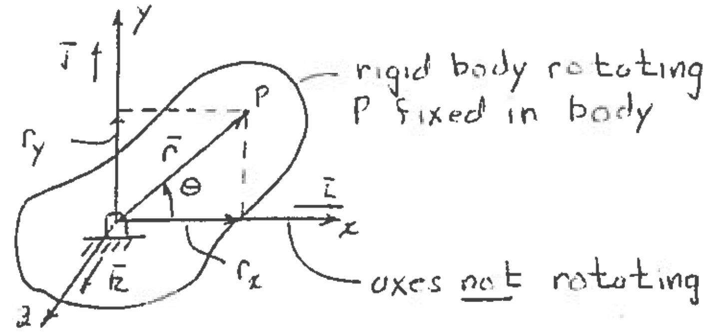
MAE – 275

Derivative of a Position Vector Fixed in a Rotating Body



$\bar{r} = r_x\bar{i} + r_y\bar{j} = (r\cos\theta)\bar{i} + (r\sin\theta)\bar{j}$

$\frac{d\bar{r}}{dt} = (-r\dot{\theta}\sin\theta)\bar{i} + (r\dot{\theta}\cos\theta)\bar{j} = -\dot{\theta}r_y\bar{i} + \dot{\theta}r_x\bar{j}$

let $\bar{\omega} = \dot{\theta}\bar{k}$

$$\bar{\omega}\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & \dot{\theta} \\ r_x & r_y & 0 \end{vmatrix}$$

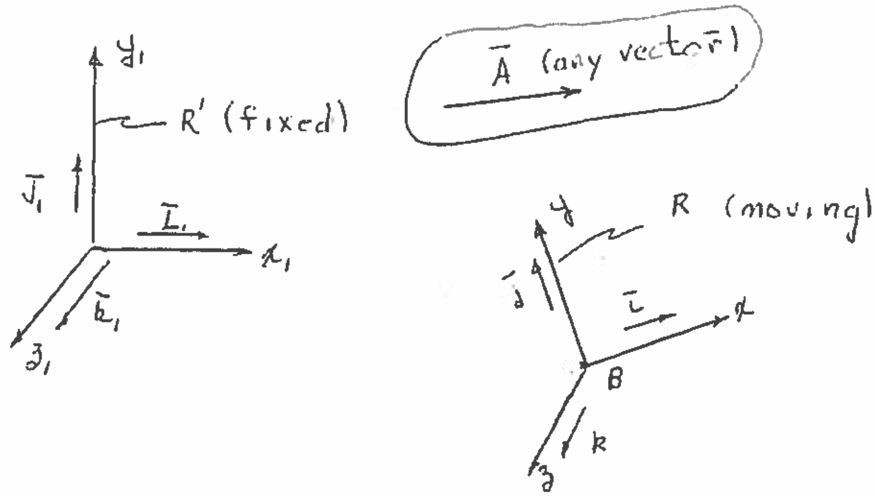
$$= \bar{i}(-\dot{\theta}r_y) - \bar{j}(-\dot{\theta}r_x) + 0\bar{k}$$

$$\bar{\omega}\times\bar{r} = -\dot{\theta}r_y\bar{i} + \dot{\theta}r_x\bar{j}$$

$\therefore$ $\boxed{\frac{d\bar{r}}{dt} = \bar{\omega}\times\bar{r}}$ $\bar{r}$ only rotating, not changing length

Note: on the following page, I use the result above in expressions like ${}^{R'}\frac{d\bar{i}}{dt} = {}^{R'}\bar{\omega}^{R}\times\bar{i}$, etc.

Derivative of a General Vector in Two Frames of Reference



$\bar{A}$ can be <u>expressed</u> in either frame R or R'

$$\bar{A} = \underbrace{A_x \bar{i} + A_y \bar{j} + A_z \bar{k}}_{\bar{A}\text{ expressed in } R} = \underbrace{A_{x_1} \bar{i}_1 + A_{y_1} \bar{j}_1 + A_{z_1} \bar{k}_1}_{\bar{A}\text{ expressed in } R'}$$

Consider $\bar{A} = A_x \bar{i} + A_y \bar{j} + A_z \bar{k}$

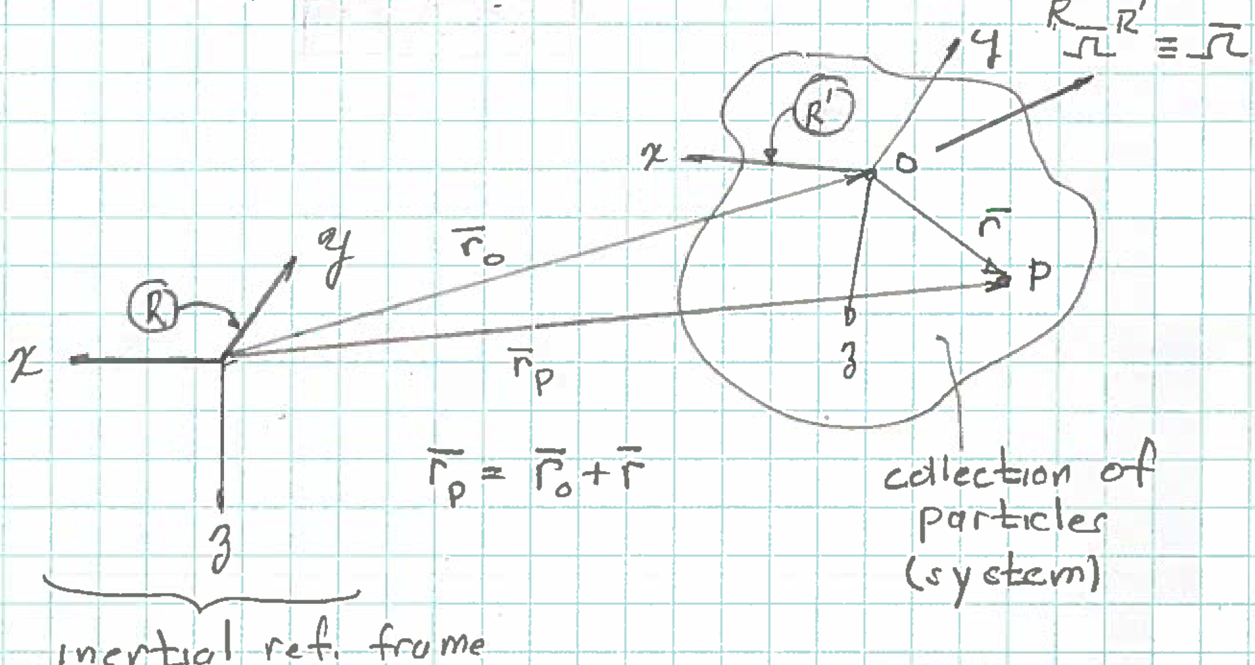
$$\frac{{}^{R'}d\bar{A}}{dt} = \underbrace{\dot{A}_x \bar{i} + \dot{A}_y \bar{j} + \dot{A}_z \bar{k}}_{\frac{{}^{R}d\bar{A}}{dt}} + A_x \underbrace{\frac{{}^{R'}d\bar{i}}{dt}}_{{}^{R'}\bar{\omega}^{R} \times \bar{i}} + A_y \underbrace{\frac{{}^{R'}d\bar{j}}{dt}}_{{}^{R'}\bar{\omega}^{R} \times \bar{j}} + A_z \underbrace{\frac{{}^{R'}d\bar{k}}{dt}}_{{}^{R'}\bar{\omega}^{R} \times \bar{k}}$$

$$\frac{{}^{R'}d\bar{A}}{dt} = \frac{{}^{R}d\bar{A}}{dt} + {}^{R'}\bar{\omega}^{R} \times \underbrace{(A_x \bar{i} + A_y \bar{j} + A_z \bar{k})}_{\bar{A}}$$

$\bar{i}, \bar{j}, \bar{k}$ only rotating; not changing length

$$\boxed{\frac{{}^{R'}d\bar{A}}{dt} = \frac{{}^{R}d\bar{A}}{dt} + {}^{R'}\bar{\omega}^{R} \times \bar{A}}$$

Equations of Motion



$$ {}^R\frac{d\bar r_p}{dt} = {}^R\frac{d\bar r_o}{dt} + {}^R\frac{d\bar r}{dt} = {}^R\frac{d\bar r_o}{dt} + \bar\Omega\times\bar r + {}^{R'}\frac{d\bar r}{dt} \triangleq \bar v_p $$

$$ {}^R\frac{d^2\bar r_p}{dt^2} = {}^R\frac{d^2\bar r_o}{dt^2} + {}^R\frac{d\bar\Omega}{dt}\times\bar r + \bar\Omega\times{}^R\frac{d\bar r}{dt} + {}^R\frac{d}{dt}\left({}^{R'}\frac{d\bar r}{dt}\right) \triangleq \bar a_p $$

$$ = \underbrace{{}^R\frac{d^2\bar r_o}{dt^2}}_{{}^R\frac{d\bar v_o}{dt}} + \dot{\bar\Omega}\times\bar r + \bar\Omega\times\left(\bar\Omega\times\bar r + {}^{R'}\frac{d\bar r}{dt}\right) + \bar\Omega\times{}^{R'}\frac{d\bar r}{dt} + {}^{R'}\frac{d^2\bar r}{dt^2} $$

$$ {}^R\frac{d^2\bar r_p}{dt^2} = \underbrace{{}^R\frac{d\bar v_o}{dt} + \dot{\bar\Omega}\times\bar r + \bar\Omega\times(\bar\Omega\times\bar r) + \underbrace{2\bar\Omega\times{}^{R'}\frac{d\bar r}{dt}}_{\text{coriolis accel}}}_{\bar a'} + \underbrace{{}^{R'}\frac{d^2\bar r}{dt^2}}_{{}^{R'}\frac{d}{dt}(\vec v_{rel})} \qquad (1) $$

multiplying by mass of particle p (dm)

$$\bar{a}_p\,dm = \bar{a}'\,dm + \frac{{}^{R'}d}{dt}(\bar{v}_{rel}'\,dm) = d\bar{F} \qquad (2)$$

do above for each particle in system & sum

$$\iiint_{sys} \bar{a}_p\,dm = \bar{F} = \iiint_{sys} \bar{a}'\,dm + \iiint_{sys} \frac{{}^{R'}d}{dt}(\bar{v}_{rel}\,dm)$$

$$= \iiint_{sys} \bar{a}'\,dm + \frac{{}^{R'}d}{dt}\underbrace{\iiint_{sys} \bar{v}_{rel}\,dm}_{\bar{P}_{rel}} \qquad (3)$$

(linear momentum of system)

Reynold's Transport Theorem

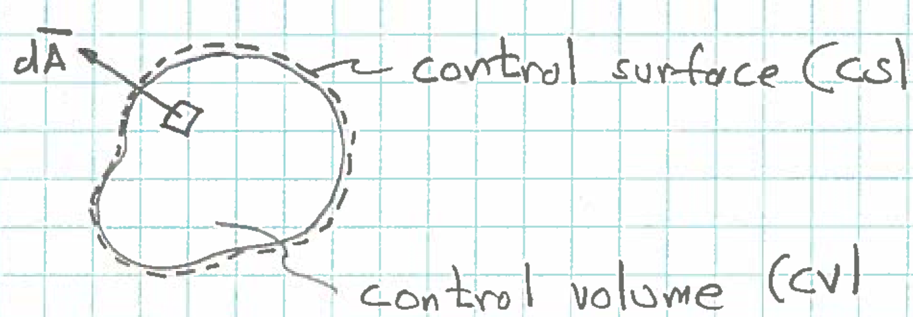
Control Volume & Control Surface moving & deforming

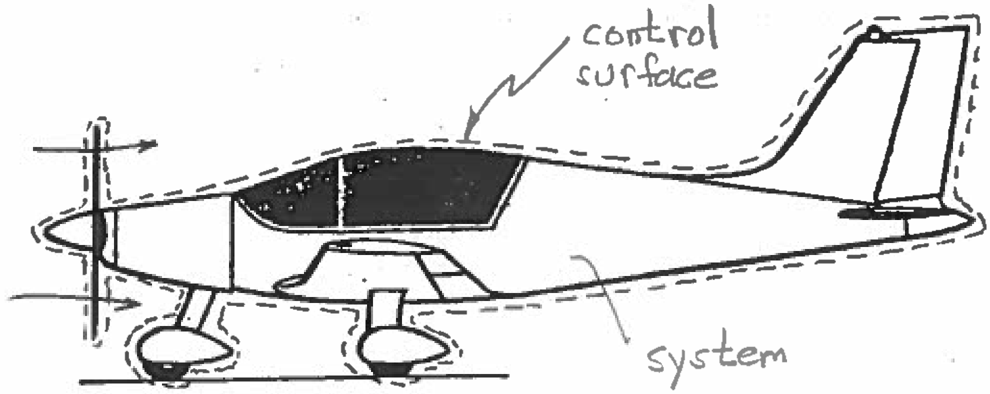
N = extensive system property; e.g., mass M or linear momentum $\bar{P}$

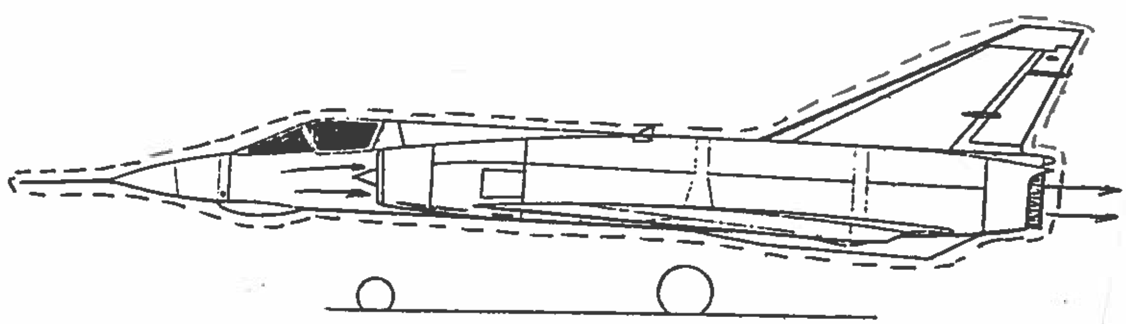
η = N/unit mass, e.g., if $N = m$, $\eta = 1$

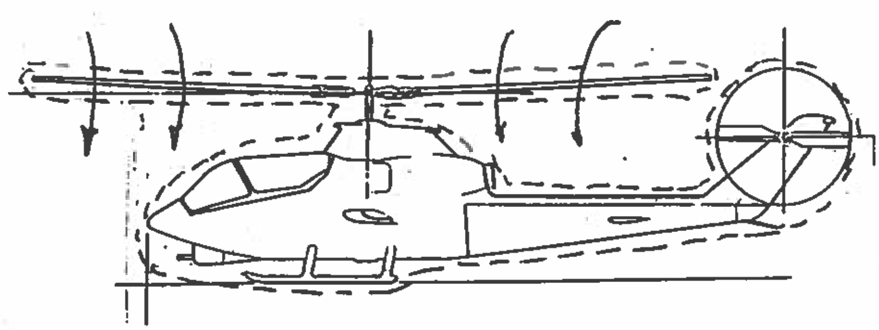
if $N = \bar{P}$, $\eta = \bar{v}$

$$\overset{R \text{ or } R'}{\frac{dN}{dt}}\Bigg)_{sys} = \oiint_{cs} \rho\eta\,\bar{v}_{r_{cs}} \cdot d\bar{A} + \overset{R \text{ or } R'}{\frac{d}{dt}} \iiint_{cv} \eta\rho\,d\forall \qquad (4)$$




control
surface
system





$\bar{v}_{r_{cs}}$ = velocity of particle of system relative to moving & deforming control surface & on the control surface

$d\forall$ = differential volume

$\oiint_{cs}$ = integral over control surface

$\iiint_{cv}$ = integral over control volume

if $N \equiv \bar{P}_{rel}$

$$\left.\frac{^{R'}d}{dt}\bar{P}_{rel}\right)_{sys} = \oiint_{cs} \rho\, \bar{v}_{rel}\,(\bar{v}_{r_{cs}})\cdot d\bar{A} + \frac{^{R'}d}{dt}\iiint_{cv} \rho\, \bar{v}_{rel}\, d\forall \qquad (5)$$

$\therefore$ Using Eq (5), Eq (3) becomes **completely general equation of motion, e.g., flexible aircraft, ejecting mass for propulsion, moving people/mechanisms,**

$$\boxed{\bar{F} - \oiint_{cs} \rho\, \bar{v}_{rel}\,(\bar{v}_{r_{cs}})\cdot d\bar{A} = \iiint_{sys} \bar{a}'\, dm + \frac{^{R'}d}{dt}\iiint_{cv} \rho\, \bar{v}_{rel}\, dm} \qquad (6)$$

Assume rigid body, with rotors spinning @ constant angular velocity & steady fluid flow

then $\dfrac{^{R'}d}{dt}\iiint_{cv} \rho\, \bar{v}_{rel}\, dm = 0$

Define $T_e \triangleq - \oiint_{cs} \rho\, \bar{v}_{rel}\,(\bar{v}_{r_{cs}})\cdot d\bar{A}$ = Thrust

$$\bar{F} + \bar{T}_e = \iiint_{sys} \bar{a}'\, dm \qquad (7)$$

Now place origin of R' @ center of gravity of system;

Then $\iiint_{sys} \bar{r}\, dm \equiv 0$ (8)

Then using Eq (5) and $\bar{a}'$ from Eq. (1), becomes

$$\iiint_{sys} \bar{r}\, dm = 0\,; \quad \iiint_{sys} \dot{\bar{\Omega}} \times \bar{r}\, dm = \dot{\bar{\Omega}} \times \iiint_{sys} \bar{r}\, dm = 0\,;$$

$$\iiint_{sys} \bar{\Omega} \times (\bar{\Omega} \times \bar{r})\, dm = \bar{\Omega} \times \Big(\bar{\Omega} \times \iiint_{sys} \bar{r}\, dm\Big) = 0\,; \quad \iiint_{sys} \Big(2\bar{\Omega} \times \frac{{}^{R'}d\bar{r}}{dt}\Big) dm = 2\bar{\Omega} \times \frac{{}^{R'}d}{dt} \underbrace{\iiint_{sys} \bar{r}\, dm}_{=0}$$

$$\bar{F} + \bar{T}_e = \iiint_{sys} \frac{{}^{R}d\bar{U}_0}{dt}\, dm = \frac{{}^{R}d\bar{U}_0}{dt} \iiint_{sys} dm = m \frac{{}^{R}d\bar{U}_0}{dt}.$$

Let $\bar{F} + \bar{T}_e = X_T \bar{i} + Y_T \bar{j} + Z_T \bar{k}$

$(\bar{i}, \bar{j}, \bar{k})$ are unit vectors in frame R'

$$\bar{U}_0 = U\bar{i} + V\bar{j} + W\bar{k}$$

$$\bar{\Omega} = P\bar{i} + Q\bar{j} + R\bar{k}$$

$$\frac{{}^{R}d\bar{U}_0}{dt} = \frac{{}^{R'}d\bar{U}_0}{dt} + \bar{\Omega} \times \bar{U}_0$$

This leads to

$$\boxed{\begin{aligned} X_T &= m(\dot{U} + QW - RV) \\ Y_T &= m(\dot{V} + RU - PW) \\ Z_T &= m(\dot{W} + PV - QU) \end{aligned}} \quad (9)$$

Now cross $\bar{r}$ into both sides of Eq (2)

$$\bar{r} \times \bar{a}_p\, dm = \bar{r} \times \bar{a}'\, dm + \bar{r} \times \frac{{}^{R'}d}{dt}(\bar{v}_{rel})\, dm = \bar{r} \times d\bar{F} = d\bar{M}_o$$

differential moment

but
$$\frac{{}^{R'}d}{dt}(\bar{r} \times \bar{v}_{rel}) = \frac{{}^{R'}d}{dt}\bar{r} \times \bar{v}_{rel} + \bar{r} \times \frac{{}^{R'}d}{dt}\bar{v}_{rel}$$

$$= \underbrace{\bar{v}_{rel} \times v_{rel}}_{0} + \bar{r} \times \frac{{}^{R'}d\bar{v}_{rel}}{dt}$$

$$\therefore \quad d\bar{M} = \bar{r} \times \bar{a}'\, dm + \frac{{}^{R'}d}{dt}(\bar{r} \times \bar{v}_{rel})$$

Now writing above eqn for every particle in system and summing:

$$\bar{M}_o = \iiint_{sys} (\bar{r} \times \bar{a}')\, dm + \frac{{}^{R'}d}{dt} \underbrace{\iiint_{sys} \bar{r} \times \bar{v}_{rel}\, dm}_{\bar{H}_{rel}}$$

$\bar{H}_{rel}$ (angular momentum of system)

Using Eq (4) with $N \equiv \bar{H}_{rel}$, $\eta = \bar{r} \times \bar{v}_{rel}$

$$\boxed{\bar{M}_o - \oiint_{cs} \rho(\bar{r} \times \bar{v}_{rel})\bar{v}_{r_{cs}} \cdot d\bar{A} = \iiint_{sys} (\bar{r} \times \bar{a}')\, dm + \frac{{}^{R'}d}{dt} \iiint_{cv} \rho(\bar{r} \times \bar{v}_{rel})\, d\forall}$$

completely general equation of motion, e.g., flexible aircraft, ejecting mass for propulsion, moving people/mechanisms

Define $\overline{M}_{T_o} = -\oiint_{CS} \rho(\bar{r}\times\bar{v}_{rel})\bar{v}_{r_{CS}}\cdot dA$ = Moment due to thrust

Again Assume rigid body, with rotors spinning @ constant angular velocity & steady fluid flow

then $\displaystyle {}^{R'}\frac{d}{dt}\iiint_{CV}\rho(\bar{r}\times\bar{v}_{rel})\,d\forall = 0$

Then using $\bar{a}'$ from Eq (1):

$$\overline{M}_o + \overline{M}_{T_o} = \iiint_{sys}(\bar{r}\times\bar{a}_o)\,dm + \iiint_{sys}\bar{r}\times(\dot{\bar{\Omega}}\times\bar{r})\,dm + \iiint_{sys}\bar{r}\times[\bar{\Omega}\times(\bar{\Omega}\times\bar{r})]\,dm + \iiint_{sys}2\bar{r}\times(\bar{\Omega}\times\bar{v}_{rel})\,dm \quad (9)$$

Again, assume origin of R' @ center of gravity of system: then $\iiint_{sys}\bar{r}\,dm = 0$. (repeat of Eq (8)). Using Eq. (8) in Eq (9), it simplifies (somewhat) to

$$\overline{M}_o + \overline{M}_{T_o} = \iiint_{sys}2\bar{r}\times(\bar{\Omega}\times\bar{v}_{rel})\,dm + \iiint_{sys}\bar{r}\times(\dot{\bar{\Omega}}\times\bar{r})\,dm + \iiint_{sys}\bar{r}\times[\bar{\Omega}\times(\bar{\Omega}\times\bar{r})]\,dm \quad (10)$$

define $M_{IR_o} = -\iiint_{sys}2\bar{r}\times(\bar{\Omega}\times\bar{v}_{rel})\,dm$

= inertia torque due to relative motion } spinning rotors

Vector calculus allows

$$\bar{r}\times(\dot{\bar{\Omega}}\times\bar{r}) = \underbrace{\bar{r}^2}_{\bar{r}\cdot\bar{r}}\dot{\bar{\Omega}} - (\bar{r}\cdot\dot{\bar{\Omega}})\bar{r}$$

and

$$\bar{r}\times[\bar{\Omega}\times(\bar{\Omega}\times\bar{r})] = \bar{r}\times(\bar{\Omega}\cdot\bar{r}) - \underbrace{\bar{r}\times\bar{\Omega}^2\bar{r}}_{\bar{\Omega}^2\,\bar{r}\times\bar{r}\,=\,0}$$

Eq (10) becomes

$$\bar{M}_o + \bar{M}_{T_o} + \bar{M}_{IR_o} = \iiint_{sys} \bar{r}^2\dot{\bar{\Omega}}\,dm - \iiint_{sys}(\bar{r}\cdot\dot{\bar{\Omega}})\bar{r}\,dm + \iiint_{sys}(\bar{\Omega}\cdot\bar{r})(\bar{r}\times\bar{\Omega})\,dm \qquad (11)$$

Now let $\bar{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\bar{\Omega} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\dot{\bar{\Omega}} = {}^{R'}\frac{d}{dt}\bar{\Omega} + \bar{\Omega}\times\bar{\Omega} = {}^{R'}\frac{d\bar{\Omega}}{dt}$$

$$= \dot{P}\vec{i} + \dot{Q}\vec{j} + \dot{R}\vec{k}$$

Now $\displaystyle\iiint_{sys}\bar{r}^2\dot{\bar{\Omega}}\,dm = \dot{\bar{\Omega}}\iiint_{sys}(x^2+y^2+z^2)\,dm$

$$= (\dot{P}\vec{i} + \dot{Q}\vec{j} + \dot{R}\vec{k})\iiint_{sys}(x^2+y^2+z^2)\,dm$$

Likewise

$$-\iiint_{sys} (\bar{r}\cdot\dot{\bar{\Omega}})\bar{r}\,dm = -\iiint_{sys} x\dot{P}(x\bar{i}+y\bar{j}+z\bar{k})\,dm - \iiint_{sys} y\dot{Q}(x\bar{i}+y\bar{j}+z\bar{k})\,dm - \iiint_{sys} z\dot{R}(x\bar{i}+y\bar{j}+z\bar{k})\,dm$$

and

$$\iiint_{sys} (\bar{\Omega}\cdot\bar{r})(\bar{r}\times\bar{\Omega})\,dm = \iiint_{sys} (Px+Qy+Rz)(Ry-Qz)\,\bar{i}\,dm + \iiint_{sys} (Px+Qy+Rz)(Pz-Rx)\,\bar{j}\,dm + \iiint_{sys} (Px+Qy+Rz)(Qx-Py)\,\bar{k}\,dm$$

For simplicity, let's concentrate on just the x-component of the Eq (i) (L = rolling moment)

$$\bar{M}_o + \bar{M}_{T_o} + \bar{M}_{IR_o} = L\bar{i} + M\bar{j} + N\bar{k}$$

$$L = \cancel{\dot{P} \iiint_{sys} x^2\, dm} + \dot{P} \iiint_{sys} (y^2+z^2)\, dm - \cancel{\dot{P} \iiint_{sys} x^2\, dm} - \dot{Q} \iiint_{sys} xy\, dm -$$

$$\dot{R} \iiint_{sys} xz\, dm + PR \iiint_{sys} xy\, dm - PQ \iiint_{sys} yz\, dm +$$

$$QR \iiint_{sys} y^2\, dm - Q^2 \iiint_{sys} yz\, dm + R^2 \iiint_{sys} yz\, dm -$$

$$RQ \iiint_{sys} z^2\, dm$$

Now, add & subtract $QR \iiint_{sys} x^2\, dm$ to the right hand side above

$$L = \dot{P} \iiint_{sys} (y^2+z^2)\, dm - \dot{Q} \iiint_{sys} xy\, dm - \dot{R} \iiint_{sys} xz\, dm + PR \iiint_{sys} xy\, dm -$$

$$PQ \iiint_{sys} yz\, dm + QR \iiint_{sys} (\underbrace{x^2}_{\text{add}} + y^2)\, dm - Q^2 \iiint_{sys} yz\, dm +$$

$$R^2 \iiint_{sys} yz\, dm \underbrace{-}_{\text{subtract}} QR \iiint_{sys} (\underbrace{x^2}_{\text{subtract}} + z^2)\, dm$$

Recognizing geometric terms

$$\iiint_{sys} (y^2+z)\, dm = I_x \text{ (or } I_{xx}\text{)} \quad \text{(moment of inertial)}$$

$$\iiint_{sys} xy\, dm = I_{xy} \quad \text{(product of inertial)}$$

$$L = \dot{P} I_x - \dot{Q} I_{xy} - \dot{R} I_{xz} + PR I_{xy} - PQ I_{yz} + QR I_z - Q^2 I_{yz} + R^2 I_{yz} - QR I_y$$

or rearranging:

$$\boxed{L = \dot{P} I_x + QR(I_z - I_y) - (PQ + \dot{R}) I_{xz} + (PR - \dot{Q}) I_{xy} - (Q^2 - R^2) I_{yz}}$$

By considering the y & z components of Eq (11)

$$M = \dot{Q} I_y + PR(I_x - I_z) - (QR + \dot{P}) I_{xy} + (PQ - \dot{R}) I_{yz} - (R^2 - P^2) I_{xz}$$ (12)

$$N = \dot{R} I_z + PQ(I_y - I_x) - (PR + \dot{Q}) I_{yz} + (QR - \dot{P}) I_{xz} - (P^2 - Q^2) I_{xy}$$

repeating Eqs. (9)

$$\boxed{\begin{aligned} X_T &= m(\dot{U} + QW - RV) \\ Y_T &= m(\dot{V} + RU - PW) \\ Z_T &= m(\dot{W} + PV - QU) \end{aligned}}$$

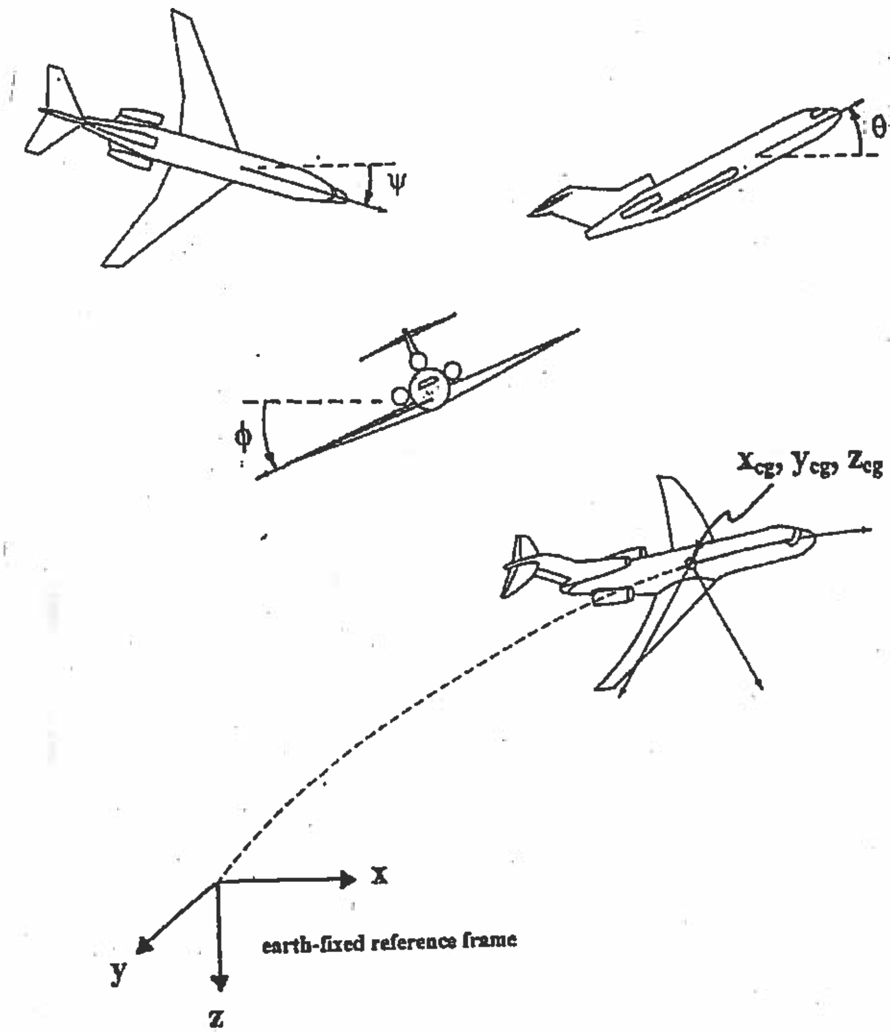


Figure 1 The six degrees of freedom of an aircraft. The three values x_{cg}, y_{cg}, z_{cg} represent the coordinates of the aircraft center of gravity relative to a reference frame fixed in the earth. The remaining degrees of freedom (ψ, θ, ϕ) describe the angular orientation of the aircraft relative to the earth-fixed frame.

An aircraft is a dynamic system whose motion in three-dimensional space is described by differential equations, obtained by application of Newton's second law of motion. This law describes how the linear and angular

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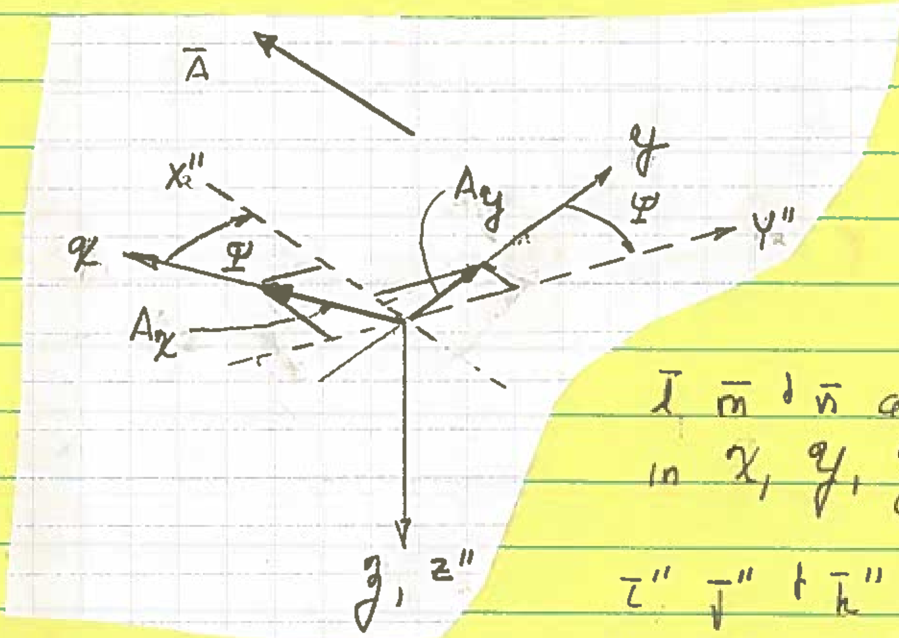
Axes Transformations

Rotation Ψ about z



$\bar{l}$ $\bar{m}$ & $\bar{n}$ are unit vectors in x, y, z axes

$\bar{i}''$ $\bar{j}''$ & $\bar{k}''$ are unit vectors in x'', y'', z'' axes

One can show

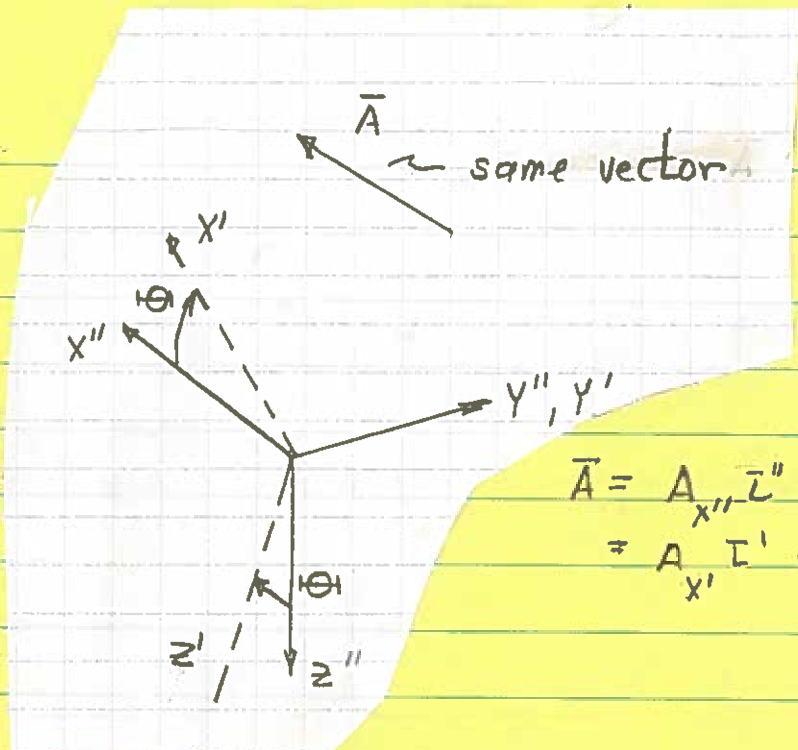
$$\bar{A} = A_x \bar{l} + A_y \bar{m} + A_z \bar{n}$$
$$= A_{x''} \bar{i}'' + A_{y''} \bar{j}'' + A_{z''} \bar{k}''$$

$$\begin{Bmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{Bmatrix} = [\Psi] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}$$

where

$$[\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Θ about Y''



$$\bar{A} = A_{x''}\bar{i}'' + A_{y''}\bar{j}'' + A_{z''}\bar{k}''$$
$$= A_{x'}\bar{i}' + A_{y'}\bar{j}' + A_{z'}\bar{k}'$$

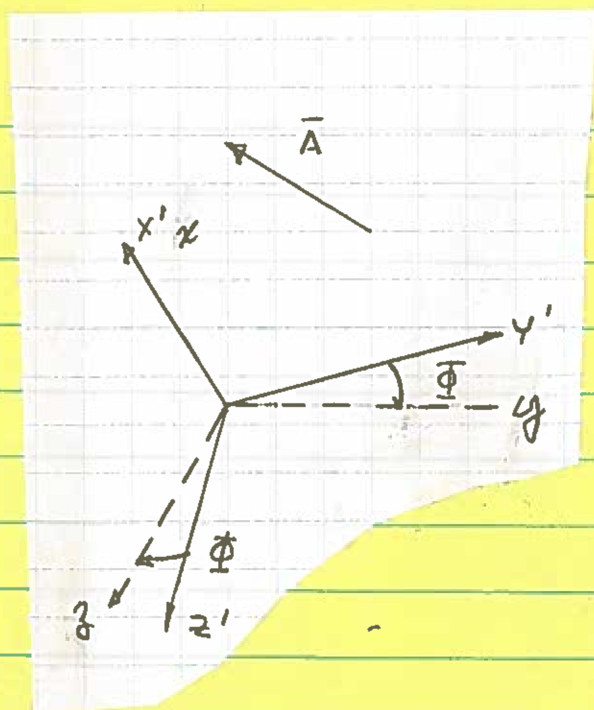
One can show

$$\begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = \left[\Theta\right] \begin{Bmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{Bmatrix}$$

where

$$\left[\Theta\right] = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}$$

Rotation Φ about x'



xyz are body-fixed axes

$$\bar{A} = A_{x'}\bar{I}' + A_{y'}\bar{J}' + A_{z'}\bar{K}'$$
$$= A_x \bar{I} + A_y \bar{J} + A_z \bar{K}$$

One can show

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [\Phi] \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix}$$

where

$$[\Phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix}$$

Finally

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = \underbrace{\overset{(\overline{DC})}{[\Phi][\Theta][\Psi]}}_{\text{matrix of direction cosines}} \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}$$

body-fixed axis components

earth-fixed axis components

DC is a unitary matrix, so

$$(\overline{DC})^{-1} = (\overline{DC})^{T}$$

earth-fixed axes to body-fixed axes

	l	m	n
i	$\cos\psi \cos\Theta$	$\sin\psi \cos\Theta$	$-\sin\Theta$
j	$\cos\psi \sin\Theta \sin\Phi - \sin\psi \cos\Phi$	$\sin\psi \sin\Theta \sin\Phi + \cos\psi \cos\Phi$	$\cos\Theta \sin\Phi$
k	$\cos\psi \sin\Theta \cos\Phi + \sin\psi \sin\Phi$	$\sin\psi \sin\Theta \cos\Phi - \cos\psi \sin\Phi$	$\cos\Theta \cos\Phi$

$= (DC)$

body-fixed axes to earth-fixed axes

	i	j	k
l	$\cos\Psi \cos\Theta$	$\cos\Psi \sin\Theta \sin\Phi - \sin\Psi \cos\Phi$	$\cos\Psi \sin\Theta \cos\Phi + \sin\Psi \sin\Phi$
m	$\sin\Psi \cos\Theta$	$\sin\Psi \sin\Theta \sin\Phi + \cos\Psi \cos\Phi$	$\sin\Psi \sin\Theta \cos\Phi - \cos\Psi \sin\Phi$
n	$-\sin\Theta$	$\cos\Theta \sin\Phi$	$\cos\Theta \cos\Phi$

$= (DC)^{-1}$

what if $\bar{A} = \bar{v}_0 = u\bar{i} + v\bar{j} + w\bar{k}$

Then $\overline{DC} \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$

or

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = (\overline{DC}^{-1}) \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

called "navigation eqns"

$$v_x = \frac{dx}{dt} = u\,c\Psi c\Theta + v(c\Psi s\Theta s\Phi - s\Psi c\Phi) + w(c\Psi s\Theta c\Phi + s\Psi s\Phi)$$

$$v_y = \frac{dy}{dt} = u\,s\Psi c\Theta + v(s\Psi s\Theta s\Phi + c\Psi c\Phi) + w(s\Psi s\Theta c\Phi - c\Psi s\Phi)$$

$$v_z = \frac{dz}{dt} = -u\,s\Theta + v(c\Theta s\Phi) + w(c\Theta c\Phi)$$

Now, with the Euler angles, we can consider

$$\underbrace{\begin{matrix} X_T = X + X_{grav} \\ Y_T = Y + Y_{grav} \\ Z_T = Z + Z_{grav} \end{matrix}}_{\text{Total force components}}$$

but $\overline{DC} \begin{Bmatrix} 0 \\ 0 \\ mg \end{Bmatrix} = \begin{Bmatrix} X_{grav} \\ Y_{grav} \\ Z_{grav} \end{Bmatrix} = \begin{Bmatrix} -mg\,s\Theta \\ mg\,c\Theta s\Phi \\ mg\,c\Theta c\Phi \end{Bmatrix}$

$$\therefore \quad X_T = X - mg\, s\Theta$$
$$Y_T = Y + mg\, c\Theta\, s\Phi$$
$$Z_T = Z + mg\, c\Theta\, c\Phi$$

In general, finite rotations cannot be considered as vector quantities, since they are not commutative with respect to addition, i.e., if **A** and **B** represent finite rotations, **A** + **B** ≠ **B** + **A**, in general. However, infinitesimal rotations can be represented as vectors. Let **ΔC** represent a small rotation.

$$\overline{\Delta C} = \Delta\Psi\, \bar{n} + \Delta\Theta\, \bar{j}'' + \Delta\Phi\, \bar{i}' \quad \leftarrow \text{see p 12}$$

or

$$\overline{\Delta C} = \Delta\Psi \underbrace{\left[-s\Theta\, \vec{i} + c\Theta\, s\Phi\, \vec{j} + c\Theta\, c\Phi\, \vec{k}\right]}_{(\overline{DC}^{-1})_{3,:} * \{\bar{i}\ \bar{j}\ \bar{k}\}} +$$

$$\Delta\Theta \underbrace{\left[c\Phi\, \vec{j} - s\Phi\, \vec{k}\right]}_{\{([\Theta][\Phi])^{-1}\}_{2,:} * \{\bar{i}\ \bar{j}\ \bar{k}\}} +$$

$$\Delta\Phi \underbrace{(1.0\, \bar{i})}_{([\Phi]^{-1})_{1,:} * \{\bar{i}\ \bar{j}\ \bar{k}\}}$$

or

$$\overline{\Delta C} = \left[-s\Theta\, \Delta\Psi + \Delta\Phi\right] \vec{i} +$$
$$\left[c\Theta\, s\Phi\, \Delta\Psi + c\Phi\, \Delta\Theta\right] \vec{j} +$$
$$\left[c\Theta\, c\Phi\, \Delta\Psi - s\Phi\, \Delta\Theta\right] \vec{k}$$

$$\lim_{\Delta t \to 0} \frac{\overline{\Delta C}}{\Delta t} \triangleq \bar{\Omega} = \underbrace{\left[-\dot{\Psi}\, s\Theta + \dot{\Phi}\right]}_{P} \vec{i} +$$
$$\underbrace{\left[\dot{\Psi}\, c\Theta\, s\Phi + \dot{\Theta}\, c\Phi\right]}_{Q} \vec{j} +$$
$$\underbrace{\left[\dot{\Psi}\, c\Theta\, c\Phi - \dot{\Theta}\, s\Phi\right]}_{R} \vec{k}$$

THE COLLECTED EQUATIONS

$$X = m\left[\dot{u} + QW - RV + g\sin\Theta\right]$$

$$Y = m\left[\dot{v} + RU - PW - g\cos\Theta\sin\Phi\right]$$

$$Z = m\left[\dot{w} + PV - QU - g\cos\Theta\cos\Phi\right]$$

$$L = \dot{P} I_x - \dot{R} I_{xz} + QR(I_z - I_y) - PQ I_{xz}$$

$$M = \dot{Q} I_y + PR(I_x - I_z) - R^2 I_{xz} + P^2 I_{xz}$$

$$N = \dot{R} I_z - \dot{P} I_{xz} + PQ(I_y - I_x) + QR I_{xz}$$

$$P = -\dot{\Psi}\sin\Theta + \dot{\Phi}$$

$$Q = \dot{\Psi}\cos\Theta\sin\Phi + \dot{\Theta}\cos\Phi$$

$$R = \dot{\Psi}\cos\Theta\cos\Phi - \dot{\Theta}\sin\Phi$$

$$\frac{dx}{dt} = U\cos\Psi\cos\Theta + V(\cos\Psi\sin\Theta\sin\Phi - \sin\Psi\cos\Phi) + W(\cos\Psi\sin\Theta\cos\Phi + \sin\Psi\sin\Phi)$$

$$\frac{dy}{dt} = U\sin\Psi\cos\Theta + V(\sin\Psi\sin\Theta\sin\Phi + \cos\Psi\cos\Phi) + W(\sin\Psi\sin\Theta\cos\Phi - \cos\Psi\sin\Phi)$$

$$\frac{dz}{dt} = -U\sin\Theta + V(\cos\Theta\sin\Phi) + W(\cos\Theta\cos\Phi)$$

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The Equations Collected

(1) $X = m[\dot{U} + QW - RV + gs\Theta]$

(2) $Y = m[\dot{V} + RU - PW - gc\Theta s\Phi]$

(3) $Z = m[\dot{W} + PV - QU - gc\Theta c\Phi]$

(4) $L = \dot{P}I_X - \dot{R}I_{XZ} + QR(I_Z - I_Y) - PQI_{XZ}$

(5) $M = \dot{Q}I_Y + PR(I_X - I_Z) - R^2 I_{XZ} + P^2 I_{XZ}$

(6) $N = \dot{R}I_Z - \dot{P}I_{XZ} + PQ(I_Y - I_X) + QRI_{XZ}$

(7) $P = -\dot{\Psi}s\Theta + \dot{\Phi}$

(8) $Q = \dot{\Psi}c\Theta s\Phi + \dot{\Theta}c\Phi$

(9) $R = \dot{\Psi}c\Theta c\Phi - \dot{\Theta}s\Phi$

or

(10) $\dot{\Theta} = Qc\Phi - Rs\Phi$

(11) $\dot{\Phi} = P + Qs\Phi t\Theta + Rc\Phi t\Theta$ $\qquad t(\ ) = TAN(\ )$

(12) $\dot{\Psi} = (Qs\Phi + Rc\Phi)/c\Theta$

(13) $v_x = U(c\Psi c\Theta) + V(c\Psi s\Theta s\Phi - s\Psi c\Phi) + W(c\Psi s\Theta c\Phi + s\Psi s\Phi)$

(14) $v_y = U(s\Psi c\Theta) + V(s\Psi s\Theta s\Phi + c\Psi c\Phi) + W(s\Psi s\Theta c\Phi - c\Psi s\Phi)$

(15) $v_z = -U(s\Theta) + V(c\Theta s\Phi) + W(c\Theta c\Phi)$