On the existence of certain generic and weakly generic arrangements in $\mathbb{F}_q^d$

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Let K be a geometric n-dimensional simplicial complex embedded in $\mathbb{R}^n$ triangulating the domain Ω to solve the boundary value problem

$$-\Delta u = f \tag{1}$$

$$u|_{\partial\Omega} = 0. \tag{2}$$

Using linear FEM we calculate

$$\langle \nabla\phi_u, \nabla\phi_v \rangle = \begin{cases} 0 & : \{u,v\} \notin K \\ \beta_{uv} & : \{u,v\} \in K^{(2)} \\ \alpha_u & : u = v \end{cases}. \tag{3}$$

Using barycentric coordinates in the n-simplex $\sigma \in K^{(n)}$ one gets

$$z = \sum_{u\in\sigma} \frac{\det(z - \sigma \setminus \{u\})}{\det(u - \sigma \setminus \{u\})} u = \sum_{u\in\sigma} \lambda_\sigma^u(z) u \tag{4}$$

by interpolation of the identity. If $\{u,v\} \in K^{(2)}$ we get

$$\beta_{uv} = \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \int_\sigma \nabla\phi_u \nabla\phi_v = \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \int_\sigma \nabla\lambda_\sigma^u(z) \nabla\lambda_\sigma^v(z) \,\mathrm{d}\, z. \tag{5}$$

Calculating the summands reveals

$$\beta_{uv} = \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \int_\sigma \sum_{i=1}^n \lambda_\sigma^u(e_i)\lambda_\sigma^v(e_i) \,\mathrm{d}\, z = \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \mathrm{vol}(\sigma) \sum_{i=1}^n \lambda_\sigma^u(e_i)\lambda_\sigma^v(e_i) \tag{6}$$

$$= \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \frac{(n-1)!^2 \,\mathrm{vol}(\sigma\setminus\{u\})\,\mathrm{vol}(\sigma\setminus\{v\})\,\langle \nu_\sigma(\sigma\setminus\{u\}), \nu_\sigma(\sigma\setminus\{v\})\rangle}{n!^2\,\mathrm{vol}(\sigma)} \tag{7}$$

$$= \sum_{\substack{\sigma\in K^{(n)} \\ \{u,v\}\subseteq\sigma}} \frac{\mathrm{vol}(\sigma\setminus\{u\})\,\mathrm{vol}(\sigma\setminus\{v\})\,\langle \nu_\sigma(\sigma\setminus\{u\}), \nu_\sigma(\sigma\setminus\{v\})\rangle}{n^2\,\mathrm{vol}(\sigma)}. \tag{8}$$

where $\nu_\sigma(\tau)$ denotes the outer unit normal vector of the face τ with respect to σ. Now, observe that $\langle \nu_\sigma(\sigma\setminus\{u\}), \nu_\sigma(\sigma\setminus\{v\})\rangle = -\cos\angle(\sigma\setminus\{u\}, \sigma\setminus\{v\})$ (the angle between the faces which is $< \pi$). Moreover, it is true that $\frac{n\,\mathrm{vol}(\sigma)}{\mathrm{vol}(\sigma\setminus\{u\})} = \mathrm{vol}(\{u,v\})\sin\angle(\{u,v\},\sigma\setminus\{u\})$ (and the analog identity for v holds as well). Inserting this in the equation gives

$$\beta_{uv} = -\frac{1}{\mathrm{vol}(\{u,v\})^2}\sum_{\substack{\sigma\in K^{(n)}\\ \{u,v\}\subseteq\sigma}} \frac{\mathrm{vol}(\sigma)\cos\angle(\sigma\setminus\{u\},\sigma\setminus\{v\})}{\sin\angle(\{u,v\},\sigma\setminus\{u\})\sin\angle(\{u,v\},\sigma\setminus\{v\})}. \tag{9}$$

For the case $n = 2$ this formula simplifies by using that for $\{u,v,w\}\in K^{(2)}$

$$2\,\mathrm{vol}(\{u,v,w\}) = \mathrm{vol}(\{u,w\})\,\mathrm{vol}(\{v,w\})\sin\angle(\{u,w\},\{v,w\}) \tag{10}$$

$$= \mathrm{vol}(\{u,v\})^2\frac{\sin\angle(\{u,v\},\{u,w\})\sin\angle(\{u,v\},\{v,w\})}{\sin\angle(\{u,w\},\{v,w\})}. \tag{11}$$

The result is then

$$\beta_{uv} = -\frac{1}{2}\sum_{w\in K^{(0)}:\{u,v,w\}\in K^{(2)}}\frac{\cos\angle(\{u,w\},\{v,w\})}{\sin\angle(\{u,w\},\{v,w\})} \tag{12}$$

$$= -\frac{1}{2}\sum_{w\in K^{(0)}:\{u,v,w\}\in K^{(2)}}\cot\angle(\{u,w\},\{v,w\}). \tag{13}$$

This last sum has at most two summands (depending on if $\{u,v\}$ lies in the boundary).

Definition 0.1 (weakly diagonally dominant matrix)**.** Let $A = (a_{ij})_{i,j=1,\dots,n}\in\mathbb{F}^{n\times n}$ where $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ then M is called *weakly diagonally dominant* if

$$|a_{ii}| \geq \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}| \tag{14}$$

and the inequality is strict for at least one $j\in\{1,\dots,n\}$.

Lemma 0.2 (regularity of irreducible weakly diagonally dominant matrices)**.** *Let $A\in\mathbb{F}^{n\times n}$ be a weakly diagonally dominant irreducible matrix where $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ then A is regular.*

Proof. Assume $Ax = 0$ and choose $i\in\{1,\dots,n\}$ such that $|x_i| = \max\{|x_j| : j = 1,\dots,n\}$. Then we have that

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \tag{15}$$

together with the fact that A is weakly diagonally dominant implying that

$$|a_{ii}x_i| \geq \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}x_i| = \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}x_j| \tag{16}$$

$$\geq \left|\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}x_j\right| = |a_{ii}x_i| \tag{17}$$

whence all the inequalities must reach equality. If $x_i = 0$ it follows that $x_j = 0$ for $j \in \{1, \ldots, n\}$ and thus $x = 0$. Otherwise, we may divide the above by $|x_i|$. If $a_{ii} = 0$ then it follows that $a_{ij} = 0$ as A is weakly diagonally dominant. But then we have that $A[\mathrm{lin}\{\boldsymbol{e}_j : (j = 1, \ldots, n) \wedge j \neq i\}] \subseteq \mathrm{lin}\{\boldsymbol{e}_j : (j = 1, \ldots, n) \wedge (j \neq i)\}$ from which it would follow that A is reducible. Thus $a_{ii} \neq 0$. From this it follows that for each j for which $a_{ij} \neq 0$ we have $|x_i| = |x_j|$.

Choosing a new $i \in \{1, \ldots, n\}$ for which $|x_i| = \max\{|x_j| : j = 1, \ldots, n\}$ we can spread these equalities to reach by the assumption that A is irreducible such that $|x_i| = |x_j|$ for all $i, j \in \{1, \ldots, n\}$ (this is basically interpreting A as a weighted adjacency matrix of graph). However, recalling the above inequalities we then notice that

$$|a_{ii}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{18}$$

for all $i \in \{1, \ldots, n\}$ (as we assume $x_i \neq 0$) contradicting the fact that A is weakly diagonally dominant. Thus $x_i = 0$ for all $i \in \{1, \ldots, n\}$ showing that $x = 0$. Thus A is regular. □

Lemma 0.3 (M-matrix criterion for weakly diagonally dominant matrices)**.** *Let $A \in \mathbb{R}^{n \times n}$ be a weakly diagonally dominant irreducible matrix such that $a_{ij} \leq 0$ for $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Then A is an M-matrix.*

Proof. Choose $x \in \mathbb{F}^n$ such that $Ax \leq 0$ and choose $i \in \{1, \ldots, n\}$ such that $x_i = \max\{x_j : j = 1, \ldots, n\}$. We then obtain using that A is weakly diagonally dominant and satisfies the additional condition in the lemma and assuming that $x_i > 0$

$$-\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j \geq a_{ii} x_i \tag{19}$$

$$\geq -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j \tag{20}$$

implying that all inequalities must reach equality. We can divide by x_i to get that for each $j \in \{1, \ldots, n\} \setminus \{i\}$ where $a_{ij} \neq 0$ that $x_j = x_i$. As A is irreducible this process reaches all $j \in \{1, \ldots, n\}$ (by repeatedly choosing a new i with $x_i = \max\{x_j : j = 1, \ldots, n\}$). Thus we get that x must be a positive multiple of $v = (1, \ldots, 1)$. But in this case, plugging in the vector v yields by A being weakly diagonally dominant a vector $Av > 0$ (i.e. with at least one non-zero coordinate) contradicting the assumption. Thus we have that $x_i \leq 0$ for all $i \in \{1, \ldots, n\}$ and thus $x \leq 0$. Hence, A is inverse monotone and together with the additional assumption it is an M-matrix. □