1 Task 1

Let $(\alpha_n)_{n\in\mathbb{N}}$ a complex valued sequence such that its generating series $A(z) := \sum_{n\in\mathbb{N}} \alpha_n z^n$ has radius of convergence $\rho > 0$. Furthermore, assume $|A(z)-\alpha_0| < 1$ for $z \in \mathbb{C}, |z| < \rho$. Prove that the series $(\beta_n)_{n\in\mathbb{N}}$ defined by

$$\beta_n := \sum_{\substack{(n_1,\ldots,n_k)\in(\mathbb{N}\setminus\{0\})^k \\ \sum_{i=1}^k n_i = n, k\geq 0}} \prod_{i=1}^k \alpha_{n_i} \tag{1}$$

define a generating function $B(z) = \sum_{n\in\mathbb{N}} \beta_n z^n$ which has the same radius of convergence (in the above definition $k = 0$ is seen as $\beta_0 = 1$).

2 Solution 1

The sequence as defined implies $B(z)$ to satisfy the identity

$$B(z) = \sum_{n\in\mathbb{N}} (A(z) - \alpha_0)^n \tag{2}$$

This series can only converge absolutely if $|A(z) - \alpha_0| < 1$. Thus its radius of convergence is equal to ρ.

Remark. Omitting the bound for $A(z) - \alpha_0$ for $z \in \mathbb{C}, |z| < 1$ one the statement is false. For example take $A(z) = \frac{1}{1-z}$. Then one gets by elementary computation $B(z) = \frac{1}{2(1-2z)} + \frac{1}{2}$.

3 Task 2

Let $(X_i)_{i\in\mathbb{N}}$ be random variables such that $X_i \sim (1 - p(i))\delta_0 + p(i)\delta_1$ where $p : \mathbb{N} \to [0, 1]$.

Randomly, select such sequence $X = (X_i)_{i\in\mathbb{N}}$. Then define $(a_i)_{i\in\mathbb{N}}$ as

$$a_i := \min\{n \in \mathbb{N} : i \leq |\{j \in \mathbb{N} : j \leq n, X_j = 1\}|\}. \tag{3}$$

How big is the probability that $(a_i)_{i\in\mathbb{N}}$ contains arithmetic progressions of length s? For the case $p(i) = 1/i$?

(1) At first we compute the partial fraction decomposition of the function

$$f(x) = \sum_{i=0}^{\infty} \prod_{j=0}^{s-1} \frac{1}{x + (i + j)m} \tag{4}$$

This is done by residuum calculation

$$\mathrm{Res}_{-mk} f(x) = \frac{1}{m^{s-1}} \sum_{i=\max\{0,k-s+1\}}^{k} \prod_{\substack{j=0 \\ i+j\neq k}}^{s-1} \frac{1}{i + j - k} \tag{5}$$

$$= \begin{cases} 0 & : k \geq s - 1 \\ \frac{1}{m^{s-1}} \sum_{i=0}^{k} \prod_{\substack{j=0 \\ i+j\neq k}}^{s-1} \frac{1}{i+j-k} & : \text{otherwise} \end{cases} \tag{6}$$

Thus we get as partial fraction decomposition

$$f(x) = \sum_{i=0}^{\infty} \prod_{j=0}^{s-1} \frac{1}{x+(i+j)m} = \frac{1}{m^{s-1}} \sum_{k=0}^{s-2} \left(\sum_{i=0}^{k} \prod_{\substack{j=0 \\ i+j\neq k}}^{s-1} \frac{1}{i+j-k} \right) \frac{1}{x+mk}. \tag{7}$$

Now to compute the probability that $(a_i)_{i\in\mathbb{N}}$ contains an arithmetic sequence of length s and step size m we have to run x through $1, \dots, m$ (therefore we computed the above term which gives the an upper bound probability for an arithmetic sequence of length s and step size m and starting element $\geq x$ which has the same congruence class as x mod m). We obtain (where $A_{m,s}$ denotes the event of the occurrence of a an arithmetic progression of length s and step size m in $(a_i)_{i\in\mathbb{N}}$) by subadditivity

$$\mathbb{P}(A_{m,s}) \leq \frac{1}{m^{s-1}} \sum_{k=0}^{s-2} \left(\sum_{i=0}^{k} \prod_{\substack{j=0 \\ i+j\neq k}}^{s-1} \frac{1}{i+j-k} \right) \left(\sum_{x=1}^{m} \frac{1}{x+mk} \right). \tag{8}$$

And finally we get

$$\mathbb{P}(A_s) \tag{9}$$