Dynamics of Spacecraft

MAE 243a – Spacecraft Engineering

Quick review of angular momentum

Why use the center of gravitational acceleration?

Why use (and what exactly is) the moment of momentum?

Can we accurately approximate the angular momentum (H_0) with the moment of momentum?

What effects could be missed?

Angular momentum of rigid bodies, H_C

$$H_C = [I_c]\omega$$

$$[I_C] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}; \ \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$I_{zz} = \int (y^2 + z^2) dm \, (moment\ of\ inertia)$$

$$I_{yz} = \int yz\, dm = I_{zy} (product\ of\ inertia)$$

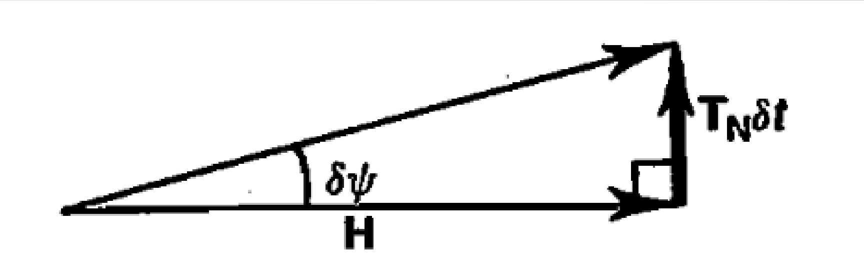
$$H_C = \begin{bmatrix} (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z) \\ (I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z) \\ (I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y) \end{bmatrix}$$

Rate of change if angular momentum, H

$$d(H)/dt = T$$

A torque in the same direction as the angular momentum H will only change the magnitude of the momentum H

A torque orthogonal to the angular momentum will rotate the momentum vector H



Rotational kinetic energy

$$E = \frac{1}{2}(H_C \cdot \omega) \ or \ \frac{1}{2}([I_C]\omega \cdot \omega)$$

Work done by a torque is at a rate of $T \cdot \omega$

Although dissipative mechanisms may lead to a loss of kinetic energy, the total angular momentum is conserved.

This can lead to a tumble about the highest inertia axis.

$$E = \frac{1}{2}\begin{bmatrix}(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\\(I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z)\\(I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)\end{bmatrix} \cdot \begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix} = \frac{1}{2}\begin{bmatrix}(I_{xx}\omega_x^2 - I_{xy}\omega_y\omega_x - I_{xz}\omega_z\omega_x)\\(I_{yy}\omega_y^2 - I_{yx}\omega_x\omega_y - I_{yz}\omega_z\omega_y)\\(I_{zz}\omega_z^2 - I_{zx}\omega_x\omega_z - I_{zy}\omega_y\omega_z)\end{bmatrix}$$

For example lets take the case of angular momentum and kinetic energy about principle axis for simplicity

$$H_C = \begin{bmatrix}I_{xx}\omega_x\\I_{yy}\omega_y\\I_{zz}\omega_z\end{bmatrix}; \mathrm{E} = \frac{1}{2}\begin{bmatrix}I_{xx}\omega_x^2\\I_{yy}\omega_y^2\\I_{zz}\omega_z^2\end{bmatrix}$$

$$|H_C| = \sqrt{I_{xx}^2\omega_x^2 + I_{yy}^2\omega_y^2 + I_{zz}^2\omega_z^2} = Constant \quad ; \quad |E| \sim \sqrt{I_{xx}^2\omega_x^4 + I_{yy}^2\omega_y^4 + I_{zz}^2\omega_z^4}$$

Assume $I = \begin{bmatrix}a\\b\\c\end{bmatrix}; \omega = \begin{bmatrix}A\\B\\C\end{bmatrix}$

$$|H_C|^2 = Constant = a^2A^2 + b^2B^2 + c^2C^2 = H'$$

Transfer all momentum to x axis to get ω'

$A'^2 = H'/a^2$

New energy is then $\sim a^2 * A'^4 = a^2 * \frac{H'^2}{a^4} = H'^2/a^2$

Simply stated the new energy state is proportional to a constant/a^2 where a is the inertia of the axis that the majority of the rotation goes to

Three axis stabilized craft

For a craft with attitude control along the principal axis of the craft

$$\left.\begin{aligned} I_{xx}\dot{\omega}_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z = T_x \\ I_{yy}\dot{\omega}_y - \left(I_{xx} - I_{zz}\right)\omega_x\omega_z = T_y \\ I_{zz}\dot{\omega}_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y = T_z \end{aligned}\right\}$$

Note that is the spacecraft is initially stationary then there will be no cross coupling effects

Also keep in mind that if your attitude control is implemented with rotational mechanisms then your new angular momentum is

$$H_C = \begin{bmatrix} (I_{xx}\omega_x + H_x) \\ (I_{yy}\omega_y + H_y) \\ (I_{zz}\omega_z + H_z) \end{bmatrix}$$

Where H_i is the momentum cause by your attitude control mechanisms relative to each principal axis

Spin Stabilized Craft

Using the equation set for a three axis stabilized craft we can build equations for a spinning craft by simply assuming it is spinning primarily along the z axis (ω_z=S)

This makes our equations take the form of

$$I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y S = T_x$$
$$I_{yy}\dot{\omega}_y - (I_{xx} - I_{zz})\omega_x S = T_y$$
$$I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x \omega_y = T_z$$

Taking a look at the first two equations, it can bee seen that they are both linear functions of for S

$$Axs - ByS = 0$$

$$Cys - DxS = 0$$

The characteristic equation in terms of the laplace operator s is

s^2+(D/A)(B/C)S^2=0

s^2+(1-I_{zz}/I_{xx})(1-I_{zz}/I_{yy})S^2 = 0

$$s = S\sqrt{\frac{I_{xx}I_{yy} - I_{xx}I_{zz} - I_{yy}I_{zz} + I_{zz}^2}{I_{xx}I_{yy}}} = S\sqrt{\frac{I_{zz}}{I_{yy}} + \frac{I_{zz}}{I_{xx}} - \frac{I_{zz}^2}{I_{xx}I_{yy}} - 1}$$

For stability I_{zz} must be the maximum or minimum moment of inertia

Oscillatory Modes

Rigid Body Modes

Nutation –

Oscillatory motion in addition to precession

Libration –

Oscillation due to disturbance of locally vertical momentum vector

Flexure Modes

Oscillatory bending due to modal frequency resonance

Important for things like truss, boom, or solar array design

Frequency estimates can be found using the following equation

$$f = \left(\frac{K^2\pi}{8L^2}\right)\sqrt{\frac{EI}{\rho A}}Hz$$

Where

K ~ 1,2,3,5,7, … for different modes	L = array/boom length [m]
E = Young's Modulus [N/m^2]	I = second moment of area [m^4]
ρ = density [kg/m^3]	A = cross-sectional area [m^2]

Rotations and Euler angles

A rotation is a matrix that "rotates" a vector. This means that it changes the direction of the vector while conserving it's magnitude. For example if two 3x1 vectors V_1 and V_2 have the same magnitude but different directions, then there exists a 3x3 matrix R that can be used to rotate V_1 to become V_2

$$V_2 = [R]V_1$$

Euler angles are used to describe a rotation as a sequence of separate rotations about the coordinate axis (i.e. $X(\phi)$, $Y(\theta)$, $Z(\psi)$)

This convention can be used to accurately describe any orientation, however problems can occur depending on the sequence of rotations required or limitation on the sequence in which the rotations can be performed

In other words, Euler angles are always good to describe a static orientation, however problems can arise in dynamic situations

Brief intro to quaternions

Quaternions have many uses to keep track of different navigational telemetry. Due to their higher complexity over a standard vector described by 3 values, they provide additional capability in specific situations.

For example one of the paces that they are most useful is keeping track of your crafts attitude and performing rotations.

First, the basis for quaternions must be established. They are developed with a concept similar to imaginary numbers. But instead of just $i^2 = -1$, we have

$$i^2 = j^2 = k^2 = -1$$

We also must establish that $ijk = -1$

Using $ijk = -1$ and multiplying both sides by different combination of I,j, and k we see some interesting outcomes

$ij = k$ while $ji = -k$ $\quad$ $jk = i$ while $kj = -I$ $\quad$ $ki = j$ while $ik = -j$

With the new rules established, we define the quaternion in the vector form we will be using it as

$$q = a + bi + cj + dk \text{ or } [a,v]$$

Where a is considered the scalar part and $bi + cj + dk$ is considered the vector part.

The scalar part of the quaternion is real and the vector part is imaginary. Every quaternion can be viewed as a vector in four dimensional space, it is common to define a vector to mean a pure imaginary quaternion. With this convention an imaginary quaternion in R^4 space becomes a real vector in R^3 space.

This means that the system can be over defined. This additional information can be useful especially when using it for keeping track of attitude.

Using quaternions

Rules for adding and subtracting quaternions

$$(r_1, v_1) + (r_2, v_2) = (r_1 + r_2, v_1 + v_2)$$

$$(r_1, v_1)(r_2, v_2) = (r_1 r_2 - v_1 v_2, r_1 v_2 + r_2 v_1 + v_1 \times v_2)$$

We can see that addition is similar to vectors, however multiplication is not communicative due to the cross product in the vector part

The standard convention to keep track of your attitude is with a unit quaternion defining an axis and a rotation about that axis

$$\boldsymbol{q} = \cos\frac{\theta}{2} + (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})\sin\frac{\theta}{2}$$

This not only can define an absolute attitude from a reference but it can also be used as an operator to rotate a quaternion to a new state

When used as a rotation operator a quaternions advantages can really be seen. Quaternions rotations are independent of the order in which they are applied as opposed to Euler angles which can be the source of problems. Quaternion transformations are also immune to gimbal lock.