Dept. of Mechanical and Aeronautical Engineering

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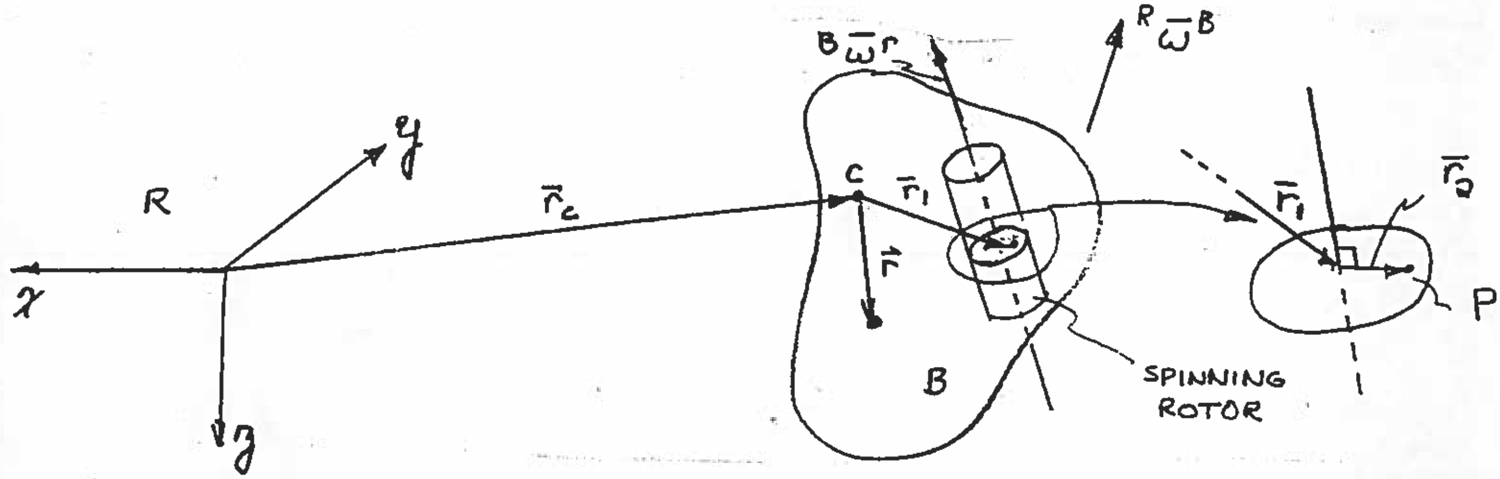
Effect of Spinning Rotors on Calculation of Angular Momentum

Consider a rigid body B with a single spinning rotor "r". The angular velocity of the rigid portion of B as measured in the inertial frame R is ${}^R\bar{\omega}^B$



$\bar{r}$ = vector from center of mass of body + rotor to any point in body + rotor

Now, the angular momentum of any particle in the body is defined as

$$\delta\bar{h} = (\bar{r}\times\bar{v})\delta m = \bar{r}\times\left(\bar{v}_c + \frac{{}^R d\bar{r}}{dt}\right)\delta m$$

Summing over the entire body, including the spinning rotor, gives

$$\bar{h} = \Sigma\delta\bar{h} = \sum_{rigid}\delta\bar{h} + \sum_{rotor}\delta\bar{h}$$

$$= \sum_{rigid}\bar{r}\times\left(\bar{v}_c + \frac{{}^R d\bar{r}}{dt}\right)\delta m + \sum_{rotor}\bar{r}\times\left(\bar{v}_c + \frac{{}^R d\bar{r}}{dt}\right)\delta m$$

$$= \quad '' \quad + \sum_{rotor}(\bar{r}_1+\bar{r}_2)\times\left(\bar{v}_c + \frac{{}^R d}{dt}(\bar{r}_1+\bar{r}_2)\right)\delta m$$

The first summation above is take over the body B but <u>not</u> including the rotor. The second summation is taken over the rotor <u>only</u>. A point "p" is located in the rotor but no necessarily on the periphery. Vector $\bar{r}_2$ is drawn from (and perpendicular to) the axis of rotation to point "p". The vector $\bar{r}_1$ is drawn from the center of mass of the rigid body + rotor to the point where $\bar{r}_2$ intersects the axis of rotation. The equation above can be rewritten

$$\bar{h} = \sum_{rigid} \bar{r} \times \left(\bar{v}_c + {}^{R}\frac{d\bar{r}}{dt}\right)\delta m + \sum_{rotor} (\bar{r}_1 + \bar{r}_2) \times \Big(\bar{v}_c + \underbrace{{}^{R}\bar{\omega}^{B} \times \bar{r}_1}_{{}^{R}\frac{d\bar{r}_1}{dt}} + \underbrace{{}^{B}\frac{d}{dt}\bar{r}_2 + {}^{R}\bar{\omega}^{B} \times \bar{r}_2}_{{}^{R}\frac{d\bar{r}_2}{dt}}\Big)\delta m$$

or

$$\bar{h} = \sum_{rigid} \bar{r} \times \left(\bar{v}_c + {}^{R}\frac{d\bar{r}}{dt}\right)\delta m + \sum_{rotor} (\bar{r}_1 + \bar{r}_2) \times \left(\bar{v}_c + {}^{R}\bar{\omega}^{B} \times (\bar{r}_1 + \bar{r}_2)\right)\delta m +$$

$$\sum_{rotor} (\bar{r}_1 + \bar{r}_2) \times (\underbrace{{}^{B}\bar{\omega}^{r} \times \bar{r}_2}_{{}^{B}\frac{d\bar{r}_2}{dt}})\delta m$$

Now, only the last summation in the equation above involves rotor motion relative to the body (${}^{B}\bar{\omega}^{r}$). In fact, one can see that the first two summations in the equation above define the angular momentum of the entire body + rotor as if the rotor were not moving. We can call this $\bar{h}_{rigid}$ where "rigid" refers to the $\bar{h}$ that was calculated assuming no spinning rotors. The last summation can be called $\bar{h}'$ which includes the effect of the rotor spinning relative to the body B. Thus

$$\bar{h} = \bar{h}_{rigid} + \bar{h}' = \bar{h}_{rigid} + \sum_{rotor} (\bar{r}_1 + \bar{r}_2) \times ({}^{B}\bar{\omega}^{r} \times \bar{r}_2)\delta m$$

Finally, it can be shown that the summation in the last equation can be further simplified by showing that

$$\sum_{rotor} (\bar{r}_1 + \bar{r}_2) \times ({}^{B}\bar{\omega}^{r} \times \bar{r}_2)\delta m = \sum_{rotor} \bar{r}_2 \times ({}^{B}\bar{\omega}^{r} \times \bar{r}_2)\delta m$$

This leads to the final equation

$$\bar{h} = \bar{h}_{rigid} + \bar{h}' = \bar{h}_{rigid} + \sum_{rotor} \bar{r}_2 \times ({}^{B}\bar{\omega}^{r} \times \bar{r}_2)\delta m$$

Note that $\bar{h}'$ can be easily determined once we know the geometry of the rotor and how fast it is spinning relative to the body B (${}^{B}\bar{\omega}^{r}$).

$$\bar{h}' = \sum_{rotor} (\bar{r}_2 \cdot \bar{r}_2)\, {}^{B}\bar{\omega}^{r}\, \delta m - \sum_{rotor} \cancelto{0}{(\bar{r}_2 \cdot {}^{B}\bar{\omega}^{r})}\, \bar{r}_2\, \delta m$$

$$= {}^{B}\bar{\omega}^{r} \sum_{rotor} (\bar{r}_2 \cdot \bar{r}_2)\, \delta m$$

$$= {}^{B}\bar{\omega}^{r}\, I_r = \left({}^{B}\omega^{r}_{x}\, \bar{i} + {}^{B}\omega^{r}_{y}\, \bar{j} + {}^{B}\omega^{r}_{z}\, \hat{k}\right) I_r$$

where I_r is the moment of inertia of the rotor about it's axis of rotation and terms like ${}^{B}\omega^{r}_{x}$ represent the x,y,z, body-fixed axis components of ${}^{B}\bar{\omega}^{r}$

Now it's the time rate of change of $\bar{h}'$ that must be taken into account in our moment equations, i.e., recall that the external moment applied to a body is equal to the time rate of change of the angular momentum. Here

$$\frac{{}^{R}d\bar{h}'}{dt} = \cancelto{0}{\frac{{}^{B}d\bar{h}'}{dt}} + {}^{R}\bar{\omega}^{B} \times \bar{h}' = {}^{R}\bar{\omega}^{B} \times \left({}^{B}\omega^{r}_{x}\, \bar{i} + {}^{B}\omega^{r}_{y}\, \bar{j} + {}^{B}\omega^{r}_{z}\, \hat{k}\right) I_r$$

$$= (p\,\hat{i} + q\,\bar{j} + r\,\hat{k}) \times \left({}^{B}\omega^{r}_{x}\, \bar{i} + {}^{B}\omega^{r}_{y}\, \bar{j} + {}^{B}\omega^{r}_{z}\, \hat{k}\right) I_r$$

Finally, the following additional terms are added to the right hand sides of our linearized, stability axis moment equations on the following page.

Note that the longitudinal and lateral equations are no longer decoupled!

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State-space Form of Linearized Aircraft Equations of Motion
Stability Axis System

<u>With</u> <u>Spinning</u> <u>Rotor</u>

longitudinal equations

$$\dot{u} = X_u \cdot u + X_w \cdot w - g\cos\theta_0 \cdot \theta + \sum_{i=1}^{n} X_{\delta_i} \cdot \delta_i$$

$$\dot{w} = \frac{Z_u}{1-Z_{\dot{w}}} \cdot u + \frac{Z_w}{1-Z_{\dot{w}}} \cdot w + \frac{Z_q + u_0}{1-Z_{\dot{w}}} \cdot q - \frac{g\sin\theta_0}{1-Z_{\dot{w}}} \cdot \theta + \frac{1}{1-Z_{\dot{w}}} \sum_{i=1}^{n} Z_{\delta_i} \cdot \delta_i$$

$$\dot{q} = \left[M_u + \frac{M_{\dot{w}} Z_u}{1-Z_{\dot{w}}}\right] \cdot u + \left[M_w + \frac{M_{\dot{w}} Z_w}{1-Z_{\dot{w}}}\right] \cdot w + \left[M_q + \frac{M_{\dot{w}}(Z_q + u_0)}{1-Z_{\dot{w}}}\right] \cdot q - \left[\frac{M_{\dot{w}} g\sin\theta_0}{1-Z_{\dot{w}}}\right] \cdot \theta +$$

$$\frac{M_{\dot{w}}}{1-Z_{\dot{w}}} \sum_{i=1}^{n} Z_{\delta_i} \cdot \delta_i + \sum_{i=1}^{n} M_{\delta_i} \cdot \delta_i - \left[r\left({}^B\omega_x^r\right) - p\left({}^B\omega_z^r\right)\right] \cdot \frac{I_r}{I_y}$$

$$\dot{\theta} = q$$

$$\dot{x} = (u_0 + u)\cos\theta_0 + w\sin\theta_0 - u_0 \cdot \theta\sin\theta_0$$

$$\dot{z} = -(u_0 + u)\sin\theta_0 + w\cos\theta_0 - u_0 \cdot \theta\cos\theta_0$$

lateral equations

$$\dot{v} = Y_v \cdot v + Y_p \cdot p + [Y_r - u_0] \cdot r + g\cos\theta_0 \cdot \phi + \sum_{i=1}^{n} Y_{\delta_i} \cdot \delta_i$$

$$\dot{p} = L'_v \cdot v + L'_p \cdot p + L'_r \cdot r + \sum_{i=1}^{n} L'_{\delta_i} \cdot \delta_i - \left[q\left({}^B\omega_z^r\right) - r\left({}^B\omega_y^r\right)\right] \cdot \frac{I_r}{I_x}$$

$$\dot{r} = N'_v \cdot v + N'_p \cdot p + N'_r \cdot r + \sum_{i=1}^{n} N'_{\delta_i} \delta_i - \left[p\left({}^B\omega_y^r\right) - q\left({}^B\omega_x^r\right)\right] \cdot \frac{I_r}{I_z}$$

$$\dot{\phi} = p + r\tan\theta_0$$

$$\dot{\psi} = r\sec\theta_0$$

$$\dot{y} = u_0 \cdot \psi\cos\theta_0 + v$$

where the so-called *prime stability derivatives* appearing above are introduced for notational convenience and are defined as

$$L'_{()} = \frac{L_{()}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}} + \frac{I_{xz}}{I_{xx}} \frac{N_{()}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}} \qquad N'_{()} = \frac{I_{xz}}{I_{zz}} \frac{L_{()}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}} + \frac{N_{()}}{1 - \frac{I_{xz}^2}{I_{xx} I_{zz}}}$$