DESCRIPTION OF RANDOM PROCESSES

3.1 Introduction

Often it is necessary to study the behavior of systems which are excited or disturbed by time functions which can only be specified in statistical or probabalistic terms. The response of an aircraft to atmospheric turbulence would be one example of this.

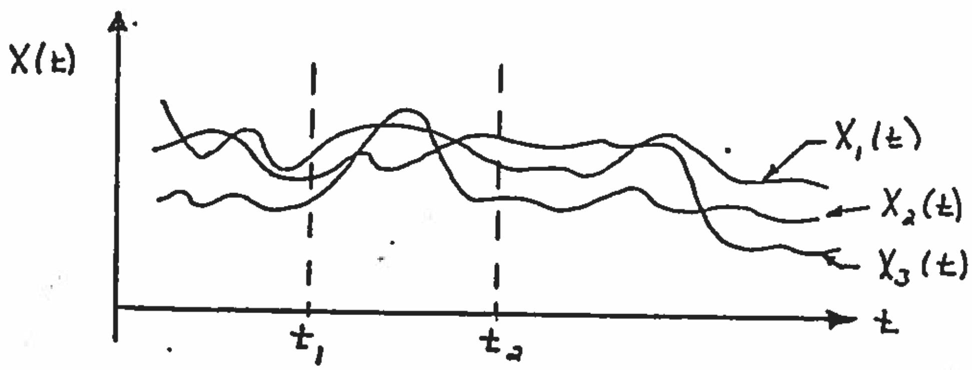
Consider for a moment an experiment whose outcome is determined by chance, e.g., the rolling of a die. We may associate with each basic possible outcome (called a sample point ζ_i) a probability of its occurence. In the case of rolling a die, we readily associate the sample points ζ_i $i = 1, 2, \cdots, 6$ with the number of dots on the upward face. Now numerical valued outcomes or numerical valued functions of outcomes of experiments of chance are called random variables, $X(\zeta)$ For the die, a random variable $X(\zeta_i)$ can be defined as being numerically equal to the dots on the upward face, i.e., $X(\zeta_i) = i, \; i = 1, 2, \cdots, 6$. Now a numerically valued function of both the sample point, ζ , and time, t , is called a random or stochastic process.

$$X = X(\zeta, t)$$

This definition can include functions which we do not ordinarily think of as "random". Consider a sine wave whose phase angle is determined by chance

$$X(\varphi, t) = \sin(t + \varphi)$$

Normally a random process consists of a family of random time functions. Such a family is called an ensemble. Any single time function from the ensemble is called a sample function. In order to describe the random process, the behavior of the ensemble must be described in some manner. The most fundamental means of doing this utilize so-called <u>probability density functions</u>. Consider the figure below.



Shown are three sample functions from an ensemble of N such functions for a particular random process. At any given time t_1, we can define a random variable, $X(t_1)$, whose sample points consist of all the values of the ensemble of sample functions at $t = t_1$.

At t_1, the probability that the random variable will be greater than some number x and smaller than $x + \Delta x$ can be computed as

$$P_R\left[x < X(t) < x + \Delta x,\ t = t_1\right] = \frac{\text{No. of functions } X_i(t_1) \text{ with values between } x \text{ and } x+\Delta x}{N}$$

If Δx is very small, this probability will be roughly proportional to Δx,

$$P_R\left[x < X(t) < x + \Delta x,\ t = t_1\right] \doteq p_1(x, t_1)_N \cdot \Delta x$$

$p_1(x, t_1)_N$ is the <u>first probability density function</u> for the random variable $X(t_1)$ for a finite ensemble of size N. $p_1(x, t_1)_N$ will depend, in general, on N, x and t_1 .

To be of any practical use as a means to describe the average amplitude characteristics of a typical sample function, the number of sample functions, N, must be sufficiently large so that $p_1(x, t_1)_N$ is not materially changed when the ensemble is made even larger. This can be accomplished and the "roughly proportional" restriction removed if the first probability density function is defined as

$$p_1(x,t_1) \triangleq \lim_{\substack{N \to \infty \\ \Delta x \to 0}} \frac{\text{Number of values lying between } x \text{ and } x + \Delta x}{N \cdot \Delta x}$$

Now one can calculate various probabilities of interest using the calculus; since the random variable $X(t)$ is a continuous one as $N \to \infty$.

$$P_R\left[x_1 < X(t) < x_2, t_1\right] = \int_{x_1}^{x_2} p_1(x, t_1)\, dx$$

When $x_1 = -\infty$ and $x_2 = x$

$$P_R\left[X(t) < x, t_1\right] = \int_{-\infty}^{x} p_1(x, t_1)\, dx = P_1(x, t_1)$$

where $P_1(x, t_1)$ is called the <u>first probability distribution function</u> for the random variable $X(t_1)$.

3.2 Statistical Averages

Suppose the average value of some arbitrary function, $f[X(t_1)]$ of the random variable $X(t_1)$ is desired. It can be shown that

$$\text{average value of } f[X(t_1)] = E[f(X)] = \int_{-\infty}^{\infty} f(X)\, p_1(x, t_1)\, dx$$

where the notation $E[\ \ ]$ means "expected value" or average. The averages computed where

$$f[X(t_1)] \equiv X^n(t_1)$$

are particularly important and are called moments of the distribution. They have the form

$$m_n(t_1) \triangleq E[X^n(t_1)] = \int_{-\infty}^{\infty} x^n p_1(x, t_1)\, dx$$

The two lowest order moments are

a. First Moment (arithmetic mean or simply mean)

$$m_1(t_1) = E[X(t_1)] = \int_{-\infty}^{\infty} x\, p_1(x, t_1)\, dx$$

b. Second Moment (mean square value)

$$m_2(t_1) = E[X^2(t_1)] = \int_{-\infty}^{\infty} x^2 p_1(x, t_1)\, dx$$

If the mean value $E[X(t_1)]$ is subtracted from all values of in the ensemble, the moments derived become central moments

$$\mu_n(t_1) \triangleq E[X(t_1) - m_1(t_1)]^n$$
$$= \int_{-\infty}^{\infty} [x - m_1(t_1)]^n p_1(x, t_1)\, dx$$

Note that

$$\mu_1(t) \equiv 0$$
$$\mu_2(t) = \int_{-\infty}^{\infty} [x - m_1(t_1)]^2 p_1(x, t_1)\, dx$$
$$= m_2(t_1) - [m_1(t_1)]^2$$
$$= E[X^2(t_1)] - \{E[X(t_1)]\}^2$$

$\mu_2(t_1)$ is called the variance of the random variable $X(t_1)$ and is often denoted $[\sigma(t_1)]^2$. The square root of the variance is called the standard deviation of the random variable and is denoted $\sigma(t_1)$.

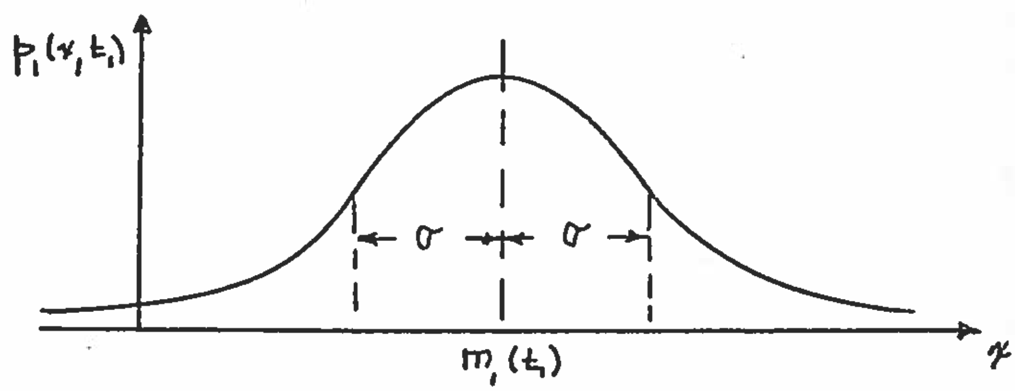
Note that when the mean is zero, the variance and mean square are identical.

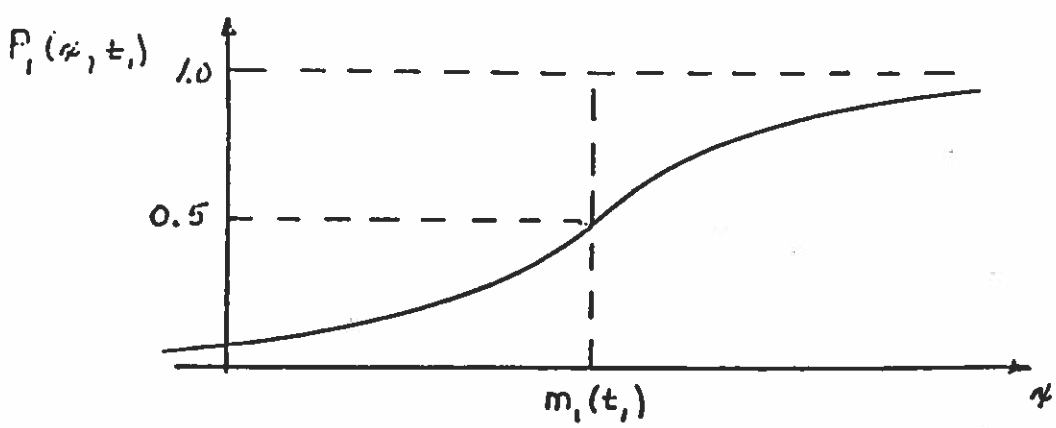
Example 3.1

A very important class of random variables are those which possess normal or Gaussian probability density functions. These density functions are of the form

$$p_1(x, t_1) = \frac{1}{\sqrt{2\pi}\,\sigma(t_1)} e^{-[x - m_1(t_1)]^2 / 2\sigma^2(t_1)}$$

The density and distribution functions for such a random variable are sketched below. The shape of the density function has earned it the name "bell-curve".





3.3 Higher Order Density Functions

Thus far, the first probability density function has provided information about the statistical averages associated with the random variable $X(t_1)$. A <u>second probability density function</u> $p_2(x_1,t_1;x_2,t_2)$ can be defined, which, when multiplied by $dx_1 \cdot dx_2$ gives the probability that $X(t)$ will be within the bounds x_1 and x_1+dx_1 , at time t_1 <u>and</u> that the same $X(t)$ will be between x_2 and x_2+dx_2 at time t_2 , i.e.,

$$Pr\left[x_1 < X(t_1) < x_1+dx_1 ; x_2 < X(t_2) < x_2+dx_2\right] = p_2(x_1, t_1; x_2, t_2)\, dx_1 \cdot dx_2$$

The second probability density function provides the tools to find average values such as

$$E\left[X(t_1)\cdot X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, p_2(x_1, t_1; x_2, t_2)\, dx_1\, dx_2$$

Higher order probability density functions can also be defined. Note that to completely define a random process in a probabalistic manner, an infinite order density function is necessary. Conceptually, such a density function would give the probability of a sample function taking on <u>any</u> specific waveform!

(Ex 3.1a)

3.4 Time Averages

In the preceding sections, the density functions and averages derived all may be functions of the times of observation, t_i . However, in many applications, this time dependence may not exist. For example, the first probability density function may not take on different shapes for different observations times, i.e.,

Ex. 3.1 a

A BIVARIATE NORMAL (GAUSSIAN) DISTRIBUTION:

$$p(x,y) = \frac{1}{2\pi(\mu_{20}\mu_{02}-\mu_{11}^2)^{1/2}} \text{EXP}\left[\frac{-\mu_{02}x^2+2\mu_{11}xy-\mu_{20}y^2}{2(\mu_{20}\mu_{02}-\mu_{11})^2}\right]$$

$$\mu_{20} = E\left[(x-E[x])^2\right] \qquad \mu_{11} = E\left[(x-E[x])(y-E[y])\right]$$

$$\mu_{02} = E\left[(y-E[y])^2\right]$$

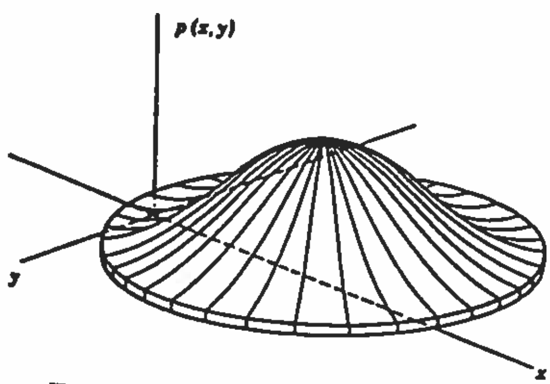


Figure 5.12 The bivariate normal distribution.

$$p_1(x, t_1) = p_1(x, t_2) = p_1(x, t_i) = p_1(x)$$

In addition, the second probability density function may turn out to depend only on $(t_2 - t_1)$ and not upon t_1 and t_2 themselves, i.e.,

$$p_2(x_1, t_1; x_2, t_2) = p_2(x_1, x_2; \tau)$$

where $\tau = t_2 - t_1$

When all the statistics describing the ensemble are not dependent upon the absolute time of observation, the random process is said to be stationary in the strict sense.

Practically speaking, strict sense stationarity is usually impossible to prove. If it can be shown that the first k probability density functions are not dependent upon the time of observation then the process is said to be stationary of order k. If the process is only shown to be stationary of order 1 and if a special time average, $E[x_1(t_1) \cdot x_1(t_2)]$ depends only upon $t_2 - t_1$, then the process is said to be stationary in the wide sense.

When a random process exhibits stationarity (at least wide sense) an alternate possiblity for finding averages exists. Instead of averaging across the ensemble at fixed times, one could consider a single sample function and compute averages over all time. Such averages would be

$$\overline{X(t)} \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, dt \quad = \text{mean value}$$

$$\overline{X^2(t)} \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)\, dt \quad = \text{mean square value}$$

$$\varphi_{xx}(\tau) = \overline{X(t)\, X(t+\tau)} \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, X(t+\tau)\, dt =$$

autocorrelation function

3.5 Ergodicity

When time averages are equal to statistical averages, a random process is said to be <u>ergodic</u>. Ergodicity is very difficult to prove, in general. When processes are wide sense stationary and the lower statistical and time averages (mean, mean square etc.) appear to be equal, ergodicity is often assumed by hypothesis. With this hypothesis,

$$E[X(t)] = \overline{X(t)}$$

$$E[X^2(t)] = \overline{X^2(t)}$$

$$E[X(t_1)X(t_2)] = \overline{X(t_1)X(t_2)} = \varphi_{xx}(\tau)$$

Note the enormous practical advantage that is obtained when the ergodic hypothesis is employed. Statistical properties dealing with an entire ensemble of sample functions can be gleaned from measurements of a single sample function.

3.6 Multiple Random Processes

Often in analyzing physical systems, more than one random process is encountered. For example, in treating aircraft response to turbulence, one may wish to consider the turbulence at a point as consisting of three mutually perpendicular wind velocity components, $u_g(t)$, $v_g(t)$, $w_g(t)$. Each of these components are considered as sample functions from three different random processes. One can define a vector

$$\begin{Bmatrix} u_g(t) \\ v_g(t) \\ w_g(t) \end{Bmatrix} \triangleq \underline{g(t)}$$

and form averages like

$$E\left[\underline{g}(t_1)\,\underline{g}^T(t_2)\right]^{*} = \begin{bmatrix} E[u_g(t_1)u_g(t_2)] & E[u_g(t_1)v_g(t_2)] & E[u_g(t_1)w_g(t_2)] \\ E[v_g(t_1)u_g(t_2)] & E[v_g(t_1)v_g(t_2)] & E[v_g(t_1)w_g(t_2)] \\ E[w_g(t_1)u_g(t_2)] & E[w_g(t_1)v_g(t_2)] & E[w_g(t_1)w_g(t_2)] \end{bmatrix}$$

The matrix on the right hand side of the equation above is given the name <u>covariance matrix</u>. If the random processes involved are each ergodic and jointly ergodic, the statistical expectations can be replaced by autocorrelation and crosscorrelation functions (see Appendix I), i.e.,

$$E\left[\underline{g}(t)\,\underline{g}^T(t+\tau)\right] = \begin{bmatrix} \varphi_{u_g u_g}(\tau) & \varphi_{u_g v_g}(\tau) & \varphi_{u_g w_g}(\tau) \\ \varphi_{v_g u_g}(\tau) & \varphi_{v_g v_g}(\tau) & \varphi_{v_g w_g}(\tau) \\ \varphi_{w_g u_g}(\tau) & \varphi_{w_g v_g}(\tau) & \varphi_{w_g w_g}(\tau) \end{bmatrix}$$

The auto and crosscorrelation functions can be determined using any three sample functions from the random processes (one from each process).

3.7 Spectral Analysis

Under the ergodic hypothesis, time domain analysis of individual sample functions is of considerable interest. Such studies come under the name of <u>spectral</u> or <u>harmonic analyses</u>. Appendix I summarizes spectral analysis mathematical tools for periodic, aperiodic (or transient) and random functions of time. In what follows it will be assumed that the reader has aquainted himself with the material of Appendix I , Section~~s~~ C

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* The expected value of a matrix $\underline{Q}(t)$ is defined as the matrix $E[\underline{Q}(t)]$ whose elements are the expected values of the elements of $\underline{Q}(t)$

S.A.

As mentioned above, one of the important time domain averages is the autocorrelation function, defined as

$$\varphi_{xx}(\tau) \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)\, dt$$

The Fourier transform of the autocorrelation function is called the <u>power spectral density</u> of the function $X(t)$ and is defined as

$$\Phi_{xx}(\omega) \triangleq \int_{-\infty}^{\infty} \varphi_{xx}(\tau) e^{-j\omega\tau} d\tau$$

Conversely, the inverse Fourier transform of the power spectral density is the autocorrelation function, i.e.,

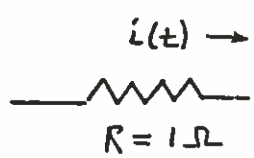
$$\varphi_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega) e^{j\omega\tau} d\tau$$

Note that another important average, the mean square value of the function $X(t)$ can be written

$$\overline{X^2(t)} \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)dt = \varphi_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega$$

Example 3.2

Consider the following 1 ohm resistor through which is flowing a random current $i(t)$. The current is a sample function from an ergodic random process.



The instantaneous power being dissipated in the resistor is

$$P(t) = i^2(t) \cdot R = i^2(t)$$

The average power being dissipated can be obtained from

$$P_{AV} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt$$

But from our spectral analysis results

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt = \varphi_{ii}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{ii}(\omega)\, d\omega = \overline{i^2(t)}$$

This last equation is, conceptually, quite important. It states

a. The mean square value of the current is numerically equal to the average power being dissipated in the resistor (of 1 Ω resistance).

b. By integrating the function $\Phi_{ii}(\omega)$ over all frequencies and dividing by 2π one abtains the average power being dissipated. Stated another way, $\Phi_{ii}(\omega)$ describes the rate at which one "accumulates" power in integrating from $\omega = -\infty$ to $\omega = +\infty$, hence the name power spectral density.

c. The square root of the mean square value of $i(t)$, called the root mean square value (or RMS value), can be thought of as the constant current value which would result in the same average power being dissipated as the actual random current $i(t)$.

The autocorrelation function and power spectral density of sample functions from ergodic random processes have certain properties which will be stated here without proof.

autocorrelation:

a. The autocorrelation function is an even function of its argument; $\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)$

b. The maximum value of the autocorrelation function occurs at the origin; $|\varphi_{xx}(0)| \geq |\varphi_{xx}(\tau)| \quad \tau \neq 0$

c. The magnitude of the autocorrelation function approaches the square of the mean value as τ approaches $\pm\infty$; $\lim_{\tau \to \pm\infty} |\varphi_{xx}(\tau)| = [\overline{X(t)}]^2$

power spectral density

a. The power spectral density is an even function of its argument $\Phi_{xx}(\omega) = \Phi_{xx}(-\omega)$

b. The power spectral density is nonnegative $\Phi_{xx}(\omega) \geq 0$

c. The power spectral density is real.

d. Any sample function from a random process which is stationary in the wide sense will have a power spectral density amenable to spectrum factorization, i.e., $\Phi_{xx}(\omega)$ can be written as

$$\Phi_{xx}(\omega) = G(j\omega) \cdot G(-j\omega)$$

where $G(j\omega)$ is a ratio of polynomials in $(j\omega)$.

It should be emphasized that when one deals with cross correlation functions and cross power spectral densities only one of the preceding properties holds, in general: the cross power spectral density is nonnegative.

3.8 Input-Output Relations for Linear Time Invariant Systems

The problem of primary interest in the analysis of linear systems is this: Given the statistical (or spectral) properties of the input(s) to a linear time invariant system, what are the statistical (or spectral) properties of the output(s)? One can be more specific and ask: Given the power spectral density of the input to a linear time invariant system, what is the power spectral density of the output? To answer this, consider the single input, single output linear time invariant system with transfer function $H(s)$

$$X(s) \longrightarrow \boxed{H(s)} \xrightarrow{\;*\;} Y(s)$$

It can be shown that $\Phi_{yy}(\omega)$ is given by

$$\Phi_{yy}(\omega) = \left| H(s) \right|^2_{s=j\omega} \cdot \Phi_{xx}(\omega)$$

i.e., the power spectral density of the output is equal to the absolute value of the transfer function squared, with s replaced by $j\omega$, times the power spectral density of the input.

Example 3.3

Given a first order system with tranafer function

$$H(s) = \frac{5}{s+5} = \frac{Y(s)}{X(s)}$$

and a random input (ergodic hypothesis) with power spectral density

$$\Phi_{xx}(\omega) = \frac{10}{\omega^2+10}$$

what is the power spectral density of the output? What is the mean square value of the output?

$$\Phi_{yy}(\omega) = \left| \frac{5}{s+5} \right|^2_{s=j\omega} \cdot \frac{10}{\omega^2+10^2} = \left| \frac{5}{j\omega+5} \right|^2 \cdot \frac{10}{\omega^2+10^2}$$

$$= \frac{5}{j\omega+5} \cdot \frac{5}{-j\omega+5} \cdot \frac{10}{\omega^2+10} = \frac{250}{(\omega^2+25)(\omega^2+10)}$$

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* We will retain the Laplace notation for system input and output despite the fact that the Laplace transform of a random function of time is undefined. This is merely notational convenience.

$$\Phi_{yy}(\omega) = \frac{250}{\omega^4 + 35\omega^2 + 250}$$

The mean square value of the output $Y(t)$ can be obtained from

$$\overline{Y^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{yy}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{250}{\omega^4 + 35\omega^2 + 250}\,d\omega$$

The evaluation of the integral on the right hand side of the equation can be done analytically, using tables, or numerically. However, a much preferable way is to utilize a special formulation offered by Newton, Gould and Kaiser in their text, <u>Analytical Design of Linear Feedback Controls</u>, Wiley, 1957. Appendix II gives a partial listing of the table from the text. For this example, referring to Appendix II

The integrand must be written in the form

$$\frac{250}{\omega^4 + 35\omega^2 + 250} = \frac{5}{j\omega + 5}\cdot\frac{\sqrt{10}}{j\omega + \sqrt{10}}\cdot\frac{5}{(-j\omega) + 5}\cdot\frac{\sqrt{10}}{(-j\omega) + \sqrt{10}}$$

$$= \frac{5\sqrt{10}}{(j\omega)^2 + (5+\sqrt{10})(j\omega) + 5\sqrt{10}}\cdot\frac{5\sqrt{10}}{(-j\omega)^2 + (5+\sqrt{10})(-j\omega) + 5\sqrt{10}}$$

Thus,

$n = 2$

$c_1 = 0 \qquad d_2 = 1$

$c_0 = 5\sqrt{10} \qquad d_1 = 5 + \sqrt{10}$

$d_0 = 5\sqrt{10}$

$$I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{yy}(\omega)\,d\omega = \frac{250}{2(5\sqrt{10})(5+\sqrt{10})}$$

$$\overline{Y^2(t)} = .969$$

MAE-275

ATMOSPHERIC DISTURBANCES

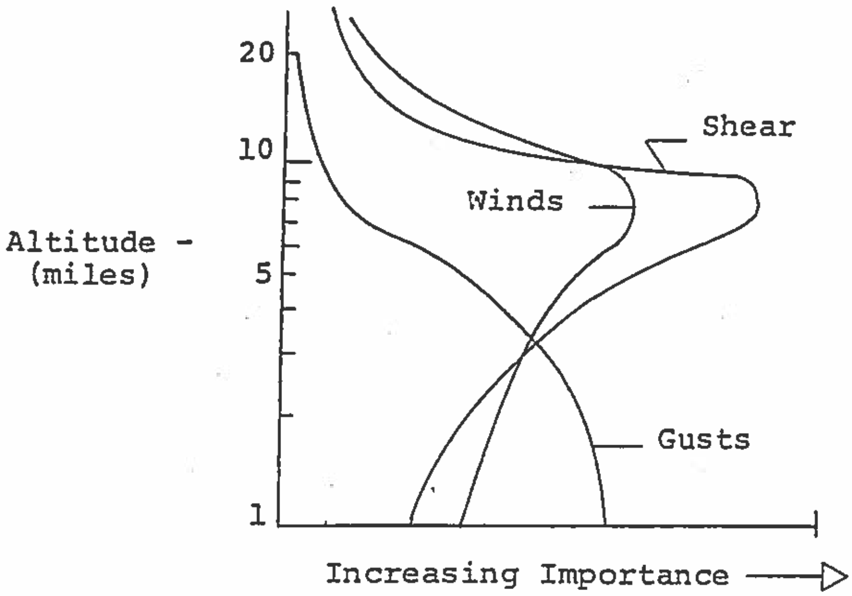
I. Introduction

The total wind force acting on any aerodynamic vehicle can be thought of as the result of several different, related conditions, acting individually or collectively.

1. Mean Wind Flow, characterized by a slowly varying average velocity.
2. Localized Wind Shear, defined as the difference in wind velocities at an upper and lower altitude divided by the altitude increment, Δh. This is the average vertical wind gradient.
3. Gusts, described as the relatively rapid wind intensity fluctuations with zero mean.

Below is a chart of the relative importance of these disturbances to flight control problems vs. altitude. One or all of these disturbances may have to be taken into account in a particular problem depending on the particular vehicle and its mission.

In aircraft stability and control analyses, gusts often play the important role. The response of an aircraft to these gusts determine, among other things, the comfort and safety of passengers and crew, the maintenance of a specific flight path, the accuracy of the aircraft as a weapons platform, the structural integrity of the aircraft and its fatigue life.



II. The Spectral Approach

Assumptions

The most sophisticated and realistic treatment of gust disturbances to date is the stochastic approach in which the gust velocity in any direction at any fixed point in space is considered to be random in nature.

The statistical description of the random, three dimensional turbulence (gust) velocity field is exceedingly complex, so much so that certain simplifying assumptions are nearly always involved. These are: stationarity, homogeneity, and isotropy .

Stationarity has to do with the temporal properties of the turbulence. If the turbulence is stationary, its statistical properties at any point in space are independent of time.

Homogeneity is the corresponding spatial property; that is, the statistical properties of a homogeneous turbulence field are not affected by a translation of the coordinate system used to describe it.

Finally, isotropy means that the statistical properties of the turbulence are not changed by a rotation of the coordinate system used to describe it.

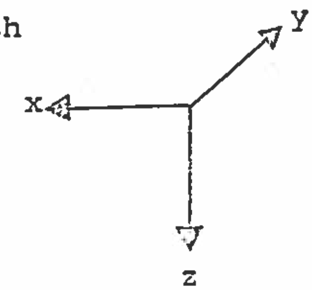
In addition to the above assumptions, the concept of a "frozen" turbulence field is often used. This concept reduces the temporal and spatial properties of the turbulence to merely spatial ones and implies that the observer is moving through the field at a velocity considerably larger than the RMS turbulence velocity. The assumption of a "frozen" field of turbulence is often referred to as Taylor's Hypothesis.

Normality

Gust velocities have been experimentally found to possess Gaussian or normal probability distributions. Although some experimental work has presented ample evidence to invalidate the Gaussian assumption, the assumption will be utilized in this work because of the simplicity it affords.

Upwash Fields

A vector can be associated with each point in a gust field to represent the velocity of the fluid at that point. If one considers just the vertical component of this vector, an upwash field results. With the help of a fixed cartesian coordinate system one can speak of one, two, and three dimensional upwash



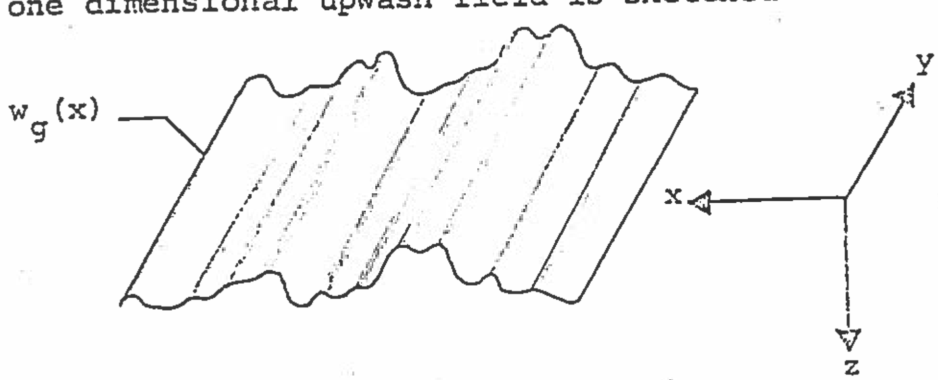
fields where the upwash velocities are, respectively

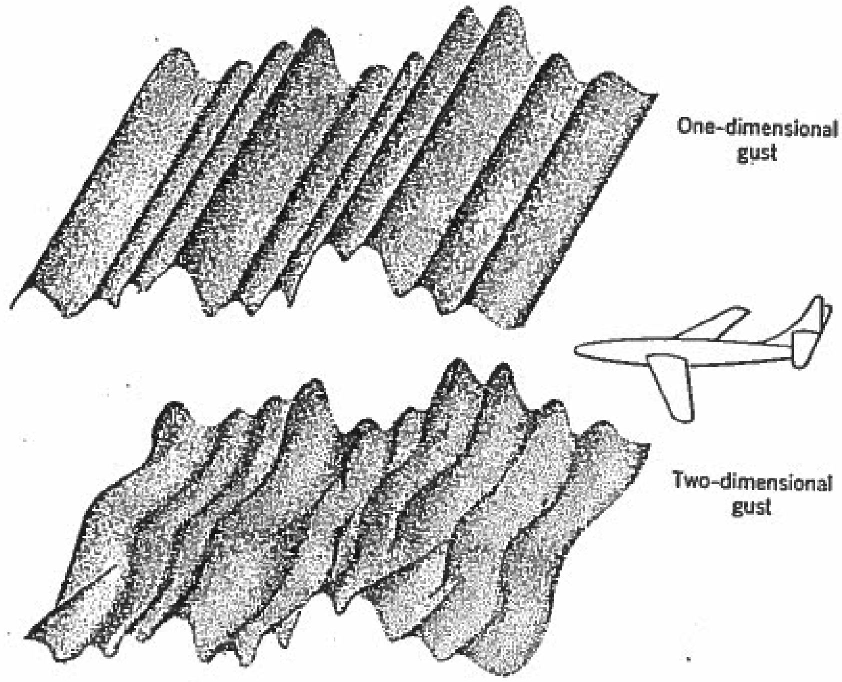
$w_g = w_g(x)$ one dimensional

$w_g = w_g(x, y)$ two dimensional

$w_g = w_g(x, y, z)$ three dimensional

A one dimensional upwash field is sketched below.





One-dimensional gust
Two-dimensional gust

Spectral Forms for Random Atmospheric Turbulence

Von Karman and Dryden Spectra

Two spectral forms widely used in turbulence analyses are the von Karman and Dryden spectra(A1),(A6). Both of these mathematical forms, named for the scientists who first proposed them, have been used in the past. The trend is to adopt the von Karman form because it exhibits a high frequency characteristic which matches certain theoretical evidence.

Von Karman form (vertical gust velocity)

$$\Phi_{w_g}(\Omega) = \sigma_w^2 L_w \frac{1 + 8/3(1.339 L_w \Omega)^2}{[1 + (1.339\ L_w \Omega)^2]^{11/6}}$$

Dryden form (vertical gust velocity)

$$\Phi_{w_g}(\Omega) = \sigma_w^2 L_w \frac{1 + 3(L_w \Omega)^2}{[1 + (L_w \Omega)^2]^2}$$

where

$$\sigma_w^2 = \text{mean square intensity}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{w_g}(\Omega)\, d\Omega$$

$$L_w = \text{characteristic length}$$

The figure on page 3 shows plots of these two spectral forms for several values of the characteristic length, L_w.

Houbolt, et al[A4], have shown the von Karman and Dryden spectra to agree fairly well with spectra measured in cumulus clouds, thunderstorms, and clear air. In all these studies, the turbulence velocity field is assumed to have zero mean.

Simplified Spectrum

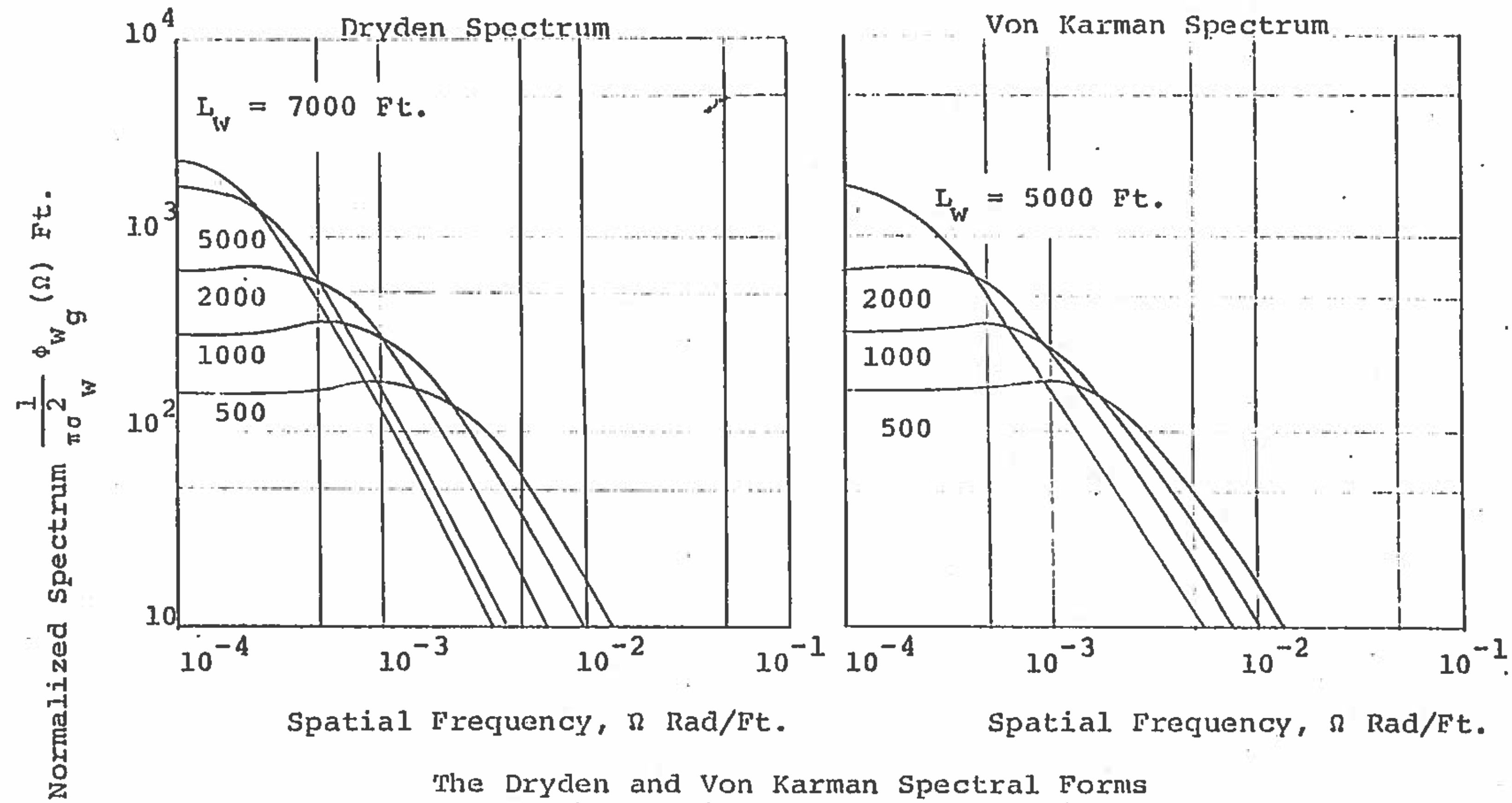
Reference (A1) suggests the use of a simplified spectrum for flight control system analysis:

$$\Phi_{w_g}(\Omega) = 2\sigma_w^2 L_w \frac{1}{1 + (L_w \Omega)^2}$$

There is no theoretical basis for this form; but its agreement with measured spectra, the preservation of the parameter, L_w, and its simplicity suggest its use. For these reasons this spectrum was chosen for the study presented here.

The values of mean square intensity, σ_w, and characteristic length vary, of course, with terrain, altitude, and atmospheric conditions. It is this variation, in fact, which forms the basis of this research.

Speaking very generally, the literature shows, L_w, ranging from 500 to 6000 feet, and σ_w from 0 to 40 feet per second.



The Dryden and Von Karman Spectral Forms
For The Vertical Turbulent Velocity

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Dryden

MAE 275

Flight in Moving Atmosphere

Assumptions:

1.) Frozen turbulence or frozen moving air mass field.

2.) Aircraft velocity >> than the root-mean-square (rms) velocity of the gust field or average velocity of air mass.

3.) For purposes of determining moving air mass effects on aircraft, consider u_g, w_g, v_g, p_g, q_g and r_g variables in the <u>aerodynamic</u> terms (not <u>inertial</u> terms) in the equations of motion.

4.) For u_g, w_g and v_g, consider the entire aircraft experiencing the air mass velocity that exists at the center of gravity in the frozen field. For p_g, q_g and r_g use the slope of the frozen airmass field at the aircraft center of gravity interpreted aerodynamically as equivalent roll-rate, pitch-rate and yaw-rate.

5.) Fundamental assumption for the validity of the above is that the spatial variation (or spatial "period") of the frozen field is large compared to the length of the aircraft.

See "ATMOSPHERIC DISTURBANCES" handout for details.

e.g. Let $\Phi_{xx}(\omega) = \dfrac{a^2}{\omega^2 + b^2}$

$$\overline{x^2(t)} = \,? = \frac{1}{2\pi}\int \frac{a^2}{\omega^2+b^2}\,d\omega$$

$$= \frac{1}{2\pi}\int \frac{a}{(j\omega)+b} \cdot \frac{a}{(-j\omega)+b}\,d\omega$$

$$\therefore \boxed{\overline{x^2(t)} = I_1 = \frac{a^2}{2b}}$$

this comes in handy if you want to set up augmented state eqns. assume you want $x(t)$ to be colored noise with a break freq of 2 rad/sec and a mean square value of .5

$$b = 2 \text{ rad/sec}$$

$$\overline{x^2(t)} = I_1 = 0.5 = \frac{a^2}{4}$$

$$\text{or} \quad a = \sqrt{2} = 1.414$$

now $x(s) = \dfrac{1.414}{s+2} \cdot w(t)$

zero-mean white noise with <u>covariance</u> of unity

$x = \rho(t)$

$\dot{\rho} = -2\rho + 1.414\,w(t)$

$E[w(t) \cdot w(t+\tau)] = 1.0$

Appendix I

Spectral Analysis

The brief review which follows is intended to summarize the tools of spectral or harmonic analysis.

A. Periodic Signals

1. Fourier Series

A periodic signal $x(t)$, with fundamental frequency ω_1 which satisfies the Dirichlet conditions can be represented by a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X(n)e^{jn\omega_1 t} \quad . \tag{1}$$

$$X(n) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_1 t}\, dt \tag{2}$$

$$T = \frac{2\pi}{\omega_1}$$

2. Autocorrelation and power spectal density

The autocorrelation function for the periodic signal $x(t)$ is defined as

$$\varphi_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{3}$$

It can be shown that

$$\varphi_{xx}(\tau) = \sum_{n=-\infty}^{\infty} |X(n)|^2 e^{jn\omega_1\tau} \tag{4}$$

Now,

$$\Phi_{xx}(n) = |X(n)|^2 \tag{5}$$

Where $\Phi_{xx}(n)$ is referred to as the power spectral density of the signal $x(t)$. It can be shown that

$$\Phi_{xx}(n) = \frac{1}{T}\int_{-T/2}^{T/2} \varphi_{xx}(\tau)e^{-jn\omega_1\tau}\, d\tau \tag{6}$$

and from eqn. (4)

$$\varphi_{xx}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{xx}(n)e^{jn\omega_1\tau} \tag{7}$$

3. Crosscorrelation and cross power spectral density

The crosscorrelation function for two periodic signals $x(t)$ and $y(t)$ with identical fundamental frequencies ω_1 and which satisfy the Dirichlet conditions is defined as

$$\varphi_{xy}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t+\tau)dt \tag{8}$$

It can be shown that

$$\varphi_{xy}(\tau) = \sum_{n=-\infty}^{\infty} \bar{X}(n)Y(n)e^{jn\omega_1\tau} \tag{9}$$

where $\overline{X}(n)$ denotes the complex conjugate of $X(n)$. Now,

$$\Phi_{xy}(n) = \overline{X}(n)Y(n) \tag{10}$$

Where $\Phi_{xy}(n)$ is referred to as the cross power spectral density of the signals $x(t)$ and $y(t)$. One can show

$$\Phi_{xy}(n) = \frac{1}{T}\int_{-T/2}^{T/2} \varphi_{xy}(\tau)e^{-jn\omega_1\tau}\, d\tau \tag{11}$$

and from eqn. (9)

$$\varphi_{xy}(\tau) = \sum_{n=-\infty}^{\infty} \Phi_{xy}(n)e^{jn\omega_1\tau} \tag{12}$$

Two periodic signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

B. Transient Signals

1. Fourier Integral

A signal $x(t)$ is said to be transient if

$$\lim_{t\to\infty} x(t) = 0$$

If such a signal

1) satisfies the Dirichlet conditions in any finite interval

2) satisfies the inequality

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

Then the signal can be expressed as a Fourier integral

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \tag{13}$$

where

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{14}$$

2. Autocorrelation and energy spectral density

The autocorrelation function for the transient signal $x(t)$ is defined as

$$\varphi_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{15}$$

It can be shown that

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 e^{j\omega\tau}\,d\omega \tag{16}$$

Now,

$$\Phi_{xx}(\omega) = |X(j\omega)|^2 \tag{17}$$

Where $\Phi_{xx}(\omega)$ is referred to as the energy spectral density of the signal $x(t)$. It can be shown that

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-j\omega\tau}\,d\tau \tag{18}$$

and from eqn. (16)

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{j\omega\tau}\,d\omega \tag{19}$$

3. Crosscorrelation and cross energy density spectra

The crosscorrelation function for two transient signals $x(t)$ and $y(t)$ each of which satisfies the Dirichlet conditions in every finite interval and which satisfy

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

$$\int_{-\infty}^{\infty} |y(t)|\,dt < \infty$$

is defined as:

$$\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)dt \qquad (20)$$

It can be shown that

$$\varphi_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X}(j\omega)Y(j\omega)e^{j\omega\tau}\,d\omega \qquad (21)$$

Now

$$\Phi_{xy}(\omega) = \overline{X}(j\omega)Y(j\omega) \qquad (22)$$

Where $\Phi_{xy}(\omega)$ is referred to as the cross energy spectral density of the signals $x(t)$ and $y(t)$.

It can be shown that

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} \varphi_{xy}(\tau)e^{-j\omega\tau}\,d\tau \qquad (23)$$

and from eqn. (21)

$$\varphi_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xy}(\omega)e^{j\omega\tau}\, d\omega \tag{24}$$

Two transient signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

C. Random Signals

1. Fourier Integral

Consider a random signal x(t) as a sample function from an ergodic random process. Since, in general,

$$\int_{-\infty}^{\infty} |x(t)|\, dt$$

is not finite, one cannot write a Fourier integral representation for x(t).

2. Autocorrelation and Power Spectral Density

The autocorrelation function for the random signal x(t) is defined

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt \tag{25}$$

Now $\varphi_{xx}(\tau)$ can be represented by a Fourier integral since it satisfies the two conditions of section B. 1. Hence, it can be shown

$$\varphi_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{xx}(\omega)e^{j\omega\tau}\, d\omega \tag{26}$$

where $\Phi_{xx}(\omega)$ is referred to as the power spectral density of the signal $x(t)$. Here

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau)e^{-j\omega\tau}\, d\tau \tag{27}$$

3. Crosscorrelation and Cross Power Spectral Density

The crosscorrelation function of two signals $x(t)$ and $y(t)$ which are sample functions from two different random processes, each of which are stationary and ergodic and jointly ergodic, is defined

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)dt \tag{28}$$

Now it can be shown that

$$\varphi_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xy}(\omega)e^{j\omega\tau}\, d\omega \tag{29}$$

where $\Phi_{xy}(\omega)$ is referred to as the cross power spectral density of the two random signals $x(t)$ and $y(t)$. Here

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} \varphi_{xy}(\tau)e^{-j\omega\tau}\, d\tau \tag{30}$$

Two random signals are said to be linearly uncorrelated when

$$\varphi_{xy}(\tau) = 0 \quad \text{for all } \tau.$$

Appendix II

Tabulated Values of the Integral Form

$$I_n = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c(j\omega)c(-j\omega)}{d(j\omega)d(-j\omega)}\,d\omega$$

Note: we have said that

$$\overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(\omega)\,d\omega$$

but since power spectral densities are even functions of frequency, they can always be subject to "spectrum factorization" i.e.,

$$\Phi_{xx}(\omega) = \frac{c(j\omega)\cdot c(-j\omega)}{d(j\omega)\cdot d(-j\omega)}$$

The table on the next pages gives closed-form solutions to the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{xx}(\omega)\,d\omega$$

Given the integral form

$$I_n = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{c(j\omega)c(-j\omega)}{d(j\omega)d(-j\omega)}\,d\omega$$

where

$$c(j\omega) = c_{n-1}(j\omega)^{n-1} + c_{n-2}(j\omega)^{n-2} + \cdots + c_0$$

$$d(j\omega) = d_n(j\omega)^n + \cdots + d_0$$

The integral values are

<u>$n = 1$</u>

$$I_1 = \frac{c_0^2}{2d_0d_1}$$

<u>$n = 2$</u>

$$I_2 = \frac{c_1^2d_0 + c_0^2d_2}{2d_0d_1d_2}$$

<u>$n = 3$</u>

$$I_3 = \frac{c_2^2d_0d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2d_2d_3}{2d_0d_3(-d_0d_3 + d_1d_2)}$$

<u>$n = 4$</u>

$$I_4 = \frac{N_1 + N_2}{D}$$

$$N_1 = c_3^2(-d_0^2d_3 + d_0d_1d_2) + (c_2^2 - 2c_1c_3)d_0d_1d_4$$

$$N_2 = (c_1^2 - 2c_0c_2)d_0d_3d_4 + c_0^2(-d_1d_4^2 + d_2d_3d_4)$$

$$D = 2d_0d_4(-d_0d_3^2 - d_1^2d_4 + d_1d_2d_3)$$

For I_5, I_6 etc, see Newton, Gould and Kaiser, <u>Analytical Design of Linear Feecback Controls</u>, Wiley, 1957.

Incorporating Large Scale Atmospheric Turbulence Effects in Aircraft Equations of Motion

We consider a "frozen" turbulence field, in which the three velocity components in an earth-fixed axis system are a function of only one's position in that axis system, i.e.

$$u_g = u_g(X,Y,Z);\; v_g = v_g(X,Y,Z);\; w_g = w_g(X,Y,Z);$$

Where X,Y,Z represents a position in the earth-fixed axis system.

Now the linearized aircraft equations of motion, split into two independent groups, with one group describing *longitudinal* motion in which the aircraft's y body axis remains horizontal, and *lateral-directional* motion, in which the aircraft's x-body axis remains within the plane defined by the equilibrium position of the x-y body axes.

We wish to approximate the effects of u_g, v_g and w_g, in the equations of motion already derived by simply modifying the linear and angular velocity components in the aerodynamic terms in theses equations.

To this end, for the effects of u_g, we will consider that it effects only the longitudinal equations, and we will consider the entire aircraft enveloped in the gust velocity $u_g(X_{cg},Y_{cg},Z_{cg})$, i.e., the instantaneous u_g occurring at the aircraft's center of mass. This is equivalent to considering the aircraft as a point.

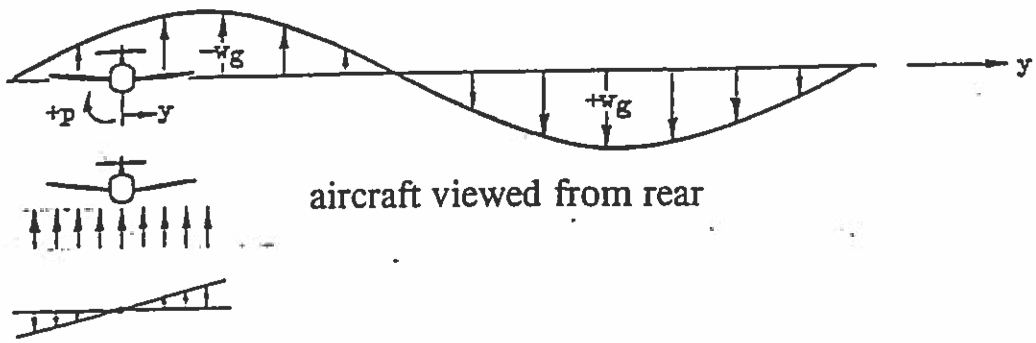
For the effects of v_g, we will consider that it affects only the lateral-directional equations, but we will take into account how v_g varies along the aircraft's x-body axis, i.e. consider the aircraft enveloped in $v_g(X,Y_{cg},Z_{cg})$ where X is now the coordinate of any point along the aircraft's x-body axis, but expressed in the earth-fixed coordinate frame.

Finally, for the effects of w_g, we will consider that it effects both the longitudinal and lateral-directional equations. For the longitudinal equations, we will consider the aircraft enveloped in $w_g(X,Y_{cg},Z_{cg})$ where again, X is the coordinate of any point along the aircraft's x-body axis, but expressed in the earth-fixed coordinate frame. For the lateral-directional equations, we will consider the aircraft enveloped in $w_g(X_{cg},Y,Z_{cg})$ where Y is the coordinate of any point along the aircraft's y-body axis, but expressed in the earth-fixed coordinate frame.

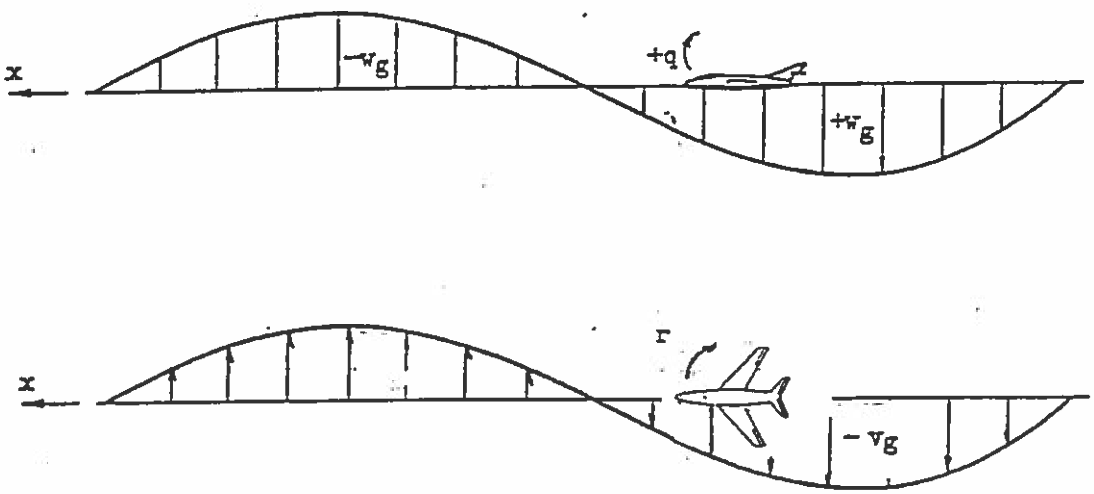
We will first write the longitudinal and lateral-directional equations (now independent) in first-order form. Then in each *aerodynamic* term (as opposed to an *acceleration* or *gravitational* term) we will replace linear velocities by u-u_g, v-v_g and w-w_g, and the angular velocities by p-p_g, q-q_g and r-r_g. The p_g, q_g, and r_g terms seem to have appeared out of nowhere here, but they will be used to account for the linear variation of v_g and w_g

along the aircraft's x and y body axes, as will be seen. Implicit in this treatment is that we will be considering gust wavelengths that are long with respect to the aircraft's length and wing span.

The figures below are from the McRuer, Ashkenas and Graham report and indicate how the p_g, q_g, and r_g terms are obtained.



aircraft viewed from rear



In the first figure above, we write

aircraft viewed from above

$$p_g = \frac{\partial w_g(X_{cg}, Y)}{\partial Y}$$

where $X_{cg} = U_0 t$

In the second figure above, we write

$$q_g = -\frac{\partial w_g(X, Y_{cg})}{\partial X} = -\frac{\partial w_g / \partial t}{\partial X / \partial t} = -\frac{\dot{w}_g}{U_0}$$

Finally, in the last figure, we write

$$r_g = \frac{\partial v_g(X, Y_{cg})}{\partial X} = \frac{\partial v_g / \partial t}{\partial X / \partial t} = \frac{\dot{v}_g}{U_0}$$

Note, the apparent sign reversal in these gust terms is due to the fact that in the equations of motion we employ the terms as $p\text{-}p_g$, etc., i.e., with a minus sign.

Below, the resulting longitudinal equations with gust terms are presented.

$$\dot{u} = [X_u + aX_{\dot{w}}Z_u](u-u_g) + [X_w + aX_{\dot{w}}Z_w](w-w_g) + [aX_{\dot{w}}(U_0+Z_q)+X_q](q-q_g) +$$

$$g[-\cos\theta_0 - aX_{\dot{w}}\sin\theta_0]\theta + \sum_i [aX_{\dot{w}}Z_{\delta_i} + X_{\delta_i}]\delta_i$$

$$\dot{w} = a\left\{Z_u(u-u_g) + Z_w(w-w_g) + U_0 q + Z_q(q-q_g) + [-g\sin\theta_0]\theta + \sum_i Z_{\delta_i}\delta_i\right\}$$

$$\dot{q} = [M_u + aM_{\dot{w}}Z_u](u-u_g) + [M_w + aM_{\dot{w}}Z_w](w-w_g) + [M_q + aM_{\dot{w}}(U_0+Z_q)](q-q_g) +$$

$$[-aM_{\dot{w}}g\sin\theta_0]\theta + \sum_i [aM_{\dot{w}}Z_{\delta_i} + M_{\delta_i}]\delta_i$$

$$\dot{\theta} = q$$

where $a = \dfrac{1}{1-Z_{\dot{w}}}$

If needed, $\alpha = \dfrac{w}{U_0}$ = perturbation angle of attack

NOTE: These equations have been simplified somewhat from their counterparts on pp. 4-61 in McRuer, Ashkenas and Graham. First, the equations without gust terms were put in first-order form, and then gust terms like $u-u_g$, etc. were inserted in the appropriate place in term on the right hand side. Note the second term on the right hand side of the $\dot{w}$ equation is slightly different, i.e. it is $U_0 q + Z_q(q-q_g)$. This is because the $U_0 q$ term is really an "inertial" term rather than an "aerodynamic" one, and therefore, should not be multiplied by $q\text{-}q_g$.

Below, the resulting lateral-directional equations with gust terms are presented.

$$\dot{v} = Y_v(v-v_g) + Y_p(p-p_g) + Y_r(r-r_g) - U_0 r + g\cos\theta_0\phi + \sum_j Y_{\delta_j}\delta_j$$

$$\dot{p} = [L'_v+bL'_{\dot{v}}Y_v](v-v_g) + [L'_p+bL'_{\dot{v}}Y_p](p-p_g) + [L'_r+bL'_{\dot{v}}Y_r](r-r_g) - [L'_r+bL'_{\dot{v}}U_0]r_g +$$
$$[bL'_{\dot{v}}g\cos\theta_0]\phi + \sum_j [L'_{\delta_j}+bL'_{\dot{v}}Y_{\delta_j}]\delta_j$$

$$\dot{r} = [N'_v+bN'_{\dot{v}}Y_v](v-v_g) + [N'_p+bN'_{\dot{v}}Y_p](p-p_g)+ +[N'_r+bN'_{\dot{v}}Y_r](r-r_g) - [N'_r+bN'_{\dot{v}}U_0]r_g +$$
$$[bN'_{\dot{v}}g\cos\theta_0]\phi + \sum_j [N'_{\delta_j}+bN'_{\dot{v}}Y_{\delta_j}]\delta_j$$

$$\dot{\phi} = p + \tan\theta_0 r$$

$$\psi = \sec\theta_0 r$$

$$b = \frac{1}{1-Y_{\dot{v}}} \qquad L'_x = \frac{L_x+\frac{I_{xz}}{I_x}N_x}{(1-\frac{I_{xz}^2}{I_xI_z})} \qquad N'_x = \frac{N_x+\frac{I_{xz}}{I_z}L_x}{(1-\frac{I_{xz}^2}{I_xI_z})}$$

If needed, $\beta = \frac{v}{U_0}$ = perturbation sideslip angle

Changing Turbulence Spectra from Spatial Frequency to Temporal Frequency

In order to be of use in aircraft stability and control analyses the power spectral density of the frozen turbulence field needs to be expressed in terms of temporal frequency ω rather than spatial frequency Ω. This can be done by appealing to the expression for the mean square value of the turbulence velocity in any direction, say w.

$$\sigma_w^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{w_g}(\Omega)\,d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{w_g}(\omega)\,d\omega$$

with

$$\omega = \Omega\cdot U_0$$

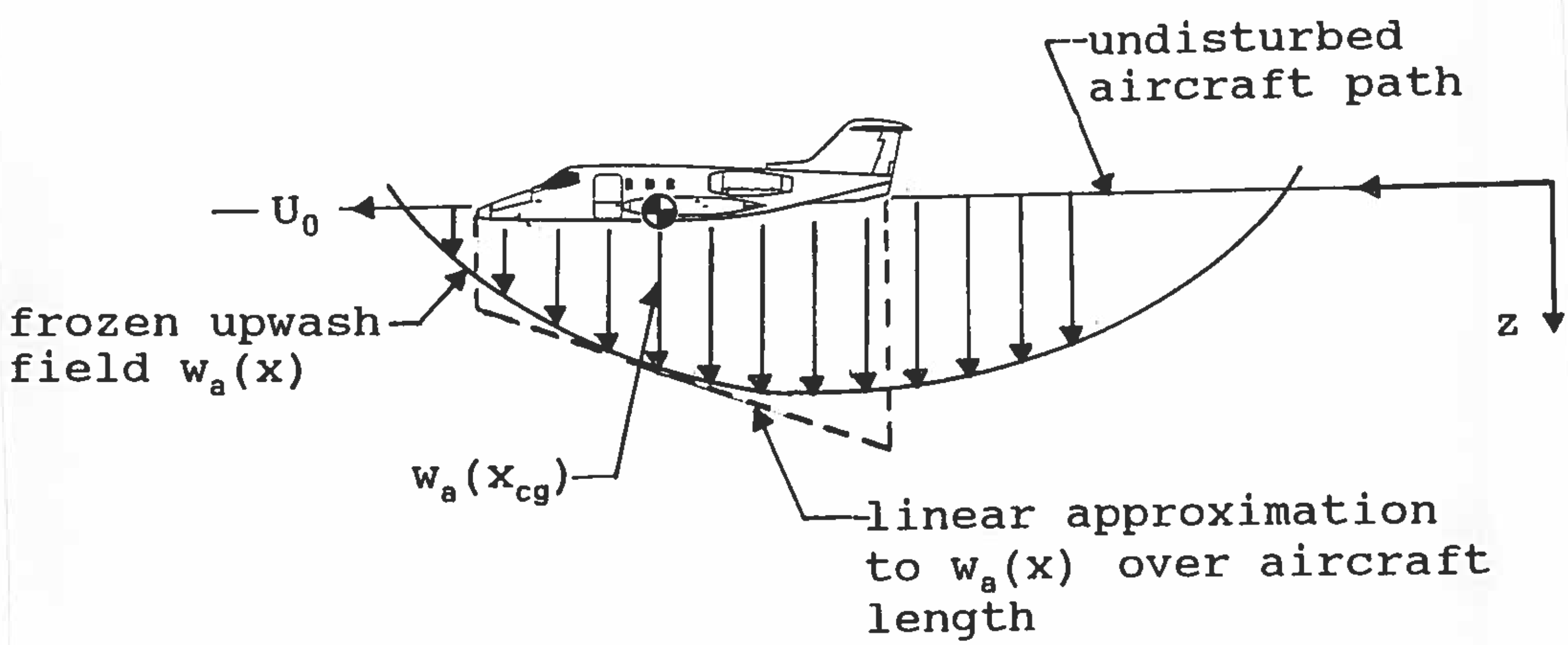
one obtains

$$\Phi_{w_g}(\omega) = \frac{1}{U_0}\Phi_{w_g}(\Omega)\,\Big|_{\Omega = \omega/U_0}$$

Thus, for example, the spectral relation on page 2 becomes

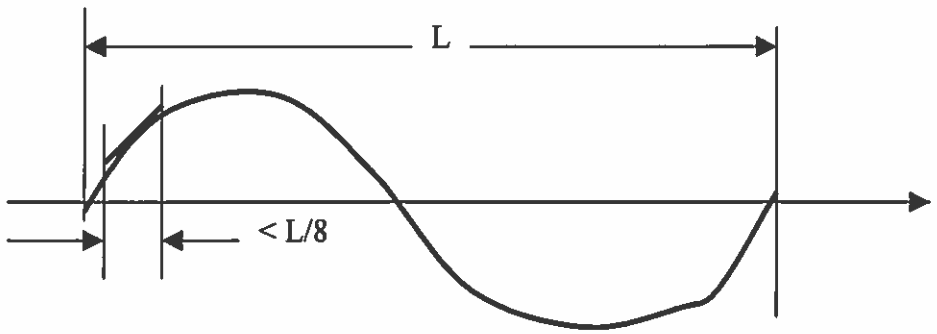
$$\Phi_{w_g}(\omega) = \frac{2\sigma_w^2 L_w}{U_0}\cdot\frac{1}{1+\left[\frac{L_w\omega}{U_0}\right]^2}$$

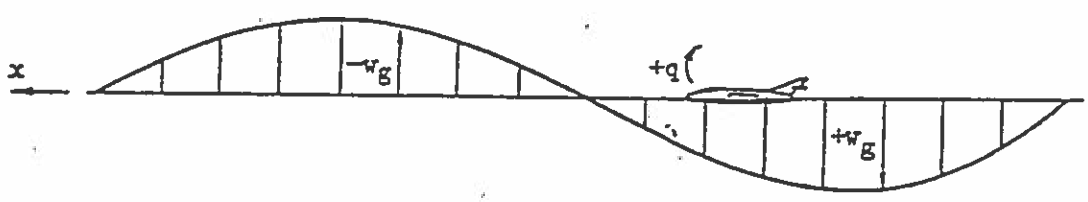
The following page demonstrates the ability of an autopilot ($\dot{\theta}$ and θ) feedback to reduce the power spectral density of normal acceleration at the aircraft cg when the aircraft encounters a vertical turblence field given by the last equation above with $\sigma_{w_g} = 10$ *ft*/sec and $L_w = 2000$ *ft*. Also note that most of the turbulence power is located at temporal frequencies below the spatial frequency corresponding to the spatial wavelength that is greater than 8 times the fuselage length, l_f.

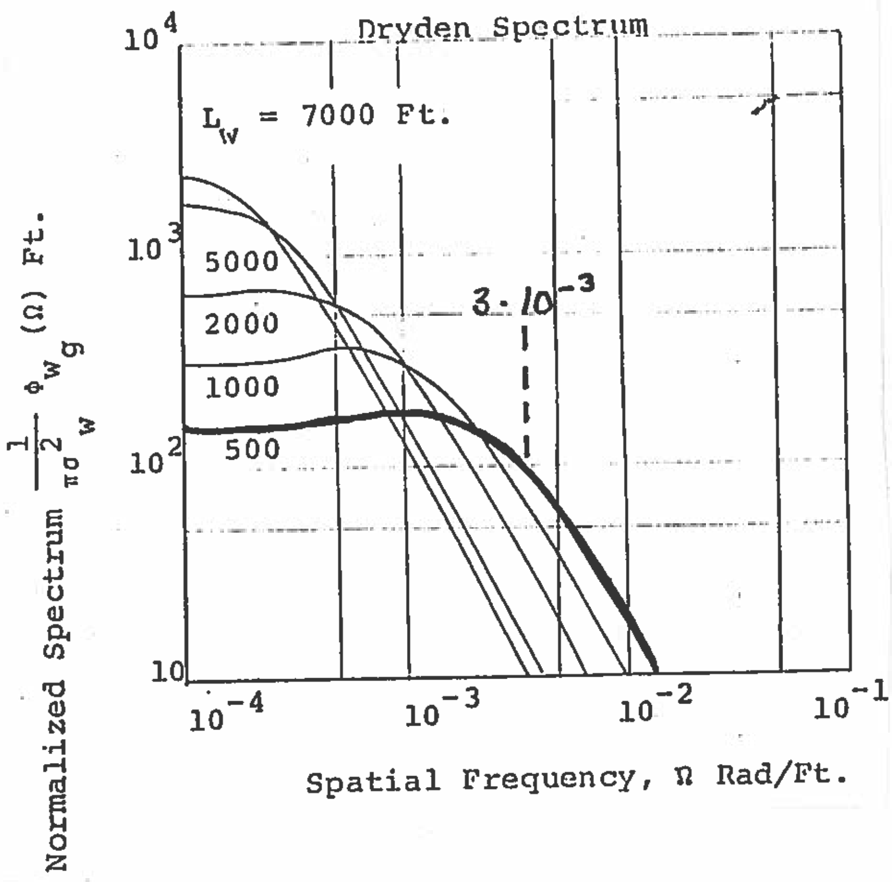


N.B:

In order to approximate portions of a sine wave by a straight line, the linear distance line over which you wish to make the approximation must be less than 1/8th the period of the sinusoid.







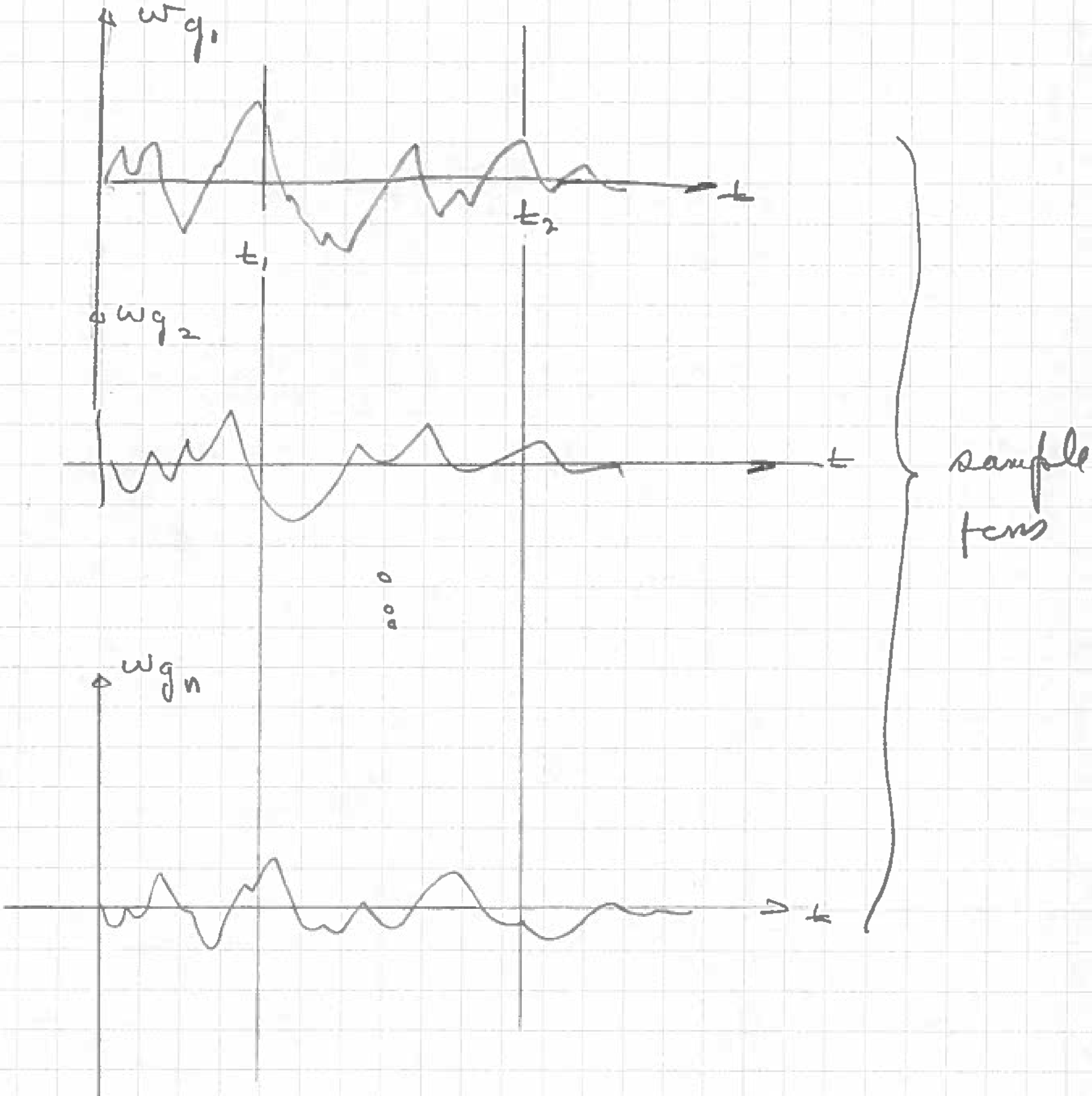
$\Omega_B = 3 \cdot 10^{-3}$

"PERIOD" $= \dfrac{2\pi}{\Omega_B}$

$\cong 2{,}100$ ft

$\dfrac{2100}{8} \approx 262$ ft

Vertical gust at a pt in space



collection of values of w_{g_i} @ $t = t_1$ defines a random variable, same for w_{g_i} @ $t = t_2$

If turb is stationary statistical properties of each RV is independent of the time at which it is defined.

VOL. 11, NO. 4, APRIL 1973 AIAA JOURNAL

Dryden Research Lecture

Atmospheric Turbulence

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Atmospheric turbulence and its influence on the flight and design of aircraft is reviewed. Coverage includes accidents and costs incurred as a result of turbulence, the mechanisms of turbulence measurements that are made, aircraft response and design procedures due to turbulence encounter, and loads alleviation devices. Basic research notions associated with the interpretation of turbulence data, some of which are controversial, and areas of weakness, are also discussed.

Table 2 Classification of turbulence severity

	w, fps	$\sigma_{\Delta n}$	Δn	Wind shear fps/1000 ft	Clouds
	5	0.05	0.15	2.5	
Light					Cumulus
	20	0.2	0.6	10	
Moderate					Alto-cumulus, thunderstorms
	35	0.35	1.05	17.5	
Severe					Mature thunderstorms
	50	0.5	1.5		
Extreme					Severe thunderstorms

$\sigma_{\Delta n}$ = RMS vertical accel
Δn = peak " "

Light: Occupants may feel a slight strain against seat belts or shoulder straps. Unsecured objects may be displaced slightly. Food services may be conducted and little or no difficulty is encountered in walking.

Moderate: Occupants feel definite strains against seat belts or shoulder straps. Unsecure objects are dislodged. Food services and walking are difficult.

Severe: Occupants are forced violently against seat belts or shoulder straps. Unsecure objects are dislodged. Food services and walking are impossible. Airplane may be momentarily out of control.

Extreme: Aircraft is violently tossed about and is practically impossible to control. Structural damage may occur.

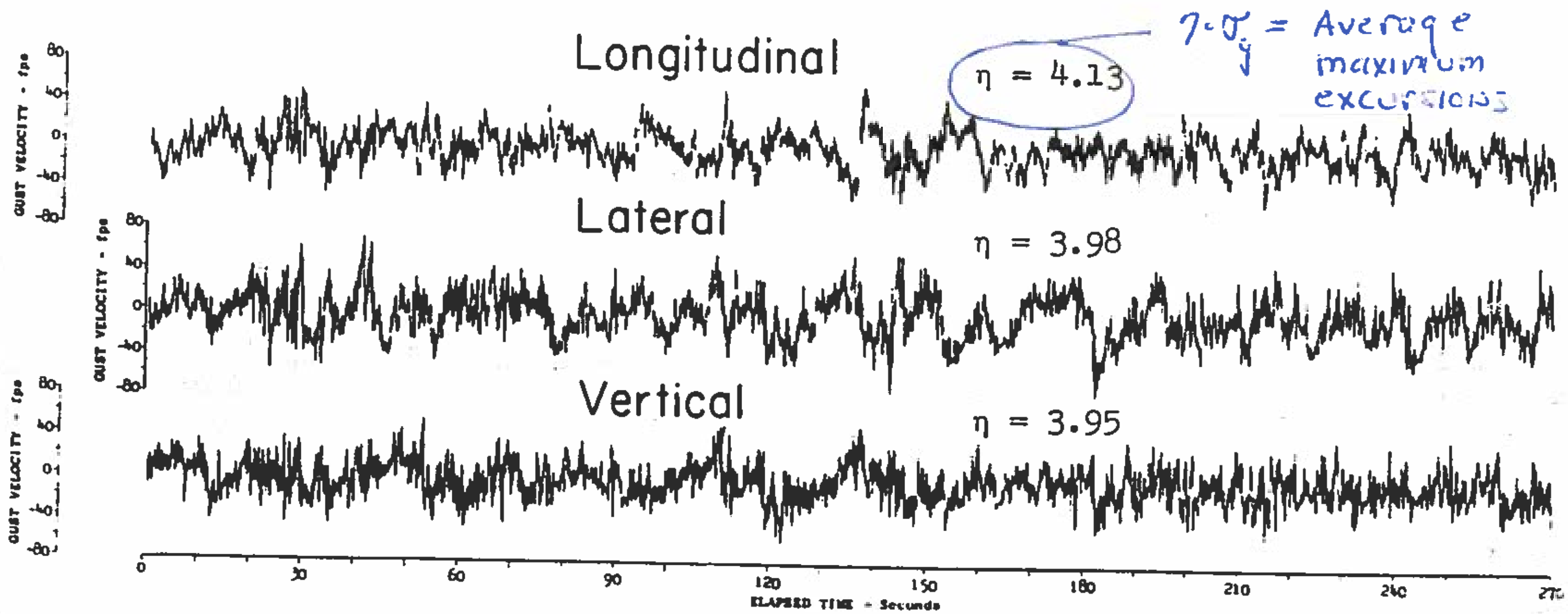


Fig. 4 Illustrations of measured turbulence velocities (severe).

[13] Jones, J. W., Mielke, R. H., Jones, G. W., et al., "Low Altitude Atmospheric Turbulence, LO-LOCAT Phase III," Vol. I, Pt. I, Data Analysis, AFFDL-TR-70-10, Nov. 1970, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.

Spectrum factorization: Dryden[redacted]

conjugate

$$\Phi_{w_g}(\Omega) = \frac{\sigma_w^2 L_w \left[\sqrt{3}\, L_w (j\Omega) [redacted] 1\right]\left[\;[redacted]\; \cdot \;\right]}{\left[(L_w)^2 (j\Omega)^2 + 2 L_w (j\Omega) [redacted] 1\right]\left[\; \cdot \;[redacted]\right]}$$

conjugate

$n = 2$

σ_w^2 [redacted] $I_2 = c \cdot \dfrac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2}$

c_1 [redacted]

[redacted]

[redacted] $= L_w^2$

$d_1 = 2L_w$

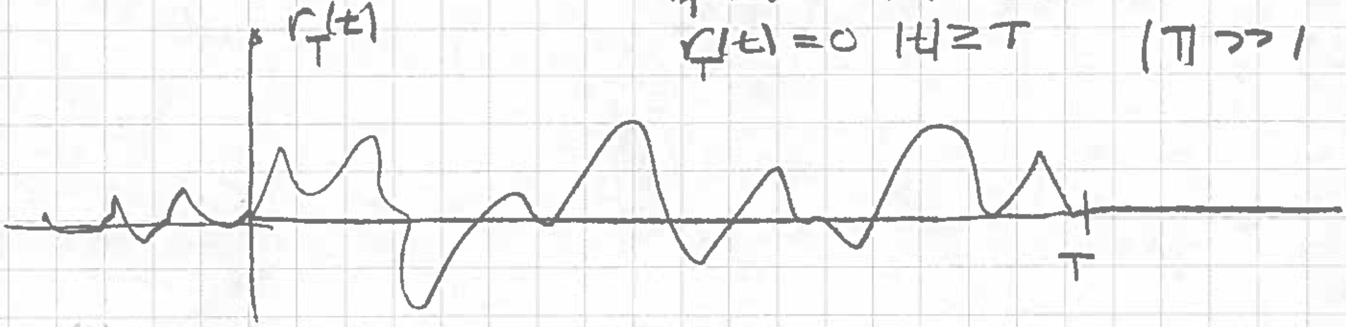
$d_0 = 1$

$$\sigma_w^2 = \sigma_w^2 L_w \frac{(3L_w)^2(1) [redacted] (1)(L_w)^2}{2(1)(2L_w)L_w^2}$$

[redacted] $\dfrac{4L_w^2}{4L_w^3} =$ [redacted] consider a random signal and truncate it at a finite, but large time T

$r_T(t) = r(t)$ -- $-T \le t \le T$

$r_T(t) = 0 \quad |t| \ge T \qquad |T| >> 1$



now $r_T(t)$ possesses a Fourier Transform

$$R_T(j\omega) = \int_{-\infty}^{\infty} r(t)\, e^{-j\omega t}\, dt$$

and an inverse transform

$$r_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_T(j\omega)\, e^{j\omega t}\, d\omega$$

$$\doteq \frac{1}{2\pi}\sum_{-\infty}^{\infty} R_T(nj\Delta\omega)\, e^{-jn(\Delta\omega)t}\, \Delta\omega$$

$$= \frac{1}{2\pi} R_T(0)\,\Delta\omega + \frac{1}{2\pi}\sum_{1}^{\infty}\left[R_T(nj\Delta\omega)\, e^{-jn(\Delta\omega)t} + R_T(-nj\Delta\omega)\, e^{+jn(\Delta\omega)t}\right]\Delta\omega$$

now $R_T(-nj\Delta\omega) = \overline{R_T(nj\Delta\omega)}$ complex conjugate

a <u>single</u> <u>term</u> in the RHS can be written

$$\frac{\Delta\omega}{\pi}\left|R_T(jn\Delta\omega)\right| \sin\left(n(\Delta\omega)t + \varphi\right)$$

$$\varphi = \tan^{-1}\frac{Re\left[R(j\omega)\right]}{Im\left[R(j\omega)\right]}$$

Let $nj\Delta\omega = x$ $\qquad R_T(-x) = \overline{R(x)}$

$$r_T(t) \approx \frac{1}{2\pi} R_T(0) + \frac{1}{2\pi}\sum_{1}^{\infty}\Big[R_T(\cos x - j\sin x) +$$

$$\overline{R_T}(\cos x - j\sin x)\Big]\Delta\omega$$

Look @ 1 term in summation:

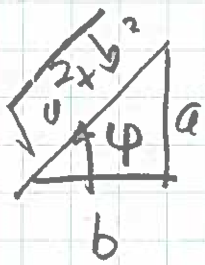
Let $R_T = a + bj$

$\overline{R_T} = a - bj$

$$(a+bj)(\cos x - j\sin x) + (a-bj)(\cos x + j\sin x)$$

$$= 2a\cos x + 2b\sin x$$

$$2\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right]$$



$$\therefore \frac{a}{\sqrt{\ }} = \sin\varphi$$

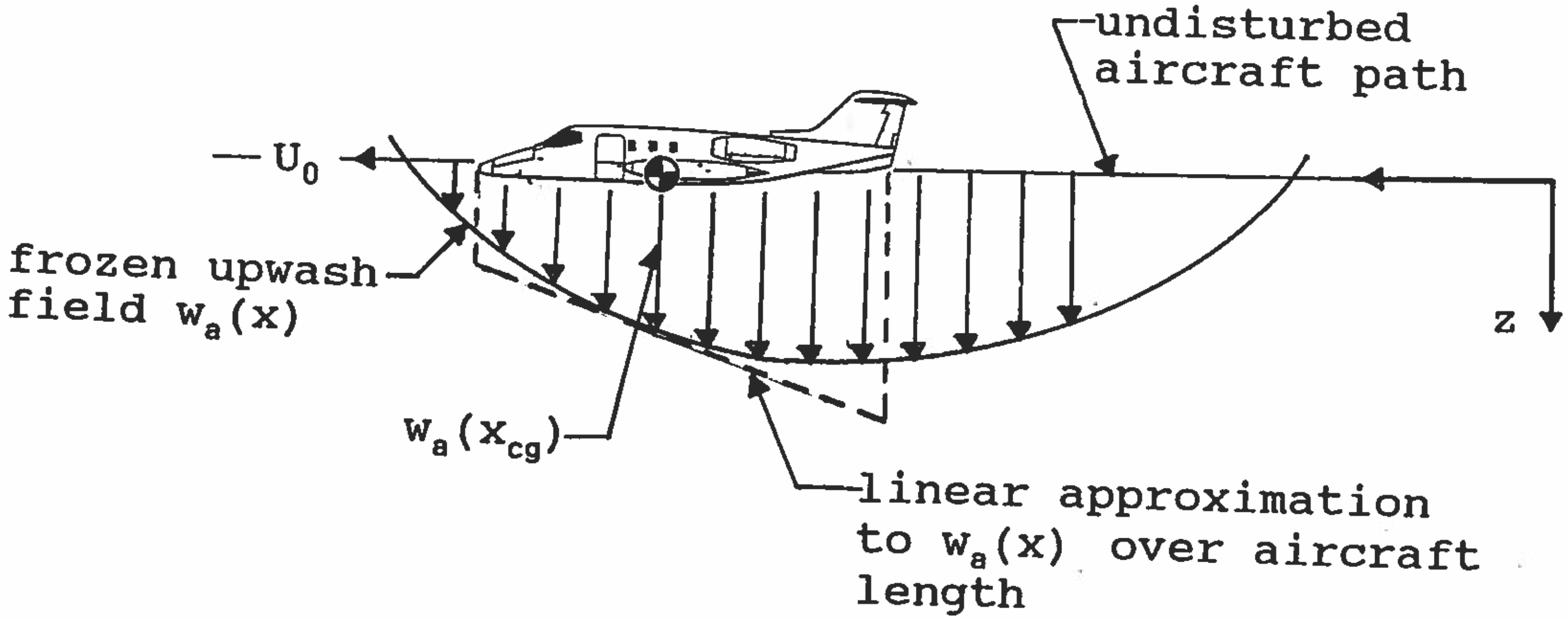
$$\frac{b}{\sqrt{\ }} = \cos\varphi$$

$$2\sqrt{a^2+b^2}\left[\sin(x+\varphi)\right]$$

$$r_T(t) \triangleq \frac{1}{2\pi} R_T(0) + \frac{1}{\pi} \sum_{i=1}^{\infty} \left| R_T(jn\Delta\omega) \right| \sin(nj\Delta\omega + \varphi)\, \Delta\omega$$

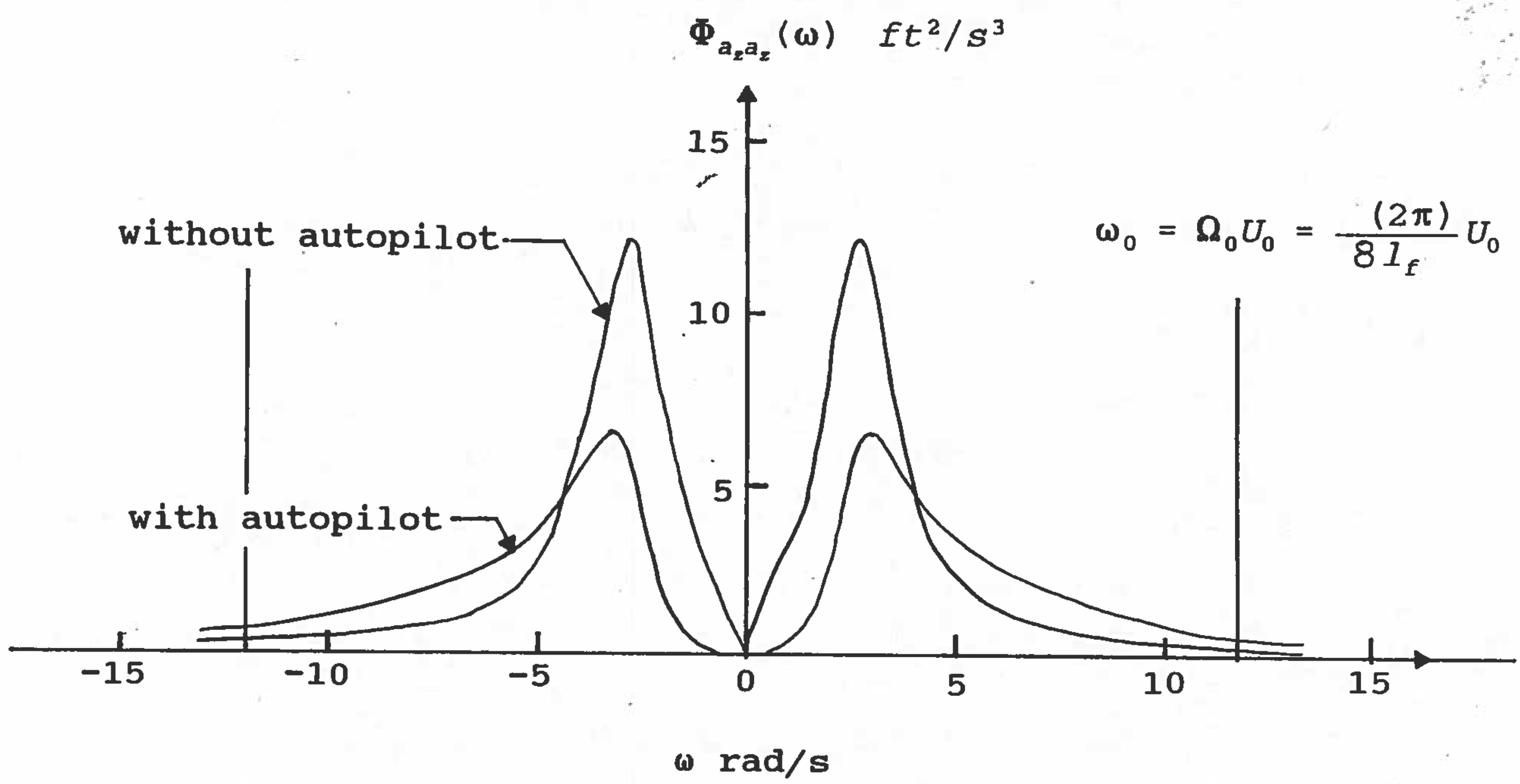
one term: $$\frac{\Delta\omega}{\pi} \left| R_T(jn\Delta\omega) \right| \sin(nj\Delta\omega + \varphi)$$

$$\varphi = \tan^{-1} \frac{R_e[R_T(j\omega)]}{I_m[R_T(j\omega)]}$$



undisturbed aircraft path
$-U_0$
frozen upwash field $w_a(x)$
z
$w_a(x_{cg})$
linear approximation to $w_a(x)$ over aircraft length

Fig. 2



$$\sigma_{a_{z_{cg}}} = \sqrt{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} a_{z_{cg}}^2(t)\,dt} = \sigma_{a_{z_{cg}}} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{a_z a_z}(\omega)\,d\omega}$$

$$\Phi_{a_z a_z}(\omega) = \left|\frac{a_{z_{cg}}}{w_g}(j\omega)\right|^2 \Phi_{w_g w_g}(\omega) = \left|\frac{a_{z_{cg}}}{w_g}(j\omega)\right|^2 \left[\frac{1}{U_0}\Phi_{w_g w_g}(\Omega)\Big|_{\Omega=\frac{\omega}{U_0}}\right]$$

Dept. of Mechanical and Aeronautical Engineering

EME-275

Creating Sample Functions from Random Processes with Simulink

The following pages demonstrate the creation of a sample function from a random process by passing an approximation of "white" noise through a shaping filter to obtain "colored" noise with a desired power spectral density.

White noise is a fictitious random signal whose autocorrelation function is a weighted impulse, i.e.

$$\varphi_{ww}(\tau) = K\delta(\tau) \tag{1}$$

where w(t) = white noise signal.

The power spectral density (PSD) of white noise is constant at all frequencies, i.e.,

$$\Phi_{ww}(\omega) = K \tag{2}$$

Recalling that the mean square value of w(t) would be given by

$$\overline{w}^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ww}(\omega)d\omega = \infty \tag{3}$$

a white noise signal would imply one with infinite power! Hence the fictitious nature of a true white noise signal. However, a white noise signal can be approximated by one with a constant PSD over a frequency range from zero to well beyond the bandwidth of the system to which it is serving as an input.

When white noise (or an approximation thereof) is passed through a filter, the resulting noise is often referred to as "colored"noise. The Simulink diagram and simulation results which follow demonstrate this. Indeed, this is the way we can create a sample function from a random process with a desired PSD.

Assume you wish to create a sample function from a random process with a power spectral density given by

$$\Phi_{cc}(\omega) = \frac{1}{\omega^4 + 2\omega^2 + 1} \tag{4}$$

This can be written as

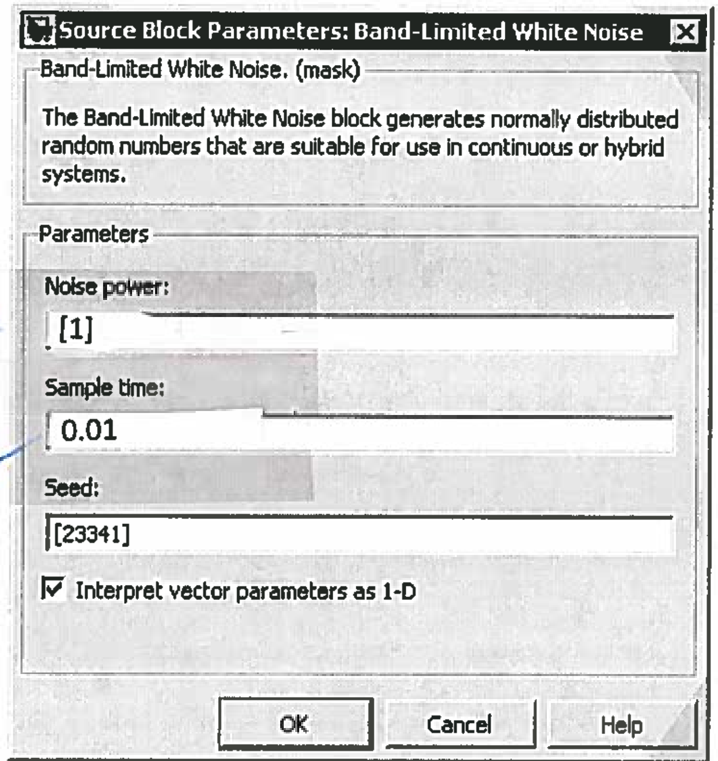
$$\Phi_{cc}(\omega) = \left|G(s)G(-s)\right|_{s=j\omega} \tag{5}$$

Equations (4) and (5) lead to

$$G(s) = \frac{1}{s^2 + 2s + 1} \tag{6}$$

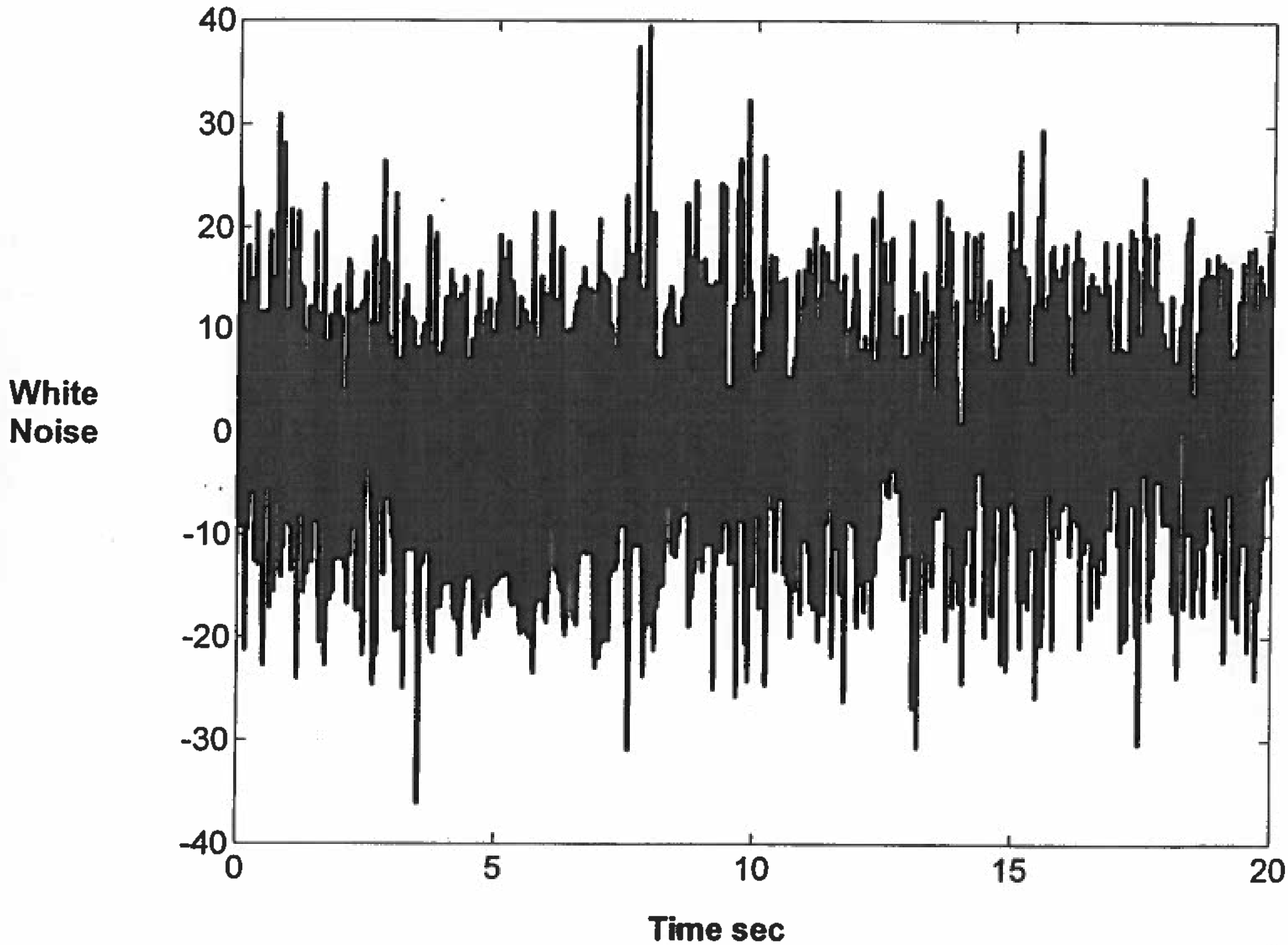
$\left| \Phi_{ww}(\omega) \right|$

w = white noise



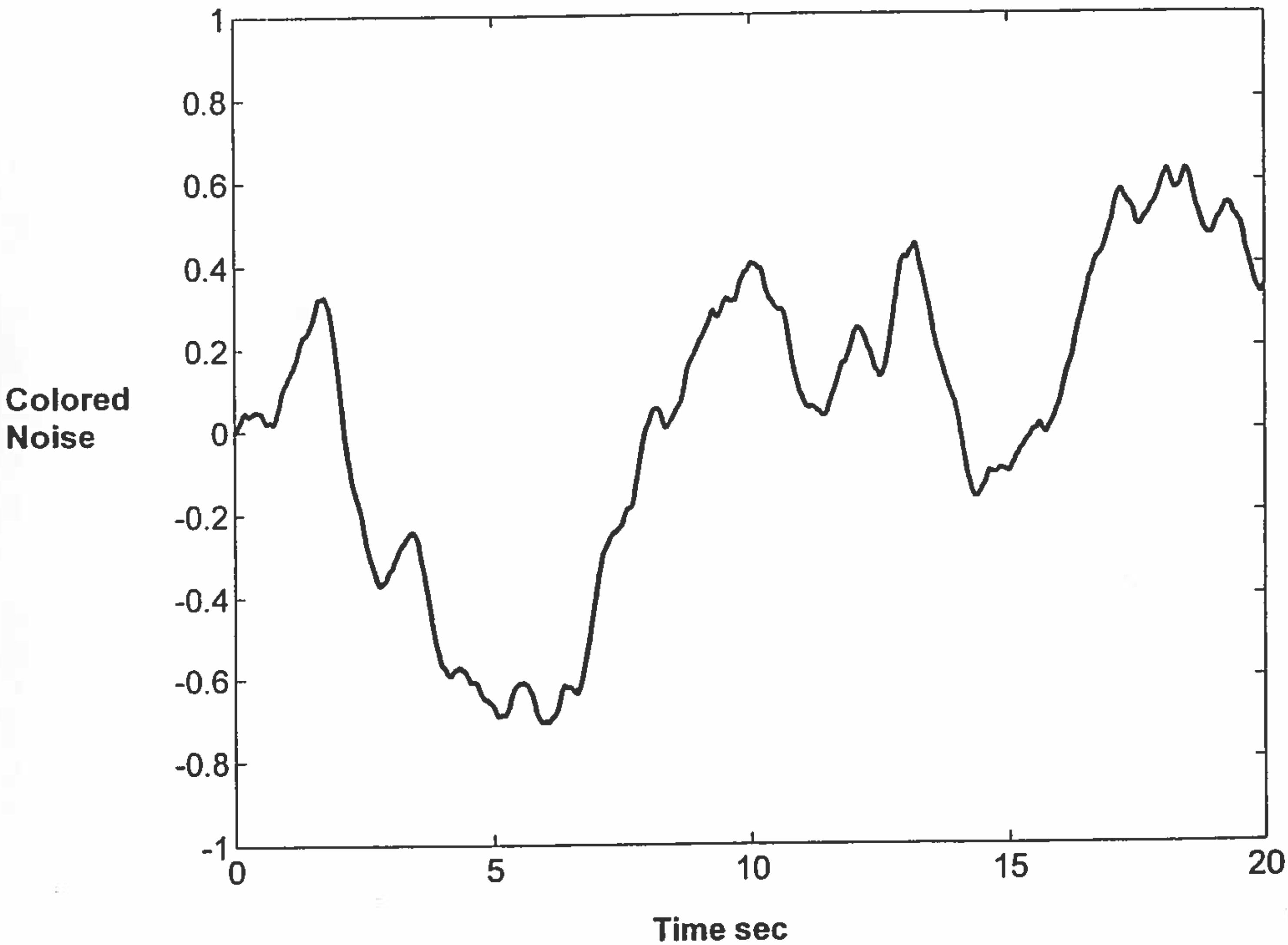
$t_s < \frac{2\pi}{100} \cdot \frac{1}{\omega_{BW}}$

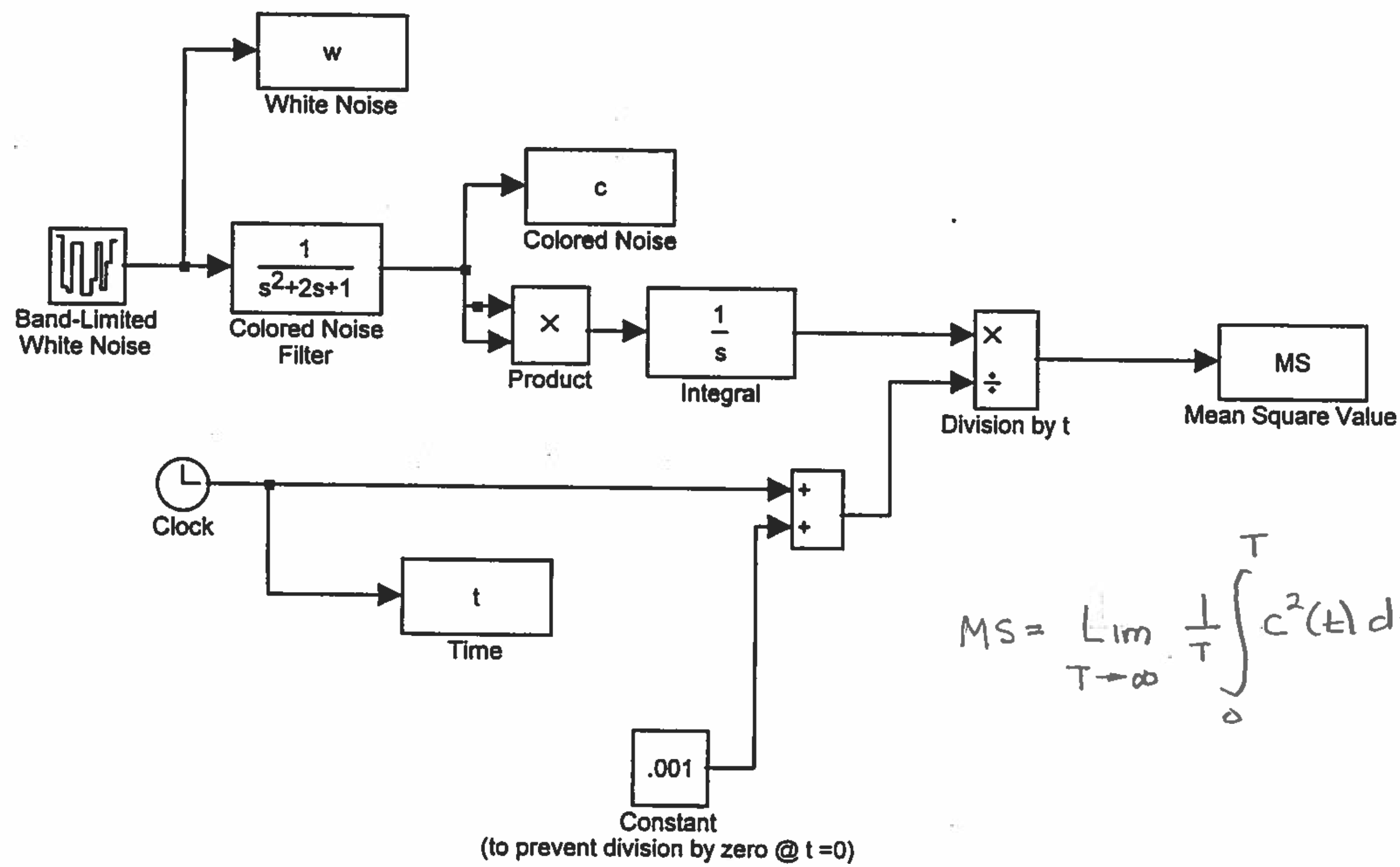
ω_{BW} = bandwidth of colored noise filter



40
30
20
10
0
-10
-20
-30
-40
White
Noise
0
5
10
15
20
Time sec

shaping filter → $\frac{1}{s^2+2s+1}$



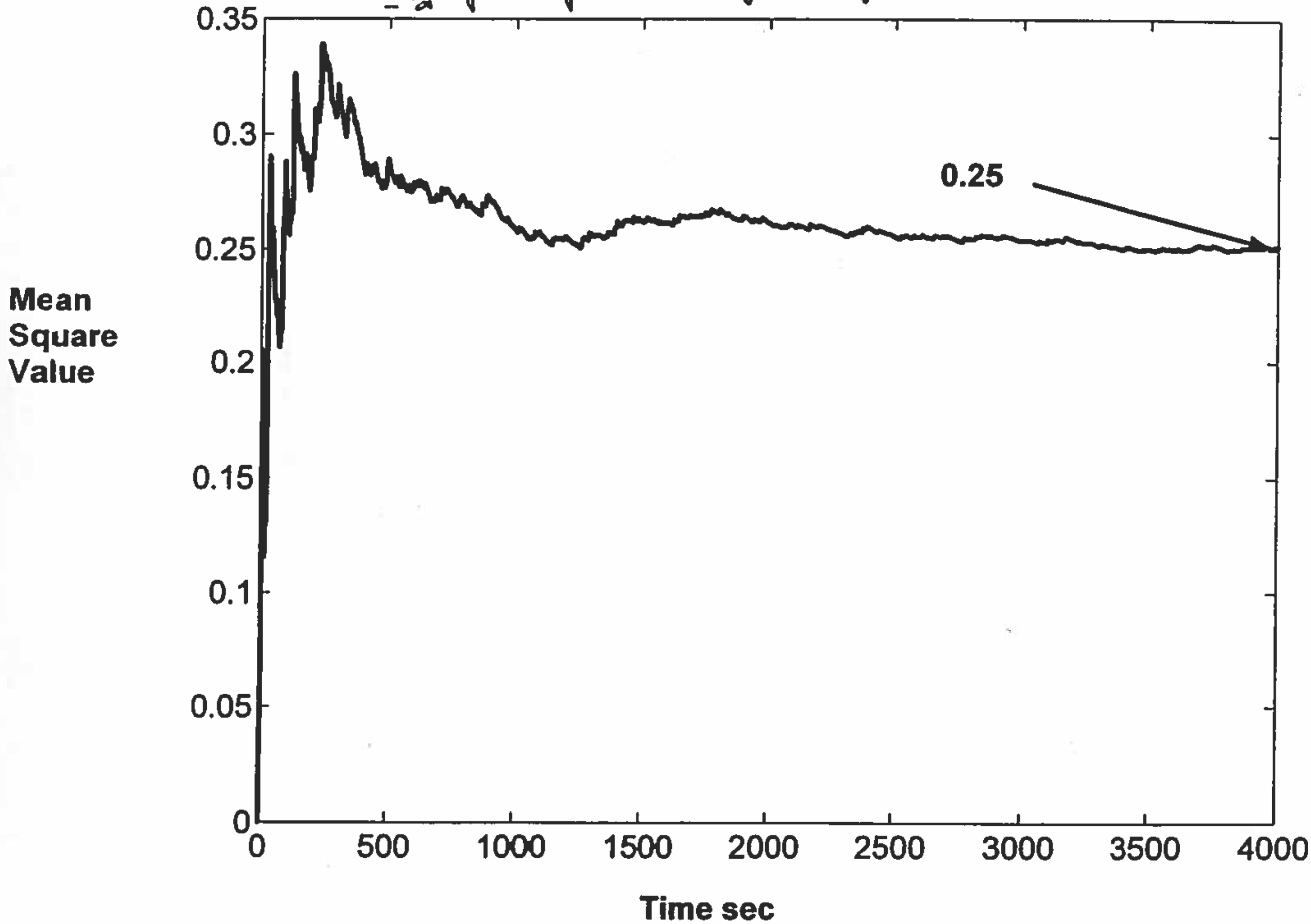


$$MS = \lim_{T \to \infty} \frac{1}{T} \int_0^T c^2(t)\, dt$$

$$\overline{c^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{cc}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ww}(\omega)\cdot\left|\frac{1}{s^2+2s+1}\right|^2_{s=j\omega} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{ww}(\omega)\cdot\frac{1}{(j\omega)^2+2(j\omega)+1}\cdot\frac{1}{(-j\omega)^2+2(-j\omega)+1}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(j\omega)^2+2(j\omega)+1}\cdot\frac{1}{(-j\omega)^2+2(-j\omega)+1}\cdot d\omega$$



$n = 2$

$c_0 = 1$

$d_0 = 1$

$d_2 = 2$

$$I_2 = \frac{c_0^2}{2 d_0 d_1} = \frac{1}{2(2)(1)} = \underline{\underline{0.25}}$$