^2} \rightarrow r\text{-loop}$$

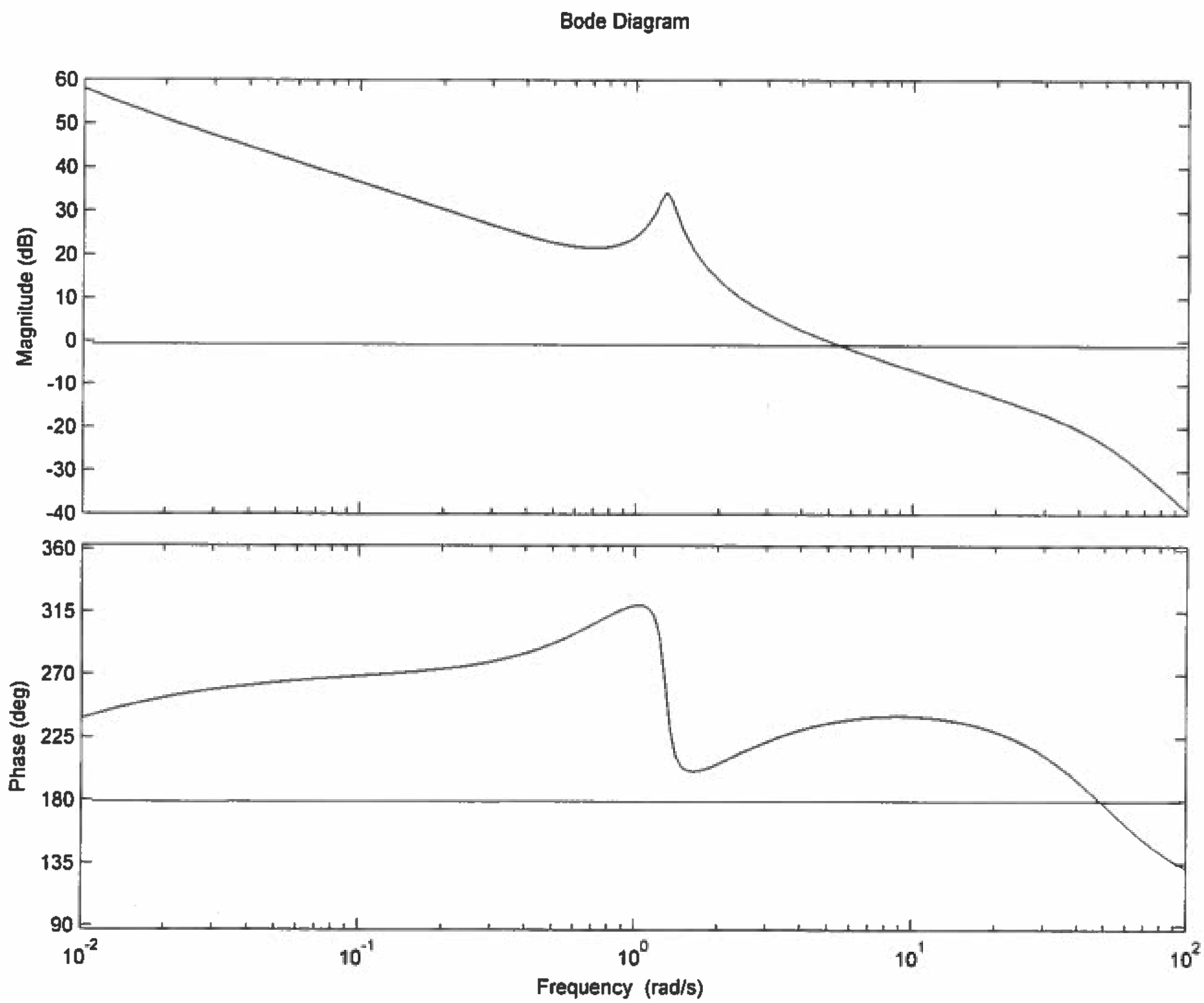
$$G_2(s) = \frac{1.5(s+1)^3}{s^3} \rightarrow p\text{-loop}$$

$\omega_c = 5$ rad/sec

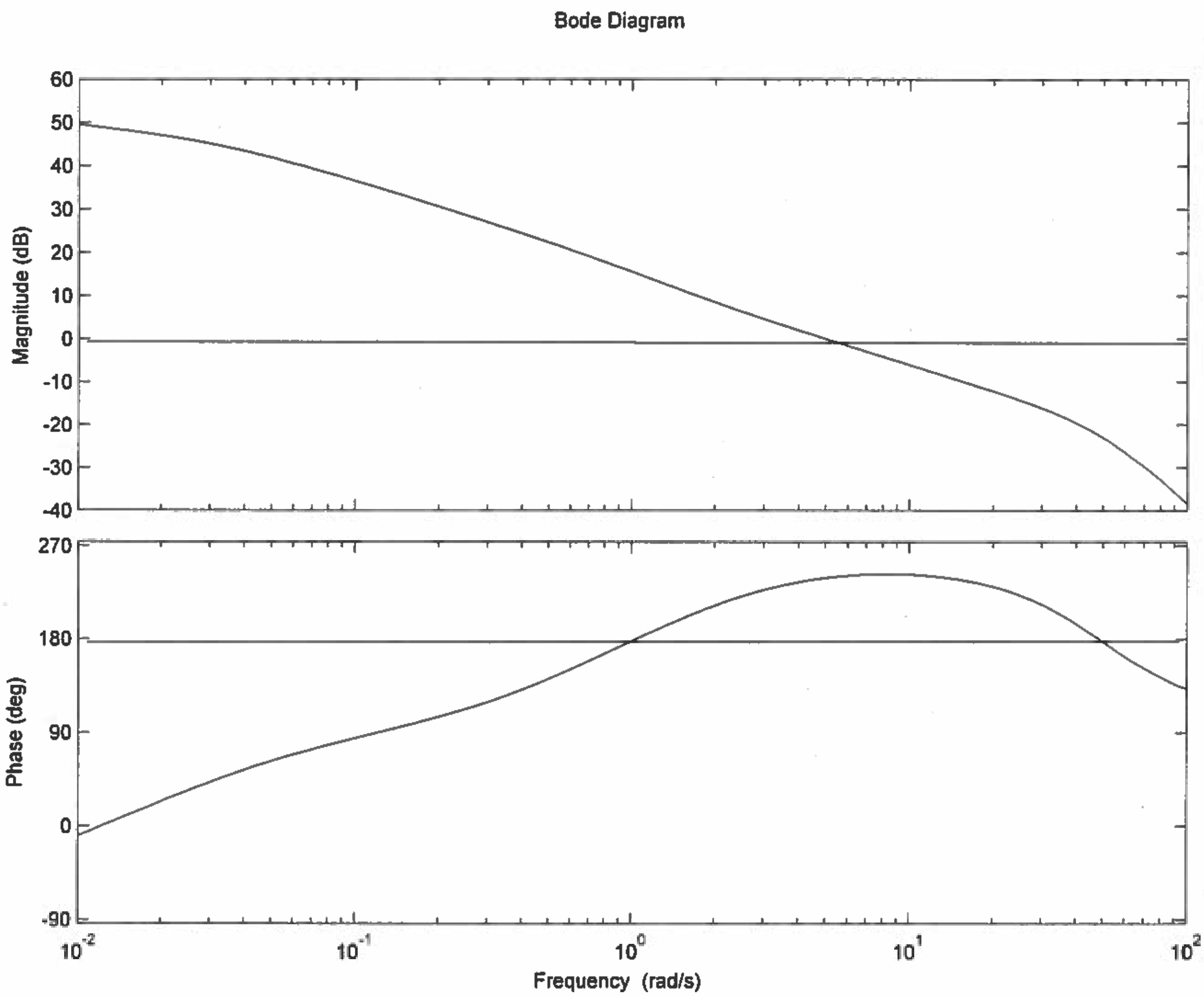
3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

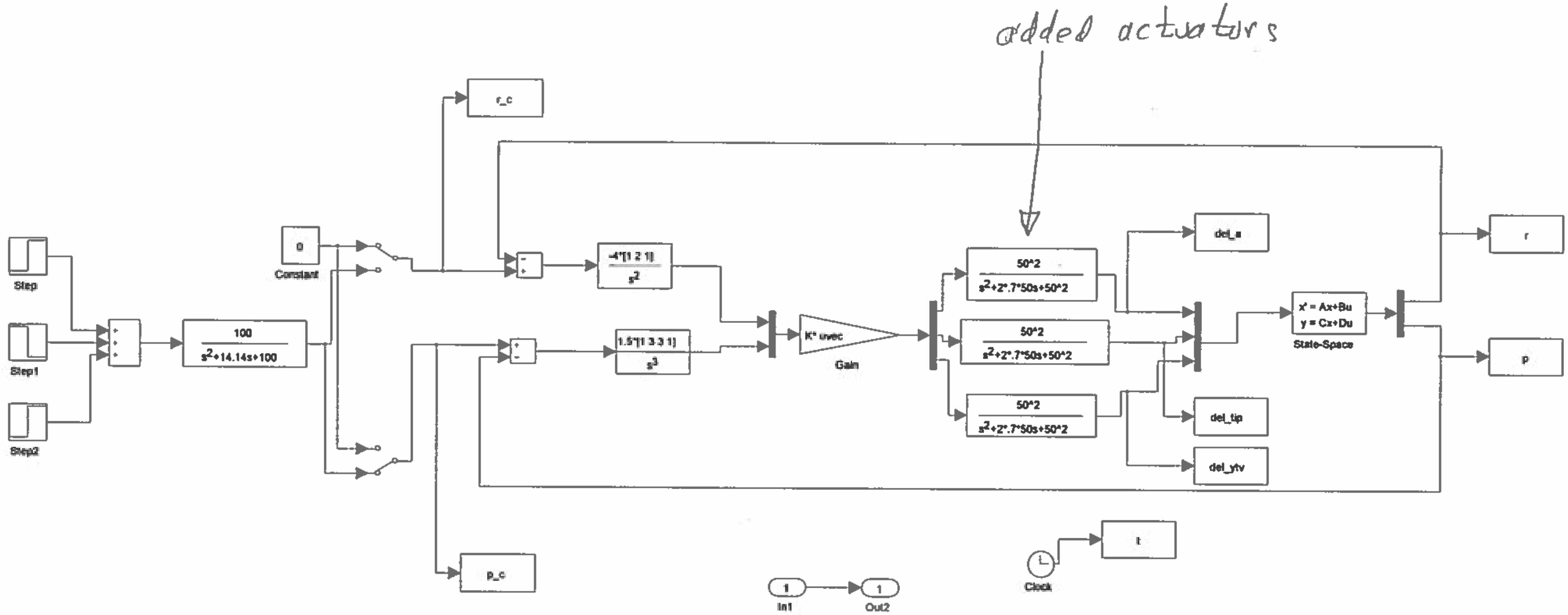
$\frac{p}{p_e}$ with r-loop closed



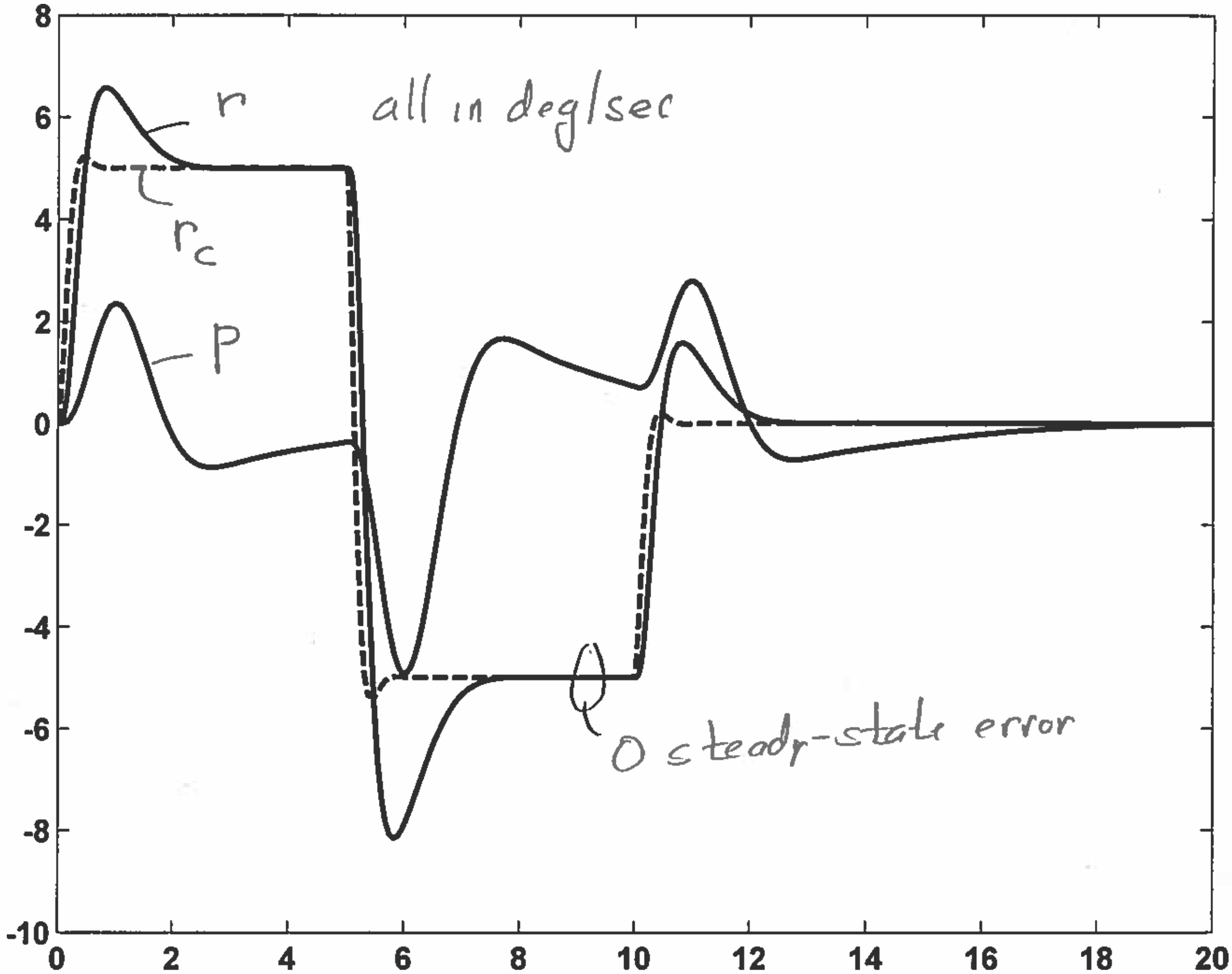
$\frac{r}{r_c}$ with p-loop closed



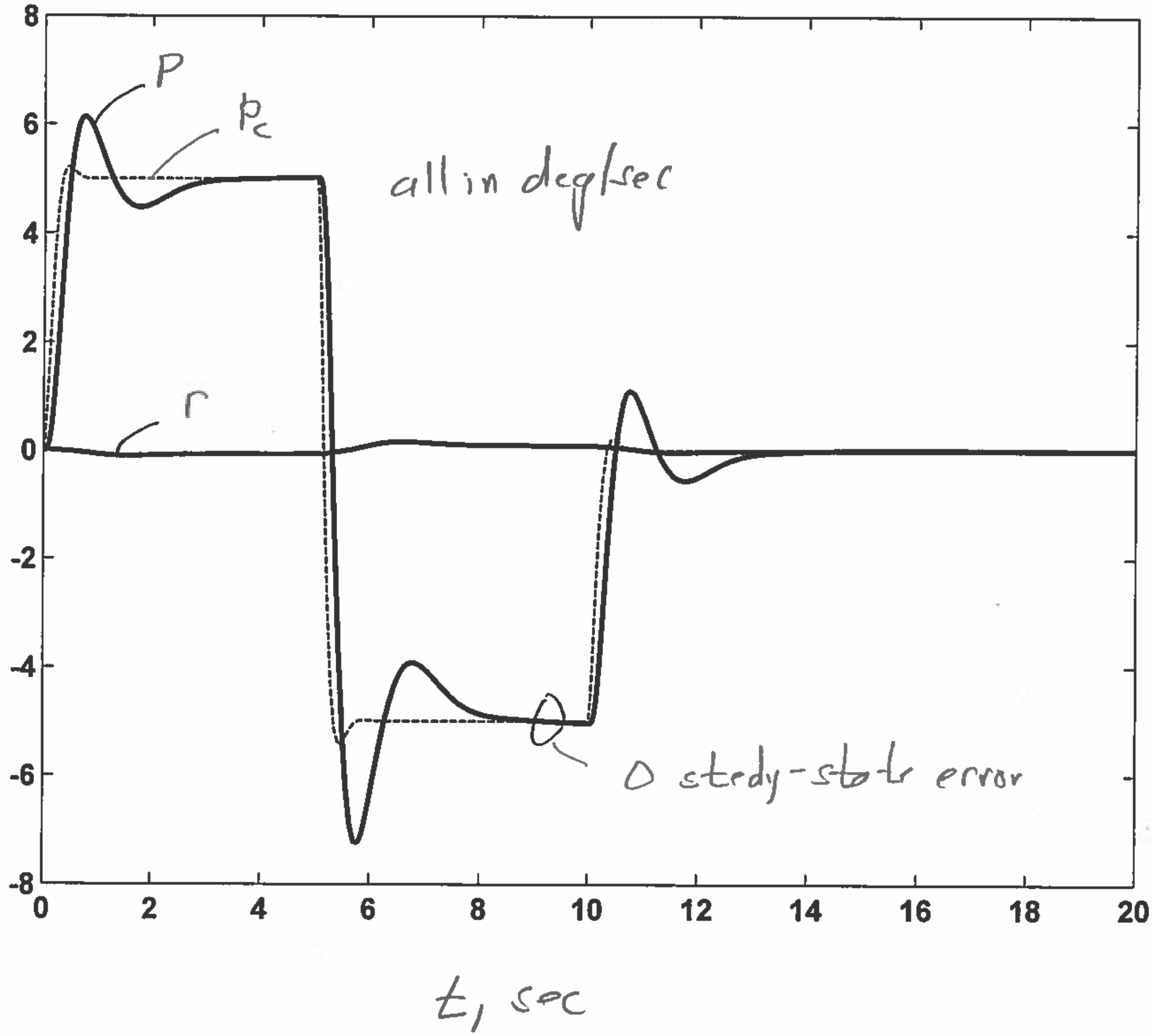
14



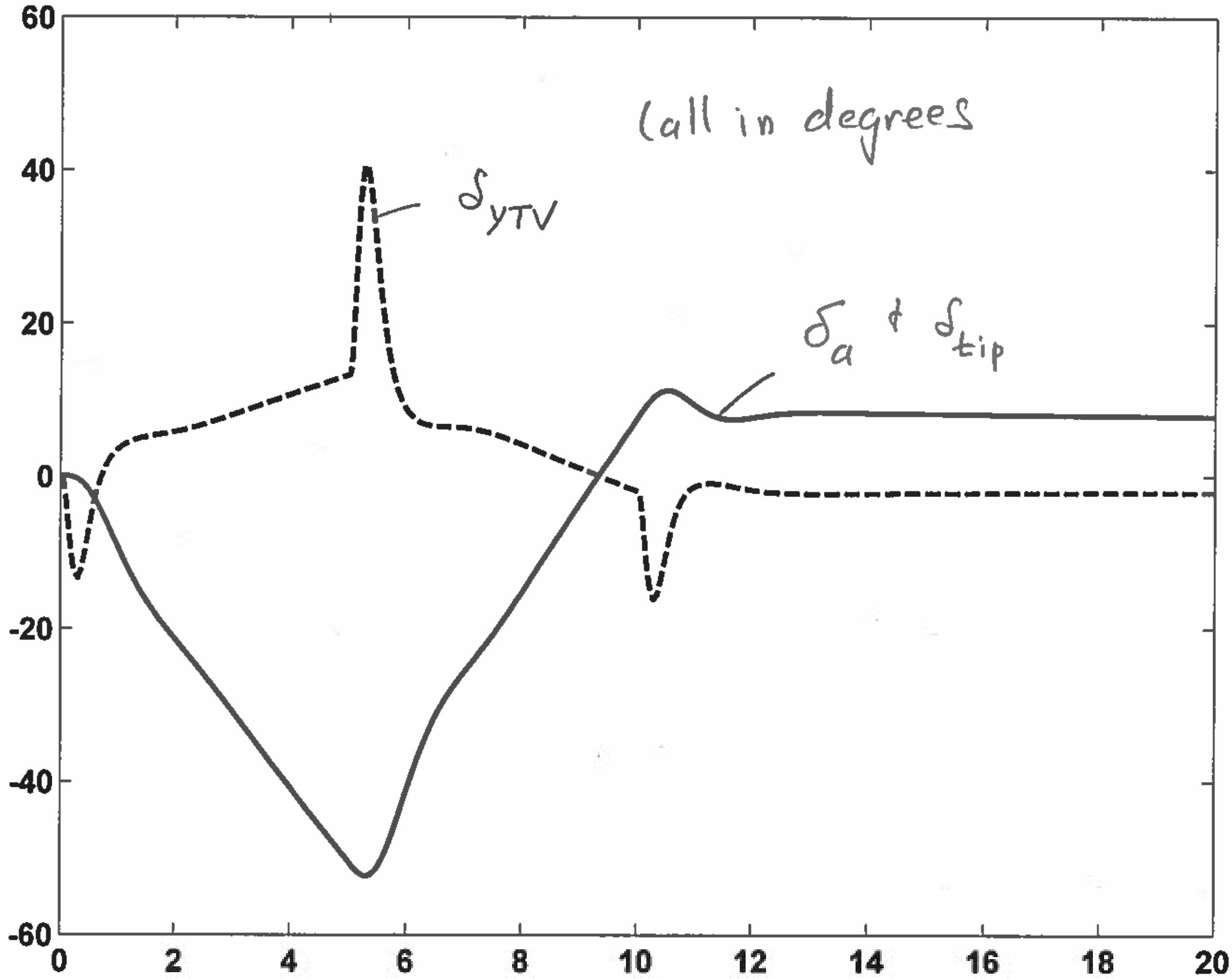
added actuators
Step
Step1
Step2
100
s^2+14.14s+100
0
Constant
r_c
p_c
-4*[1 2 1]
s^2
1.5*[1 3 3 1]
s^3
K* uvec
Gain
50^2
s^2+2*.7*50s+50^2
50^2
s^2+2*.7*50s+50^2
50^2
s^2+2*.7*50s+50^2
del_a
del_tip
del_ytv
x' = Ax+Bu
y = Cx+Du
State-Space
r
p
1
In1
1
Out2
Clock
t


8
6
4
2
0
-2
-4
-6
-8
-10
0
2
4
6
8
10
12
14
16
18
20
r
all in deg/sec
r_c
p
0 steady-state error

t, sec

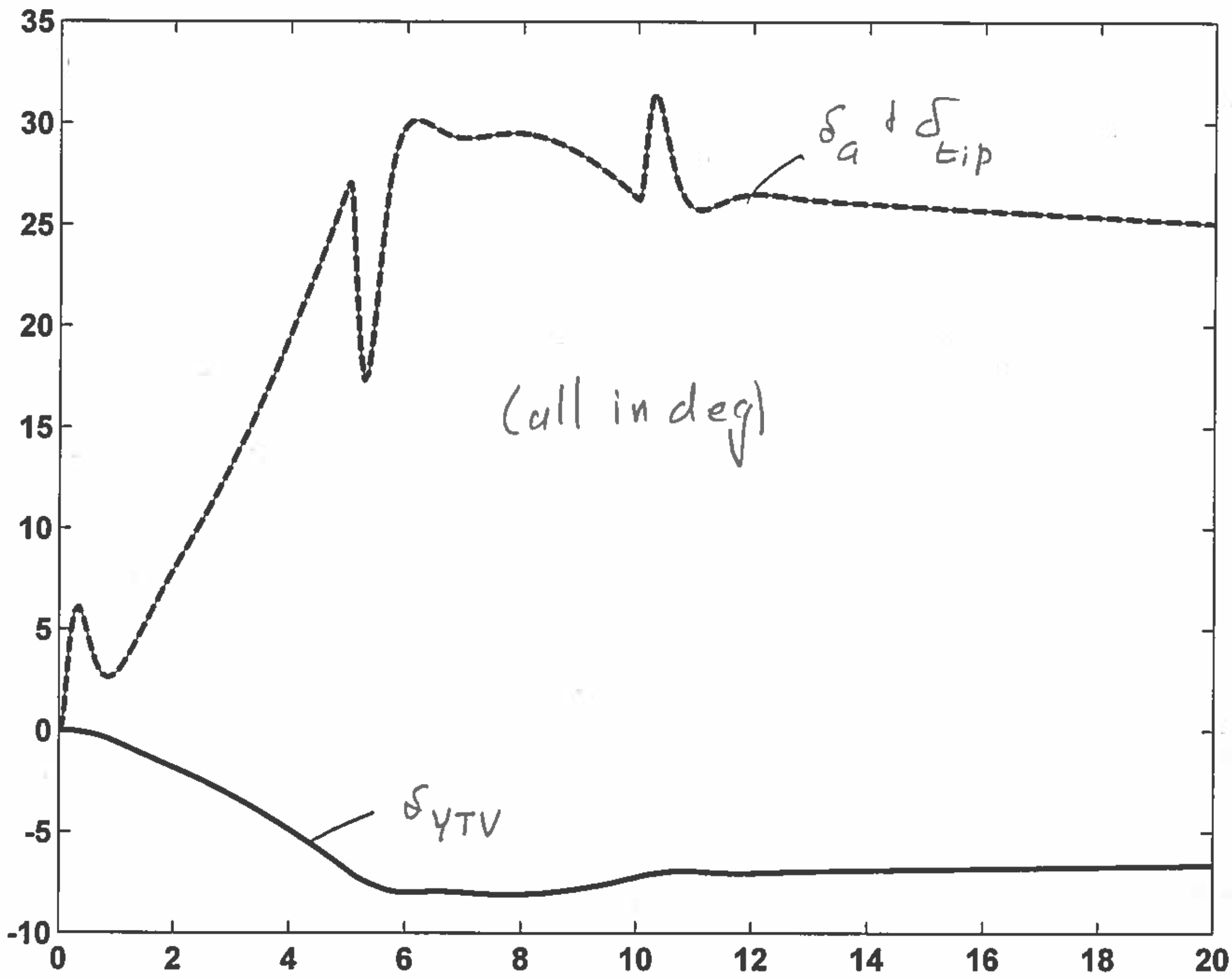

P
p_c
all in deg/sec
r
0 stedy-state error
8
6
4
2
0
-2
-4
-6
-8
0
2
4
6
8
10
12
14
16
18
20
t, sec

yaw-rate command (r_c)



t, sec

roll rate input (p_c)



t, sec