Perturbed Orbits (non-Keplerian)

Method 1

① initial value problem: start with state vector $\vec{r}_0, \vec{v}_0$

② add perturbation acceleration vector $\vec{b}$ (functional form)

③ numerically integrate

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{b}$$

$\vec{b}$:
- nonspherical attractor
- atmospheric drag
- propulsive thrust
- solar radiation pressure
- Nth body gravity (Sun, moon, etc)

to get $\vec{r}(t), \vec{v}(t)$ at any later time

④ compute orbital elements at any time from state vector

Method 2

① get orbital elements as a function of time by integrating Lagrange Planetary Equations ("Method of Variation of Orbital Elements")

② compute state vector at time t from instantaneous orbital elements

Encke's Method for Variation of Orbital Elements

- Consider motion due to primary body separately from motion due to perturbation accelerations $\vec{b}$.

- Concept of Osculating Orbit

Osc. Orbit ≡ 2-body Keplerian trajectory that would be followed after time t, if at that instant the perturbation accel $\vec{b}$ were to suddenly vanish.

So, $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$ describes every osculating orbit

(but with different $\vec{r}_0, \vec{v}_0$)

Each Osculating Orbit has its' own 6 Orbital Elements, so orbital elements are a function of t.

- Thus solution of eqn. of motion

$$\vec{r}_{osc} = f(t)$$

Enke

- Two-body osculating orbit $\vec{r}_{osc}(t)$ used as a reference orbit upon which unknown perturbation $\delta\vec{r}(t)$ is superposed to get complete perturbed orbit. So,

$$\underbrace{\vec{r}(t) = \vec{r}_{osc}(t)}_{\text{Keplerian}} + \underbrace{\delta\vec{r}(t)}_{\text{net effect of perturbation}}$$

- Let $(\vec{r}_0, \vec{v}_0)$ = known SV at time t_0.

Osculating orbit at t_0: $\ddot{\vec{r}}_{osc} = -\mu \dfrac{\vec{r}_{osc}}{r^3}$; IC's @ t_0: r_0, v_0 12.2

- At subsequent times $t > t_0$, trajectory will diverge from initial, osculating orbit: $\vec{r}_{osc} = \vec{r} - \delta\vec{r}$

$$\vec{r}(t) = \vec{r}_{osc}(t) + \delta\vec{r}(t) \qquad 12.6$$

↑ sub in eqn 12.4

Think of $\vec{r}_{osc}, \vec{v}_{osc}$ as local IC's

or,

$$\delta\ddot{\vec{r}} = \delta\vec{a} = \ddot{\vec{r}} + \mu\frac{\vec{r} - \delta\vec{r}}{r^3_{osc}}$$

↑

sub in $\ddot{\vec{r}} = -\mu\dfrac{\vec{r}}{r^3} + \vec{b}$ 12.2

⋮

- SV at t_0 : $(\vec{r}_0, \vec{v}_0)$

Osculating orbit @ t_0 governed by

$$\ddot{\vec{r}}_{osc} = -\frac{\mu}{r_{osc}^3}\vec{r}_{osc} \qquad \text{with IC's } \vec{r}_0, \vec{v}_0$$

- After t_0, the perturbed trajectory will diverge from the initial osculating path:

$$\vec{r}(t) = \vec{r}_{osc}(t) + \delta\vec{r}(t) \quad \text{or} \quad \underbrace{\vec{R}_{osc} = \vec{r} - \delta\vec{r}}$$

plug into eqn of motion

$$\ddot{\vec{r}} - \delta\ddot{\vec{r}} = -\frac{\mu}{r_{osc}^3}(\vec{r} - \delta\vec{r})$$

or

$$\delta\ddot{\vec{r}} = \ddot{\vec{r}} + \frac{\mu}{r_{osc}^3}(\vec{r} - \delta\vec{r})$$

describes geometric divergence from osculating orbit, but not forcing function of perturbation $\vec{b}$

sub in $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{b}$

to get

$$\delta\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{b} + \frac{\mu}{r_{osc}^3}(\vec{r} - \delta\vec{r})$$

$$= -\frac{\mu}{r_{osc}^3}\left[\delta\vec{r} - \left(1 - \frac{r_{osc}^3}{r^3}\right)\right] + \vec{b}$$

- Change in accel due to perturbing forces is thus:

$$\delta \ddot{\vec{r}} = \delta \vec{a} = -\frac{\mu}{r_{osc}^3}\left[\delta \vec{r} - \left(1 - \frac{r_{osc}^3}{r^3}\right)\vec{r}\right] + \vec{b} \quad 12.7$$

known function of time ☆

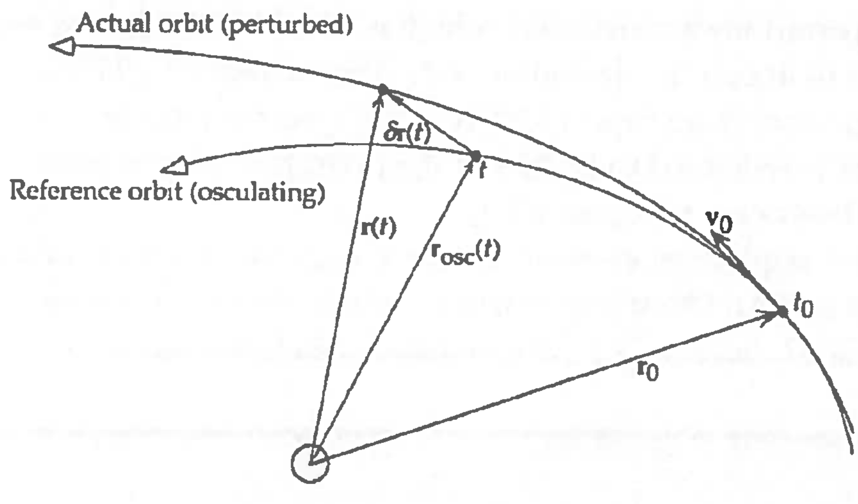
- Fig 12.3

- Algorithm for computing perturbed trajectory at time t:

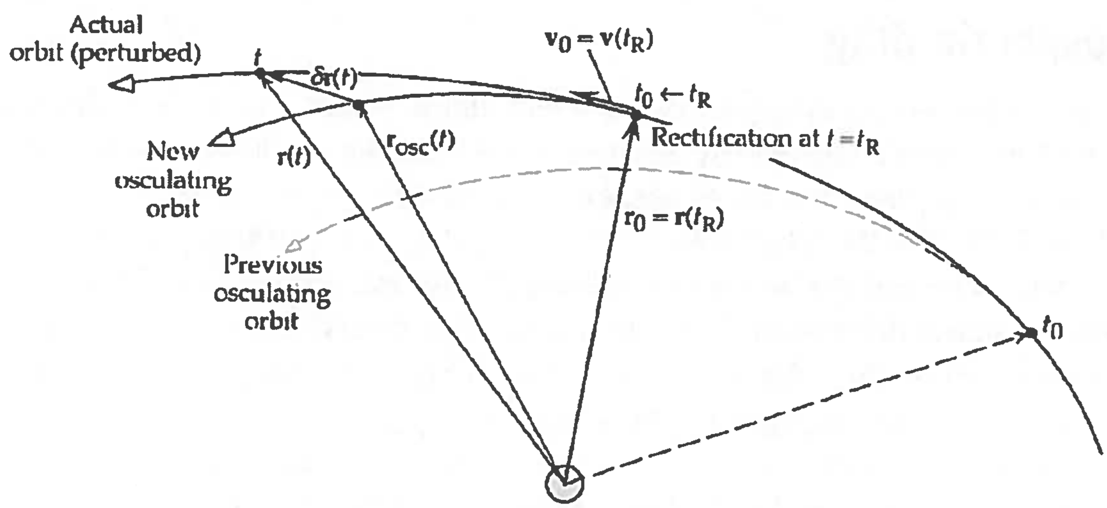
① integrate 12.7 to get $\delta \vec{r}(t)$

② add to get inst. perturbed trajectory: $\vec{r}(t) = \underbrace{\vec{r}_{osc}(t)}_{IC} + \delta \vec{r}(t)$ 12.6

③ next time step: $\underbrace{\text{let } \vec{r}(t), \vec{v}(t) \xrightarrow[\text{become}]{} \vec{r}_0, \vec{v}_0}_{\text{"rectification"}}$


Actual orbit (perturbed)
δr(t)
t
Reference orbit (osculating)
r(t)
r_osc(t)
v_0
t_0
r_0

12.2


Actual orbit (perturbed)
t
δr(t)
v_0 = v(t_R)
t_0 ← t_R
Rectification at t = t_R
New osculating orbit
r(t)
r_osc(t)
r_0 = r(t_R)
Previous osculating orbit
t_0

12.3

Gravitational Perturbations

Fig 12.6

- Earth's gravitational field not quite spherically symmetric

ideal : $U = -\frac{\mu}{r}$ ← GM (Spherically symmetric, so independent of azimuth ϕ)

U: potential energy per unit mass

r: distance from Earth center

(and recall $\ddot{\vec{r}} = -\vec{\nabla} U$ force on a mass in the gravitational potential field)

actual : $U = -\frac{\mu}{r} + \Phi(r, \phi)$ 12.19

Φ: perturbation of gravitational potential due to oblateness (non-spherical)

Note: rotationally symmetric $\neq f(\theta)$

- Rotationally symmetric perturbation of grav. potential:

Represent with infinite series:

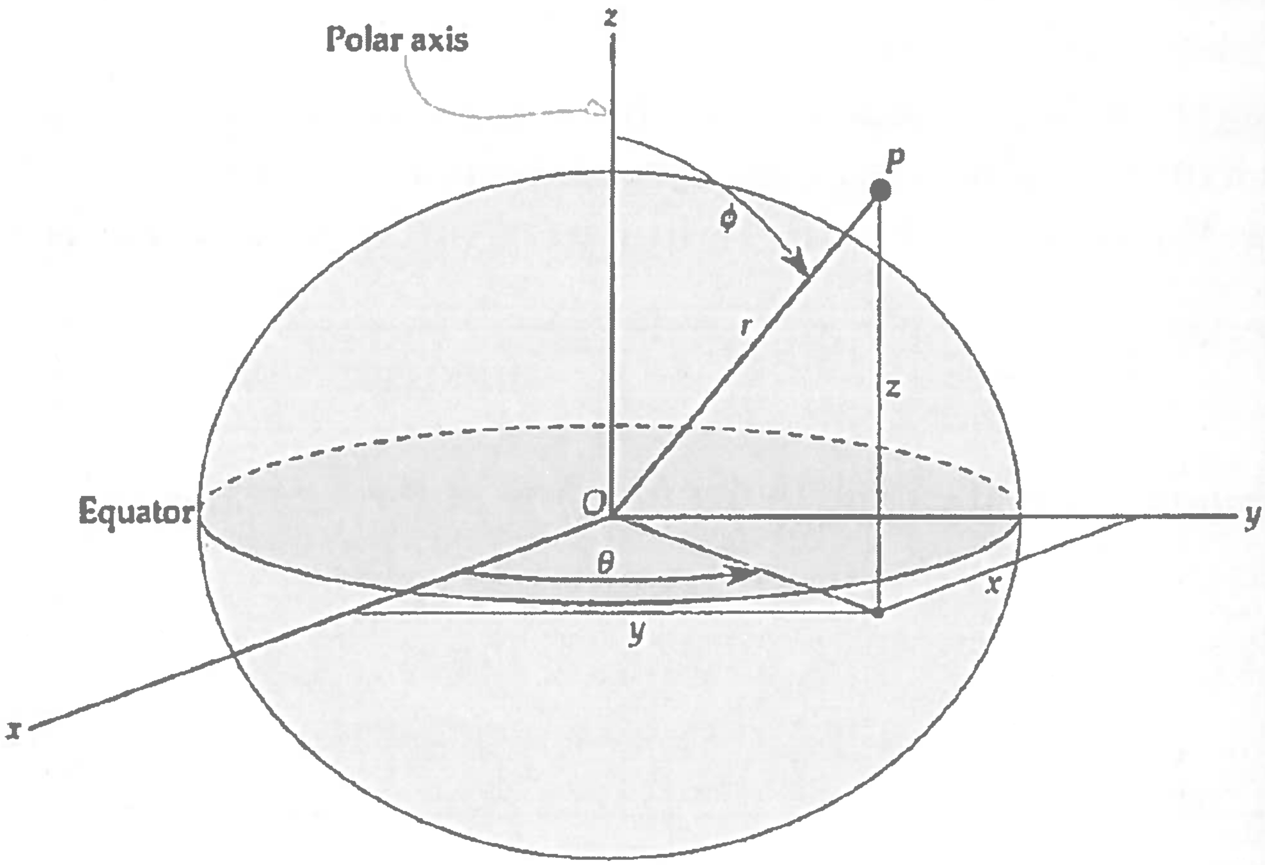
$$\Phi(r,\phi) = \frac{\mu}{r} \underbrace{\sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k(\cos\phi)}_{\text{dimensionless}}$$ (Richard Battin) 12.20

J_k: dimensionless zonal harmonics of the planet
— mass distribution
— experimental results

R = equatorial radius

P_k: Legendre polynomials

12.6



- Earth zonal harmonics J_k :

$$J_1 = 0$$

$$J_2 = 0.00108263 \longrightarrow \text{Dominant term}$$

$$J_3 = -2.33936E-3\,(J_2)$$

$$J_4 = -1.49601E-3\,(J_2)$$

$$J_5 = -0.20995E-3\,(J_2)$$

$$J_6 = 0.49941E-3\,(J_2)$$

$$J_7 = 0.32547E-3\,(J_2)$$

- Legendre Polynomials:

$$P_k(x) = \frac{1}{2^k k!} \frac{d}{dx^k} (x^2 - 1)^k$$ 12.2

Fig. 12.7

- Keep only J_2 term:

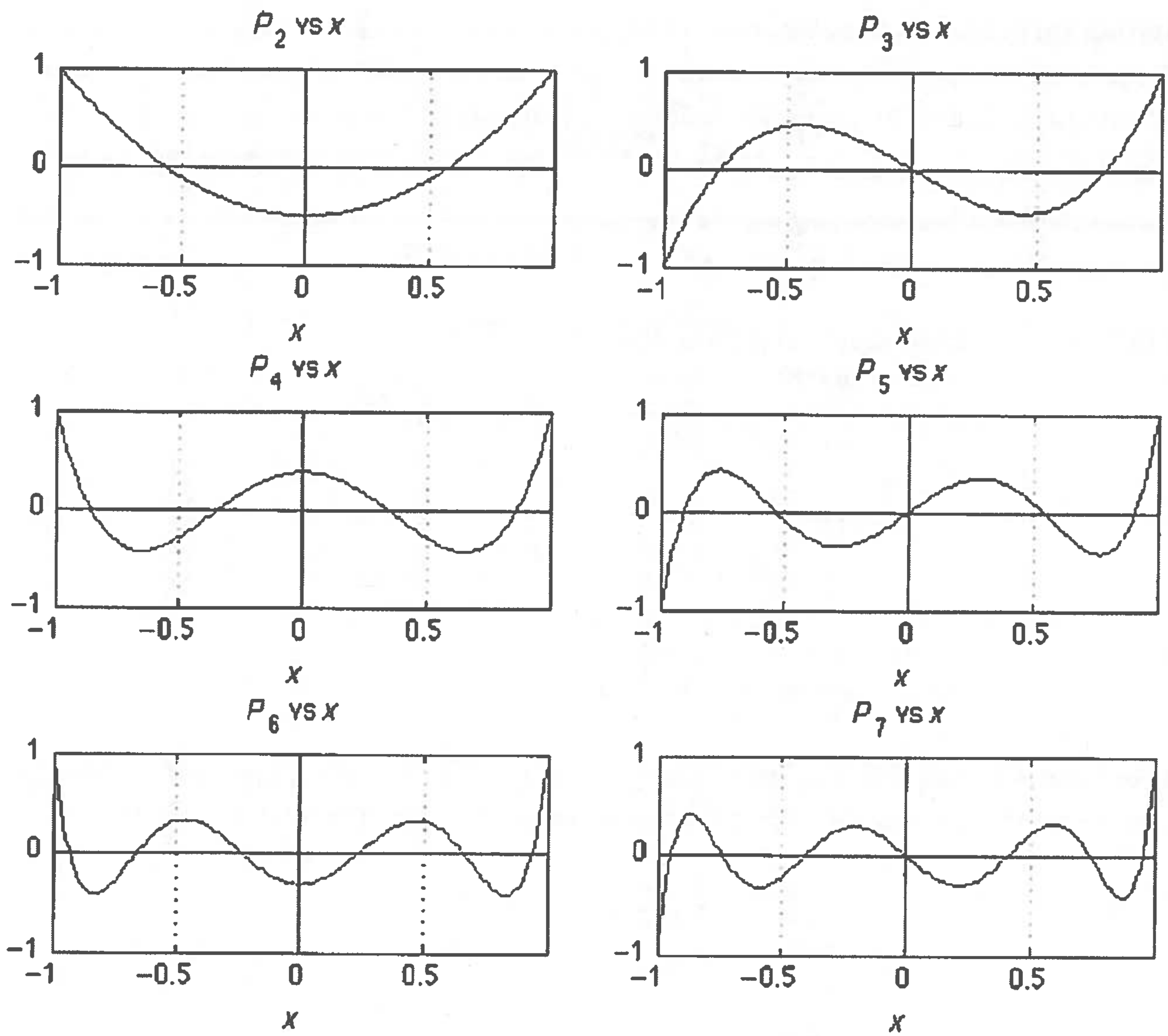
$$\boxed{\Phi(r, \phi) = \frac{J_2}{2} \frac{\mu}{r} \left(\frac{R}{r}\right)^2 (3\cos^2\phi - 1)}$$ 12.24

rotationally symmetric, $\neq f(\theta)$

Legendre Polynomials: solutions to

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}P_n(x)\right] + n(n+1)\,P_n(x) = 0$$

12.7



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- Functional form of perturbing accel due to Earth's grav. field:

- Since $\partial \Phi / d\theta = 0$ (Φ has no dependence on θ); $\Phi(r, \phi)$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial y} \qquad \text{(chain rule)}$$ 12.2

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial \Phi}{\partial \phi}\frac{\partial \phi}{\partial z}$$

- From prev. page,

$$\Phi(r, \phi) = \frac{J_2}{2}\frac{\mu}{r}\left(\frac{R}{r}\right)^2 (3\cos^2\phi - 1) \quad \left(\begin{array}{l}\text{perturbed} \\ \text{grav.} \\ \text{potential}\end{array}\right)$$ 12.2

differentiate by r and ϕ

$$\frac{\partial \Phi}{\partial r} = -\frac{3}{2} J_2 \frac{\mu}{r^2}\left(\frac{R}{r}\right)^2 (3\cos^2\phi - 1)$$ 12.27

$$\frac{\partial \Phi}{\partial \phi} = -\frac{3}{2} J_2 \frac{\mu}{r}\left(\frac{R}{r}\right)^2 \sin\phi\cos\phi$$

• Note that $\phi = \tan^{-1} \dfrac{\sqrt{x^2+y^2}}{z}$ converting from spherical to Cartesian coord system   12.1

• Recall: we are looking for functional expression for the perturbing accel: $\vec{b} = -\vec{\nabla}\,\Phi$

↑ so want (Cartesian) spatial gradients of Φ

• Convert from r, ϕ derivatives to x, y, z derivatives:

$$\frac{\partial \phi}{\partial x} = \frac{xz}{(x^2+y^2+z^2)} \frac{1}{\sqrt{x^2+y^2}} = \frac{xz}{r^3 \sin\phi}$$

$$\frac{\partial \phi}{\partial y} = \frac{yz}{r^3 \sin\phi}$$ 12.28

$$\frac{\partial \phi}{\partial z} = -\frac{\sin\phi}{r}$$

- Gradient of perturbing potential Φ:

Subs. 12.27 + 12.28 into 12.26:

$$\frac{\partial \Phi}{\partial x} = -\frac{3}{2} J_2 \frac{\mu}{r^2}\left(\frac{R}{r}\right)^2 \frac{x}{r}\left[5\left(\frac{z}{r}\right)^2 - 1\right]$$ 12.29

$$\frac{\partial \Phi}{\partial y} = \text{"} \qquad \text{"} \frac{y}{r} \text{"} \qquad \text{"}$$

$$\frac{\partial \Phi}{\partial z} = \text{"} \qquad \text{"} \frac{z}{r}\left[5\left(\frac{z}{r}\right)^2 - 3\right]$$

- Sub these into $\vec{b} = -\vec{\nabla}\Phi$ to get:

$$\vec{b} = \frac{3}{2}\frac{J_2 \mu R^2}{r^4}\left[\frac{x}{r}\left(5\frac{z^2}{r^2} - 1\right)\vec{i} + \frac{y}{r}\left(5\frac{z^2}{r^2} - 1\right)\vec{j} + \frac{z}{r}\left(5\frac{z^2}{r^2} - 3\right)\vec{k}\right]$$ 12.30

and $\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{b}$ at an instantaneous position (x, y, z, r)

fig 12.6

Now can numerically integrate for each time step.

- Similar for zonal harmonics $J_3 \dots J_n$

Atmospheric Drag (important for LEO)

- Continuum assumption for fluid physics = FALSE
  Significant mean free path
  no shock wave formed → no wave drag
  drag due to molecule/surface impact only

- Drag force:
  unit vector

$$\vec{D} = \frac{1}{2} \rho S C_D V_r^2 \left(\frac{-\vec{V}_r}{|V_r|} \right) \quad 4.41$$

where $\vec{V}_r$ = velocity vector of spacecraft rel. to atmosphere

ρ = atmos. density (function of alt, solar activity)

S = surface area exposed to "wind" (attitude dependent)

C_D = coefficient of drag

$\cong$ 2.2 – 2.5 (difficult to determine)

AAS 05-199 Rev 2

An Analysis of State Vector Propagation Using Differing Flight Dynamics Programs*

David A Vallado†

Since the demonstration of the first numerically generated space catalog by the United States Navy in 1997, the issue of how to transition from the two-line element sets (TLEs), to routine use of numerical vectors in satellite flight dynamics operations is generating some unique challenges. Specifically, how will organizations efficiently interact with and use orbital data from programs outside their control? The historical TLE operations used analytically generated datasets for a majority of their calculations which required strict adherence to a specific mathematical technique Use of numerical techniques presents different challenges even though the underlying mathematical technique is the same. This paper provides results of an experiment in which various initial state vectors, representing a cross-section of the existing satellite population, were propagated from several days to a month. The ephemerides, created by several legacy flight dynamics programs, are compared to ephemerides from Analytical Graphics Inc.'s Satellite Tool Kit (STK). There is no assertion of right or wrong answers within the comparisons; rather, the relative differences are shown to gauge the effectiveness of the setup for each case. Most of the comparisons show that mm to cm-level comparisons are possible with careful attention to parameters Differences are discussed including potential error sources. One goal is to present a format that simplifies transmission and use of state vector information between programs, seeking a standard for better integration of interoperability This will avoid significant expenses in using entirely new, or unavailable software. Tables are presented to demonstrate the effect of various force models and their contribution to the satellite orbit. Finally, sample ephemeris information, potential new formats to exchange data, and STK scenario setups are included to initiate a community forum on numerical ephemeris propagations.

INTRODUCTION

The use of numerically generated state vectors for satellite operations is not new. However, with the first numerically generated space catalog by the Navy in 1997 (Coffey and Neal, 1998), the potential to replace the existing TLEs with numerical results now poses some unique challenges for the astrodynamics community. To effectively make this transition, several things must occur. Vallado (1999) proposed a fundamental question for all space surveillance functions.

What observations and processing are needed to achieve a certain level of accuracy on a particular satellite, now, and at a future time?

The answer involves tracking and surveillance functions, orbit determination, propagation, and standards. Also implied are the formats to effectively transmit the information to various organizations that will make operational decisions. Vallado and Carter (1997) showed that significantly more observational data is required than is currently being taken on some satellites, and Vallado and Alfano (1999) outlined many of the issues with obtaining and distributing data from a tracking and surveillance system. This paper answers some of the issues surrounding the propagation, interoperability, standards, and transfer of information. All these functions will be needed to transition from TLE data to numerical processing.

For several decades, many organizations have relied on TLEs to perform various flight dynamics operations. This implied the use of certain mathematical theories, and resulted in limited accuracy in analyses‡ Numerical state vectors are clearly the current choice for many of these operations, but they are only now beginning to gain mainstream acceptance in some routine space surveillance operations. To accurately propagate numerical satellite state vectors between programs, four primary types of information are required:

- the initial state vector and detailed satellite parameters
- a standard mathematical approach from which various applications can be implemented
- specific details of any tailoring or assumptions made to the processing

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* This paper was originally presented at the AAS/AIAA Space Flight Mechanics conference at Copper Mountain, CO in January 2005. This version has been expanded to include all the figures of the orbits examined. Additional information has been added to the background and discussion sections.

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‡ Cefola and Fonte (1996) showed that even the AFSPC analytical theories could achieve order-of-magnitude accuracy increases by adopting features of semi-analytical satellite theories.

ATMOSPHERIC DRAG SENSITIVITY ANALYSIS

Atmospheric drag is probably the most elusive of the force models examined. There are several reasons for this. Before discussing the potential sources for the differences, it's useful to review the basic acceleration equation.

$$\bar{a}_{drag} = -\frac{1}{2}\rho\frac{c_D A}{m}v_{rel}^2\frac{\bar{v}_{rel}}{|\bar{v}_{rel}|}$$

ρ The density usually depends on the atmospheric model, EUV, $F_{10.7}$, k_p, a_p, prediction capability, atmospheric composition, etc. There is wide variability here, and many parameters that can cause significant changes. The popular parameters to examine today are the density and the exospheric temperatures. This single parameter represents the largest contribution to error in any orbit determination application.

c_D The coefficient of drag is related to the shape, but ultimately a difficult parameter to define. Gaposchkin (1994) discusses that the c_D is affected by a complex interaction of reflection, molecular content, attitude, etc. It will vary, but typically not very much as the satellite materials usually remain constant.

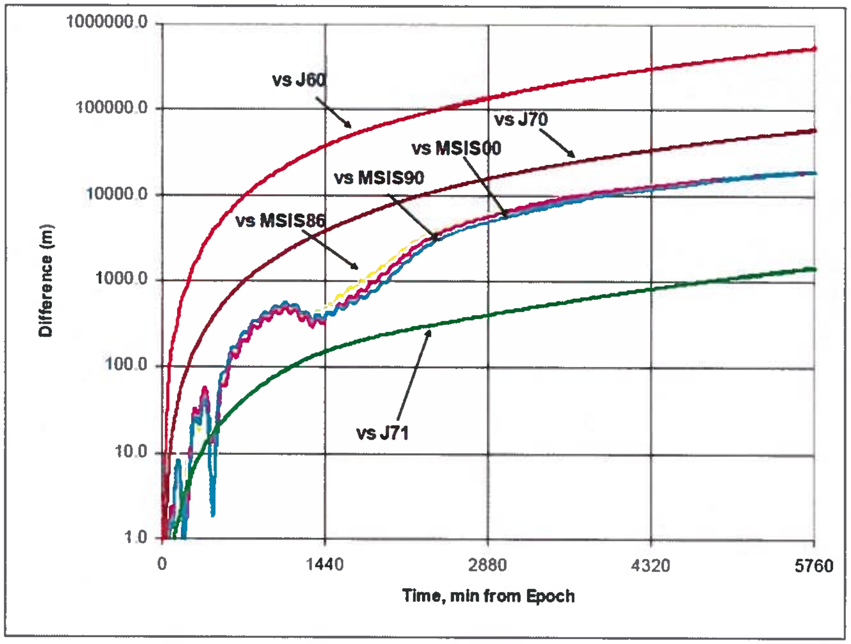
A The cross sectional area changes constantly (unless there is precise attitude control, or the satellite is a sphere). This variable can change by a factor of 10 or more depending on the specific satellite configuration. Macro models are often used for modeling solar pressure accelerations, but seldom if ever, for atmospheric drag.

m The mass is generally constant, but thrusting, ablation, etc., can change this quantity.

BC The ballistic coefficient (m/c_DA – a variation is the inverse of this in some systems) is generally used to lump the previous values together. It *will* vary, sometimes by a large factor. Several initiatives are examining the time-rate of change for this parameter, but not looking at the variable area, and its effect in this combined factor. It's probably best not to model this parameter because it includes several other time-varying parameters that are perhaps better modeled separately.

$\bar{v}_{rel}$ The velocity relative to the rotating atmosphere depends on the accuracy of the *a-priori* estimate, and the results of any differential correction processes. Because it's generally large, and squared, it becomes a *very* important factor in the calculation of the acceleration.

The primary inputs in any program are the atmospheric density (handled via a specified model), and the *BC*. The mass and cross-sectional area are usually well known, and an estimate of the drag coefficient permits



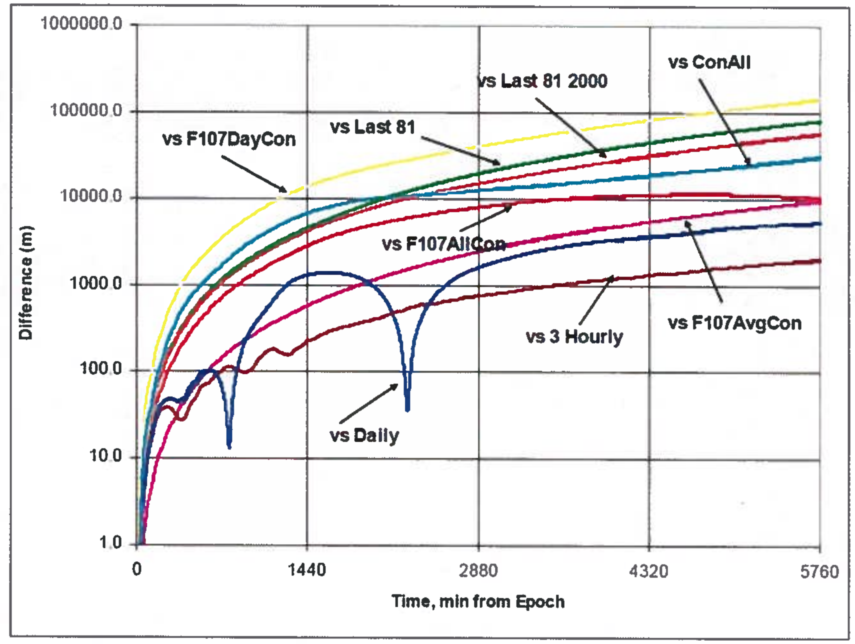


Figure 11: Sample Atmospheric Drag Sensitivity: Positional differences are shown for satellite 21867. Jacchia-Roberts is the baseline for all runs with 3-hourly interpolation. The left-hand graph shows the variations by simply selecting different atmospheric models. The right-hand graph shows the effect of various options for treating solar weather data. Specific options are discussed in the text. Note that the scales are the same, the relative effect of different models and solar data options are about the same, and any transient effects quickly disappear as the effect of drag overwhelms the contributions.

Solar Radiation Pressure

- Perturbing force on satellite due to SRP: (in sunlight phase)

$$\vec{F} = -\frac{S}{c} C_R A_s \vec{u}$$ C 12.96

where $\vec{u}$ = vector pointing from satellite to sun (or E to S)

A_s = absorbing area of spacecraft

C_R = radiation pressure coefficient (1 to 2)

= 1 black body (fully absorbtive)

2 (fully reflective)

} spacecraft

$S = S_0 \left(\frac{R_0}{R}\right)^2$ = radiation intensity at Earth

S_0 = intensity of radiated power from Sun surface

$= \sigma T^4$

σ = 5.670 E-8 W/m²K⁴

T = 5777 K temp of solar photosphere

R_0 = radius of photosphere — visible surface of sun (black-body radiation)

R = distance from sun center

c = speed of light

Variation of Orbital Elements

- Lagrange planetary equations in terms of true anomaly θ:

$$\frac{da}{d\theta} = \frac{2pr^2}{\mu(1-e^2)^2}\left[e\sin\theta\, S + \frac{p}{r}T\right]$$

p: semi-latus rectum
e = eccentricity
r = radius
θ = true anomaly

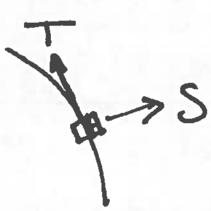
$$\frac{de}{d\theta} = \frac{r^2}{\mu}\left[\sin\theta\, S + \left(1+\frac{r}{p}\right)\cos\theta\, T + e\frac{r}{p}T\right]$$

$$\frac{di}{d\theta} = \frac{r^3}{\mu p}\left[\cos(\theta+\omega)\, W\right]$$

$$\frac{d\Omega}{d\theta} = \frac{r^3\sin(\theta+\omega)}{\mu p\sin i}W$$

$$\frac{d\omega}{dt} = \frac{r^2}{\mu e}\left[-\cos\theta\, S + \left(1+\frac{r}{p}\right)\sin\theta\, T\right] - \cos i\left(\frac{d\Omega}{d\theta}\right)$$

$$\frac{dt}{d\theta} = \frac{r^2}{\sqrt{\mu p}}\left[1 + \frac{r^2}{\mu e}\left[\cos\theta\, S - \left(1+\frac{r}{p}\right)\sin\theta\, T\right]\right]$$



S, T, W = perturbing accels on spacecraft (many sources)