Lecture 9: Factorial ANOVA

• 9. 1 Introduction to Factorial ANOVA

- The one-way ANOVA is used when at least three groups are compared in terms of mean level

- The "effect" exists if the means vary more than we would expect them to vary by chance

$$\text{ANOVA} = \frac{Between - Group\,Variability}{Within - Group\,Variability}$$

- A significant effect implies that the means vary at more than chance levels

- A significant *F*-value implies a difference <u>somewhere</u>, we simply don't know where without performing further analyses

- We usually perform planned contrasts or post hoc (multiple comparison) tests after finding a significant *F*

- Called "one-way" ANOVA because there is only one independent variable, or *factor*

- In psychological research we are usually interested in more than just one factor

  - Example: effectiveness of teaching ANOVA based on the method (i.e., lecture, discussion, study) and previous experience (i.e., yes, no)

  - This is an example of a 2 x 3 ANOVA design…but we are not limited to 2 factors

- ***Advantages*** of using factorial ANOVA

  - *Generalizability* – In one-way ANOVA, conclusions can be applied only to the groups (levels) of one factor (e.g., teaching method). With factorial ANOVA we can make finer distinctions (e.g., for teaching method and previous experience)

  - *Efficiency* – We can address several questions with one study (e.g., examining the effects of teaching method and previous experience simultaneously)

  - It is more powerful – MS_W is derived from all the cell variances, not just the ones related to only one factor (as it would be the case in a simple ANOVA)…the error variance is reduced

  - *Interactions* – Factorial designs allow examination of *interaction effects* (e.g., does the effect of one IV on the DV depend on the level of the other IV?)

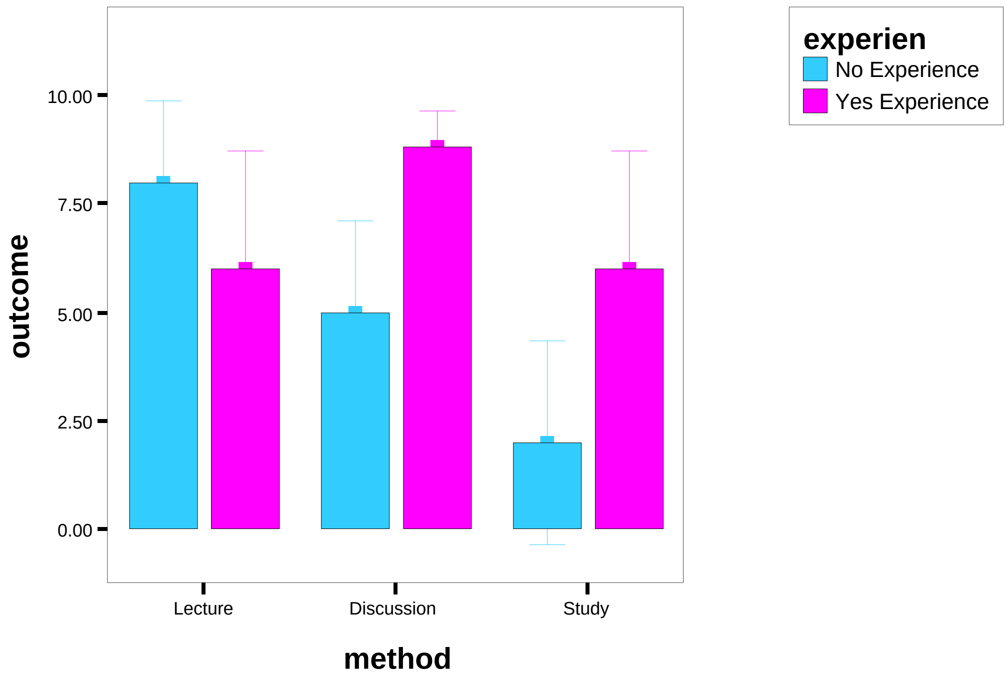
- ***Example***: effectiveness of teaching ANOVA based on the method and previous experience

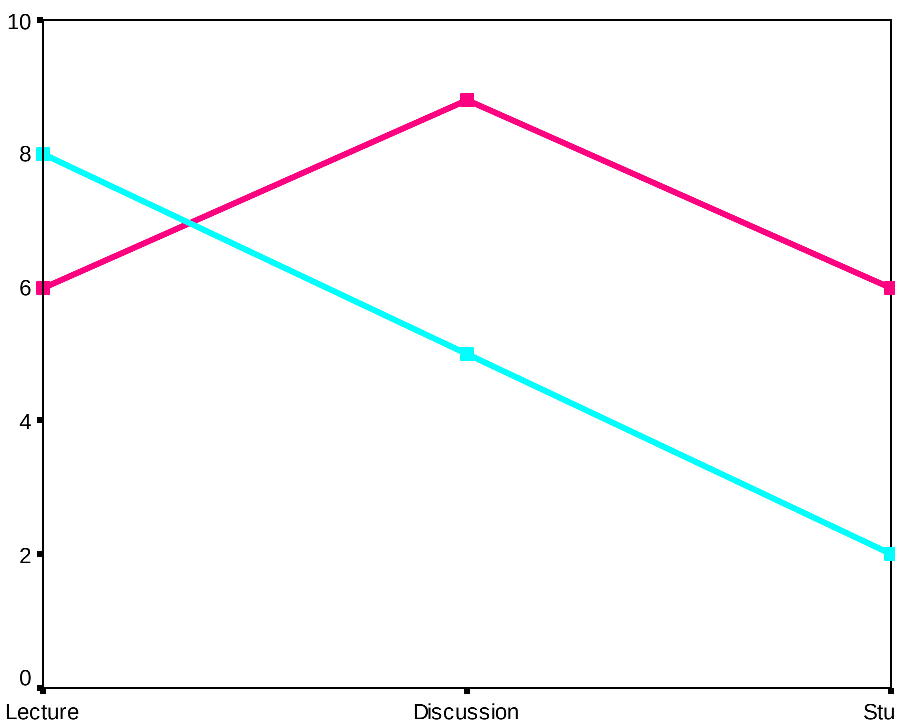
	Method		
	Lecture	*Discussion*	*Study*
No	5	2	0
	8	4	0
	8	5	1
	9	7	5
	10	7	4
Yes	4	8	4
	3	8	3
	6	9	6
	7	9	7
	10	10	10

- Let's first look at the means to get a sense of what's going on

	L	*D*	*S*	*Row Means*
N	8	5	2	$\bar{X}_N = 5$
Y	6	9	6	$\bar{X}_Y = 7$
Column Means	$\bar{X}_L = 7$	$\bar{X}_D = 7$	$\bar{X}_S = 4$	$\bar{X}_{.} = 6$

- Let's now plot the means





- What can we conclude?

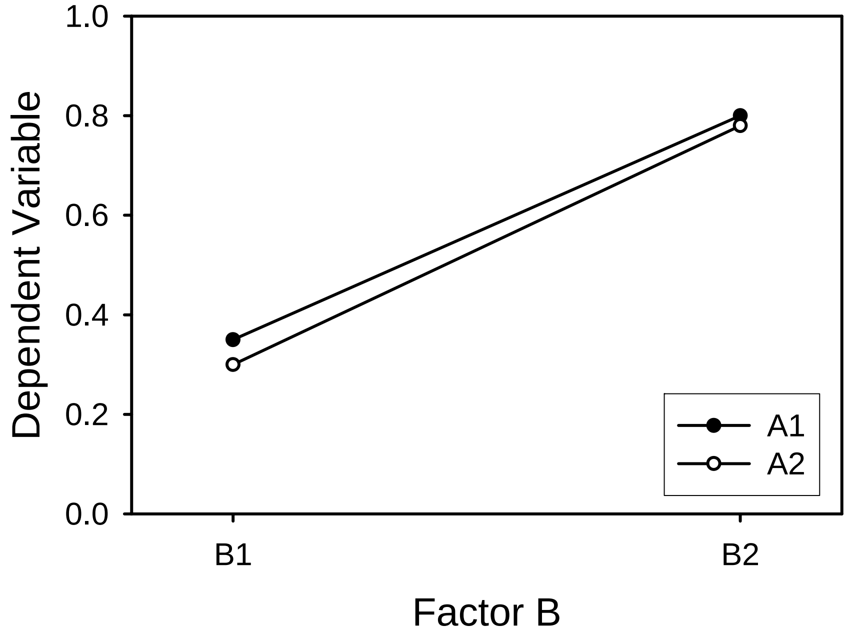
    - Not much…but we can observe that, under the lecture method, individuals with experience do worse than individuals without experience, whereas the reverse is true for the other methods…this is called an *interaction effect*

    - The effect of one of the factors is not constant over all the levels of the other factor

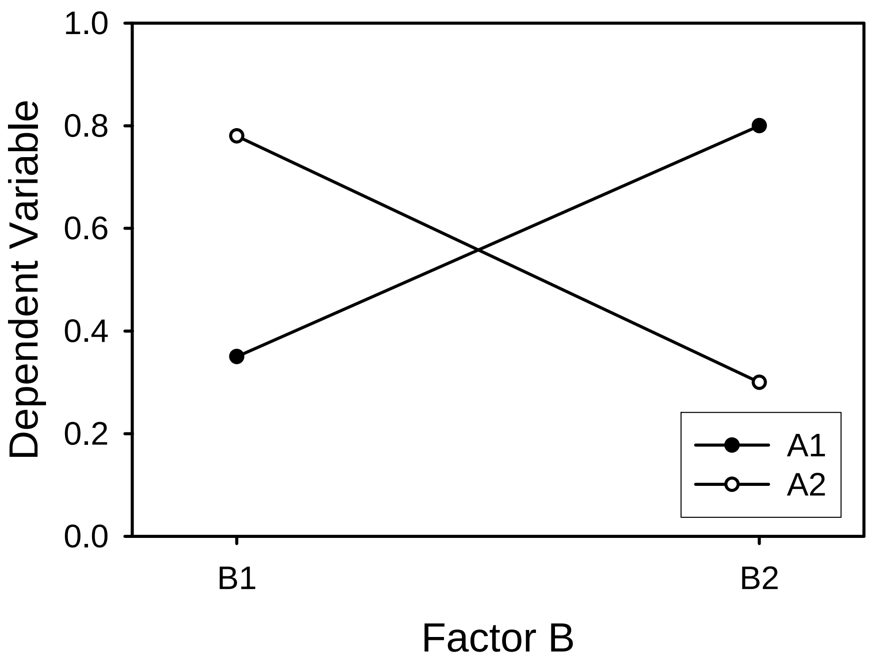
    - Such an interaction effect would not be detected using a simple ANOVA

- **9. 2 Terminology and Plots of Possible Effects**

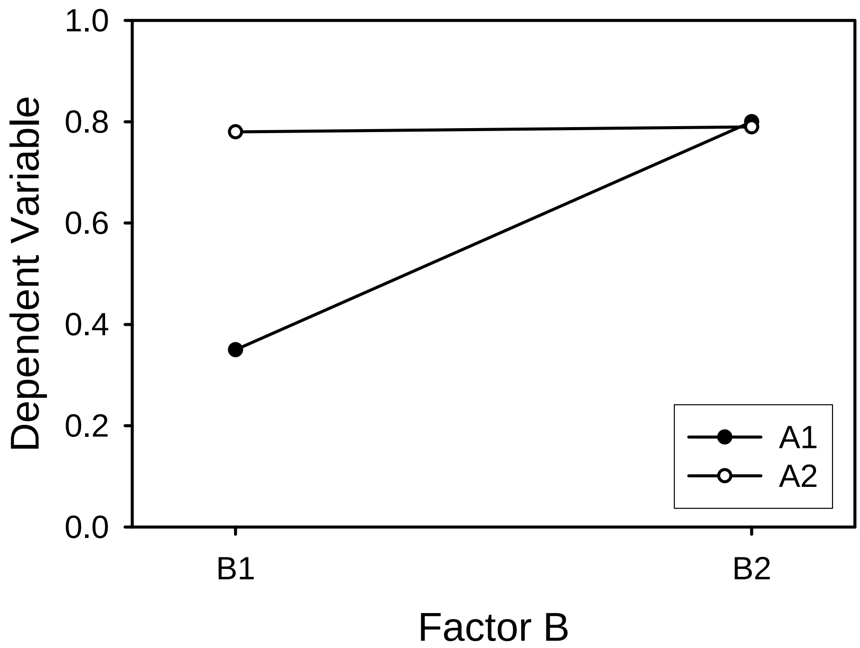
- One main effect, no interaction



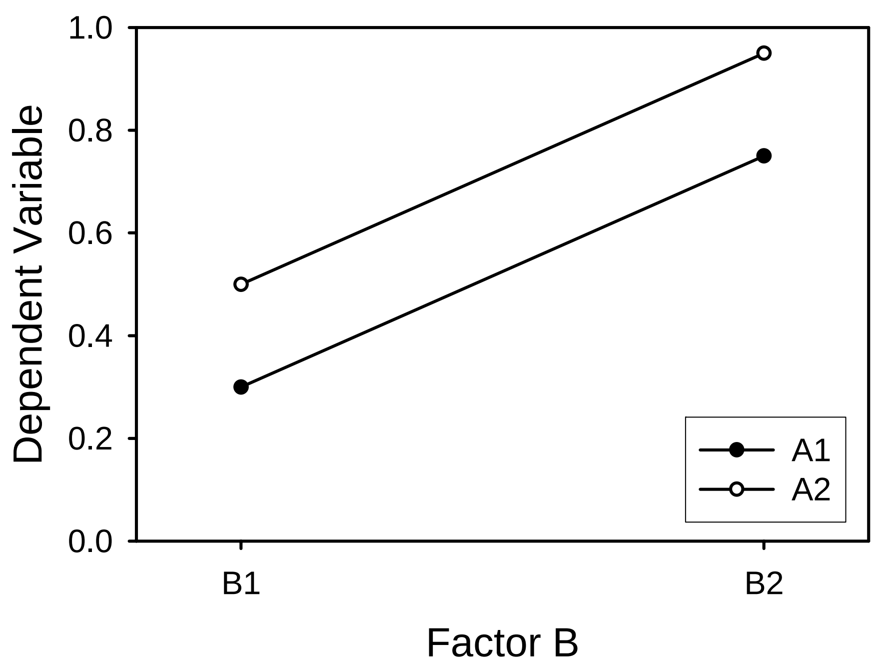
- Interaction, no main effects



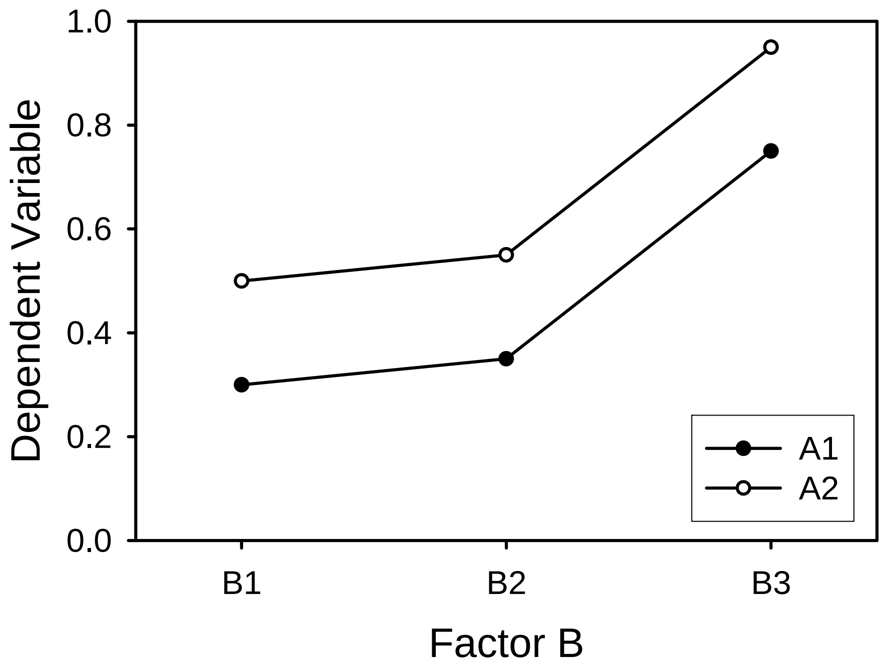
- Two mai



- Two main effects, no interaction



- Two main eff


1.0
0.8
0.6
0.4
0.2
0.0
Dependent Variable
A1
A2
B1
B2
B3
Factor B

- ## 9. 3 Linear Model

$X_{ijk} = \mu + \alpha_j + \beta_k + \alpha\beta_{jk} + \varepsilon_{ijk}$

X_{ij} = score for person i in row j and column k

μ = population mean

α_j = effect of row j ($\alpha_j = \mu_j - \mu$)

β_k = effect of column k ($\beta_k = \mu_k - \mu$)

$\alpha\beta_{jk}$ = interaction effect ($\alpha\beta_{jk} = \mu_{jk} - \mu_j - \mu_k + \mu$)

ε_{ijk} = error – or residual – for score X_{ijk}

($\varepsilon_{ij} = X_{ij} - \mu - \alpha_j - \beta_k - \alpha\beta_{jk}$)

$\varepsilon_{ijk} \sim$ NID $(0, \sigma^2)$

- Because we have the model, we can now estimate the model parameters

	L	*D*	*S*	
N	8	5	2	$\overline{X}_N = 5$
Y	6	9	6	$\overline{X}_Y = 7$
	$\overline{X}_L = 7$	$\overline{X}_D = 7$	$\overline{X}_S = 4$	$\overline{X}. = 6$

$\hat{\mu} = \overline{X}.$

$\hat{\alpha}_j = \hat{\mu}_j - \hat{\mu} = \overline{X}_j - \overline{X}.$

$\hat{\beta}_k = \hat{\mu}_k - \hat{\mu} = \overline{X}_k - \overline{X}.$

$\hat{\alpha}\hat{\beta}_{jk} = \hat{\mu}_{jk} - \hat{\mu}_j - \hat{\mu}_k + \hat{\mu} = \overline{X}_{jk} - \overline{X}_j - \overline{X}_k + \overline{X}.$

- Properties

$\sum \hat{\alpha}_j = 0$; $\sum \hat{\beta}_k = 0$;

$\sum \hat{\alpha}\hat{\beta}_{jk} = 0$; $\sum \hat{\varepsilon}_{ijk} = 0$;

• 9. 4 Estimated Values

- Estimated Main Effects (Effects that do not take into account the interaction)

	L	*D*	*S*	*Row Means*
N				-1
Y				1
Row Means	1	1	-2	

- e.g., not having experience gives individuals a value of -1 over the grand mean

- Estimated Interactions

	L	*D*	*S*	*Row Means*
N	2	-1	-1	
Y	-2	1	1	
Row Means				

- e.g., for the first cell, 8 – 5 – 7 + 6 = 2; there is a particular feature in this cell that gives individuals in this cell a value of 2 over the grand mean

- This is the unique effect of that cell that is not predicted of knowing the effects of rows and columns

- If there is no interaction, we could predict the values of the cells (= 0)

• Model without Interaction

- Two main effects

- Suppose we have a theory that says

$X_{ijk} = \mu + \alpha_j + \beta_k + \varepsilon_{ijk}$

we can estimate the cell means under this model as

$\overline{X}_{jk} = \overline{X}_j + \overline{X}_k - \overline{X}. = \hat{\alpha}_j + \hat{\beta}_k - \hat{\mu}$

so the expected cell means (predicted from model without interaction) are

	L	*D*	*S*	
N	6	6	3	
Y	8	8	5	

and the observed *minus* predicted values are

	L	*D*	*S*	
N	2	-1	-1	
Y	-2	1	1	

What do these values represent?

• 9. 5 Factorial ANOVA Calculations

- Two main effects

$$F_A = \frac{MS_A}{MS_{error}} \qquad F_B = \frac{MS_B}{MS_{error}}$$

- Interaction Effect

$$F_{AB} = \frac{MS_{AB}}{MS_{error}}$$

- Mean Squares

$$MS_A = \frac{SS_A}{df_A} \qquad MS_B = \frac{SS_B}{df_B} \qquad MS_{AB} = \frac{SS_{AB}}{df_{AB}}$$

- Degrees of Freedom

$df_{\text{A}} = (a - 1)$

$df_{\text{B}} = (b - 1)$

$df_{\text{AB}} = (a - 1)\,(b - 1) = df_{\text{A}} \times df_{\text{B}}$

$df_{error} = (N - 1) - df_{\text{A}} - df_{\text{B}} - df_{\text{AB}}$

$\qquad = N - (a \cdot b)$

$df_{total} = (N - 1)$

- Sum of Squares Between

$SS_B = SS_R + SS_C + SS_{INT}$

$$SS_R = nK\sum_j (\bar{X}_j - \bar{X}.)^2 = nK\sum_j \hat{\alpha}_j^2$$

$$=15((-1)^2 + (1)^2) = 30$$

$$SS_C = nJ\sum_k (\bar{X}_k - \bar{X}.)^2 = nJ\sum_k \hat{\beta}_k^2$$

$$=10((1)^2 +(1)^2 +(-2)^2) = 60$$

$$SS_I = n\sum_k \sum_j (\bar{X}_{jk} - \bar{X}_j - \bar{X}_k + \bar{X}.)^2 = n\sum_k \sum_j \hat{\beta}\hat{\alpha}_{jk}^2$$

$$= 5((2)^2 + (-1)^2 + (1)^2 + (-2)^2 + (1)^2 + (1)^2) = 60$$

- Sum of Squares Within

$$SS_W = \sum_k \sum_j \sum_i (X_{ijk} - \bar{X}_{ij})^2$$

= 120 (squared deviations of row scores from cell means)

This will give an estimator of σ^2

- Sum of Squares Total

$$SS_T = \sum_k \sum_j \sum_i (X_{ijk} - \bar{X}.)^2$$

= 270 (in the example)

$SS_T = SS_B + SS_W$

$SS_T = SS_R + SS_C + SS_{INT} + SS_W$

$270 = 30 + 60 + 60 + 120$

- Degrees of Freedom

Rows = $J - 1$ (2 – 1 = 1)

Columns = $K - 1$ (3 – 1 = 2)

Interaction = $(J - 1)(K - 1)$ (1 x 2 = 2)

Within = $N - (J \times K)$ (30 – 6 = 24)

Total = $N - 1$ (30 – 1 = 29)

- Mean Squares

Rows = $SS_R / J - 1$ (30/1 = 30)

Columns = $SS_C / K - 1$ (60/2 = 30)

Interaction = $SS_I / (J - 1)(K - 1)$ (60/2 = 30)

Within = $SS_W / N - (J \times K)$ (120/24 = 5)

These are four independent estimates of σ^2. If H_0 is true, all estimates should be the same. If H_0 is false, the estimates will be different. In reality there are 3 H_0 that we test, not only a global one

- Hypotheses

Rows

H_0: $\alpha_1 = \alpha_2 = \ldots = \alpha_j$

H_1: $\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_j$ (for at least one α)

Columns

H_0: $\beta_1 = \beta_2 = \ldots = \beta_k$

H_1: $\beta_1 \neq \beta_2 \neq \ldots \neq \beta_k$ (for at least one β)

Interaction

H_0: all $\alpha\beta_{jk} = 0$

H_1: at least one $\alpha\beta_{jk} \neq 0$

- F-Tests

Row Effect

$F = MS_R / MS_W$ (30/5 = 6) crit $F(1,24) = 4.3$

Column Effect

$F = MS_C / MS_W$ (30/5 = 6) crit $F(2,24) = 3.4$

Interaction Effect

$F = MS_I / MS_W$ (30/5 = 6) crit $F(2,24) = 3.4$

- Source Table

Source	*SS*	*df*	*MS*	*F*	*Sig.*
Experience	30	1	30	6	<.05
Method	60	2	30	6	<.01
Interaction	60	2	30	6	<.01
Within	120	24	5		
Total	270	29			

• 9. 6 Simple Effects

- Given an interaction one may want to examine whether an effect exists for all the levels of the other factor (and vice-versa)

- These tests are called ***simple effects***

  - Is there an "experience" effect for each level of method?

  - Is there a "method" effect for each level of experience?

- Simple Effects for Experience

  - Row effects (variability due to the rows and variability due to the interaction)

$SS_R + SS_I = \Sigma_k SS_{R\ at\ k} = n\Sigma_k\Sigma_j (X_{jk} - X_{k\cdot})^2$

$= n\Sigma_j (X_{j1} - X_{\cdot 1})^2 + n\Sigma_j (X_j^2 - X_{\cdot 2})^2 + n\Sigma_j (X_{j3} - X_{\cdot 3})^2$

SS_R(at lecture) = $5((8–7)^2 + (6–7)^2) = 10$

SS_R(at discuss) = $5((5–7)^2 + (9–7)^2) = 40$

SS_R(at study) = $5((2–4)^2 + (6–4)^2) = 40$

$\therefore SS_{R\ at\ k} = 90$

$SS_R + SS_I = 30 + 60 = 90$

- Simple Effects F – Tests

Lecture

$F = MS_R(\text{lecture}) / MS_W$ (10/5 = 2) $p > .05$

Discussion

$F = MS_R(\text{discussion}) / MS_W$ (40/5 = 8) $p < .01$

Study

$F = MS_R(\text{study}) / MS_W$ (40/5 = 8) $p < .01$