Department of Mechanical and Aeronautical Enginernig

EME - 172

The Bode Diagram

Consider the following transfer function

$$G(s) = K\,\frac{\left(\frac{s}{z}+1\right)\left(\frac{s^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}s+1\right)}{s\left(\frac{s}{p}+1\right)\left(\frac{s^2}{\omega_p^2} + \frac{2\zeta_p}{\omega_p}s+1\right)} \tag{1}$$

This G(s) contains all the different types of factors one can find in the rational polynomial transfer functions. They are:

1.) a gain, K

2.) a simple zero $(s/z+1)$

3.) a simple pole $1/(s/p+1)$

4.) a pole at the origin $1/s$

5.) a quadratic zero $\left(\frac{s^2}{\omega_z^2}+\frac{2\zeta_z}{\omega_z}s+1\right)$

6.) a quadratic pole $1/\left(\frac{s^2}{\omega_p^2}+\frac{2\zeta_p}{\omega_p}s+1\right)$

Equation (1) is written so that the coefficients of s^0 (constant terms) are all unity. This is called the <u>Bode form</u> of the transfer function, as opposed to the <u>root locus form</u>.

Now the algebra of complex variables tells us that the magnitude of the complex number $G(j\omega)$ can be written

$$|G(j\omega)| = |K|\cdot\frac{\left|\frac{j\omega}{z}+1\right|\cdot\left|\frac{(j\omega)^2}{\omega_z^2}+\frac{2\zeta_z}{\omega_z}(j\omega)+1\right|}{|j\omega|\cdot\left|\frac{j\omega}{p}+1\right|\cdot\left|\frac{(j\omega)^2}{\omega_p^2}+\frac{2\zeta_p}{\omega_p}j\omega+1\right|} \tag{2}$$

By defining the magnitude in decibels (dB) as

$$|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)| \tag{3}$$

Equation 2 becomes

$$|G(j\omega)|_{dB} = 20\left[\log|K| + \log\left|\frac{j\omega}{z}+1\right| + \log\left|\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega)+1\right| - \log|j\omega| - \log\left|\frac{j\omega}{p}+1\right| - \log\left|\frac{(j\omega)^2}{\omega_p^2} + \frac{2\zeta_p}{\omega_p}(j\omega)+1\right|\right] \tag{4}$$

Notice that the <u>multiplicative</u> terms in Eqn. 2 become <u>additive</u> terms when the magnitude is expressed in dB. If G(s) were more complicated than that shown in Eqn. 1, e.g., if it possessed more than one simple zero, the additional zero would simply appear as an additional term in Eqn. 4.

The algebra of complex variables also tells us that the phase of the complex number $G(j\omega)$ can be written as:

$$\angle G(j\omega) = \angle K + \angle\left(\frac{j\omega}{z}+1\right) + \angle\left(\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega)+1\right) - \angle j\omega - \angle\left(\frac{j\omega}{p}+1\right) - \angle\left(\frac{(j\omega)^2}{\omega_p^2} + \frac{2\zeta_p}{\omega_p}(j\omega)+1\right) \tag{5}$$

Once again note the additive nature of the terms in Eq. 5.

A Bode diagram consists of two plots:

(1) $|G(j\omega)|$ in dB vs $\log\omega$ or vs ω on a log scale

(2) $\angle G(j\omega)$ in deg vs $\log\omega$ or vs ω on a log scale.

Equations 4 and 5 indicate that the Bode diagram of Eq. 1 can be obtained by a superposition of the Bode diagrams of the 6 elements listed below Eq. 1. We can now turn our attention toward obtaining the Bode diagrams of these constituent elements.

1.) <u>a gain</u>, K

Here

$$|K|_{dB} = 20\log|K| \tag{6}$$

$$\angle K = 0^\circ \quad K > 0$$
$$= \pm 180^\circ \quad K < 0 \tag{7}$$

Both Eqs. 6 and 7 describe horizontal lines on the Bode diagram. See Fig. 1.

2.) <u>a simple zero</u>, $\left(\frac{s}{z}+1\right)$

$$\left|\frac{j\omega}{z}+1\right|_{dB} = 20\log\left|\frac{j\omega}{z}+1\right| \tag{8}$$

Now for $\omega \ll z$

$$\left|\frac{j\omega}{z}+1\right| \doteq 1 \quad \& \quad 20\log\left|\frac{j\omega}{z}+1\right| \doteq 0 \text{ dB} \tag{9}$$

and for $\omega \gg z$

$$\left|\frac{j\omega}{z}+1\right| \doteq \frac{\omega}{z} \quad \& \quad 20\log\left|\frac{j\omega}{z}+1\right| \doteq 20\log\left(\frac{\omega}{z}\right) \tag{10}$$

Equations 9 and 10 define two straight lines on the Bode diagram, with the line of Eq. 9 being horizontal and the line of Eq. 10 having a slope of 20 decibels per decade (dB/dec). Here, a <u>decade</u> is the distance between any two points on the ω axis which differ by a factor of 10, e.g. 1.0 and 10., .25 and 2.5, etc. The straight lines intersect at the <u>break frequency</u>

$$\omega = z \tag{11}$$

These straight lines are referred to as the <u>asymptotes</u> for the Bode diagram for the simple zero. Except around the break frequency, the asymptotes are an excellent approximation to the actual magnitude. Even at the break frequency, the maximum deviation between the straight lines and the actual magnitude curve is only 3 dB.

For $\omega \ll z$ the phase angle is given by

$$\angle\left(\frac{j\omega}{z}+1\right) \doteq \angle(1) = 0^\circ \tag{12}$$

and for $\omega \gg z$

$$\angle\left(\frac{j\omega}{z}+1\right) \doteq \angle\left(\frac{j\omega}{z}\right) = 90^\circ \tag{13}$$

A linear approximation to the phase curve can be made by drawing a straight line with slope 65.96 degrees per decade (deg/dec) which intersects the 0 deg and 90 deg horizontal lines of Eqs. (12) and (13) at

$$0.208 \cdot z \quad \text{and} \quad 4.81 \cdot z \quad \text{respectively.} \tag{14}$$

The Bode diagram for a simple zero is shown in Fig. 2, with the straight line asymptotes.

3.) <u>a simple pole</u>, $1/(s/p + 1)$

By referring to Eqs. 4 and 5 it should be apparent that the Bode diagram for a simple pole is merely the Bode diagram for a simple zero reflected about the 0 dB line for the magnitude and the 0 deg line for the phase. See Fig. 3.

4.) <u>a quadratic zero</u>, $\left(\frac{s^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z} s + 1\right)$

For $\omega << \omega_z$

$$\left|\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega) + 1\right|_{dB} \doteq 20 \log |1| = 0 dB \tag{15}$$

and for $\omega >> \omega_z$

$$\begin{aligned} \left|\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega) + 1\right|_{dB} &\doteq 20 \log \left|\frac{-\omega^2}{\omega_z^2}\right| \\ &\doteq 20 \log \left(\frac{\omega^2}{\omega_z^2}\right) \\ &\doteq 40 \log \left(\frac{\omega}{\omega_z}\right) \end{aligned} \tag{16}$$

Equations 15 and 16 represent two straight lines on the Bode diagram, with the line of Eq. 15 being horizontal, and the line of Eq. 16 having a slope of 40dB/dec. The two straight lines intersect at the <u>break frequency</u>,

$$\omega = \omega_z \tag{17}$$

Again, except at $\omega = \omega_z$ the asymptotes are an excellent approximations to the actual magnitude. However, as opposed to the simple zero and pole, the magnitude for the complex zero (and pole) can differ considerably from the asymptotes around the break frequency. The amount of this discrepancy depends upon the damping ratio ζ_z .

For $\omega << \omega_z$ the phase angle is given by

$$\angle\left[\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega) + 1\right] \doteq \angle(1) = 0^\circ \tag{18}$$

and for $\omega >> \omega_z$

$$\angle\left[\frac{(j\omega)^2}{\omega_z^2} + \frac{2\zeta_z}{\omega_z}(j\omega) + 1\right] \doteq \tan^{-1}\frac{2\zeta}{-\omega/\omega_z} \doteq 180^\circ \tag{19}$$

A linear approximation to the phase curve can be made by drawing a straight line with slope $131.92/\zeta_z$ degrees per decade (deg/dec) which intersects the 0 deg and 180 deg horizontal lines of Eqs. (18) and (19) at

$$e^{-\zeta_z(\pi/2)} \cdot \omega_z \quad \text{and} \quad e^{\zeta_z(\pi/2)} \cdot \omega_z \quad \text{respectively}$$

Figure 4 shows the Bode diagram for a quadratic zero.

5.) <u>a quadratic pole</u>, $1/\left(\frac{s^2}{\omega_p^2} + \frac{2\zeta_p}{\omega_p}s + 1\right)$

By referring to Eqs. 4 and 5 it should be apparent that the Bode diagram for a quadratic pole is merely the Bode diagram for a simple zero reflected about the 0 dB line for the magnitude and the 0 deg line for the phase. See Fig. 5.

6.) <u>a pole at the origin</u>, $1/s$

Here,

$$\left|\frac{1}{j\omega}\right|_{dB} = -20 \log |j\omega| \tag{20}$$

Equation 20 represents a straight line with a slope of -20 dB/dec which intersects the 0 dB line at $\omega = 1$ rad/sec.

Likewise the phase for a pole at the origin is given by

$$\angle\left(\frac{1}{j\omega}\right) = -90^\circ \tag{21}$$

Both straight lines given by Eqs. 20 and 21 are <u>exact</u> representations for the magnitude and phase plots. Figure 6 shows the Bode diagram for a pole at the origin.

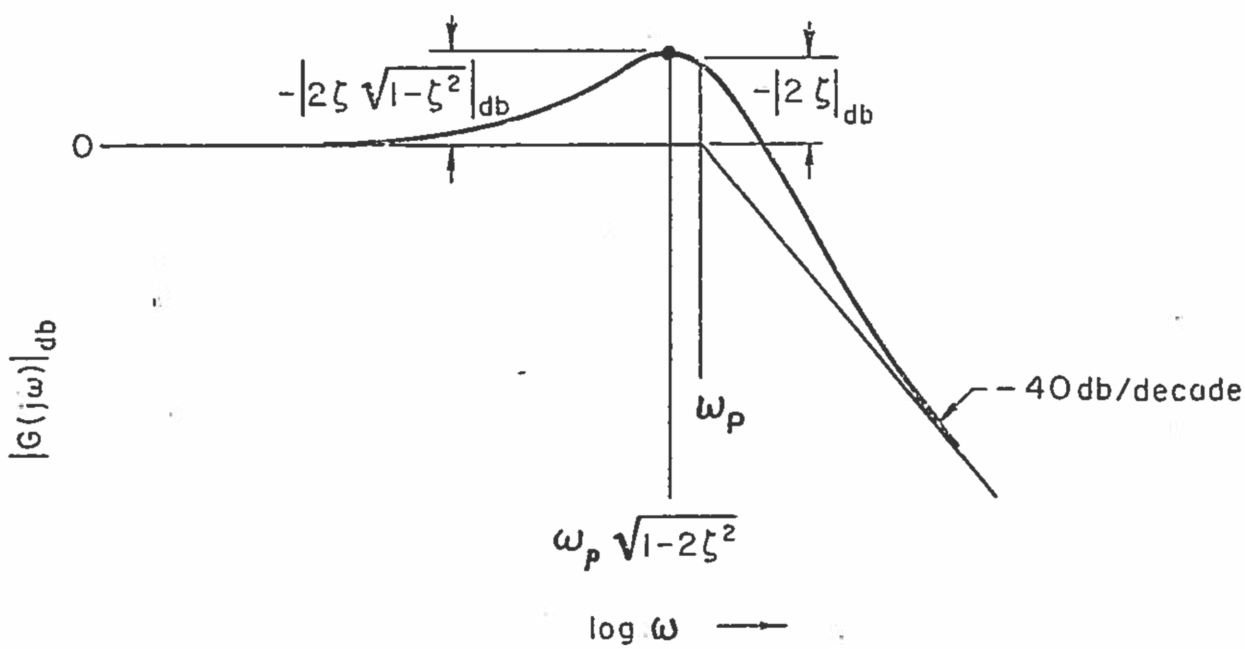
Straight line asympototic approximations for the Bode diagrams of a transfer function consisting of a combination of the elemental terms can be obtained by superposition of the asymptotes of the elemental forms. In doing this the <u>gain</u> term is not plotted, but left until last. Then, rather than shifting the entire magnitude portion of the diagram up or down by a new 0 dB line is simply defined on the diagram.

Figure 7 shows the Bode diagram for a transfer function

$$G(s) = \frac{(5s+1)}{s\left(\frac{s^2}{2^2} + \frac{2(.2)}{2}s + 1\right)} \qquad \zeta_p = 0.2 \quad \omega_p = 2.0 \text{ rad/sec} \tag{22}$$

<u>Magnitude Correction for Quadratic Terms</u>

As mentioned above, the straight line asympotic approximation for the magnitude portion of the Bode diagram for quadratic zeros and poles can differ significantly from the actual magnitude around the break frequency. The sketch below shows how to find the point of maximum deviation and the frequency at which it occurs for a quadratic pole. The result is merely rotated about the 0 dB line for a quadratic zero.



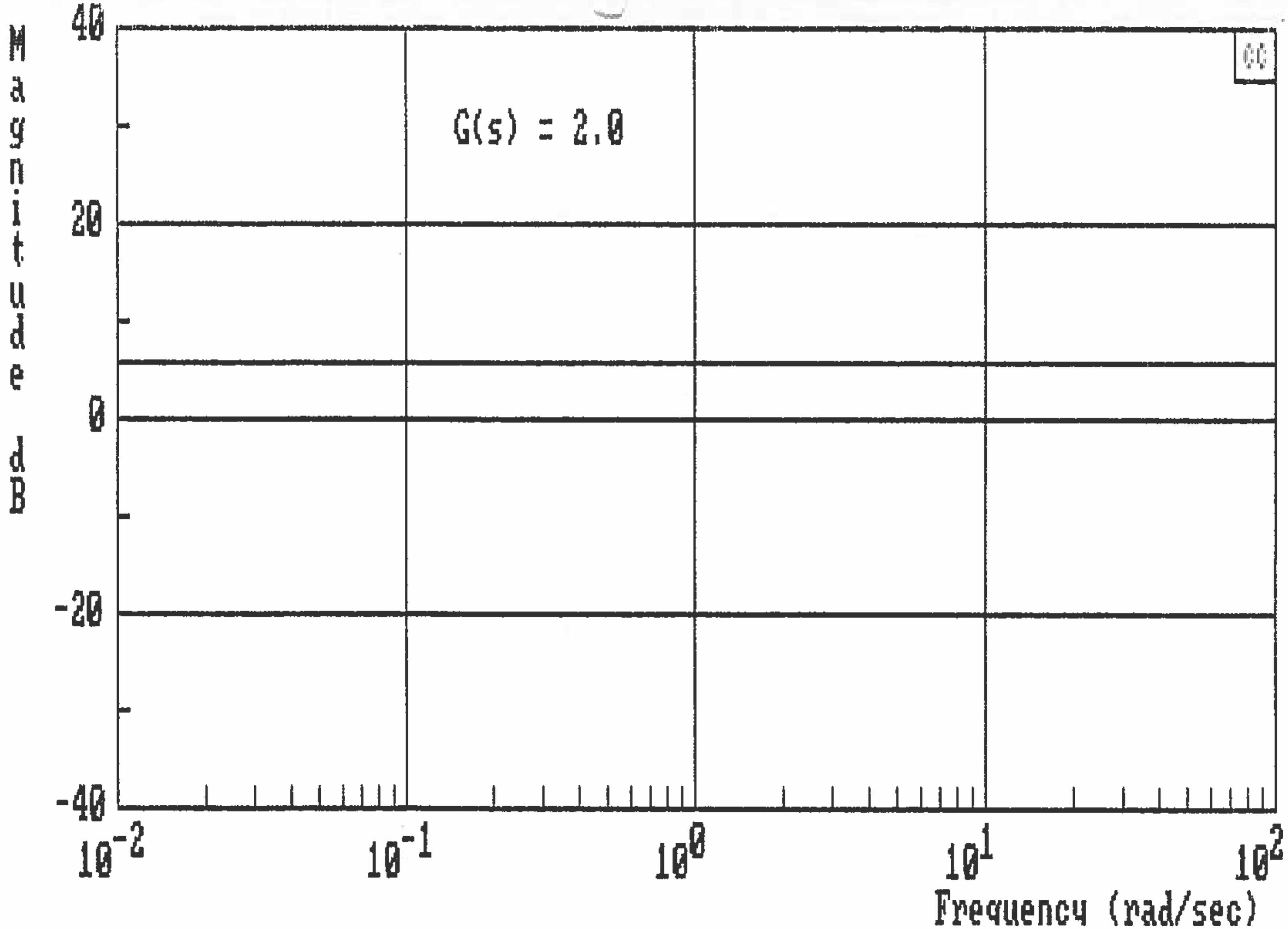
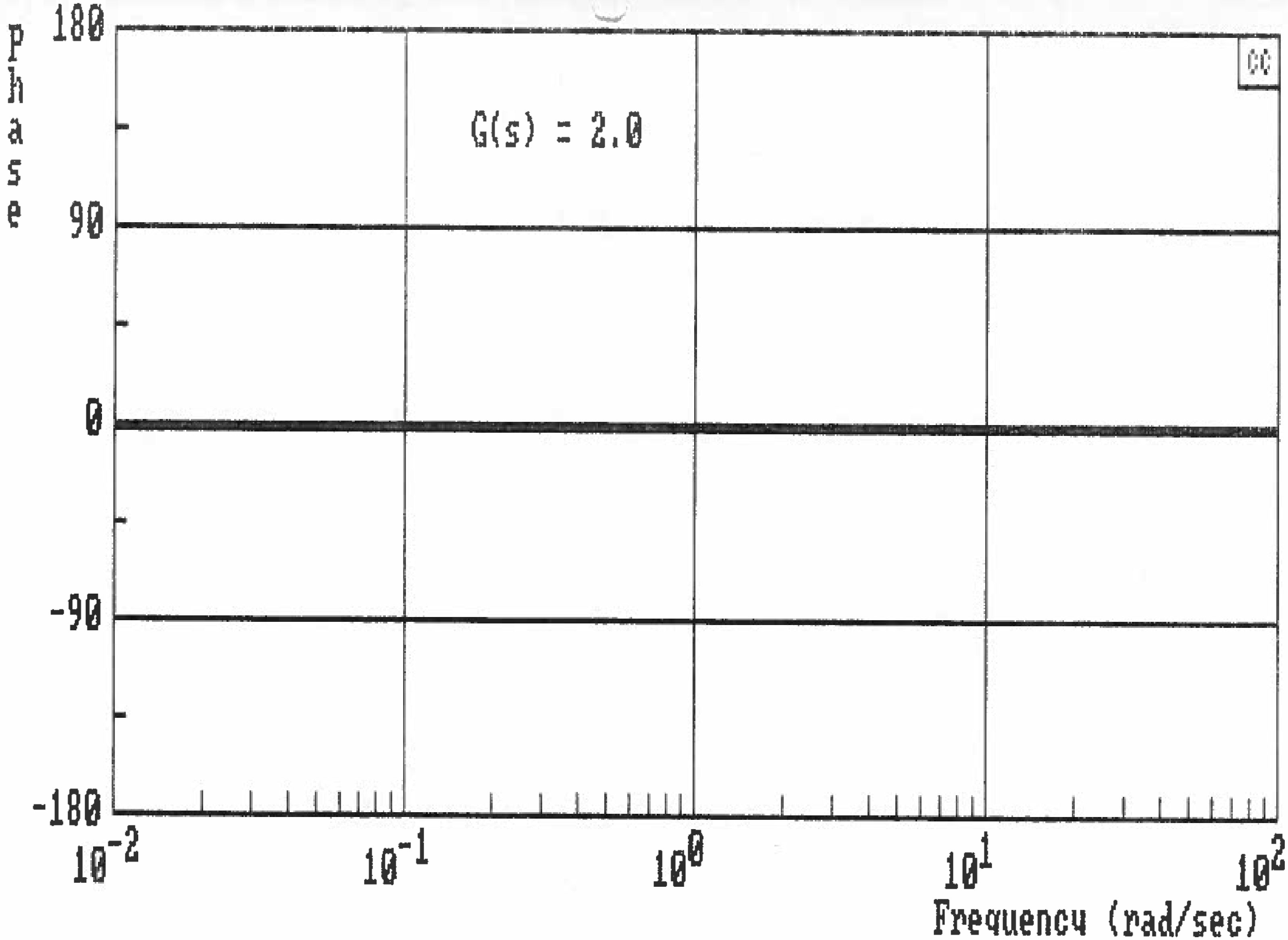

G(s) = 2.0
Magnitude dB
40
20
0
-20
-40
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 1



Phase
180
90
0
-90
-180
G(s) = 2.0
CC
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 1

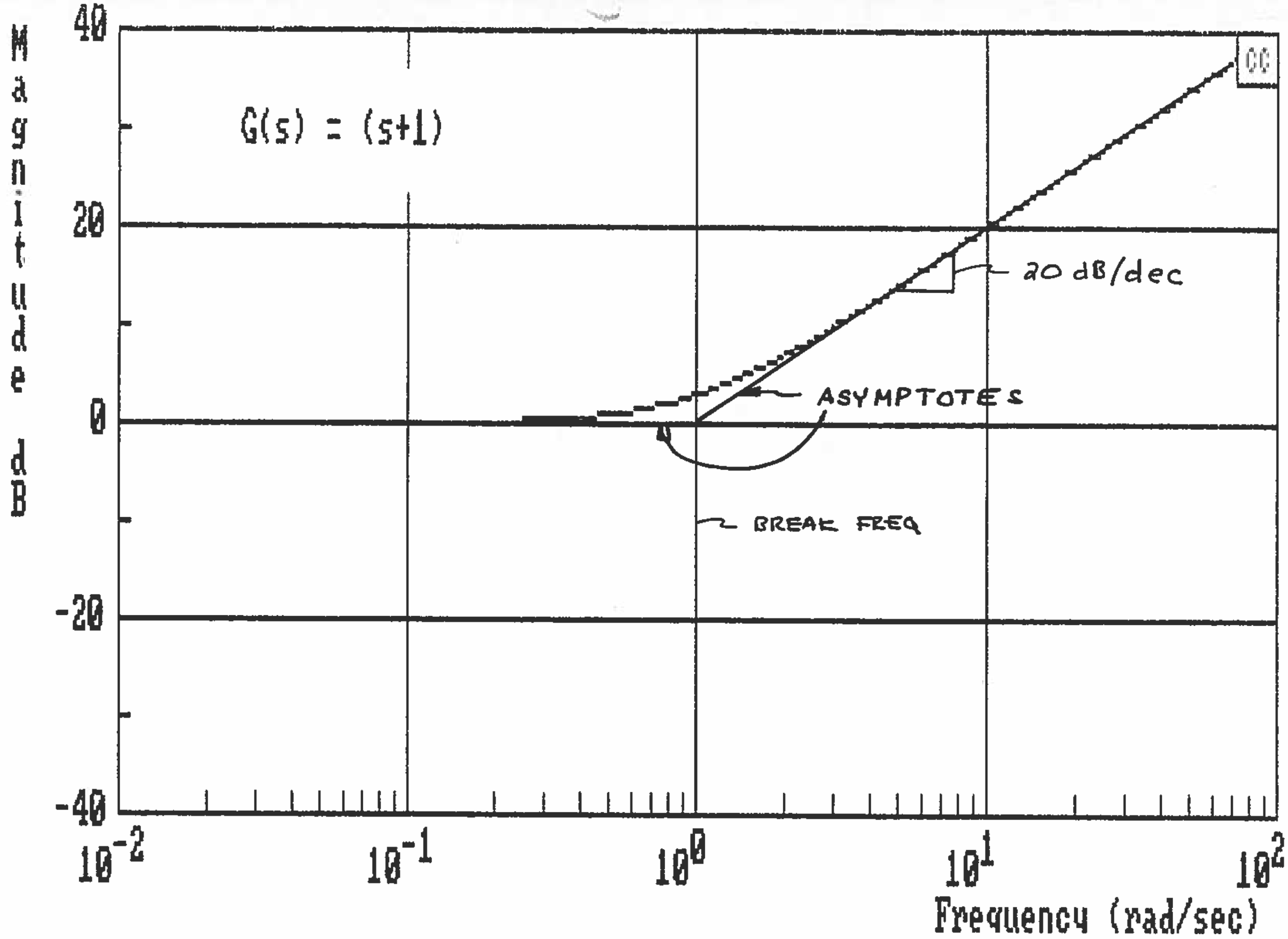

G(s) = (s+1)
Magnitude dB
40
20
0
-20
-40
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
20 dB/dec
ASYMPTOTES
BREAK FREQ
CC

FIG. 2

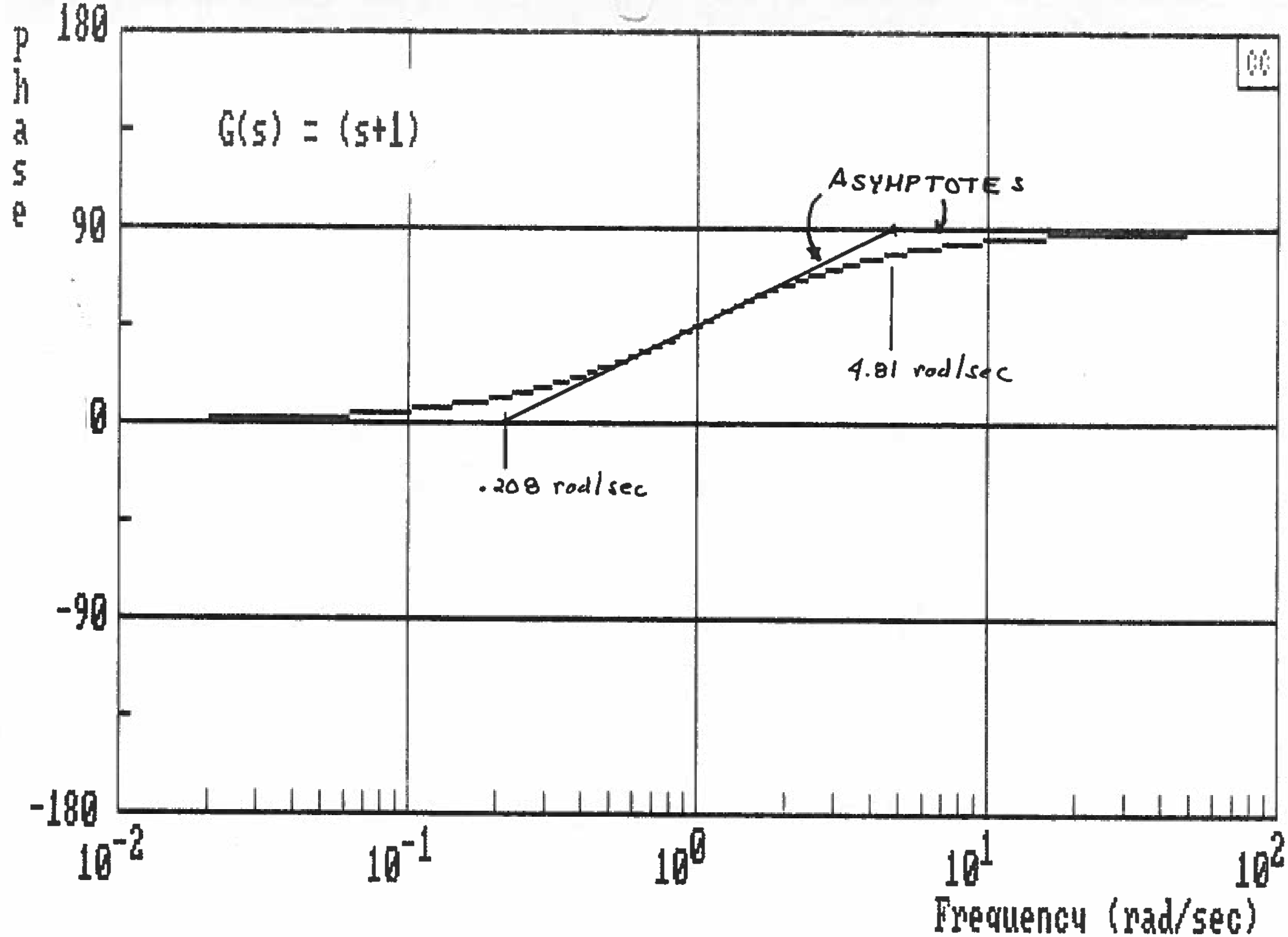

phase
180
90
0
-90
-180
G(s) = (s+1)
ASYMPTOTES
4.81 rad/sec
.208 rad/sec
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 2

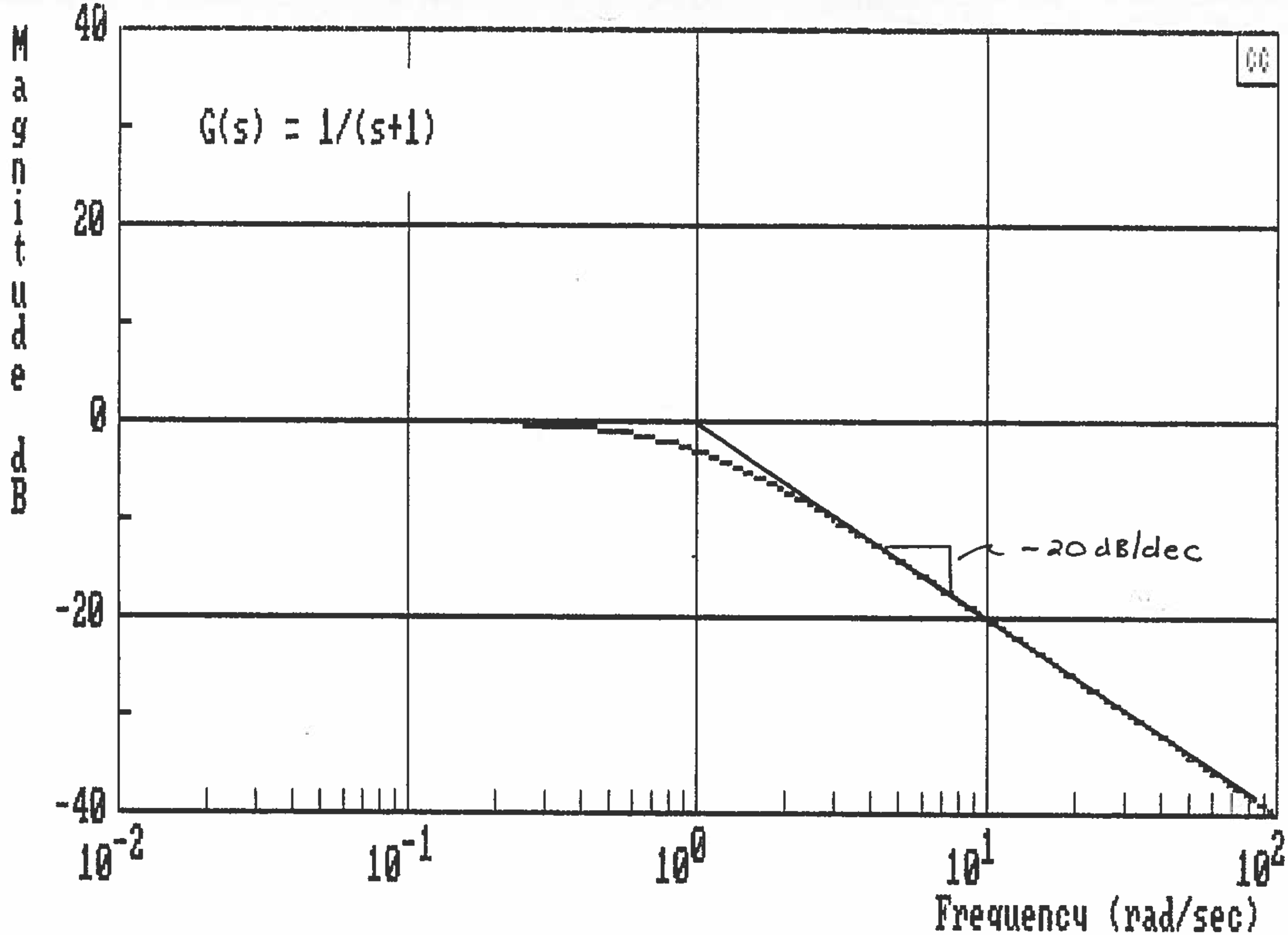
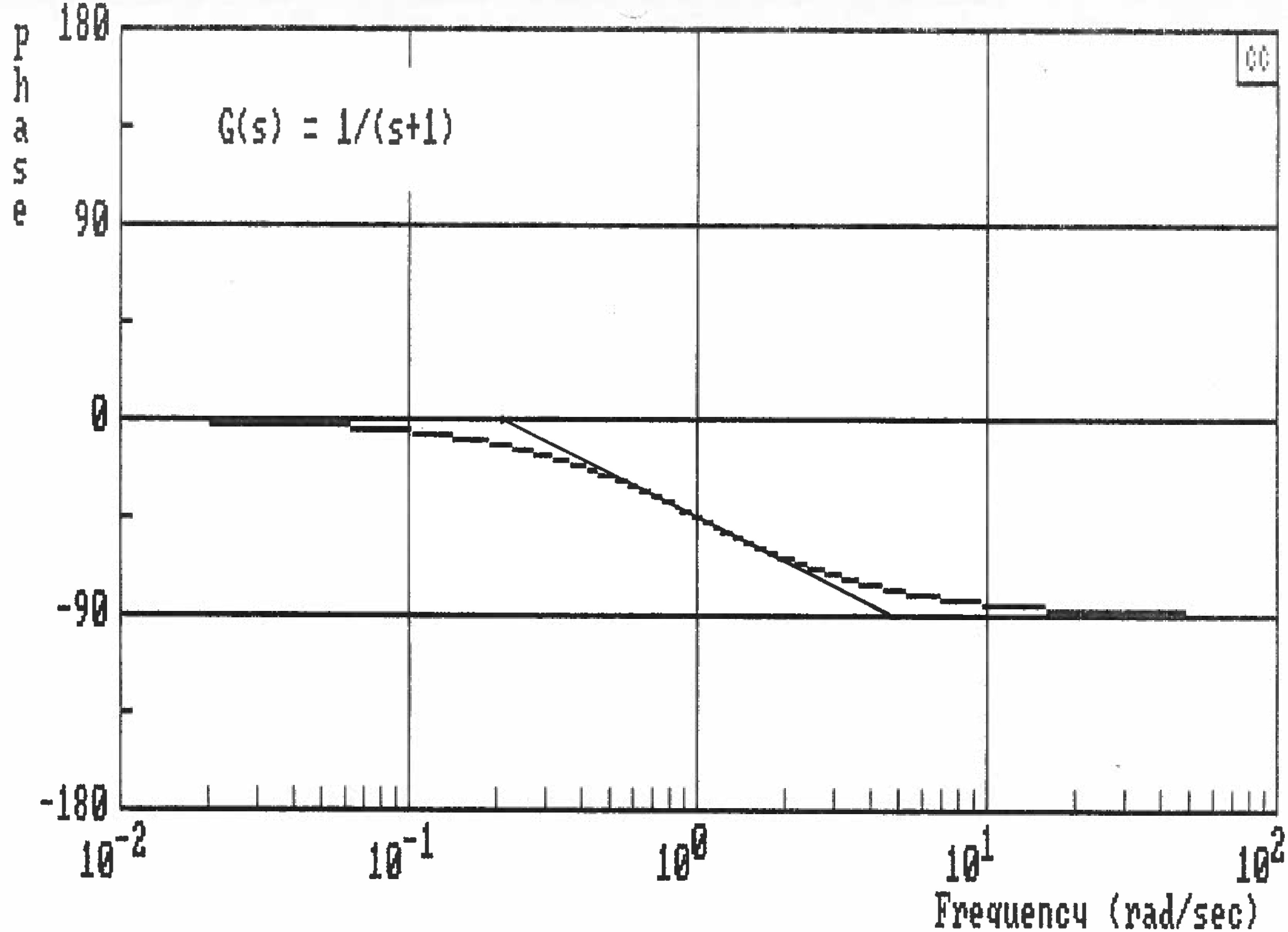
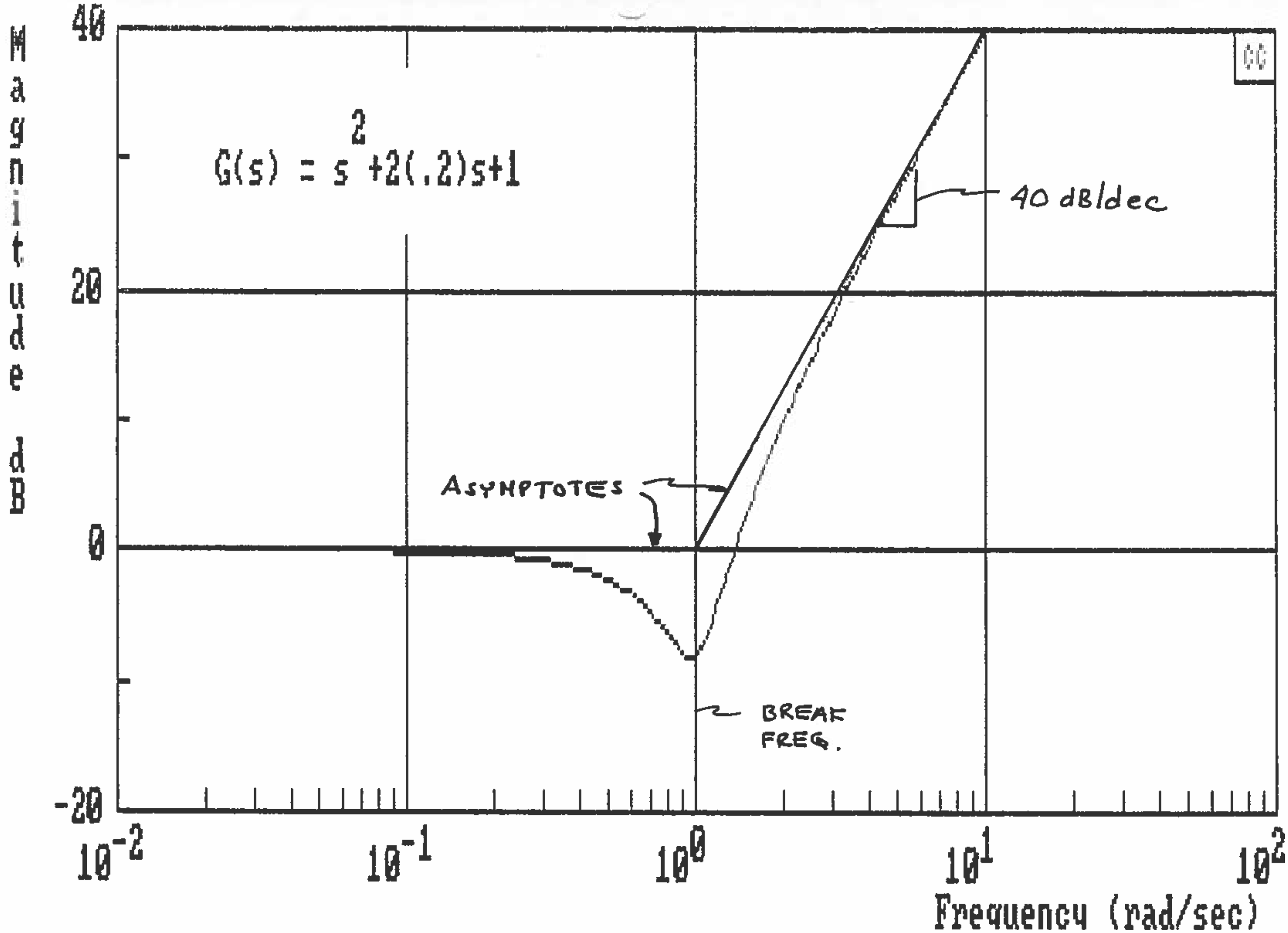

Magnitude dB
40
20
0
-20
-40
G(s) = 1/(s+1)
-20dB/dec
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 3



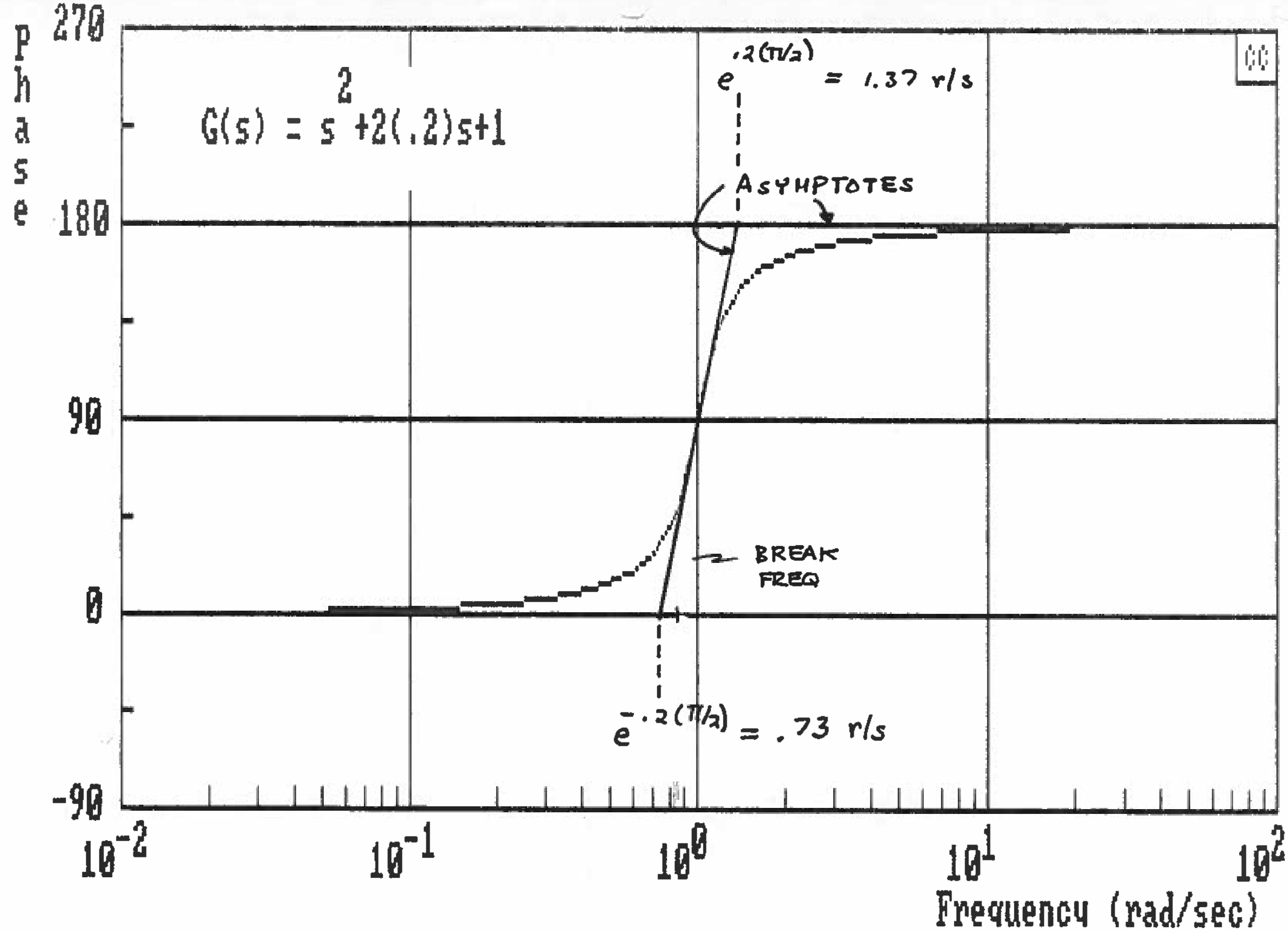
phase
180
90
0
-90
-180
G(s) = 1/(s+1)
CC
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

FIG. 3



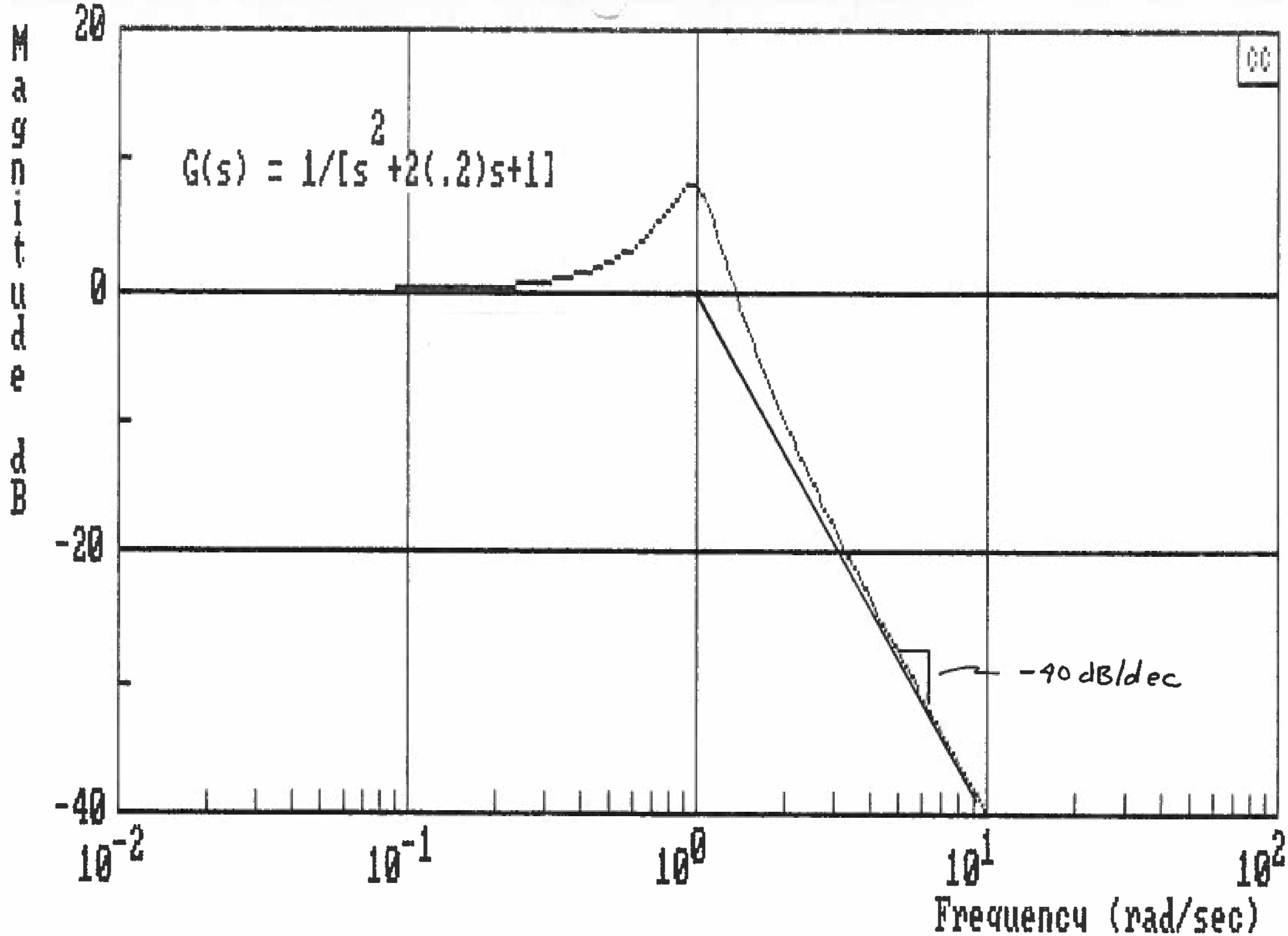
Magnitude dB
40
20
0
-20
$G(s) = s^2 + 2(.2)s + 1$
40 dB/dec
ASYMPTOTES
BREAK FREQ.
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

FIG. 4



Phase
270
180
90
0
-90
$G(s) = s^2 + 2(.2)s + 1$
$e^{.2(\pi/2)} = 1.37$ r/s
ASYMPTOTES
BREAK FREQ
$e^{-.2(\pi/2)} = .73$ r/s
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

FIG. 4



Magnitude dB
20
0
-20
-40
G(s) = 1/[s^2 +2(.2)s+1]
CC
-40 dB/dec
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 5

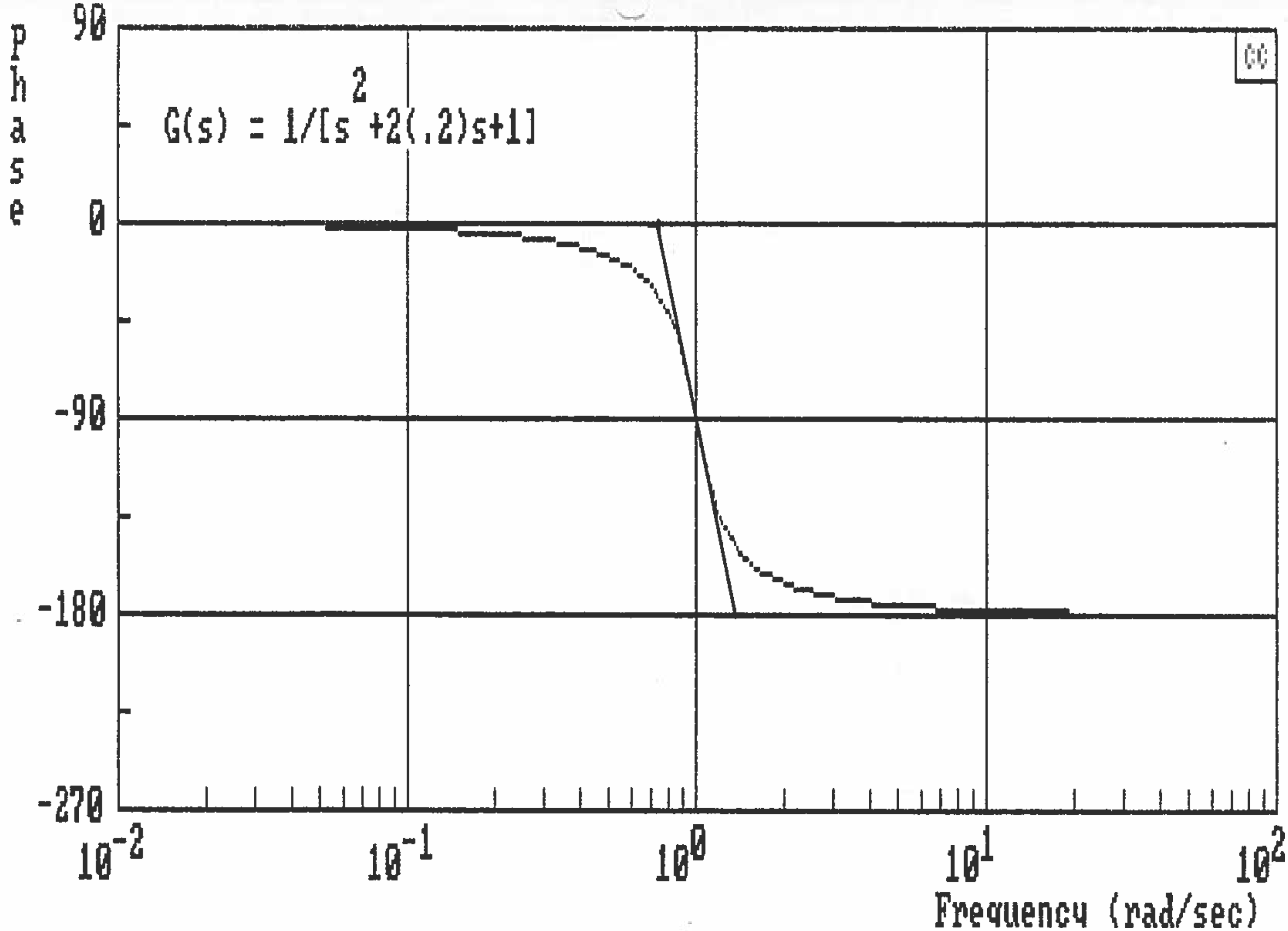
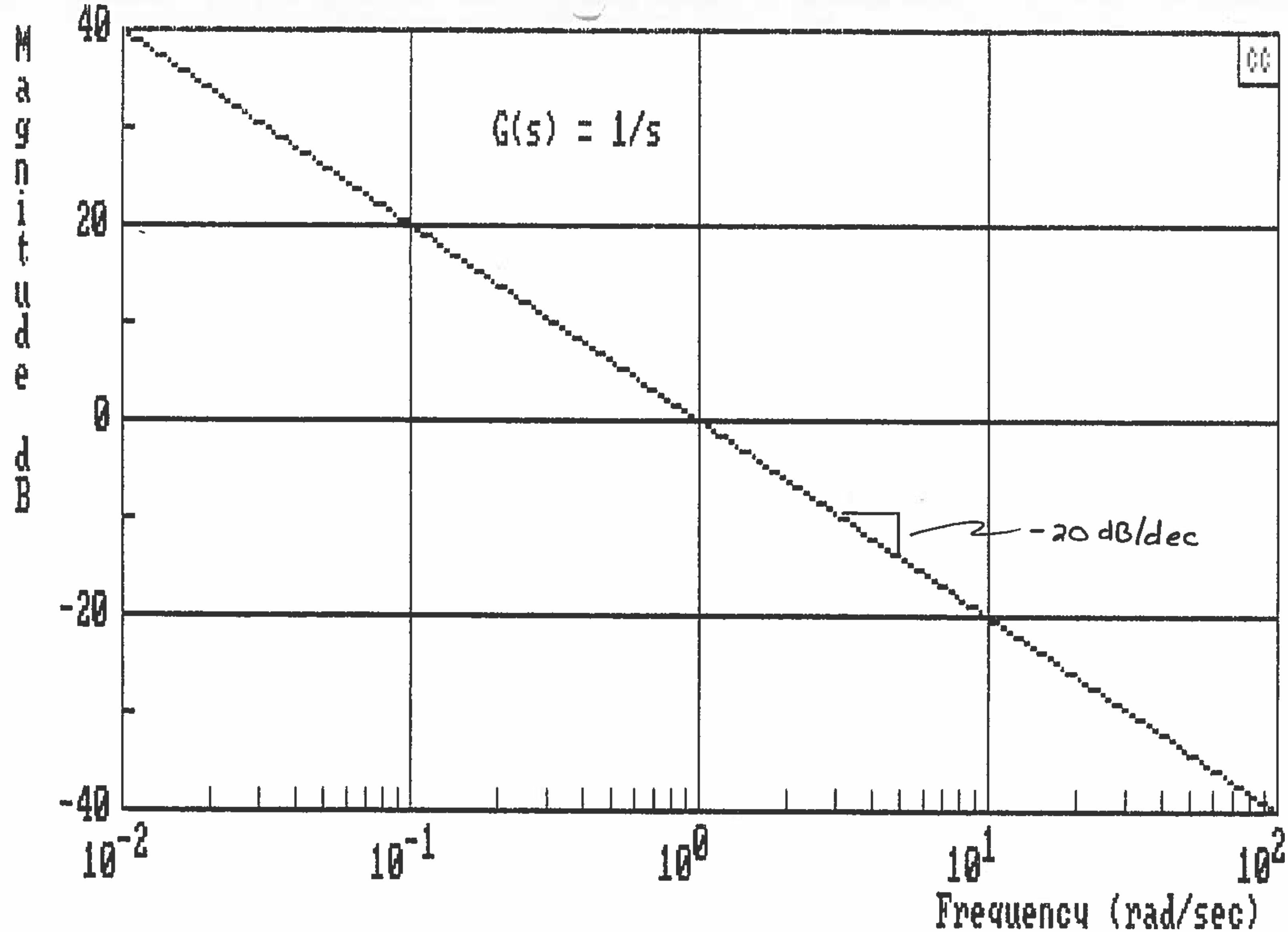
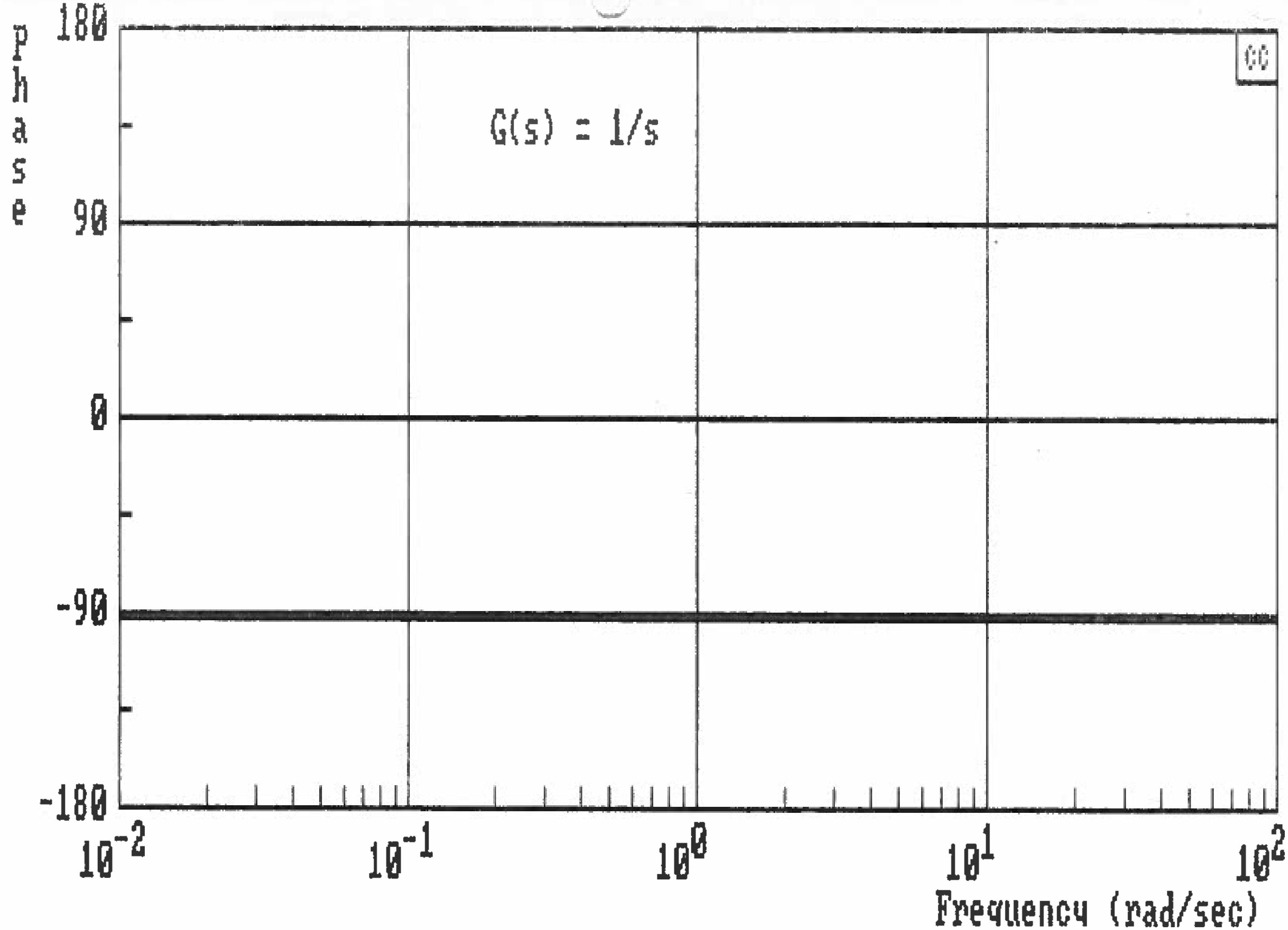

Phase
G(s) = 1/[s^2 +2(.2)s+1]
90
0
-90
-180
-270
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)
CC

FIG. 5



Magnitude dB
40
20
0
-20
-40
G(s) = 1/s
-20 dB/dec
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 6



phase
180
90
0
-90
-180
G(s) = 1/s
CC
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

FIG. 6

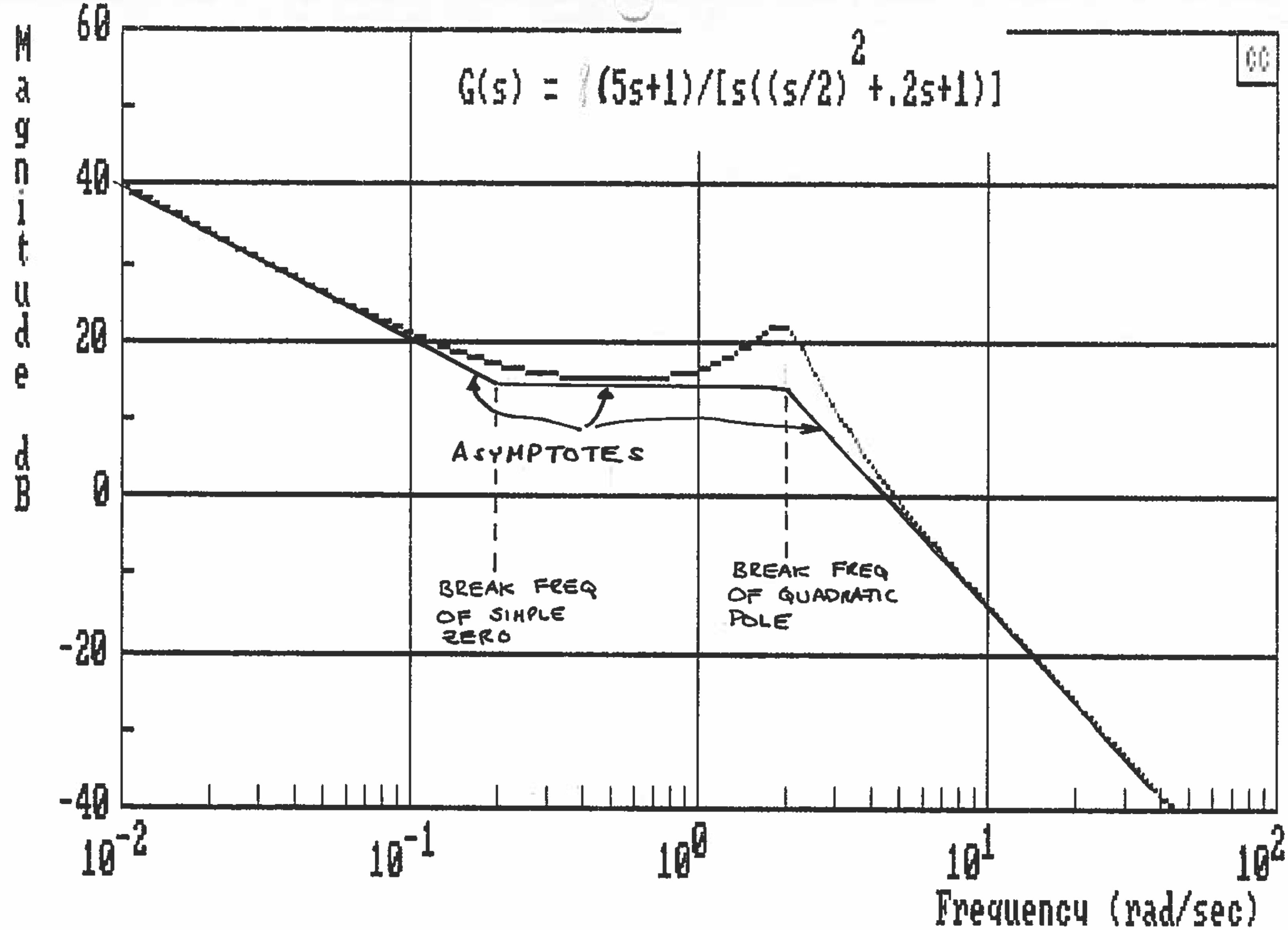

G(s) = (5s+1)/[s((s/2)^2 +.2s+1)]
CC
Magnitude dB
60
40
20
0
-20
-40
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)
ASYMPTOTES
BREAK FREQ OF SIMPLE ZERO
BREAK FREQ OF QUADRATIC POLE

FIG. 7

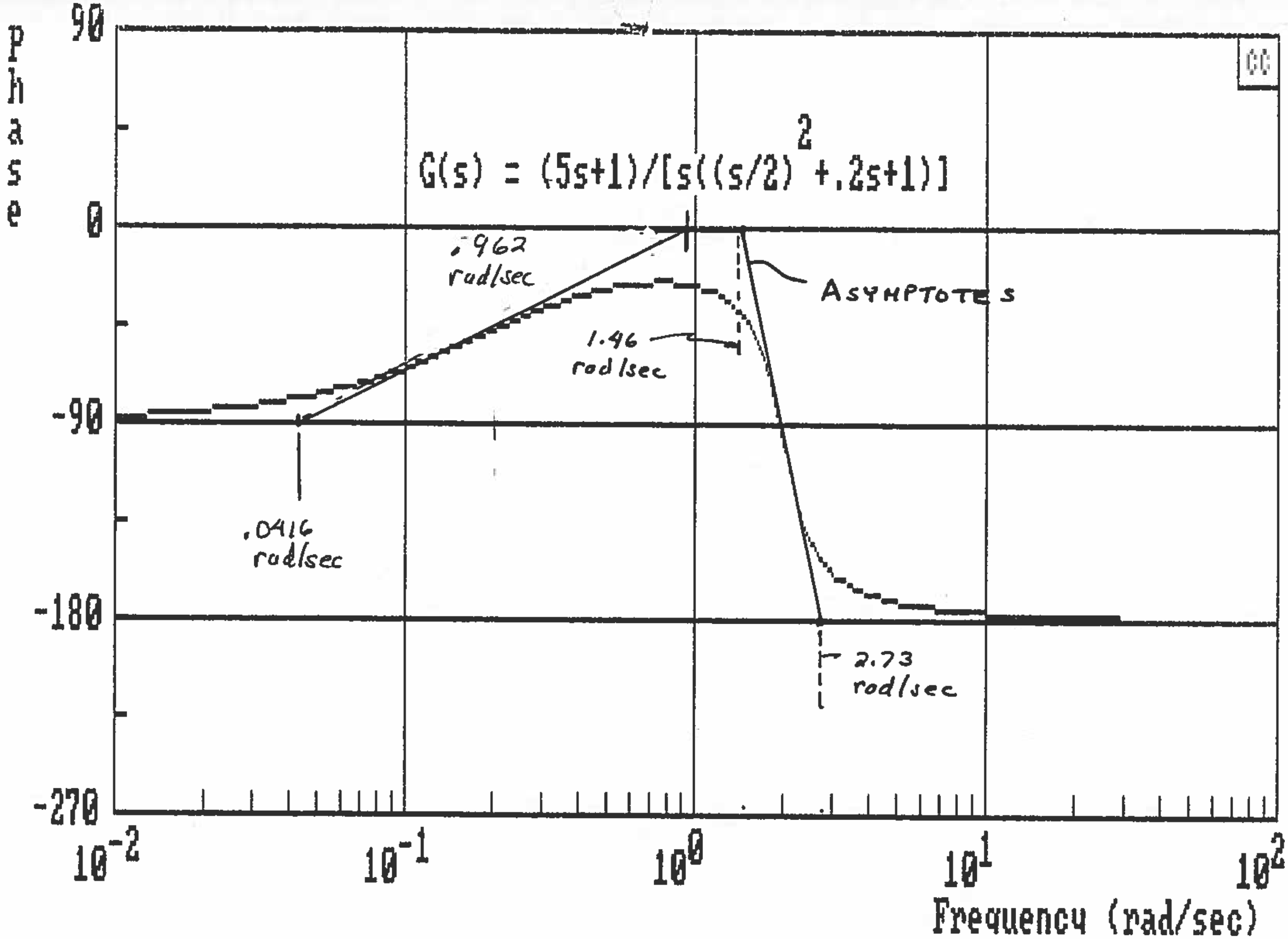

Phase
90
0
-90
-180
-270
$G(s) = (5s+1)/[s((s/2)^2+.2s+1)]$
.962 rad/sec
1.46 rad/sec
ASYMPTOTES
.0416 rad/sec
2.73 rad/sec
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

FIG. 7