A Geometric Approach on Perron-Frobenius Theorem (and some applications in graph theory)

Jakob Schneider

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Abstract

The theorem of Perron-Frobenius in a finite-dimensional space is usually proven using means like Brouwers fixed point theorem or metrics in which the corresponding mapping is somehow contractive (Banachs fixed point theorem). We will give a geometric proof on the subject for polyhedral cones using a different technique which is based on the finiteness of vertices of the corresponding polytopes. This shall be done for ideal irreducible and primitive operators. In the last section, some graph-theoretical application of the theorem is presented.

1 Symbols

The following symbols will be used through this paper:

- V denotes some ordered vector space over $\mathbb{R}$ with generating cone V^+ admitting base B[1] (the symbol B will only be used for such base). If stated, it is finite-dimensional (dimension is usually denoted by n) or its cone has polyhedral base.
- P denotes some compact polytope in V.
- T denotes some positive linear operator from V to V.
- $\|\cdot\|$ denotes the Euclidean norm in finite-dimensional space.
- $\varrho(A)$ the spectral radius of operator A.
- $\sigma(A)$ the spectrum of operator A.

2 Introduction

2.1 Some definitions around polytopes

As the theorem of Perron-Frobenius is especially interesting when stated for polyhedral cones, we will need the following

Definition 2.1 (compact finite-dimensional polytope)**.** Let V be a linear space[2] and $P \subset V$. Then P is called a *compact*[3] *finite-dimensional polytope* if $P = \operatorname{conv} S$ for a finite set $S \subset V$. The *dimension* of P as $\dim \operatorname{lin} P$.[4]

Next, we introduce k-dimensional faces and the k-dimensional boundary of a polytope as they are heavily used in our proof.

Definition 2.2 (degree, boundary, face)**.** Let P be a finite-dimensional compact polytope and let $p \in P$. We define the *degree*[5] of p (with respect to P) as

$$\deg_P p := \max\{|Q| : Q \subset P, Q \text{ affinely independent } \wedge p \in \operatorname{int}\operatorname{conv} Q\} - 1. \tag{1}$$

Then we define the *k-dimensional boundary* $\partial_k P$ by[6]

$$\partial_k P := \deg_P^{-1}\{l \in \mathbb{N}_0 : l \leq k\}. \tag{2}$$

Furthermore, any maximal[7] convex subset of $\partial_k P$ is called a *k-dimensional face*.

[1] Recall that a base $B \subset V^+$ of a cone V^+ is a convex set such that for all $v > 0$ there is a unique $\lambda > 0$ and $b \in B$ such that $\lambda b = v$. For any base B there exists a strictly positive linear functional $f : V \to \mathbb{R}$ such that $B = f^{-1}\{1\} \cap V^+$.

[2] This will always mean a vectorspace over $\mathbb{R}$.

[3] All topological expressions are meant in Euclidean topology.

[4] Clearly $\dim \operatorname{lin} P < |S|$.

[5] Roughly speaking, this is the maximum of the dimension of the faces in whose interior p lies.

[6] By $\mathbb{N}$ we denote the set $\{0, 1, 2 \ldots\}$

[7] with respect to inclusion

Remark 2.3. The set $\partial_0 P$ will also be denoted as *vertices*. It is an easy exercise to show that for any n-dimensional compact convex polytope P ($n \in \mathbb{N}$) the sets $\partial_0 P, \ldots, \partial_n P$ are not empty.

Remark 2.4. One may take $Q \subset \partial_0 P$ in the above definition.

Remark 2.5. If $P = V^+$ is a cone then $\deg_{V^+} v = \dim I_v$ where I_v denotes the ideal generated by v.

We will see that the spectral properties of the linear maps we are going to consider become especially nice when we have a lattice cone. Thus we give the following definition.

Definition 2.6 (simplex). [8] Let V be a linear space and $S \subset V$ a convex set such that for any two homotheties[9] $\alpha, \beta : V \to V$ the intersection $\alpha(S) \cap \beta(S)$ is either empty or the image $\gamma(S)$ of S under another homothety $\gamma : V \to V$. Then we call S a *simplex*[10].

Now it is a routine matter to check that a finite-dimensional compact simplex in the sense of these two definition coincides with the notion of an n-simplex (that is a convex hull of $n+1$ affinely independent points).

Remark 2.7. The reader might ask why we use such general definition instead of just the version for finite-dimensional spaces. The reason for this is that from this definition one should directly see that the intersection of a chain of compact simplices is itself a simplex. This property will be of importance.

2.2 The idea of the proof

The following simple fact about simplices and finite-dimensional polytopes shall be mentioned at this point as they were the initial idea for our proof of Perron-Frobenius theorem.

Lemma 2.8 (Intersections of polytopes and simplices). *Let $(S_m)_{m \in \mathbb{N}}$ be a chain of sequentially compact*[11] *simplices in a linear Hausdorff space V (that is $S_m \supset S_n$ for $m \leq n$). Then $\bigcap_{m \in \mathbb{N}} S_m$ is itself a simplex*[12]*. The same holds for compact finite-dimensional polytopes with uniformly bounded number of vertices.*

Proof. ***For simplices.*** Consider the set $S := \bigcap_{m \in \mathbb{N}} S_m$ where $(S_m)_{m \in \mathbb{N}}$ is the given chain of simplices. Then for any non-degenerate homotheties α, β we have[13]

$$\alpha(S) \cap \beta(S) = \bigcap_{m \in \mathbb{N}} \alpha(S_m) \cap \beta(S_m) \tag{3}$$

where this is due to injectivity of α, β. However, this is again a chain of simplices and we find homotheties γ_m such that

$$\alpha(S) \cap \beta(S) = \bigcap_{m \in \mathbb{N}} \gamma_m(S_m). \tag{4}$$

[8] this definition is also given in [3].

[9] By a homothety we denote any map of the form $x \mapsto \lambda x + y$ acting on a vectorspace ($\lambda \geq 0$).

[10] A simplex S does not need to have 'finite-dimensional faces' (in the sense that there is some $s \in S$ in definition 2.2 such that $\deg_S s$ is a finite cardinal number). The reader is invited to think of such simplices (hint: consider the base of the Archimedean vector lattice $C[0,1]$ with the pointwise order).

[11] This condition is needed - otherwise any based Archimedian lattice space would have suprema for any totally ordered non-decreasing bounded net $x_\alpha \uparrow$ which is obviously false for $C[0,1]$.

[12] This is what was mentioned in remark 2.7.

[13] For degenerate homotheties there is nothing to prove as the $\alpha(S) \cap \beta(S)$ would be empty or a singleton.

The obvious goal is now to find a homothety γ satisfying $\gamma(S) = \bigcap_{m\in\mathbb{N}} \gamma_m(S_m)$. Let us distinguish two cases.

Case 1. If S is a singleton[14] there is nothing to prove (as $\alpha(S)\cap\beta(S)$ is either a singleton - in which case we choose the corresponding constant homothety - or empty). The same holds if $\alpha(S)\cap\beta(S)$ is a singleton or empty and S does not behave trivial.

Case 2. Thus the only interesting case is when $\dim \operatorname{lin} S, \dim \operatorname{lin}[\alpha(S)\cap\beta(S)] > 0$.

In this case due to sequential compactness of $S_0, \gamma_0(S_0)$ and non-trivial behavior of $\alpha(S)\cap\beta(S)$ and S the real number

$$b := \sup\{\lambda \geq 0 : \exists y \in V : \lambda(S) + y \subset \gamma_0(S_0)\} \tag{5}$$

must exist (otherwise the former sets $S_0, \gamma_0(S_0)$ would not be bounded[15] and thus could not be sequentially compact, respectively) This last argument uses the Hausdorff property. (Assume $b = \infty$. Then for any $x \in S - S$ there exists sequences $(y_m)_{m\in\mathbb{N}}$ and $(\lambda_m)_{m\in\mathbb{N}}$ with $y_m \in S, \lambda_m > 0, \lambda_m \to \infty$ as $m \to \infty$ such that $\operatorname{conv}\{y_m, y_m + \lambda_m x\} \subset \gamma_0(S_0)$. But as S is sequentially compact we find a convergent subsequence $(y_{m_j})_{j\in\mathbb{N}}$ convergent to $y^* \in S$. It is clear that then $y^* + \mathbb{R}^+ x \subset \gamma_0(S_0)$, but this contradicts sequential compactness of $\gamma_0(S_0)$ as it implies that $\mathbb{R}^+ x \subset N$ for any neighborhood of 0 contradicting V being Hausdorff.)

Thus the scaling constants λ_m of γ_m (that means $\gamma_m : V \to V, x \mapsto \lambda_m x + y_m$) are uniformly bounded by b as for $y \in V, m \in \mathbb{N}$ we have

$$\lambda(S_m) + y \subset \alpha(S)\cap\beta(S) \Rightarrow \lambda(S) + y \subset \gamma_0(S_0). \tag{6}$$

Moreover, we may assume w.l.o.g. that $0 \in S$[16]. We then notice that the translation parameters y_m all must lie in the sequentially compact set S_0 for $m \in \mathbb{N}$. Thus the parameters of the γ_m lie in the sequentially compact space $(\lambda_m, x_m) \in [0, b] \times S_0$ and thus have an accumulation point. This shows actually that there is a subsequence $(\gamma_{m_j})_j$ of the homotheties which converges pointwise. Now assume any accumulation point (λ^*, x^*) satisfies $\lambda^* = 0$. But in this case, $\beta(S)\cap\beta(S)$ shrinks down to a point (which was excluded) as the following identity shows

$$\bigcap_{m\geq m_0} \gamma_m(S_m) \subset \bigcap_{m\in\mathbb{N}} \gamma_m(S_0) = \{x\} \text{ for some } x \in V. \tag{7}$$

This identity can be proven by the following argument: Define for a sequentially compact set B and a vector $x \in V$ the number

$$L(B, x) := \sup\{\lambda \geq 0 : \exists y \in B : \operatorname{conv}\{x, x + \lambda y\} \subset B\} \tag{8}$$

(it is well-defined as B is sequentially compact). From this definition one observers that in equation (7) we must have

$$L\left(\bigcap_{m\in\mathbb{N}} \gamma_m(S_0), x\right) \leq \limsup_{m\in\mathbb{N}} \lambda_m L(S_0, x) = 0 \tag{9}$$

[14] Singleton means a set of the form $\{x\}$.

[15] Von-Neumann boundedness: a set $B \subset V$ is called *bounded* if for any neighborhood N of 0 there is $\xi > 0$ such that $B \subset \xi N$.

[16] The property being simplex is translation invariant.

which implies the set being singleton.

Thus we may assume $\limsup_{m\in\mathbb{N}} \lambda_m > 0$. This implies (by previous compactness argument) there exists a subsequence $((\lambda_{m_j}, x_{m_j}))_{j\in\mathbb{N}}$ converging to some pair (λ^*, x^*) with $\lambda^* > 0$. For any point $s \in S$ we now have $\gamma_{m_i}(s) \in \gamma_{m_j}(S_{m_j})$ for all $i, j \in \mathbb{N}$ with $i \geq j$ which implies by sequential compactness for the limit homothety γ^* of γ_{m_j} that $\gamma^*(s) \in \bigcap_{j\in\mathbb{N}} \gamma_{m_j}(S_{m_j}) = \alpha(S) \cap \beta(S)$ (here we use the property of the $\gamma_m(S_m)$ being a chain with respect to inclusion). Thus γ^* satisfies $\gamma^*(S) \subset \alpha(S) \cap \beta(S)$.

It is now clear that we may apply the exact same argument to the operators $\gamma_{m_j}^{-1}$ converging to γ^{*-1}. From symmetric argument (interchanging S_m and $\gamma_m(S_m)$ as well as γ_m and γ_m^{-1}, respectively) we see that γ^* is a bijection and

$$\alpha(S) \cap \beta(S) = \gamma^*(S). \tag{10}$$

Thus we are done.

For polytopes: For finite-dimensional compact polytopes the proof is slightly different. Consider such a chain of polytopes $(P_m)_{m\in\mathbb{N}}$ and define $P := \bigcap_{m\in\mathbb{N}} P_m$. Then let $\sup_{m\in\mathbb{N}} |\partial_0 P_m| =: k$ ($< \infty$ by condition). Consider the compact[17] space $[0,1]^k \times P_0^k$ equipped with the product topology. In this space the map $\eta : [0,1]^k \times P_0^k \to P_0$ which assigns the linear combination to a given pair $(\mu, x) \mapsto \sum_{i=1}^k \mu_i x_i$ is L-continuous (where $\mu = (\mu_1, \ldots, \mu_k) \in [0,1]^k$ and $x = (x_1, \ldots, x_k) \in P_0^k$). This can be seen from the following equation (we may calculate with the product metric which arises from the norm in the spanned vectorspace at this point as the product topology is precisely the one induced by any norm in finite-dimensional space)

$$\begin{aligned}
\|\eta(\mu, x) - \eta(\nu, y)\|_{\operatorname{lin} P_0} &= \left\| \sum_{i=1}^k \mu_i x_i - \sum_{i=1}^k \nu_i y_i \right\|_{\operatorname{lin} P_0} && (11)\\
&\leq \sum_{i=1}^k |\mu_i - \nu_i| \, \|x_i\|_{\operatorname{lin} P_0} + \sum_{i=1}^k |\nu_i| \, \|x_i - y_i\|_{\operatorname{lin} P_0} && (12)\\
&\leq \sup_{p\in P_0} \|p\|_{\operatorname{lin} P_0} \sum_{i=1}^k |\mu_i - \nu_i| + \sum_{i=1}^k \|x_i - y_i\|_{\operatorname{lin} P_0} && (13)\\
&\leq \max\left\{1, \sup_{p\in P_0} \|p\|_{\operatorname{lin} P_0}\right\} \|(\mu, x) - (\nu, y)\|_{\operatorname{lin}[0,1]^k \times P_0^k} && (14)
\end{aligned}$$

Now consider a point $p \in P$. Then there exists a sequence $((\mu_m, x_m))_{m\in\mathbb{N}}$ of parameters of convex combinations of p such that the entries of x_m contain the vertices P_m (this holds as $p \in P \subset P_m = \operatorname{conv} \partial_0 P_m$). Obviously, this sequence $((\mu_m, x_m))_{m\in\mathbb{N}}$ has an accumulation point (μ^*, x^*) which is also a convex combination as its underlying space is compact. Due to the shown continuity of the evaluation function η we have $\eta(\mu^*, x^*) = p$.

We thus showed that if X^* denotes the set of entries of x^* then $\operatorname{conv} X^* \supset P$.

The other direction is much faster. Obviously, the sequence $(x_m)_{m\in\mathbb{N}}$ is eventually in $P_{m_0}^k$ for $m_0 \in \mathbb{N}$. Thus $x^* \in P^k$ (as $P^k = \bigcap_{m\in\mathbb{N}} P_m^k$). Thus $X^* \subset P$ and we obtain the

[17]This is due to Tychonoff's telling us that any product of compact spaces is compact. Indeed, in finite-dimensional case this is obvious. In infinite-dimensional case it is equivalent to the axiom of choice.

desired equation conv $X^* = P$ (where X^* is a finite set). It is clear that the polytope P has at most $\limsup_{m\in\mathbb{N}} |\partial_0 P|$ vertices.

□

Another fact which shall be mentioned at this point (without proof) is the following

Lemma 2.9. *A face of a finite-dimensional compact simplex is itself a simplex (the same for polytopes).*

2.3 Irreducible and primitive operators

We give a definition on primitive operators and generalize the notion of a positive irreducible matrices or operator for our purposes

Definition 2.10 (primitive operator)**.** Let $A : V \to V$ be a linear operator on a finite-dimensional ordered vectorspace V admitting a closed cone. Then A is called *primitive* if $A \geq 0$ and there is some $m \in \mathbb{N}$ such that $A^m[V^+ \setminus \{0\}] \subset \mathrm{int}V^+$.

Definition 2.11 (ideal irreducible operator)**.** [18] Let V be an ordered vector space of finite dimension n with V^+ admitting polyhedral (compact) base B and let $A : V \to V$ be a linear operator. A is called *ideal irreducible*[19] if $A[F] \not\subset F$ for any k-dimensional face F of V^+ $(k \leq n-1)$.

Remark 2.12. Normally, definition 2.11 is given for $V = \mathbb{R}^n$ and the standard cone $\mathbb{R}^n_+$. An *irreducible matrix* is a matrix A for which there is no permutation matrix P such that $P^{-1}AP = \mathrm{diag}(A_1, A_2) + B$ where A_1, A_2 are non-empty square matrices and B is an upper triangular matrix. The notion of these definitions arise from graph theory as we will see in the last sections.

3 The theorem

The theorem of Perron-Frobenius can be stated for both types of operators (where we introduced the definition 2.11 because it is the most general definition which preserves the most facts of the theorem as originally stated for positive matrices).

Theorem 3.1 (Perron-Frobenius theorem for primitive operators)**.** *Let V be an n-dimensional Archimedean vector space admitting compact base B with generating cone*[20] *and T be some primitive operator. Then the following facts hold*

1. *There is a unique (up to positive multiples) positive eigenvector $v \in V^+$ which lies in the interior of V^+.*

2. *The corresponding eigenvalue λ of v is simple*[21] *and the strict maximum in modulus among all complex eigenvalues of T, thus $\lambda = \varrho(T)$.*

[18] This definition is mostly given in the form that the operator A is ideal irreducible if $A[I] \not\subset I$ for any non-trivial ideal $I \neq \{0\}$. However, our equivalent definition emphasizes the geometric structure.

[19] this definition is newly introduced

[20] This condition is needed, because otherwise we cannot state maximility in modulus.

[21] algebraic multiplicity one

Theorem 3.2 (Perron-Frobenius theorem for ideal irreducible operators)**.** *Let V be an n-dimensional ordered vector space with V^+ admitting polyhedral generating base B with q vertices and T a linear positive, ideal irreducible operator. Then the following facts hold*

1. *There is a unique (up to positive multiples) positive eigenvector $v \in V^+$ which lies in the interior of V^+.*
2. *The corresponding eigenvalue λ of v is simple and the greatest eigenvalue in modulus*[22] *among all complex eigenvalues of T, that is $\lambda = \varrho(T)$.*
3. *For any other complex eigenvalue λ' of maximal modulus we have $\lambda' = \varrho(T)\zeta$ where ζ is some root of unity with $\operatorname{ord}\zeta \leq q$. In the case B is a simplex*[23] *than $\sigma(T) \cap \{c \in \mathbb{C} : |c| = \lambda\} = \langle \zeta_{q'} \rangle \lambda$ for unique q' with $q' \leq q$.*[24] *Moreover, for all these λ' the algebraic and geometric multiplicity coincide.*

Example 3.3. Consider the operator (which is ideal irreducible)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi/q) & -\sin(2\pi/q) \\ 0 & \sin(2\pi/q) & \cos(2\pi/q) \end{pmatrix} \tag{15}$$

acting on $\mathbb{R}^3$ with a cone

$$K := \operatorname{conv}\left(\left\{ \begin{pmatrix} 1 \\ \cos(2\pi j/q) \\ \sin(2\pi j/q) \end{pmatrix} : j = 0, \ldots, q-1 \right\} \cup \{0\}\right). \tag{16}$$

We see that the eigenvalues of the complexified operator are $1, \boldsymbol{e}^{\pm 2\pi \boldsymbol{i}/q}$. The positive eigenvector $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ is an internal point of V^+.

4 Collecting some ideas

4.1 Proof of existence.

The main goal of this section is to establish Perron-Frobenius using the argument presented in lemma 2.8. The following lemma would directly follow from Brouwers fixed point theorem.

Lemma 4.1. *Let V be an n-dimensional vector space ($n \in \mathbb{N}$) with a cone V^+ admitting polyhedral base B (i.e. B is a compact polytope). Let T be a strictly positive operator (i.e. $\forall v \in V^+ \setminus \{0\} : T(v) > 0$). Define the equivalence relation $\sim := \{(u, v) \in V \setminus \{0\} : \exists \lambda > 0 : u = \lambda v\}$. Then the map $T|_{V^+ \setminus \{0\}}/\sim =: \chi$ (that is $\chi([x]_\sim) = [Tx]_\sim$) has a fixed point (i.e. T has a eigenvector in V^+).*

Remark 4.2. If T is regular the space $V \setminus \{0\}/\sim$ is isomorphic to S^{n-1}. For n even any continuous map from S^{n-1} to S^{n-1} admits a fixed point (corollary of Lefschetz fixed point theorem).

[22] it is not necessarily the strict maximum in modulus
[23] That is V is a lattice.
[24] here $\langle \zeta_{q'} \rangle$ means the multiplicative group generated by $\zeta_{q'} = \boldsymbol{e}^{2\pi \boldsymbol{i}/q'}$

Remark 4.3. Since T was defined to be strictly positive, it is an easy matter to verify that χ is well-defined. However, the statement about the eigenvector without this restriction since one may otherwise T has a positive eigenvector corresponding to the eigenvalue 0.

Remark 4.4. In our further considerations B is a generating polyhedral base, specialized when needed to a simplicial (lattice) base.

During the proof of this lemma we also want to establish some kind of characterization. The (short) proof of the lemma will be a minor part of that and will be given later. At first we impose some more general considerations.

4.2 The map ρ

Definition 4.5. Introduce the map $\rho : B \to B$ by

$$\rho = \phi^{-1}\chi\phi \tag{17}$$

where ϕ is the *natural projection map* $\phi : B \to V^+/\sim$ with $\phi(b) := [b]_\sim$ for $b \in B$.

Remark 4.6. By definition of ρ, T admits a positive eigenvector if and only if ρ has a fixed point.

Now we are ready for the following important

Lemma 4.7. *The map ρ has the following three properties*

1. *ρ is continuous.*
2. *For any set $A \subset B$ we have* $\operatorname{conv} \rho[A] = \rho[\operatorname{conv} A]$.[25]
3. *If $P \subset B$ is a compact convex polytope such that* $\dim \operatorname{aff} P = \dim \operatorname{aff} \rho[P]$ *then ρ preserves* deg, *i.e.* $\deg_P p = \deg_{\rho[P]} \rho(p)$ *(for $p \in P$).*

Proof.

1. Clearly, as V is a finite-dimensional Euclidean space, we have T being continuous. Now taking the quotient topology and conjugating with the natural projection map ϕ does not destroy anything.

2. Let us pick a set $A \subset B$. By definition any point $a \in \operatorname{conv} A$ admits a representation $a = \sum_{i=1}^{l} \alpha_i a_i$ with $a_i \in A, \alpha_i > 0$ for all $i = 1, \dots, l$ and $\sum_{i=1}^{l} \alpha_i = 1$. Let f be a strictly positive linear functional such that $f^{-1}\{1\} \cap V^+ = B$. We then have by

$$\rho(a) = \frac{\sum_{i=1}^{l} \alpha_i T(a_i)}{f\left(\sum_{i=1}^{l} \alpha_i T(a_i)\right)} = \frac{\sum_{i=1}^{l} \alpha_i \lambda_i \rho(a_i)}{f\left(\sum_{i=1}^{l} \alpha_i T(a_i)\right)} \tag{18}$$

that $\rho(a) \in \operatorname{conv} \rho[A]$ as we may obtain a convex combination of $\rho(a)$ in $\rho(a_i)$ $(i = 1, \dots, l)$ by considering the fact that there exist (unique) $\lambda_i > 0$ such that $\lambda_i \rho(a_i) = T(a_i)$ $(i = 1, \dots, l)$. Thus $\operatorname{conv} \rho[A] \supset \rho[\operatorname{conv} A]$.

[25] The analogue identity $\operatorname{aff} \rho[A] = \rho[\operatorname{aff} A]$ does not hold in general.

For the other inclusion consider $\tilde{a} = \sum_{i=1}^{l} \alpha_i \rho(a_i) \in \operatorname{conv} \rho[A]$ for $a_i \in A, \alpha_i > 0$ for all $i = 1, \ldots, l$ and $\sum_{i=1}^{l} \alpha_i = 1$. By taking the λ_i $(i = 1, \ldots, n)$ as in the previous case, we obtain that

$$\tilde{a} = \sum_{i=1}^{l} \frac{\alpha_i}{\lambda_i} T(a_i) = T\left(\sum_{i=1}^{l} \frac{\alpha_i}{\lambda_i} a_i\right) = \rho\left(\frac{\sum_{i=1}^{l} \frac{\alpha_i}{\lambda_i} a_i}{f\left(\sum_{i=1}^{l} \frac{\alpha_i}{\lambda_i} a_i\right)}\right). \tag{19}$$

Thus we obtain the desired equality $\operatorname{conv} \rho[A] = \rho[\operatorname{conv} A]$.

3. We will show now: a subset $C \subset P$ is linearly (affinely) independent if and only if $\rho[C]$ is linearly independent (linearly and affinely independence are equivalent for such C as $0 \notin P$).

   $\Leftarrow$**:** So let us suppose C is linearly dependent, i.e. for some numbering of the elements of C by $c_1, \ldots, c_p, \overline{c}_1, \ldots, \overline{c}_q$ we have $\delta_1, \ldots, \delta_p, \gamma_1, \ldots, \gamma_q \geq 0$ with $\sum_{i=1}^{p} \delta_i = \sum_{i=1}^{q} \gamma_i = 1$ (this condition is no restriction as $f(0) = 0$ and from $f(c_i) = f(\overline{c}_j) = 1$ for all i, j) such that

$$\sum_{i=1}^{p} \delta_i c_i = \sum_{i=1}^{q} \gamma_i \overline{c}_i. \tag{20}$$

   Plugging the c_i and $\overline{c}_i$ in the role of the a_i in equation (18) one obtains that $\rho[C]$ is affinely dependent.

   $\Rightarrow$**:** For the other direction we will need the dimension condition.

   Let $k := \dim \operatorname{aff} P$. Suppose there is a set C of linearly independent points such that $\rho[C]$ is linearly dependent. Clearly, $|C| \leq k+1$ as $C \subset P$ and $\dim \operatorname{aff} P = k$. We may extend C to a $(k+1)$-element set $\tilde{C}$ such that $\rho[\tilde{C}]$ is still linearly dependent. If we now pick some point $p \in P$ we get a unique representation of it as $p = \sum_{i=1}^{k+1} \epsilon_i \tilde{c}_i$ (with $\sum_{i=1}^{k+1} \epsilon_i = 1$). Now, shifting the negative ϵ_i to the side of p and applying equation 18 we get that $\rho(p) \in \operatorname{aff} \rho[\tilde{C}]$ a contradiction to the condition $\dim \operatorname{aff} \rho[P] = k$ (as $p \in P$ was arbitrary).

   As we now have shown that affinely independence and lying in the interior of a convex set are preserved (to see this consider equations 18 and 19), we may deduce the desired result (as the definition of deg only works with these two).

$\square$

Remark 4.8. In the case where T does not preserve the dimension of P, deg may increase and decrease. This can be seen from a tetrahedron which degrades to a triangle (i.e. the degree of one vertex increases) and a tetrahedron which degrades to a line (where two antipodal edges degrade to points - here the degree of the internal points of theses edges decreases).

4.3 The ρ-invariant polytope (or simplex) $\tilde{B}$

The first thing we do is to simplify the problem a bit by the following consideration. Define

$$\tilde{B} := \bigcap_{m \in \mathbb{N}} \rho^m[B]. \tag{21}$$

Then from lemma 2.8 we know that $\tilde{B}$ is a compact polytope (or simplex if B was one, respectively). Note that if ρ has a fixed point in B then it has to be in $\tilde{B}$.

The degraded case. If $\tilde{B}$ has lower dimension than B we may consider the 'projected' problem in the ordered subspace $\operatorname{lin} \tilde{B} \subset \bigcap_{i \in \mathbb{N}} \operatorname{im} T^i$ equipped with the order induced by the cone $V^+ \cap \operatorname{lin} \tilde{B}$ on which T acts as a bijection. Maybe, we shall mention at this point that the 'initial' case where V is one-dimensional (if $\dim V = 0$ we had no strictly positive operator and no base B) is trivial.

4.4 The case T being regular.

Thus let us assume that $\dim \operatorname{lin} \tilde{B} = n \geq 1$ (actually this implies T being regular). Then it is immediately clear that $\rho[\partial_0 \tilde{B}] = \partial_0 \tilde{B}$ by considering the fact that $\tilde{B}$ is fixed under ρ and ρ preserves deg as T is regular (cf. lemma 4.7 part 3). Thus ρ acts on the set of vertices of $\tilde{B}$ as a finite permutation which has some representation of the form $\rho|_{\partial_0 \tilde{B}} = c_1 \circ \cdots \circ c_l$ (where c_i for $i = 1, \ldots, l$ are disjoint cycles). Let us denote the vertices of $\tilde{B}$ by $v_1, \ldots, v_m$ and define τ implicitly by $\rho|_{\partial_0 \tilde{B}}(v_i) = v_{\tau(i)}$ for $i = 1, \ldots, m$.

From the following we will see that the number of fixed points of ρ is at least $l \geq 1$ and thus establish lemma 4.1.

The case τ being itself a cycle. Let us now consider the case that $\rho|_{\partial_0 \tilde{B}}$ is itself a cycle (which can be trivial). We then have some positive $\lambda_1, \ldots, \lambda_m$ such that $T(v_i) = \lambda_i v_{\tau(i)}$ for $(i = 1, \ldots, m)$. It is now an easy matter to solve the arising equation (for an eigenvector of T in $\tilde{B}$)

$$T\left(\sum_{i=1}^{m} \mu_i v_i\right) = \sum_{i=1}^{m} \mu_i \lambda_i v_{\tau(i)} = \lambda \sum_{i=1}^{m} \mu_i v_i \tag{22}$$

from which one obtains $\frac{\mu_i}{\mu_{\tau(i)}} = \frac{\lambda}{\lambda_i}$ $(i = 1, \ldots, m)$, which obviously has the solution $\lambda = \sqrt[m]{\prod_{i=1}^{m} \lambda_i}$ and $\mu_{\tau^k(1)} = \frac{1}{\lambda^k} \prod_{i=0}^{k-1} \lambda_{\tau^i(1)} \mu_1$ (all solutions $(\mu_1, \ldots, \mu_m)$ are a multiple of each other). Thus ρ in this special case has a unique fixed point in the interior of $\tilde{B}$ (and thus in the interior of V^+).

Remark 4.9. In this case the operator T^m coincides with the homothety $v \mapsto \lambda^m v$. From this it follows that any eigenvalue must be of the form $\lambda\zeta$ where ζ is an m-th root of unity. If $m = n$ we may explicitly calculate the eigenvalues of T by taking $\partial_0 \tilde{B}$ as the representation base and setting w.l.o.g. the map τ to $i \mapsto i + 1 \pmod{n}$:

$$\mathcal{M}^{\partial_0 \tilde{B}}_{\partial_0 \tilde{B}}(T) = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{23}$$

This yields $\chi_T(\mu) = (-1)^n(\mu^n - \lambda^n)$ from which one deduces that the eigenvalues of T are exactly those of the form $\lambda\zeta$ where ζ is any n-th root of unity.

The general case. Let us now reconsider the case where $\tau = c_1 \circ \cdots \circ c_l$. Then for each c_i we may apply the observations from the last paragraph in the corresponding subspace. We thus may deduce that there exists a unique eigenvector v in V^+ (up to positive multiples) if and only if τ is a single cycle.

5 Some final lemmas

We need to recall the following elementary results from linear algebra

Lemma 5.1 (characterization of uniform boundedness of the powers of an operator)**.** *Let $A : V \to V$ some linear operator where $\dim V = n < \infty$. Then the powers A^m $(m \in \mathbb{N})$ are uniformly bounded if and only if all complex eigenvalues μ of A satisfy $|\mu| \leq 1$ and if $|\mu| = 1$ then the algebraic multiplicity and the geometric multiplicity coincide.*

Proof. By an appropriate choice of the (complex) base we may assume A can be represented as some Jordan matrix

$$\mathcal{M}_B^B(A) = \operatorname{diag}(J_1, \ldots, J_l) \tag{24}$$

such that J_i $(i = 1, \ldots, l)$ are Jordan blocks. We than have

$$\mathcal{M}_B^B(A^k) = \operatorname{diag}(J_1^k, \ldots, J_l^k) \tag{25}$$

for $k \in \mathbb{N}$ from which we see that no complex eigenvalue can have larger modulus than one (otherwise A^m would not be bounded). Moreover if some eigenvalue μ with $|\mu| = 1$ has a non-trivial Jordan block J then by

$$J^k = (\mu I + N)^k = \sum_{i=0}^{k} \binom{k}{i} \mu^i N^{k-i} \tag{26}$$

we see that A^m would also not be uniformly bounded (here I denotes the identity matrix and N is the nilpotent matrix with ones on the secondary diagonal). Thus we are done (the other direction is trivial). □

Lemma 5.2 (internal eigenvector)**.** *Suppose T has an internal point v of V^+ as an eigenvector. Then the corresponding eigenvalue $\lambda > 0$ is the largest eigenvalue of T in modulus among all complex eigenvalues of T. Moreover, if λ' is a complex eigenvalue of T such that $|\lambda'| = |\lambda|$ then the algebraic and geometric multiplicity of λ' coincide.*

Proof. Consider the strictly positive operator $S := T/\lambda$. We prove that the powers of S are uniformly bounded and apply the previous lemma. Let $u = u_1 - u_2 \in V$ where $u_1, u_2 \in V^+$ (this is possible because V^+ is generating). From this representation it is clear that it suffices to prove that the powers of S are uniformly bounded on V^+. But this follows from the fact that $S[0, v] \subset [0, S(v)] = [0, v]$ (as S is positive operator) because $[0, v]$ is also generating (in the sense that for any $u \in V$ there are $u_1, u_2 \in [0, v], \mu_1, \mu_2 \geq 0$ such that $v = \mu_1 u_1 - \mu_2 u_2$) as v is an internal point of V (and thus an order unit). □

Remark 5.3. However, this lemma implies that any eigenvector in the interior of V^+ must have corresponding eigenvalue $\lambda = \varrho(A)$.

Lemma 5.4 (symmetry property of the spectrum $\sigma(T)$)**.** *Let V have polyhedral cone. Suppose T has an internal point v of V^+ as eigenvector (corresponding eigenvalue λ). Then any other eigenvalue λ' of maximal modulus (i.e. by previous lemma $|\lambda'| = |\lambda|$) is of the form $\lambda\zeta$ where ζ is a root of unity of order smaller than the number q of vertices of the cone.*

Proof. We already proved a stronger fact for simplicial (lattice cones) which is stated in remark 4.9.

By the previous lemma we know that all Jordan chains of such λ' behave trivial. W.l.o.g. we may assume that $\lambda = 1$ (otherwise let us define $S := T/\lambda$ as previously and do the argument on S). Then, for some $\varepsilon > 0$ we obviously find a number $m \in \mathbb{N}$ such that $|\lambda'^m - 1| < \varepsilon$ holds for all λ' of modulus one. From this we see that there is some $m \in \mathbb{N}$ such that the pair of vectors $(v_1, v_2) = (\Re\tilde{v}, \Im\tilde{v})$ (where $\tilde{v}, \overline{\tilde{v}}$ are the associated complex eigenvectors of $\lambda', \overline{\lambda'}$) is nearly left invariant under S^m, i.e. for some $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\|v_i - S^m(v_i)\| < \varepsilon$ $(i = 1, 2)$.

Let

$$U := \mathrm{lin}(\{v_1, v_2 : v_1 \pm \boldsymbol{i} v_2 \text{ associated complex eigenvectors of } \lambda', \overline{\lambda'} \text{ with } |\lambda'| = 1\} \cup \mathrm{Eig}_\lambda(T)). \tag{27}$$

We then have $\tilde{B} \supset U \cap B$ from the last property. But from this (and from the fact that v is an internal point of V^+) we get that $S^{\operatorname{ord}\tau}$ acts as the identity on U and thus all λ' must be roots of unity of order smaller than the number of vertices q of V^+ (from remark 4.9). □

Remark 5.5. This does not hold for non polyhedral cones. To see this consider the ordered vector space $\mathbb{R}^3$ with the cone $K := \{x \in \mathbb{R}^3 : x_1 \geq \sqrt{x_2^2 + x_3^2}\}$ and the operator given by the matrix

$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \tag{28}$$

where $a + b\imath = e^{2\pi \boldsymbol{i} r}$ for some $r \in \mathbb{R} \setminus \mathbb{Q}$.

6 The proof

We have collected all facts for the proof of the two theorems. Let us reassemble them.

The first thing which is of importance is that in both cases there cannot be an eigenvector on the boundary of the cone (simply by the properties of T in the theorems). Thus we know by lemma 4.1 that there exists a positive eigenvector in the interior of V^+.

For non-polyhedral cone in theorem 3.1 needs some easy approximation argument. Assume B is the base of a non-polyhedral cone. We then observe that in the case where T is primitive there is a number $m \in \mathbb{N}$ such that for the induced map ρ we have

$$\mathrm{dist}(\rho^m[B], \partial_{n-2}B) =: \varepsilon > 0 \tag{29}$$

as the sets are compact[26]. We then may build a polytope $P := \operatorname{conv} S$ ($S \subset \partial_{n-2}B$ finite) such that any $(n-2)$-simplex in $\partial_{n-2}P$ has diameter at most $\varepsilon/2$ (this is not done explicitly here, but is based on precompactness arguments).

[26] Where $\mathrm{dist}(A, B) := \inf_{a \in A, b \in B} \|a - b\|$ for compact sets A, B.

Then it is evident that $\operatorname{dist}(\rho^m[B], \partial_{n-2}P) > \varepsilon/2$ using triangle inequality. To see this, consider a point $p \in \partial_{n-2}P$ which lies in the $(n-2)$-dimensional face F of P. Moreover, let $b \in \rho^m[B]$ and $v \in \partial_0 B$ be arbitrary. Then $\|p - n\| \geq |\, \|p - v\| - \|v - b\|\, | > \varepsilon/2$ as $v \in \partial_{n-2}B$ and $\operatorname{diam}(F) < \varepsilon/2$. We thus see that $\operatorname{dist}(\rho^m[B], \partial_{n-2}P) > \varepsilon/2$ is true implying $\operatorname{dist}(\rho^m[P], \partial_{n-2}P)$ (as $\rho^m[P] \subset \rho^m[B]$). This implies that T is primitive with respect to the order induced by the polyhedral cone generated by P. Thus we also happy and done in this case.

Algebraic multiplicity one and unicity. Now, let us prove that the algebraic multiplicity of the corresponding eigenvalue λ is one. Suppose it would be larger than one. Then by lemma 5.2 we woulds have $\dim \operatorname{Eig}_\lambda(T) > 1$. But then $\operatorname{Eig}_\lambda(T)$ has non-empty intersection with the boundary of B. To see this take another eigenvector $v' \in \operatorname{Eig}_\lambda(T)$ which is linearly independent from v the line $v + \mathbb{R}v'$ must intersect with the boundary of V^+ otherwise V would not be Archimedean and would thus not admit a base). We thus may also deduce that there is no other positive eigenvector as all internal eigenvectors v' (with respect to V^+) must satisfy $Tv' = \lambda v'$.

Strict maximum. Let us prove that λ is the strict maximum in modulus in $\sigma(T)$ in the case T is primitive (theorem 3.1). Otherwise, from lemma 5.4 we know that there is a power of T^m which has no other eigenvalues of modulus λ^m (e.g. take $T^{q!}$). But this power is also primitive. Thus λ^m must have algebraic multiplicity one. From this it follows that the number of eigenvalues of modulus λ of T is also one.

Symmetry properties of $\sigma(T)$. We are left with the 3rd point of theorem 3.2. But this was already proven by lemma 5.4 and remark 4.9.

7 Connection to weighted adjacency matrices of graphs

The theorem of Perron-Frobenius has many applications. One which is of significant importance is its usage in graph theory.

Definition 7.1 (weighted adjacency matrix, induced graph)**.** Let $G = (V, E)$ a directed (or undirected) graph. Then a weight on the edges of G is some map $\omega : E \to \mathbb{R}^+$. The weighted adjacency matrix of G is then a non-negative matrix $\mathbb{R}^{n\times n}$ ($n = |V|$) which has as entries the values of ω and zeros where no edges exist (via some numbering of the vertices). On the other hand, given any non-negative A one defines the induced weighted graph (up to isomorphism) as the pair (G, ω) such that A is a weighted adjacency matrix.

We then have the following connections

- A non-negative matrix A is periodic with period p (that is $p = \gcd\{n \in \mathbb{N} : \exists i \in [n] : (A^n)_{ii} > 0\}$) if and only if for the induced graph G we have $p = \gcd\{|C| : C \text{ cycle in } G\}$.

- A non-negative matrix A is irreducible (that is there is no permutation matrix P and non-vanishing square matrices A, B such that $P^{-1}AP = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$[27]) if and only if the induced graph G is strongly connected (that is any two vertices are joined by a path - or topologically: the graph as topological (even metric) space has only one path component).

- A non-negative matrix A is primitive if and only if A is aperiodic (that is $p = 1$) and irreducible (the induced graph is aperiodic and strongly connected).

- A graph G is periodic with period p if and only if there p is the maximal natural number such that there exists some partition $\{V_1, \ldots, V_p\} =: \mathcal{P}$ of its vertices such that any edge starting in V_i ends in V_{i+1} (mod p)[28].

Note the following lemma on periodic matrices, which is very similar (but even stronger) to the spectral properties of ideal irreducible operators outlined in our proof (whereas these are satisfied in a more general situation as they do not only work for lattice cones). For lattice cones, the notion of an ideal irreducible and an irreducible matrix coincide.

Lemma 7.2. *Let A be a periodic matrix with the representation*

$$A = \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_{p-1} \\ A_p & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{30}$$

where $A_i \in \mathbb{C}^{n_i \times n_{i+1}}$ (i mod p) (where n_i can be interpreted as the cardinalities of V_i where $\{V_i : i = 1, \ldots, p\}$ is the maximal periodic partition). Then the set of pairs of eigenvalues and corresponding eigenvectors (λ, v) are invariant under the following actions (let $v^{\mathrm{T}} = (v_1, \ldots, v_p)$ with $v_i \in \mathbb{C}^{n_i}$):

$$\phi_\zeta(\lambda, v) = \left(\zeta\lambda, \begin{pmatrix} v_1 \\ \zeta v_2 \\ \cdots \\ \zeta^{p-1} v_p \end{pmatrix} \right). \tag{31}$$

Here $\zeta \in \mathbb{C}$ satisfies $\zeta^p = 1$. Moreover, the Jordan submatrices corresponding to the eigenvalues $\zeta\lambda$ (ζ is p-th root of unity) all have the same structure.

Proof. As one directly verifies we have for $v \in V, \lambda \in \mathbb{C}$ with $Av = \lambda v$ that

$$A\phi_{\zeta,2}(\lambda, v) = \zeta\lambda\phi_{\zeta,2}(\lambda, v) = \phi_{\zeta,1}(\lambda, v)\phi_{\zeta,2}(\lambda, v) \tag{32}$$

where $\phi_{\zeta,i}$ denote the components of ϕ_ζ.

[27] irreducible is defined for general matrices
[28] such partition is unique up to cyclic permutation

We obtain a similar fact if $u, v \in V$ are some consecutive vectors in a Jordan chain of A of the eigenvalue λ (that is $Au = \lambda u + v$):

$$A\phi_{\zeta,2}(\lambda, u) = \zeta(\lambda\phi_{\zeta,2}(\lambda, u) + \phi_{\zeta,2}(\lambda, v)) \tag{33}$$

From this one obtains that some Jordan chain $v_1, \ldots, v_k$ satisfying $Av_1 = \lambda v_1$ and $Av_j = \lambda v_j + v_{j-1}$ (for $1 < j \leq k$) is mapped to the chain $\tilde{v}_j = \zeta^{k-j}\phi_{\zeta,2}(\lambda, v_j)$ $(1 \leq j \leq k)$. $\square$

Remark 7.3. Thus the spectrum $\sigma(A)$ is invariant under multiplication with ζ (p-th root of unity).

A little exercise. At this point, we want the reader to prove the following simple fact about graphs using Perron-Frobenius theorem.

Task. Let $G = (V, E)$ be a graph of period p such that $V = \bigcup_{i=1}^{p} V_i$ (V_i,V_j disjoint for $i \neq j$) and for any $v \in V_i$ we have $d(v) = c_i$ $(i = 1, \ldots, p)$ where d denotes the degree[29] of v. Prove that $\rho(A) = \sqrt[p]{c_1 \cdots c_p}$ (where A denotes the adjacency matrix of G)[30].

8 An appetizer at the end: The spectral radius of the infinite d-regular tree

At the end we want to prove a familiar fact in graph theory which makes a statement about the Perron-Frobenius eigenvalue of trees of uniformly bounded degree.

The example. It is a well known theorem due to Nilli that any d-regular connected graph G of diameter m has its biggest non-trivial eigenvalue greater than $2\sqrt{d-1}(1 - 1/m)$. We want to prove another well known fancy result.

Lemma 8.1. *Let $G = (V, E)$ be a connected tree with $\max_{v \in V} d(v) \leq d$. Then for the adjacency matrix M of G we have $\varrho(M) \leq 2\sqrt{d-1}$.*

The proof of this result can be done elementary (or using extremal graph theory). We will give an elementary proof.

Proof. Let us extend G to the infinite d-regular tree G'. Then the adjacency matrix M' of G' can be interpreted as an operator on $\boldsymbol{l}^2$. As the operators M and M' are self-adjoint it holds that $\|M\|_2 = \varrho(M)$ and $\|M'\|_2 = \varrho(M')$.

Moreover, as M is an initial square submatrix of M' we have $\varrho(M) = \|M\|_2 \leq \|M'\|_2 = \varrho(M')$. Hence, we just need to show that $\rho(M') = 2\sqrt{d-1}$.

To see this, consider the powers of M and its behavior on the main diagonal. Basically, $(M^m)_{ii}$ counts the paths of length m starting and ending at vertex i. Let us denote this number by β_m $(m \in \mathbb{N})$.

We will calculate β_m with some help of combinatorics.

[29] that is the number of outgoing edges of v

[30] One often calls this the spectral radius of the graph G

Let us denote by β'_m the number of paths of length m from vertix i to i which do not return to i in between.

Such paths can be parameterized by two sequences $(a_j)_{j=1}^l$ and $(b_j)_{j=1}^l$ satisfying $\sum_{j=1}^m a_j > \sum_{j=1}^m b_j$ for $m < l$ and $\sum_{j=1}^l a_j = \sum_{j=1}^l b_j$.

At first we walk from i along some path of length a_1. Then we backtrack this path by b_1 steps. Again we start a new path of length a_2 (which does not backtrack) etc.

During each step in the sequence (a_j) we may choose from $d-1$ possible directions (disregarding the beginning where we may choose from d directions).

It is clear that for m odd the number of circles starting and ending at i is zero. So let $m = 2k$ ($k \in \mathbb{N}$).

We already constructed a (not very formal) bijection between the set of such sequences (a_j) and (b_j) times some set of the size $d(d-1)^{k-1}$ to the number of such paths β'_{2k}. Let us denote the former number by C'_k. It is easy to see that this number is exactly the same as the number C_{k-1} of Dyck-paths of length $2(k-1)$ (that are paths on the discrete positive axis of length $2(k-1)$ starting and ending at 0). But thus number is well known by its recurrence relation

$$C_{k+1} = \sum_{i=0}^{k} C_i C_{k-i}. \tag{34}$$

This shows us that C_k are actually the Catalan numbers, thus $C_n = \frac{1}{n+1}\binom{2n}{n}$. From this we get for $k > 0$

$$\beta'_{2k} = d(d-1)^{k-1}\frac{1}{k}\binom{2(k-1)}{k-1} \tag{35}$$

Now we may compute β_{2k} by the fact that any returning path from vertex i to i can be uniquely partitioned in the subpaths described by β'_{2k}.

$$\beta_{2k} = \sum_{k_1+\ldots+k_l=k, k_i>0} \beta'_{2k_1}\ldots\beta'_{2k_l} \tag{36}$$

(Here the sum is taken over all positive ordered partitions, i.e. permuted partitions are not considered equal.) In this case we are lucky to know z-transformation. We thus may compute the generating sequence (from the recursion formula of C_n) of β'_{2k} which we will denote by α'_k:

$$A'(z) = \sum_{k=0}^{\infty} \alpha'_k z^k = d\left(\frac{1-\sqrt{1-4(d-1)z}}{2(d-1)}\right) \tag{37}$$

(this can easily be derived from the generating function of the Catalan numbers C_n which is $C(z) = \frac{1-\sqrt{1-4z}}{2z}$). Now from this last equation we get the convergence radius of $A'(z)$ in 0 which is $\varrho_{A'} = \frac{1}{4(d-1)}$. Simple argument shows that this is also the convergence radius of $A(z) = \sum_{k\in\mathbb{N}} a_k z^k$ which is $\frac{1}{\limsup_{k\in\mathbb{N}} \sqrt[k]{a_k}}$.

Let us now consider any initial square submatrix $\tilde{M} \in \mathbb{R}^{n\times n}$ of M'. We then have $\lim_{n\to\infty} \varrho(\tilde{M}) = \varrho(M')$ (which is true as we are in a separable Hilbert space). Moreover, the diagonal entries of $\tilde{M}^{2k}$ can at most be β_{2k}. Thus we get

$$\varrho(\tilde{M}) = \limsup_{k\to\infty} \sqrt[2k]{|\operatorname{tr} \tilde{M}^{2k}|} \leq \limsup_{k\to\infty} \sqrt[2k]{\beta_{2k}} = 2\sqrt{d-1} \tag{38}$$

This last we already have computed as $\varrho_A = \frac{1}{\limsup_{k\in\mathbb{N}} \sqrt[k]{a_k}}$. This finishes the proof. $\square$

Remarks. Some papers that inspired me are given in the references.

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