Added-variable plots require the estimator $\hat{\boldsymbol{\beta}}$ of the parameters $\boldsymbol{\beta}$ to have the form

$$\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}.$$

where $\tilde{\mathbf{X}} = \mathbf{P}(\mathbf{X})$ and $\tilde{\mathbf{y}} = \mathbf{P}(\mathbf{y})$ are transformations of the right- and left-hand-side variables $\mathbf{X}$ and $\mathbf{y}$.

Partitioning $\tilde{\mathbf{X}}$ into $\tilde{\mathbf{X}} = [\tilde{\mathbf{x}}_1\ \tilde{\mathbf{X}}_2]$ and $\hat{\boldsymbol{\beta}} = [\hat{\beta}_1\ \hat{\boldsymbol{\beta}}_2]'$, one can calculate $\mathbf{e}_{\tilde{\mathbf{x}}_1} = \tilde{\mathbf{M}}_2\tilde{\mathbf{x}}_1$ and $\mathbf{e}_{\tilde{\mathbf{y}}} = \tilde{\mathbf{M}}_2\tilde{\mathbf{y}}$ where $\tilde{\mathbf{M}}_2 = \mathbf{I} - \tilde{\mathbf{X}}_2(\tilde{\mathbf{X}}_2'\tilde{\mathbf{X}}_2)^{-1}\tilde{\mathbf{X}}_2'$.

The added-variable plot of variable $\mathbf{x}_1$ is a graph of $\mathbf{e}_{\tilde{\mathbf{y}}}$ versus $\mathbf{e}_{\tilde{\mathbf{x}}_1}$ which has a prediction line with slope equal to $\hat{\beta}_1 = (\mathbf{e}_{\tilde{\mathbf{x}}_1}'\mathbf{e}_{\tilde{\mathbf{x}}_1})^{-1}\mathbf{e}_{\tilde{\mathbf{x}}_1}'\mathbf{e}_{\tilde{\mathbf{y}}}$.

0.1 The `arima` command

The ARMA(p,q) model is

$$\begin{aligned} y_t &= \mathbf{x}_t'\boldsymbol{\beta}_{\boldsymbol{x}} + \mu_t \\ \mu_t &= \sum_{i=1}^{p}\rho_i\mu_{t-i} + \sum_{j=1}^{q}\theta_j\epsilon_{t-j} + \epsilon_t. \end{aligned}$$

If the model is ARIMA(p,d,q) for $d \in 1,2,3,\ldots$ then replace y_t and $\mathbf{x}_t$ by $\Delta^d y_t$ and $\Delta^d\mathbf{x}_t$, respectively, where Δ is the difference operator.

The ARMA(p,q) model can be written as

$$\begin{aligned} y_t &= \sum_{i=1}^{p}\rho_i y_{t-i} + \mathbf{x}_t'\boldsymbol{\beta}_{\boldsymbol{x}} - \sum_{i=1}^{p}\rho_i\mathbf{x}_{t-i}'\boldsymbol{\beta}_{\boldsymbol{x}} + u_t \\ u_t &= \sum_{j=0}^{q}\theta_j\epsilon_{t-j} \end{aligned}$$

where $\theta_0 = 1$.

u_t is an MA(q) process with autocovariances

$$\gamma_j = Cov[u_t u_{t-j}] = \begin{cases} \sigma_\epsilon^2\sum_{k=0}^{q-j}\theta_k\theta_{j+k} & \text{for } j = 0,1,\ldots,q \\ 0 & \text{for } j > q \end{cases}$$

If the model is written in matrix form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u},$$

then

$$\mathbf{y} = \begin{bmatrix} y_{p+1} \\ \vdots \\ y_T \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} y_p & \cdots & y_1 & \mathbf{x}'_{p+1} & \mathbf{x}'_p & \cdots & \mathbf{x}'_1 \\ \vdots & & \vdots & \vdots & & \vdots & \\ y_{T-1} & \cdots & y_{T-p} & \mathbf{x}'_T & \mathbf{x}'_{T-1} & \cdots & \mathbf{x}'_{T-p} \end{bmatrix}, \ \boldsymbol{\beta} = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_p \\ \boldsymbol{\beta_x} \\ -\rho_1 \boldsymbol{\beta_x} \\ \vdots \\ \rho_p \boldsymbol{\beta_x} \end{bmatrix}, \ \text{and } \mathbf{u} = \begin{bmatrix} u_{p+1} \\ \vdots \\ u_T \end{bmatrix}. \tag{1}$$

The maximum likelihood estimator for this ARMA(p, q) model is

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{y}.$$

where $\hat{\boldsymbol{\Sigma}}$ is the estimated variance of $\mathbf{u}$, formed from the estimates of autocovariances γ_j using $\hat{\theta}_1, \ldots, \hat{\theta}_q$.

To put the `arima` estimator in the form of $\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}$, $\tilde{\mathbf{X}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{X}$ and $\tilde{\mathbf{y}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{y}$. The added-variable plot can then be constructed from $\mathbf{e}_{\tilde{\mathbf{y}}}$ and $\mathbf{e}_{\tilde{\mathbf{x}}_1}$.

A potential problem with this is I think the Kalman filter maximum likelihood estimate of $\boldsymbol{\beta}$ is using all T observations in the sample because the autocorrelations are addressed in the error term, whereas the formula above in terms of lagged y's and $\mathbf{x}$'s uses only $T - p$ observations, so their $\hat{\boldsymbol{\beta}}$ values will be slightly different. In other words, the slope of a regression fit of the added-variable plot values will have a slightly different slope than the coefficient estimate from the **arima** command.

As long as the difference in observations causes a minor deviation between the **arima** and the added-variable parameter estimates, we can show the added-variable plot of $\mathbf{e}_{\tilde{\mathbf{y}}}$ versus $\mathbf{e}_{\tilde{\mathbf{x}}_1}$ but draw the trend line using the **arima** $\hat{\beta}_1$ coefficient value (which is what gavplot.ado does now), but we should check how much difference there is between the estimates (which is displayed in test_gavplot.do).

When the user specifies weights, the formula for $\hat{\boldsymbol{\beta}}$ becomes

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{D}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{D}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{D}\mathbf{y}.$$

where $\mathbf{D} = \text{diag}(\text{sqrt}(\mathbf{w}))$ and $\mathbf{w}$ is a vector of weights. If the user specifies weights $\mathbf{v}$ as `fweights` or `iweights`, $\mathbf{w} = \mathbf{v}$; otherwise, $\mathbf{w} = \{\mathbf{v}/(\mathbf{1}'\mathbf{v})\}(\mathbf{1}'\mathbf{1})$.

When weights are used, $\tilde{\mathbf{X}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{D}\mathbf{X}$ and $\tilde{\mathbf{y}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{D}\mathbf{y}$.

0.2 The `arfima` command

The ARFIMA(p, d, q) model of a second-order stationary y_t, $t = 1, \ldots, T$ is written as

$$\boldsymbol{\rho}(L^p)(1 - L)^d(y_t - \mathbf{x}'_t\boldsymbol{\beta}) = \boldsymbol{\theta}(L^q)\epsilon_t$$

where

$$\boldsymbol{\rho}(L^p) = 1 - \rho_1 L - \rho_2 L^2 - \cdots - \rho_p L^p$$
$$\boldsymbol{\theta}(L^q) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$
$$(1-L)^d = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} L^j$$

and the lag operator L is defined as $L^j y_t = y_{t-j}; \epsilon_t \sim \mathsf{NID}(0, \sigma^2); \Gamma()$ is the gamma function; and $-0.5 < d < 0.5, d \neq 0$. The row vector $\mathbf{x}_t'$ contains the exogenous variables specified as *indepvars* in the `arfima` syntax. The process is stationary and invertible for $-0.5 < d < 0.5$, and the roots of $\boldsymbol{\rho}()$ and $\boldsymbol{\theta}()$ lie outside the unit circle and have no common roots.

The ARFIMA(p, d, q) model can be written as

$$y_t = \sum_{i=1}^{p} \rho_i y_{t-i} + \mathbf{x}_t' \boldsymbol{\beta}_{\boldsymbol{x}} - \sum_{i=1}^{p} \rho_i \mathbf{x}_{t-i}' \boldsymbol{\beta}_{\boldsymbol{x}} + u_t$$
$$u_t = (1-L)^{-d} \boldsymbol{\theta}(L^q) \epsilon_t.$$

Doornik and Ooms (2003, p. 5) shows that the autocovariances of u_t are equal to

$$\gamma_j = \sigma_\epsilon^2 \sum_{k=-q}^{q} \phi_k \frac{\Gamma(1-2d)}{[\Gamma(1-d)]^2} \frac{(d)_{k-j}}{(1-d)_{k-j}}$$

where

$$\phi_k = \sum_{s=|k|}^{q} \theta_s \theta_{s-|k|}$$

and

$$(d)_i = d(d+1)(d+2)\cdots(d+i-1), \quad (d)_0 = 1, \text{ so } (1)_i \text{ equals } i!.$$

The ratio

$$\frac{(d)_i}{(1-d)_i} \text{ for } i = -q-(T-1), \ldots, 0, \ldots, q$$

can be computed using forward recursion for $i > 0$:

$$(d)_i = (d+i-1)(d)_{i-1},\ i > 0$$

and backward recursion otherwise:

$$(d)_i = \frac{(d)_{i+1}}{(d+i)},\ i < 0.^1$$

1. In Doornik and Ooms (2003, p. 5) the backward recursion equation has a typo.

With $\mathbf{y}, \mathbf{X}, \boldsymbol{\beta}$ and $\mathbf{u}$ defined as in Equation 1, the maximum likelihood estimator for this ARFIMA(p, d, q) model is

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{y}.$$

where $\hat{\boldsymbol{\Sigma}}$ is the estimated variance of $\mathbf{u}$, formed from estimates of autocovariances γ_j using `arfima` estimates of the parameters.

The `arfima` model can be written as $\hat{\boldsymbol{\beta}} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}$ when $\tilde{\mathbf{X}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{X}$ and $\tilde{\mathbf{y}} = \hat{\boldsymbol{\Sigma}}^{-1/2}\mathbf{y}$ enabling construction of an added-variable plot.

0.3 The `arch` command

1 Reference

Doornik, J. A., and M. Ooms. 2003. Computational Aspects of Maximum Likelihood Estimation of Autoregressive Fractionally Integrated Moving Average Models. *Computational Statistics & Data Analysis* 42(3): 333–348.