Homework #1

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1 Orthogonal projection in a hilbert space

$$\pi_V(f) \in V;$$
$$\langle f - \pi_V(f), g\rangle = 0 \ \forall g \in V;$$

1.
$$\begin{aligned}\|f\|^2 &= \langle f, f\rangle \\ &= \langle \pi_v(f) + (f - \pi_v(f)), \pi_v(f) + (f - \pi_v(f))\rangle \\ &= \langle \pi_v(f), \pi_v(f)\rangle + 2\langle \pi_v(f), f - \pi_v(f)\rangle + \langle f - \pi_v(f), f - \pi_v(f)\rangle \\ &= \|\pi_v(f)\|^2 + \|f - \pi_v(f)\|^2 \ \textit{center term cancels since orthogonal}\end{aligned}$$

2.
$$\begin{aligned}\langle f, \pi_v(g)\rangle &= \langle (f - \pi_v(f)) + \pi_v(f), \pi_v(g)\rangle \\ &= \langle f - \pi_v(f), \pi_v(g)\rangle + \langle \pi_v(f), \pi_v(g)\rangle \\ &= \langle \pi_v(f), \pi_v(g)\rangle \ \textit{by orthogonality}\end{aligned}$$

$$\begin{aligned}\langle g, \pi_v(f)\rangle &= \langle (g - \pi_v(g)) + \pi_v(g), \pi_v(f)\rangle \\ &= \langle g - \pi_v(g), \pi_v(f)\rangle + \langle \pi_v(g), \pi_v(f)\rangle \\ &= \langle \pi_v(g), \pi_v(f)\rangle \ \textit{by orthogonality}\end{aligned}$$

$$\langle g, \pi_v(f)\rangle = \langle f, \pi_v(g)\rangle$$

3.

2 A reproducing kernel is positive definite

1.
$$K(x, y) = \langle K_x(y), K_y(x)\rangle \ \textit{by reproducing property}$$
$$K(y, x) = \langle K_y(y), K_x(x)\rangle \ \textit{by reproducing property}$$
$$K(x, y) = K(y, x)$$

2.
$$\begin{aligned}K(x, y) &= \langle K_x(y), K_y(x)\rangle \\ &= \langle K_y(x), K_y(x)\rangle \geq 0 \ \textit{by symmetry}\end{aligned}$$

3.
$$\begin{aligned}|f(x)| &\leq \|f\|\sqrt{K(x,x)} \\ |\langle f, K_x\rangle| &\leq \|f\|\sqrt{\langle K_x, K_x\rangle} \\ |\langle f, K_x\rangle| &\leq \|f\|\|K_x\| \ \textit{true by Cauchy} - \textit{Schwartz inequality}\end{aligned}$$

3 RKHS over a finite set

1.