Dynamics of Spacecraft

MAE 243a – Spacecraft Engineering

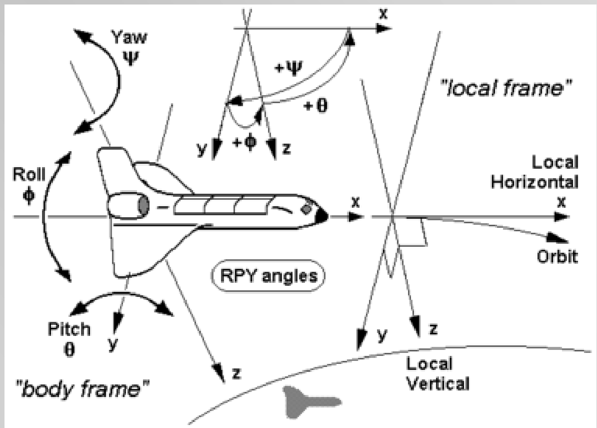
Separation of translation and rotation

A convenient aspect of spacecraft is that their translational (trajectory) motion is nearly independent of their rotational motion.

The dynamics of a craft may be described in terms of its momenta:

linear momentum L - leading to equations that describe the trajectory

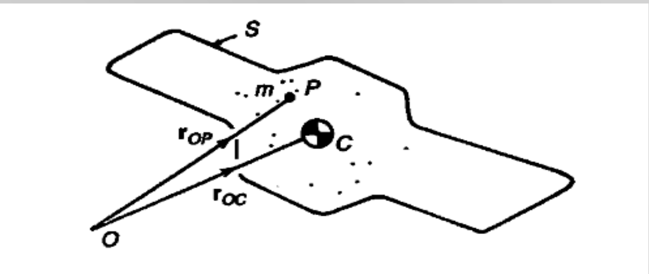
angular momentum H –leading to equations that describe the attitude motion



This means we will be separating the dynamics into two parts:

1. The motion of the centre-of-mass, C and
2. The motion relative to the centre-of-mass

The centre-of-mass, C



The centre-of-mass of the particles in S, relative to an arbitrary point O, is the point whose position vector r_{oc} obeys

$$Mr_{OC} = \sum(mr_{OP})$$

With a continuous mass distribution, an integral equivalent can be used

$$Mr_{OC} = \int r_{OP}dm$$

If we consider the centre-of-mass as the origin then $r_{OC} = 0$ and

$$\int r_{CP}dm = 0$$

Since this is always true then we also know that their derivatives must be

$$\int V_{CP}dm = 0$$

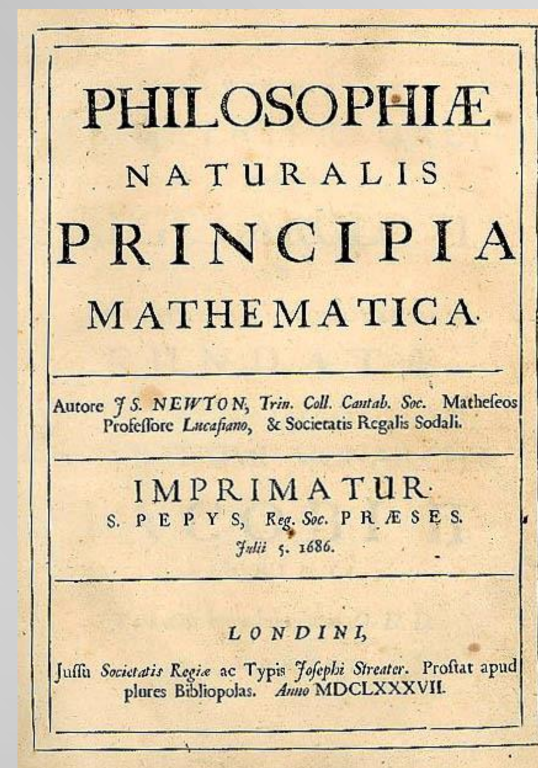
What is V*m?

Does this tell us anything about our previous assumption?

Back to the basics
Newton's Second Law

What equation best represents Newton's Second Law?

$$F = ma\,?$$



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S.* NEWTON, *Trin. Coll. Cantab. Soc.* Matheſeos
Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

A different but "original" approach

Newton's original Latin text:

"Lex II: Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur."

1729 translation by Andrew Motte:

"Law II: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd."

I. Bernard Cohen:

"<u>The change of momentum of a body is proportional to the impulse impressed on the body</u>, and happens along the straight line on which that impulse is impressed."

Momentum = mV : Change in Momentum = $m_1V_1 - m_0V_0$

Impulse = Ft : Impulse Impressed (constant Force) = $Ft_1 - Ft_0$

In difference form we can express Newton's Second Law as

$$Ft_1 - Ft_0 = m_1V_1 - m_0V_0$$

If we wanted to, from here we can get to F = ma, however we can see this is only after assuming a constant mass

Difference form: $$F = \frac{m_1V_1 - m_0V_0}{t_1 - t_0}$$

With constant mass: $$F = m\,\frac{V_1 - V_0}{t_1 - t_0}$$

t = time
X = location
m = mass
V = Velocity

$$F = m\,a$$

Force = mass x acceleration

Velocity, acceleration, momentum and force are vector quantities

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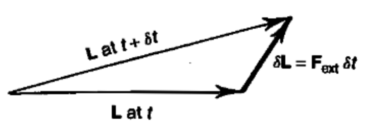
Translational motion as Newton intended

Lets go back to Newton's unaltered second law but look at it for an external force over an infinitesimal time

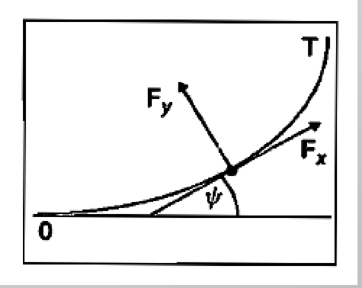
$$\delta L = F_{ext}\delta t$$



With the baseline equation and notation established we will go back to a simplified constant mass system to establish some behaviors (we will return to variable mass systems later). We can write the above equation as the Newtonian equation:

$$dL/_{dt} = \boldsymbol{d(Mv_c)}/_{\boldsymbol{dt}} = \boldsymbol{F_{ext}}$$

The force can then be decomposed into components, one parallel to the crafts momentum vector and one orthogonal to the crafts momentum vector



Translational motion as Newton intended

How does each component affect our Linear Momentum L?

F_x along the trajectory will only change the magnitude of the momentum

$$M(V_T - V_0) = \int_0^T F_x dt$$

F_y normal to the trajectory will change only the direction of the momentum

$$M_V(\psi_T - \psi_0) = \int_0^T F_y dt$$

The rate at which the direction changes is $d\psi/dt = F_y Mv$

When is a direction change most efficient?

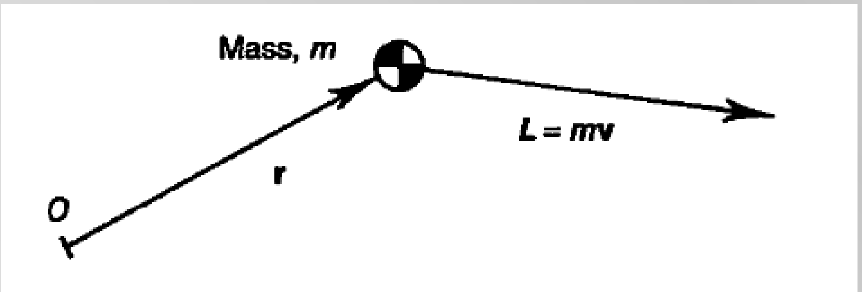
What shape would you get if F_y is constant?

All of this is for our point mass located at the centre-of-mass and is only used for linear momentum. It will be seen that for the rotational half of the problem, a torque T will affect our angular momentum H in a similar manner to the way that a force F affects our linear momentum L.

Moment-of-momentum mh

Now a concept you may not be as familiar with, the moment of momentum. We will define it as follows:

$$mh_0 = r \times mv$$



Where mh_o is the moment of momentum relative to point 0 and r is the vector from 0 to the line of action of v. The moment of momentum is useful because we can use it as a stepping stone to angular momentum.

Angular Momentum of craft in reference to gravitational center H_0

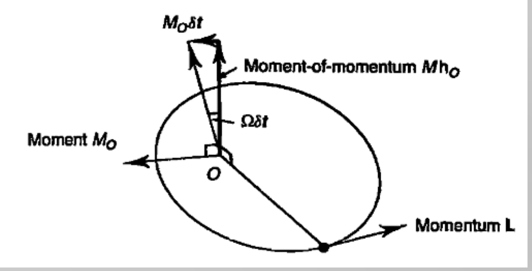
For a craft of mass M on orbit around a body, it is best to use the moment of momentum relative to the center of gravitational attraction (i.e. the center of the Earth). Its moment of momentum Mh_0 is sometimes erroneously referred to as the angular momentum H_0 however it only makes up part of the total angular momentum. The total angular momentum of the craft is:

$$H_0 = Mh_0 + H_C$$

Here H_C is the angular momentum of the craft relative to centre-of-mass. This is the type of angular momentum we are more familiar with. When performing celestial mechanics, H_C is very much les than Mh_0 and is usually neglected. H_C is usually only significant for attitude dynamics.

To the right we can see the effect of applying a moment M_0 upon the Moment-of-momentum Mh_0.

Note: the Moment-of-momentum is always normal to the orbital plane



Rate of change of moment-of-momentum

The change resulting from a force that causes a moment about 0, M_0, can be represented with the Newtonian equation:

$$d(Mh_0)/_{dt} = M_0$$

External forces on orbit can be placed into two categories, gravitational forces and perturbative forces. Gravitational forces act through 0, therefore do not impart a moment. The perturbative forces can cause moments, which means that they can cause orbital perturbations.

Looking back at our relationship between external forces and the momentum we can see some similarities in our moment of momentum equation shown above. Similarly to how we handled the forces we can also decompose the momentum change into components in the orbital plane and normal to the orbital plane.

An M_0 in the orbital plane will only change the shape of the orbit.

An M_0 normal to the orbital plane will only rotate the orbital plane.

What is an impulse?

We are used to dealing with work and forces applied over a distance, but what if no distance is covered?

Suppose you were asked to hold a sandbag at the same height all day. You would exert a constant force to hold it up but would never cover a distance since it is always at the same height. At the end of the day your boss doesn't pay you because you didn't do any work. Well then why are you so tired?

Another way to transfer energy/momentum is through an impulse. An impulse is when a force is applied over an increment of time instead of an increment of distance. It can be represented by the following equation:

$$I = \int_0^\tau F dt$$

From our earlier equation $\delta L = F_{ext}\delta t$ we can see that $I = L_\tau - L_0$ so that an impulse is equal to the change in momentum that it causes

For rotational elements there is also a similar torque impulse I_T defined as

$$I_T = \int_0^\tau T dt$$

It also follows that a torque impulse is equal to the change in angular momentum that it causes

Translational motion under propulsion

For motion under propulsion we have to consider the changing mass due to spent propellent. We can consider the mass flow rate of a rocket to be σ. This would mean that the resulting mass rate of change could be represented as:

$$dM/_{dt} = -\sigma$$

Considering the absolute velocity of the exhaust ($v_C + v_{ex}$) we can write an equation for the momentum change:

$$d(Mv_C)/_{dt} = -\sigma(v_C + v_{ex})$$

In almost all cases we will have additional external forces which gives us the complete equation:

$$d(Mv_C)/_{dt} = F_{ext} - \sigma(v_C + v_{ex})$$

So it follows that our absolute acceleration a_C is:

$$Ma_C = F_{ext} - \sigma v_{ex}$$

Tsiolkovsky rocket equation and course changes

Integrating the equation for translational motion under propulsion with no external forces produces the following equation for a incremental change in velocity due to a burn:

$$\Delta v = -v_{ex} ln(^{M_0}/_{M_1})$$

Where M_0/M_1 is the ratio between the mass before the burn and the mass after the burn. This is useful for short burns that can be considered impulsive.

If the Δv is in a direction such that the initial and final velocity vectors are equal in magnitude than it is considered a pure course change. The relationship between the angle of the plane change and the Δv imparted can be found to be:

$$\Delta\psi = 2arcsin(^{|\Delta v|}/_{2v})$$

Translational Energy

Keeping the momentum going

$$KE = \frac{1}{2}Mv_C^2 = \frac{1}{2}Mv_C v_C = \frac{1}{2}Lv_C$$

If we were to look at the change in kinetic energy we see that

$$\Delta KE = \Delta\left(\frac{1}{2}Mv_C v_C\right) = \int F_{ext} ds$$

Conservative forces result in a change in PE. Therefore in the previous equation, the RHS can be split into conservative forces (PE) and non-conservative forces, F_{nc}

$$\Delta KE + \Delta PE = \int F_{nc} ds$$

With the gravitational potential energy represented as -$\mu M / r$ and in the absence of non-conservative forces, the orbital energy equation becomes:

$$\frac{1}{2}Mv_C v_C - {}^{\mu M}/_{r} = Constant$$

From this we can see that with no non-conservative forces, the total energy does not change. The energy oscillates between kinetic and potential energy.