On the Road Coloring Problem

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Abstract

We review Trahtmans proof of the Road Coloring Conjecture published in 2009.

1 Introduction

Let G be a finite, strongly connected, directed graph where all vertices have the same out-degree $k \in \mathbb{N}$. Let Σ be the alphabet containing the letters $1, \dots, k$. A synchronizing coloring (also known as a collapsible coloring) in G is a labeling of the edges in G with letters from Σ such that

1. each vertex has exactly one outgoing edge with a given label and
2. for every vertex v in the graph, there exists a word s over Σ such that all paths in G corresponding to s terminate at v.

For such a coloring to exist at all, it is necessary that G be aperiodic. The road coloring theorem states that aperiodicity is also sufficient for such a coloring to exist. Therefore, the road coloring problem can be stated briefly as:

Conjecture 1.1. *Every finite strongly connected aperiodic directed graph of uniform out-degree has a synchronizing coloring.*

This paper adapts the (ingenious) proof presented by Trahtman published in 2009 (see [1]). It adds upon his work in explaining some details left in his proof, some of which were referenced to [2] while others were expected as base knowledge of the reader.

2 Preliminaries

At first we want to establish some notation.

Definition 2.1 (strongly connected, aperiodic)**.** A directed graph $G = (V, E)$ is called *strongly connected* if any two vertices $u, v \in V$ are joined by a directed path. We call G *aperiodic* if the gcd of the lengths of its circles is one.

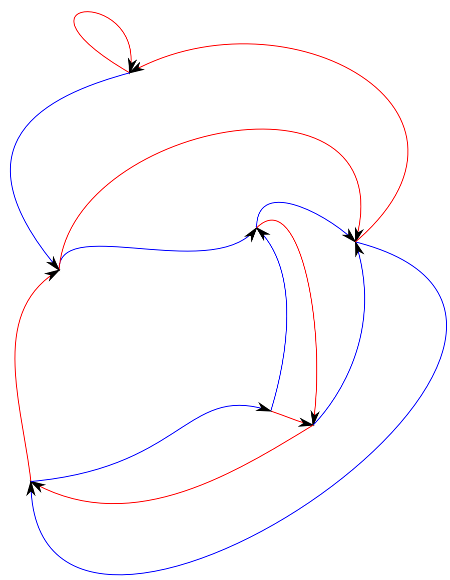

Figure 1: AGW graph with synchronizing word r^5.

At this point we want to mention that the property of strongly connectedness is equivalent to the property of the adjacency matrix A of G that for any $i, j \in \{1, \dots, |V|\}$ there exists $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$. This can be seen easily by recalling that A^n just counts the number of paths of length n from one vertex to another.

Another equivalent formulation of aperiodicty is that if there exists $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$ then there exists $n_0 \in \mathbb{N}$ such that $(A^n)_{ij} > 0$ for all $n \geq n_0$.

Definition 2.2. An AGW graph is a *strongly connected*, *aperiodic* directed finite graph which is k-regular for some $k \in \mathbb{N}$.

Combining both properties (or equivalent formulations for the adjacency matrix, respectively) implies that the adjacency matrix A of an AGW graph admits a strictly positive natural power A^n (for some $n \in \mathbb{N}$). This property is often called *primitive* with respect to matrices as well as to graphs.

Hereafter $G = (V, E)$ is an AGW graph, A the adjacency matrix of G and k its out-degree.

The case $k = 1$. This case is trivial but we shall do the short argument here (any coloring is the trivial coloring). Let G be an AGW graph with out-degree

$k = 1$. Let C be a circle in G of maximal length. If $G = C$ then C must be the trivial circle with one vertex as G is acyclic. In the opposite case G is not strongly connected (it consists of at least two disjoint cycles). Thus G is the trivial graph with one vertex.

3 Weights

We want to introduce the concept of weights of vertices.

Definition 3.1. Let u be a left eigenvector of eigenvalue k with natural entries of the adjacency matrix A of an AGW graph G, then we define the weight $w(v_i)$ as the i-th component of u (w.l.o.g. we may assume that these entries have gcd one). For a subset $L \subset V$ we define $w(L) := \sum_{v \in S} w(v)$.

The existence of such eigenvector can easily be seen from the fact that A is primitive. A more precise explanation is given in the following theorem.

Theorem 3.2. *Let $k \geq 2$[1]. All of the eigenvalues λ of the adjacency matrix A lie in the interval $[-k, k]$ and especially k is an eigenvalue of A of algebraic multiplicity one with an integer-valued positive left and right eigenvector, where the right eigenvector trivially is a multiple of* $(1, \dots, 1)^{\mathrm{T}}$.[2]

Proof. One directly notices that $\|A\|_\infty = k$ implying that for all eigenvalues λ we have $|\lambda| \leq k$. The rest is just applying Perron-Frobenius' theorem. □

Again we need to introduce some notation conventions. Let $L \subset V$ be a set of vertices of the colored AGW graph G and s be a word of Σ^*. Then we denote by Ls the set of vertices which are reached from the vertices of L by a s-path.

The papers we are referring to are mainly dealing with the following two concepts.

Definition 3.3 (F-maximal set)**.** An *F-maximal set* set L is a subset of V where $w(L)$ is maximal under the condition that $|Ls| = 1$ for some word $s \in \Sigma^*$. This last condition is also called *contractible*.

Definition 3.4 (F-clique)**.** An F-clique is a set of the form Vs where $s \in \Sigma^*$ such that every pair of Vs is not *synchronizable*, i.e. for all $u, v \in Vs$ and $t \in \Sigma^*$ we have $ut \neq vt$. Such pairs we also call *deadlocks*.

The first important step for the solution of the problem is now given by the following

Theorem 3.5 (Partition into F-maximal sets, Friedman)**.** [3] *The graph G has a partition into F-maximal sets.*

[1] This shall also be assumed for the rest of the paper as the case $k = 1$ is trivial as mentioned in the preliminaries.

[2] Actually, such weights can be found for any strongly connected graph, as Perron-Frobenius' theorem works for positive irreducible operators (which are the corresponding adjacency matrices).

[3] This theorem does also hold for strongly connected graphs.

Friedman actually proved this theorem for $k = 2$. But his proof works as well for the general case.

Proof. Let L be an F-maximal set. Then by the definition of the weight, the following identity holds for the backward images of L.

$$\sum_{s \in \Sigma} w(Ls^{-1}) = kw(L) \tag{1}$$

From this identity one observes that all the sets Ls^{-1} $(s \in \Sigma)$ must be F-maximal sets, because they are contractible (as L is) and must all have the same weight as L does. Now, if $L = V$ we are done. Otherwise, let r be a synchronizing word of L and $v_0 \in V$ the vertex such that $Lr = \{v_0\}$. Furthermore, pick $v \in V \setminus L$. By strongly connectedness there is some word w such that $v_0 w = v$. Then consider the F-maximal sets $Lw^{-1}r^{-1} =: \tilde{L}$ and L (the first one has this property by successive application of inverses of single letters). Then it is obvious that $|Lrwr| = |\tilde{L}rwr| = 1$. Furthermore, these both sets cannot be synchronized to the same vertex, since $Lr = \tilde{L}rwr = \{v_0\}$ and thus otherwise we would have $v_0 wr = vr = v_0$ which would imply $v \in v_0 r^{-1} = L$, a contradiction[4]. From this, one deduces that L and $\tilde{L}$ are disjoint. Continuing the argument inductively, one obtains the desired partition. $\square$

A second important theorem is also of significant importance for the proof of the conjecture. It is stated by Trahtman.

At first we need the following

Definition 3.6. A pair $u, v \in V$ is called *stable* if for any word $w \in \Sigma^*$ the pair $uw, vw \in V$ is synchronizable.

It is clear, that *stability* defines a congruence relation on the set V. Let us denote this relation by ρ.

Theorem 3.7 (Quotient graph, Trahtman)**.** *The graph G/ρ is an AGW graph.*

The proof of the theorem is obvious and is omitted[5]. While this theorem is quiet natural and clear by the definition of ρ, it is a very strong tool in the proof, because it ensures that the existence of one single stable pair of vertices in an AGW graph implies (inductively) the existence of a synchronizing coloring.

We now continue with a lemma on F-cliques.

Lemma 3.8 (Equal cardinality of F-cliques, Trahtman)**.** [6] *Let w be the weight of an F-maximal set of G. Then any F-clique Vs is of cardinality $w(V)/w$. Moreover, if $F_1, \ldots F_l$ is a partition of V into F-maximal sets then $|F_i \cap Vs| = 1$ (forall $i = 1, \ldots, l$).*

The proof is mainly based on Friedmans theorem.

[4] This argument uses strict positivity of weights. Otherwise we could only say $L \subset v_0 r^{-1}$.

[5] Circles and paths remain, although they might get 'glued' somewhere

[6] This lemma also holds for strongly connected graphs.

Proof. Let Vs be an F-clique ($s \in \Sigma^*$). The first thing we note is that for any F-maximal set F we have $|F \cap Vs| \leq 1$, because otherwise Vs would contain a synchronizable pair. Thus we have $|Vs| \leq w(V)/w$ (number of F-maximal sets. Besides, for any $v \in Vs$ we have

$$w(vs^{-1}) \leq w \tag{2}$$

by the definition of F-maximal sets. Furthermore, we have the following identity

$$w(V) = \sum_{v \in Vs} w(vs^{-1}) \tag{3}$$

Plugging (18) into equation (19) gives

$$w(V) \leq |Vs|w \tag{4}$$

from which (together with the other inequality) we obtain the desired result $|Vs| = w(V)/w$. The second fact mentioned follows easily. □

The next lemma is an easy one.

Lemma 3.9. *For any F-clique Vs and any word $w \in \Sigma^*$ the set Vsw is an F-clique. Moreover, any vertex v belongs to some F-clique.*

Proof. Just use the stability of the binary relation to be deadlock. The second fact is an easy consequence of strongly connectedness. □

The next lemma is essential for the final proof and is based on the idea to prove the existence of a stable couple.

Lemma 3.10. *Assume G via some coloring has no stable couples. Let A and B be distinct F-cliques ($|A| = |B| > 1$). Then $|A \setminus B| = |B \setminus A| > 1$.*

Proof. Assume $|A \setminus B| = |B \setminus A| = 1$ (the cardinalities are equal by Lemma (4.1). Then pick $u \in A \setminus B$ and $v \in B \setminus A$. By condition these two form no stable couple. Thus we find $s \in \Sigma^*$ such that the pair (us, vs) is deadlock. Thus we see immediately that $(A \cup B)s$ must also be an F-clique which contradicts Lemma (4.1).[7] □

Finally, we end up the preparation with an easy one for which we need the following

Definition 3.11. Let $B = \{e_1, \ldots, e_k\} \subset E$ a set of k edges starting from a vertex u and ending in $v \in V$. Then we call B a bunch.

Lemma 3.12. *Let G have a vertex v with two incoming bunches. Then G has a stable couple.*

Proof. The two starting vertices of the two incoming bunches are obvioulsy stable. □

[7] Attention. If $|A \setminus B| \geq 2$ one cannot use the same argument repetitively (twice), because it is not assured that a pair $(u's, v's)$ is unstable for $u' = A \setminus B \setminus \{u\}$ and $v' = B \setminus A \setminus \{v\}$ are unstable ($s \in \Sigma^*$ chosen for u and v as above).

4 Spanning subgraphs, trees and cycles

This section gives the main part of the proof. For the understanding of this, the reader is allowed to forget everything about the weights of the graph G. They were only a mean to proof the facts about F-cliques and F-maximal sets.

At first we need some terminology.

Definition 4.1 (Spanning subgraph)**.** Let R be a subgraph of G. We call R a *spanning subgraph* of G if the vertex sets of R and G coincide and every vertex of R has out-degree one.[8]

Definition 4.2 (Rooted tree)**.** A graph T of the spanning subgraph R is called a *rooted tree* of R if T is a maximal tree of R having no edges with cycles of R in common and its root in a cycle of R.

Definition 4.3 (Level of a vertex)**.** Let T be a rooted tree in a spanning subgraph R and $v \in V$. Then the *level* of v is defined as the length of the path from v to the root of T and is denoted by $\boldsymbol{l}(v)$. Furthermore let $\boldsymbol{l}(T) := \max\limits_{v \in T} \boldsymbol{l}(v)$

From these three definitions it should be clear that any spanning subgraph R of G can be uniquely partitioned into disjoint cycles and rooted trees. The level for each vertex $v \in V$ is then defined by the particular tree in which v lies. If v is member of a cycle, then its level is 0.

In the following, if nothing different is stated, we use the same symbols as in the definitions.

Considering such partitioned spanning subgraph R of G the following lemma should be trivial.

Lemma 4.4. *Let all edges of R have the same color c and let Vs be an F-clique then we have $|\boldsymbol{l}^{-1}[n] \cap Vs| \leq 1$ ($s \in \Sigma^*$).*

Proof. If $\boldsymbol{l}^{-1}[n] \cap Vs$ had more than one member, Vs would contain a synchronizable pair (synchronizing word is c^n). □

For the rest of the section let us assume, that G has more than one vertex.[9]

Lemma 4.5. *Let R be some spanning subgraph of G consisting only of disjoint cycles. Then there is another spanning subgraph R' of G with exactly one vertex of maximal level.*

Proof. If R is a single cycle then R has (not necessarily distinct) vertices $u, v, w \in V$ such that $u \to w \in E_G$, $u \to v \in E_R$[10] and $v \neq w$[11]. We then may take the new spanning subgraph $R' = (V_R, E_R \setminus \{u \to v\} \cup \{u \to u\})$.

In the second case, where R is not a single cycle we may apply the same argument. In the end we have a single tail (that is a tree which is a path) rooted in w and thus a single vertex of maximal length.

[8] If G is colored then any monochromatic subgraph is a spanning subgraph. Moreover, for any spanning subgraph there exists such coloring.

[9] This case is sufficiently trivial.

[10] In this proof we use the subscripts to indicate to which subgraph vertices belong.

[11] Here we use $k \geq 2$ and the fact that if all edges of G would belong to bunches then we could assign any coloring to G and deal with it like $k = 1$.

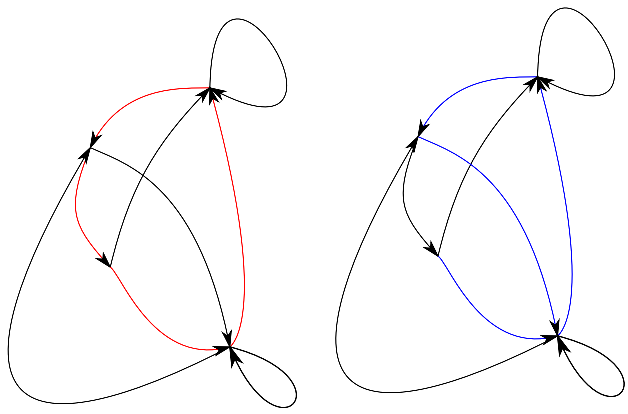

Figure 2: The graphs R (red) and R' (blue) as constructed.

□

The rest of the proof is ugly for guys who dislike dealing with 'pointers'.

Lemma 4.6. *Every AGW graphs without two incoming bunches has a spanning subgraph such that all of its vertices of maximal level belong to one non-trivial tree.*

Proof. For this proof we utilize severe case-by-case analysis. Denote by $C_1, \ldots, C_n$ the cycles of a spanning subgraph R with the trees $T_{i,1}, \ldots, T_{i,l_i}$ on the cycle C_i and $l_i \leq |C_i|$. To be more precise let R be a spanning subgraph for which the sum of the number of vertices in each cycle $\sum |C_i|$ is maximal.
Let us further name some important parts of R. Let T be a tree of maximal level $L = \boldsymbol{l}(T) > 0$, i.e. $\boldsymbol{l}(T) \geq \boldsymbol{l}(T_{i,j})$ for all i, j and C the corresponding cycle. In addition let $p \in T$ have the level L, the root vertex of T be denoted by r with C having a vertex c and an edge $c \to r$ and b be the vertex on the path $a \to \ldots \to b \to r$ which points to r. With G being strongly connected, there exists also an edge $a \to p$, which is not in R.

Throughout the proof we want to check, if we can create a new spanning subgraph satisfying the conditions by changing some of the edges of R. The edges which are subject to change are $a \to p$, $b \to r$ and $c \to r$. As L is the maximal level, it is sufficient to construct a tree with level $L+1$ (or greater) in one circle.

First we want to see what happens if we include $a \to p$ in our spanning subgraph R and delete the edge $a \to x \in R$, (which exists as a has out-degree one). If $a \notin T \cup C$ then T is enlarged to a tree with a level greater then L (as

$\boldsymbol{l}(a) = L+1$), so the new tree with root r includes every vertex with maximal level, else if $a \in T$ a new cycle (of length greater or equal to one) is added to R, which contradicts the construction of R as subgraph with maximal number of vertices in cycles. So we can assume in further considerations that $a \in C$. There is a new cycle $C' = a \to p \to \ldots \to b \to r \to \ldots \to a$. Note that if we exchange $a \to x$ with $a \to p$ in R, we remove the cycle C from R and add C' (without touching any other cycle). By construction of R, the number of vertices in C' is lesser or equal to the number of vertices in the cycle C. Both cycles have a common path from r to a, so we can conclude, that the path $a \to x \to \ldots \to c \to r$ (of length $L'+1$) in C is longer then or equally long as the path $a \to p \to \ldots \to b \to r$ of length $L+1$ in T, thus $L' \geq L$. If $L' > L$, then C' has a tree of length L', so by exchanging C with C' we have a new spanning subgraph with the desired properties. So we can assume that $L = L'$.

Now we want to examine the case when we replace $b \to r$ by another edge $b \to r'$, $r \neq r'$, (we consider the case that were is no such edge, i.e. that there is a bunch from b to r in the last case). Of course, if b has an outgoing edge to a vertex of another tree $T_{i,j}$ (for some i, j), which is not the root of the tree, we would end up having a single tree which contains every vertex of maximum length. If b has an edge to a vertex in a circle C_i, we can combine both methods of changing the outgoing edges of a and b to construct a new spanning subgraph. As previously showed, there is an edge $a \to x$ in C and an edge $a \to p$ in G. So if b has an outgoing edge to a vertex r' of C_i, we can change edges, such that C_i has a tree with the path $x \to \ldots \to c \to r \to \ldots \to a \to p \to \ldots \to b \to r'$, which contains a path of length $L+1$ ($a \to p \to \ldots \to b \to r$).

The case that b has an outgoing edge to a vertex in T contradicts the definition of R as seen above. So finnally we must consider whether b has an edge to a vertex r' in C. As we had seen previously the path $a \to x \to \ldots\ldots \to c \to p$ is of length $L+1$, we can use the same argumentation that either the path $a \to x \to \ldots r'$ is of length $L+1$ or we can find a new spanning subgraph which meet the requirements. But as these paths are both part of the cycle C, we see that $r = r'$, contradiction.

In the last case we can assume that were is a bunch from the vertex b to the vertex r. As were are no two incoming bunches in r, there is an edge $c \to r''$, $r \neq r''$. If $r'' \in T$, we could replace C by a new cycle of greater length, contradicting maximality of R. If $r'' \notin C \cup T$, we add a path of length greater then L to another cycle or tree, which would in return have all vertices of maximal level, so $r'' \in C$. There is a path from r'' to r in C, which leads to the observation, that there is a new circle C'' containing edges from C and the edge $r \to r''$. As $r \notin C''$ the shortest path (in C) from r to a vertex of C'' is of non-zero length, thus the tree of C'' which contains T has a maximal level greater then L, proving the claim. $\square$

The last part is now the final theorem.

Theorem 4.7. *The graph G always has a coloring with a stable couple.*

Proof. We see immediately that by Lemma 5.1 G has a stable couple, if it contains a vertex with two incoming bunches. In the other case, Lemma **??** ensures that G has a spanning subgraph R with all vertices of maximal (positive) level belonging to one tree of R. Let $\mathcal{C} = \bigcup C_i$ be the set of all vertices from cycles of R, (C_i is defined as in Lemma **??**). We want to color all edges in R with one color (for example $c \in \sigma$). It is not of importance how every other edge is colored. In regard of Lemma 4.6, there is a non-empty intersection between the set of vertices of maximal level N and some F-clique F. By reviewing 5.2, we observe that $|F \cap N| = 1$. The word $c^{L-1} \in \Sigma^*$ (with L as the level of the vertices of N) translates F to another F-clique F' of the same size. With L being the maximal level of all vertices it follows, that $F' \setminus \mathcal{C}$ contains exactly one vertex (i.e. $F' \setminus \mathcal{C} = (F \cap N)c^{L-1})$), as every vertex of $V \setminus N$ is mapped to $\mathcal{C}$. From this we see, that $|Nc^{L-1} \cap F'| = |F' \setminus \mathcal{C}| = 1$ and $|\mathcal{C} \cap F'| = |F'| - 1$.

Let $m = \mathrm{lcm}(|C_1|, \ldots, |C_n|)$ be the least common multiple of the lengths of all cycles, so for $v \in \mathcal{C}$ we have $vc^m = v$. We conclude, that for the F-clique $F'' := F'c^m$ holds $F'' \subseteq \mathcal{C}$ and $\mathcal{C} \cap F' = F' \cap F''$. It follows that F' and F'' have $|F'| - 1$ common vertices, which contradicts Lemma **??**, if the examined coloring has no stable couple. ☐

Corollary 4.8. *Any AGW graph is synchronizable.*

Proof. Let G be the smallest counter-example. Then the number of vertices is greater then one, as a graph with one vertex is obviously synchronizable. Then we see from Theorem **??** that there exists a stable couple, with that we have a non-trivial congruence class of more than one element. Therefore G has at least one further vertex then G/ρ, which in return has a synchronizing coloring and implies that G also has a synchronizing coloring. ☐

References

[1] A. N. Trahtman. The road coloring problem. *CoRR*, abs/0709.0099, 2007.

[2] Joel Friedman. On the road coloring problem. *Proc. Amer. Math. Soc*, 110:1133–1135, 1990.