Topology

Jakob Schneider

April 15, 2014

Contents

Definition 0.1. Let $X \in \mathbf{Top}$. A cover of X is a set $\mathcal{U} \subseteq \mathrm{Sub}(X)$ having the following properties

(I) U covers X, i.e. $\bigvee \mathcal{U} = X$.

(II) For each $x \in X$ there exists an element $U \in \mathcal{U}$ such that $U \in \mathrm{N}|_x$, or alternatively, for all $x \in X$ we have $\mathrm{N}|_x X \cap \mathcal{U} \neq \emptyset$.

Definition 0.2. A space X is called *compact* if for any cover $\mathcal{U}$ of X there is a finite subcover $\mathcal{F} \in \mathrm{Sub}_{\mathrm{fin}}(\mathcal{U})$.

Remark 1. An equivalent formulation is that any *open* cover has the above property.

Definition 0.3. Let X be a topological space. Define $\mathrm{Sub}_{\mathrm{cpt}}(X)$ as the topological space having as underlying set the compact subspaces of X and as topology the sets $\langle\{\mathrm{Sub}(U) \cap \mathrm{Sub}_{\mathrm{cpt}}(X) : U \in \tau\}\rangle$ where τ is the topology of X.

Remark 2. In the above system $\{\mathrm{Sub}(U) \cap \mathrm{Sub}_{\mathrm{cpt}}(X) : U \in \tau\}$ is a basis of the topology since it is obviously closed under finite intersections.

FALSE?

Lemma 0.1. *Let $\mathcal{X}$ be locally compact, then* $\mathrm{Sub}_{\mathrm{cpt}} X$ *is locally compact.*

Proof. Let $C \in \mathrm{Sub}_{\mathrm{cpt}}(\mathcal{X})$ be compact. Choose compact neighboorhoods $C_c \in \mathrm{N}|_c \mathcal{X}$ for each point of $c \in C$. Then by compactness of C there is a finite subcover $\{C_c : c \in F\}$ ($F \in \mathrm{Sub}_{\mathrm{fin}}(C)$). Define $D := \bigcup_{c \in F} C_c$ — then D is compact. We claim that $\mathrm{Sub}_{\mathrm{cpt}}(D)$ is a compact neighboorhood of C in $\mathrm{Sub}_{\mathrm{cpt}}(\mathcal{X})$. It is clear that $\mathrm{Sub}_{\mathrm{cpt}}(D) \in \mathrm{N}|_C \mathrm{Sub}_{\mathrm{cpt}}(\mathcal{X})$ (by definition of the topology for $\mathrm{Sub}_{\mathrm{cpt}}(\mathcal{X})$). We are left to show that it is compact. To do this, choose an open cover $\mathcal{U}$ of $\mathrm{Sub}_{\mathrm{cpt}}(D)$. By definition, for each $U \in \mathcal{U}$ there exist open sets $V \in \tau$ such that $U = \mathrm{Sub}_{\mathrm{cpt}}(V)$. Collect these V in set $\mathcal{V} \subseteq \tau$. As $\mathcal{U}$ is an open cover of $\mathrm{Sub}_{\mathrm{cpt}}(D)$, for each compact subset $F \subseteq D$ there must exist a set $V \in \mathcal{V}$ such that $F \subseteq V$. As the one-element subset of D are compact, since D is compact and $\mathcal{X}$ is HAUSDORFF we have that $\mathcal{V}$ is an open cover of D. ... □

Lemma 0.2. *Let X be a locally compact space, then X is a* BAIRE *space.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be dense open subspaces of X and V be an open subspace of X. Then we define $V_0 := V$ and choose inductively V_n as a non-empty relatively compact open subspace of X such that $V_n \subseteq \mathrm{cl}\, V_{n-1} \cap U_{n-1}$ for $n \geq 1$. This is possible since U_n is dense in X for all n. We thus obtain $V := \bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \mathrm{cl}\, V_n \neq \emptyset$ as a compact set such that $V \subseteq \bigcap_{n \in \mathbb{N}} U_n \cap V$. This shows that $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X. □

Lemma 0.3. *The following are equivalent*

(I) Axiom of choice.

(II) ZORN*'s lemma.*

(III) Wellordering theorem.

Proof.

first to second Let P be preordered and for any subset $Q \subseteq P$ define $Q^\vee$ as the set of upper bounds. We then have by axiom of choice a function $F : \{Q^\vee : Q \subseteq P\} \setminus \{\emptyset\} \to P$ with $F(X) \in X$.

$\square$