How to draw masses from the Chabrier IMF:

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The cumulative distribution of a Chabrier IMF is

$$N\left(<M\right) \quad = \quad A\int_{M_{\min}}^{M}\frac{1}{m}\exp\left[-\left(\frac{\log m-\log\mu}{\sqrt{2}\sigma}\right)^{2}\right]\mathrm{d}m$$

Where we typically set $\mu=0.08$ and $\sigma=0.69$ and $A=0.15$. Let's make a substitution. $x=\left(\log m-\log\mu\right)/\sqrt{2}\sigma$ and $\mathrm{d}m=\sqrt{2}\ln\left(10\right)\sigma m\,\mathrm{d}x$. Then:

$$N_{\mathrm{cum}} \quad = \quad \sqrt{2}\ln\left(10\right)\sigma\times A\int_{\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma}^{\left(\log M-\log\mu\right)/\sqrt{2}\sigma}\exp\left(-x^{2}\right)\mathrm{d}x$$

We can express this analytically using the error function:

$$\mathrm{erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{z}\exp\left(-t^{2}\right)\mathrm{d}t$$

We can write

$$\begin{aligned}\int_{\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma}^{\left(\log M-\log\mu\right)/\sqrt{2}\sigma}\exp\left(-x^{2}\right)\mathrm{d}x &= \int_{0}^{\left(\log M-\log\mu\right)/\sqrt{2}\sigma}\exp\left(-x^{2}\right)\mathrm{d}x-\int_{0}^{\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma}\exp\left(-x^{2}\right)\mathrm{d}x\\ &= \frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\left[\left(\log\mathrm{M}-\log\mu\right)/\sqrt{2}\sigma\right]-\mathrm{erf}\left[\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma\right]\right)\end{aligned}$$

So we have

$$N_{\mathrm{cum}}=\frac{A}{2}\times\sqrt{2\pi}\ln\left(10\right)\sigma\left(\mathrm{erf}\left[\left(\log\mathrm{M}-\log\mu\right)/\sqrt{2}\sigma\right]-\mathrm{erf}\left[\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma\right]\right)$$

Now we want to invert this. Let's solve for M in terms of N_{cum}.

$$M \quad = \quad 10^{\left(\sqrt{2}\sigma\mathrm{erfinv}\left[N_{m}\left(\mathrm{erf}\left[\left(\log\mathrm{M_{max}}-\log\mu\right)/\sqrt{2}\sigma\right]-\mathrm{erf}\left[\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma\right]\right)+\mathrm{erf}\left[\left(\log M_{\min}-\log\mu\right)/\sqrt{2}\sigma\right]\right]+\log\mu\right)}$$

So if we draw masses from $\mathcal{U}(0,1)$ and pass them through the above function for M, they'll be drawn from a Chabrier IMF.