On Functions of Bounded Variation

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1 Introduction

It is a very interesting subject to generalize or weaken the property of a function being differentiable. It is a well known theorem that functions from $\mathbb{R}$ to $\mathbb{R}$ of locally bounded variation are almost everywhre differentiable. We will comment on this thing a bit.

Theorem 1.1 (Vitali's Covering Lemma)**.** *Let $\mathcal{B}$ be a collection of closed non-degenerate balls in the $\mathbb{R}$-metric*[1] *space (X, d) with uniformly bounded radii. Then for $\varepsilon > 0$ there exists a subcollection $\mathcal{B}'$ of $\mathcal{B}$ of pairwise disjoint balls such that $\bigcup \rho_{3+\varepsilon}[\mathcal{B}'] \supset \bigcup \mathcal{B}$ where ρ_α maps a ball to a ball with the same center but scaled with α.*

Proof. Let R be an upper bound of the radii. Let us write $r(B)$ for the radius of a ball B (and consider r as a map from the balls[2] to the reals). The idea of the proof is to construct a maximal subcollection which satisfies the desired properties. Consider the following partition of $\mathcal{B}$:

$$\mathcal{B} = \bigcup_{k \in \mathbb{N}_0} \mathcal{B} \cap r^{-1} \left(C \left(1 + \frac{\varepsilon}{2}\right)^{-(k+1)}, C \left(1 + \frac{\varepsilon}{2}\right)^{-k} \right]. \tag{1}$$

Let us denote these sets by $\mathcal{B}_k$ for $f \in \mathbb{N}_0$. We now 'inductively' choose a maximal disjoint subcollection $\mathcal{C}$ in the following manner.

- Assume we have chosen some subcollections $\mathcal{C}_i$ of $\mathcal{B}_i$ for $i < k$ ($k \in \mathbb{N}_0$) such that $\bigcup_{i<k} \mathcal{C}_i$ is a maximal disjoint subcollection of $\bigcup_{i<k} \mathcal{B}_i$.
- Then chose $\mathcal{C}_k$ such that $\bigcup_{i \le k} \mathcal{C}_i$ is a maximal disjoint subcollection of $\bigcup_{i \le k} \mathcal{B}_i$.

[1] that is the metric is real valued

[2] or more generally bounded sets

This procedure is possible due to Zorn's lemma. We finally get $\mathcal{C} = \bigcup_{k \in \mathbb{N}_0} \mathcal{C}_k$. Now to prove the desired property, assume that there is a ball $B \in \mathcal{B}$ with $B \not\subset \bigcup \mathcal{C}$. Then B must lie in some $\mathcal{B}_k$ ($k \in \mathbb{N}_0$). Thus due to maximality there exists $C \in \mathcal{C}_k$ such that $B \cap C \neq \emptyset$. Now pick points $p \in B \setminus C, q \in B \cap C$ and let o be the center of C. We then have by triangle inequality

$$d(o,p) \leq d(o,q) + d(q,p) \leq r(C) + 2r(B) \leq r(C)(3+\varepsilon) \tag{2}$$

where the last is due to the fact that $B, C \in \mathcal{B}_k$. □

Remark 1.2. The value $3+\varepsilon$ is best possible.

Remark 1.3. If X is σ-finite and non-degenerate balls have measure greater than zero the constructed collection is countable.

Theorem 1.4 (Vitali's Covering Theorem)**.** *Let $\mathcal{B}$ be a collection of closed non-degenerate balls in the metric space (X, d) such that if $b \in \bigcup \mathcal{B}$ then for $\varepsilon > 0$ there exists a ball $B \in \mathcal{B}$ with $r(B) < \varepsilon$ and $b \in B$. Let a Borel-measure be assigned on X. Then there exists a disjoint subcollection $\mathcal{C}$ fo $\mathcal{B}$ such that $\bigcup \mathcal{B} \setminus \bigcup \mathcal{C}$ has locally finite measure.*