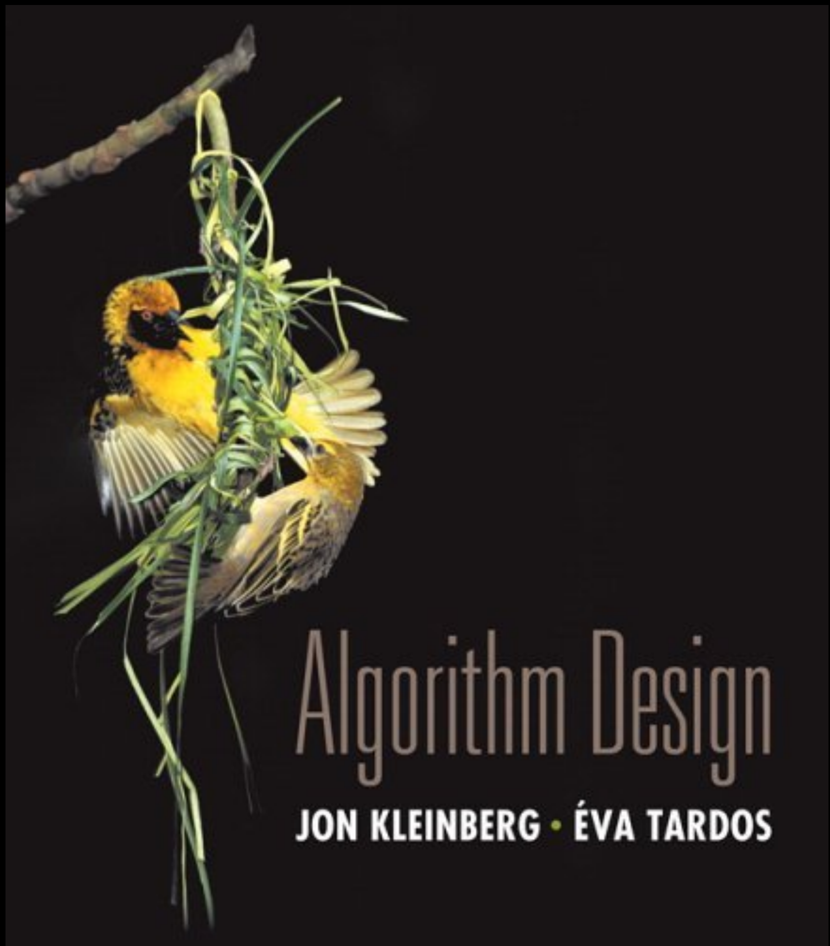


Chapter 1

Introduction:
Some Representative Problems

Slides by Kevin Wayne.

Last update: Zehra Cataltepe, Feb 19, 2013

1.1 A First Problem: Stable Matching

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:
- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.
- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?
A. No. Bertha and Xavier will hook up.

favorite ↓ (1st) — least favorite ↓ (3rd)

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ (1st) — least favorite ↓ (3rd)

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not obvious a priori.

is core of market *(a housing term)* nonempty?

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Demo (from: http://sephlietz.com/gale-shapley/)

m

m0	w0	w1	w2
m1	w0	w1	w2
m2	w0	w1	w2

w

w0	m0	m1	m2
w1	m1	m0	m2
w2	m2	m1	m0

Match ☐ Verbose Output?

Results

20: m0 is paired with w0

21: m1 is paired with w1

22: m2 is paired with w2

m

m0	w0	w1	w2
m1	w0	w1	w2
m2	w0	w1	w2

w

w0	m0	m1	m2
w1	m0	m1	m2
w2	m0	m1	m2

Match ☐ Verbose Output?

Results

17: m0 is paired with w0

18: m1 is paired with w1

19: m2 is paired with w2

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

- Note: Please read the following to review proof of contradiction:
- http://zimmer.csufresno.edu/~larryc/proofs/proofs.contradict.html
- Proofs in general: http://zimmer.csufresno.edu/~larryc/proofs/proofs.html

Proof of Correctness: Stability

Claim. No unstable pairs.
Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

- Case 1: Z never proposed to A. ← men propose in decreasing order of preference
  - ⇒ Z prefers his GS partner to A.
  - ⇒ A-Z is stable.

- Case 2: Z proposed to A.
  - ⇒ A rejected Z (right away or later)
  - ⇒ A prefers her GS partner to Z. ← women only trade up
  - ⇒ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- X-A, Y-B, Z-C.
- Y-A, X-B, Z-C.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

man optimal matching

woman optimal matching

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Define: S* ={(m,best(m)): m in M} where best(m) is the best valid partner of m

Examples for S* = {(m,best(m)): m in M}

An instance with two stable matchings.

- X-A, Y-B, Z-C.
- Y-A, X-B, Z-C.

S*={(X,A),(Y,B),(Z,C)}

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

- X-A, Y-B, Z-C

S*={(X,A),(Y,B),(Z,C)}

	1st	2nd	3rd
Xavier	A	B	C
Yancey	A	B	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	X	Y	Z
Bertha	X	Y	Z
Clare	X	Y	Z

man optimal matching

woman optimal matching

Man Optimality

Claim. Every execution of the GS algorithm results in the (man-optimal) set S* !

Pf. (by contradiction)

- Suppose an execution E of GS resulted in some man paired with someone who is not his best valid partner.
- Since men propose in decreasing order of preference, then there must be some man who is rejected by a valid partner during E.
- Consider the first moment during E in which some man (m) is rejected by a valid partner (w).
- men propose in decreasing order of preference AND this is the first time such a rejection occurred
  - Therefore it must be that best(m)=w
- w may have rejected m,
  - either because m proposed and w turned it down because she was already engaged with someone she prefers more, or
  - w broke her engagement to m in favor of a better proposal.
  - Let m' be the man whom w prefers to compared to m.

	1st	2nd	3rd	4th	5th
m	w				

	1st	2nd	3rd	4th	5th
w			m'		m

Man Optimality

- Since w is a valid partner of m, there exists a stable matching S′ containing the pair (m,w).
- Let m′ be matched with some w′ ≠w in that matching S′.
  - S′={(m,w),(m′,w′),...}
- Rejection of m by w was the first rejection in THEREFORE m′ had not been rejected by any valid partner at the point in E when he became engaged to w.
- Since m′ proposed in decreasing order of preference AND w′ is a valid partner of m′ THEREFORE m′ prefers w to w′.
- But we have already seen that w prefers m′ to m, because in E she rejected m in favor of m′.
- Since (m′,w) is not in S′, then (m′,w) is an unstable pair in S′ (because both m′ and w are willing to leave their current partners and get engaged, see below)!
- This contradicts our claim that S′ is stable, hence it contradicts our initial assumption.

Matching S′ in which (m,w) happens

	1st	2nd	3rd	4th	5th
m	w				
m′			w	w′	

	1st	2nd	3rd	4th	5th
w			m′		m
w′					

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, Z-A is an unstable pair in S. ▪

S
Yancey-Amy
Zeus-Bertha
. . .

Extensions: Matching Residents to Hospitals

Ex: Men $\approx$ hospitals, Women $\approx$ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)

- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Stable Marriage Interesting Notes

other stable marriages possible?
-can be many

More questions, rich theory

do better by lying? boys -No! girls -Yes!

CC Huang,
<u>How Hard is it to Cheat in the Gale-Shapley ...</u>
To our knowledge, ours is the first attempt in proposing men-*lying*

Stable Matching Recent Publications:

- **Local search algorithms on the stable marriage problem: Experimental studies**
- Gelain, Pini, Rossi, Venable… - 2010

- **Stable marriage with ties and bounded length preference lists**
- Irving, Manlove… - Journal of Discrete Algorithms, 2009

- **Approximation algorithms for hard variants of the stable marriage and hospitals/residents problems**
- RW Irving… - Journal of Combinatorial Optimization, 2008

- **A 1.875: approximation algorithm for the stable marriage problem**
- Iwama, S Miyazaki… - Proceedings of the eighteenth …, 2007

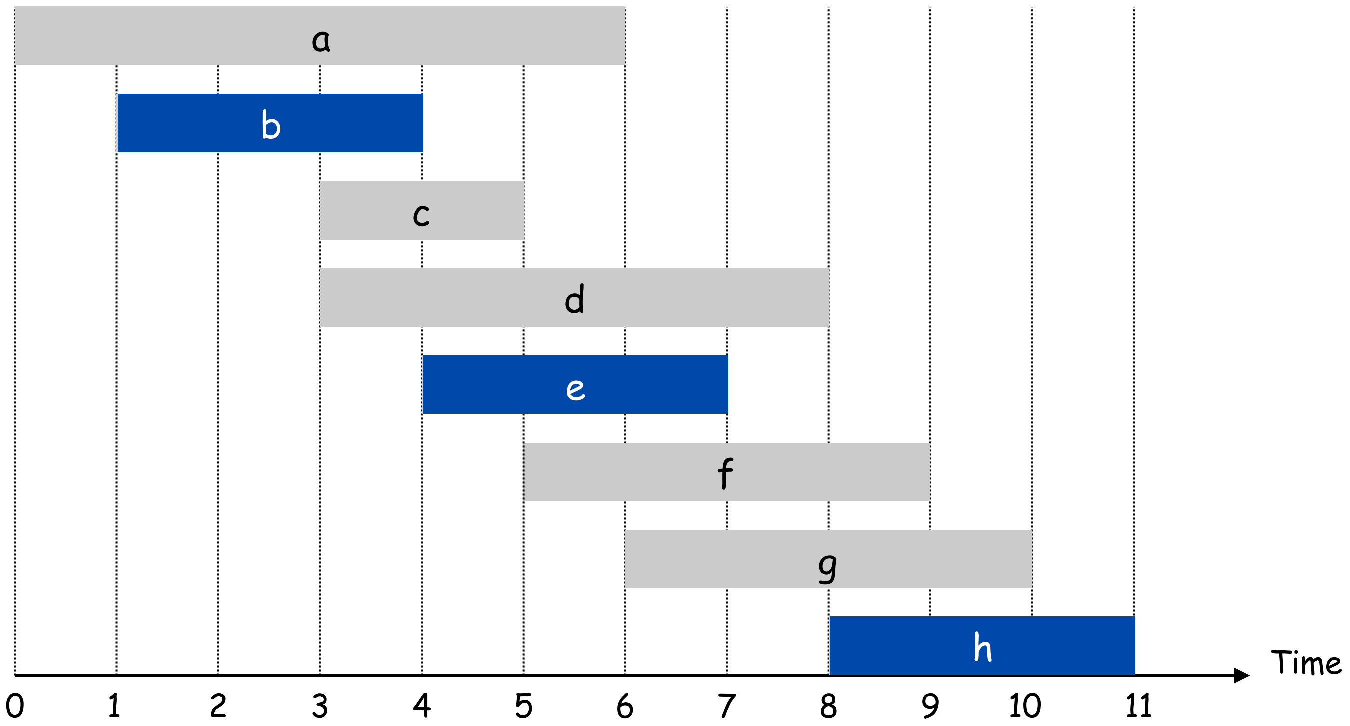
1.2 Five Representative Problems

Interval Scheduling

Input. Set of jobs with start times and finish times.

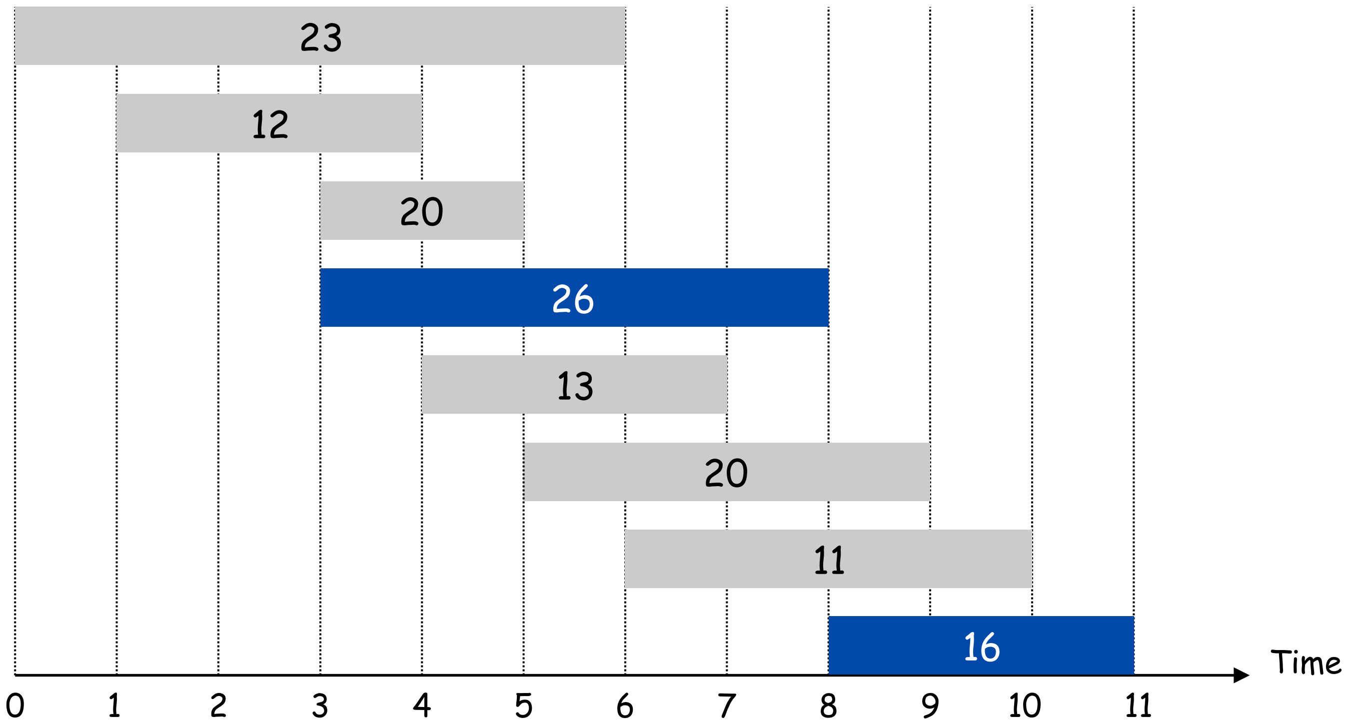
Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



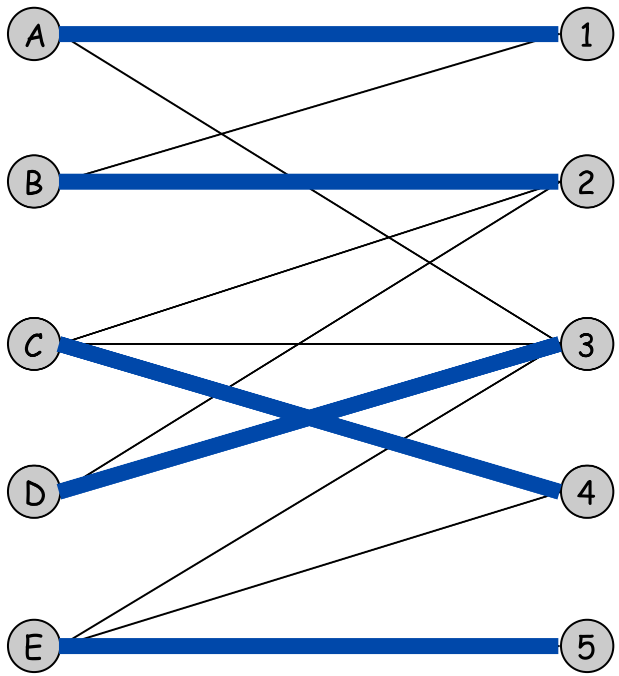
Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

Input. Bipartite graph.
Goal. Find maximum cardinality matching.

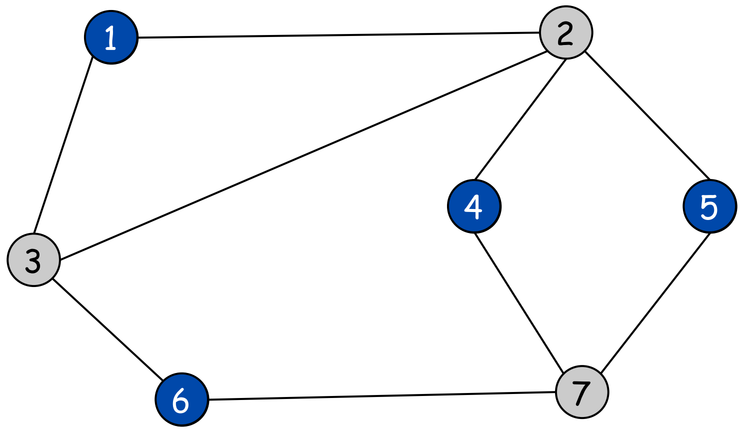


Independent Set

Input. Graph.

Goal. Find maximum cardinality independent set.

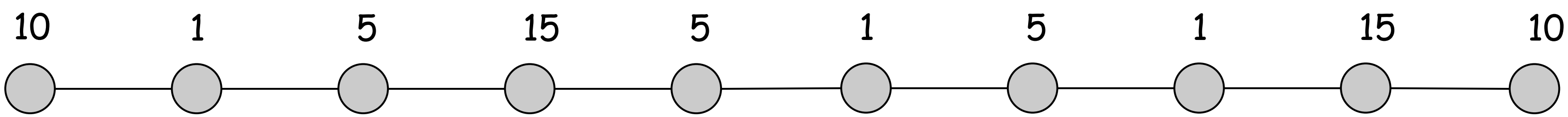
subset of nodes such that no two joined by an edge



Competitive Facility Location

Input. Graph with weight on each each node.
Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.
Weighted interval scheduling: n log n dynamic programming algorithm.
Bipartite matching: n^k max-flow based algorithm.
Independent set: NP-complete.
Competitive facility location: PSPACE-complete.

PSPACE: The set of all problems that can be solved by an algorithm with polynomial space complexity (Chapter 9).
P $\subseteq$ PSPACE (in poly time an algorithm can only consume poly space.)
NP $\subseteq$ PSPACE (There is an algorithm that can solve 3-SAT using only a polynomial amount of space.