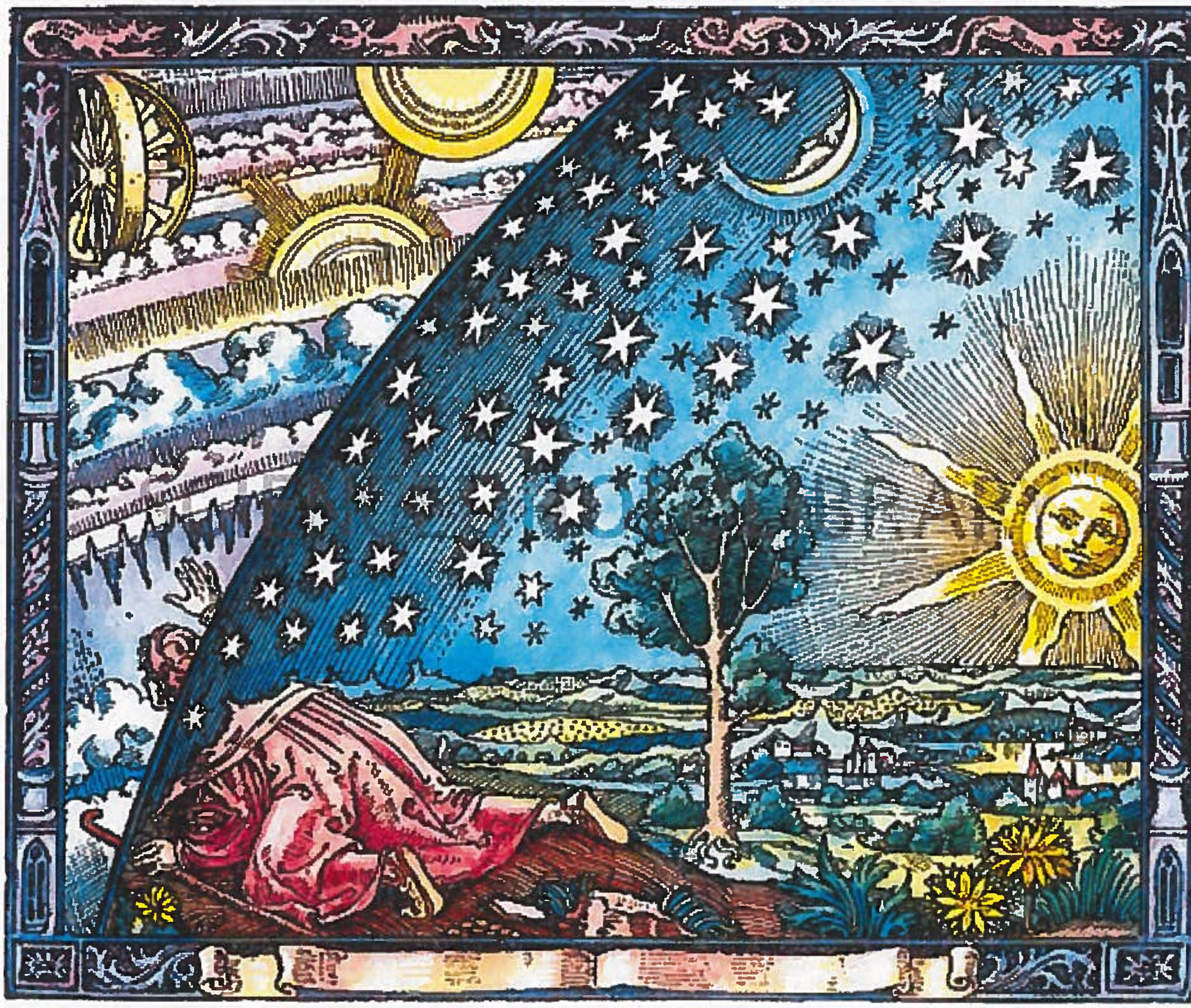
Celestial Mechanics



Ideal Equation of Orbital Motion – "2-Body Problem"

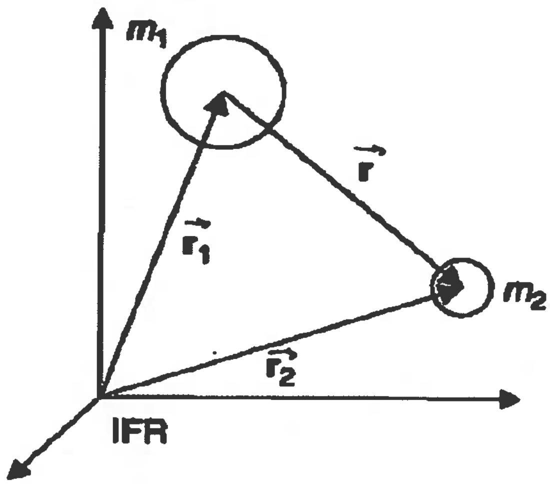


Figure 4.2 The two-body problem geometry in an inertial frame of reference (IFR)

- Kepler's 3 Laws : Kepler – observation ($\Delta t \cong 65\,yr$)
  Newton – analysis (laws of mechanics)

- 2 bodies : sole force is mutual gravitational attraction :

Newton's Law of Universal Gravitation :

$$\vec{F}_1 = \underbrace{\frac{G m_1 m_2}{r^2}}_{\text{attractive force due to mass}} \underbrace{\left(\frac{\vec{r}}{r}\right)}_{\text{unit vector in direction of other mass}} \quad ; \quad \vec{F}_2 = \frac{G m_1 m_2}{r^2}\left(\frac{-\vec{r}}{r}\right) \qquad 4.2$$

$G \equiv$ Univ. Constant of Gravitation ($6.670\text{E-}11\ Nm^2/kg^2$)

- Newton's 2^{nd} Law (constant mass): $\vec{F}_1 = m_1 \ddot{\vec{r}}_1$

$\vec{r}_i$ is position wrt inertial ref. frame (stars) $\quad \vec{F}_2 = m_2 \ddot{\vec{r}}_2$ 4.3

- Change ref frame to relative motion between bodies:

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \quad \Rightarrow \text{position vector of } m_2 \text{ rel. to } m_1$$

- Combine our 2 laws of mechanics (4.2 and 4.3) to get equation of motion of m_2 relative to m_1:

$$\ddot{\vec{r}} + \frac{G(m_1 + m_2)}{r^2}\left(\frac{\vec{r}}{r}\right) = 0$$

- "Restricted" 2-body problem: $\underbrace{m_1}_{\text{planet}} \gg \underbrace{m_2}_{\text{satellite}}$

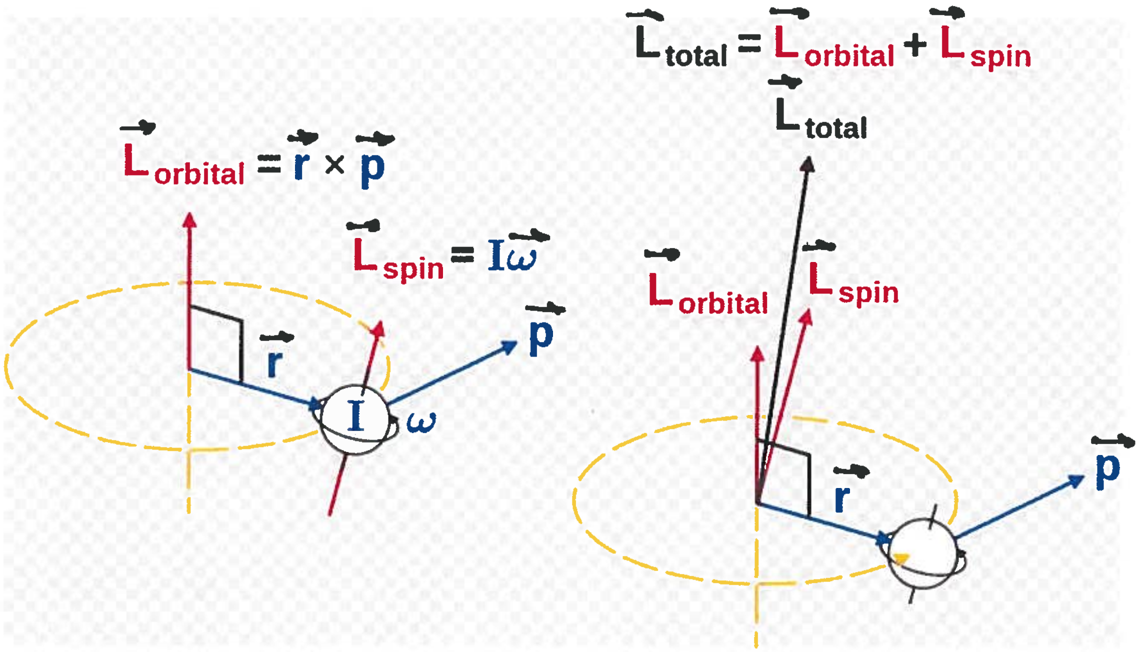
$m_1 \to M$

Eqn. of Motion of "massless" satellite about planetary mass M:

$$\boxed{\ddot{\vec{r}} + \frac{\mu}{r^2}\left(\frac{\vec{r}}{r}\right) = 0 \quad \text{where } \mu \equiv GM \text{ grav. constant for } M} \quad \text{or} \quad \boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}} \qquad 4.4$$

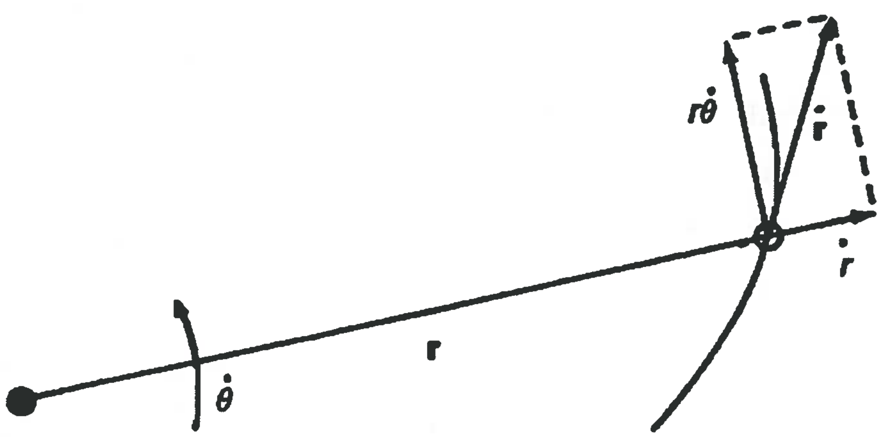
First Constant of Ideal Orbital Motion:

Orbital Angular Momentum (Moment of Momentum)



$$\left.\begin{array}{l}\text{Orbital Ang Momentum}\\ \text{Moment of Momentum}\end{array}\right) = \vec{r} \times m\vec{v} = m\vec{h}$$

$$\text{Specific angular momentum} = \vec{h} = \vec{r} \times \dot{\vec{r}}$$



$$|\vec{h}| = r\,(r\omega)$$
$$= r\,(r\dot{\theta})$$
$$= r^2\dot{\theta}$$

Figure 4.3 The particle's position and velocity vectors

- No reason for M or M to change, since only force acts through center of primary mass M.

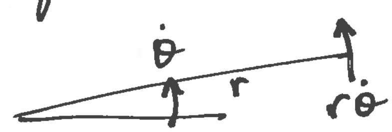
$$\therefore \quad \vec{r} \times m\vec{v} = \text{constant} \Rightarrow \text{both magnitude and direction}$$

- Direction: $\vec{h} = \vec{r} \times \dot{\vec{r}} \Rightarrow \perp$ to both position + velocity vec

$$= \underline{\text{normal to orbital plane}} = \text{constant (plane)}$$

- Magnitude: $|\vec{h}| = \text{constant} \Rightarrow$ an expression of Kepler #2

$$= r(V_{normal}) = r(r\omega) = r(r\dot{\theta}) = r^2\dot{\theta}$$



- Proof that $\vec{h} = \vec{r} \times \dot{\vec{r}} = \text{constant}$:

$$\frac{d\vec{h}}{dt} = \cancelto{0}{(\dot{\vec{r}} \times \dot{\vec{r}})} + (\vec{r} \times \ddot{\vec{r}}) \quad \text{and} \quad \ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

$$= \vec{r} \times \left(-\frac{\mu}{r^3}\vec{r}\right)$$

$$= -\frac{\mu}{r^3}\cancelto{0}{(\vec{r} \times \vec{r})} = 0$$

$$\therefore \boxed{\vec{h} = \text{constant}}$$

Potential Fields

- A mass has gravitational potential energy / unit mass

$$U \equiv -\frac{GM}{r} \quad \left(\frac{Nm}{kg}\right)$$

$\rightarrow$ work (energy)
$\rightarrow$ unit mass

- $\dfrac{\text{Force}}{\text{unit mass}}$ in a potential energy field:

$$\vec{F} = -\vec{\nabla} U \quad \text{(spatial gradient)}$$

$$= \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}$$

- $F = ma$ for spacecraft of constant mass may be written as

$$\ddot{\vec{r}} = \underbrace{-\vec{\nabla} U}_{\frac{\text{force}}{\text{unit mass}}}$$

$\leftarrow$ grav. potential field

Second Constant of Ideal Orbital Motion:

Total Mechanical Energy (kinetic + potential)

- Energy is conserved because only external force on the orbiter (in an ideal, 2-body, Keplerian universe) is due to conservative gravitational field, with potential energy/unit mass of $-\mu/r$.

- Derive expression for satellite energy:

eqn of motion:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$ 4.4

Dot each term with $\dot{\vec{r}}$ (or $\vec{v}$) to get scalar energy terms:

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = -\frac{\mu}{r^3}\left(\dot{\vec{r}} \cdot \vec{r}\right)$$

or $$v\frac{dv}{dt} = -\frac{\mu}{r^2}\frac{dr}{dt}$$

$$v\,dv = -\frac{\mu}{r^2}\,dr$$

Integrate ...

- Integrate manipulated eqn of motion to get energy expression:

$$V\,dV = -\frac{\mu}{r^2}\,dr$$

$$\downarrow \int$$

Vis-Viva equation "Living Force"

$$\boxed{\underbrace{\underbrace{\frac{V^2}{2}}_{KE} - \underbrace{\frac{\mu}{r}}_{PE}}_{\text{"specific energies"}} = \text{const} \equiv \epsilon \quad \frac{\text{total energy}}{\text{unit mass}}}$$ 4.5

where V = velocity magnitude of satellite relative to planet

$-\dfrac{\mu}{r} = \dfrac{\text{potential energy}}{\text{unit mass}}$ of satellite in gravitational field of planet M

(note datum: $PE = 0$ at $r = \infty$)

Solution to Satellite Equation of Motion: instaneous position at time t

- Eqn of motion of satellite orbiting M:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \qquad 4.4$$

- Take cross product of $\dot{\vec{v}}$ with constant $\vec{h}$ and integrate once:

$$\dot{\vec{r}} \times \vec{h} = \mu\left(\frac{\vec{r}}{r} + \vec{e}\right) \qquad 4.6$$

vector const. of integration "eccentricity vector" lies in orbital plane

- Dot 4.6 with $\vec{r}$ to get scalar eqn for instantaneous position:

Orbit Equation $\boxed{r = \frac{h^2/\mu}{1 + e\cos\theta}}$ Eqn. of a conic section with parameter e 4.7

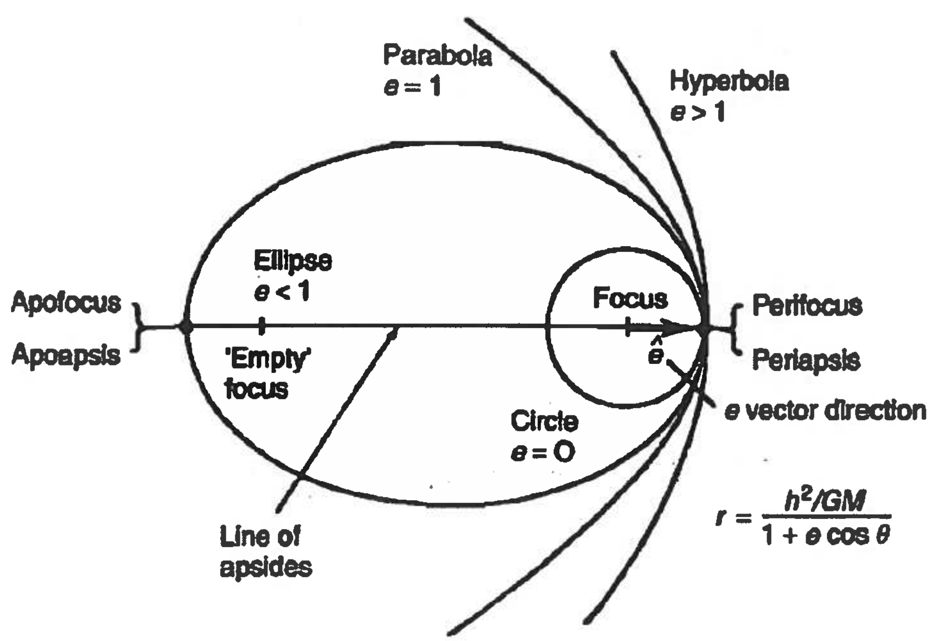
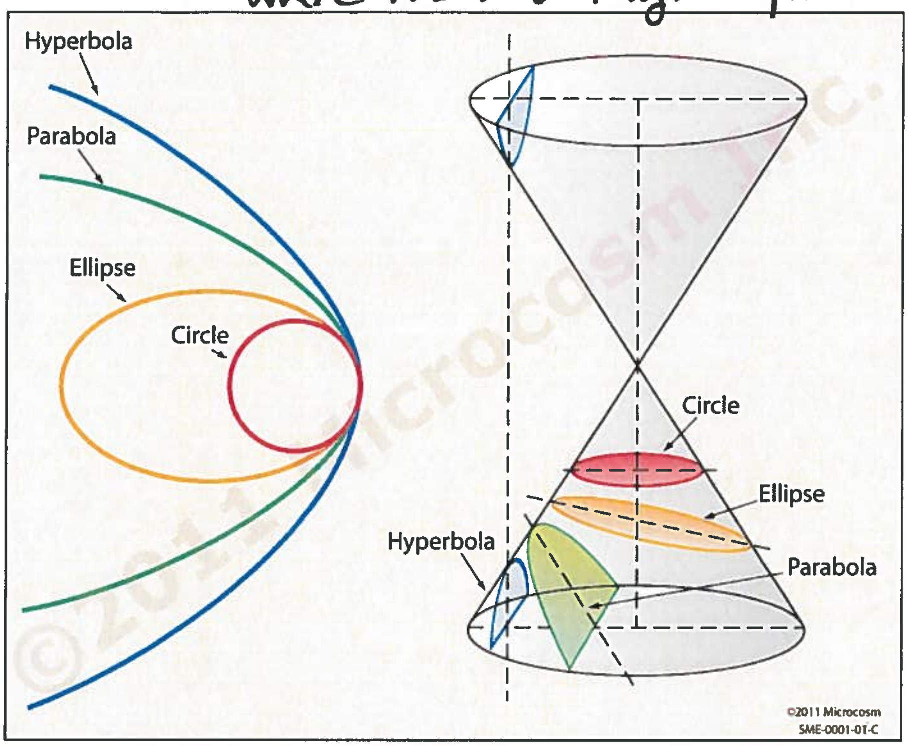


Figure 4.4 Conic sections

Keplerian Orbits are Conic Sections:

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$ 4.7

where $h = r^2\dot{\theta}$ mag. of specific orbital ang. momentum



e = eccentricity

$e = 0$ circle

$0 < e < 1$ ellipse

$e = 1$ parabola

$e > 1$ hyperbola

Fig. 9-1. The 4 Conic Sections Result from the Intersection of a Plane and a Right Circular Cone. Two special cases occur when the angle between the plane and axis of the cone is either 90 deg (resulting in a circle) or equal to the angular radius of the cone (resulting in a parabola).

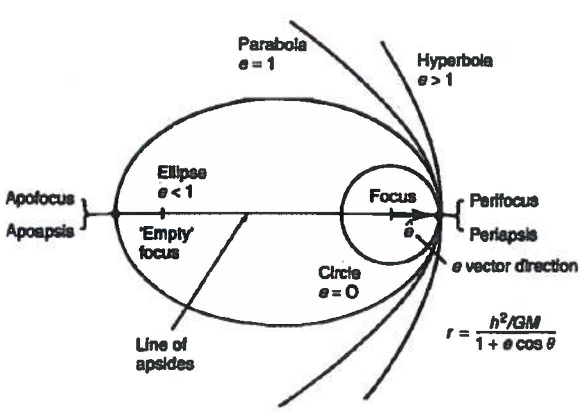
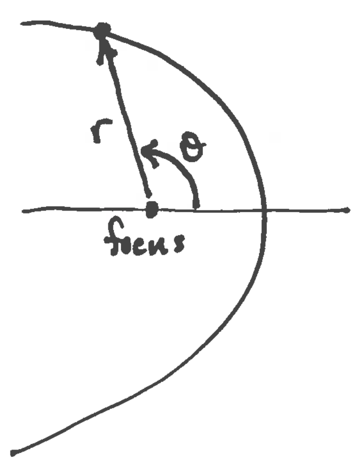


Figure 4.4 Conic sections



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Summary of Keplerian Elliptical Orbits:

a and e define ellipse

(SMAD Section 9.1)

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{p}{1+e\cos\theta} = \frac{b^2/a}{1+e\cos\theta}$$

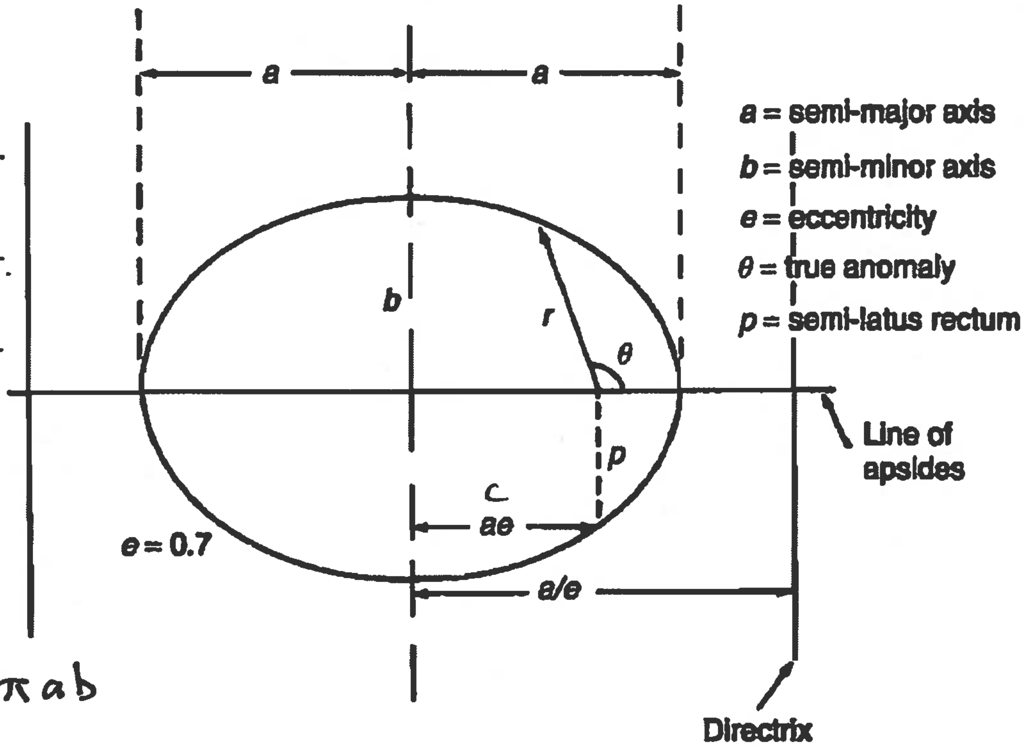


Figure 4.5 Ellipses

area: $A_{ellipse} = \pi a b$

parametric equation: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

eccentricity: $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2+b^2}}{a}$

velocity: $V = \sqrt{\mu\left(\dfrac{2}{r} - \dfrac{1}{a}\right)}$

specific energy: $\epsilon = -\dfrac{\mu}{2a} = \dfrac{1}{2}V^2 - \dfrac{\mu}{r}$ = constant

Period: $P = \dfrac{2\pi}{\sqrt{\mu/a^3}} = \tau$

Position: $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$

Specific Angular Momentum: $h = \sqrt{\mu a(1-e^2)}$
$= Vr$ (circ)

Perigee: $r_p = a(1-e)$
Apogee: $r_a = a(1+e)$
} measured from Earth center ; $r_p + r_a = 2a$

Position vs Time Relationships – Elliptical Orbits $(0<e<1)$

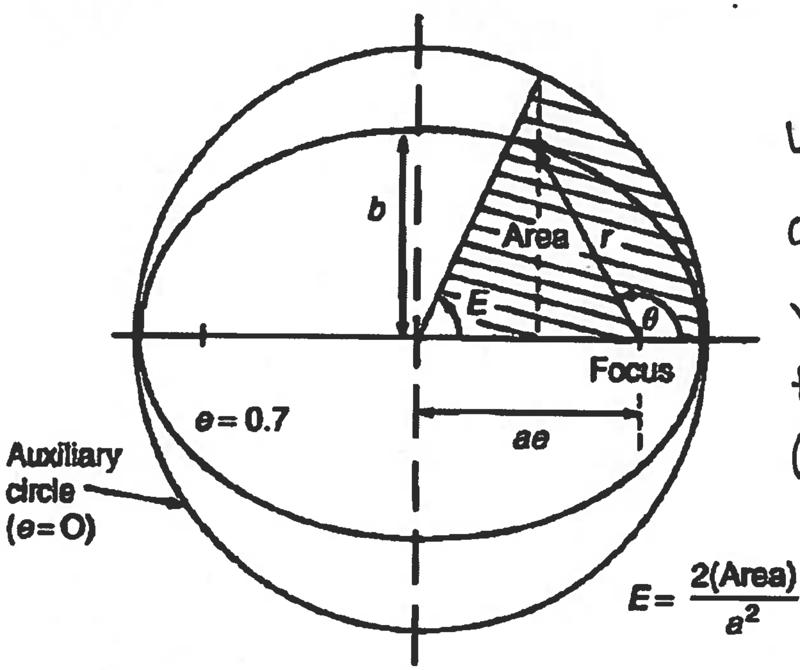


Figure 4.6 Eccentric anomaly definition

Use "Auxiliary Circle" concept to transform variable θ (true anomaly) to new variable E (eccentric anomaly)

- Geometry gives position on ellipse or circle:

$$r = a(1 - e\cos E)$$ position on ellipse in terms of E 4.13

$$\tan\left(\frac{\theta}{2}\right) = \tan\left(\frac{E}{2}\right)\sqrt{\left(\frac{1+e}{1-e}\right)}$$ relationship between the anomalies θ and E 4.14

- Position vs time relationship:

Two eqns for position on ellipse:

$$r = \frac{h^2/\mu}{1+e\cos\theta}$$ 4.7

$$r = a(1-e\cos E)$$ 4.13

Differentiate both vs time and equate to get:

$$\frac{dE}{dt} = \frac{1}{r}\sqrt{\mu/a}$$

Separate vars:

$$a(1-e\cos E)\,dE = \sqrt{\mu/a}\,dt$$

Integrate:

$$E - e\sin E = \sqrt{\mu/a^3}\,(t-t_p)$$ 4.16

time of perifocal passage

- We can let $t_p = 0$, then

$$E - e\sin E = \sqrt{\mu/a^3}\,t$$

Let $n \equiv \frac{2\pi}{\tau} = \sqrt{\mu/a^3}$ "Mean Motion" $\left(\frac{1}{t}\right)$

$\tau =$ orbital period

and $M \equiv nt$ "Mean Anomaly" unitless time

So now $\boxed{E - e\sin E = M}$ Kepler's Equation 4.18

relates time (M) to position (E)

Time/Position Recipes for a Known Elliptical Orbit: know $\begin{pmatrix} a \\ e \\ \mu \end{pmatrix}$

- Compute position (θ) given time since perifocal passage (t):

t = given

a = known major axis

μ = known grav. constant of planet

e = known eccentricity of orbit

$n = \sqrt{\mu / a^3}$, $M = n t$ 4.17 4.1

$E - e \sin E = M \Rightarrow$ solve for E (iterative) 4.1

1.14 $\tan\left(\frac{\theta}{2}\right) = \tan\left(\frac{E}{2}\right)\sqrt{\left(\frac{1+e}{1-e}\right)} \Rightarrow$ solve for $\boxed{\theta}$ position at time t

- Compute time (t) to reach a given position (θ):

θ = given

$\tan\left(\frac{E}{2}\right) = \tan\left(\frac{\theta}{2}\right)\left(\frac{1+e}{1-e}\right)^{-\frac{1}{2}} \Rightarrow$ solve for E 4.14

$M = E - e \sin E$ 4.18

$n = \sqrt{\mu / a^3}$ 4.17

$\boxed{t = \frac{M}{n}}$ time since perifocal passage at position θ (or E) 4.19

Velocity Relationships for Elliptic Orbits:

- From Vis-Viva and momentum eqns.

$$\frac{1}{2}V^2 - \frac{\mu}{r} = \mathcal{E} = -\frac{\mu}{2a}$$

or, $$\boxed{V_{ellip} = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}}$$ Velocity of satellite relative to planet for elliptical orbits

- Circular orbits: $r = a$

$$\boxed{V_{circ} = \sqrt{\mu/r}}$$ 4.22

- Surface orbit: Earth radius $\cong$ 6378 km (equator)

  $V =$ ~~[illegible]~~ 7.91 km/s

  $V_{\substack{Earth\\surface}} = 0.46$ km/s (equator)

  $\Delta V = 7.45$ km/s to orbit at sea level!

- For elliptical orbits, $|PE| > |KE|$, ~~[illegible]~~, so $\mathcal{E} < 0$ due to $PE = 0$ at $r = \infty$, by definition (so any $r < \infty$ must have negative PE)

Two Ways to Specify an Orbit: State Vectors and Orbital Elements

State Vectors: Keplerian Eqn of Motion : $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$

2nd order vector equation

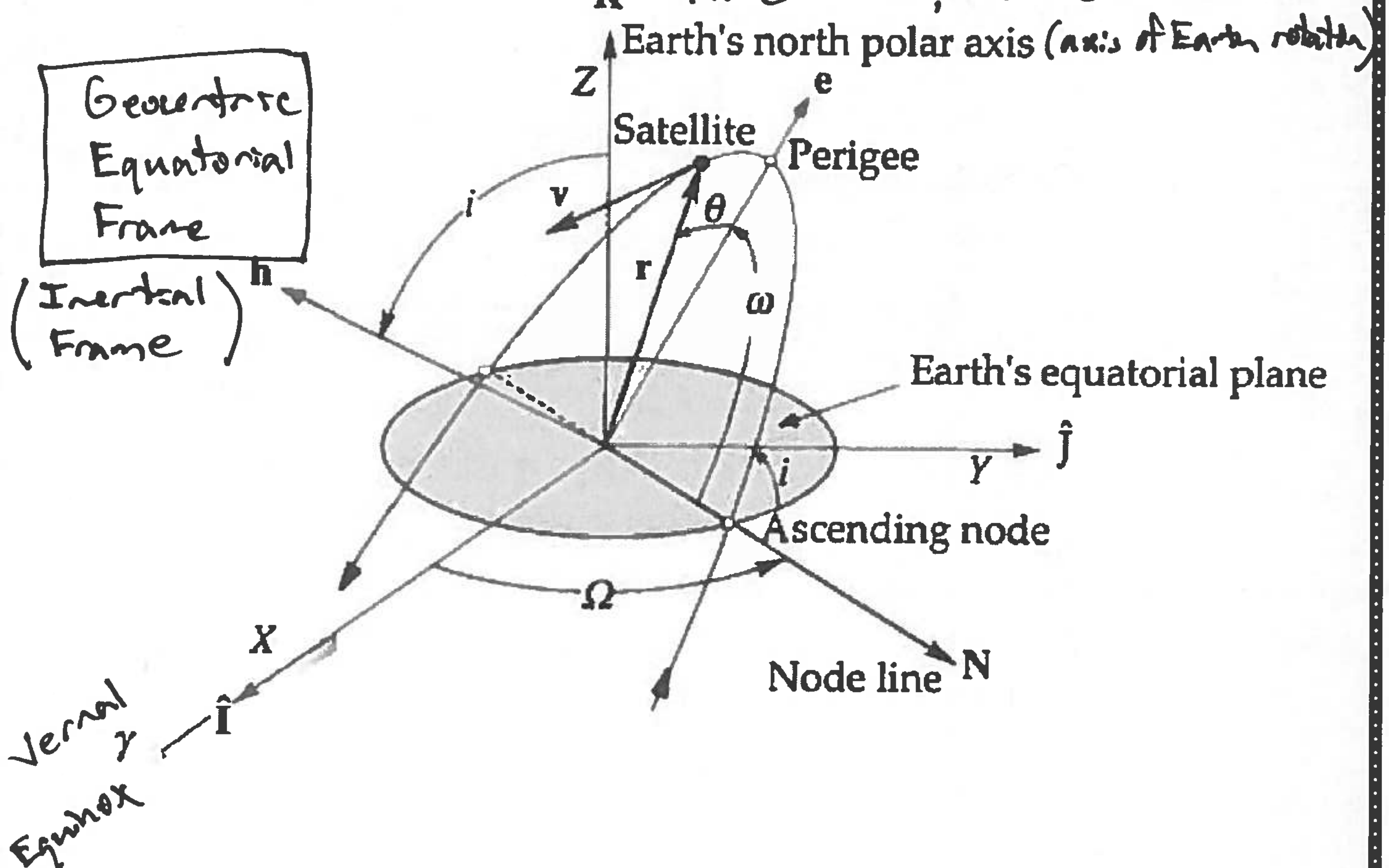
or 6 1st order scalar equations → 6 constants of integration

So need 6 independent parameters to define an orbit :

- Two ways : ① State Vector = velocity $\vec{v}$, position $\vec{r}$ } at an instant of time

  ② "Orbital elements"

- Must measure position and its derivatives in a non-rotating frame of ref, attached to Earth



Keplerian Orbit Elements and Terminology (class demo)

Text 9.1.5: Keplerian Orbit Elements & Terminology

- Orbital Elements = numerical specification of an orbit

- For Keplerian orbits, two approaches:

① If one position (vector) and velocity (vector) known, can integrate equations of orbital motion to get the entire orbit – good for computational approach, not so much for conceptualizing!

② For conceptualization, use Keplerian Elements to calculate motion of a satellite over time:

A. • orbit size and shape (2 parameters)

B. • orientation of orbital plane in space (2 parameters)

C. • rotational orientation of the semi-major axis within the orbital plane (1 parameter)

D. • where the satellite is on the orbit

• (value of $\mu = GM$ for Central Body)

Text 9.1.5: Keplerian Orbit Elements & Terminology - Orbit Size and Shape $\boxed{A}$

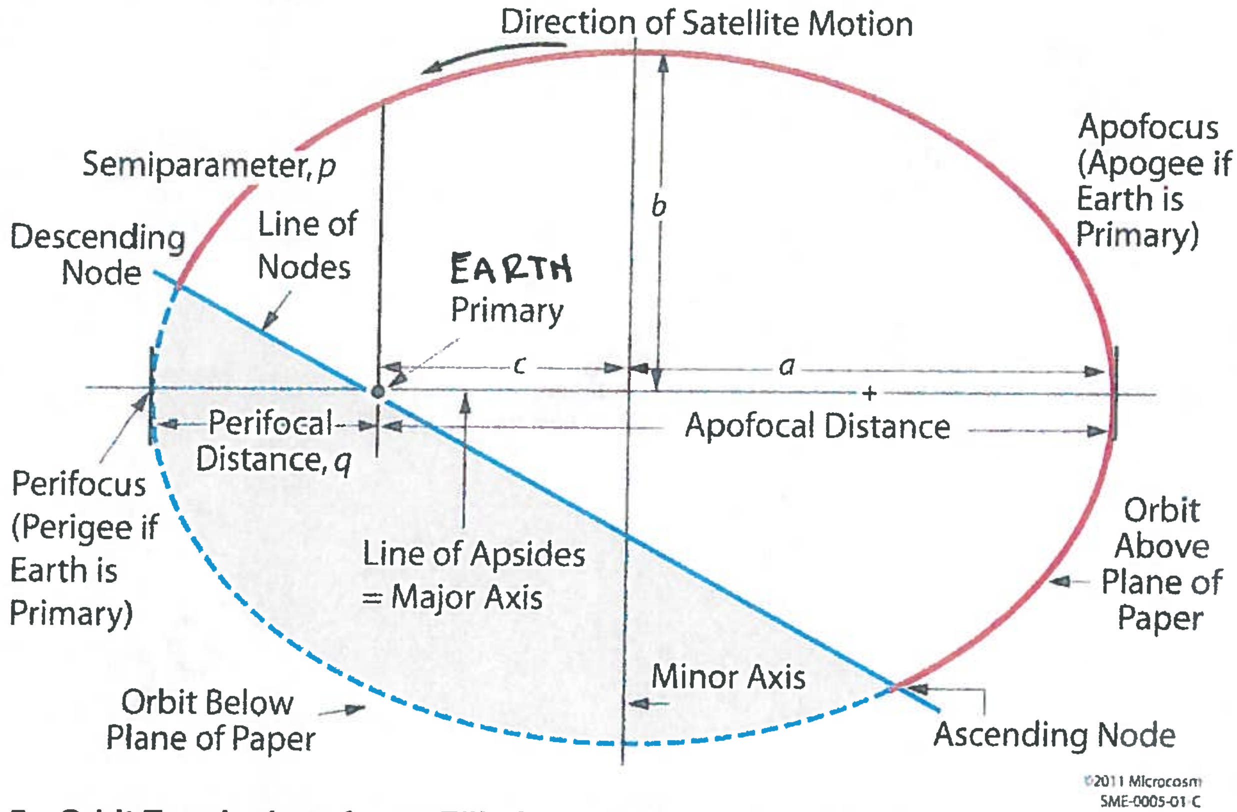


Fig. 9-5. Orbit Terminology for an Elliptical Orbit. The orbit is tilted, or inclined, with respect to the plane of the paper such that the dashed segment is below the paper which is assumed to be the reference plane.

- Size + shape of elliptical Keplerian orbit can be completely defined by either

① a = semi-major axis

e = eccentricity $= \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$

or

② h_p = perigee height (perifocal distance for Earth)

h_A = apogee height (apofocal distance " ")

Text 9.1.5: Keplerian Orbit Elements & Terminology - Orientation of the Orbit Plane

B

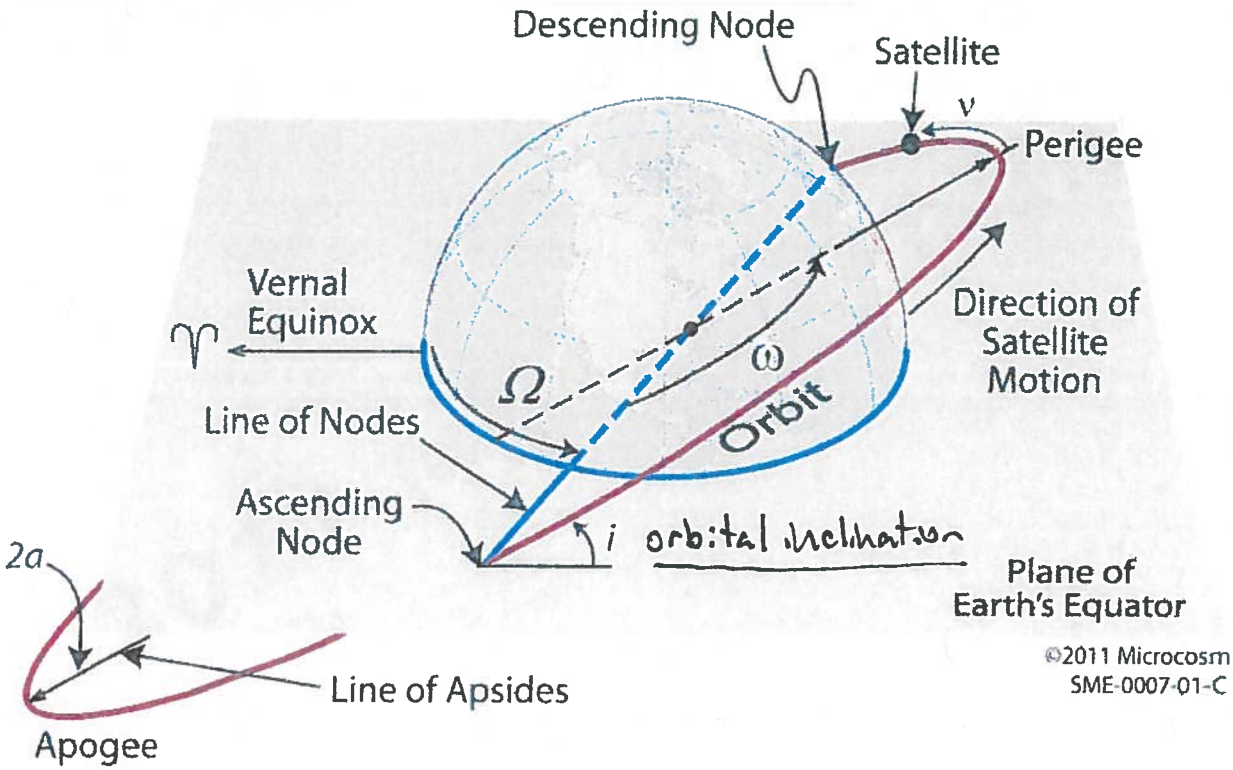


Fig. 9-7. Keplerian Orbit Elements. ♈ marks the direction of the vernal equinox. The line of nodes is the intersection between the equatorial plane and the orbit plane. Ω is measured in the equatorial plane, and ω is measured in the orbit plane.

- Orbital inclination i referred to Equatorial Plane
- Prograde orbit : $i = 0$ to $90°$, satellite travels same direction as Earth rotation

Retrograde orbit : $i = 90$ to $180°$

- Line of Apsides : connects perigee and apogee

Line of Nodes : line between intersection of orbit and equatorial plane

Text 9.1.5: Keplerian Orbit Elements & Terminology

B

- Recall we need to define orientation of the orbital plane in inertial space:

  inclination i

  orientation of Line of Nodes

- Keplerian orbits are approximately fixed in inertial space, so should be defined w/o ref to Earth (at least rotation of orbital plane should be).

  Standard origin for inertial ref frame is Vernal Equinox, location of Sun in sky on first day of Spring (but that's not constant either!)

So rotational orientation of orbital plane is Ω, measured in Equatorial plane, from Vernal Equinox

Text 9.1.5: Keplerian Orbit Elements & Terminology - Orientation of the Orbit w/in the Plane

C

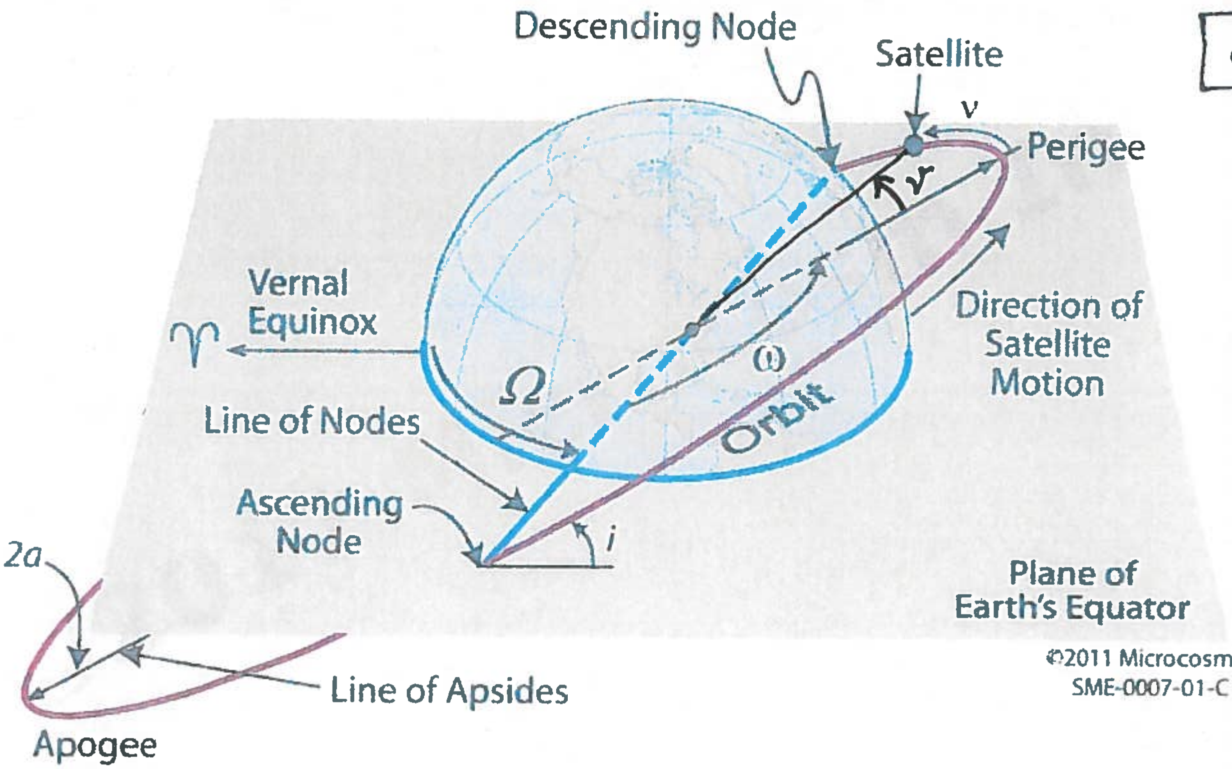


Fig. 9-7. Keplerian Orbit Elements. ♈ marks the direction of the vernal equinox. The line of nodes is the intersection between the equatorial plane and the orbit plane. Ω is measured in the equatorial plane, and ω is measured in the orbit plane.

$\boxed{C}$ • Define rotational orientation of major axis (Line of Apsides):

$$\omega = \text{"argument of perigee"}$$

$\boxed{D}$ • Position of satellite w/in orbit:

$$\nu = \text{"true anomaly"} \quad (\text{or } \theta \text{ in Fortescue})$$
$$= \text{angle between perigee and satellite}$$

Text 9.1.5: Keplerian Orbit Elements Summary

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Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis ($a < 0$) b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 - b^2}/a \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a(1 - e)$	$q = p/2$	$q = a(1 - e)$
Velocity, V, at Distance, r, from Focus	$V^2 = \mu/r$	$V^2 = \mu(2/r - 1/a)$	$V^2 = 2\mu/r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, ε	$\varepsilon = -\mu/2a < 0$	$\varepsilon = -\mu/2a < 0$	$\varepsilon = 0$	$\varepsilon = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu/a^3}$	$n = 2\sqrt{\mu/p^3}$	$n = \sqrt{\mu/(-a)^3}$
Period, P	$P = 2\pi/n$	$P = 2\pi/n$	$P = \infty$	$P = \infty$
Anomaly	$\nu = M = E$	Eccentric anomaly, E $\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{\nu}{2} = D/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e\sin E$	$M = qD + (D^3/6)$	$M = (e\sinh F) - F$
Distance from Focus, $r = q(1+e)/(1+e\cos\nu)$	$r = a$	$r = a(1 - e\cos E)$	$r = q + (D^2/2)$	$r = a(1 - e\cosh F)$
$r\,dr/dt = r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\nu}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu(1-e^2)}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu(1-e^2)}$ © 2011 Microcosm, Inc.

Note: μ = GM is the gravitational constant of the central body; ν is the true anomaly, and $M = n(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion. See App. C for additional formulas and a discussion and listing of terminology and notation.

Keplerian Elements define an orbit:

A Size + shape: semimajor axis a, eccentricity e 2

B Orientation of orbital plane: inclination i 2

right ascension of ascending node Ω

C Rotation of orbit w/in plane: argument of perigee ω 1

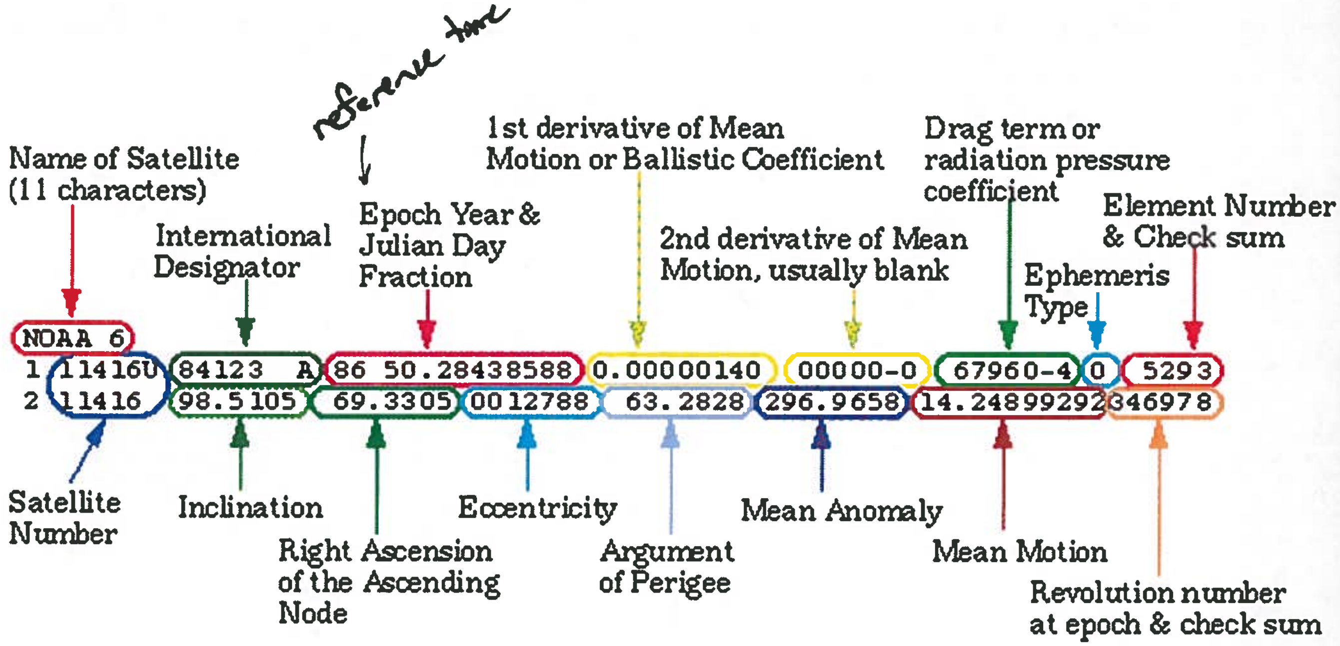
D Position of satellite in its orbit: true anomaly ν 1

mean

eccentric

6

NORAD Two-Line Elements: TLE's



Detailed Definitions:

- http://spaceflight.nasa.gov/realdata/sightings/SSapplications/Post/JavaSSOP/SSOP_Help/tle_def.html
- http://www.celestrak.com/NORAD/documentation/tle-fmt.asp
- https://celestrak.com/columns/v04n03/

Current TLE's:

http://www.celestrak.com/NORAD/elements/

Coordinate Systems:

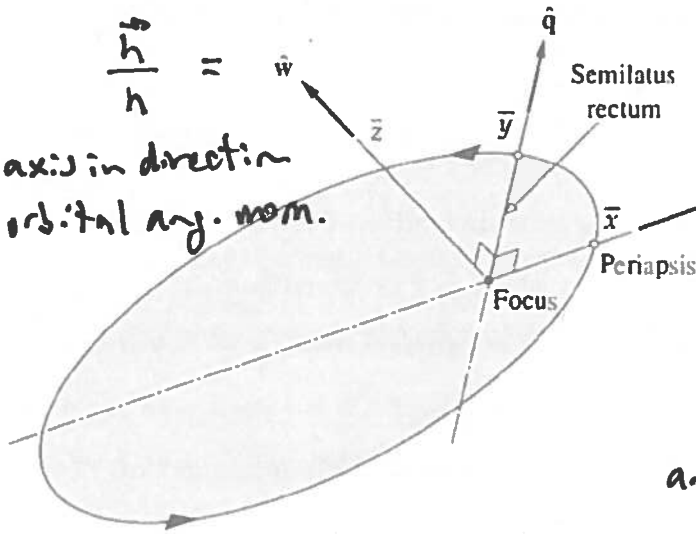
2D: Perifocal Coordinate System

Cartesian
fixed in space
centered at focus of ellipse

$\frac{\vec{h}}{h} = \hat{w}$

$\bar{z}$ axis in direction of orbital ang. mom.

$\hat{p}$ unit vector



Position vector

$$\vec{r} = \bar{x}\vec{p} + \bar{y}\vec{q} + 0\vec{w}$$ (2D

and $\bar{x} = r\cos\theta$, $\bar{y} = r\sin\theta$

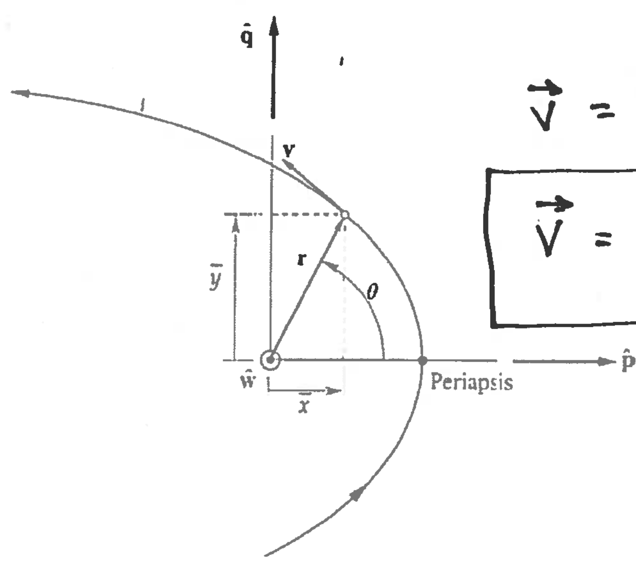
• Kinematics: start with Orbit Equation

$$r = \left(\frac{h^2}{\mu}\right)\frac{1}{1+e\cos\theta}$$

$$\vec{r} = \left(\frac{h^2}{\mu}\right)\frac{1}{1+e\cos\theta}\left(\overset{\bar{x}}{\cos\theta}\,\vec{p} + \overset{\bar{y}}{\sin\theta}\,\vec{q}\right)$$

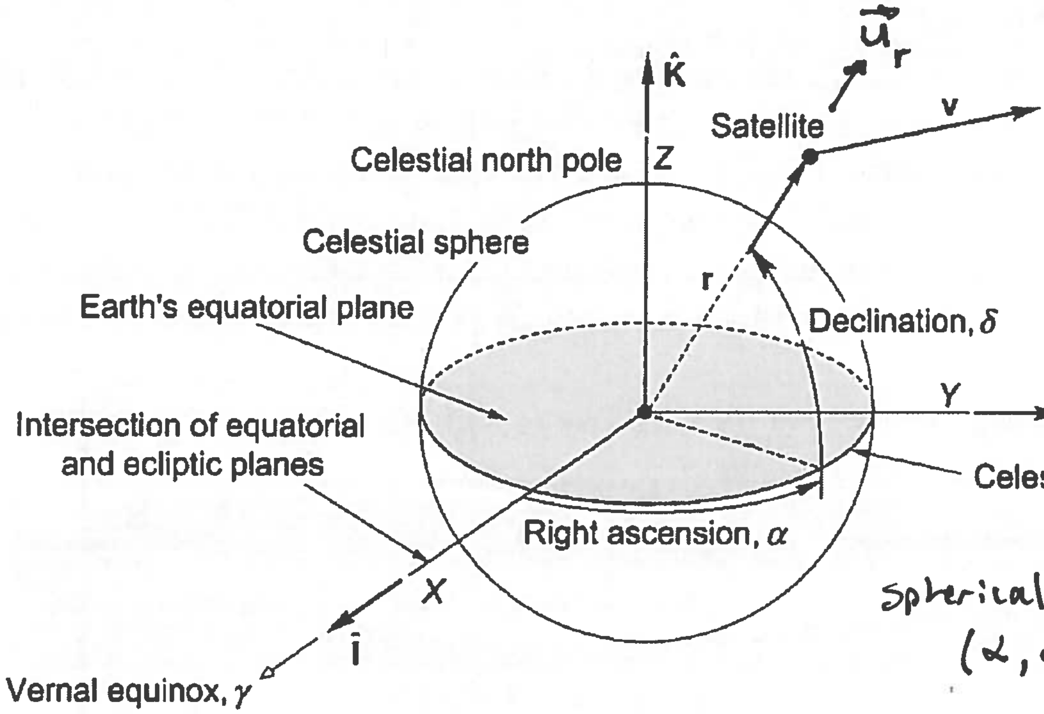
$$\vec{V} = \dot{\vec{r}} = \dot{\bar{x}}\vec{p} + \dot{\bar{y}}\vec{q}$$

$$\boxed{\vec{V} = \frac{\mu}{h}\left[-\sin\theta\,\vec{p} + (e+\cos\theta)\,\vec{q}\right]}$$



2D description of
orbital velocity
in perifocal plane

3D: Geocentric Equatorial Inertial Coordinate System



spherical coord system (α, δ)

- In this frame, State Vector (SV) is

$$\vec{r} = X\vec{I} + Y\vec{J} + Z\vec{K}$$

$$\vec{V} = V_x\vec{I} + V_y\vec{J} + V_z\vec{K}$$

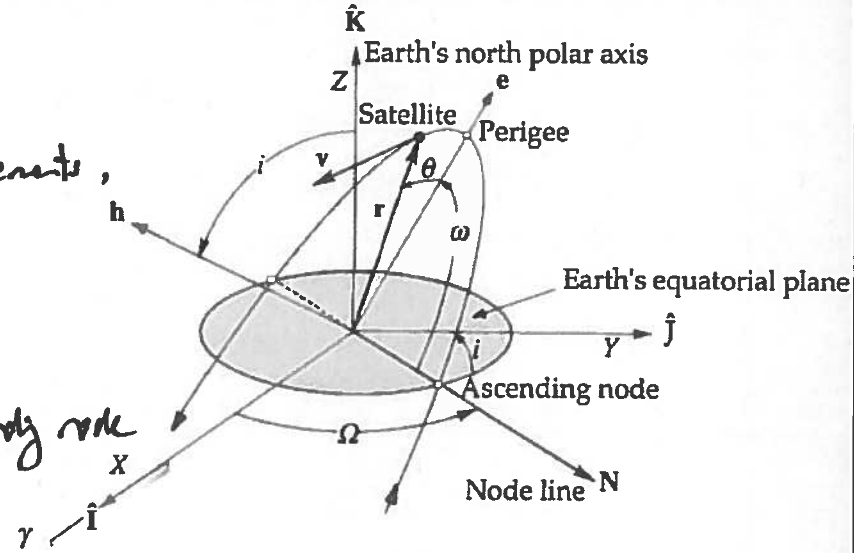
- If r = magnitude of position vector, then

$$\vec{r} = r\vec{u}_r \quad \text{where } \vec{u}_r = \text{unit vector in } \vec{r} \text{ direction}$$

$$= \underbrace{\cos\delta\cos\alpha}\,\vec{I} + \underbrace{\cos\delta\sin\alpha}\,\vec{J} + \underbrace{\sin\delta}\,\vec{K}$$

"direction cosines" of $\vec{u}_r$

Compute Orbital Elements from State Vector



- Given $\vec{r}$ and $\vec{v}$,

  want the 6 orbital elements,

  - h = specific ang. mom.
  - i = inclination
  - Ω = right ascension of ascending node
  - e = eccentricity
  - ω = argument of perigee
  - θ = true anomaly

- note that it is common to substitute:

  semimajor axis (a) for angular momentum (h)

  mean anomaly (M) for true anomaly (θ)

- Need a step-by-step algorithm to convert from State Vector to the equivalent Orbital Elements.

  Use Curtis Algorithm 4.2, pp 197-199

ALGORITHM 4.2

Obtain orbital elements from the state vector. A MATLAB version of this procedure appears in Appendix D.18. Applying this algorithm to orbits around other planets or the sun amounts to defining the frame of reference and substituting the appropriate gravitational parameter μ.

1. Calculate the distance:

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{X^2 + Y^2 + Z^2}$$

2. Calculate the speed:

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_X^2 + v_Y^2 + v_Z^2}$$

3. Calculate the radial velocity:

$$v_r = \mathbf{r}\cdot\mathbf{v}/r = (Xv_X + Yv_Y + Zv_Z)/r.$$

Note that if $v_r > 0$, the satellite is flying away from perigee. If $v_r < 0$, it is flying toward perigee.

4. Calculate the specific angular momentum:

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}$$

5. Calculate the magnitude of the specific angular momentum:

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}}$$

the first orbital element.

6. Calculate the inclination:

$$i = \cos^{-1}\left(\frac{h_Z}{h}\right) \tag{4.7}$$

This is the second orbital element. Recall that i must lie between 0° and 180°, which is precisely the range (principle values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If $90° < i \leq 180°$, the angular momentum **h** points in a southerly direction. In that case, the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to earth's rotation.

7. Calculate:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix} \tag{4.8}$$

This vector defines the node line.

8. Calculate the magnitude of **N**:

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}}$$

9. Calculate the right ascension of the ascending node:

$$\Omega = \cos^{-1}(N_X/N)$$

the third orbital element. If $(N_x/N) > 0$, then Ω lies in either the first or fourth quadrant. If $(N_x/N) < 0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane $(0 \leq \Omega < 180°)$ if $N_Y > 0$. On the other hand, the ascending node lies on the negative side of the XZ plane $(180° \leq \Omega < 360°)$ if $N_Y < 0$. Therefore, $N_Y > 0$ implies that $0 < \Omega < 180°$, whereas $N_Y < 0$ implies that $180° < \Omega < 360°$. In summary,

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases} \tag{4.9}$$

10. Calculate the eccentricity vector. Starting with Eqn (2.40):

$$\mathbf{e} = \frac{1}{\mu}\left[\mathbf{v}\times\mathbf{h} - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\mathbf{v}\times(\mathbf{r}\times\mathbf{v}) - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2 - \mathbf{v}(\mathbf{r}\cdot\mathbf{v})}^{\text{bac}\ \ \text{cab rule}} - \mu\frac{\mathbf{r}}{r}\right]$$

so that

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - r v_r \mathbf{v}\right] \tag{4.10}$$

11. Calculate the eccentricity:

$$e = \sqrt{\mathbf{e} \cdot \mathbf{e}}$$

the fourth orbital element. Substituting Eqn (4.10) leads to a form depending only on the scalar obtained thus far,

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)} \tag{4.11}$$

12. Calculate the argument of perigee:

$$\omega = \cos^{-1}\left(\frac{\mathbf{N}}{N} \cdot \frac{\mathbf{e}}{e}\right)$$

the fifth orbital element. If $\mathbf{N} \cdot \mathbf{e} > 0$, then ω lies in either the first or fourth quadrant. If $\mathbf{N} \cdot \mathbf{e} < 0$ then ω lies in either the second or third quadrant. To place ω in the proper quadrant, observe tha perigee lies above the equatorial plane ($0 \leq \omega < 180°$) if $\mathbf{e}$ points up (in the positive Z direction and that perigee lies below the plane ($180° \leq \omega < 360°$) if $\mathbf{e}$ points down. Therefore, $e_Z \geq 0$ implies that $0 < \omega < 180°$, whereas $e_Z < 0$ implies that $180° < \omega < 360°$. To summarize,

$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N} \cdot \mathbf{e}}{Ne}\right) & (e_Z \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{N} \cdot \mathbf{e}}{Ne}\right) & (e_Z < 0) \end{cases} \tag{4.12}$$

13. Calculate the true anomaly:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e} \cdot \frac{\mathbf{r}}{r}\right)$$

the sixth and final orbital element. If $\mathbf{e} \cdot \mathbf{r} > 0$, then θ lies in the first or fourth quadrant. If $\mathbf{e} \cdot \mathbf{r} < 0$ then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if the satellite is flying away from perigee ($\mathbf{r} \cdot \mathbf{v} \geq 0$), then $0 \leq \theta < 180°$, whereas if the satellite is flying toward perigee ($\mathbf{r} \cdot \mathbf{v} < 0$), then $180° \leq \theta < 360°$. Therefore, using the results of Step 3 above

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e} \cdot \dfrac{\mathbf{r}}{r}\right) & (v_r \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{e}}{e} \cdot \dfrac{\mathbf{r}}{r}\right) & (v_r < 0) \end{cases} \tag{4.13a}$$

Substituting Eqn (4.10) yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \geq 0) \\ 360° - \cos^{-1}\left[\frac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases} \tag{4.13b}$$

Transform from Perifocal to Geocentric Equatorial

- General coordinate transformations from one Cartesian system to another:

Consider position vector $\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{K}$ sys

$= r_x' \vec{i}' + r_y' \vec{j}' + r_z' \vec{K}'$ sys

- If you know vector components in one system, you can express the vector in the other as:

$$[r'] = [Q][r] \text{ and } [r] = [Q]^T [r']$$

where $[r'] = \begin{bmatrix} r_x' \\ r_y' \\ r_z' \end{bmatrix}$, $[r] = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$

$[Q]$ = matrix of direction cosines of $\vec{i}', \vec{j}', \vec{K}'$ relative to $\vec{i}, \vec{j}, \vec{K}$

$$= \begin{bmatrix} (\vec{i}' \cdot \vec{i}) & (\vec{i}' \cdot \vec{j}) & (\vec{i}' \cdot \vec{K}) \\ (\vec{j}' \cdot \vec{i}) & (\vec{j}' \cdot \vec{j}) & (\vec{j}' \cdot \vec{K}) \\ (\vec{K}' \cdot \vec{i}) & (\vec{K}' \cdot \vec{j}) & (\vec{K}' \cdot \vec{K}) \end{bmatrix}$$

Direction Cosine Matrix (DCM)

- To transform a state vector from perifocal → geo. equatorial, you are going from 2D to 3D.

  2D: need 3 params to define orbit: h, e, θ

- Recall perifocal frame:

$$\vec{r} = \bar{x}\,\vec{p} + \bar{y}\,\vec{q} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\left(\cos\theta\,\vec{p} + \sin\theta\,\vec{q}\right)$$

$$\vec{v} = \dot{\bar{x}}\,\vec{p} + \dot{\bar{y}}\,\vec{q} = \frac{\mu}{h}\left[-\sin\theta\,\vec{p} + (e+\cos\theta)\,\vec{q}\right]$$

- Vector form:

$$\{r\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix} \quad 4.4'$$

zeroes because 2D defn

$$[v]_{\bar{x}} = \frac{\mu}{h}\begin{bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{bmatrix} \quad 4.41$$

- Expand orbital description to 3D by adding 3 more params: ω, i, and Ω

- Transform from $\bar{x}$ (2D perifocal) to X (3D geo. eq.) via classical Euler angle sequence:

$$[Q]_{\bar{x}X} = \left[R_3(\Omega)\,R_1(i)\,R_3(\omega)\right] \quad \text{(order matters)}$$

Transformation Matrix

direction cosine matrices

- Thus transformation from perifocal to geometric equatorial components is:

$$\underset{\substack{\uparrow \\ \text{3D geo. eq.}}}{[r]_X} = \underset{\substack{\uparrow \\ \text{direct} \\ \text{cosine} \\ \text{transformation} \\ \text{matrix}}}{[Q]_{\bar{x}X}} \underset{\substack{\uparrow \\ \text{2D perifocal}}}{[r]_{\bar{x}}} \; ; \; [v]_X = [Q]_{\bar{x}X} [v]_{\bar{x}} \qquad 4.51$$

where

$$[r]_X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad [r]_{\bar{x}} = \begin{bmatrix} \bar{x} \\ \bar{y} \\ 0 \end{bmatrix}$$

$$[v]_X = \begin{bmatrix} v_X \\ v_Y \\ v_Z \end{bmatrix}, \quad [v]_{\bar{x}} = \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ 0 \end{bmatrix}$$

$$[Q]_{\bar{x}X} = \begin{bmatrix} (-\sin\Omega \cos i \sin\omega + \cos\Omega \cos\omega) & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$$

Curtis 4.49

p. 217

Compute State Vector from Orbital Elements

- Curtis Algorithm 4.5, pg 218, App. D22 Matlab

- Given orbital elements h, e, i, Ω, ω, and θ, find the equivalent state vector in the geo. eq. frame

① Calc position vector in perifocal coords (4.45)

② Calc velocity vector in perifocal coords (4.46)

③ Calc transformation matrix $[Q]_{\bar{x}X}$ (4.49)

④ Calc state vector in geo. eq. coord frame

$$[r]_X = [Q]_{\bar{x}X} [r]_{\bar{x}}$$

$$[v]_X = [Q]_{\bar{x}X} [v]_{\bar{x}}$$

Orbit Perturbations (text 4.4)

- Keplerian orbits : 2-body motion, free from perturbations

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

Valid for:
- only 2 objects in space
- their gravity fields are spherically symmetric (can't impose torques on each other)
- only source of interaction between them is their gravitational fields

- "Perturbation" : any effect that causes motion to deviate from a Keplerian trajectory

- Perturbed eqn of orbital motion:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{b}$$

← net perturbation accel

- Example : effect of drag → work done on atmosphere by spacecraft → total energy of satellite reduced

$$\mathcal{E} = -\frac{\mu}{2a}$$

↑ if total energy changes

↖ then shape of orbit also changes

Perturbation Effects:

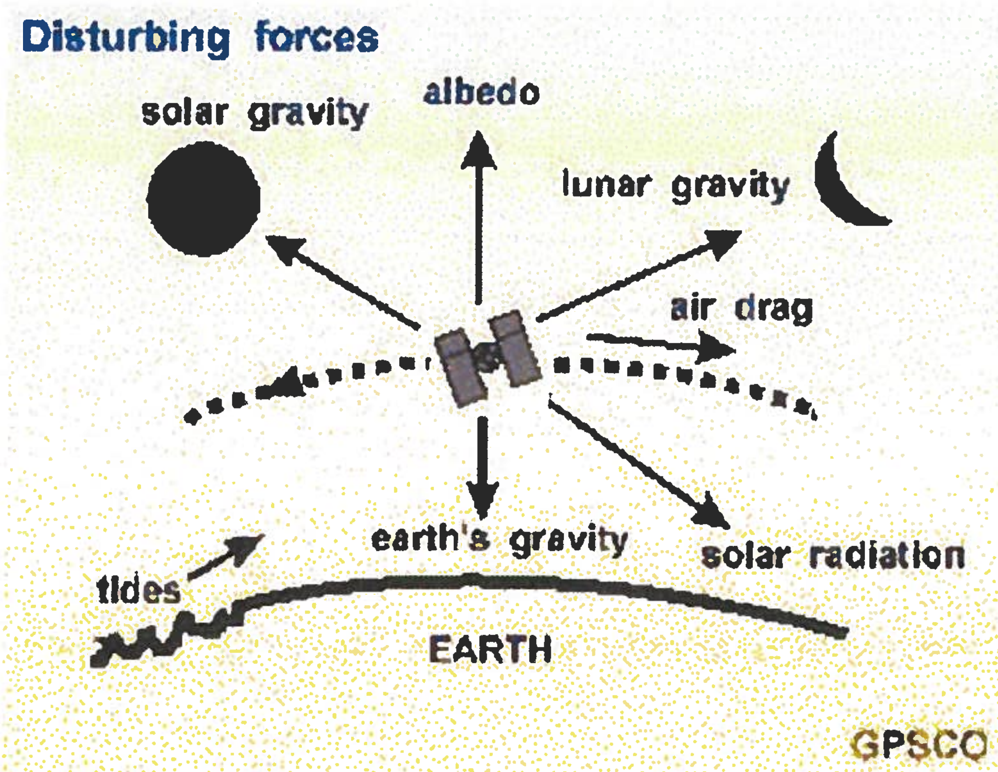


Table 4.2 Magnitude of disturbing accelerations acting on a space vehicle whose area-to-mass ratio is *A/M*. Note that A is the projected area perpendicular to the direction of motion for air drag, and perpendicular to the Sun for radiation pressure

Source	Acceleration (m/s²)	
	500 km	Geostationary orbit
Air drag*	$6 \times 10^{-5} A/M$	$1.8 \times 10^{-13} A/M$
Radiation pressure	$4.7 \times 10^{-6} A/M$	$4.7 \times 10^{-6} A/M$
Sun (mean)	5.6×10^{-7}	3.5×10^{-6}
Moon (mean)	1.2×10^{-6}	7.3×10^{-6}
Jupiter (max.)	8.5×10^{-12}	5.2×10^{-11}

*Dependent on the level of solar activity

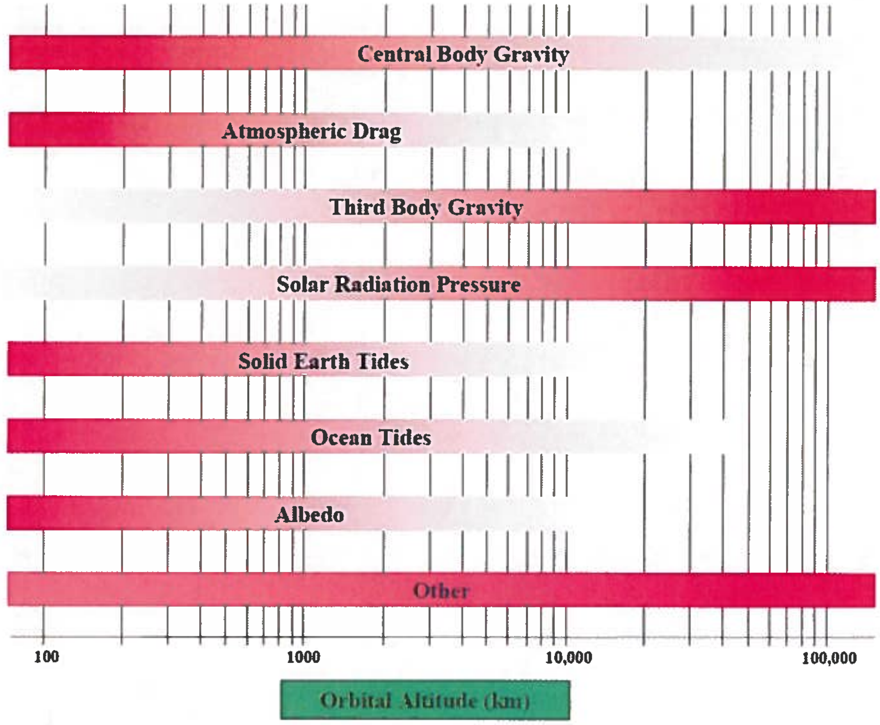


Figure 2 : **Generic Force Model Setup:** This figure shows approximate force model setups for various orbital altitudes. Note that specific accuracy requirements may extend the areas of applicability, and hence the faded color bars.