Definition 0.1 (Algebraic winding number)**.** Let $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ be a continuous curve ($a, b \in \mathbb{R}, a \leq b$). Then there exists a unique (modulo 2π) continuous function $\arg\gamma$ such that

$$\gamma = |\gamma| \exp(\boldsymbol{i} \arg\gamma) = |\gamma| \operatorname{sgn}\gamma \tag{1}$$

The *algebraic winding number* of γ with respect to the origin 0 is then defined as

$$\omega_0(\gamma) := \arg\gamma|_a^b = \arg\gamma(b) - \arg\gamma(a). \tag{2}$$

Remark 0.2. If $\operatorname{sgn}\gamma : I \to \mathbb{C}$ is a rectifiable function, one has the identity

$$\omega_0(\gamma) = \frac{1}{2\pi\boldsymbol{i}} \int_{\operatorname{sgn}\gamma} \frac{1}{z} \,\mathrm{d}\, z. \tag{3}$$

A natural idea is now to store the rotational information (i.e. the information of the function $\arg\gamma$) in another function which then can be analyzed using generalized Sturm chains and Cauchy indices. However, the 'function' which is introduced may have some gaps in its domain.

Definition 0.3 (Cauchy function)**.** Let $\gamma : I \to \mathbb{C}\setminus\{0\}$ be a continuous curve for a compact interval I. The Cauchy *function* $\operatorname{Cf}(\gamma)$ of γ is then defined as

$$\operatorname{Cf}(\gamma) : I \to \mathbb{R} \setminus \{x \in I : \operatorname{Re}\gamma(x) = 0\}, x \mapsto \frac{\operatorname{Im}\gamma(x)}{\operatorname{Re}\gamma(x)}. \tag{4}$$

Any partially defined function which can be obtained in such manner is called a Cauchy *function* over I.

Now, one needs to redefine the operations standard arithmetic operations for such partial functions.

Definition 0.4 (Cauchy function inverse)**.** Let f be a Cauchy be a function on the compact interval I corresponding to the path $\gamma : I \to \mathbb{C} \setminus \{0\}$. Then the Cauchy function $1/f$ is defined as

$$1/f : \operatorname{dom}(f) \setminus f^{-1}\{0\} \cup \bigcap_{n\in\mathbb{N}} f^{-1}[\mathbb{R} \setminus (-n,n)] \to \mathbb{R}, x \mapsto \lim_{\xi\to x} \frac{1}{f(x)} \tag{5}$$

which gives the Cauchy-function corresponding to the path $\boldsymbol{i}\bar{\gamma}$.

Definition 0.5 (Cauchy function of the product path)**.** Let f, g be Cauchy be a function on the compact interval I corresponding to paths $\gamma, \eta : I \to \mathbb{C} \setminus \{0\}$. Then the Cauchy function $f \star g = \frac{f+g}{1-fg}$ is defined as ... which gives the Cauchy-function corresponding to the path product $\gamma\eta$.

Definition 0.6 (maximal rotational interval, rotational domain of a partial function)**.** Let $f : J \to \mathbb{R}$ be continuous where $J \subset I$ is open in the subspace I for a compact interval $I \subset \mathbb{R}$. Let $\{I_n\}_n$ be the partition of J into maximal (with respect to inclusion) intervals (which exists as J lies relatively open in I). An interval $I_i \in \{I_n\}_n$ is called *rotational* if it is not an interval of the form (a,b) ($a, b \in I, a < b$) which satisfies

$$\lim_{\xi\downarrow a} f(\xi) = \lim_{\xi\uparrow b} f(\xi) = \pm\infty. \tag{6}$$

The *rotational domain* of f is defined as $\operatorname{conv}\left(\bigcup\{\bar{I}_i\}_i\right)$ where $\{\bar{I}_i\}_i$ are all rotational intervals among $\{I_n\}_n$.

Lemma 0.7 (Characterization of CAUCHY functions). *Let $I \subset \mathbb{R}$ be a compact interval and $J \subset I$ be a an open set in the subspace topology I. Then for a continuous function $f : J \to \mathbb{R}$ the following two are equivalent*

1. *The function f is a* CAUCHY *function over I.*
2. *Let $\{I_n\}_{n\in\mathbb{N}}$ be the partition of J into maximal (with respect to inclusion) intervals (which exists as J lies relatively open in I). Then there are only finitely many rotational intervals $\{\bar{I}_i\}_{i=0}^{l}$ among them.*
3. *There exist continuous functions $g, h : I \to \mathbb{R}$ with no common zero $\xi \in I$ with $g(\xi) = h(\xi)$ such that $\frac{g}{h}\big|_{\mathrm{dom}(f)} = f$ (especially the expression on the left side is well-defined on $\mathrm{dom}(f)$).*

Proof. $1 \Rightarrow 2$. Let f be a CAUCHY function corresponding to the path $\gamma : I \to \mathbb{C} \setminus \{0\}$. Now assume there would be infinitely many maximal rotational intervals. Then at most two of them are half-open intervals (at most one at each bound a and b). Thus let $\{\bar{I}_i\}_{i\in\mathbb{N}} \subset \mathcal{P}(\mathrm{dom}(f)) = I \setminus \{x \in I : \mathrm{Re}\,\gamma(x) = 0\}$ be all maximal open rotational intervals. Let $\{L_i\} = \{(\hat{a}_i, \hat{b}_i)\} \subset \{\bar{I}_i\}_{i\in\mathbb{N}}$ be the set of maximal open rotational intervals for which

$$\lim_{\xi\downarrow\hat{a}_i} f(\xi) = -\infty \vee \lim_{\xi\uparrow\hat{b}_i} f(\xi) = \infty \tag{7}$$

and $\{R_i\} = \{(\check{a}_i, \check{b}_i)\} \subset \{\tilde{I}_i\}_{i\in\mathbb{N}}$ such that

$$\lim_{\xi\downarrow\hat{a}_i} f(\xi) = \infty \vee \lim_{\xi\uparrow\hat{b}_i} f(\xi) = -\infty. \tag{8}$$

(The '$\vee$' is just needed to include the half-open intervals at the boundary of I which might exist.) It is then clear that in each interval L_i which is not half-open the curves γ and $\mathrm{sgn}\,\gamma$ rotates the angle π around 0 staying in one of the half planes $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ or $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$, counterclockwise. In the intervals R_i which is not half-open happens the same but the angle of rotation is $-\pi$ (clockwise rotation). Thus we obtain for the unique (modulo 2π) continuous function $\arg\gamma$ (defined by $\exp(\boldsymbol{i}\arg\gamma) = \mathrm{sgn}\,\gamma$) an infinite monotone sequence $(\xi_n)_{n\in\mathbb{N}} \subset I$ of distinct boundary points of these intervals such that

$$|\arg\gamma(\xi_i) - \arg\gamma(\xi_{i+1})| = \pi. \tag{9}$$

But this cannot happen as such sequence would converge and thus as $\arg\gamma$ is continuous $\lim_{i\to\infty} |\arg\gamma(\xi_i) - \arg\gamma(\xi_{i+1})|$.

$2 \Rightarrow 1$. Let f have the desired property. Then one directly constructs a path $\gamma : I = [a, b] \to \mathbb{C} \setminus \{0\}$ such that $\mathrm{Cf}(\gamma) = f$. Let $\{L_i\}$ and $\{R_i\}$ be defined as in the previous part. We then define γ as

$$\gamma(x) := \exp\left(\boldsymbol{i}(\arctan f(x) + \pi|\{L_i\} \cap \mathcal{P}[a, x]| - |\{R_i\} \cap \mathcal{P}[a, x]|)\right) \tag{10}$$

for $x \in I \cap \mathrm{dom}(f)$. For the other points $x \in I \setminus \mathrm{dom}(f)$ set $\gamma|_{I\setminus\mathrm{dom}(f)}$ constant on path components (values $\pm\boldsymbol{i}$) such that γ is continuous.

It is routine to check that $\mathrm{Cf}(\gamma) = f$ is fulfilled. $\square$

Remark 0.8. Any function $\tilde{\gamma}$ with $\mathrm{Cf}(\tilde{\gamma}) = f$ satisfies $\tilde{\gamma} = \alpha\gamma$ where $\alpha : I \to \mathbb{R}$ is continuous and of constant sign as one easily verifies.

Definition 0.9 (CAUCHY index via paths)**.** Let f be a CAUCHY be a function on the compact interval $I = [a, b]$ corresponding to the path $\gamma : I \to \mathbb{C} \setminus \{0\}$ and let $\arg \gamma$ be the unique (modulo 2π) continuous function such that

$$\gamma = |\gamma| \exp(\boldsymbol{i} \arg \gamma). \tag{11}$$

Then its CAUCHY index is defined as

$$\operatorname{Ind}(f) := \operatorname{sgn}(\arg \gamma(a) - \arg \gamma(b)) \left| \operatorname{conv}\{\arg \gamma(a), \arg \gamma(b)\} \cap \left(\pi\mathbb{Z} + \frac{\pi}{2}\right) \right|. \tag{12}$$

Remark 0.10. From this we see that the Cauchy index basically counts how many times the curve γ traverses the imaginary line clockwise, where counterclockwise traverses are counted negative (roughly speaking).

Lemma 0.11 (CAUCHY index inversion formula, most general)**.** *Let f be a Cauchy function over $I = [a, b]$ and let $\{\bar{I}_i\}_{i=0}^l$ be the maximal rotational intervals of f. Moreover, let $\bar{I} = \operatorname{conv}\left(\bigcup\{\bar{I}_i\}_{i=0}^l\right)$ be the* rotational domain *of f and $\bar{a} < \bar{b}$ the bounds of the interval $\bar{I}$ (which can be open or closed at each side). Then it holds*

$$\operatorname{Ind}(f) + \operatorname{Ind}\left(\frac{1}{f}\right) = \frac{1}{2}\left(\lim_{\xi \uparrow \bar{b}, \xi \in \bar{I}} \operatorname{sgn} f(\xi) - \lim_{\xi \downarrow \bar{b}, \xi \in \bar{I}} \operatorname{sgn} f(\xi)\right). \tag{13}$$

Proof. Let $\tilde{\gamma} : \bar{I} \to \mathbb{C} \setminus \{0\}$ be a corresponding continuous curve to $f|_{\bar{I}}$ (i.e. $\operatorname{Cf}(\tilde{\gamma}) = f|_{\bar{I}}$). Then $\operatorname{Cf}(\boldsymbol{i}\bar{\tilde{\gamma}}) = (1/f)|_{\bar{I}}$. It is clear that $\operatorname{Ind}(f|_{\bar{I}}) = \operatorname{Ind}(f)$ from the definition of the CAUCHY index since if $\gamma : I \to \mathbb{C} \setminus \{0\}$ such that $\operatorname{Cf}(\gamma) = f$ then $\gamma(a) = \tilde{\gamma}(\bar{a})$ and $\gamma(b) = \tilde{\gamma}(\bar{b})$. Thus in the definition of the CAUCHY indices of f and $1/f$ we get

$$\begin{aligned}
\operatorname{Ind}(f) + \operatorname{Ind}\left(\frac{1}{f}\right) &= \operatorname{sgn}(\arg \tilde{\gamma}(\bar{a}) - \arg \tilde{\gamma}(\bar{b})) \left| \operatorname{conv}\{\arg \tilde{\gamma}(\bar{a}), \arg \tilde{\gamma}(\bar{b})\} \cap \left(\pi\mathbb{Z} + \frac{\pi}{2}\right) \right| && (14) \\
&\quad + \operatorname{sgn}(\arg \boldsymbol{i}\bar{\tilde{\gamma}}(\bar{a}) - \arg \boldsymbol{i}\bar{\tilde{\gamma}}(\bar{b})) \left| \operatorname{conv}\{\arg \delta(\bar{a}), \arg \delta(\bar{b})\} \cap \left(\pi\mathbb{Z} + \frac{\pi}{2}\right) \right| && (15) \\
&= \operatorname{sgn}(\arg \tilde{\gamma}(\bar{a}) - \arg \tilde{\gamma}(\bar{b})) \left| \operatorname{conv}\{\arg \tilde{\gamma}(\bar{a}), \arg \tilde{\gamma}(\bar{b})\} \cap \left(\pi\mathbb{Z} + \frac{\pi}{2}\right) \right| && (16) \\
&\quad - \operatorname{sgn}(\arg \tilde{\gamma}(\bar{a}) - \arg \tilde{\gamma}(\bar{b})) \left| \operatorname{conv}\{\arg \tilde{\gamma}(\bar{a}), \arg \tilde{\gamma}(\bar{b})\} \cap \pi\mathbb{Z} \right| && (17)
\end{aligned}$$

Now set $\eta := \max\{\arg \tilde{\gamma}(\bar{a}), \arg \tilde{\gamma}(\bar{b})\}$ and $\nu := \min\{\arg \tilde{\gamma}(\bar{a}), \arg \tilde{\gamma}(\bar{b})\}$. Then we obtain from our previous calculation

$$\operatorname{Ind}(f) + \operatorname{Ind}\left(\frac{1}{f}\right) = \operatorname{sgn}(\arg \tilde{\gamma}(\bar{a}) - \arg \tilde{\gamma}(\bar{b})) \begin{cases} 1 & : \eta \in \pi\mathbb{Z} + \left[-\frac{\pi}{2}, 0\right) \wedge \nu \in \pi\mathbb{Z} + \left(0, \frac{\pi}{2}\right] \\ -1 & : \eta \in \pi\mathbb{Z} + \left[0, \frac{\pi}{2}\right) \wedge \nu \in \pi\mathbb{Z} + \left(-\frac{\pi}{2}, 0\right] \\ 0 & : \text{ otherwise} \end{cases} \tag{18}$$

Now let L be the rotational interval whose lower bound is $\bar{a}$ and U be the rotational interval whose upper bound is $\bar{b}$ (these exist as there are only finitely many rotational intervals as f is CAUCHY function). Then there are two cases to be distinguished at each bound namely whether L or U are boundary intervals (and thus half-open) or if they are open intervals. In the second case it is clear that $\arg \tilde{\gamma}(a) \in \pi\mathbb{Z} + \pi/2$ or respectively $\arg \tilde{\gamma}(b) \in \pi\mathbb{Z} + \pi/2$. $\square$

1 Some remarks on Routh-Hurwitz-Theorem

Definition 1.1 (HAAR space)**.** Let $K \subset \mathbb{R}$ be a non-empty set. Then a n-dimensional linear subspace H of $\mathrm{C}(K)$ is called HAAR *space* if any function $g \in H \setminus \{0\}$ has at most $n-1$ zeros.

Definition 1.2 (CHEBYSHEV system)**.** A system $\{\varphi_i\}_{i=1}^n$ is called CHEBYSHEV *system* or HAAR *system* if its spanned vector space is a HAAR space.

Definition 1.3 (FM space)**.** Let K be a compact space. Then any linear subspace V of $\mathrm{C}(K)$ is called an *FM space* if there exists a number $N \in \mathbb{N}$ such that any function $g \in V$ has at most N maxima.

Definition 1.4 (DZ space)**.** Let $K \subset \mathbb{R}$ be a non-empty set. Then a linear subspace H of $\mathrm{C}(K)$ is called *DZ space* if any function $g \in H \setminus \{0\}$ has only isolated zeros.

2 Cauchy's Argument principle

In this section we will deal with the question, how many zeros and poles of a meromorphic function function lie in a open set $D \subset \mathbb{C}$ which is bounded by a rectifiable curve γ.

Lemma 2.1 (Cauchy's Argument Principle)**.** *Let $f : D \to \mathbb{C}$ be meromorphic inside a set C and have no zeros on its contour γ. Moreover, let P be the number of poles and Z be the number of zeros of f inside C. Then*

The theorem of Routh-Hurwitz is of remarkable importance in system and control theory taught in virtually any undergraduate course. In this section we want to take a closer look of the proof of this theorem.

The statement. To check the stability of a rational transfer function it is necessary to check weather all roots of its denominator lie in the left half-plane of $\mathbb{C}$.

Such polynomials having all its roots in the left half-plane are often called HURWITZ-polynomials.

The check is often formulated as follows

Check. Given a monic polynomial $P \in \mathbb{R}[X]$ where $P = X^n + \cdots + a_0$. Such polynomial is HURWITZ if and only if

1. All its coefficients are of the same sign (that is $\forall i \in [n] : a_i > 0$).
2. All principal minors of the HURWITZ-Matrix $H_n(P) \in \mathbb{R}^{n \times n}$ have positive determinant where

$$H_n(P) := \begin{pmatrix} a_{n-1} & a_{n-3} & \cdots & 0 & \cdots & 0 \\ 1 & a_{n-2} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & a_{n-3} & \cdots \\ 0 & \cdots & 0 & 1 & a_{n-2} & \cdots \end{pmatrix}. \tag{19}$$

The actual statement. The theorem of ROUTH-HURWITZ as stated originally gives the above as an easy consequence. The idea is to count the difference between the number of zeros of in the left half-plane L and the number of zeros in the right half-plane R of $\mathbb{C}$ of a polynomial $P \in \mathbb{C}[X]$.

The idea. A really simple but useful idea is here to consider the behavior of $\arg P$ on the imaginary axis.

To do this one needs the condition that P has no root on the imaginary line. Furthermore, we assume that P is monic ($a_n = 1$).

Assume P factors over $\mathbb{C}$ such that

$$P(z) = (z - l_1) \cdots (l - l_L)(z - r_1) \cdots (z - r_R) \tag{20}$$

clearly this holds by fundamental theorem of algebra and by condition $R+L = n$. In the following equations let us abbreviate the operator $\lim_{\omega\to\infty} - \lim_{\omega\to-\infty}$ by Δ^ω where $\omega \in \mathbb{R}$.

$$\Delta^\omega \arg P(\omega \boldsymbol{i}) = \sum_{i=1}^{L} \Delta^\omega \arg(\omega \boldsymbol{i} - l_i) + \sum_{i=1}^{R} \Delta^\omega \arg(\omega \boldsymbol{i} - r_i) \tag{21}$$

Here arg is of course considered as continuous and takes values in $\mathbb{R}$.

Let us evaluate each of the two sums separately. Thus, consider a left zero $l_i = a_i^l + b_i^l \boldsymbol{i}$ with $a_i^l < 0$. One obtains that

$$\arg(\omega \boldsymbol{i} - l_i) = \arctan\left(\frac{\omega - b_i^l}{-a_i^l}\right) - \pi \bmod 2\pi \tag{22}$$

As a_i^l is negative, the above term increases as ω increases. Thus

$$\Delta^\omega \arg(\omega \boldsymbol{i} - l_i) = \pi \tag{23}$$

and analogously for any r_i we have

$$\Delta^\omega \arg(\omega \boldsymbol{i} - r_i) = -\pi. \tag{24}$$

From this we get in equation (21) that

$$\Delta^\omega \arg P(\omega \boldsymbol{i}) = \pi(L - R). \tag{25}$$

Proceeding the argument. A natural question is now, how we can get back from this last condition to the original polynomial. However, this is also not very difficult. Note that P can be uniquely decomposed into polynomials R and S such that

$$P(\omega \boldsymbol{i}) = R(\omega) + S(\omega)\boldsymbol{i}. \tag{26}$$

Now, it is a natural question to translate the term $\Delta^\omega \arg P(\omega \boldsymbol{i})$ in a term depending on the rational function $\frac{S}{R}$ as the argument arg is connected to this function via arctan.

Therefore one introduces the following definition.

3 The Cauchy index

At this point it becomes reasonable to introduce a special parameter for rational functions from $\mathbb{R}$ to $\mathbb{R}$

Definition 3.1 (Cauchy index)**.** Let $I \subset \mathbb{R}$ be an open interval and $F : I \setminus P \to \mathbb{R}$ be a continuous function where P is a finite set of poles of F, i.e. for each $p \in P$ we have

$$\lim_{x\uparrow p} F(x) = \pm\infty \tag{27}$$

and

$$\lim_{x \downarrow p} F(x) = \pm\infty, \tag{28}$$

respectively. For $r \in \mathbb{R}$ one defines the pointwise CAUCHY *index* of F as

$$\mathrm{Ind}_r(F) := \begin{cases} 1 & : \lim_{x\uparrow r} F(x) = -\infty \wedge \lim_{x\downarrow r} F(x) = \infty \\ -1 & : \lim_{x\uparrow r} F(x) = \infty \wedge \lim_{x\downarrow r} F(x) = -\infty \\ 0 & : \text{otherwise} \end{cases}. \tag{29}$$

Then the Cauchy index of F on the interval I is defined as

$$\mathrm{Ind}(F) := \sum_{r \in I} I_r(F). \tag{30}$$

Starting from this definition it is obvious that the operator $\mathcal{I}$ has several nice properties.

Lemma 3.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be rational and non-zero. Then one has*

$$\mathcal{I}(F) = -\mathcal{I}(F^{-1}). \tag{31}$$

Moreover, if $P, Q \in \mathbb{R}[X]$ have no common zero then one has

$$\mathcal{I}\left(\frac{P}{Q}\right) = \mathcal{I}\left(\frac{\mathrm{res}(P,Q)}{Q}\right). \tag{32}$$

Proof. **The first statement:** Let $p_1, \dots, p_l \in \mathbb{R} \cup \{-\infty, \infty\}$ be the poles of F on the real line ordered by ascending order. Consider the interval (p_i, p_{i+1}) for $i \in \{1, \dots, l-1\}$. If

$$\mathrm{sgn}(\lim_{r\downarrow p_i} F(r)) = \mathrm{sgn}(\lim_{r\uparrow p_{i+1}} F(r)) \tag{33}$$

then one gets that there must be equally many zeros r_0 of F in (p_i, p_{i+1}) with $F'(r_0) < 0$ as with $F'(r_0) > 0$. Thus it directly follows that than

$$\mathcal{I}_{(p_i,p_{i+1})}(F^{-1}) = 0. \tag{34}$$

In the opposite case where

$$\mathrm{sgn}(\lim_{r\downarrow p_i} F(r)) = -\,\mathrm{sgn}(\lim_{r\uparrow p_{i+1}} F(r)) \tag{35}$$

one gets analogously that

$$\mathcal{I}_{(p_i,p_{i+1})}(F^{-1}) = \mathrm{sgn}(\lim_{r\downarrow p_i} F(r)). \tag{36}$$

Moreover, we then may compute $\mathcal{I}(F^{-1})$ by

$$\mathcal{I}(F^{-1}) = \sum_{i=1}^{l-1} \mathcal{I}_{(p_i,p_{i+1})}(F^{-1}) + \mathcal{I}_\infty(F^{-1}) \tag{37}$$

From this consideration it is now clear, that we may assume there are no intervals (p_i, p_{i+1}) of the first type. To see this one checks that deleting them does not change $\mathcal{I}(F^{-1})$ and $\mathcal{I}(F)$.

Analogously, we may delete two consecutive intervals of the second type (p_i, p_{i+1}) and (p_{i+1}, p_{i+2}) ($i \in \{1, \dots, l-2\}$) where one has

$$\lim_{r \downarrow p_{i+1}} F(r) = \pm\infty. \tag{38}$$

These arguments can also be done if one bound of an interval is $\pm\infty$. Proceeding this argument, one is left with the case that all intervals are of the second type and that for all $i, j \in \{1, \ldots, l\}$ we have

$$\operatorname{sgn}(\lim_{r \downarrow p_i} F(r)) = \operatorname{sgn}(\lim_{r \downarrow p_j} F(r)) \wedge \operatorname{sgn}(\lim_{r \uparrow p_i} F(r)) = \operatorname{sgn}(\lim_{r \uparrow p_j} F(r)). \tag{39}$$

It is then a routine matter to check the statement holds true.

The second statement is obvious. □

Remark 3.3. One can prove the first statement much faster considering the functions $\omega \mapsto P(\omega) \pm Q(\omega)\boldsymbol{i}$ where $F = P/Q$. The number of rotations of the curves of these functions in the complex plane around 0 is the same and directly connected with the two CAUCHY-indices.

Definition 3.4 (Generalized STURM chain)**.** Let $I \subset \mathbb{R}$ be an open interval. A *generalized* STURM *chain* over I is a finite sequence of continuous functions $(f_i)_{i=0}^n$ such that

1. The functions $f_i : I \to \mathbb{R}$ have only finitely many zeros ($i = 0, \ldots, n$).
2. The function $\operatorname{sgn} f_n$ is constant on I.
3. If $f_i(\xi) = 0$ for $\xi \in I$ and $i \in \{1, \ldots, n-1\}$, then $\operatorname{sgn}(f_{i-1}) = -\operatorname{sgn}(f_{i+1})$.

Definition 3.5 (Number of sign changes for a finite sequence)**.** Let $(x_i)_{i=0}^n$ be a finite sequence. Let $(x_{i_j})_{j=0}^m$ be a subsequence of $(x_i)_{i=0}^n$. Then we define the *number of sign changes* of $x = (x_i)_{i=0}^n$ by

$$\sigma(x) := \frac{1}{2} \sum_{i=1}^{m} \left| \operatorname{sgn}(x_{i_j}) - \operatorname{sgn}(x_{i_{j-1}}) \right|. \tag{40}$$

Theorem 3.6 (Calculation of the CAUCHY index by a generalized STURM chain)**.** *Let $I = (a, b) \subset \mathbb{R}$ be an open interval and $F : I \setminus P \to \mathbb{R}$ be a continuous function where P is a finite set of poles of F, i.e. for each $p \in P$ we have*

$$\lim_{x \uparrow p} F(x) = \pm\infty \tag{41}$$

and

$$\lim_{x \downarrow p} F(x) = \pm\infty, \tag{42}$$

respectively. Let $(f_i)_{i=0}^n$ be a generalized STURM *chain such that*

$$F(\xi) = \frac{f_1(\xi)}{f_0(\xi)} \tag{43}$$

for $\xi \in I \setminus P$. Then we have

$$\operatorname{Ind}(F) = \sigma(f_i(b))_{i=0}^n - \sigma(f_i(a))_{i=0}^n. \tag{44}$$

Proof. □

Explanation. The obvious intention of this definition is to count how many times a function $P(\omega\boldsymbol{i}) = R(\omega) + S(\omega)\boldsymbol{i}$ rotates around the origin 0 of $\mathbb{C}$. Therefore one may plugin $F = \frac{S}{R}$ are its reciprocal.

But we want to give some more detailed explanation.

Details. Let $\omega_1^{-1}, \dots, \omega_d^{-1} \in \mathbb{R}$ be the numbers where the function $P(\cdot\boldsymbol{i}) : \mathbb{R} \to \mathbb{C}$ traverses the imaginary line counterclockwise and $\omega_1^1, \dots, \omega_e^1 \in \mathbb{R}$ the numbers where $P(\cdot\boldsymbol{i})$ traverses the imaginary line clockwise with respect to the origin. By condition, $P(\cdot\boldsymbol{i})$ has no zero, thus for all these ω's we have $P(\omega\boldsymbol{i}) \neq 0$.

Formally, this means when $f : \mathbb{R} \to \mathbb{R}$ denotes the differentiable function $\omega \mapsto \arg P(\omega\boldsymbol{i})$

$$R(\omega_j^i) = 0 \wedge \operatorname{sgn} f'(\omega_j^i) = -i \tag{45}$$

for $i = \pm 1$ and j in the appropriate index set. More, from this we see that

$$I_{\omega_j^i}\left(\frac{S}{R}\right) = -i. \tag{46}$$

One thus may deduce the following equation

$$\Delta^\omega \arg P(\omega\boldsymbol{i}) = -\pi\mathcal{I}\left(\frac{S}{R}\right) + \Delta^\omega \arctan\left(\frac{S}{R}\right). \tag{47}$$

Now, at this point, the most interesting argument comes in. Obviously, we have $-\Delta^\omega \arg \boldsymbol{i}\overline{P}(\omega\boldsymbol{i}) = \Delta^\omega \arg P(\omega\boldsymbol{i})$ and $\boldsymbol{i}\overline{P}(\omega\boldsymbol{i}) = S(\omega) + P(\omega)\boldsymbol{i}$. We thus may deduce from (47) that

$$-\pi\mathcal{I}\left(\frac{S}{R}\right) + \Delta^\omega \arctan\left(\frac{S}{R}\right) = \pi\mathcal{I}\left(\frac{R}{S}\right) - \Delta^\omega \arctan\left(\frac{R}{S}\right). \tag{48}$$

Now, let us consider the arctan-terms. Indeed, here only the leading coefficients of R and S matter and it is easy to observe that

BSBS

$$\Delta^\omega \arctan\left(\frac{S}{R}\right) = \pi(-1)^{\deg R - \deg S} \tag{49}$$

From this definition one sees that if $F = \frac{S}{R}$ for polynomials R and S then

$$-\pi\mathcal{I}(F) = -\pi\mathcal{I}\left(\frac{S}{R}\right) = \Delta^\omega \arg(R(\omega) + S(\omega)\boldsymbol{i}). \tag{50}$$

From equations (25) and (50) we obtain that

$$L - R = -\mathcal{I}_J\left(\frac{S}{R}\right) \tag{51}$$