Lecture 5: Hypothesis Tests Applied to Means

- ## 5. 0 Recap Hypotheses about Means with Known σ^2.

- See examples in book

- ## 5. 1 Large-Sample Problems with Unknown Population Variance

- So far, we have made inferences about population mean assuming that the σ^2 was known

- This is not typically the case … but there are some exceptions

- Instead, we must rely on s^2 as an estimate of σ^2

    - This is OK for large sample sizes ($N \geq 100$ or so)

    - So, s can be used instead of σ to estimate the standard error of the mean, as

$$\sigma_M = \frac{s}{\sqrt{N}} = \frac{S}{\sqrt{N-1}}$$

- The standardized score corresponding to the sample mean is referred to the normal distribution

    - This is justified by the central limit theorem when N is large

- Example:

    - Research shows that the mean of conservatism values for the general population has historically been around 9 (on a 0-15 scale). A researcher is interested in examining if the current political turmoil has had any effect on conservatism values. She takes a poll from a random sample of 175 individuals and the resulting mean is 8.8 with a $S = 2.5$. What can the researcher conclude?

    - Hypotheses:

        - H_0: $\mu = 9$
        - H_1: $\mu \neq 9$ (there is not a clear hypothesis about the direction of a possible shift)

    - Alpha level is chosen as $\alpha = .01$
        - Results will be significant if the sample mean falls in either extreme .005 of all possible results
        - z-scores = -2.58 and 2.58. If obtained $z > |2.58|$, reject H_0

    - If H_0 is true, $E(\overline{X}) = 9$

    - Given $N = 175$, est.$\sigma_M = \frac{s}{\sqrt{N}}$ or $\frac{S}{\sqrt{N-1}}$ should be close to σ_M

- $\text{est.}\sigma_M = \dfrac{2.5}{\sqrt{175-1}} = \dfrac{2.5}{13.19} = .189$

$z_M = (\bar{X} - \mu) / \text{est.}\sigma_M = (8.8 - 9) / .189 = -1.06$

- Obtained $z <$ critical z, therefore, retain H_0, $p > .01$ or $.05$ (on the basis of this sample)

- CI can also be constructed (assuming large N, given that σ^2 is unknown)

 99% CI = $\bar{X} \pm 2.58\,(\text{est.}\sigma_M) = 8.8 \pm .488 = (8.31, 9.29)$, the p is about .99 that the true value of μ is covered by this interval

• 5. 2 The Distribution of *t*

- When making inferences about the population mean μ, the ratio we evaluate in reference to a normal sampling distribution is the standardized value

$$z_M = \frac{\bar{X} - \mu}{\sigma_M}$$

- this indicates how far a sample mean is from the population mean expressed in terms of variability of sample means

- When σ^2 is unknown and we only have an estimate of σ_M, the ratio is not a normal standardized score (although it has a similar form). Instead, the ratio is

$$t = \frac{[\bar{X} - E(\bar{X})]}{est.\sigma_M} = \frac{[\bar{X} - E(\bar{X})]}{s / \sqrt{N}}$$

- this indicates how far a sample mean is from the population mean expressed in terms of the *estimated* variability of sample means

- In both cases, the numerator is the same but the denominator is not

- For z_M, the denominator is a constant value of σ_M (the same regardless of the particular sample of size N we observe). Over different samples of size N, the same value of $\bar{X}$ will give the same z_M

- For t, the denominator is only an estimate of σ_M, which will take different values for different samples. Over different samples of size N, the same value of $\bar{X}$ will give different t values

- Similar intervals of z_M and t should have different probabilities of occurrence

- Solution: examine the sampling distribution of t as a random variable

    - Draw every conceivable sample of N independent observations from some normal distribution with mean μ. Each sample would provide some value of t,

$$t = \frac{[\bar{X} - E(\bar{X})]}{est.\sigma_M}$$

    - Over the different samples, the value of t will vary and can be considered a random variable and, more specifically, a ***test statistic*** that has a sampling distribution, just like any other statistic

    - What is the sampling distribution of t?

        - Recall that sampling distribution of z is normal because of CLT

            CLT $\rightarrow \bar{x} \sim N$

            z is a linear transformation of $\bar{x} \therefore z \sim N$

        - The distribution for t is also normal because of CLT

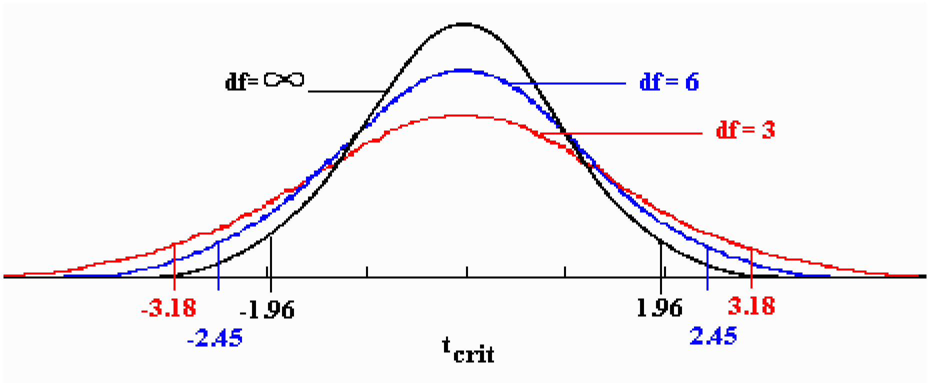
            CLT $\rightarrow \bar{x} \sim N$, regardless of the shape of the parent distribution, but t is not a linear transformation of $\bar{x}$:

                σ^2 is a constant but s^2 is a variable that will vary across samples

        - Assume the basic population distribution is normal so

            $\bar{x} \sim N$, regardless of the sample size

            $\bar{x}$ and s are independent (only possible under normal distribution)



- The *t* distribution is a theoretical probability distribution. It is symmetrical, bell-shaped, and similar to the standard normal curve

- It differs from the standard normal curve in that it has an additional parameter ν (degrees of freedom), which changes its shape

- ***Degrees of freedom***, usually symbolized by ν or df, is a parameter of the *t* distribution that can be any real number greater than zero. Setting the value of df defines a particular member of the family of *t* distributions

- Estimates of parameters can be based upon different amounts of information. The number of independent pieces of information that go into the estimate of a parameter is called the degrees of freedom (df)

- In general, the degrees of freedom of an estimate is equal to the number of independent scores that go into the estimate minus the number of parameters estimated as intermediate steps in the estimation of the parameter itself

    - Example: if the variance, σ^2, is to be estimated from a random sample of N independent scores, then the degrees of freedom is equal to the number of independent scores (N) minus the number of parameters estimated as intermediate steps. One such parameter is μ, which is estimated by $\bar{x}$, so df is equal to N-1

- The smaller the df, the larger the variance, the flatter the shape of the distribution, resulting in greater area in the tails of the distribution

- As ν is made large, the variance of the *t* distribution approaches 1.00, which is precisely the variance of the standardized normal distribution

- Relationships to the normal curve

    - The t distribution looks similar to the normal curve

    - As ν increases, the t distribution approaches a standard normal N ($\mu = 0$, $\sigma^2 = 1$)

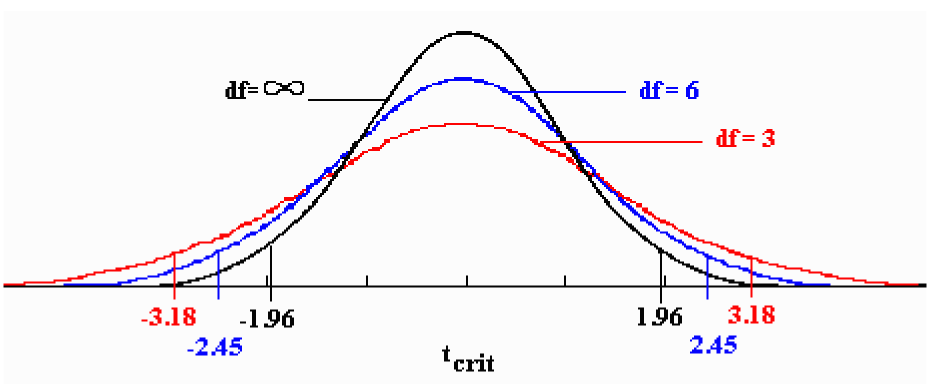
    - The standard normal curve is a special case of the t distribution when df = ∞. For practical purposes, the t distribution approaches the standard normal distribution relatively quickly, such that when df = 30 the two are almost identical

    - The $\bar{x}$ of the t distribution is 0 for ν > 1 but the variance of t is > 1 when ν > 1. Thus, for any extreme interval on either tail of the t distribution, the p associated with this interval will be larger than that for the corresponding normal distribution. The smaller the ν, the larger the discrepancy

    - Danger of using t for z_M, except when ν is large:

        - if ν is small, extreme values of t are more likely to occur than those for z_M

        - as ν becomes large (sample size increases, N - 1) the distribution of t approaches the normal distribution and the exact p of intervals in the t distribution can be approximated closely by normal probabilities



• 5. 3 Significance Test and Confidence Limits for Means Using *t*

- Example:

  - A poll is conducted to determine the number of hours per week that psychology graduate students devote to TA work. It has always been the case that this number of hours is 40. A researcher recently asked 8 graduate students (independently drawn) and found out that the sample mean was 49, with a standard deviation of 9.8. What can the researcher conclude?

  - Hypotheses:

    - H_0: $\mu = 40$
    - H_1: $\mu \neq 40$

  - Alpha level is chosen as $\alpha = .05$
    - Results will be significant if the sample mean falls in either extreme .025 of all possible results

  - $\upsilon = N - 1 = 7$; $Q = .025$; $t = 2.365$. If obtained $t > |2.365|$, reject H_0

  - If H_0 is true, $E(\bar{x}) = 40$

  - Given $N = 8$, est.$\sigma_M = \dfrac{S}{\sqrt{N-1}}$

  - est.$\sigma_M = \dfrac{9.8}{\sqrt{8-1}} = \dfrac{9.8}{\sqrt{7}} = 3.7$

  $t = [\bar{x} - E(\bar{x})]$ / est.$\sigma_M = (49 - 40)/\ 3.7 = 2.43$

  - Obtained $t >$ critical t, therefore, reject H_0, $p < .05$ (on the basis of this sample)

  - 95% CI = $\bar{x} \pm t_{(\alpha/2;\upsilon)} \cdot$ (est.σ_M) = $49 \pm (2.365) \cdot (3.7) = (40.25\ ,\ 57.75)$

  - Over all random samples of $N = 8$, the p is about .95 that the true value of μ is covered by this interval